COTORSION THEORIES
AND $\mathcal{F}$-COVERS

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ABSTRACT

In this thesis we intend to investigate the Flat Cover Conjecture of Enochs, in different contexts and give some applications of the conjecture. We introduce Torsion theories and redefine the injective projective and flat classes as they exist taken over a given torsion theory. Finally, we define a cotorsion theory $(\mathcal{F},\mathcal{C})$ in order to ascertain whether a cotorsion theory duality can be defined between the existence of $\mathcal{F}$-covers and $\mathcal{C}$-envelopes.

ÖZET

Bu tezde, yeni ispatı düşünerek, Enochs'ın 'Düz Örtü Sav'ın araştırılması amaçlandı ve buna yönelik uygulamalar ele alındı. Bir torsiyon kuramı tanımlanarak, injektif, projektif ve düz sınıflar, verilmiş bir torsiyon kuramı üzerinden tekrar tanımlandı. Son olarak, bir kotorsiyon kuramı ikileminin $\mathcal{F}$–örtüleri ve $\mathcal{C}$-zarflarının varlığı arasında tanımlanıp tanımlanamayacağını belirlemek için, bir kotorsiyon kuramını $(\mathcal{F}, \mathcal{C})$'ı tanımlandı.

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CHAPTER ONE
INTRODUCTION

The motivation for this thesis was the search for a more elegant duality statement. Classical theorems of Baer give us the existence of injective envelopes (hulls) for all modules over any ring. However, classically the 'dual' of the injective hull is seen as the projective cover and whereas injective envelopes always exist it can be easily seen that projective covers exist for all modules only when taken over a perfect ring. This lead to our investigation of Enochs' conjecture and the existence question for flat covers as a larger class containing the projective modules. However, here we see that the dual of flat covers has to be a larger class, that of the cotorsion modules, containing the injective modules. This lead us to a fuller investigation of the notion of 'duality' using the concept of a cotorsion theory as a basis.

In chapter one we introduce the basic notation and definitions that will be used together with a section on homological algebra which forms a vital tool for the work. We conclude the chapter with a discourse on the properties of cotorsion groups and how these properties translate over to cotorsion modules.

Chapter two introduces the recently proved Enochs' Conjecture. We explore the various recent developments in the research on new full and partial proofs of the conjecture and conclude, using a theorem of Xu both the existence and 'duality' of the flat covers and cotorsion envelopes.

Chapter three is an applications section where we explore the effects of the conjecture on certain modules. What happens when a module has two flat (pre)covers? When do two modules have the same flat cover?

The 'duality' that we would like to establish is considered in chapter four. In this theoretical duality there are two classes $\mathcal{F}$ and $\mathcal{C}$, and the existence of an $\mathcal{F}$-cover would be necessary and sufficient to conclude the existence of a $\mathcal{C}$-envelope. The main results of chapter four go some way towards providing this duality by the introduction of the cotorsion theory and the concepts of having 'enough injectives and projectives'.

1.1 Preliminaries

Throughout this work all rings are assumed to be associative right coherent rings with identities, and all modules are unitary. Terminology is primarily that of (Anderson & Fuller 1992), with more recent terms taken from Xu 1996.

A linear map (homomorphism) $f: M \to N$ is such that $f(ax+by) = af(x) + bf(y)$ $\forall a,b \in R$ and $x, y \in M$. A module P is called *projective* if for epic $M \xrightarrow{f} N$ and linear map $P \xrightarrow{g} N$, g can be factored through f. That is there exists a linear map $h: P \to M$ such that $f \circ h = g$ and the associated diagram (1.1.1) is commutative.

$$\begin{array}{ccc} & & P \\ & \overset{\exists h}{\swarrow} & \downarrow g \\ M & \xrightarrow{f} & N \to 0 \end{array} \qquad (1.1.1)$$

Dually a module E is taken to be *injective* if for monic $M \xrightarrow{f} N$ and linear map $M \xrightarrow{g} E$ there exists linear map $h: N \to E$ commutatively completing the associated diagram (1.1.2).

$$\begin{array}{ccc} 0 \to M & \xrightarrow{f} & N \\ g \downarrow & \overset{\exists h}{\swarrow} & \\ E & & \end{array} \qquad (1.1.2)$$

Put more succinctly for any projective module P, the functor $Hom(P, -)$ preserves exact sequences and similarly for any injective module E, the functor $Hom(-,E)$ preserves exact sequences. A module F is called *flat* if the functor $Tor(F, -)$ preserves exact sequences. The classes of injective, projective and flat modules are denoted by $\mathcal{E}$, $\mathcal{P}$ and $\mathcal{F}$ respectively.

1.2 $\mathcal{X}$-envelopes and $\mathcal{X}$-covers

By an injective (hull) envelope of a module M it is classically understood that we mean an injective module E which contains the module M essentially. That is, there is an exact $0 \rightarrow M \xrightarrow{i} E$ such that $\mathrm{Im}(i) \cap L$ for any submodule L of E, implies that $L = 0$.

In a similar vein, a projective cover of a module M is understood to be a module P such that there is an epimorphism $P \xrightarrow{\pi} M \rightarrow 0$ where $Ker(\pi)$ is superfluous in P. That is for any submodule L in P $Ker(\pi) + L = P$ implies that $L = P$.

For our part however, the notions of an $\mathcal{X}$-(pre)envelope and an $\mathcal{X}$-(pre)cover of a module M for a given class $\mathcal{X}$ of modules over the hereditary ring R are defined according to Xu 1996 for generality. Using this definition it is not difficult to show that the classical definitions of injective hulls (envelopes) and projective covers agree with the generalized definition. The reader is referred to Xu for the proof.

Let $\mathcal{X}$ be a class of left R-modules closed under isomorphisms, finite direct sums and direct summands.

Definition 1.2.1 For a left R-module M, a module X in $\mathcal{X}$ is called an $\mathcal{X}$-envelope of M if there is a linear map $\phi : M \rightarrow X$ such that the following hold:

(1) For any linear map $\phi' : M \rightarrow X'$ with $X' \in \mathcal{X}$, there exists a linear map $\varphi : X \rightarrow X'$ such that $\phi' = \varphi \circ \phi$.

(2) If an endomorphism $\varphi : X \to X$ is such that $\phi = \varphi \circ \phi$ then φ is an automorphism.

In the case of (1) holding but not necessarily (2) then X is a preenvelope.

$$\begin{array}{ccc} M & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle \phi'} & \swarrow{\scriptstyle \varphi} & \\ X' & & \end{array} \qquad (1.2.1)$$

Definition 1.2.2 For a left R-module M, a module X in $\mathcal{X}$ is called an $\mathcal{X}$-cover of M if there is a linear map $\phi : X \to M$ such that the following hold.

(1) For any linear map $\phi' : X' \to M$ with $X' \in \mathcal{X}$, there exists a linear map $\varphi : X' \to X$ such that $\phi' = \phi \circ \varphi$.

(2) If an endomorphism $\varphi : X \to X$ is such that $\phi = \phi \circ \varphi$ then φ is an automorphism.

In the case of (1) holding but not necessarily (2) then X is a precover.

$$\begin{array}{ccc} & & X' \\ & \swarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle \phi'} \\ X & \xrightarrow{\phi} & M \end{array} \qquad (1.2.2)$$

Many distinguished authorities have considered $\mathcal{X}$-covers and $\mathcal{X}$-envelopes where $\mathcal{X}$ is given as a certain class, for example torsion covers or injective envelopes.

1.3 Enoch's Conjecture

After finding that every module has a torsion-free cover in his paper "Torsion free covering modules 1963, Enochs turned his attention to the existence of flat covers and in 1985 posed his conjecture:

"Every module has a flat cover over a ring *R*"

This conjecture remained open for almost fifteen years until it was proved independently by El Bashir & Bican (preprint August 1999) and Trlifaj (preprint August 1999). The proof is of particular interest because it considers an arbitrary class $\mathcal{X}$ and states that $\mathcal{X}$-covers exist for all modules if the class $\mathcal{X}$ satisfies certain criterion which they then proceed to show that flat modules satisfy. The general nature of this proof is of interest when taken to the last chapter on cotorsion theories. The details of the proofs will be considered more deeply in chapter three. As a necessary tool for the proofs of these theorems the functor *Ext* should be considered, functors are essentially elements of Homology theory but have far reaching consequences in other branches such as this. Readers are referred to (Cartan Eilenberg 1999) for a classical introduction to the subject.

1.4 The Functor *Ext*

Taking an injective resolution (1.4.1) of an R-module A as;

$$0 \to A \to E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} E_2 \to \cdots \qquad (1.4.1)$$

Ext can be defined as follows:

$$Ext(C,A) = \frac{Ker Hom(C,d_1)}{\operatorname{Im} Hom(C,d_0)} \qquad (1.4.2)$$

the construction is independent of the choice of injective resolution. The same functor is achieved using a projective resolution of C.

The following is a list of some properties of *Ext*

(1) Obviously if A is injective or alternatively C is projective then $Ext(C,A) = 0$.

This can easily be seen since *Ext* is independent of the choice of injective or projective resolution and therefore using $0 \to A \to A \to 0$ or $0 \to C \to C \to 0$ as respectively the injective and projective resolutions $Ext(C,A) = 0$ is established.

(2) If $Ext(C,A) = 0$, then every extension $0 \to A \to B \to C \to 0$ is splitting.

The event of $Ext(C,A)$ repairs the loss of right exactness in the functor $Hom(C,A)$ via a homomorphism $\partial : Hom(A,A) \to Ext(C,A)$ and where $Ext(C,A)$ = 0, the functor *Hom* applied on the above short exact sequence gives a short exact sequence also. Then $Hom(B,A) \to Hom(A,A)$ is an epimorphism and the extension is splitting.

(3) If $Ext(C,A) = 0$ for all R-modules A then C is projective (since it necessitates a projective resolution of the above form).

(4) If $Ext(C,A) = 0$ for all R-modules C then A is injective (since it necessitates an injective resolution of the above form).

Eklov and Trlifaj (preprint 1999) held that it is possible to construct the module A from a given module C in order to force $Ext(C,A) = 0$. This is an important result since, as their paper establishes, using this several previously open questions on splitters (R-modules C for which $Ext(C,C) = 0$), cotorsion theories and saturated rings can be answered. Further to this however, as will be seen in the next section we use the definition $Ext(F,C) = 0$ for all F flat to mean that C is cotorsion therefore being able to force $Ext(F,C) = 0$ has obvious advantages.

1.5 Cotorsion Groups and Modules

Definition 1.5.1 Given the classes $\mathcal{X}$ and $\mathcal{Y}$ of right R-modules, define the associated classes:

$$\mathcal{X}^{\perp} = \{Y \mid Ext(X,Y) = 0 \quad \forall X \in \mathcal{X}\} = Ker(Ext(X,-)) \tag{1.5.1}$$

$$^{\perp}\mathcal{Y} = \{X \mid Ext(X,Y) = 0 \quad \forall Y \in \mathcal{Y}\} = Ker(Ext(-,Y)) \tag{1.5.2}$$

as respectively, the right orthogonal class of $\mathcal{F}$ and the left orthogonal class of $\mathcal{C}$.

As will be seen, cotorsion modules form an essential part of the study of flat covers. A module is called *cotorsion* if it is in the class orthogonal to the class of flat modules, that is C is cotorsion if $Ext(F,C)=0$ for all F flat. The class of cotorsion modules is therefore denoted $\mathcal{F}^{\perp}$ or simply by $\mathcal{C}$. It should be noted that the earlier definition by (Matlis 1972) in his extensive work on torsion free modules over an integral domain is slightly different in that it includes the condition that $Hom(Q,C)=0$ where Q is the field of fractions of the ring R. This means that C must be, along our definition, not only cotorsion but reduced as well. For consistency we generalize Matlis' definition and call a module reduced cotorsion if it is cotorsion and there are no non zero divisible submodules of C.

For a cotorsion group we can simplify our definition somewhat. Flat groups are simply the torsion-free groups so the class of cotorsion groups is the class orthogonal to that of the torsion-free groups. Now since every torsion-free group J can be embedded as a subgroup in a direct sum of copies of Q thus: $0 \to J \to \oplus Q$, by the exactness of $Ext(\oplus Q, G) \cong \prod Ext(Q,G) \to Ext(J,G) \to 0$, we see that $Ext(Q,G)=0$ guarantees G is cotorsion. So we can define the cotorsion class as that class right orthogonal to Q.

In Fuchs' 1970 version of "Infinite Abelian Groups" there is a section on cotorsion groups since they were first defined within this context. The class of cotorsion modules is closed under extensions, finite direct sums and direct summands. Many of the results of Fuchs can be applied to modules over rings with homological dimension less than or equal to one and some can even be applied to modules over a general ring. Let us continue in the vein of Fuchs with this list of "more or less elementary results" on cotorsion modules:

Property 1.5.2 The epimorphic image of a cotorsion group is cotorsion.

Take $0 \to K \to G \to H \to 0$ with G cotorsion and apply *Ext*(*F*,-) to get $0 = Ext(F,G) \to Ext(F,H) \to Ext^2(F,K) = 0$. This holds of a module over any ring of homological dimension less than or equal to one but not in the general case, since the argument relies on having $Ext^2(F,K) = 0$.

This property will be considered in more detail in chapter four under the heading of cotorsion theories for a general cotorsion theory. In particular we will see that this holds for cotorsion modules if the kernel *K* is also taken to be cotorsion.

Property 1.5.3 Let *G* a be reduced cotorsion group, for $K \subseteq G$ to be cotorsion it is necessary and sufficient for *G/K* to be reduced.

Take the short exact sequence $0 \to K \to G \to G/K \to 0$. From (Fuchs 1970) this holds for groups because in this case given the fact that cotorsion groups can be classified by $Ext(Q,C) = 0$ as well, we have;

$$0 = Hom(Q,G) \to Hom(Q,G/K) \to Ext(Q,K) \to Ext(Q,G) = 0 \qquad (1.5.3)$$

giving the isomorphism $Ext(Q,K) \cong Hom(Q,G/K)$ then the former is zero whenever the latter vanishes that is whenever *G/K* is reduced.

Property 1.5.4 If G is a reduced cotorsion group and $\theta : G \to G$ is an endomorphism then both $Ker\theta$ and $\operatorname{Im}\theta$ are cotorsion.

This follows from the above for groups.

Property 1.5.5 If K is a submodule of G such that both K and G/K are cotorsion then G is also cotorsion.

This holds in the general case, see (Xu 1995).

Property 1.5.6 If K is a submodule of G such that both K and G are cotorsion then G/K is also cotorsion.

This holds in the general case, see (Xu 1995).

Property 1.5.7 A direct product $\prod_{i \in I} G_i$ is cotorsion if and only if every summand G_i is cotorsion.

This is an immediate consequence of the isomorphism $Ext\left(F, \prod_{i \in I} G_i\right) \cong \prod_{i \in I} Ext(F, G_i)$.

Property 1.5.8 The inverse limit of a reduced cotorsion group is reduced cotorsion, follows from (6) and (2).

Property 1.5.9 If G is a reduced cotorsion group then $Hom(A,G)$ is cotorsion for any A.

Given a free resolution of A, $0 \to H \to F \to A \to 0$ and applying $Hom(\text{-},G)$ we get, $0 \to Hom(A,G) \to Hom(F,G) \to Hom(H,G)$ the middle group is a product of copies of G hence reduced cotorsion and the final group being reduced by the above property 1.5.3 $Hom(A,G)$ is cotorsion.

From properties 1.5.5 and 1.5.6 we have the following lemma of Xu.

Lemma 1.5.10 (Xu 1996) Let $O \to C_1 \to C_2 \to C_3 \to O$ be a short exact sequence, then:

a) $C_1, C_2 \in \mathcal{F}^{\perp} \Rightarrow C_3 \in \mathcal{F}^{\perp}$

b) $C_1, C_3 \in \mathcal{F}^{\perp} \Rightarrow C_2 \in \mathcal{F}^{\perp}$

What one might consider to be the natural part c to this lemma, that is,

c) $C_2, C_3 \in \mathcal{F}^{\perp} \Rightarrow C_1 \in \mathcal{F}^{\perp}$ (1.5.3)

is not necessarily true for cotorsion modules over a general ring R, but the class of rings for which this property is satisfied is not empty as can be shown with the obvious example of modules over a semisimple ring, the exact sequence would split and applying part a) would give the required result.

CHAPTER TWO
σ-PURITY AND ENOCHS' CONJECTURE WITH APPLICATIONS

"Every module over an arbitrary ring has a flat cover"

This fifteen-year-old problem has recently been solved independently by El Bashir & Bican, and by Trlifaj & Enochs. This chapter considers a generalization and overview of the methods that can be used to affirm the conjecture. The concept of σ-purity is developed since in order to prove that over any ring, every module has a flat cover it is first necessary to generalize the concept and prove that over any ring given a σ-purity there exists an $\mathcal{F}_\sigma$-cover. Further to that we show that if the class $\mathcal{F}_\sigma$ is the usual flat modules over the usual purity then flat covers exist. As a corollary, we see that if $\mathcal{F}_\sigma = \mathcal{RD}_\sigma$, the class of relative divisible modules, then there always exists a $\mathcal{RD}_\sigma$-cover for any module.

In the later sections we are mainly concerned with applications of Enochs' conjecture. Now that we have an affirmative proof, it can be put to use in certain situations. Some questions come to mind, the two most obvious being: What happens when a module has two (different) flat (σ-coprojective) covers?; and when might it occur that two non-isomorphic modules have the same flat (σ -coprojective) precovers?

2.1 σ-purity

Definition 2.1.1 Let $\mathcal{A}$ be a class of modules, then the class of all short exact sequences for which all elements of $\mathcal{A}$ are projective is referred to as the *purity projectively generated by* $\mathcal{A}$, and is denoted by σ.

Lemma 2.1.2 In Abelian groups the purity σ is the purity generated by the class of finitely generated groups.

Proof For the usual purity in an Abelian group we usually say that a subgroup H of G is pure if $nH = H \cap nG$. It is always true that $nH \subseteq H \cap nG$. Take $ng \in H \cap nG$ with $g \in G$, and consider the diagram (2.1.1) with short exact rows:

$$\begin{array}{ccccccccc} 0 & \to & nZ & \xrightarrow{\alpha} & Z & \to & Z/nZ & \to & 0 \\ & & \downarrow{\scriptstyle\beta} & \swarrow{\scriptstyle\gamma'} & \downarrow{\scriptstyle\delta} & \swarrow{\scriptstyle\gamma} & \downarrow{\scriptstyle\varepsilon} & & \\ 0 & \to & H & \xrightarrow{\alpha'} & G & \longrightarrow & L & \to & 0 \end{array} \tag{2.1.1}$$

Here $\delta_g : Z \to G$ is the multiplicative Z-homomorphism defined by $z \mapsto zg$. Since Z/nZ is finitely generated it is projective with respect to the exact sequence by hypothesis so γ exists, and by the Homotopy Lemma (or Serpent Lemma) there exists $\gamma' : Z \to H$ then $\beta(n) = \gamma' \circ \alpha(n) = n.\gamma'(1) = n.h$ for some $h \in H$. However $ng = \delta_g \circ \alpha(n) = \alpha' \circ \beta(n) = \beta(n) = nh$ so $ng = nh$ and $ng \in nH$ which gives the desired result $H \cap nG = nH$. □

Lemma 2.1.3 In R-mod the classical Cohn σ-purity is generated by the class of all finitely presented modules.

Remark on the class of finitely presented modules $\mathcal{FP}$. If $P_0 \to P_1 \to F \to 0$ is a finite presentation of F then there exists a finitely presented R-mod F' such that $P^0 \to P^1 \to F' \to 0$ is a finite presentation where $P^i = Hom(P_i, R)$. Considering that for any finitely presented M, $(M')' = M$, there exists an N in $\mathcal{FP}$ such that $M = N'$ for all M in $\mathcal{FP}$. In this case (Sklyarenko 1978) there exists a monomorphism $Ext^1(M,A) \to N \otimes A$ and an epimorphism $Hom(N,A) \to Tor_1(A,M)$.

Proof of Lemma 2.1.3 Cohn's purity is that $IA = A \cap IB$ for all right ideals I in R. That this purity is equivalent to the condition that $0 \to R/I \otimes A \xrightarrow{\alpha} R/I \otimes B$ be exact is easily seen from diagram (2.2.2) which comes from the fact that $IA = \mathrm{Im}(\gamma)$ where $\gamma : I \otimes A \to R \otimes A = A$

$$\begin{array}{ccccccccc} 0 & \to & IA & \to & A & \xrightarrow{\beta} & R/I \otimes A & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow & \alpha & \downarrow & & \\ 0 & \to & IB & \to & B & \xrightarrow{\beta'} & R/I \otimes B & \to & 0 \end{array} \qquad (2.2.2)$$

Taking $x \in R/I \otimes A$, $\exists y \in A$ such that $\beta(y) = x$. Then $\alpha \circ \beta(y) = \alpha(x) = \beta' \circ \alpha'(y) = \beta'(y)$. So $x \in Ker\alpha \Leftrightarrow y \in Ker\beta'$ i.e. α is monic $\Leftrightarrow y \in A \cap IB$, and also $y \in Ker\beta$ i.e. α is monic $\Leftrightarrow y \in IA$.

In this case obviously $N \otimes A \to N \otimes B$ is monic for all N in $\mathcal{FP}$ and we have the diagram (2.2.3) by the above remark.

$$\begin{array}{ccc} Ext^1(M,A) & \xrightarrow{\alpha} & Ext^1(M,B) \\ \downarrow & monic & \downarrow \quad monic \\ N \otimes A & \xrightarrow{monic} & N \otimes B \end{array} \qquad (2.2.3)$$

So that $Hom(M,C) \to Ext(M,A)$ can be factorized through zero, giving that M is projective with respect to all short exact sequences $0 \to A \to B \to C \to 0$. Conversely if $Hom(M,C) \to Ext(M,A)$ is exact then by the remark and a similar argument to the above it can be shown that $N \otimes A \to N \otimes B$ is monic. □

Definition 2.1.4 For a given purity σ, the *class of σ-coprojective modules* is the class denoted $\mathcal{F}_\sigma$ of modules for which whenever F is in $\mathcal{F}_\sigma$ the short exact sequence $0 \to M \to N \to F \to 0$ remains in σ.

To continue the above example in Lemma 2.1.2, the class $\mathcal{F}_\sigma$ is the class of torsion-free groups. (To see this remember that a subgroup L of M is pure if and only if $nx = l$ solvable in M means that it is solvable in L so the factor group M/L is obviously torsion-free.) Similarly, in Lemma 2.1.3 $\mathcal{F}_\sigma$ is the class of flat modules.

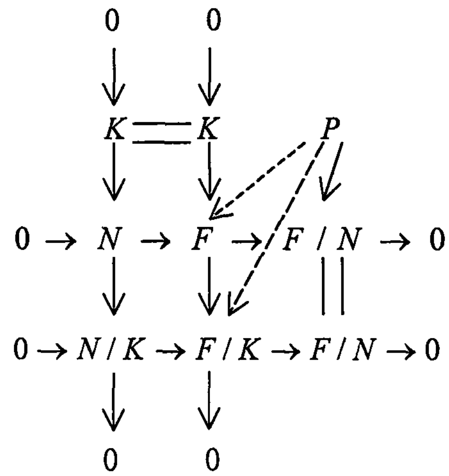
If a class is σ-coprojective then it contains the projective class $\mathcal{P}$ since then the short exact sequence would split. From a generalization of the Flat Test Lemma it is not difficult to show that it is closed under extensions, and for P in $\mathcal{F}_\sigma$ and short exact sequence $0 \to K \to P \to M \to 0$ then the factor module M remains in $\mathcal{F}_\sigma$ if and only if the exact sequence is in σ.

2.2 Proofs of the Conjecture

The following section examines the proof of the conjecture due to Bican and El Bashir 1999, The original proof can be found in their paper, and though the proof here is slightly different, the underlying theme has been kept consistent. We assume Theorem B&EB found in the appendix and begin with a note on σ-pure submodules of factor modules.

Lemma 2.2.1 Let σ be a purity projectively generated by a class $\mathcal{A}$ of modules. If N/K is a σ-pure submodule of F/K and K is a σ-pure submodule of F, then N is a σ-pure submodule of F.

Proof Consider diagram (2.2.1) below:



(2.2.1)

Given any P in $\mathcal{A}$ and homomorphism $P \to F/N$ by the σ-purity of N/K in F/K the homomorphism $P \to F/K$ exists and by the σ-purity of K in F the homomorphism $P \to F$ exists. Therefore for any P in $\mathcal{A}$ and homomorphism $P \to F/N$, P is projective with respect to the short exact sequence $0 \to N \to F \to F/N \to 0$ which means that N is σ-pure in F. □

Theorem 2.2.2 (El Bashir) Let σ be a purity projectively generated by the class of finitely presented modules over a ring. Then every module has a σ-coprojective cover.

Proof By Xu 1996 it is sufficient to prove that every module has a σ-coprojective precover since σ-coprojective modules are closed under direct limits. Let M be an arbitrary R-module, and take a linear map $F \xrightarrow{f} M$ where F is an arbitrary σ-coprojective module. There exists a maximal σ-pure submodule K of F such that K is in the kernel of f. Setting $|M| = \kappa$, $|F/K| < \kappa$, since otherwise i.e. if $|F/K| \geq \kappa$ applying Theorem E&EB we have, $\left|{}^{F/K}\!\big/_{Kerf/K}\right| = \left|{}^{F}\!\big/_{Kerf}\right| < \lambda$ which means that there is a N/K submodule of $Kerf/K$ σ-pure in F/K. This implies by Lemma 2.2.1 that N is a σ-pure submodule of F which is larger than the maximal K. Then there exists an isomorphism $\psi : F/K \to G$ for some G in G_σ (defined as the subclass of the σ-

coprojective modules where cardinalities are less than the λ inducing $|M| = \kappa$). Consider diagram (2.2.2) below:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & F & \xrightarrow{\pi} & F/K & \to & 0 \\ & & & & g\downarrow & \searrow^{f} \ \swarrow^{\psi} & \downarrow \bar{f} & & \\ 0 & \to & L & \to & G & \xrightarrow{\varphi} & M & & \end{array} \qquad (2.2.2)$$

Where $\pi : F \to F/K$ is the canonical homomorphism and f induces $\bar{f} : F/K \to M$ with $\bar{f} \circ \pi = f$.

So there exists a G in G_σ isomorphic to F/K for which there exists a linear map along $G \xrightarrow{\varphi} M$ by $\varphi = \bar{f} \circ \psi^{-1}$ whereby, for any σ-coprojective F for which there exists linear map $F \xrightarrow{f} M$, $\varphi \circ g = \bar{f} \circ \psi^{-1} \circ g = \bar{f} \circ \psi^{-1} \circ \psi \circ \pi = \bar{f} \circ \pi = f$. In this way G forms a σ-coprojective precover of M. □

As has been pointed out in the paper, classical purity is projectively generated by finitely presented modules, and in this case $\mathcal{F}_\sigma$ is the class of flat modules therefore according to Theorem 2.2.2 every module has a flat cover. Similarly, relative divisibility (that is, a submodule K of L is relatively divisible if $rK = K \cap rL$ for all $r \in R$) is projectively generated by the class of cyclically presented modules, so every module has a $\mathcal{RD}_\sigma$-cover.

Corollary 2.2.4 Over any ring every module has a flat cover.

Proof Since the class of flat modules, is closed with respect to direct limits, apply Xu 1996 to show that the existence of a precover implies the existence of a cover, then apply Theorem 2.3.2 to show that a precover exists for any module. □

Corollary 2.2.5 Over any ring every module has an $\mathcal{RD}_\sigma$-cover.

The approach of Enochs and subsequently that of the synopsis by Trlifaj is different, the above approach involved finding a set for each R-module M such that a σ-coprojective module in this set had the required properties for a σ-coprojective precover for M, whereas the following theorem considers the question as a whole. Using an interesting cardinality argument (see appendix), they prove that the *Ker*(*Tor*(-,*N*)) is a cover class for any left R-module N and considering the Flat Test Lemma this coincides with the class of all flat right R-modules and so the flat cover conjecture is seen to hold.

Lemma 2.2.6 Let N be the direct sum of a representative set of all cyclic left R-modules, then *Ker*(*Tor*(-,*N*)) coincides with the class of all flat R-modules $\mathcal{F}$.

Proof Let $F \in$ *Ker*(*Tor*(-,*N*)), then *Tor*(F,*N*) $= F \otimes N = 0$, so that the Z-epimorphism $\mu_N : F \otimes N \to FN$ via $\mu_N(f \otimes n) \mapsto fn$ is the zero epimorphism and therefore monic. Since N is a finitely generated ideal in R, we can apply (c)$\Leftrightarrow$(a) of the Flat Test Lemma (see Anderson Fuller p227).

Lemma 2.2.7 (Trlifaj) *Ker*(*Tor*(-,*N*)) is a covering class for any left R-module N.

Proof Let κ be as in Theorem T1 (see appendix) taking $B = N$. Let H be the direct sum of the elements of a representative set of the class $\{A \mid card(A) \leq \kappa \wedge Tor(A,N) = 0\}$, clearly $K^{\perp} \subseteq H^{\perp}$. For the converse take $C \in H^{\perp}$, by the above any A is $A = \bigcup_{\beta \leq \sigma} A_{\beta}$ then *Ext*(*A*,*C*) = 0 (see Eklov and Trlifaj 1999 or Appendix Theorem E&T). So $K \subseteq {}^{\perp}C$ whence $C \in K^{\perp}$, since *Tor* commutes with direct limits, K is closed under direct limits and using Corollary 4.4 K is seen to be a covering class.

Corollary 2.2.8 (Trlifaj) Every module over any ring has a flat cover.

The proof of the corollary is a direct result of lemmas 2.3.6 and 2.3.7. In addition to this proof of the flat conjecture Trlifaj was able to obtain results for pure-injective modules significantly for *dual modules*.

Definition 2.2.9 A right R-module A, is called a *dual module* if there exists a ring S, an R-S-bimodule B and an injective cogenerator Q for all right S-modules, such that $M \cong Hom(B,Q)$ as right R-modules.

Theorem 4.4.1 under the heading of Cotorsion Theories relates the fact that $M^{\perp}$ is a preenvelope class for any module *M*. It is however, still an open question as to whether ${}^{\perp}M$ is a precover class for any module *M*. This question has at least been solved in the case of dual modules.

Theorem 2.2.10 (Trlifaj) ${}^{\perp}M$ is a precover class for any dual right R-module *M*.

Proof By definition $M \cong Hom(B,Q)$ for an R-S-bimodule B and an injective cogenerator *Q*. By (Cartan &Eilenberg 1956) ${}^{\perp}M = Ker(Tor(-,N))$, so ${}^{\perp}M$ is a covering class by lemma 2.3.7.

In order to go further for pure-injective modules Trlifaj uses the analogue of his Theorem T1 see appendix Theorem T2.

Corollary 2.2.11 (Trlifaj) Let R be a ring and C be a pure-injective right R-module of injective dimension ≤ 1, then ${}^{\perp}C$ is a precover class.

Of course this means that if *R* is right hereditary then ${}^{\perp}C$ is a precover class for any pure-injective module *C*.

2.3 A Module with two σ-coprojective Covers

We can go some way toward answering the first question, 'What happens when a module has two (different) flat (σ-coprojective) covers?', via a generalisation of the version of Schanuel's Lemma found in Xu 1995. Since in the definition of an $\mathcal{X}$-cover of a module M we rely both on the cover module F and the cover homomorphism $f : F \to M$ it seems acceptable that the link between the two covers would involve both the cover module and the kernel of the cover map. Hence:

Lemma 2.3.1 Generalized Schanuel's Lemma Let $0 \to K \to P \to M \to 0$ and $0 \to K' \to P' \to M \to 0$ be exact with both P and P' σ-coprojective-precovers of M, then $P \oplus K' \cong P' \oplus K$

Proof Consider the following second quadrant pullback diagram (2.3.1),

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K' & = & K' & & \\
 & & & & \downarrow & & \downarrow h & & \\
0 & \to & K & \to & Q & \to & P' & \to & 0 \\
 & & \| & & \downarrow & & \downarrow k & & \\
0 & \to & K & \xrightarrow{g} & P & \xrightarrow{f} & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (2.3.1)$$

Since P and P' are both σ-coprojective-precovers of M, the linear maps $p : P \to P'$ and $q : P' \to P$ exist.

Now define the maps $\varphi : P \oplus K' \to P' \oplus K$ and $\psi : P' \oplus K \to P \oplus K'$ by:

$$\varphi(x, y') = (p(x) - y', -q(y') - (1 - qp)(x))$$

$$\psi(x',y)=\left(q(x')-y,-p(y)-(1-pq)(x')\right)$$

From the above diagram we can se that $f = kp$ and $k = fq$ which means that $f = f_o q_o p$ and $k = k_o p_o q$ so that $f(1- qp) = 0$ and $k(1- pq) = 0$ so that:

$$\psi\circ\varphi(x,y')=\left(q(p(x)-y')+q(y')-(1-qp)(x),-p(-q'(y')-(1-qp)(x))-(1-pq)(p(x)-y')\right)$$
$$=1$$

and similarly $\varphi\circ\psi=1$ so we have $P\oplus K'\cong P'\oplus K$ as required. □

Furthermore if the R-module M has two different σ-coprojective covers then we have the following result:

Theorem 2.3.2 Let M be an R-module with two different σ-coprojective covers P and P' , then $P \cong P'$.

Proof Given that $p:P\to M$ and $p':P'\to M$ are both covers we can form the diagram (2.3.2) below:

$$\begin{array}{ccc} P & & \\ \downarrow & \searrow & \\ P' & \longrightarrow & M \\ \downarrow & \nearrow & \\ P & & \end{array} \qquad (2.3.2)$$

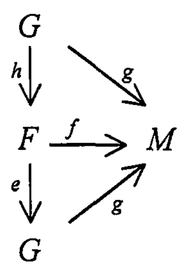
Where φ exists by the cover $p':P'\to M$ and ϕ exists by the cover $p:P\to M$. Then $\varphi\circ\phi=1_{P'}$ and $\phi\circ\varphi=1_P$ so clearly both φ and ϕ are equal and isomorphisms.

2.4 Modules Sharing a Common Flat (σ-coprojective) Precover

Here we consider $\mathcal{F}_\sigma$ to be the class of flat left R-modules and focus our attention on the question of the effect of the existence of a flat cover of a given module on another module connected to it via a certain epimorphism.

Lemma 2.4.1 Let $F \xrightarrow{f} M$ be an epimorphism where $F \in \mathcal{F}_\sigma$. If $K_f \in \mathcal{F}_\sigma^\perp$ then $F \xrightarrow{f} M$ is a flat precover of *M*. Moreover, if there is no $S \subset K_f$ such that *S* is a summand of *F* except for $S = 0$, then $F \xrightarrow{f} M$ is a flat cover.

Proof Consider the short exact sequence $O \to K_f \to F \to M \to O$. Let $K_f \in \mathcal{F}_\sigma^\perp$ and apply $Hom(F',\cdot)$ to the exact sequence giving $Hom(F',F) \to Hom(F',M) \to Ext(F',K_f) = O$ since $K_f \in \mathcal{F}_\sigma{}^\perp$, then $F \xrightarrow{f} M$ is a flat precover of *M*. Now assume the second property, there is no $S \subset K_f$ such that *S* is a summand of *F* except for $S = 0$. By Theorem A we know that *M* has a flat cover - say $G \xrightarrow{g} M$. Then there is the commutative diagram (2.4.1) below.



(2.4.1)

Since $G \xrightarrow{g} M$ s a flat cover, $e_o h$ is an automorphism of *G*. But then if $S = Kerg$ we can see that $S \subset K_f$ and *S* is a summand of *F*. So by hypothesis, $S = 0$. Since *g* is clearly an epimorphism we get that in fact *g* is an isomorphism, and so $F \xrightarrow{f} M$ is a flat cover. □

Theorem 2.4.2 If there exists a flat cover $F \xrightarrow{f} M$ and an epimorphism $M \xrightarrow{p} N$ then $K_p \in \mathcal{F}_\sigma{}^\perp$ implies that F is also a flat precover of N.

Proof Consider the third quadrant pullback diagram (3.2.2) from $F \xrightarrow{f} M$ and $K_p \to M$. We have by hypothesis that $K_p \in \mathcal{F}_\sigma{}^\perp$ and since $F \xrightarrow{f} M$ is a flat cover $K_f \in \mathcal{F}_\sigma{}^\perp$ then it follows that $K_{p \circ f} \in \mathcal{F}_\sigma{}^\perp$.

$$
\begin{array}{ccccccccc}
 & & O & & O & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K_f & \leftrightarrow & K_f & & & & \\
 & & \downarrow & & \downarrow & & & & \\
O & \to & K_{p \circ f} & \to & F & \to & N & \to & O \\
 & & \downarrow & & \downarrow & & \| & & \\
O & \to & K_p & \to & M & \to & N & \to & O \\
 & & \downarrow & & \downarrow & & & & \\
 & & O & & O & & & &
\end{array}
\qquad (2.4.2)
$$

Now we have a short exact sequence $O \to K_{p \circ f} \to F \to N \to O$ where $K \in \mathcal{F}_\sigma{}^\perp$ and F is flat which means that F is a flat precover of N. □

The converse of this theorem does not necessarily hold that is if $F \xrightarrow{f} M$ and $M \xrightarrow{p} N$ are as in Theorem 2.4.2 and N has an arbitrary flat cover then we cannot extrapolate that $K_p \in \mathcal{F}_\sigma{}^\perp$ since it is not possible to form any link between the flat cover of N and F. However with a logical constriction there is a valid converse in the form of the following theorem.

Theorem 2.4.3 Let $F \xrightarrow{f} M$ and $M \xrightarrow{p} N$ be as in Theorem 2.4.2, if $F \xrightarrow{p \circ f} N$ is a flat (pre)cover then $K_p \in \mathcal{F}^\perp$

Proof Again consider the above diagram then by hypothesis $K_f \in \mathcal{F}_\sigma{}^\perp$ since $F \xrightarrow{f} M$ is a flat cover and $K \in \mathcal{F}_\sigma{}^\perp$ since $F \to N$ is a flat (pre)cover. By (Xu 1996) we have that $K_p \in \mathcal{F}_\sigma{}^\perp$ as required. □

On the other hand there is a very interesting question converse to the above, which has so far not been fully answered. That is if F is a flat cover of N and there exists an epimorphism form M to N then is there a connection between the flat cover of N and the R-module M? This is answered in the affirmative below for the case when the ring R satisfies part (c) of Lemma 1.5.10.

Theorem 2.4.4 Let R be a ring satisfying Lemma 1.5.10 (c) If $F \xrightarrow{g} N$ is a flat cover and $M \xrightarrow{p} N$ is a superfluous epimorphism then $K_p \in \mathcal{F}_\sigma{}^\perp$ implies that F is a flat pre-cover of M.

Proof For the first part of the theorem, that R satisfies (c) is not necessary. An epimorphism $F \xrightarrow{f} M$ can be established such that $g = p \circ f$. From the $\mathcal{F}_\sigma{}^\perp$ nature of K_p, applying $Hom(F,.)$ to the short exact sequence $0 \to K_p \to M \xrightarrow{p} N \to 0$ gives $Hom(F,M) \to Hom(F,N) \to Ext(F,K_p)$ where $Ext(F,K_p) = 0$ thus the existence of the homomorphism f is seen.

Furthermore, f is epic since for all $m \in M$, it is seen that $p(m) = g(x) = p \circ f(x)$ for some $x \in F$, this implies $f(x) - m \in Kerp$ giving that $m = f(x) - (f(x) - m)$ that is, $m \in \operatorname{Im} f + Kerp$ therefore $M = \operatorname{Im} f + Kerp$, however since $Kerp << M$, $M = \operatorname{Im} f \Rightarrow f$ is epic. However for the epimorphism $F \xrightarrow{f} M$ to form a flat precover for M it is necessary to prove that $K_f \in \mathcal{F}_\sigma{}^\perp$.

Consider the pullback diagram (3.2.2); Since $F \to N$ is a flat cover A is cotorsion and also by hypothesis $K_p \in \mathcal{F}_\sigma{}^\perp$. Then by (c), $K_f \in \mathcal{F}_\sigma{}^\perp$ too. This gives the short exact sequence $0 \to K_f \to F \xrightarrow{f} M \to 0$ where K_f is cotorsion and F is flat which by (Xu 1996) gives us that $F \xrightarrow{f} M$ is a flat precover of M. □

2.5 Conclusions

According to the note following Lemma 1.5.10, the class of rings satisfying condition (c) is not empty, therefore the study of this question may give further rings for which Theorem 2.4.4 is valid.

There are interesting avenues of thought, for example following the definition of cotorsion due to Matlis, if R is taken to be an integral domain and we restrict, in this case, K_p to being cotorsion in the Matlis sense i.e. reduced cotorsion, we can formulate the proposition below.

Proposition 2.5.1 Let R be a P.I.D. then in $0 \to C_1 \to C_2 \to C_3 \to 0$, where C_2 is cotorsion and C_3 is reduced cotorsion then C_1 is also cotorsion.

Proof In fact it is sufficient for C to be reduced. Given that R is a P.I.D. where Q is the field of fractions of R. Applying Hom(Q,-) to the exact sequence above we have;

$$\cdots \to Hom(Q,C_3) \to Ext(Q,C_1) \to Ext(Q,C_2) \to \cdots \qquad (2.5.1)$$

and since C_3 is reduced the first term is zero, and since C_2 is cotorsion the flatness of Q gives us that the last term is also zero, so we have that $Ext(Q,C_1) = 0$. Remembering that in an integral domain all flat modules are torsion free we can form a sequence,

$$0 \to F \to \oplus Q \to D \to 0 \qquad (2.5.2)$$

for any flat module F. To (3.2.4) we apply $Ext(\text{-},C_1)$ giving,

$$0 = \prod Ext^1(Q,C_1) = Ext^1(\oplus Q,C_1) \to Ext^1(F,C_1) \to Ext^2(D,C_1) = 0 \qquad (2.5.3)$$

where the right hand zero comes from the homological dimension of the P.I.D. being 1, so that $Ext(F,C_1) = 0$ and C_1 is cotorsion in the Xu definition. □

Corollary 2.5.2 Let R be a P.I.D. If if $F \xrightarrow{f} N$ is a flat cover and $M \xrightarrow{p} N$ is a superfluous epimorphism then K_p is reduced cotorsion implies that F is a flat precover of M.

CHAPTER THREE
TORSION THEORIES

3.1 Introduction

The concept of a Torsion Theory was first introduced by S. E. Dickson in 1966 in his paper 'A Torsion Theory for Abelian Categories'. Since that time a great deal of research has been done on the subject and it has become clear that torsion theories exist in many forms in ring and module theory, the reader is referred to P. E. Bland 'Topics in Torsion Theory'.

The definition of a torsion theory can be written in different forms but we shall use the following so that the analogous nature of the cotorsion theory can be seen more easily later.

Definition 3.1.1 A *torsion theory* over the ring R is a pair $(\mathcal{T},\mathcal{F})$ of classes of modules such that $\mathcal{T}=\mathcal{F}^*$ and $\mathcal{F}=\mathcal{T}^*$. That is the classes satisfy the following conditions.

(1) $Hom(T,F)=0$ for all T in $\mathcal{T}$ and F in $\mathcal{F}$.

(2) $Hom(X,F)=0$ implies that X is in $\mathcal{T}$, whenever F is in $\mathcal{F}$.

(3) $Hom(T,X)=0$ implies that X is in $\mathcal{F}$, whenever T is in $\mathcal{T}$.

It should be noted that the universal existence of injective envelopes implies that every module in $\mathcal{F}$ has an injective envelope however as we have previously mentioned the existence of projective covers is far from universal, only existing when taken over a right perfect ring, so it may be that not all of the modules in $\mathcal{T}$ have a projective cover. A torsion theory is called *balanced* if projective covers exist universally over $\mathcal{T}$.

A torsion theory is called hereditary if $\mathcal{T}$ is closed under submodules, and cohereditary if $\mathcal{F}$ is closed under homomorphic images. In his book, Bland gives the following theorem;

Theorem 3.1.2 For a torsion theory $(\mathcal{T},\mathcal{F})$ we have:

(i) $(\mathcal{T},\mathcal{F})$ is hereditary if and only if $\mathcal{T}$ is closed under injective envelopes, and furthermore if $(\mathcal{T},\mathcal{F})$ is *balanced*,

(ii) $(\mathcal{T},\mathcal{F})$ is cohereditary if and only if $\mathcal{T}$ is closed under projective covers,

There are general properties that hold for any torsion theory, firstly the torsion class is closed under homomorphic images , direct sums and extensions, and the torsion-free class is closed under submodules, isomorphic images, direct products and extensions.

Many of the concepts that we have been considering can be redefined within the context of a torsion theory, the following definitions form a basis for the discussion. Given a torsion theory $(\mathcal{T},\mathcal{F})$;

Definition 3.1.3 An R-module M is said to be *τ-injective* if it is injective with respect to any exact sequence $0 \to A \to B \to C \to 0$ where C is in $\mathcal{T}$.

Definition 3.1.4 An R-module M is said to be *τ-projective* if it is projective with respect to any exact sequence $0 \to A \to B \to C \to 0$ where A is in $\mathcal{F}$.

Definition 3.1.5 An R-module M is said to be *τ-flat* if it is flat with respect to any exact sequence $0 \to A \to B \to C \to 0$ where C is in $\mathcal{T}$. That is, M is τ-flat if and only if $Tor_1^R(N,M) = 0$ for all $N \in \mathcal{T}$.

It is a simple observation that τ-injective coincides with injective only when the τ in question is $(\mathcal{T}, 0)$ where every module is torsion we can see that τ-flat modules coincide with the flat modules in this case also; and that τ-projective coincides with projective when the torsion theory is $(0, \mathcal{F})$ where every module is torsion-free.

Definition 3.1.6 A submodule N of M is called *τ-dense* if $M/N \in \mathcal{T}$. A *filter of τ-dense submodules of M*, is the set $F_\tau(R)$ of submodules N of M such that $M/N \in \mathcal{T}$. The filter set $F_\tau(M)$ is closed under submodules and intersections.

When we come to the classification question, we have certain results in classical literature which give us methods of distinguishing modules we can adapt these results so that we obtain methods of distinguishing the τ-modules as well when τ is a given torsion theory $(\mathcal{T},\mathcal{F})$. The first of these is given by Bland 1998, the generalised Baer criterion

Baer's Criterion gives us a method of distinguishing injective modules using the right ideals as a test set, a module M is injective if and only if for each right ideal K of R and every $f: K \to M$ there is an element $m \in M$ such that $f(k) = mk$ for all $k \in K$. The following theorem of Bland gives a similar test set for τ-injective modules.

Lemma 3.1.7 The Generalised Baer Criterion. A module M is injective if and only if for each $K \in F_\tau(R)$ and every $f: K \to M$ there is an element $m \in M$ such that $f(k) = mk$ for all $k \in K$.

In the same way we can adapt the flat test lemma.

Lemma 3.1.8 The Flat Test Lemma. (Anderson and Fuller) The following statements about a right R-module V are equivalent.

1. V is flat,
2. V is flat relative to $_R R$,
3. For each (finitely generated) left ideal $I \leq_R R$ the $\boldsymbol{Z}$-epimorphism $\mu : V \oplus_R I \to VI$ with $\mu : (v \otimes a) = va$ is monic.

Lemma 3.1.9 The τ-Flat Test Lemma. For a given torsion theory $(\mathcal{T},\mathcal{F})$, the following statements about a right R-module V are equivalent.

1. V is τ–flat,
2. V is τ–flat relative to $_R R$,
3. For each (finitely generated) $K \in F_\tau(R)$ the $\boldsymbol{Z}$-epimorphism $\mu : V \oplus_R K \to VK$ with $\mu : (v \otimes k) = vk$ is monic.

In defining this τ-flatness property in this way we have also defined a new kind of purity, let us call this the τ-purity. This τ-purity is generated by the class $\mathcal{A}$ such that all elements of $\mathcal{A}$ are projective with respect to the short exact sequences $0 \to A \to B \to C \to 0$ where $C \in \mathcal{T}$. That is, $\mathcal{T}$ becomes the class of τ-coprojective modules.

Theorem 3.1.10 For a short exact sequence $0 \to A \to B \to C \to 0$ the following are equivalent:

1. C is τ-flat
2. $KA \cap B = KB$ for each $K \in F_\tau(R)$
3. $EA \cap B = EB$ for each τ-essential right ideal E of R.
4. A is τ-pure in B

3.2 τ-injective envelopes and τ-projective covers

Now we come to the envelope and cover concepts as they apply to torsion theories. We need to make a few alterations to basic classical definitions so that they apply under the torsion theory in question. A submodule N is called τ-essential in M if it is essential in M and $N \in F_\tau(M)$, and a mapping $f: L \to M$ is called τ-minimal if Ker$f$ is small in L and contained in the class $\mathcal{F}$. A τ-injective envelope of a module M is a τ-injective module N with an injective mapping $f: M \to N$ such that $f(M)$ is a τ-essential submodule of M. Similarly we can define a τ-projective cover of a module M as a τ-projective module P with an epimorphic mapping $f: P \to M$ such that *Kerp* is τ-essential in P. Consider Xu's general definition of an $\mathcal{X}$-envelope and an $\mathcal{X}$-cover as they might be applied to a torsion theory $(\mathcal{T},\mathcal{F})$, Here the definition is as it appears in the preliminaries section of chapter one however in this definition we now take the class $\mathcal{X}$ to be that of the τ-injective modules for the $\mathcal{X}$-envelope and that of the τ-projective modules for the $\mathcal{X}$-cover.

We must show that the concepts of a $\mathcal{E}_\tau$-envelope in the sense of Xu, taking the class $\mathcal{X}$ to be that of the τ-injective modules, is in fact equivalent to that of the τ-injective envelope, in the Eckmann, Schopf Bland sense as cited above; and that the same holds for the τ-projectives.

Theorem 3.2.1 Let M be an R-module and $(\mathcal{T},\mathcal{F})$ a torsion theory where $(\mathcal{T}, \mathcal{E}_\tau)$ form a cotorsion theory, then the following are equivalent.

1. $f: M \to E$ is an $\mathcal{E}_\tau$-envelope in the sense of (Xu 1996)
2. $f: M \to E$ the τ-injective envelope, in the sense of Eckmann, Schopf and Bland

Proof $(1 \Rightarrow 2)$ Let $f: M \to E$ be an $\mathcal{E}_\tau$-envelope in the sense of (Xu 1996). Firstly the linear mapping must be monic by the following argument. Take a monomorphism $e: M \to I$ where I is an injective module, since I is also a τ-injective module we have the following diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \dashrightarrow & M & \xrightarrow{f} & E \\ & & \downarrow f' & \swarrow & \\ & & I & & \end{array} \qquad (3.2.1)$$

By Lemma 2.1.2. in (Xu 1996) $Ext\left(E/f(M), X\right) = 0$ for every $X \in \mathcal{E}_\tau$. Then $E/f(M) \in \mathcal{T}$, therefore $f(M) \in F_\tau(E)$. To prove that $f(M)$ is a essential submodule of E let L be a submodule of E such that $f(M) \cap L = 0$. By the third isomorphism theorem

$$E/(f(M) \oplus L) \cong \left(E/f(M)\right) \Big/ \left(f(M) \oplus L/f(M)\right)$$

Since $E/f(M) \in \mathcal{T}$ and $\mathcal{T}$ is closed under epimorphic images and isomorphisms, $E/f(M) \oplus L \in \mathcal{T}$. Define $p: f(M) \oplus L \to E$ by $p(x+l) = x$. Since E is τ-injective, the diagram (3.2.2)

$$\begin{array}{ccc} 0 \to f(M)\oplus L & \xrightarrow{i} & E \to E/(f(M)\oplus L) \to 0 \\ p\downarrow & \swarrow g & \\ E & & \end{array} \qquad (3.2.2)$$

Where i is an inclusion map, can be completed commutatively by some linear mapping g such that $g \circ i = p$. Define $h: M \to f(M)\oplus L$ by $h(m) = f(m) + 0$ Then $f = p \circ h = g \circ i \circ h = g \circ f$. Since $f: M \to E$ is an $\mathcal{E}_{\mathcal{T}}$-envelope, in the diagram (3.2.3) below:

$$\begin{array}{ccc} M & \xrightarrow{f} & E \\ f\downarrow & \swarrow g & \\ E & & \end{array} \qquad (3.2.3)$$

g must be an automorphism but $L \subseteq Ker g$ therefore $L = 0$.

$(2 \Rightarrow 1)$ Let $f: M \to E$ be a τ-injective envelope, in the sense of Eckmann, Schopf and Bland, $E/f(M) \in \mathcal{T}$. For every $X \in$, $Ext\left(E/f(M), X\right) = 0$, therefore for the exact sequence $0 \to M \to E \to E/f(M) \to 0$ we have the exact sequence $Hom(E, X) \to Hom(M, X) \to Ext\left(E/f(M), X\right) = 0$ from which we conclude that X is injective with respect to the monomorphism $f: M \to E$.

It remains only to show that any endomorphism g of E, with $f = g \circ f$, is an automorphism. g is monic since f is essential. Then $g(E) \cong E$, therefore $g(E)$ is τ-injective. Now by the third isomorphism theorem $E/g(E) \cong \left(E/f(M)\right) \Big/ \left(g(E)/f(M)\right) \in \mathcal{T}$ and therefore the sequence $0 \to g(E) \to E \to E/g(E) \to 0$ is splitting, that is $E = g(E) \oplus K$ for some submodule K of E. If K is nonzero then there is a $0 \neq k \in K$ Since $f(M)$ is essential in E there is an $r \in R$ such that $0 \neq rk \in f(M) \subseteq g(E)$. So $0 \neq rk \in g(E) \cap K = 0$ gives us a contradiction and $K = 0$, giving that g is an automorphism. □

Theorem 3.2.2 Let M be an R-module and $(\mathcal{T},\mathcal{F})$ a torsion theory where $(\mathcal{P}_\tau,\ \mathcal{F})$ form a cotorsion theory, then the following are equivalent.

1. $f:P\to M$ is a $\mathcal{P}_\tau$-cover in the sense of (Xu 1996)
2. $f:P\to M$ the τ-projective cover, in the sense of Eckmann, Schopf and Bland

Proof $(1\Rightarrow 2)$ Let $f:P\to M$ be a $\mathcal{P}_\tau$-cover in the sense of (Xu 1996), then $K = Kerf \in \mathcal{P}_\tau^{*} = \mathcal{F}$ by Lemma 2.1.1 in (Xu 1996). Take any epimorphism $g:F\to M$ with projective F. Since $F\in\mathcal{P}_\tau$, there is a homomorphism $h:F\to P$ such that $f\circ h = g$. Therefore f is an epimorphism. It remains only to prove that K is a small submodule of P.

Assume that there is a submodule L of P such that $K+L=P$. The restriction $f|_L : L\to M$ is an epimorphism since f itself is an epimorphismm and $Kerf|_L = K\cap L \le K$. Since $\mathcal{F}$ is closed under submodules, $Kerf|_L \in \mathcal{F}$. Since P is τ-projective, there is a linear mapping $e:P\to L$ such that $f|_L\circ e = f$. Then for the endomorphism $s = i\circ e : P\to P$, where i is an inclusion mapping, we have that $f\circ s = f$. By the second condition of $\mathcal{P}_\tau$-covers s is an automorphism. Then i is an epimorphism and therefore $P = L$.

$(2\Rightarrow 1)$ Let $f:P\to M$ be a τ-projective cover, in the sense of Eckmann, Schopf and Bland, then $K = Kerf \in \mathcal{F}$ and therefore $Ext(X,K)=0$ for every X in $\mathcal{P}_\tau$. For the short exact sequence $0\to K\to P\to M\to 0$ we have the exact sequence $Hom(X,P)\to Hom(X,M)\to Ext(X,K)=0$ So $Hom(X,f)$ is an epimorphism, that is $f:P\to M$ is a $\mathcal{P}_\tau$-precover.

Suppose $:P\to P$ is a linear mapping with $f\circ g = f$, g is an epimorphism since f is small. $Kerg \le Kerf \in \mathcal{F}$. Since P is τ-projective, the sequence $0\to Kerg\to P\xrightarrow{g} P\to 0$ is splitting, that is $P = Kerg\oplus T$ for some submodule T in P. Since $Kerg\le Kerf$ and $Kerf$ is small in P, $T = P$, therefore $Kerg = 0$ and so g is an automorphism.

CHAPTER FOUR
COTORSION THEORIES

4.1 Introduction to Cotorsion Theories

Cotorsion theories were originally explored by (Salce 1979) in his innovative paper "Cotorsion theories for Abelian groups". Following the work of (Richman, Walker. C., and Walker. E. 1968), he makes the following observations (for Abelian groups) which we extend for R-modules over any ring. Given a class $\mathcal{A}$ of R-Modules one can associate to that class, as in Chapter 2 the class of σ-pure exact sequences $E(\mathcal{A})$ for which every module in $\mathcal{A}$ is projective, and given any class $\mathcal{D}$ of short exact sequences one can similarly associate the class $P(\mathcal{D})$ of R-modules which are projective with respect to the class $\mathcal{D}$.

Classical results show us that $E(P(E(\mathcal{A}))) = E(\mathcal{A})$ and that $P(E(P(\mathcal{D}))) = P(\mathcal{D})$. The class $P(E(\mathcal{A}))$ is called the projective closure of the class $\mathcal{A}$ and has been vigorously studied. Again as in Chapter 2, one can associate with the class $\mathcal{A}$ the class of σ-coprojective R-modules written $\mathcal{F}(\mathcal{A})$, similarly to any class $\mathcal{B}$, one can associate the class of σ-coinjective R-modules, which can be defined as the class of modules for which all short exact sequences containing the module in the first nonzero position remain σ-pure, this class is written $\mathcal{C}(\mathcal{B})$. It is also well known that $\mathcal{F}(\mathcal{C}(\mathcal{F}(\mathcal{A}))) = \mathcal{F}(\mathcal{A})$ and $\mathcal{C}(\mathcal{F}(\mathcal{C}(\mathcal{B}))) = \mathcal{C}(\mathcal{B})$. The class $\mathcal{C}(\mathcal{F}(\mathcal{A}))$ is called the cotorsion closure of $\mathcal{A}$ and $\mathcal{F}(\mathcal{C}(\mathcal{B}))$ the cotorsion-free closure of $\mathcal{B}$.

Definition 4.1 A cotorsion theory is a pair $(\mathcal{F}, \mathcal{C})$ of classes such that $\mathcal{F} = \mathcal{C}^{\perp}$ and $\mathcal{C} = \mathcal{F}^{\perp}$. That is the classes satisfy the following conditions.

(1) $Ext(F,C) = 0$ for all F in $\mathcal{F}$ and C in $\mathcal{C}$.

(2) $Ext(X,C) = 0$ implies that X is in $\mathcal{F}$, whenever C is in $\mathcal{C}$.

(3) $Ext(F,X) = 0$ implies that X is in $\mathcal{C}$, whenever F is in $\mathcal{F}$.

This is a dualization of the torsion theory defined by Dickson (1966) where instead of the functor *Ext*, the functor *Hom* is used. The reader is referred to Bland "Topics in Torsion Theories" for further information.

By far the easiest and most employed method of constructing a cotorsion theory is to generate or cogenerate it by some class $\mathcal{A}$, (or even set $\mathcal{A}$, see Göbel & Shelah preprint 1999). A cotorsion theory is said to be generated by a class $\mathcal{A}$ if it is of the form $(\mathcal{F}(\mathcal{A}), \mathcal{C}(\mathcal{F}(\mathcal{A})))$, that is the class $\mathcal{F}(\mathcal{A})$ is the cotorsion-free class generated by $\mathcal{A}$ and $\mathcal{C}(\mathcal{F}(\mathcal{A})$ is the cotorsion class generated by $\mathcal{A}$. Dually a cotorsion theory is said to be cogenerated by the class $\mathcal{A}$ if it is of the form $(\mathcal{F}(\mathcal{C}(\mathcal{A})), \mathcal{C}(\mathcal{A}))$. The most studied cotorsion theory is the pair $(\mathcal{F}, \mathcal{C})$ of torsion-free / cotorsion groups found in Fuchs. For this example it is not difficult to verify that the class $\mathcal{C}$ comes from the rationals Q, where $\mathcal{C} = \{C \mid Ext(Q,C) = 0\}$ and so $(\mathcal{F}, \mathcal{C})$ is seen to be cogenerated by the rationals.

Some examples of cotorsion theories:

(1) $(\mathcal{P},\mathcal{M})$ forms a cotorsion theory where $\mathcal{M}$ is the class of all modules and $\mathcal{P}$ is the class of projectives.

(2) $(\mathcal{M},\mathcal{E})$ is also a cotorsion theory where $\mathcal{E}$ is the class of injective modules.

(3) $(\mathcal{B},\mathcal{A})$ where $\mathcal{A}$ is the class of absolutely pure modules and $\mathcal{B}$ is the left orthogonal class of $\mathcal{A}$. Since here $\mathcal{B} = {}^{\perp}\mathcal{A}$ and ${}^{\perp}\mathcal{A}^{\perp} = \mathcal{A}$.

(4) $(\mathcal{F},\mathcal{C})$ is a cotorsion theory over **Z** where $\mathcal{F}$ is the class of all torsion-free abelian groups and $\mathcal{C}$ is the class of cotorsion abelian groups.

(5) $(\mathcal{F},\mathcal{C})$ is again a cotorsion theory over any ring R where $\mathcal{F}$ is the class of all flat modules and $\mathcal{C}$ is the class of cotorsion modules.

The above examples are easily seen but are not the only genre of cotorsion theories. It is possible to define a cotorsion theory over homological entities as well for example if $\mathcal{E}$ is taken to be the class of all complexes of R-modules then it is not difficult to see that $({}^{\perp}\mathcal{E}, \mathcal{E})$ and $(\mathcal{E}, \mathcal{E}^{\perp})$ form cotorsion theories.

4.2 Having 'Enough Injectives or Projectives'

The following definition posed first by Salce for Abelian groups forms the essence of this study, it is necessary to know if a cotorsion theory has enough projectives and injectives since then, under various conditions the existence of cotorsion envelopes or cotorsion-free covers can be addressed.

Definition 4.2.1 A cotorsion theory is said to have enough injectives if for every module M there exists a short exact sequence (4.2.1) where $F \in \mathcal{F}$ and $C \in \mathcal{C}$.

$$0 \to M \to C \to F \to 0 \qquad (4.2.1)$$

Definition 4.2.2 A cotorsion theory is said to have enough projectives if for every module M there exists a short exact sequence (4.2.2) where $F \in \mathcal{F}$ and $C \in \mathcal{C}$.

$$0 \to C \to F \to M \to 0 \qquad (4.2.2)$$

4.3 Properties of Cotorsion Theories

Rational cotorsion theories are those cogenerated by subgroups of Q, they formed the main objects of Salce' study and were used in Göbel and Shelah's 1999 preprint. Before rational cotorsion theories though we must look at cotorsion theories as they exist is group theory.

The class of cotorsion theories in *Mod-Z* becomes a complete lattice given the partial order $(\mathcal{F}, \mathcal{C}) \leq (\mathcal{F}', \mathcal{C}')$ defined by $\mathcal{C} \subseteq \mathcal{C}'$ or equivalently $\mathcal{F} \subseteq \mathcal{F}'$ The supremum is called the *maximal* cotorsion theory and denoted ($\mathcal{L}$, *Mod-Z*) where $\mathcal{L}$ is the class of free groups, and the infimum is the *minimal* cotorsion theory denoted

(Mod-Z, $\mathcal{D}$), where $\mathcal{D}$ is the class of divisible groups. Obviously every cotorsion class contains $\mathcal{D}$, and every cotorsion-free class contains $\mathcal{L}$.

Cotorsion-free classes are closed with respect to isomorphisms, direct sums, and extensions; similarly every cotorsion class is closed with respect to isomorphisms, direct products, and extensions, see "Properties of cotorsiòn modules" - chapter one.

We have already noted one example of a cotorsion theory cogenerated by the class Q, that is, the usual torsion-free groups / cotorsion groups cotorsion theory, apart from this there are other simple examples. Take $\mathcal{T}$, the class of torsion groups, then we can show that this cogenerates the minimal cotorsion theory (Mod-Z,$\mathcal{D}$) by the following construction: in the cotorsion theory, $(\mathcal{F}(\mathcal{C}(\mathcal{T})),\mathcal{C}(\mathcal{T}))$,

$$\mathcal{C}((\mathcal{T})) = \{X \mid Ext(T,X) = 0, \forall T \in \mathcal{T}\ \}$$

and it is easily seen that if $0 = Ext(Z_p, X) \cong X/pX$ then X is p-divisible, So where $Ext(T, X) = 0$, we see that X is p-divisible for all p that is, divisible. Of a less trivial nature is that the class $\mathcal{T}$ generates the maximal cotorsion theory, this is a corollary to Griffith's solution of Baer's splitting problem and will not be considered here. Lemma 4.3.1 gives us a fundamental fact for cotorsion theories over groups which we will use and investigate more fully later for cotorsion theories over arbitrary rings.

Lemma 4.3.1 (Salce 1972) For a cotorsion theory over groups, having enough injectives implies and is implied by having enough projectives.

Proof Given an arbitrary group G we have a free resolution of G of the form: $0 \to L \to L' \to G \to 0$. If the cotorsion theory has enough injectives then $\forall L$ there exists a short exact sequence $0 \to L \to C \to F \to 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. using the pushout diagram (4.3.1) of these two sequences we have,

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & L & \to & L' & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & C & \to & X & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & F & = & F & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \tag{4.3.1}$$

Using the Flat Lemma, since L' and F are both in $\mathcal{F}$, X is also in $\mathcal{F}$, and we have the necessary short exact sequence as in (4.2.2) for $(\mathcal{F},\mathcal{C})$ to have enough projectives. Conversely if the cotorsion theory has enough projectives then we can take the injective envelope $E(G)$ of G and the projective sequence containing $E(G)/G$ in its third position, and form the pullback to X which is cotorsion by property five, preliminary section. Giving the required sequence as in (4.2.1) for $(\mathcal{F},\mathcal{C})$ to have enough injectives:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & C & = & C & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & G & \to & X & \to & F & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & G & \to & E(G) & \to & E(G)/G & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{4.3.2}$$

□

The following list of properties is well known for general cotorsion theories over groups.

Property 4.3.2 (Salce) The following facts are equivalent for the cotorsion theory ($\mathcal{F}$, $\mathcal{C}$),

(a) $\mathcal{F}$ contains a non-trivial p-group

(b) $\mathcal{F}$ contains all the p-groups

(c) Every group in $\mathcal{C}$ is p-divisible.

Proof If F contains a non-trivial p-group for some prime p, then $Z(p) \in \mathcal{F}$ it then follows that if $C \in \mathcal{C}$ then $0 = Ext(Z(p), C) \cong C/pC$ hence C is p-divisible. Conversely if C is p-divisible and G is a p-group then $Ext(G, C) = 0$, see Fuchs. □

Definition 4.3.3 Let the *prime set of* $\mathcal{F}$ refer to the set $P(\mathcal{F}) = \{p \in P \mid Z(p) \in \mathcal{F}\}$ then we have the following property:

Property 4.3.4 (Salce) Let ($\mathcal{F}$, $\mathcal{C}$) be a cotorsion theory generated by torsion groups only, then C consists of all those groups which are p-divisible for each $p \in P(\mathcal{F})$.

Proof It is enough to show that for each cardinal α the exact sequence $0 \to \bigoplus_\alpha Z_\alpha \to C \to F \to 0$ with C p-divisible for each $p \in P(\mathcal{F})$ and $F \in \mathcal{F}$. Let R_α be the subgroups of Q generated by $\{1/p^n \mid p \in P(\mathcal{F}), n \in N\}$. R_α is clearly p-divisible for each p prime in F. Put $C = \bigoplus_\alpha R_\alpha$ then $F \cong \bigoplus_\alpha R/Z \in \mathcal{F}$ because it is a torsion group with non trivial p-components only for $p \in P(\mathcal{F})$. □

Property (4.3.2) shows that when considering the cotorsion theory ($\mathcal{F}$, $\mathcal{C}$), it is enough to investigate the behavior of torsion-free and mixed groups, and further the following shows that we can actually disregard the mixed groups

Property 4.3.5 (Salce) Every cotorsion theory is cogenerated by torsion and torsion-free groups. It is sufficient to show that in a cotorsion theory $(\mathcal{F}, \mathcal{C})$, $F \in \mathcal{F}$ if and only if $tF \in \mathcal{F}$ and $F/_{tF} \in \mathcal{F}$

Proof By closure properties of $\mathcal{F}$ it is enough to prove $F \in \mathcal{F} \Rightarrow F/_{tF} \in \mathcal{F}$. Take a cotorsion group C, it is enough to consider the reduced part K, K is p-divisible for each $p \in P(\mathcal{F})$ hence $t_pK = 0$ for each $p \in P(\mathcal{F})$. From the exact sequence $Hom(tF,C) \to Ext(F/tF,C) \to Ext(F,C) = 0$, where $Hom(tF,C) = 0$ since $t_pF \neq 0$ implies that $p \in P(\mathcal{F})$. Hence $Ext(F/tF,C) = 0$ and $F/_{tF} \in \mathcal{F}$ as required.

Property 4.3.6 (Salce) There exists a bijection between the class of cotorsion theories, from the classical to the minimal one, and the set of all subsets of $P(\mathcal{F})$.

For a general cotorsion theory over rings there are a few results that we can obtain by generalizing results for the flat / cotorsion cotorsion theory. However there is an important result that we must state first because the flat / cotorsion cotorsion theory satisfies a certain criterion, that of theorem 4.3.7 which may not be satisfied by a general cotorsion theory, we will consider cotorsion theories which satisfy the theorem for the remainder of this section.

Theorem 4.3.7 For any cotorsion theory $(\mathcal{F}, \mathcal{C})$ the following are equivalent:

(i) In any $0 \to A \to B \to C \to 0$, $A, B \in \mathcal{C} \Rightarrow C \in \mathcal{C}$

(ii) In any $0 \to X \to Y \to Z \to 0$, $Y, Z \in \mathcal{F} \Rightarrow X \in \mathcal{F}$

(iii) For any $C \in \mathcal{C}$ and $F \in \mathcal{F}$, $Ext^2(F,C) = 0$

Proof The proof is in four parts where we prove that both of the properties are equivalent to $Ext^2(F,C) = 0$ $\forall F \in \mathcal{F}$ and $\forall C \in \mathcal{C}$.

(a) Assume (i), take any $F \in \mathcal{F}$ and $C \in \mathcal{C}$, and consider the short exact sequence formed from the injective envelope of C: $0 \to C \to E \to K \to 0$, by (i) $K \in \mathcal{C}$.

Now apply *Ext(F,-)* to it giving $0 = Ext(F,K) \to Ext^2(F,C) \to Ext^2(F,E) = 0$. Then $Ext^2(F,C) = 0 \ \forall F \in \mathcal{F}$ and $\forall C \in \mathcal{C}$.

(b) Assume $Ext^2(F,C) = 0 \ \forall F \in \mathcal{F}$ and $\forall C \in \mathcal{C}$ and take any short exact sequence $0 \to A \to B \to C \to 0$, where $A, B \in \mathcal{C}$ then apply *Ext(F,-)* to give $0 = Ext(F,B) \to Ext(F,C) \to Ext^2(F,A) = 0$ by assumption so *Ext(F,C)* = 0 and we see that $C \in \mathcal{C}$.

(c) Assume (ii), take any $F \in \mathcal{F}$ and $C \in \mathcal{C}$, and consider the short exact sequence formed from the projective resolution of *F* by cutting off at the first projective element and passing to the kernel of the epimorphism: $0 \to K \to P \to F \to 0$, by (ii) $K \in \mathcal{F}$. Now apply *Ext(-,C)* to it giving $0 = Ext(K,C) \to Ext^2(F,C) \to Ext^2(P,C) = 0$. Then $Ext^2(F,C) = 0 \ \forall F \in \mathcal{F}$ and $\forall C \in \mathcal{C}$.

(d) Assume $Ext^2(F,C) = 0 \ \forall F \in \mathcal{F}$ and $\forall C \in \mathcal{C}$ and take any short exact sequence $0 \to X \to Y \to Z \to 0$, where $Y, Z \in \mathcal{F}$ then apply *Ext(-,C)* to give $0 = Ext(Y,C) \to Ext(X,C) \to Ext^2(Z,C) = 0$ by assumption so *Ext(X,C)* = 0 and we see that $C \in \mathcal{C}$.

As an aside it should be noted that the proposition, for any cotorsion theory $(\mathcal{F},\mathcal{C})$ in $0 \to A \to B \to C \to 0$ ($B, C \in \mathcal{C} \Rightarrow A \in \mathcal{C}$) $\Leftrightarrow$ ($A, B \in \mathcal{F} \Rightarrow C \in \mathcal{F}$), is clearly not correct and this can be seen by the simple example below.

Example Consider the cotorsion theory over groups $(\mathcal{G},\mathcal{D})$ formed by the class of all abelian groups $\mathcal{G}$ and all divisible groups $\mathcal{D}$. Then obviously for all short exact sequences $0 \to A \to B \to C \to 0$ where $A, B \in \mathcal{G}$ we have that $C \in \mathcal{G}$ also since it is the class of all abelian groups. However, since every group has a divisible envelope, given a non-divisible group *H* we can form its divisible resolution via its divisible envelope; $0 \to H \to D \to E \to 0$ with *D* and *E* divisible, forming a counterexample.

For the remainder of this section we will assume that the cotorsion theory in question satisfies the above theorem 4.3.7. The first theorem of this nature shows that if for any cotorsion theory we have that all modules are in the cotorsion class then this cotorsion theory is equivalent to the maximal one over that ring.

Theorem 4.3.8 For any cotorsion theory $(\mathcal{F},\mathcal{C})$ over a ring R, the following are equivalent.

(i) Every left R-module is in $\mathcal{C}$.

(ii) Every left R-module in $\mathcal{F}$ is also in $\mathcal{C}$.

(iii) Every left R-module in $\mathcal{F}$ is also in $\mathcal{P}$.

Proof

(i)$\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii) Given any cotorsion-free left R-module F we can take its projective resolution to the first element, $0 \to K \to P \to F \to 0$ then $P \in \mathcal{P} \subseteq \mathcal{F}$ and $F \in \mathcal{F}$, which since $(\mathcal{F},\mathcal{C})$ is assumed to satisfy theorem 4.3.7, gives us that $K \in \mathcal{F}$ also. By (ii) this implies that $K \in \mathcal{C}$ so the sequence is split and $F \in \mathcal{P}$.

(iii) $\Rightarrow$ (i) Since every left cotorsion-free R-module is projective, for any left R-module M we have that $Ext(F,M) = 0$ so all left R-modules are cotorsion.

Next we generalize certain results known for the cotorsion theory formed by the flat and cotorsion classes.

Theorem 4.3.9 Let $(\mathcal{F},\mathcal{C})$ be a cotorsion theory and $0 \to A \to B \to C \to 0$ be exact. If A and C have special $\mathcal{F}$-precovers then B has a special $\mathcal{F}$-precover. Moreover if $\mathcal{F}$ is closed under direct limits then B has a special $\mathcal{F}$-cover.

Proof Since C has a special $\mathcal{F}$-precover $0 \to C_1 \to F_1 \to C \to 0$ with $C_1 \in \mathcal{C}$ and $F_1 \in \mathcal{F}$ we have the following pullback diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C_1 & = & C_1 & & \\
 & & & & \downarrow & & \downarrow & & \\
E_1 : 0 & \rightarrow & A & \rightarrow & X & \rightarrow & F_1 & \rightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A & \rightarrow & B & \rightarrow & C & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

A has a special $\mathcal{F}$-precover $0 \rightarrow C_2 \rightarrow F_2 \rightarrow A \rightarrow 0$ with $C_2 \in \mathcal{C}$ and $F_2 \in \mathcal{F}$. In the exact sequence:

$$\cdots \rightarrow Ext^1(F_1, F_2) \xrightarrow{g_*} Ext^1(F_1, A) \rightarrow Ext^2(F_1, C_2) \rightarrow \cdots \tag{4.3.1}$$

$Ext^2(F_1, C_2) = 0$ by assumption, therefore g^* is an epimorphism. So there is an exact sequence $E : 0 \rightarrow F_2 \rightarrow Y \rightarrow F_1 \rightarrow 0$ such that $g^*(E) = E_1$, that is we have a commutative exact diagram (4.3.2):

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C_2 & = & C_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
E : 0 & \rightarrow & F_2 & \rightarrow & Y & \rightarrow & F_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
E_1 : 0 & \rightarrow & A & \rightarrow & X & \rightarrow & F_1 & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{4.3.2}
$$

Since $F_1, F_2 \in \mathcal{F}$, $Y \in \mathcal{F}$ also. From these two diagrams we have the following commutative exact diagram (4.3.3):

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & C_2 & = & C_2 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & Z & \to & Y & \xrightarrow{f \circ h} & B & \to & 0 \\ & & \downarrow & & \downarrow & & // & & \\ 0 & \to & C_1 & \to & X & \xrightarrow{f} & B & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \qquad (4.3.3)$$

$Z \in \mathcal{C}$ since $C_1, C_2 \in \mathcal{C}$. So $0 \to Z \to Y \xrightarrow{f \circ h} B \to 0$ is a special $\mathcal{F}$-precover for B. The second part of the Theorem comes from Theorem A, see appendix.

Lemma 4.3.10 (Xu) If $\varphi_i : X_i \to M_i$ is an $\mathcal{X}$-cover for $i = 1,2,...n$ then $\oplus \varphi_i : \oplus X_i \to \oplus M_i$ is an $\mathcal{X}$-cover.

Corollary 4.3.11 If $0 \to A \to B \to C \to 0$ is a splitting exact sequence and A, B have $\mathcal{F}$-covers then B has an $\mathcal{F}$-precover. Furthermore if $\mathcal{F}$ is closed under direct limits then B has an $\mathcal{F}$-cover

Proof Since the sequence is splitting we can replace B by $A \oplus C$ and using the $\mathcal{F}$–covers F and G of respectively A and C we can show that $F \oplus G$ is an $\mathcal{F}$–precover of $A \oplus C$, see diagram (4.3.4) :

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J & \to & J \oplus K & \to & K & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F & \to & F \oplus G & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & A \oplus C & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad (4.3.4)$$

Since $J, K \in \mathcal{C}$ in the top exact sequence we know that $J \oplus K \in \mathcal{C}$ and we can apply lemma 3.2.1 to show that $F \oplus G$ is an $\mathcal{F}$–precover of $A \oplus C$, since by taking the induced short exact sequence $0 \to J \oplus K \to F \oplus G \to A \oplus C \to 0$ and applying *Ext*(*F'*,-) for any cotorsion-free *F'* we see that *F'* is projective with respect to this sequence. Further, if $\mathcal{F}$ is closed under direct limits then we can apply Theorem A (see appendix) to show that *B* has an $\mathcal{F}$–cover.

Lemma 4.3.12 For any cotorsion theory $(\mathcal{F}, \mathcal{C})$ where $\mathcal{F}$ is closed under direct limits, if *M* is a submodule of a module *G* where *G* has an $\mathcal{F}$–cover and $G/M \in \mathcal{F}$ then *M* has an $\mathcal{F}$–cover.

Proof Consider the diagram (4.3.4) formed by the pullback of the $\mathcal{F}$–cover of *G* and the map $M \to G$. Since we have assumed that the cotorsion theory satisfies theorem 4.3.7, we have that $\varphi^{-1}(M) \in \mathcal{F}$ which gives us the induced short exact sequence $0 \to K \to \varphi^{-1}(M) \to M \to 0$. Now $K \in \mathcal{C}$ by Wakamatsu's Lemma, and $\varphi^{-1}(M) \in \mathcal{F}$, so using Lemma 3.2.1 we have that $\varphi^{-1}(M) \to M$ is an $\mathcal{F}$–precover. Then since by hypothesis the cotorsion-free class is closed under direct limits, using Theorem A we see that *M* has an $\mathcal{F}$–cover.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \rightarrow & \varphi^{-1}(M) & \rightarrow & F & \rightarrow & F/\varphi^{-1}(M) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \rightarrow & M & \rightarrow & G & \rightarrow & G/M & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (4.3.5)$$

□

4.4 The Flat Cover Conjecture

The interest that cotorsion theories hold with regard to this thesis is in that the flat cover conjecture is equivalent to the conjecture that the relevant cotorsion theory has enough projectives. This is an obvious corollary to lemma 2.3. and will be proved later through Wakamatsu's Lemmas. Salce (1979) proved that for a cotorsion theory over an abelian group having enough projectives was synonymous with having enough injectives so that for example the existence of injective hulls gives that for the cotorsion theory in example 2 every group is the epimorphic image of some other group. This is not a very interesting result but along the same argument, theorem 58.1 Fuchs (1970) shows that the cotorsion theory in example 5 has enough projectives, that is every abelian group has a torsion-free cover. This is a new proof of a theorem of Enochs (1979).

The above result of Salce has not been proved to hold over every ring, but another point of interest is theorem 3.4.6 Xu (1995). This says that over any ring, every module has a flat cover if and only if it has a cotorsion envelope, which is just an alternative statement of Salce's theorem that the cotorsion theory has enough projectives if and only if it has enough injectives, at least for the cotorsion theory in example 6.

Bearing these facts in mind, an interesting avenue of research lies in taking an arbitrary cotorsion theory and seeing whether it has enough projectives (respectively injectives) and whether this fact implies that there are $\mathcal{F}$-covers (respectively $\mathcal{C}$-envelopes), in conjunction with the above question of whether the existence of such $\mathcal{F}$–covers (or the existence of enough projectives) is equivalent to the existence of $\mathcal{C}$-envelopes (or enough injectives).

While Eklov & Trlifaj were undertaking research into Enoch's conjecture they also considered Salce's 1979 result as it applies to modules rather than abelian groups. Göbel and Shelah (Theorem 6.1 (1999)) showed that a rational cotorsion theory, that is a cotorsion theory cogenerated by a subgroup of Q, has enough projectives and injectives if and only if it is cogenerated by a set of finite rank groups satisfying certain criterion. Eklov & Trlifaj took the concept and applied it to the case of modules essentially via the next theorem. This is a powerful theorem because it not only gives us the synonymy that we want for enough projectives and injectives but it also gives the existence of such.

Theorem 4.4.1 (Eklov & Trlifaj) Every cotorsion theory which is cogenerated by a set of modules has enough projectives. Furthermore enough projectives implies and is implied by enough injectives.

Proof Take the cotorsion theory $(\mathcal{F},\mathcal{C})$ cogenerated by the set S of modules. Let B be the direct sum of the modules of S. Given an arbitrary module L, choose κ such that $\kappa \geq |B|+|L|+|R|$ and let $\lambda = 2^{\kappa}$. Take the A from Theorem E&T, A contains L and A is in $\mathcal{C}$ since $Ext(B,A) = 0$. Now define $F = A/L$, to show that $(\mathcal{F},\mathcal{C})$ has enough injectives it is sufficient to show that F is in $\mathcal{F}$, as then we would have a short exact sequence $0 \to A \to F \to M \to 0$ as required. $F = \bigcup_{\alpha<\lambda} F_\alpha = \bigcup_{\alpha<\lambda} A_\alpha$ so $F_0 = 0$ and $\forall \alpha < \lambda \quad {F_{\alpha+1}}/{F_\alpha} = {A_{\alpha+1}}/{A_\alpha} \cong B \Rightarrow Ext(F,X) = 0$ whenever $Ext(B,X) = 0$, giving that F is in $\mathcal{F}$. The property of having enough projectives can be show from the above result, and similarly the inverse implication, as in Salce's 1979 paper. □

The seemingly natural corollary to this theorem would be that if $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set of modules then it admits an $\mathcal{F}$-cover and indeed a $\mathcal{C}$-envelope. However the proof of this corollary is not quite as straightforward as it may seem, since we are assuming an equivalence of the concept of enough projectives (injectives) with that of admitting $\mathcal{F}$-covers ($\mathcal{C}$-envelopes). This equivalence has only so far been shown to hold for groups and for certain cotorsion theories, notably those whose cotorsion-free class is closed with respect to direct limits, it has not yet been expressly proved or disproved in the general case.

Proposition 4.4.2 A cotorsion theory $(\mathcal{F}, \mathcal{C})$ where $\mathcal{F}$ is closed with respect to direct limits has enough projectives (injectives) if and only if it admits an $\mathcal{F}$-cover ($\mathcal{C}$-envelope).

Proof

(i) Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory with enough projectives, then for any R-module M there is a short exact sequence $0 \to C \to F \to M \to 0$ with C in $\mathcal{C}$ and F in $\mathcal{F}$. Then by Xu $F \to M$ is an $\mathcal{F}$–precover. And by Theorem A since $\mathcal{F}$ is closed with respect to direct limits there is an $\mathcal{F}$-cover.

(ii) Assume every R-module M has an $\mathcal{F}$-cover, and let $(\mathcal{F}', \mathcal{C})$ be the cotorsion theory where $\mathcal{C} = \mathcal{F}^{\perp}$ and $\mathcal{F}' = {}^{\perp}\mathcal{C}$. Wakamatsu's Lemma as applied to this cotorsion theory shows that $K \in \mathcal{C}$ whenever $K = Ker f$ the $\mathcal{F}$-cover. Thus we have the required short exact sequence for any R-module.

(iii) Let $(\mathcal{F}, \mathcal{C})$ have enough injectives, then for any R-module M there is a short exact sequence. $0 \to M \to C \to F \to 0$ with $C \in \mathcal{C}$ and $F \in \mathcal{F}$. In the notation of Xu this sequence becomes a *generator for Ext*$(\mathcal{F}, M)$. To see this take another extension $0 \to M \to C' \to F' \to 0$ then we have the diagram below.

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & F & \to & 0 \\ & & \| & & \uparrow g & & \uparrow f & & \\ 0 & \to & M & \xrightarrow{\alpha'} & C' & \xrightarrow{\beta'} & F' & \to & 0 \end{array}$$

Obviously the map $g: C' \to C$ exists by applying *Hom*(-, *C*) to the original sequence, giving $\cdots \to Hom(C',C) \xrightarrow{\alpha'^*} Hom(M,C) \to Ext(F,C) = 0$ where α'^* is epic so $\exists g: C' \to C$ such that $\alpha'^*(g) = \alpha$ that is $g \circ \alpha' = \alpha$. The map $f: F' \to F$ exists by a simple diagram chasing argument. β is epic so define $f(x) = \beta' \circ g(c)$ where $\beta(c) = x$ to preserve commutativity of the diagram. This is a well defined map since if we take another $c' \in C$ where $\beta(c') = x$ then $(c - c' \in Ker\beta = \operatorname{Im}\alpha) \Rightarrow \exists m \in M$ *s.t.* $\alpha(m) = c - c'$ so $g \circ \alpha(m) = \alpha'(m) = g(c - c') \in \operatorname{Im}\alpha' = Ker\beta') \Rightarrow \beta' \circ g(c - c') = 0$

(iv) Assume every *R*-module has a *C*-envelope and let $(\mathcal{F}, \mathcal{C}')$ be the cotorsion theory where $\mathcal{F} = {}^{\perp}\mathcal{C}$ and $\mathcal{C}' = \mathcal{F}^{\perp}$. Then the short exact sequence $0 \to M \to C \to F \to 0$ exists with *C* in $\mathcal{C}$ and by Wakamatsu's Lemma, $F \in \mathcal{F}$.

Corollary 4.4.3 If $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set of modules where $\mathcal{F}$ is closed with respect to direct limits then it admits an $\mathcal{F}$-cover ($\mathcal{C}$-envelope) for all modules.

Proof A direct result of Theorem 4.4.1 and Proposition 4.4.2.

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APPENDIX

For the most part the work contained in this thesis is new in not only content but reference, many references are from preprints and conference lecture notes. With this in mind the following appendix section has been included containing fundamental theorems which we have assumed but are nevertheless too new to be contained in contemporary literature.

Theorem A (*Enochs* see *Xu 'Flat Covers of Modules'*- 1996) Let $\mathcal{X}$ be a class of modules closed under direct limits then the existence of an $\mathcal{X}$-precover implies the existence of an $\mathcal{X}$-cover.

Theorem B&EB (*Bican & El Bashir* "Over any ring, every module has a flat cover." – preprint 1999) Let R be an arbitrary ring and σ a purity over that ring projectively generated by the class A. For each cardinal λ module M and submodule $L<M$ such that $|M/L|<\lambda$ there exists a cardinal κ such that $|M|>\kappa$ and the submodule L contains a non-zero submodule L' which is σ-pure in M.

Theorem E&T (*Eklov & Trlifaj* "How to make *Ext* Vanish" – preprint 1999) Let $A = A_\mu$ be the union of a continuous chain of submodules, $A = \bigcup_{\alpha<\mu} A_\alpha$ such that $Ext(A_0, C) = 0$ and for all $\alpha < \mu$, $Ext\left(A_{\alpha+1}/A_\alpha, C\right) = 0$. Then $Ext(A, C) = 0$.

Theorem T1 (*Trlifaj* "A Generalization of the Flat Cover Conjecture" – preprint 1999) Let R be a ring and B be a left R-module. There exists a cardinal κ such that the following conditions are equivalent for any right R-module A.

(i) $Tor(A,B) = 0$

(ii) There is an ordinal σ and a strictly increasing sequence, $(A_\alpha \mid \alpha \leq \kappa)$ consisting of submodules of A such that $A_0 = 0$, $A_\alpha = \bigcup_{\beta\leq\alpha} A_\beta$ for all limit ordinals less that σ, $A_\sigma = A$, $\operatorname{card}(A_{\alpha+1}/A_\alpha) \le \kappa$ and $Tor(A_{\alpha+1}/A_\alpha, B) = 0$ for all $\alpha < \sigma$

Theorem T2 (*Trlifaj* "A Generalization of the Flat Cover Conjecture" – preprint 1999) Let R be a ring and C be a pure-injective left R-module. There exists a cardinal κ such that the following conditions are equivalent for any right R-module A.

(iii) $Ext(A,C) = 0$

(iv) There is an ordinal σ and a strictly increasing sequence, $(A_\alpha \mid \alpha \le \kappa)$ consisting of pure R-submodules of A such that $A_0 = 0$, $A_\alpha = \bigcup_{\beta \le \alpha} A_\beta$ for all limit ordinals less that σ, $A_\sigma = A$, $\operatorname{card}(A_{\alpha+1}/A_\alpha) \le \kappa$ and $Ext(A/A_\alpha, C) = 0$ for all $\alpha < \sigma$

Lemma W1 (Wakamutsu's Lemma – *Xu 'Flat Covers of Modules'*-1996) Let $\varphi : X \to M$ and assume that $\mathcal{X}$ is closed under extensions. Set $K = Ker(\varphi)$. Then $Ext(X', K) = 0$ for any $X' \in \mathcal{X}$.

Lemma W2 (Wakamutsu's Lemma – *Xu 'Flat Covers of Modules'*- 1996) Let $\varphi : M \to X$ and assume that $\mathcal{X}$ is closed under extensions. Set $D = Co\ker(\varphi)$. Then $Ext(D, X') = 0$ for any $X' \in \mathcal{X}$.