A New Keynesian Model with Commodity Input in Production

Murat Unur

Submitted in partial fulfilment of the requirements for the degree of the Master of Philosophy in Economics

St. Cross College

University of Oxford

May 2016

24426 words

Acknowledgements

I would like to thank Professor Francesco Zanetti for his genuine support and supervision without which I would not be able to complete this project. I would also like to thank my parents, Mehmet Sabri Unur and Meral Unur for their patience, care and support throughout my education for which I will always be deeply grateful.

Abstract

This thesis aims to explore the effects of having a commodity input in production at a deeper level than treating it as an exogenous cost push shock would allow. This is done by explicitly modelling an extraction sector that supplies a commodity to intermediate good producers which use it alongside differentiated household labour in production. Commodity price inflation emerges as the most volatile one. Whether an output expansion or contraction is associated with rising or falling commodity prices depends on the type of shock hitting the economy. Moreover, the existence of a commodity as an additional input in production strengthens expansions and deepens recessions. Nominal rigidities in goods and labour markets have significant effects on commodity price and level even though real commodity price adjusts to clear the commodity market every period. Finally, price inflation targeting is a suboptimal policy rule. Hybrid rules, composite inflation targeting and output gap targeting perform nearly as well as the optimal policy and should be considered as alternatives.

Word Count: A typical page contains 354 words. There are 69 pages which makes 24426 words in total.

Contents

List of Figures

1 Introduction

Some of the most important economic events in recent history, like the passing of the corn laws or the restriction of oil supply by OPEC are related to commodities in one way or another. In addition to their historical importance, their large impact on the economy despite their relatively small shares, the volatility of their prices, the uncertainty associated with their supply due to political events or climate conditions and their asymmetric distribution across countries make commodities an interesting subject to study.

This thesis focuses on another important aspect of commodities: their role as productive factors. Although no single commodity has a very large share in production, their joint impact can be significant given their volatile prices and prevalence in production. Oil and natural gas prices influence energy costs firms face. Different types of metals are used across a variety of sectors like steel in construction and copper in electronics. Even agricultural based commodities like wheat or coffee go through some processing before reaching the end customer. Commodities are an important part of production.

Economists have chosen different ways to model how commodities affect the economy, ranging from ad hoc cost push shocks to full on DSGE models describing the commodity extraction and trade processes. The former approach can be easier to implement and help us understand the effects of a commodity price hike but is silent on why commodity prices move in the direction they do in the first place. It also provides very little insight about the transmission mechanisms at work. I adopt the latter approach and model an economy with an explicit commodity extraction sector that supplies intermediate good producers with a commodity to use alongside differentiated household labour in production. The commodity price is endogenous and adjusts to clear the commodity market every period. With this approach, it is possible to analyse not only how the commodity price affects the economy but also how the commodity price itself is affected by other shocks. The

transmission channels are easier to understand and richer comparative statics exercises can be performed.

The thesis is organised as follows. Section 2 provides a brief literature review and relates the thesis to the already existing research in the field. Section 3 presents the model, its solution and its representation in terms of gaps from natural levels. Section 4 describes the calibration, illustrates the responses of the model to various shocks and analyses the effects of nominal rigidities on the economy. Section 5 discusses the optimal policy and various other policy rules. Section 6 concludes. A bibliography and an appendix showing the details of derivations and approximations follow.

2 Literature Review

There are many interlinked strands of research when it comes to modelling commodities. For the sake of brevity, this review focuses on the relationship between commodities and price shocks, business cycles and economic policy. It is provided to relate the thesis to the ongoing research in the field and is in no way a comprehensive review.

An initial and important question brought to the centre of macroeconomic research by OPEC restricting the oil supply in 1970s is whether oil price shocks can cause recessions. The seminal paper by Hamilton (1983) started an important debate by arguing that energy prices can account for a large fraction of US recessions. This result was challenged by Bernanke et al. (1997) who conducted a more structured VAR analysis to find that it is actually the endogenous monetary policy response that causes the downturns. The debate has continued with Hamilton and Herrera (2004) showing the costly effects of oil price shocks and Bernanke et al. (2004) continuing to emphasise the endogenous response of monetary policy.

All these papers used econometric methods to explore this issue but more recent papers use microfounded DSGE models to come up with more refined answers. Rotemberg and Woodford (1996) show that the large impact of energy prices on the economy despite their small share can be explained with the existence of imperfect competition due to collusion among producers. Finn (2000) argues that it is actually possible to build a model with perfect competition that can still explain how energy price shocks lead to recessions once energy is tied to capital capacity utilisation. Leduc and Sill (2004) also build a model where oil use is tied to capacity utilisation to find a more detailed answer: that 40% of the downturns are caused by the endogenous response of the monetary policy whereas the rest is caused by oil price hikes. It is important to note that oil price shocks are useful in improving existing models as well. For instance, Kim and Loungani (1992) build an RBC model with an energy input where energy prices follow an exogenous process. They manage to create a model relying less on unobserved technology changes and obtain correlations closer to what is actually observed.

A related strand of literature is on what inflation target to choose. If monetary policy is responsible for some of the downturn associated with price hikes, how it is crafted can have important consequences for welfare. Bodenstein et al. (2008) note that headline and core inflation can diverge significantly when there are energy price shocks and advocate the targeting of core inflation. Aoki (2001) reaches a similar conclusion and recommends putting more weight on the inflation of sectors that have stickier prices.

Another feature of commodities is their asymmetric distribution across countries with some economies relying heavily on commodity exports. Therefore commodity prices may have very significant effects for these countries. Chen and Rogoff (2003) find a strong relationship between real exchange rates and commodity price shocks in New Zealand and Australia. Similarly, Cashin et al. (2004) examine the same phenomenon in a large number of commodity exporting countries and find a significant relationship in one third

of their sample. Bidarkota and Crucini (2000) find that a large portion of the terms of trade variation of developing countries is explained by the changes in the prices of three or fewer key exported commodities. Finally, Borenzstein and Reinhart (1994) emphasise the importance of supply side factors in the analysis of commodity price determination.

This is exactly what some newer generation models do. Ferrero and Seneca (2015) model a small, resource rich economy after Norway with a specified oil extraction process and a sovereign wealth fund. Their focus is on how monetary policy should react to oil price shocks given that oil extraction heavily influences the demand for domestic goods. Catao and Chang (2012) build a model of an economy that is both an importer and exporter of commodities. They find that risk sharing is an important determinant of the optimal policy rule, favouring PPI targeting under portfolio autarky and real exchange rate targeting under perfect risk sharing. Hevia and Nicolini (2013) also model an economy that is both an importer and exporter and emphasise the importance of the interaction between fiscal and monetary policies.

The thesis builds on the sticky price, sticky wage model of Erceg et al. (2000) by adding a commodity extraction sector that supplies intermediate good producers with a commodity to use in production alongside labour. This commodity can be interpreted broadly as a commodity bundle. An extraction firm uses up materials (final output) to extract this commodity, as in Ferrero and Seneca (2015). The production function is like its counterpart in Hevia and Nicolini (2013) but simpler since there is not one imported and exported commodity as in their model but just a single representative one.

These additions produce a model closely related to Bodenstein et al. (2008) as both are closed economy models with similar intermediate good production functions. However, their focus is on energy which is consumed by both households and firms. Energy supply is exogenous in their model whereas commodity supply is endogenous, determined by a profit maximising commodity extractor in the thesis model. Since there is no world

supply to compete or world demand to absorb any extracted amount, the commodity price is endogenous and adjusts to clear the commodity market, similar to Bodenstein et al. (2008) where energy price adjusts to clear the energy market but markedly different from the rest of the literature where commodity or energy prices are taken to be exogenous. The commodity also has a larger share in production in the thesis model.

This approach has various benefits. Explicitly having a commodity input in production reveals a lot more about the transmission mechanisms in comparison to using an ad hoc cost push shock. It becomes possible to model the economy at a deeper level and perform more involved comparative statics analysis. The knowledge of transmission mechanisms can be informative about policy as well. As Kilian (2006) notes, there are various types of shocks that may affect oil prices. With an exogenous commodity price, the analysis can only be one way by nature. However, when the commodity sector is explicitly modelled, it becomes possible to analyse how commodity prices are determined in the first place and affected by other shocks or parameters. In sum, the thesis model differs from the rest of the literature with the commodity price and level being determined endogenously as intermediate good producers and a commodity extractor optimise. This makes a more insightful model and leads to a richer analysis.

3 The Model

The model is built on the sticky price, sticky wage model of Erceg et al. (2000) by adding a profit maximising commodity extraction firm supplying a commodity to intermediate good producers which require it as a production input in addition to labour.

The economy is composed of differentiated households, intermediate good producers, a commodity extraction firm, a final good producer and a central bank. A continuum of households on the unit interval choose how much to consume and what wage to set to

maximise their discounted expected utilities. The commodity extraction firm chooses how much material (final output) to use to extract a commodity which it sells to intermediate good producers. The problem of the commodity extraction firm is identical to the one in Ferrero and Seneca (2015) in the sense that both involve a commodity extractor taking the commodity price as given and choosing how much material to use to maximise its profits. Since this is a closed economy model, there is no world demand to absorb all the commodity supply. Therefore, the commodity price is endogenous and adjusts to clear the commodity market. The continuum of monopolistically competitive intermediate good producers on the unit interval employ labour and commodity to produce differentiated goods and set prices to maximise their expected discounted profits. These intermediate goods are combined by a final good producer which is essentially a CES aggregator. There is also a central bank that can set the short term nominal interest rate. It follows a Taylor type rule, possibly reacting to one or a combination of price, wage or commodity price inflation and output gap. Figure 1 summarises the model.

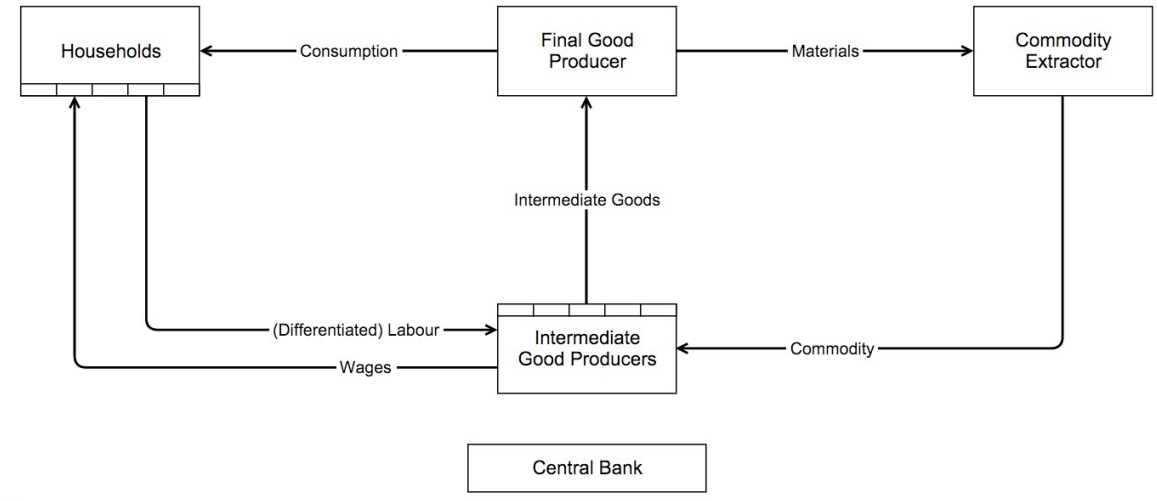

Figure 1: Model Structure

Markets are complete. This assumption greatly simplifies calculations in the household problem when combined with the assumption that utility is separable in consumption and labour.

For any variable X_t, x_t denotes its deviation from the steady state and $\tilde{x_t}$ the gap from its natural level. Subscript j refers to a household and subscript i refers to an intermediate good firm unless otherwise indicated. Detailed solutions can be found in the appendix. All loglinearizations are done around a zero inflation steady state.

3.1 Households

There is a continuum of households on the unit interval. They choose how much final good to consume, how many risk free bonds to buy and what wage to set to maximise their discounted expected utility subject to their budget constraints. Wage setting is a la Calvo (1983) so a household has a $(1-\theta_w)$ probability of resetting its wage in any given period, independent of how many periods have passed since it last reset its wage. Households will provide any amount of labour demanded when paid the wage they have set. Only the problem in which a household gets to set its wage once is considered because it will reoptimise using the process already described if it gets to readjust. Household j's utility maximisation problem when it gets to reset its wage in period t is

$$\max_{C_{jt+k|t},D_{jt+k|t},W_t^*} E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k U(C_{jt+k|t},N_{jt+k|t})$$

subject to

$$P_{t+k}C_{jt+k|t} + E_{t+k}Q_{t+k,t+k+1}D_{jt+k+1|t} = D_{jt+k|t} + T_{t+k} + \int_0^1 W_t^* N_{jit+k|t}di \quad \forall\, k \geq 0$$

$$N_{jit+k|t} = \left(\frac{W_t^*}{W_{t+k}}\right)^{-\varepsilon_w} N_{it+k} \quad \forall\, k \geq 0$$

where $W_{t+k} = \left(\int_0^1 W_{jt+k}^{1-\varepsilon_w} dj \right)^{\frac{1}{1-\varepsilon_w}}$

with a no Ponzi condition of the form $\lim_{k\to\infty} E_t Q_{t,t+k} D_{t+k} \geq 0$.

The $jt+k|t$ subscript denotes the period $t+k$ value of a variable of household j who last got to set its wage in period t. P_t denotes the price of consumption in period t and $E_t Q_{t,t+1}$ the price of a bond that yields a single unit of consumption in period $t+1$. D_t is the number of bonds bought in period $t-1$. ε_w is the elasticity of substitution between any two households' labour services. T_t represents transfers (profits from monopolistically competitive intermediate good producers and commodity extraction firm net of lump sum taxes) and $N_{jit+k|t}$ is the amount of labour firm i demands from household j in period $t+k$ given that the household has last set its wage in period t. W_t is a wage index.

The optimal wage for a household to set when it is able to do so is given by

$$W_t^* = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k W_{t+k}^{\varepsilon_w} N_{t+k} U_c(C_{t+k|t}, N_{t+k|t}) MRS_{t+k|t}}{E_t \sum_{k=0}^{\infty} (\beta\theta_w)^k W_{t+k}^{\varepsilon_w} \frac{N_{t+k}}{P_{t+k}} U_c(C_{t+k|t}, N_{t+k|t})}$$

Note that in the absence of wage rigidities ($\theta_w = 0$), a household will set its wage to be a constant mark up over its marginal rate of substitution.

$$\frac{W_t^*}{P_t} = \frac{\varepsilon_W}{\varepsilon_w - 1} MRS_t$$

The evolution of the wage index is given by

$$W_t = \left[\theta_w W_{t-1}^{1-\varepsilon_w} + (1-\theta_w) W_t^{*1-\varepsilon_w} \right]^{\frac{1}{1-\varepsilon_w}}$$

Combining the loglinearized form of the above equation with the loglinearized optimal wage setting equation yields a concise wage inflation equation.

$$\pi_t^w = \beta \pi_{t+1}^w - \lambda_w \mu_t^w$$

where μ_t^w is the deviation of wage mark up from its steady state value and $\lambda_w = \frac{(1-\theta_w)(1-\beta\theta_w)}{\theta_w(1+\varepsilon_w\phi)}$

A standard utility function of the following form

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\phi}}{1-\phi}$$

is used. σ is the coefficient of relative risk aversion and ϕ is the inverse Frisch elasticity of labour. The separability of utility in consumption and labour greatly simplifies our calculations when markets are complete. Under these two assumptions, households will make their consumption decisions based on their life history earnings, independent of when they last set their wages as Erceg et al. (2000) notes. Formally, $C_{t+k|t} = C_{t+k}$ will be the case. Optimal consumption choice leads to the standard Euler equation.

$$(1+i_t)\beta E_t \left[\frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right] = 1$$

where $(1+i_t)^{-1} = E_t Q_{t,t+1}$

Loglinearizing this expression around a zero inflation steady state gives

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(i_t - \rho - E_t \pi_{t+1}^p)$$

where $-\rho = log(\beta)$

The wage inflation equation and the Euler equation describe the aggregate household behaviour. Since the key differences of this model are in the production sector of the economy, it is not surprising to find households behaving in a manner typical of models characterised by wage setting.

3.2 Final Good Producer

The final good producer is just a CES aggregator of intermediate goods. The amount of final good produced is given by

$$Y_t = \left[\int_0^1 Y_{it}^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di \right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}}$$

where ε_p is the elasticity of substitution between any two different intermediate goods. This formulation implies that the demand for firm i's production will be

$$Y_{it} = \left(\frac{P_{it}}{P_t}\right)^{-\varepsilon_p} Y_t$$

where the price index P_t is

$$P_t = \left[\int_0^1 P_{it}^{1-\varepsilon_p} di\right]^{\frac{1}{1-\varepsilon_p}}$$

A unit of Y_t is sold for P_t. The final good producer does not make any profits. This may be rationalised by there being many identical and small final good firms in a perfectly competitive setting that are represented by a single final good producer.

3.3 Intermediate Good Producers

Each intermediate good producer uses differentiated household labour and commodity to produce an intermediate good according to the following production function

$$Y_{it} = A_t N_{it}^{\alpha} Z_{it}^{\gamma} \qquad \alpha + \gamma \leq 1$$

A_t, α and γ are common across all intermediate good producers. A_t is the production technology, Z_{it} is the amount of commodity used in production and N_{it} is a CES aggregate of differentiated household labour services where

$$N_{it} = \left[\int_0^1 N_{jit}^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dj\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}}$$

The profit maximisation problem of firms can be solved in several stages. Suppose they initially choose the optimal mix of N_{jit} by maximising N_{it} for any given level of wage payments. This gives rise to the labour demand equation in the household problem. Since all firms face the same profile of wages and have the same CES labour aggregator, they will have the same optimal mix of differentiated household labour with

$$N_{jit} = \left(\frac{W_{jt}}{W_t}\right)^{-\varepsilon_w} N_{it} \quad \forall\, j \in [0,1]$$

$$W_t = \left[\int_0^1 W_{jt}^{1-\varepsilon_w} dj\right]^{\frac{1}{1-\varepsilon_w}}$$

After choosing the optimal mix of N_{jit}, a firm chooses N_{it} and Z_{it} to minimise its costs given a certain level of production Y_{it}. Firm i's optimal labour demand is

$$N_{it} = \left(\frac{Y_{it}}{A_t}\right)^{\frac{1}{\alpha+\gamma}} \left(\frac{\alpha}{\gamma}\frac{Q_t^r}{W_t^r}\right)^{\frac{\gamma}{\alpha+\gamma}}$$

where W_t^r is the real wage rate and Q_t^r is the real commodity price. This equation has an intuitive appeal. Amount of labour demanded increases with the level of production and labour productivity (α) but decreases with wages. Higher commodity productivity (γ) and lower commodity prices lead firms to substitute commodity for some labour and decreases labour demand. Firm i's commodity demand is given by a symmetric expression

$$Z_{it} = \left(\frac{Y_{it}}{A_t}\right)^{\frac{1}{\alpha+\gamma}} \left(\frac{\gamma}{\alpha}\frac{W_t^r}{Q_t^r}\right)^{\frac{\alpha}{\alpha+\gamma}}$$

which has the same intuitive appeal. Demand for both factors of production fall with a higher level of productivity (A_t) for a given level of production.

Loglinearizing these expressions around the steady state and aggregating them across firms give us

$$n_t = \frac{1}{\alpha+\gamma}(y_t - a_t) + \frac{\gamma}{\alpha+\gamma}(q_t^r - w_t^r)$$

$$z_t = \frac{1}{\alpha+\gamma}(y_t - a_t) + \frac{\alpha}{\alpha+\gamma}(w_t^r - q_t^r)$$

where $x_t = log(X_t) - log(X) = log(\int_0^1 X_{it} di) - log(X)$ for $X_{it} = N_{it}$, Z_{it} , Y_{it}

Finally, firms set prices to maximise their expected discounted profits taking into account that only a $(1-\theta_p)$ fraction of them will be able to reset their price in any given period. Firm i's price setting problem is to

$$\max_{P_t^*} E_t \sum_{k=0}^{\infty} \theta_p^k Q_{t,t+k}(P_t^* Y_{it+k|t} - TC(Y_{it+k|t}))$$

subject to

$$Y_{it+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon_p} Y_{t+k}$$

where $TC(Y_{it+k|t})$ is the total cost of producing $Y_{it+k|t}$ units.

The optimal price for a firm to choose is given by

$$P_t^* = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{E_t \sum_{k=0}^{\infty} (\theta_p \beta)^k P_{t+k}^{\varepsilon_p - 1} (C_{t+k}^{1-\sigma} + C_{t+k}^{-\sigma} M_{t+k}) MC_{it+k|t}}{E_t \sum_{k=0}^{\infty} (\theta_p \beta)^k P_{t+k}^{\varepsilon_p - 1} (C_{t+k}^{1-\sigma} + C_{t+k}^{-\sigma} M_{t+k})}$$

where M_t is the amount of material used by the commodity extractor. $E_t Q_{t,t+k}$ is the discount rate between period $t+k$ and t. It is a function of consumption and prices. Note that intermediate goods are not only aggregated to meet households' consumption demand but also commodity extractor's materials demand. Therefore a wedge between consumption and output arises. This gives rise to the additional M_t term in the above expression. Otherwise, this a standard solution to the price setting problem.

Note that in the absence of price rigidities $(\theta_p = 0)$, price will just be a constant mark up over marginal cost.

$$P_t^* = \frac{\varepsilon_p}{\varepsilon_p - 1} MC_t$$

This implies that real marginal cost will just be the inverse of the price mark up.

$$MC_t^r = \frac{\varepsilon_p - 1}{\varepsilon_p}$$

The evolution of prices is described by

$$P_t = \left[\theta_p P_{t-1}^{1-\varepsilon_p} + (1-\theta_p) P_t^{*1-\varepsilon_p}\right]^{\frac{1}{1-\varepsilon_p}}$$

Using the loglinearized version of the above expression with the loglinearized optimal price setting equation yields

$$p_t^* = \mu^p + (1-\theta_p\beta)E_t\sum_{k=0}^{\infty}(\theta_p\beta)^k(mc_{it+k|t}^r + p_{t+k})$$

where $\mu^p = log(\frac{\varepsilon_p}{\varepsilon_p - 1})$

Manipulating this equation further by using labour, commodity and intermediate good demand equations gives a more concise price inflation equation.

$$\pi_t^p = \beta\pi_{t+1}^p + \lambda_p mc_t^r$$

where $\lambda_p = \frac{(1-\theta_p\beta)(1-\theta_p)}{\theta_p}\frac{(\alpha+\gamma)}{(1-\varepsilon_p)(\alpha+\gamma)+\varepsilon_p}$

Finally, production technology is subject to shocks and follows an AR(1) process.

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a$$

where $\varepsilon_t^a \sim N(0, \sigma_a^2)$

3.4 Commodity Extractor

The commodity extraction firm decides how much material to use in extraction, taking the commodity price Q_t and the final good price P_t as given as in Ferrero and Seneca (2015). The profit maximisation problem of the commodity extraction firm is given by

$$\max_{M_t} Q_t Z_t - P_t M_t$$

subject to the commodity extraction function

$$Z_t = S_t M_t^{\chi}$$

where Z_t is the amount of commodity extracted, S_t is a measure of extraction technology and M_t is the amount of material used up in the commodity extraction process.

Extraction technology is subject to shocks and follows an AR(1) process.

$$s_t = \rho_s s_{t-1} + \varepsilon_t^s$$

where $\varepsilon_t^s \sim N(0, \sigma_s^2)$

The optimal choice of materials is

$$M_t^* = (\chi Q_t^r S_t)^{\frac{1}{1-\chi}}$$

which will lead to a commodity level of

$$Z_t^* = S_t^{\frac{1}{1-\chi}} (\chi Q_t^r)^{\frac{\chi}{1-\chi}}$$

The loglinearized forms of these two equations are

$$m_t = \frac{1}{1-\chi}(q_t^r + s_t)$$

$$z_t = \frac{1}{1-\chi} s_t + \frac{\chi}{1-\chi} q_t^r$$

One can think of this commodity as a bundle of different commodities rather than simply taking it to be oil. Note that the commodity extraction process is a capital intensive one where materials are used up. Alternatively, it can be modelled as a labour intensive process like Catao and Chang (2012) or Hevia and Nicolini (2013) do.

3.5 Central Bank

There is a central bank that can set the nominal interest rate in the short run. It can react to price, wage or commodity price inflation as well as the output gap. It follows a Taylor type interest setting rule of the form

$$i_t = \rho + \phi_\pi \pi_t^p + \phi_w \pi_t^w + \phi_q \pi_t^q + \phi_y \tilde{y}_t + v_t$$

where $\tilde{y}_t$ is the deviation of output from its natural level and v_t is a monetary policy shock that follows and AR(1) process.

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v$$

where $\varepsilon_t^v \sim N(0, \sigma_v^2)$

This interest setting rule closes the model. The model can be solved by using the first order conditions outlined so far after imposing the market clearing conditions shown next.

3.6 Market Clearing

The final good market should clear which implies

$$Y_t = C_t + M_t$$

or in loglinearized form

$$y_t = (1-\delta)c_t + \delta m_t$$

where $\delta = \frac{M}{Y}$ is the steady state ratio of materials to output.

In open economy models of small resource rich economies, it is generally assumed that the commodity price is exogenous and world demand can absorb any amount of supply. Similarly, oil price is taken to be exogenous and world supply can always meet domestic demand in models of oil importing countries. However, this is a closed economy model in which both the demand for and the supply of the commodity are microfounded. There is a commodity market that should clear through the adjustment of the real commodity price. Therefore, Q_t^r should equate the demand and the supply in the commodity market. More formally, the sum of commodity demands of firms should match the supply.

$$\int_0^1 Z_{it} di = \int_0^1 \left(\frac{Y_{it}}{A_t}\right)^{\frac{1}{\alpha+\gamma}} \left(\frac{\gamma}{\alpha}\frac{W_t^r}{Q_t^r}\right)^{\frac{\alpha}{\alpha+\gamma}} di = S_t^{\frac{1}{1-\chi}} (\chi Q_t^r)^{\frac{\chi}{1-\chi}}$$

should hold which is

$$z_t = \frac{1}{1-\chi} s_t + \frac{\chi}{1-\chi} q_t^r = \frac{1}{\alpha+\gamma}(y_t - a_t) + \frac{\alpha}{\alpha+\gamma}(w_t^r - q_t^r)$$

in loglinearized form. This implies that the equilibrium real commodity price is

$$q_t^r = \frac{1-\chi}{\alpha+\chi\gamma}(y_t - a_t + \alpha w_t^r) - \frac{\alpha+\gamma}{\alpha+\chi\gamma} s_t$$

The term in brackets can be interpreted as the demand side factors of equilibrium real commodity price determination. Higher output requires more factor input, raising the demand for and thus the price of the commodity. An increase in the level of production technology has the opposite effect with more efficient production reducing the demand for inputs for any given level of production. Higher wages lead firms to substitute some commodity for labour, raising the demand for the commodity and its price. The second term represents the only supply side factor. Better commodity extraction technology makes extraction less costly and reduces the real commodity price.

Intermediate good suppliers are willing to supply any amount demanded when paid the prices they have set. Similarly, households are willing to supply any amount of labour demanded if they receive the wages they have set. Therefore, labour and intermediate goods markets will always clear.

3.7 Steady State, Natural Levels and Efficient Allocation

Steady State

The economy will be at its steady state in the absence of any shocks: $a_t = s_t = v_t = 0$. Steady state values of the model variables are given below. Two ratios (δ and η) are defined to ease notation.

$$\delta = \frac{M}{Y} = \frac{\varepsilon_p - 1}{\varepsilon_p}\chi\gamma \qquad \eta = \left(\frac{\varepsilon_w - 1}{\varepsilon_w}\frac{\alpha}{\chi\gamma}\right)^{\alpha}$$

$$Y = \left(\frac{\eta\delta^{(\alpha+\chi\gamma+\chi\gamma\phi)}}{(1-\delta)^{\alpha\sigma}}\right)^{\frac{1}{\kappa}} \qquad \text{where } \kappa = 1+\phi+\alpha\sigma-\alpha-\chi\gamma-\chi\gamma\phi$$

$$C = (1-\delta)Y \qquad M = \delta Y$$

$$Z = M^{\chi} = (\delta Y)^{\chi} \qquad Q^r = \frac{M^{1-\chi}}{\chi} = \frac{(\delta Y)^{1-\chi}}{\chi}$$

$$N = \left(\frac{Y}{Z^{\gamma}}\right)^{\frac{1}{\alpha}} = \left(\frac{Y^{1-\gamma\chi}}{\delta^{\gamma\chi}}\right)^{\frac{1}{\alpha}} \qquad W^r = \frac{\varepsilon_w}{\varepsilon_{w-1}}N^{\phi}C^{\sigma} = \frac{\varepsilon_w}{\varepsilon_{w-1}}\frac{(1-\delta)^{\sigma}}{\delta^{\frac{\phi\chi\gamma}{\alpha}}}Y^{(\sigma+\frac{(1-\chi\gamma)\phi}{\alpha})}$$

Real rigidities arise from the bargaining power of households and price setting power of monopolistically competitive firms. The limiting case of $\varepsilon_p \to \infty$ and $\varepsilon_w \to \infty$ can serve as an efficient benchmark in which these rigidities vanish. Observing the steady state expressions reveals that both δ and η are lower than their efficient counterparts which implies an inefficiently low level of output. It immediately follows that consumption, materials, commodity level and real commodity price are all inefficiently low. Lower consumption puts downward pressure on wages but whether this is enough to offset the effects of wage setting depends on the parameter values chosen. Similarly, the effect of real rigidities on employment is not immediately obvious. Nevertheless, both are lower than their efficient counterparts in reasonable parameterisations of the model and under the baseline calibration. In sum, it is clear that real rigidities in goods and labour markets have overarching effects on the economy.

Natural Levels

The levels that will prevail in the absence of both nominal rigidities ($\theta_p = \theta_w = 0$) are the natural levels. Both production technology (a_t) and commodity extraction technology (s_t)

matter for economic activity. Therefore, natural levels will depend on these as well as the structural parameters of the model.

The log natural level of output is given by

$$y_t^n = \frac{1+\phi}{\kappa}(a_t + \gamma s_t) + \frac{1}{\kappa} log\left(\frac{\eta \delta^{\chi\gamma+\chi\gamma\phi+\alpha}}{(1-\delta)^{\alpha\sigma}}\right)$$

The natural levels of all the remaining variables are defined in a similar form. Refer to Appendix A for a complete list.

Efficient Levels

It has been shown that real rigidities arising from the bargaining power of households and price setting power of firms lead to inefficiencies. An optimal subsidy scheme that can overcome the effects of these real rigidities and restore efficiency is proposed. The benevolent social planner problem is to

$$\max_{C_{jt},N_{jit},Z_{it}} \int_0^1 U(C_{jt},N_{jt})dj$$

subject to

$$Y_t = \left[\int_0^1 Y_{it}^{\frac{\varepsilon_p-1}{\varepsilon_p}} di\right]^{\frac{\varepsilon_p}{\varepsilon_p-1}} \qquad Y_{it} = A_t N_{it}^{\alpha} Z_{it}^{\gamma} \qquad Y_t = C_t + M_t$$

$$N_{it} = \left[\int_0^1 N_{jit}^{\frac{\varepsilon_w-1}{\varepsilon_w}} dj\right]^{\frac{\varepsilon_w}{\varepsilon_w-1}} \qquad \int_0^1 Z_{it} di = Z_t \qquad Z_t = S_t M_t^{\chi}$$

The optimality conditions are

$$C_{it} = C_t \qquad N_{it} = N_t \qquad Z_{it} = Z_t$$

$$Y_{it} = Y_t \qquad -\frac{U_n(C_t,N_t)}{U_c(C_t,N_t)} = MPN_t \qquad \frac{M_t}{Y_t} = \chi\gamma$$

Households are symmetric and labour is CES aggregated. Intermediate goods produced according to the same production function are also CES aggregated. Given these aspects of the model, the first four symmetry conditions are expected.

In the absence of any corrective subsidisation,

$$-\frac{U_n(C_t,N_t)}{U_c(C_t,N_t)} = MRS_t = \frac{\varepsilon_w - 1}{\varepsilon_w}\frac{W_t}{P_t}$$

$$\frac{W_t}{P_t}\frac{1}{MPN_t} = MC_t^r = \frac{\varepsilon_p - 1}{\varepsilon_p}$$

will be the case. These two jointly imply that

$$-\frac{U_n(C_t,N_t)}{U_c(C_t,N_t)} = \frac{\varepsilon_w - 1}{\varepsilon_w}\frac{\varepsilon_p - 1}{\varepsilon_p}MPN_t$$

violating one of the efficiency conditions. It is possible to eliminate the wedge between the marginal rate of substitution and the marginal product of labour by a wage subsidy $(1-\tau)$ where

$$\tau = 1 - \frac{1}{\frac{\varepsilon_p}{\varepsilon_p - 1}\frac{\varepsilon_w}{\varepsilon_w - 1}}$$

This wage subsidy has now become a standard corrective measure against inefficiencies caused by real rigidities. In a standard sticky price/sticky wage model, wage mark ups push wages above the optimal level and lower the demand for labour. Price setting power of monopolistically competitive firms lead to a lower than efficient level of production and as a result to an inefficiently low level of employment. A correctly chosen wage subsidy suffices to offset these effects since both act on labour, the single factor of production. However, an additional commodity subsidy is needed in this model as an inefficient level of output leads to inefficiently low levels of commodity and materials being utilised. In the absence of corrective subsidisation

$$\delta = \frac{M_t}{Y_t} = \frac{\varepsilon_p - 1}{\varepsilon_p}\chi\gamma$$

whereas the efficient materials to output ratio is

$$\delta^{eff} = \chi\gamma$$

Correcting this inefficiency is possible with a commodity subsidy of $1-\iota$ where

$$\iota = 1 - \frac{1}{\frac{\varepsilon_p}{\varepsilon_p - 1}}$$

Once this corrective subsidy scheme is in place, natural levels will be efficient. Simply, replacing δ and η with δ^{eff} and ηeff respectively, where

$$\delta^{eff} = \chi\gamma \qquad \eta^{eff} = \left(\frac{\alpha}{\chi\gamma}\right)^{\alpha}$$

will give the efficient levels. Note that the outcome obtained under the subsidy scheme corresponds to the efficient benchmark obtained by letting $\varepsilon_p \to \infty$ and $\varepsilon_w \to \infty$ which is the limiting case where firms and households have no bargaining power.

3.8 The Loglinearized Model

The following equations are obtained after loglinearizing all the first order conditions, imposing the market clearing conditions and solving for the model in terms of gaps from the natural levels.

Loglinearized Model Equations

1. $\pi_t^p = \beta E_t \pi_{t+1}^p + \kappa_{pw}\tilde{w}_t + \kappa_{py}\tilde{y}_t - \kappa_{pq}\tilde{q}_t$	Price inflation
2. $\pi_t^w = \beta E_t \pi_{t+1}^w - \kappa_{ww}\tilde{w}_t + \kappa_{wy}\tilde{y}_t - \kappa_{wq}\tilde{q}_t$	Wage inflation
3. $q_t^r = q_{t-1}^r + \pi_t^q - \pi_t^p$	Evolution of commodity price
4. $\tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{\delta}{1-\chi}E_t(\tilde{q}_{t+1} - \tilde{q}_t) - \frac{1-\delta}{\sigma}(i_t - E_t\pi_{t+1}^p - r_t^n)$	Evolution of output gap
5. $w_t^r = w_{t-1}^r + \pi_t^w - \pi_t^p$	Evolution of real wage
6. $\tilde{q}_t = \frac{1-\chi}{\alpha+\chi\gamma}(\tilde{y}_t + \alpha\tilde{w}_t)$	Commodity price gap
7. $\tilde{c}_t = \frac{\tilde{y}_t - \delta\tilde{m}_t}{1-\delta}$	Consumption gap
8. $\tilde{m}_t = \frac{1}{1-\chi}\tilde{q}_t$	Materials gap
9. $\tilde{z}_t = \chi\tilde{m}_t$	Commodity gap
10. $\tilde{n}_t = \frac{\tilde{y}_t - \gamma\tilde{z}_t}{\alpha}$	Employment gap
11. $r_t^n = \rho + \frac{\sigma}{\kappa}(1+\phi)[(\rho_a - 1)a_t + (\rho_s - 1)\gamma s_t]$	Natural interest rate

Central Bank

12. $i_t = \rho + \phi_\pi\pi_t^p + \phi_w\pi_t^w + \phi_q\pi_t^q + \phi_y\tilde{y}_t$	Policy rule

Technology and Shocks

13. $a_t = \rho_a a_{t-1} + \varepsilon_t^a$	Production technology
14. $s_t = \rho_s s_{t-1} + \varepsilon_t^s$	Extraction technology
15. $v_t = \rho_v v_{t-1} + \varepsilon_t^v$	Monetary policy shock

The complete expression for price inflation is

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p \tilde{w}_t + \frac{1-\alpha}{\alpha}\lambda_p \tilde{y}_t - \frac{\gamma}{\alpha}\frac{\chi}{1-\chi}\lambda_p \tilde{q}_t$$

It is actually possible to eliminate the commodity price gap term from price and wage inflation equations. Inflation equations are presented this way to show that this model converges to the standard sticky price, sticky wage model of Erceg et al. (2000) when the weight on the commodity in production is zero ($\gamma = 0$). When this is the case, firms will not demand any commodity since it has no contribution to production. Commodity extractor will not extract any commodity since there is no demand for it. As a result, the model will be of a closed economy characterised by price and wage setting where the only productive factor is labour. The price inflation equation will be given by

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p \tilde{w}_t + \left(\frac{1}{\alpha} - 1\right)\lambda_p \tilde{y}_t$$

which we can easily compare with the price inflation equation of our model after using the commodity price gap expression to rewrite the price inflation equation as

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \frac{\alpha}{\alpha + \chi\gamma}\lambda_p \tilde{w}_t + \left(\frac{1}{\alpha + \chi\gamma} - 1\right)\lambda_p \tilde{y}_t$$

λ_p term is unchanged if the total weight on the input(s) in the production function is the same across the two models. Suppose this is the case to make comparison easier. One sees that a given level of wage gap has a smaller effect on price inflation when firms can use the commodity in production. Firms will substitute commodity for some labour when wages are high. Marginal costs will rise less rapidly and firms will raise their prices by less. Similarly, a given level of output gap has a smaller impact on inflation when commodity is a productive factor. In the absence of the commodity input, firms can only increase employment to produce more. This drives up household marginal rates of substitution and leads them to set higher wages. Higher wages imply higher marginal costs and more price inflation. Since household disutility is convex in labour, households will demand

increasingly higher wages and this effect will get stronger. Therefore, achieving increased production through two factors raises marginal costs by less and inflation reacts less to output gaps.

The full expression for wage inflation is given by

$$\pi_t^w = \beta E_t \pi_{t+1}^w - \lambda_w \tilde{w}_t + \left(\frac{\sigma}{1-\delta} + \frac{\phi}{\alpha} \right) \lambda_w \tilde{y}_t - \left(\frac{\sigma\delta}{1-\delta} + \frac{\phi\chi\gamma}{\alpha} \right) \frac{1}{1-\chi} \lambda_w \tilde{q}_t$$

When $\gamma = 0$ which also implies $\delta = 0$, the expression simplifies to the standard wage setting equation.

$$\pi_t^w = \beta E_t \pi_{t+1}^w - \lambda_w \tilde{w}_t + \left(\sigma + \frac{\phi}{\alpha} \right) \lambda_w \tilde{y}_t$$

Once again, it is possible to rewrite the original wage inflation equation as

$$\pi_t^w = \beta E_t \pi_{t+1}^w - \left(1 + \frac{\phi\chi\gamma}{\alpha+\chi\gamma} + \frac{\alpha}{\alpha+\chi\gamma} \frac{\sigma\delta}{1-\delta} \right) \lambda_w \tilde{w}_t + \left(\sigma \frac{1 - \frac{\delta}{\alpha+\gamma\chi}}{1-\delta} + \frac{\phi}{\alpha+\gamma\chi} \right) \lambda_w \tilde{y}_t$$

Although the expression is not very concise, comparing it with the standard wage inflation equation in the absence of the commodity input is relatively easy when one remembers that all parameters are positive, $\alpha + \gamma\chi \leq 1$ and $1 - \delta > 0$.

Wage inflation reacts more strongly to a wage gap when firms use the commodity as an additional input. Wages fall more in response to a positive wage gap because in addition to households revising their wages downward, firms are substituting commodity for some labour. This reduces employment which puts a further downward pressure on wages. On the other hand, wage inflation reacts less strongly to an output gap in comparison to the no commodity case. This is because when firms are expanding output, they are not solely relying on employment. Some of the expansion comes from using more commodity. As a result, employment rises less rapidly and households that can adjust their wages raise it by less than their counterparts in a model without commodities.

In sum, the commodity sector provides an additional input for production which enables intermediate good producers to substitute one input for another if it is cost effective to do so. As a result, marginal costs and prices are less reactive to wage and output gaps. This substitution effect also makes wage inflation less responsive to an output gap but more to a wage gap.

The evolution of the output gap is described by equation 4 which is just a slightly modified IS equation. It is obtained from

$$\tilde{c}_t = E_t\tilde{c}_{t+1} - \frac{1}{\sigma}(i_t - E_t\pi^p_{t+1} - r^n_t)$$

by accounting for the difference between consumption and output, arising from some output being used as materials in the commodity extraction process. This gives us

$$\tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{\delta}{1-\chi}E_t(\tilde{q}_{t+1} - \tilde{q}_t) - \frac{1-\delta}{\sigma}(i_t - E_t\pi^p_{t+1} - r^n_t)$$

The ways in which this expression differs from a standard IS curve have an intuitive appeal. Interest rates only affect consumption which is only (1 - δ) of output. This is why the effect of real interest rate gap $(i_t - E_t\pi^p_{t+1} - r^n_t)$ is toned down by this term. Similarly, materials is a fraction δ of output and is proportional to commodity prices. Accounting for this gives rise to the change in the commodity price gap term, $E_t(\tilde{q}_{t+1} - \tilde{q}_t)/(1-\chi)$. When the real interest rate is higher than its natural level, output gap will fall as consumers decrease their consumption. An expected increase in the commodity price gap also lowers the output gap. Higher commodity prices will negatively affect production and since output is a forward looking variable, its current value is affected by a change that will take place next period.

The rest of the equations are production/extraction related expressions, market clearing conditions and identities. Equation 3 and 5 describe the evolution of real commodity price and real wage respectively. Equation 6 shows the deviation of equilibrium real

commodity price from its natural level. It is included to ensure that the commodity market clears. Equation 7 is obtained from the final good market clearing condition. Equation 8 describes the optimal materials choice of the commodity extractor. Equation 9 is just the commodity extraction function. Equation 10 is the rearranged and aggregated version of the intermediate goods production function. Equation 11 reproduced below, gives us the natural real interest rate.

$$r_t^n = \rho + \frac{\sigma}{\kappa}(1+\phi)[(\rho_a - 1)a_t + (\rho_s - 1)\gamma s_t]$$

A higher than steady state level production or extraction technology will lead to a lower natural real rate (since $\rho_a, \rho_s \leq 1$) as sustaining an increased but falling level of output requires a lower but rising interest rate to satisfy the Euler equation. The same rise in extraction technology requires a smaller fall in the interest rate (since $\gamma < 1$) compared to production technology since it only increases production indirectly by increasing the amount of commodity available. Note that a larger fall will be required with a higher γ or χ since κ is decreasing in these parameters. When more commodity is used with higher γ or it is less costly to extract the commodity with higher χ, the commodity supply will expand more to accommodate higher levels out output. The natural rate of output will be higher and a lower interest rate will be needed. Parameters related to household utility (σ, ϕ) also affect the natural real rate directly and through κ.

Equation 12 describes the nominal interest setting rule of the central bank. It is a standard one where the nominal interest rate can react to price, wage or commodity price inflation and output gap. Equation 13, 14 and 15 describe the evolution of production technology, extraction technology and monetary policy shock respectively.

Once the model parameters are appropriately calibrated, it will be possible to simulate the model by using the equations provided in this section along with the natural level equations provided in the appendix.

4 Calibration, Simulation and Analysis

Parameter calibrations are shown below.

β	α	γ	χ	σ	ϕ	ε_p	ε_w
0.99	0.6	0.4	0.3	1.5	1.5	6	6

θ_p	θ_w	ϕ_y	ϕ_p	ϕ_w	ϕ_q	ρ_a/ρ_s	ρ_v
$\frac{3}{4}$	$\frac{3}{4}$	0	1.5	0	0	0.9	0.5

With a quarterly model in mind, the discount rate is chosen to be 0.99 which implies a 4% annual return in the steady state. The production function need not have constant returns to scale but I follow Hevia and Nicolini (2013) and calibrate it to be so. Their constant returns to scale production function is richer with one domestic and one imported commodity which jointly have a weight of 0.43. Since this is a closed economy model with a single commodity, the weight on the commodity is set to 0.4 which implies a weight of 0.6 on labour. χ is calibrated to 0.3 following Ferrero and Seneca (2015) who use a value of 0.28 in their model of a resource rich economy calibrated after Norway. σ and γ are adjusted to 1.5 to get a richer response to a productivity shock which would not affect the natural level of employment in the simple case of log utility in consumption with $\sigma = 1$. ε_p and ε_w are 6 as in Bodenstein et al. (2008) which imply steady state wage and price mark ups of 20%. The Calvo parameters are chosen so that both wages and prices are expected to change annually. In this baseline calibration, the central bank is only targeting goods price inflation so $\phi_p = 1.5$ and $\phi_w = \phi_y = \phi_q = 0$ are the case. As in Gali (2008), technology shocks are calibrated to be quite persistent with $\rho_a = \rho_s = 0.9$ unlike monetary policy shocks that are only moderately persistent with $\rho_v = 0.5$. With

this baseline calibration, 10% of final output is used as materials and the rest is consumed by households.

The model is simulated with Dynare (4.4.3) using this baseline calibration if not indicated otherwise. The response of the economy to productivity, extraction and monetary policy shocks are illustrated. A comparison with a standard no commodity model is provided. The effects of price and wage rigidities are analysed. Note that the following impulse responses show the deviations of variables from their steady state values. All inflation terms are nominal. All inflation and interest rate terms are annualised.

4.1 A Productivity Shock

Figure 2 illustrates the response of the economy to a positive productivity shock that raises the production technology level a_t by 1%. Natural levels of all variables except employment and real interest rate rise. Higher productivity lowers intermediate good producers' marginal costs. Firms that can readjust, charge lower prices. This gives rise to price disinflation. The strictly price inflation targeting central bank lowers the nominal interest rate even further and the real interest rate falls.

Falling real rate stimulates consumption as households optimise according to their Euler equations. More intuitively, lower returns discourage saving and increase consumption. This drives up output by 0.72% but a negative output gap still emerges as the nominal rigidities in goods and labour markets prevent output from fully adjusting. Even though output is increasing, employment and commodity levels fall as improved productivity is more than enough to expand production without increasing the utilisation of inputs.

Commodity prices and wages differ in their response because the former continuously adjusts to clear the commodity market whereas the latter is set by households as mark ups over their marginal rates of substitution, taking into account that they might not be able

to reoptimise for some time. Higher consumption increases households' marginal rates of substitution whereas lower employment reduces it. As a fraction θ_w of households readjust their wages every period, an increasing real wage profile emerges even though wage inflation is actually negative. This is because households actually find it optimal to reduce their nominal wages since prices are already falling very rapidly.

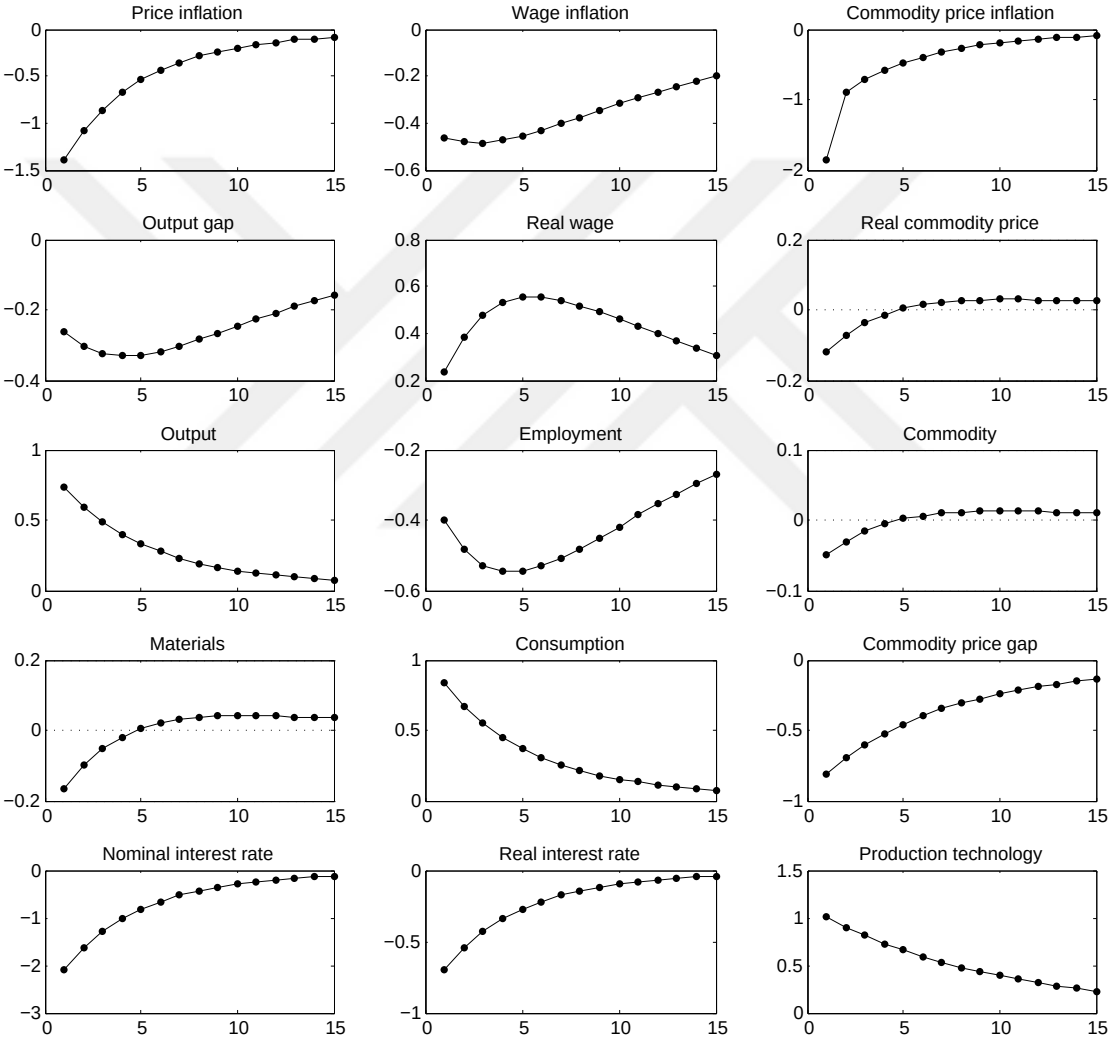

Figure 2: A Productivity Shock

When the commodity demand falls with increased productivity, the real commodity price falls too. Commodity extractor reduces the level of materials employed in extraction and cuts back the commodity supply. The change in the real commodity price is not very dramatic since supply falls alongside demand. However, there is still a large negative commodity price gap since a higher level of productivity raises the marginal product of the commodity which would result in a higher commodity price in the absence of nominal rigidities.

As productivity reverts back to its steady state level, firms increase their demand for factor inputs to meet their production demands. Facing higher levels of real wages, they choose to substitute some commodity for labour. Increased commodity demand raises the amount of commodity extracted, materials and the real commodity price. Given the nominal rigidities in the labour market, it takes a relatively long time for real wages to adjust. As a result, the increased demand for the commodity persists over a long time and there is a prolonged period where commodity price and level are slightly above their steady state values. Commodity disinflation is also very strong as a fall in real commodity price when prices are already falling rapidly requires.

Commodity price inflation has the strongest reaction to the productivity shock whereas wage inflation has the weakest. This seems to be in line with the empirical regularities. Also, the boom is associated with a higher commodity level and price a year after the impact of the shock even though they have initially fallen.

4.2 An Extraction Shock

A shock to the extraction technology s_t raises it by 1%. This increases the natural levels of all variables except employment, the real commodity price and the real interest rate. The response of the economy is illustrated in Figure 3. It is similar to the response to a

productivity shock except for the reaction of the commodity sector. It is also more toned down since the extraction shock affects output indirectly by providing the economy with more commodity at a lower price, different from a productivity shock that directly impacts production.

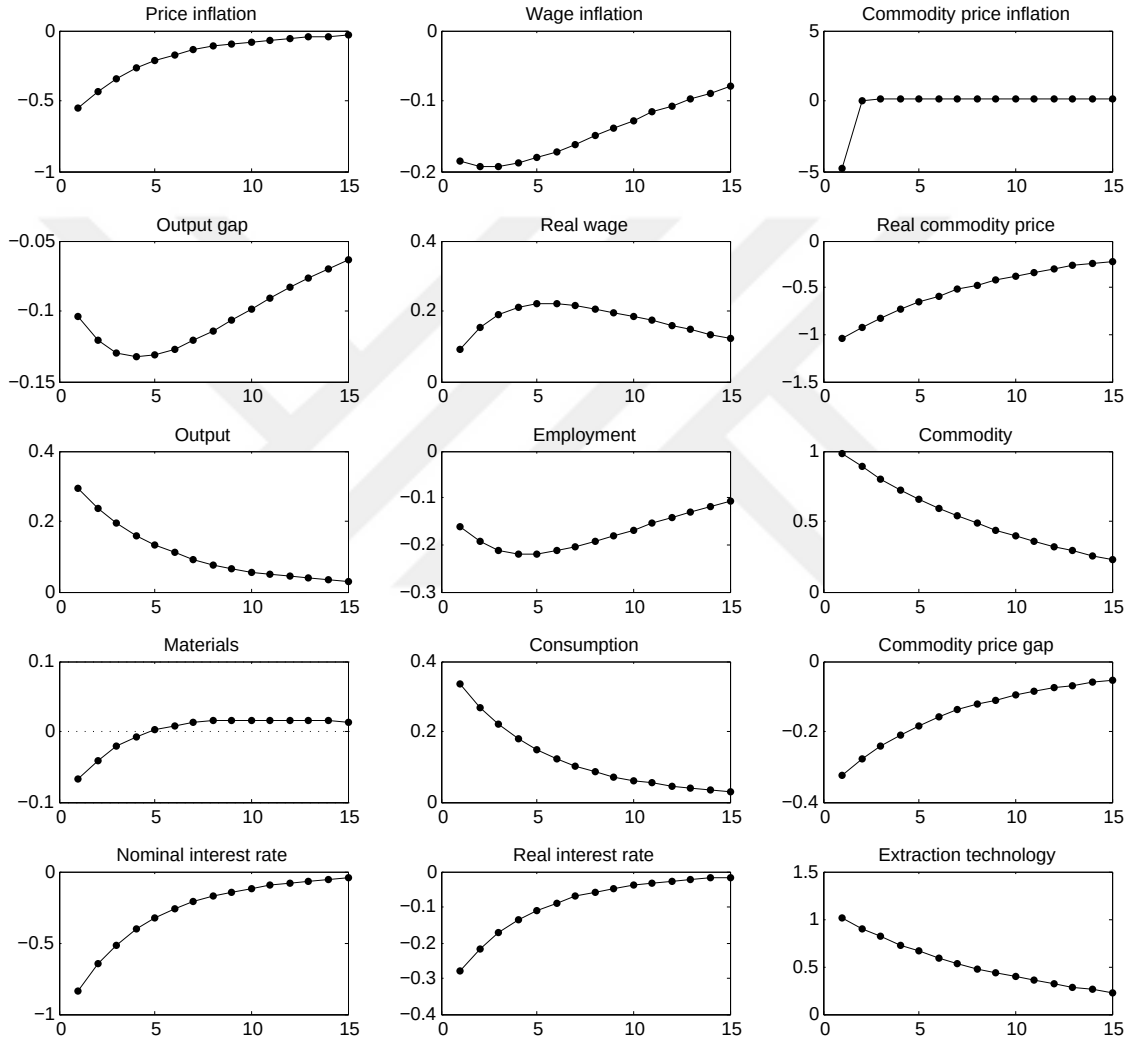

Figure 3: An Extraction Shock

Supply of the commodity increases rapidly with the shock and the real commodity price falls to clear the commodity market. Since less material is now needed to extract any given level of commodity, material level initially falls. Intermediate good producers' marginal costs fall too. Those who can readjust, charge lower prices which leads to price disinflation. Commodity price disinflation is even stronger since real commodity price itself is falling. The central bank reacts to negative price inflation by lowering the nominal interest rate even more and real interest rate falls. Households increase their consumption as returns to savings are reduced. A 0.29% rise in output follows. Nevertheless, output cannot fully adjust due to nominal rigidities and a negative output gap emerges. Also, note that the rise in output is slightly less than the rise in consumption since the demand for materials is falling with enhanced commodity extraction.

Firms substitute commodity for some labour to minimise their production costs which reduces employment. This puts a downward pressure on wages by reducing households' marginal rates of substitution but higher consumption has the opposite effect. Although there is negative wage inflation, an increasing real wage profile emerges due to rapidly falling prices. A fraction θ_w of households must find it optimal to disinflate their nominal wages for wage disinflation to take place. The rise in real wages provides another reason for employment to fall.

There are two additional details to note here. First, even though materials is initially lower with better extraction technology enabling higher amounts to be extracted, it goes slightly above its steady state value and stays there for a prolonged period of time. A shock that makes it less costly to extract the commodity actually increases the amount of materials employed because higher wages in the labour market persists a long time due to nominal rigidities, making the commodity a more preferred input even when the extraction shock is dying down. Second, the commodity price gap is negative. In the absence of nominal rigidities, output would expand further with stronger price disinflation

leading to a lower real interest rate and higher consumption. Demand for commodity would be even higher which would offset some of the fall in the real commodity price and a negative commodity price gap would not emerge. These results make it clear that nominal rigidities in other markets affect the commodity market which is not associated with any rigidities itself.

In sum, a positive extraction shock can be another source of output expansion. On the other hand, a negative extraction shock can lead to a commodity price hike and depress the economy. Recent developments like the advances in shale gas extraction activities or Iran coming back into the world oil market can be interpreted as positive extraction shocks. Finally, whether booms or busts are associated with rising or falling commodity prices depends on the types of shocks hitting the economy as illustrated by the two cases presented so far.

4.3 A Monetary Policy Shock

A mildly persistent monetary policy shock calibrated to raise the nominal interest rate by 25 basis points hits the economy as shown in Figure 4. Nominal interest rate rises a little less than this as the central bank contemporaneously reacts to negative price inflation as explained below. The annualised real interest rate ends up rising less than one per cent. Natural levels are unchanged since production and extraction technologies are unaffected by the monetary policy shock.

A higher real interest rate lowers consumption by incentivising households to save more. Output falls by 0.23% following consumption. Lower output reduces the demand for productive factors so both employment and commodity levels fall. The real commodity price decreases with the reduced demand for the commodity. This reduces firms' marginal cost, leading those that can readjust to charge lower prices. The central bank reacts to

price disinflation by reducing the nominal interest rate which is why it rises by less than 25 basis points in the first place. Commodity price inflation is very negative upon impact but recovers in a single period as the commodity price quickly readjusts due to the commodity extractor reducing the commodity supply by utilising less materials. Materials level rises alongside commodity level and price as the shock dies down.

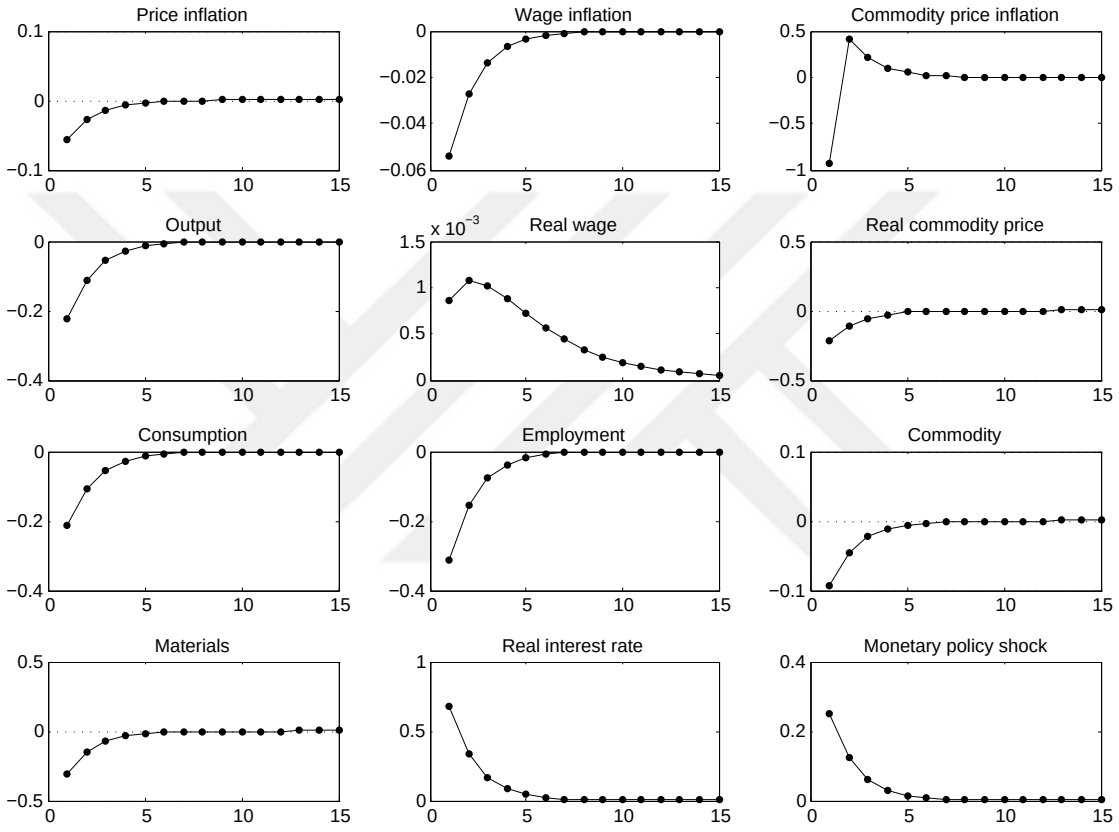

Figure 4: A Monetary Policy Shock

Households want to cut their wages with falling consumption and employment levels reducing their marginal rates of substitution. However, firms can cut down their prices faster than households can cut their wages (even though goods and labour markets are equally rigid) as firms can change their commodity/labour mix to economise on lower

commodity prices. Therefore, price disinflation is slightly stronger than wage disinflation which gives rise to a small real wage increase. As the shock is mildly persistent, it wears off quickly but the slight increase in real wages does not. This leads firms to use more commodity, causing materials, commodity level and real commodity price to be negligibly above their steady state values for a prolonged period of time.

There are several commonalities across the impulse responses to three different shocks. In all cases, commodity price inflation has the largest variance whereas wage inflation has the smallest. The real commodity price has a higher variance than real wages under extraction and monetary policy shocks as well. The standard deviations of chosen variables are illustrated in the table below.

Shocks / Variables	π^p	π^w	π^q	w^r	q^r
Productivity Shock	0.515	0.267	0.703	2.149	0.521
Extraction Shock	0.206	0.107	1.264	0.739	6.227
Monetary Policy Shock	0.060	0.033	1.186	0.006	1.311

This is a nice feature of the model as commodity prices are known to be the most volatile. Jacks et al. (2001) state that commodity prices have always been more volatile than manufactures prices after their analysis of three hundred years of data. This result is achieved naturally through the assumptions made about the price formation mechanisms rather than by assuming an exogenous process for the commodity price. It also becomes clear that although the real commodity price is flexible, it is affected by the nominal rigidities in the goods and labour markets as shown by the commodity price gaps that emerge in all cases. The effects of price and wage rigidities are analysed in more detail in the following sections but a more fundamental issue is addressed first: What effect does the inclusion of a commodity input in production have on the economy in the first place?

4.4 Commodity Input in Production

A nice feature of the thesis model is that it nests the Erceg et al. (2000) model of staggered price and wage contracts. When γ is set to zero, commodity will no longer be a productive factor and will not be demanded by intermediate good producers. No commodity will be extracted so no output will be used up as materials. Output will equal consumption. Labour will be the only productive factor. The model will converge to a sticky price, sticky wage, no commodity model.

Since the intermediate good production function is calibrated to have constant returns to scale, γ is set to 0 and α to 1 in the no commodity case. What differs between the two cases is simply the weight on the commodity in the intermediate good production function.

Baseline calibration	No commodity model
$Y_{it} = A_t N_{it}^{0.6} Z_{it}^{0.4}$	$Y_{it} = A_t N_{it}$

Figure 5 shows how the impulse responses differ across the two calibrations. Observe the natural rates first. The key difference between labour and commodity is that the expansion of the former reduces household utility whereas the expansion of the latter is profitable for the commodity extraction firm. When a productivity shock hits the economy, natural output rises by more in the baseline calibration compared to the no commodity case since commodity supply can expand to fuel production further. Higher output raises marginal rates of substitution and natural wages more which results in a lower level of natural employment. The availability of more commodity contributes to this as well. Maintaining an even higher but falling level of output requires an even lower but increasing level of natural interest rate which is the case. In sum, the availability of an additional factor of production makes the natural rates of variables react more strongly to a productivity shock without changing the direction they are moving in.

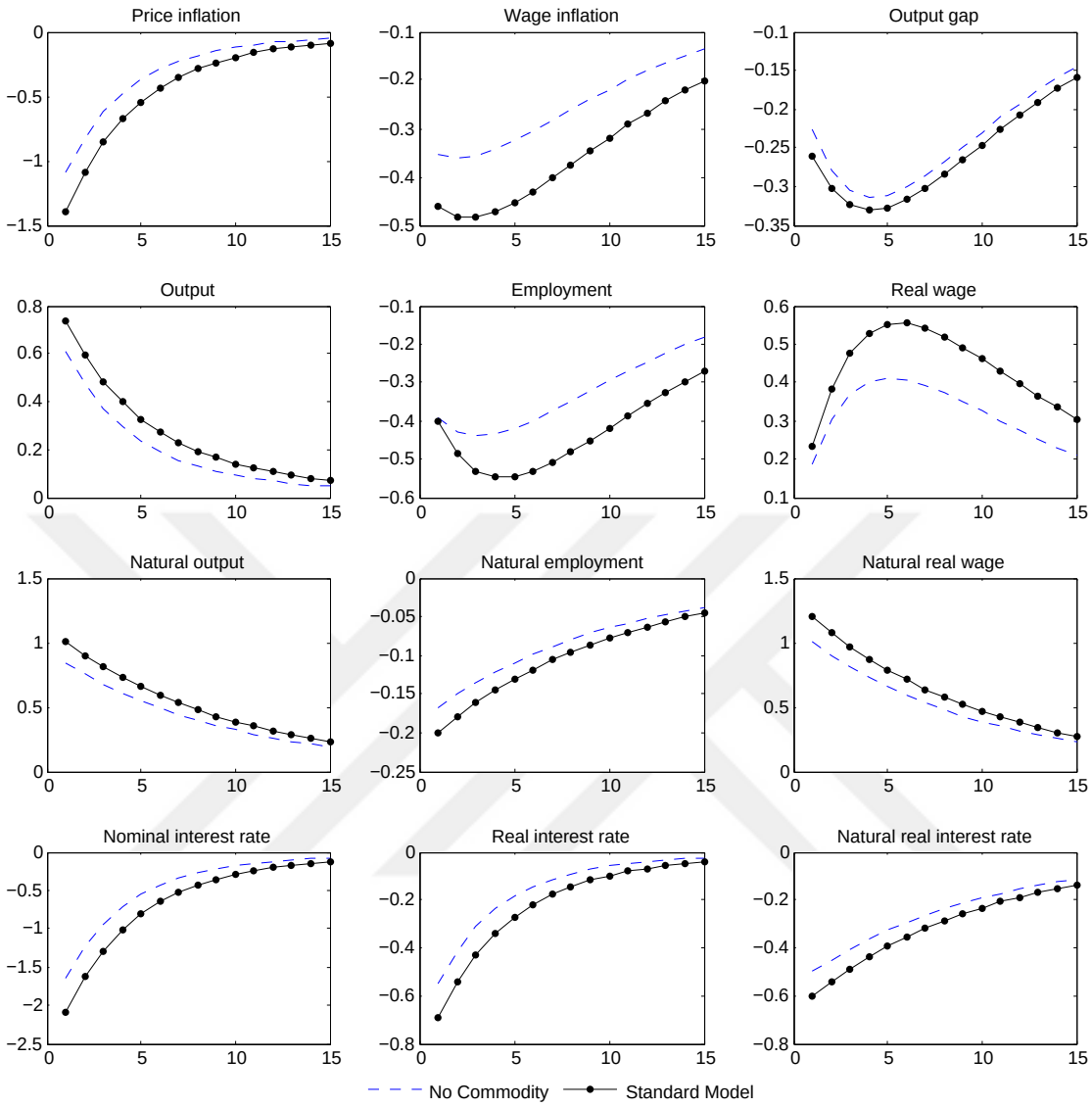

Figure 5: Commodity Input in Production

The productivity shock lowers firms' marginal costs. When commodity is an additional factor of production, firms can substitute some commodity for labour to minimise their costs when faced with an increasing real wage profile. Therefore, the availability of a commodity lowers marginal costs more and results in a stronger price disinflation. The central bank reacts to stronger price disinflation by cutting the nominal interest rate by

even more which leads to lower real interest rates. Consumption is stimulated further and output rises more to meet the demand. Nevertheless, the output gap is still more negative when the commodity is available as an additional input because the extra increase in output is not enough to compensate for the additional expansion in its natural level. Higher consumption also raises marginal rates of substitutions and real wages more. However, wage disinflation is still stronger in the baseline calibration as households find it optimal to disinflate their wages strongly when prices are already falling much more rapidly. Higher real wages coupled with firms substituting some commodity for labour result in a larger fall in employment in the baseline calibration.

In sum, the additional commodity input enhances the expansion arising from a positive productivity shock without changing the direction variables are moving in. Commodity supply can expand when the real commodity price is higher to to contribute to growth in addition to higher technology or employment. Firms now have an additional adjustment channel since they can change their commodity/labour mix. This makes prices practically more flexible although the same fraction of firms get to readjust every period across the two calibrations. These transmission mechanisms make commodities more important for the business cycle. Moreover, the response to a negative productivity shock is stronger when the commodity is needed as a productive input.

4.5 Effects of Price Rigidity

The impulse responses in Figure 6 illustrate the effect of a productivity shock under three different levels of price rigidity: $\theta_p = \frac{1}{2}, \frac{3}{4}$ and $\frac{5}{6}$ which imply expected price durations of two quarters, a year and six quarters respectively. In all cases, productivity rises by 1% and changes in natural levels are identical. The shock lowers firms' marginal costs and firms that can readjust, charge lower prices.

When prices are less rigid, a higher fraction of firms can revise their prices downward and price disinflation is stronger. The central bank lowers the nominal interest rate more. Since the Taylor principle is satisfied, the real interest rate falls more too, stimulating consumption further. Output rises to meet increased consumption demand.

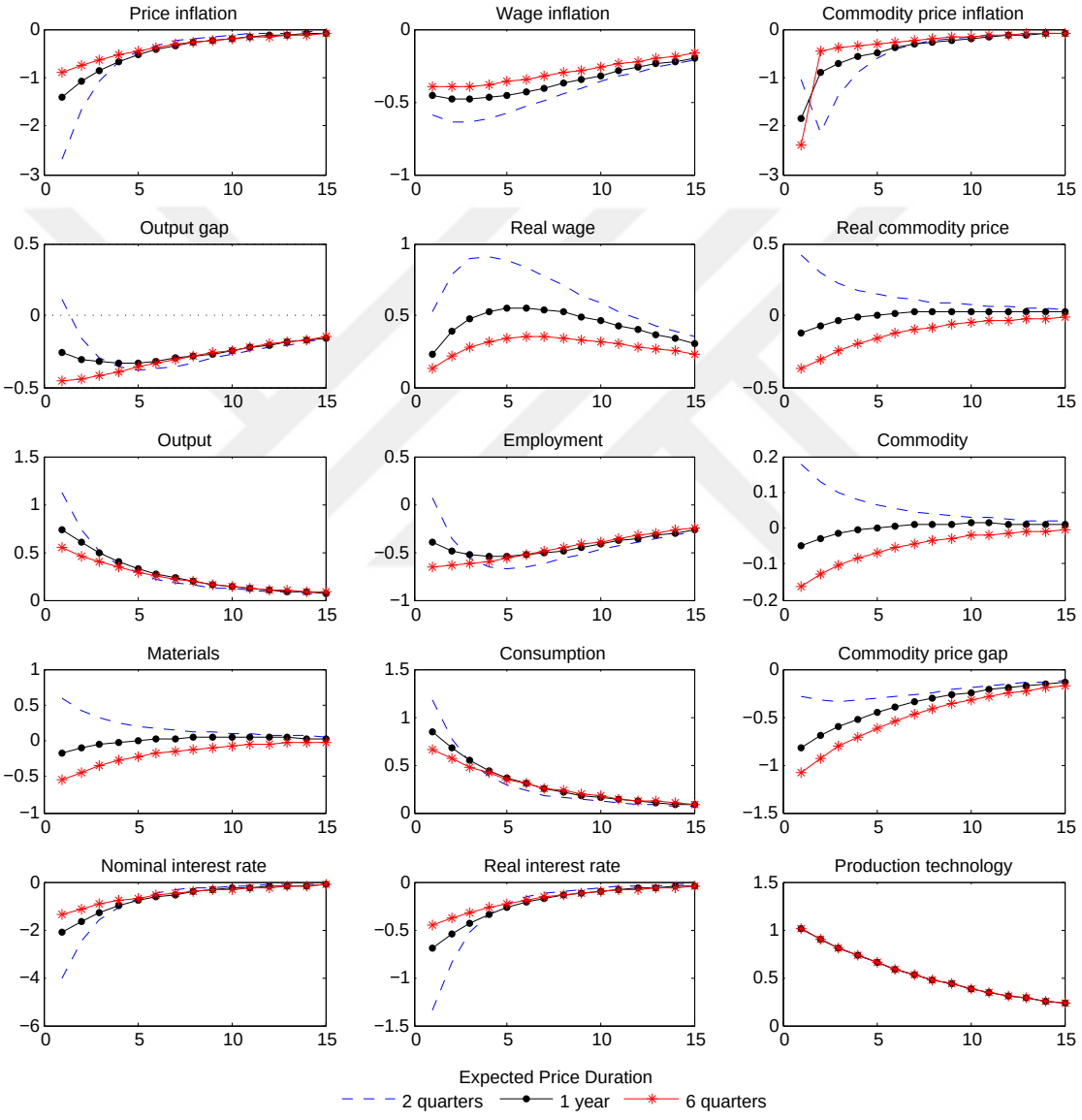

Figure 6: Effects of Price Rigidity

With less reliance on inputs due to higher productivity, employment and commodity levels fall in the baseline calibration. Under more flexible prices, output rises by more which implies a smaller fall in employment and commodity utilisation. When prices are very flexible, consumption is stimulated to strongly that meeting the consumption demand actually requires increasing employment and commodity level. The expansion in the natural level of output is the same across all cases so a higher output level implies a smaller negative output gap or even a slightly positive one under very flexible prices. With very rigid prices, output expands little and the demand for commodity falls more, bringing the real commodity price and materials level down. On the other hand, real commodity price and materials level rise as output and commodity demand expand greatly under very flexible prices. In all three cases, there is a negative commodity price gap as nominal rigidities in other markets lead firms to demand less than the natural level of commodity. This gap is smaller with more flexible prices as expected.

Increasing consumption raises households' marginal rates of substitution which lead them to demand higher wages. Since consumption rises more under more flexible prices, real wages rise more as well. Observing stronger wage disinflation when real wages are rising by more is not a surprise since this is a necessary adjustment when there is already very strong price disinflation. Note that when prices are very flexible, output gap is slightly positive and the fall in employment is small initially but then they rapidly fall, even below their counterparts under different calibrations. This can be explained by the large rises in real wage and the real commodity price. As the shock dies down, intermediate good producers start demanding more productive inputs. Under very flexible prices, they face both higher real wages and real commodity price. Increasing real marginal costs leads firms that can readjust to charge higher prices and price disinflation becomes weaker. Both nominal and real interest rates are higher as a result and the output gap is more negative. Note that this effect starts materialising a year after the impact of the shock.

This analysis shows that the degree of price rigidity has qualitative as well as quantitative effects on the responses of economic variables to a productivity shock. A higher level of output will be associated with higher real commodity prices if prices are flexible enough. Although the effects are not as strong, similar results arise with an extraction shock or when the degree of wage rigidity is varied as shown in the next section.

4.6 Effects of Wage Rigidity

Figure 7 shows the effects of a shock that raises productivity by 1% under three different levels of wage rigidity: $\theta_w = \frac{1}{2}, \frac{3}{4}$ and $\frac{5}{6}$ which imply expected wage durations of two quarters, a year and six quarters respectively. Higher productivity reduces firms' marginal costs. The initial fall in price inflation is very similar across the three parameterisations. Eventually, prices fall more when wages are less rigid.

Note that households find it optimal to set lower nominal wages as prices are already falling rapidly. Not being able to readjust means being stuck with a higher than optimal real wage which in turn implies lower employment. When wages are less rigid, more households can revise their nominal wages down and wage disinflation is stronger. This explains why real wages are lower in a more flexible labour market which may be seen as a counterintuitive result at first.

Since firms face lower real wages when wages are more flexible, their marginal costs are lower and price disinflation is stronger. This implies lower nominal and real interest rates as a price inflation targeting central bank reacts to disinflation. Consumption is stimulated more with lower real interest rates. Output rises more to meet this consumption demand. The emerging output gap and the fall in both productive factors are smaller under less rigid wages. When wages are very flexible, commodity level and price actually increase a few quarters after the shock hits as demand for productive factors recovers. This happens

for the same reasons explained in the case of very flexible prices. Materials follow the real commodity price very closely in all cases. Finally, commodity price disinflation is stronger when wages are more rigid.

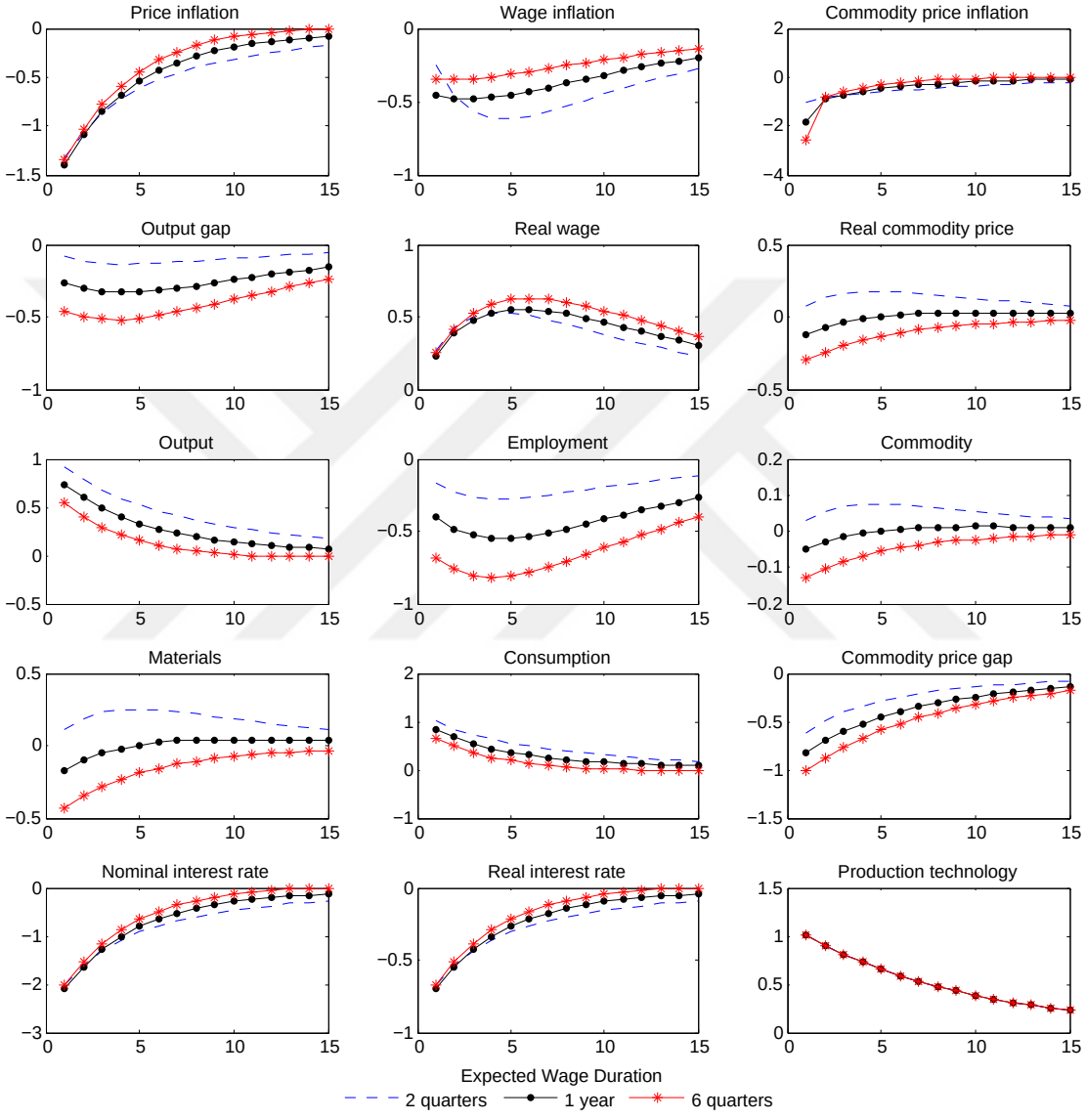

Figure 7: Effects of Wage Rigidity

Observing the impulse responses under different levels of price and wage rigidities reveals that a greater fall in real commodity price and a stronger commodity price disinflation take place when the degree of nominal rigidity in either the labour or the goods market is higher. This shows that although the commodity market does not suffer from any nominal rigidities itself, it is highly affected by the nominal rigidities in other markets. Now that the inefficiencies created by price and wage stickiness have been shown, I turn to the optimal policy problem.

5 Welfare and Optimal Policy

A second degree approximation to the utility function gives the following measure of expected welfare loss[1] when aggregated over households. Note that the optimal subsidy scheme is assumed to be in place in this analysis.

$$\begin{aligned} W &= E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{U_t - U}{U_c C} \right) \\ &\approx E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1-\sigma}{2} c_t^2 - \frac{\alpha}{2} \frac{1+\phi}{1-\delta} n_t^2 - \frac{\alpha}{2} \frac{\varepsilon_w}{\lambda_w} \frac{1}{1-\delta} (\pi_t^w)^2 - \frac{1}{2} \frac{\varepsilon_p}{\lambda_p} \frac{1}{1-\delta} (\pi_t^p)^2 \right) \end{aligned}$$

where $U_t = \int_0^1 U_{jt} dj$. The expected average period loss is

$$L = \left(\frac{\sigma - 1}{2} \right) var(c_t) + \left(\frac{\alpha}{2} \frac{1+\phi}{1-\delta} \right) var(n_t) + \left(\frac{\alpha}{2} \frac{\varepsilon_w}{\lambda_w} \frac{1}{1-\delta} \right) var(\pi_t^w) + \left(\frac{1}{2} \frac{\varepsilon_p}{\lambda_p} \frac{1}{1-\delta} \right) var(\pi_t^p)$$

The details of the derivation of this expression can be found in Appendix C. This loss function is minimised by using the *Ramsey Policy* tool of Dynare. The first thing to note is that even the optimal policy cannot completely offset the effects of shocks hitting the

[1] Note that the variables in the welfare loss function are expressed in deviations from the steady state. Ideally, these variables would be in gaps from the natural rates since with the optimal subsidy scheme in place, natural rates are Pareto optimal and deviations from these rates should be considered as losses. Therefore, this analysis should be considered as only a first step that can be extended in further analysis.

economy. The losses associated with productivity and extraction shocks are 4.87% and 0.78% of steady state consumption respectively under the optimal policy. However, a central bank can easily mimic the optimal policy with an appropriately chosen interest setting rule. The following table shows the difference between welfare losses under the specified alternative policy rules and the optimal policy in terms of steady state consumption for both a productivity and an extraction shock.

Policy rule	**Productivity shock**	**Extraction shock**
Benchmark price inflation targeting $i_t = \rho + 1.5\pi_t^p$	43.51%	7.41%
Optimal price inflation targeting $i_t = \rho + 4.51\pi_t^p$	11.10%	1.77%
Hybrid interest setting rule $i_t = \rho + 1.5\pi_t^p + 3\tilde{y}_t$	0.35%	0.05%
Composite inflation targeting $i_t = \rho + 1.5\pi_t^p + 5\pi_t^w$	0.58%	0.13%
Output gap targeting $i_t = \rho + 3.25\tilde{y}_t$	0.08 %	0.02%

The results of this analysis are in line with Erceg et al. (2000) conclusions. Benchmark price inflation targeting performs very poorly, leading to large welfare losses. The optimal price inflation targeting rule found using the *OSR (Optimal Simple Rule)* tool in Dynare performs much better in comparison to the benchmark price inflation targeting rule but still leads to large losses compared to other interest setting rules. The remaining rules all perform well. A hybrid rule that targets both price inflation and the output gap, a composite inflation targeting rule with more weight on wage inflation than price inflation and an output gap targeting rule all lead to losses less than 1% different from the optimal policy outcome.

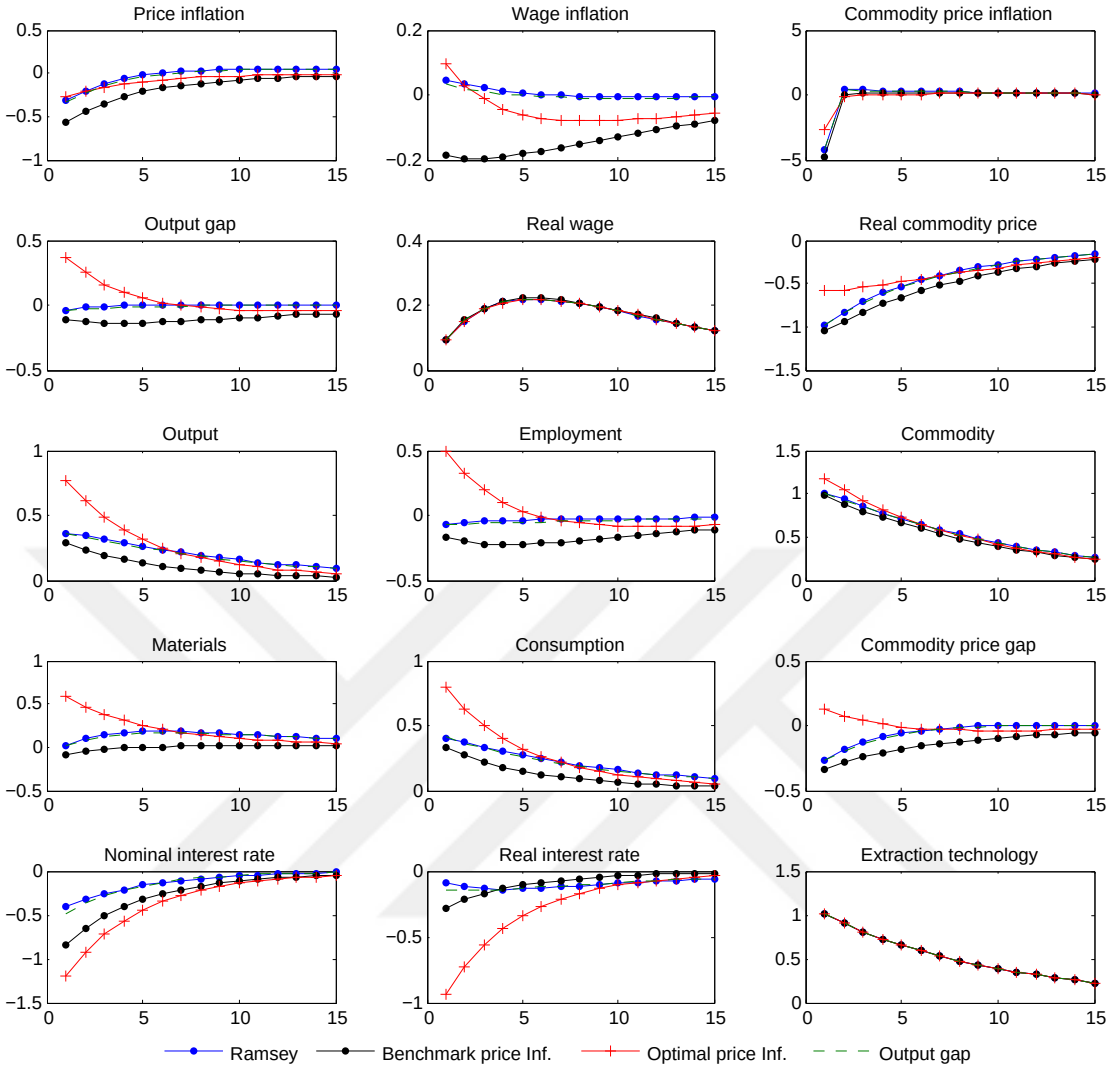

Figure 8: Optimal Monetary Policy

The difference in responses to an extraction shock under different interest setting rules is illustrated in Figure 8. Composite inflation targeting and the hybrid interest setting rule are not shown as they resemble the output gap targeting rule very closely. Note that interest rates fall more under the benchmark and the optimal price inflation targeting rules but welfare losses are still higher. This shows that simply changing the nominal interest rate in response to price inflation is not sufficient. An interest setting rule should do more.

Other rules can achieve better welfare outcomes without causing such large movements in the nominal interest rate. Composite inflation targeting achieves this by stimulating consumption by lowering the nominal interest rate in response to price disinflation on one hand and responding to wage inflation to stabilise marginal costs and to prevent inefficient dispersions in employment across households on the other. Output gap targeting works well too with its threat of adjusting the nominal interest rate strongly whenever an output gap emerges. The hybrid rule targeting price inflation and the output gap works with a similar principle.

Deviations in consumption and employment lead to welfare losses by directly affecting household utility. Price and wage inflation result in welfare losses as well because they lead to production inefficiency. Since different intermediate goods and household labour are combined with CES aggregators, dispersions in these lead to less being produced. Wage inflation also leads to costly dispersions in employment across households. As a result, a rule that can both stimulate or depress consumption enough in response to a shock and offset the productive inefficiencies arising from price or wage dispersions will perform well. Price inflation targeting rules fail in this respect, leading to significant welfare losses but composite inflation targeting, output gap targeting and other hybrid rules can achieve this and should be considered seriously as alternatives.

6 Conclusion

It has been shown that the inclusion of an additional commodity input significantly affects the dynamics of the model. The changes in the natural rates are greater. The economy reacts more strongly to shocks with output expanding or shrinking more according to the nature of the shock which also determines whether these expansions or contractions are associated with rising or falling commodity prices. Another nice feature of the model is that commodity price inflation emerges as the most volatile one and wage inflation the least, in line with empirical regularities. This result naturally emerges from the assumed model structure and the parameterisation chosen rather than an exogenous commodity price process. The emerging output and commodity price gaps increase with price or wage stickiness showing that nominal rigidities in other markets have significant effects on the economy and the commodity market. Optimal policy exercises reveal that it is important to target the right measure of inflation. Price inflation by itself is suboptimal. Output gap targeting and composite inflation targeting or other hybrid rules can perform nearly as well as the optimal rule and should be considered as alternatives.

This model can be expanded in various different directions. Firstly, the model can be developed into a two country model to see whether an extraction shock in one country can lead to significant output movements in the other, especially if one is a net commodity importer and the other an exporter. The effect of extraction shocks on real exchange rates can also be analysed in this setting. Secondly, an AR(1) process may not capture the risks associated with commodity supply. Different ways of introducing uncertainty can be considered which may matter even more if commodities can be stored or if commodity futures are traded. Finally, multiple commodities can be introduced into the model. This might be especially useful in analysing topical issues like how shale gas fracturing is affecting oil supply and demand.

7 References

[1] Aoki, K. (2001) "Optimal Monetary Policy Response to Relative Price Changes," *Journal of Monetary Economics*, 48, 55-80.

[2] Bernanke, B.S., M. Gertler and M.W. Watson (1997) "Systematic Monetary Policy and the Effects of Oil Price Shocks," *Brookings Papers on Economic Activity*, 1, 91-142.

[3] Bernanke, B.S., M. Gertler and M.W. Watson (2004) "Reply: Oil Shocks and Aggregate Macroeconomic Behaviour: The Role of Monetary Policy," *Journal of Money, Credit and Banking*, 36, 287-291.

[4] Bidarkota, P. and M.J. Crucini (2000) "Commodity Prices and the Terms of Trade," *Review of International Economics*, 8, 674-666.

[5] Bodenstein, M., J.C. Erceg and L. Guerrieri (2008) "Optimal Monetary Policy with Distinct Core and Headline Inflation Rates," *Journal of Monetary Economics*, 55, 518-533.

[6] Borensztein, E. and C.M. Reinhart (1994) "The Macroeconomic Determinants of Commodity Prices," IMF Staff Papers, 41, 236-261.

[7] Calvo, G. (1983) "Staggered rices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12, 383-398.

[8] Cashin, P., L.P. Cespedes and R. Sahay (2004) "Commodity Currencies and the Real Exchange Rate," *Journal of Development Economics*, 75, 239-268.

[9] Catao, L. and R. Chang (2012) "Monetary Rules for Commodity Traders," NBER Working Paper No. 18536.

[10] Chen, Y. and K. Rogoff (2003) "Commodity Currencies," *Journal of International Economics*, 60, 133-160.

[11] Erceg, C.J., D.W. Henderson and A.T. Levin (2000) "Optimal Monetary Policy with Staggered Wage and Price Contracts," *Journal of Monetary Economics*, 46, 281-313.

[12] Ferrero, A. and M. Seneca (2015) "Notes on the Underground: Monetary Policy in Resource-Rich Economies," Oxcarre Research Paper No. 158, University of Oxford.

[13] Finn, M.G. (2000) "Perfect Competition and the Effects of Energy Price Increases on Economic Activity," *Journal of Money, Credit and Banking*, 32, 400-416.

[14] Gali, J. (2008) *Monetary Policy, Inflation, and the Business Cycle*, Princeton; Princeton University Press.

[15] Hamilton, J.D. (1983) "Oil and the Macroeconomy Since World War II," *Journal of Political Economy*, 91, 228-248.

[16] Hamilton, J.D. and A.M. Herrera (2004) "Comment: Oil Shocks and Aggregate Macroeconomic Behaviour: The Role of Monetary Policy," *Journal of Money, Credit and Banking*, 36, 265-286.

[17] Hevia, C. and J.P. Nicolini (2013) "Optimal Devaluations," Working Paper No. 702, Federal Reserve Bank of Minneapolis.

[18] Jacks, D.S., K.H. O'Rourke and J.G. Williamson (2011) "Commodity Price Volatility and World Market Integration Since 1700," *The Review of Economics and Statistics*, 93, 800-813.

[19] Killian, L. (2006) "Not All Oil Price Shocks are Alike: Disentangling Demand and Supply Shocks in the Crude Oil Market," CEPR Discussion Paper No. 5994.

[20] Kim, I. and P. Loungani (1992) "The Role of Energy in Real Business Cycle Models," *Journal of Monetary Economics*, 29, 173-189.

[21] Leduc, S. and K. Sill (2004) “A Quantitative Analysis of Oil-Price Shocks, Systematic Monetary Policy, and Economic Downturns,” *Journal of Monetary Economics*, 51, 781-808.

[22] Rotemberg, J.J. and M. Woodford (1996) “Imperfect Competition and the Effects of Energy Price Increases on Economic Activity,” *Journal of Money, Credit and Banking*, 28, 549-577.

[23] Woodford, M. (2003) *Interest and Prices: Foundations of a Theory o Monetary Policy*, Princeton: Princeton University Press.

8 Appendix

There are three parts to the appendix. Part A shows how to solve the model in the absence of nominal rigidities to get the natural levels. Part B illustrates how to write the model in terms of gaps from these natural levels. Part C details the approximations and manipulations applied to the utility function to get a measure of welfare.

8.1 Appendix A: Natural and Steady State Solutions

In this appendix, the model is solved in the absence of any nominal rigidities. The solution gives us the natural levels. All t and i or j subscripts are removed to ease notation. Note that setting $A_t = 1$ and $S_t = 1$ or equivalently $a_t = 0$ and $s_t = 0$ will kill all the shocks and give us the steady state.

We start with the following 10 equations.

$$W = \frac{\varepsilon_w}{\varepsilon_w - 1} MRS \qquad \text{(A1)}$$

$$MRS = N^{\phi} C^{\sigma} \qquad \text{(A2)}$$

$$Z = SM^{\chi} \qquad \text{(A3)}$$

$$M^{1-\chi} = \chi QS \qquad \text{(A4)}$$

$$N^{\alpha+\gamma} = \frac{Y}{A}\left(\frac{\alpha}{\gamma}\frac{Q}{W}\right)^{\gamma} \qquad \text{(A5)}$$

$$Z^{\alpha+\gamma} = \frac{Y}{A}\left(\frac{\gamma}{\alpha}\frac{W}{Q}\right)^{\alpha} \qquad \text{(A6)}$$

$$MC = \frac{W}{MPN} = \frac{\varepsilon_p - 1}{\varepsilon_p} \qquad \text{(A7)}$$

$$Y = AN^{\alpha} Z^{\gamma} \qquad \text{(A8)}$$

$$Y = M + C \qquad \text{(A9)}$$

$$MPN = \alpha A N^{\alpha-1} Z^{\gamma} \qquad \text{(A10)}$$

We eliminate MRS using (A2), Z using (A3), MPN using (A10) and substitute for Q noting that (A4) implies $Q = \frac{M^{1-\chi}}{\chi S}$. This leave us with the following 6 equations.

$$W = \frac{\varepsilon_w}{\varepsilon_w - 1} N^{\phi} C^{\sigma} \qquad \text{(A1)}$$

$$\frac{\varepsilon_p - 1}{\varepsilon_p} = W \frac{1}{A} \frac{1}{\alpha} N^{1-\alpha} S^{-\gamma} M^{-\chi\gamma} \qquad \text{(A7)}$$

$$N^{\alpha+\gamma} = \frac{Y}{A} \left(\frac{\alpha}{\gamma\chi} \frac{1}{W} \right)^{\gamma} S^{-\gamma} M^{\gamma-\gamma\chi} \qquad \text{(A5)}$$

$$Y = A N^{\alpha} S^{\gamma} M^{\chi\gamma} \qquad \text{(A8)}$$

$$S^{\gamma} M^{\alpha+\chi\gamma} = \frac{Y}{A} \left(\frac{\gamma\chi}{\alpha} W \right)^{\alpha} \qquad \text{(A6)}$$

$$Y = M + C \qquad \text{(A9)}$$

Combining (A5) and (A8) gives us the following expression for N.

$$N = \frac{\alpha}{\gamma\chi} \frac{M}{W} \qquad \text{(A11)}$$

Using this in (A1) and (A7) gives us the following 4 equations.

$$W^{1+\phi} = \frac{\varepsilon_w}{\varepsilon_w - 1} \left(\frac{\alpha}{\gamma\chi} \right)^{\phi} C^{\sigma} M^{\phi} \qquad \text{(A1)}$$

$$\frac{\varepsilon_p - 1}{\varepsilon_p} = \frac{W^{\alpha}}{A S^{\gamma}} \frac{M^{1-\alpha-\chi\gamma}}{\alpha^{\alpha} (\gamma\chi)^{1-\alpha}} \qquad \text{(A7)}$$

$$S^{\gamma} M^{\alpha+\chi\gamma} = \frac{Y}{A} \left(\frac{\gamma\chi}{\alpha} W \right)^{\alpha} \qquad \text{(A6)}$$

$$Y = M + C \qquad \text{(A9)}$$

Note that (A6) implies $W^{\alpha} = S^{\gamma} M^{\chi\gamma+\alpha} \frac{A}{Y} (\frac{\alpha}{\gamma\chi})^{\alpha}$. Use this in (A7) to get $\frac{M}{Y}$.

$$\delta = \frac{M}{Y} = \frac{\varepsilon_p - 1}{\varepsilon_p} \gamma\chi$$

Observe (A9) which implies the following by definition.

$$C = (1 - \delta) Y \qquad M = \delta Y$$

Taking (A1) to the power α and using the result from (A6) enables us to substitute W^{α}. This gives us an expression only in terms of C, M and Y. Using the definitions obtained thus far, it is possible to get the natural level of output only in terms of model parameters and technologies. The following are used to ease the notation.

$$\eta = \left(\frac{\varepsilon_w - 1}{\varepsilon_w}\frac{\alpha}{\gamma\chi}\right)^{\alpha} \qquad \kappa = 1 + \phi + \alpha\sigma - \chi\gamma - \chi\gamma\phi - \alpha$$

The natural level of output is

$$Y_t^n = A_t^{\frac{(1+\phi)}{\kappa}} S_t^{\frac{\gamma(1+\phi)}{\kappa}} \left(\frac{\eta\,\delta^{\chi\gamma+\chi\gamma\phi+\alpha}}{(1-\delta)^{\alpha\sigma}}\right)^{\frac{1}{\kappa}}$$

and log natural level of output is given by

$$y_t^n = \tfrac{(1+\phi)}{\kappa}(a_t + \gamma s_t) + \tfrac{1}{\kappa} log(\tfrac{\eta\,\delta^{\chi\gamma+\chi\gamma\phi+\alpha}}{(1-\delta)^{\alpha\sigma}})$$

Using the $C = (1-\delta)Y$ and $M = \delta Y$ definitions give us

$$c_t^n = \tfrac{(1+\phi)}{\kappa}(a_t + \gamma s_t) + \tfrac{1}{\kappa} log\left(\eta\,\delta^{\chi\gamma+\chi\gamma\phi+\alpha}(1-\delta)^{1+\phi-\chi\gamma-\chi\gamma\phi-\alpha}\right)$$

$$m_t^n = \tfrac{(1+\phi)}{\kappa}(a_t + \gamma s_t) + \tfrac{1}{\kappa} log\left(\tfrac{\eta\,\delta^{1+\phi+\alpha\sigma}}{(1-\delta)^{\alpha\sigma}}\right)$$

Once m_t^n is obtained, (A3) and (A4) can be used to find z_t^n and q_t^n respectively.

$$z_t^n = \tfrac{\chi(1+\phi)}{\kappa} a_t + \tfrac{(1+\phi+\alpha\sigma-\alpha)}{\kappa} s_t + \tfrac{\chi}{\kappa} log\left(\tfrac{\eta\,\delta^{1+\phi+\alpha\sigma}}{(1-\delta)^{\alpha\sigma}}\right)$$

$$q_t^n = \tfrac{(1+\phi)(1-\chi)}{\kappa} a_t + \tfrac{(\alpha-\alpha\sigma)-(1-\gamma)(1+\phi)}{\kappa} s_t + \tfrac{1-\chi}{\kappa} log\left(\tfrac{\eta\,\delta^{1+\phi+\alpha\sigma}}{(1-\delta)^{\alpha\sigma}}\right) - log(\chi)$$

With y_t^n and z_t^n determined, (A8) can be used to find n_t^n which in conjunction with (A11) can then give us w_t^n.

$$n_t^n = \tfrac{(1-\sigma)}{\kappa}(a_t + \gamma s_t) + \tfrac{1}{\kappa} log\left(\tfrac{\eta^{\frac{1-\gamma\chi}{\alpha}}\,\delta^{1-\chi\gamma\sigma}}{(1-\delta)^{\sigma(1-\gamma\chi)}}\right)$$

$$w_t^n = \tfrac{(\phi+\sigma)}{\kappa}(a_t + \gamma s_t) + \tfrac{1}{\kappa} log\left(\tfrac{\eta^{\frac{\alpha+\gamma\chi-1}{\alpha}}\,\delta^{\phi+\alpha\sigma+\alpha\gamma\phi}}{(1-\delta)^{\sigma(\alpha+\gamma\chi-1)}}\right) + log(\tfrac{\alpha}{\gamma\chi})$$

Setting $a_t = 0$ and $s_t = 0$ will give the log steady state values. Level steady state values were already given in section 3.7.

8.2 Appendix B: Detailed Model Solution

Once the first order and the market clearing conditions are obtained and loglinearized around the zero inflation steady state, it is relatively straightforward to write the model in terms of gaps from natural levels.

Note that the n superscript implies natural levels. When applied to wages or commodity prices, it denotes the natural rate of real wage or the real commodity price. The tilde accent implies deviation from the natural rates. All variables subject to dispersion like wages or marginal costs refer to economy averages.

1. Price Inflation

Note the following definitions

Marginal cost	$mc_t = w_t - mpn_t$	(B1)
Marginal product of labour	$mpn_t = a_t + (\alpha - 1)n_t + \gamma z_t = y_t - n_t$	(B2)
Output:	$y_t = a_t + \alpha n_t + \gamma z_t$	(B3)

Obtain an expression for labour by rearranging (B3) and use it in (B2) to get

$$y_t - n_t = \frac{(\alpha - 1)y_t + a_t + \gamma z_t}{\alpha}$$

Using this in (B1) enables us to expresses marginal cost as

$$mc_t = w_t + \frac{1-\alpha}{\alpha} y_t - \frac{a_t}{\alpha} - \frac{\gamma}{\alpha} z_t$$

This equation will also hold in the absence of nominal rigidities.

$$mc_t^n = w_t^n + \frac{1-\alpha}{\alpha} y_t^n - \frac{a_t}{\alpha} - \frac{\gamma}{\alpha} z_t^n$$

Take the difference of the above two equations to get

$$\tilde{mc}_t = \tilde{w}_t + \frac{1-\alpha}{\alpha}\tilde{y}_t - \frac{\gamma}{\alpha}\tilde{z}_t$$

and using the materials gap (8) and commodity gap (9) equations, express the deviation of marginal cost from its natural level as

$$\tilde{mc}_t = \tilde{w}_t + \frac{1-\alpha}{\alpha}\tilde{y}_t - \frac{\gamma\chi}{\alpha(1-\chi)}\tilde{q}_t \tag{B4}$$

Note that both in the steady state and in the absence of nominal rigidities, real marginal cost is the inverse of the price mark up. Therefore, the deviation of marginal cost from its steady state is identical to its deviation from its natural level. From the optimal price setting problem of intermediate good producers, the following price inflation equation was obtained.

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p mc_t^r$$

Using (B4) in the above equation gives us the price inflation equation.

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \lambda_p \tilde{w}_t + \lambda_p \frac{1-\alpha}{\alpha}\tilde{y}_t - \frac{\gamma\chi}{\alpha(1-\chi)}\lambda_p \tilde{q}_t$$

which is expressed more concisely as,

$$\pi_t^p = \beta E_t \pi_{t+1}^p + \kappa_{pw}\tilde{w}_t + \kappa_{py}\tilde{y}_t - \kappa_{pq}\tilde{q}_t$$

2. Wage inflation

Wage mark up is given by $\varepsilon_w/(\varepsilon_w - 1)$ both in the steady state and in the absence of nominal rigidities. Therefore, the deviation of wage mark up from its steady state level can be written as

$$\mu_t^w = (w_t^r - mrs_t) - (w_t^n - mrs_t^n) = \tilde{w}_t - \tilde{mrs}_t \tag{B5}$$

and using the definition of marginal rate of substitution, $mrs_t = \sigma c_t + \phi n_t$, gives

$$\mu_t^w = \tilde{w}_t - \sigma\tilde{c}_t - \phi\tilde{n}_t$$

Use the employment gap (10) and consumption gap (7) equations to substitute for $\tilde{n}_t$ and $\tilde{c}_t$ respectively. Then, use the materials gap (8) equation to write (B5) as

$$\mu_t^w = \tilde{w}_t - \left(\frac{\sigma}{1-\delta} + \frac{\phi}{\alpha}\right)\tilde{y}_t + \left(\frac{\sigma\delta}{1-\delta} + \frac{\phi\chi\gamma}{\alpha}\right)\frac{1}{1-\chi}\tilde{q}_t$$

Substituting this into the wage inflation equation, $\pi_t^w = \beta\pi_{t+1}^w - \lambda_w\mu_t$, obtained in Section 3.1 gives the full wage inflation equation

$$\pi_t^w = \beta\pi_{t+1}^w - \lambda_w\tilde{w}_t + \lambda_w\left(\frac{\sigma}{1-\delta} + \frac{\phi}{\alpha}\right)\tilde{y}_t - \lambda_w\left(\frac{\sigma\delta}{1-\delta} + \frac{\phi\chi\gamma}{\alpha}\right)\tilde{q}_t$$

Expressed more concisely as

$$\pi_t^w = \beta\pi_{t+1}^w - \kappa_{ww}\tilde{w}_t + \kappa_{wy}\tilde{y}_t - \kappa_{wq}\tilde{q}_y$$

3. Evolution of the real commodity price

The evolution of the real commodity price is given by

$$q_t^r = q_{t-1}^r + \pi_t^q - \pi_t^p$$

Its derivation is analogous to the derivation of equation (5) which describes the evolution of real wages. More detail can be found below.

4. Evolution of output gap

Households' optimal consumption choice leads to

$$c_t = E_t c_{t+1} - \frac{1}{\sigma}(i_t - \rho - E_t\pi_{t+1}^p)$$

which can also be written as

$$y_t - \delta m_t = E_t(y_{t+1} - \delta m_{t+1}) - \frac{1-\delta}{\sigma}(i_t - \rho - E_t\pi_{t+1}^p)$$

using the market clearing condition. Using the commodity extractor first order condition to substitute for materials gives

$$y_t - \frac{\delta}{1-\chi}(q_t^r + s_t) = E_t y_{t+1} - \frac{\delta}{1-\chi}E_t(q_{t+1}^r - s_{t+1}) - \frac{1-\delta}{\sigma}(i_t - \rho - E_t\pi_{t+1}^p) \quad \text{(B6)}$$

This expression will hold in the absence of nominal rigidities as well.

$$y_t^n - \frac{\delta}{1-\chi}(q_t^n + s_t) = E_t y_{t+1}^n - \frac{\delta}{1-\chi}(q_{t+1}^n - s_{t+1}) - \frac{1-\delta}{\sigma}(r_t^n - \rho)$$

Taking the difference of the above two equations gives an expression describing the evolution of the output gap.

$$\tilde{y}_t - \frac{\delta}{1-\chi}\tilde{q}_t = E_t\tilde{y}_{t+1} - \frac{\delta}{1-\chi}\tilde{q}_{t+1} - \frac{1-\delta}{\sigma}(i_t - E_t\pi_{t+1}^p - r_t^n) \tag{B7}$$

5. Evolution of the real wage

The change in the wage gap can be expressed as

$$\begin{aligned}
\Delta\tilde{w}_t &= \tilde{w}_t - \tilde{w}_{t-1} \\
&= [(w_t^r - w_t^n) - (w_{t-1}^r - w_{t-1}^n)] \\
&= [(w_t - p_t - w_t^n) - (w_{t-1} - p_{t-1} - w_{t-1}^n] \\
&= (w_t - w_{t-1}) - (p_t - p_{t-1}) - (w_t^n - w_{t-1}^n) \\
&= \pi_t^w - \pi_t^p - (w_t^n - w_{t-1}^n)
\end{aligned}$$

which implies

$$\begin{aligned}
\tilde{w}_t &= \pi_t^w - \pi_t^p - (w_t^n - w_{t-1}^n) + \tilde{w}_{t-1} \\
w_t^r - w_t^n &= \pi_t^w - \pi_t^p - w_t^n + w_{t-1}^n + w_{t-1}^r - w_{t-1}^n
\end{aligned}$$

and this simplifies to give us an equation describing the evolution of the real wage.

$$w_t^r = w_{t-1}^r + \pi_t^w - \pi_t^p$$

6. Commodity price gap

The deviation of real commodity price from its steady state value is given by

$$q_t = \frac{(1-\chi)}{(\alpha+\chi\gamma)}(y_t + \alpha w_t - a_t) + \frac{\alpha+\gamma}{\alpha+\gamma\chi}s_t$$

Since this expression will also hold in the absence of nominal rigidities, the natural level of the real commodity price can be expressed in the same way. Taking their difference gives an equation describing the commodity price gap.

$$\tilde{q}_t = \frac{(1-\chi)}{(\alpha+\chi\gamma)}(\tilde{y}_t + \alpha\tilde{w}_t)$$

7. Consumption gap

Goods market clearing requires $y_t = (1-\delta)c_t + \delta m_t$. Subtracting

$y_t^n = (1-\delta)c_t^n + \delta m_t^n$ and rearranging it gives the consumption gap equation.

$$\tilde{c}_t = \frac{\tilde{y}_t - \delta\tilde{m}_t}{1-\delta}$$

8. Materials gap

The first order condition of the commodity extractor is $m_t = \frac{1}{1-\chi}(s_t + q_t)$ which also holds in the absence of any nominal rigidities: $m_t^n = \frac{1}{1-\chi}(s_t + q_t^n)$ and taking their difference gives us the materials gap equation.

$$\tilde{m}_t = \frac{1}{1-\chi}\tilde{q}_t$$

9. Commodity gap

Commodity gap is the difference of the commodity extraction function $z_t = s_t + \chi m_t$ and its natural version $z_t^n = s_t + \chi m_t^n$.

$$\tilde{z}_t = \chi\tilde{m}_t$$

10. Employment gap

Employment gap equation is just the rearranged version of the difference of the production function $y_t = a_t + \alpha n_t + \gamma z_t$ with itself in the absence of nominal rigidities.

$$\tilde{n}_t = \frac{\tilde{y}_t - \gamma\tilde{z}_t}{\alpha}$$

11. Natural interest rate

Start from the evolution of the output gap (B7)

$$\tilde{y}_t - \frac{\delta}{1-\chi}\tilde{q}_t = E_t\tilde{y}_{t+1} - \frac{\delta}{1-\chi}E_t\tilde{q}_{t+1} - \frac{1-\delta}{\sigma}(i_t - E_t\pi^p_{t+1} - r^n_t)$$

and rearrange it into

$$r^n_t = \frac{\sigma}{1-\delta}\left(\frac{\delta}{1-\chi}(E_t\tilde{q}_{t+1} - \tilde{q}_t) - (E_t\tilde{y}_{t+1} - \tilde{y}_t)\right) + i_t - E_t\pi^p_{t+1}$$

and rewriting the expressions that are in terms of gaps from their natural rates gives

$$r^n_t = \frac{\sigma}{1-\delta}\left[\left(E_t\Delta y^n_{t+1} - \frac{\delta}{1-\chi}E_t\Delta q^n_{t+1}\right) + \left((y_t - E_t y_{t+1}) - \frac{\delta}{1-\chi}(q_t - E_t q_{t+1})\right)\right] + i_t - E_t\pi^p_{t+1}$$

Rearrange (B6) into

$$y_t - E_t y_{t+1} - \frac{\delta}{1-\chi}(q_t - E_t q_{t+1}) = -\frac{1-\delta}{\sigma}(i_t - \rho - E_t\pi^p_{t+1}) - \frac{\delta}{1-\chi}\Delta s_{t+1}$$

to substitute for the second expression in brackets in the previous equation. This gives

$$\frac{\sigma}{1-\delta}\left(E_t\Delta y^n_{t+1} - \frac{\delta}{1-\chi}E_t\Delta q^n_{t+1}\right) + \rho - \frac{\delta}{1-\chi}\frac{\sigma}{1-\delta}E_t\Delta s_{t+1} = r^n_t$$

which can be rearranged into

$$\frac{(1-\chi)(1-\delta)}{\sigma}(r^n_t - \rho) = (1-\chi)E_t\Delta y^n_{t+1} - \delta E_t\Delta q^n_{t+1} - \delta E_t\Delta s_{t+1} \qquad \text{(B8)}$$

Using the definitions of natural levels in Appendix A, obtain

$$E_t\Delta y^n_{t+1} = \frac{1+\phi}{\kappa}E_t\Delta a_{t+1} + \frac{\gamma(1+\phi)}{\kappa}E_t\Delta s_{t+1}$$

$$\Delta E_t q^n_{t+1} = \frac{(1+\phi)(1-\chi)}{\kappa}E_t\Delta a_{t+1} + \frac{(\alpha-\alpha\sigma)-(1-\gamma)(1+\phi)}{\kappa}E_t\Delta s_{t+1}$$

and substituting these in (B8) gives

$$\frac{(1-\chi)(1-\delta)}{\sigma}(r^n_t - \rho) = \frac{(1-\delta)(1+\phi)(1-\chi)}{\kappa}E_t\Delta a_{t+1} + \frac{\gamma(1-\delta)(1-\chi)(1+\phi)}{\kappa}E_t\Delta s_{t+1}$$

which nicely simplifies into

$$r_t^n = \rho + \sigma \frac{1+\phi}{\kappa}((\rho_a - 1)a_t + (\rho_s - 1)\gamma s_t)$$

giving us the natural real interest rate.

This completes the list of equations used to simulate the model.

8.3 Appendix C: The Welfare Loss Function

In this section, a second order approximation to the utility function is used to obtain a welfare loss function. Gali (2008) is followed very closely. Terms independent of policy and of order higher than two are not considered.

The deviation of any variable X_t from its steady state can be expressed as

$$\frac{X_t - X}{X} \approx x_t + \frac{1}{2}x_t^2$$

where x_t denotes the log deviation of X_t from its steady state. $\hat{x}_t$ is used to express the deviation of a variable from its contemporaneous average. This only applies to variables that are subject to dispersion like prices, wages and employment.

The utility of household j can be approximated as follows.

$$U_{jt} \approx U + U_c(C_{jt} - C) + \frac{1}{2}U_{cc}(C_{jt} - C)^2 + U_n(N_{jt} - N) + \frac{1}{2}U_{nn}(N_{jt} - N)^2$$

which can be written as

$$\begin{aligned}
U_{jt} - U &\approx U_c C \frac{C_{jt} - C}{C} + \frac{1}{2}U_{cc}C^2 c_t^2 + U_n N \frac{N_{jt} - N}{N} + \frac{1}{2}U_{nn}N^2 n_{jt}^2 \\
&\approx U_c C(c_{jt} + \frac{1}{2}c_{jt}^2) + \frac{1}{2}U_{cc}C^2 c_{jt}^2 + U_n N(n_{jt} + \frac{1}{2}n_{jt}^2) + \frac{1}{2}U_{nn}N^2 n_{jt}^2 \\
&= U_c C(c_{jt} + \frac{1}{2}c_{jt}^2 + \frac{1}{2}\frac{U_{cc}C}{U_c}c_{jt}^2) + U_n N(n_{jt} + \frac{1}{2}n_{jt}^2 + \frac{1}{2}\frac{U_{nn}N}{U_n}n_{jt}^2) \\
&= U_c C(c_{jt} + \frac{1-\sigma}{2}c_{jt}^2) + U_n N(n_{jt} + \frac{1+\phi}{2}n_{jt}^2)
\end{aligned}$$

where the last equality uses the $\frac{U_{cc}C}{U_c} = -\sigma$ and $\frac{U_{nn}N}{U_n} = \phi$ results.

The above expression is an approximation of household j's utility only. It needs to be aggregated over all j to get a welfare measure encompassing all the households on the unit interval.

$$\int_0^1 \frac{U_{jt} - U}{U_c C} dj \approx \int_0^1 c_{jt} dj + \frac{1-\sigma}{2} \int_0^1 c_{jt}^2 dj + \frac{U_n N}{U_c C} \left(\int_0^1 n_{jt} dj + \frac{1+\phi}{2} \int_0^1 n_{jt}^2 dj \right)$$

Complete markets assumption coupled with a utility function separable in consumption and labour imply that $c_{jt} = c_t$ where c_t is the deviation of aggregate consumption from its steady state level. However, the same does not hold for labour as wage stickiness is likely to result in different households supplying different amounts of labour. The one period welfare function can be written as

$$\int_0^1 \frac{U_{jt} - U}{U_c C} dj \approx c_t + \frac{1-\sigma}{2} c_t^2 + \frac{U_n N}{U_c C} \left(\int_0^1 n_{jt} dj + \frac{1+\phi}{2} \int_0^1 n_{jt}^2 dj \right)$$

Define aggregate employment as $N_t = \int_0^1 N_{jt} dj$. As Gali (2008) notes

$$n_t + \frac{1}{2} n_t^2 \approx \int_0^1 n_{jt} dj + \frac{1}{2} \int_0^1 n_{jt}^2 dj$$

Adding $\frac{\phi}{2} \int_0^1 n_{jt}^2 dj$ to this expression will enable us to write the welfare function without any integrals which will simplify our calculations. Following Gali (2008),

$$\begin{aligned}
\int_0^1 n_{jt}^2 dj &= \int_0^1 (n_{jt} - n_t + n_t)^2 dj \\
&= n_t^2 + 2\hat{n}_t \int_0^1 (n_{jt} - n_t) dj + \int_0^1 (n_{jt} - n_t)^2 dj \\
&= n_t^2 - 2\varepsilon_w n_t E_j \hat{w}_{jt} + \varepsilon_w^2 var_j w_{jt} \\
&= n_t^2 + \varepsilon_w^2 var_j w_{jt}
\end{aligned}$$

The third equality follows from the labour demand equation $N_{jt} = (W_{jt}/W_t)^{-\varepsilon_w} N_t$ which yields $n_{jt} - n_t = -\varepsilon_w (w_{jt} - w_t) = -\varepsilon_w \hat{w}_{tj}$ when loglinearized.

The final equality follows from the $E_j\hat{w}_{jt} = \frac{\varepsilon_w - 1}{2} var_j(w_{tj})$ result from Gali (2008) which implies that the $n_t E_j \hat{w}_{jt}$ is a third order term. Using the above results yield

$$\int_0^1 n_{jt} dj + \frac{1+\phi}{2} \int_0^1 n_{jt}^2 dj \approx n_t + \frac{1+\phi}{2} n_t^2 + \frac{\phi \varepsilon_w^2}{2} var(w_{jt})$$

which enables us to write the welfare function as

$$\int_0^1 \frac{U_{jt} - U}{U_c C} dj \approx c_t + \frac{1-\sigma}{2} c_t^2 + \frac{U_n N}{U_c C} (n_t + \frac{1+\phi}{2} n_t^2 + \frac{\phi \varepsilon_w^2}{2} var(w_{jt}))$$

Suppose that the optimal subsidy scheme described previously is in place. In the steady state, wage is a mark up over the marginal rate of substitution.

$$W = -\frac{U_n}{U_c} \frac{\varepsilon_w}{\varepsilon_w - 1}$$

Steady state marginal cost is the inverse of price mark up.

$$MC = \frac{\varepsilon_p - 1}{\varepsilon_p} = \frac{(1-\tau)W}{MPN} \implies W = \frac{\varepsilon_p - 1}{\varepsilon_p} \frac{1}{1-\tau} MPN$$

which jointly imply

$$\frac{\varepsilon_p - 1}{\varepsilon_p} \frac{1}{1-\tau} MPN = -\frac{\varepsilon_w}{\varepsilon_w - 1} \frac{U_n}{U_c}$$

$$MPN = -\frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\varepsilon_p}{\varepsilon_p - 1} (1-\tau) \frac{U_n}{U_c}$$

$$MPN = \alpha \frac{Y}{N} = -\frac{U_n}{U_c}$$

$$\alpha \frac{N}{C} \frac{Y}{N} = \alpha \frac{Y}{C} = \frac{\alpha}{1-\delta} = -\frac{U_n}{U_c} \frac{N}{C}$$

This enables us to rewrite the welfare approximation as

$$\int_0^1 \frac{U_{jt}-U}{U_c C} dj \approx c_t + \frac{1-\sigma}{2} c_t^2 - \frac{\alpha}{1-\delta}\left(n_t + \frac{1+\phi}{2} n_t^2 + \frac{\phi \varepsilon_w^2}{2} var(w_{jt})\right)$$

Multiply this expression with $(1-\delta)$ to get

$$(1-\delta)\int_0^1 \frac{U_{jt}-U}{U_c C} dj \approx (1-\delta)c_t - \alpha n_t + \frac{(1-\sigma)(1-\delta)}{2} c_t^2 - \alpha\left(\frac{1+\phi}{2} n_t^2 + \frac{\phi \varepsilon_w^2}{2} var(w_{jt})\right)$$

Using the intermediate good producers' labour and commodity demand equations as well as the demand equation for intermediate goods themselves, the production function can be expressed as

$$Y_t = A_t S_t^{\gamma} N_t^{\alpha} M_t^{\gamma\chi} \Delta W_t^{-\alpha} \Delta P_t^{-(\alpha+\gamma)}$$

where $\int_0^1 N_{jt} dj = N_t$ is aggregate employment and $\int_0^1 Z_{it} di = Z_t$ is aggregate commodity level. The other terms are

$$\Delta W_t = \int_0^1 \left(\frac{W_{tj}}{W_t}\right)^{-\varepsilon_w} dj \qquad \Delta P_t = \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\frac{\varepsilon_p}{\alpha+\gamma}} di$$

The production function can be approximated as

$$y_t = (1-\delta)c_t + \delta m_t = \alpha n_t + \gamma\chi m_t - \alpha \Delta w_t - (\alpha+\gamma)\Delta p_t + t.i.p$$

where the first equality follows from the market clearing condition. Under the optimal subsidy scheme, $\delta = \chi\gamma$ so the above equation can be simplified and rearranged into

$$(1-\delta)c_t - \alpha n_t = \alpha \Delta w_t - (\alpha+\gamma)\Delta p_t + t.i.p$$

$$\begin{aligned}
\Delta W_t = \int_0^1 \left(\frac{W_{jt}}{W_t}\right)^{-\varepsilon_w} dj &= \int_0^1 e^{-\varepsilon_w \hat{w}_{jt}} dj \\
&\approx 1 - \varepsilon_w \int_0^1 \hat{w}_{jt} dj + \frac{\varepsilon_w^2}{2} \int_0^1 \hat{w}_j^2 t dj \\
&= 1 - \varepsilon_w E_j \hat{w}_{jt} dj + \varepsilon_w^2 var_j w_{jt} \\
&= 1 + \frac{\varepsilon_w}{2} var_j w_{jt}
\end{aligned}$$

where the last equality follows from a the well established result of $E_j\hat{w}_{jt} = \frac{\varepsilon_w - 1}{2} var_j w_{jt}$

Loglinearizing this expression around its steady state yields

$$\Delta w_t = log(\Delta W_t) - log(\Delta W) \approx \frac{\varepsilon_w}{2} var_j w_{jt}$$

A similar expression is obtained for ΔP_t next.

$$\begin{aligned}
\Delta P_t = \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{-\frac{\varepsilon_p}{\alpha+\gamma}} di &= \int_0^1 e^{-\frac{\varepsilon_p}{\alpha+\gamma}\hat{p}_{it}} di \\
&\approx 1 - \frac{\varepsilon_p}{\alpha+\gamma}\int_0^1 \hat{p}_{it} di + \frac{1}{2}\frac{\varepsilon_p^2}{(\alpha+\gamma)^2}\int_0^1 \hat{p}_{it}^2 di \\
&= 1 - \frac{\varepsilon_p}{\alpha+\gamma} E_i \hat{p}_{it} + \frac{1}{2}\frac{\varepsilon_p^2}{(\alpha+\gamma)^2} var_i p_{it} \\
&= 1 + \frac{\varepsilon_p}{2}\frac{1}{\alpha+\gamma}\left[1 + \varepsilon_p\left(\frac{1}{\alpha+\gamma} - 1\right)\right] var_i p_{it}
\end{aligned}$$

where the last equality follows from the well established result of $E_i\hat{p}_{it} = \frac{\varepsilon_p - 1}{2} var_i p_{it}$.

When loglinearized around the steady state, this expression leads to

$$\Delta p_t = log(\Delta P_t) - log(\Delta P) \approx \frac{\varepsilon_p}{2}\frac{1}{\alpha+\gamma}\left[1 + \varepsilon_p\left(\frac{1}{\alpha+\gamma} - 1\right)\right] var_i p_{it}$$

The two results above imply that

$$\begin{aligned}
(1-\delta)c_t - \alpha n_t &= \alpha \Delta w_t - (\alpha+\gamma)\Delta p_t \\
&\approx -\frac{\alpha\varepsilon_w}{2} var_j w_{jt} - \frac{\varepsilon_p}{2}\left[1 + \varepsilon_p\left(\frac{1}{\alpha+\gamma} - 1\right)\right] var_i p_{it}
\end{aligned}$$

Using this expression to substitute for the $(1-\delta)c_t - \alpha n_t$ term in the welfare function gives us

$$\begin{aligned}
(1-\delta)\int_0^1 \frac{U_{jt} - U}{U_c C} dj \approx{}& \frac{(1-\sigma)(1-\delta)}{2} c_t^2 - \alpha\frac{1+\phi}{2} n_t^2 \\
&- \frac{\alpha\varepsilon_w}{2}(1+\phi\varepsilon_w) var_j w_{jt} - \frac{\varepsilon_p}{2}\left[1 + \left(\frac{1}{\alpha+\gamma} - 1\right)\varepsilon_p\right] var_i p_{it}
\end{aligned}$$

Woodford (2003) has shown that

$$\sum_0^\infty \beta^t var(w_{jt}) \approx \sum_0^\infty \beta^t \frac{\theta_w}{(1-\theta_w)(1-\beta\theta_w)} \pi_t^w$$

$$\sum_0^\infty \beta^t var(p_{it}) \approx \sum_0^\infty \beta^t \frac{\theta_p}{(1-\theta_p)(1-\beta\theta_p)} \pi_t^p$$

to express price and wage variances in terms of price and wage inflation respectively, we get the following one period welfare function.

$$\int_0^1 \frac{U_{jt}-U}{U_c C} dj \approx \frac{1-\sigma}{2} c_t^2 - \frac{\alpha}{2}\frac{1+\phi}{1-\delta} n_t^2 - \frac{\alpha}{2}\frac{\varepsilon_w}{\lambda_w}\frac{1}{1-\delta}(\pi_t^w)^2 - \frac{1}{2}\frac{\varepsilon_p}{\lambda_p}\frac{1}{1-\delta}(\pi_t^p)^2$$

Its discounted sum is the welfare measure used in policy analysis.

$$\begin{aligned} W &= E_0 \sum_{t=0}^\infty \beta^t \left(\frac{U_t - U}{U_c C} \right) \\ &\approx E_0 \sum_{t=0}^\infty \beta^t \left(\frac{1-\sigma}{2} c_t^2 - \frac{\alpha}{2}\frac{1+\phi}{1-\delta} n_t^2 - \frac{\alpha}{2}\frac{\varepsilon_w}{\lambda_w}\frac{1}{1-\delta}(\pi_t^w)^2 - \frac{1}{2}\frac{\varepsilon_p}{\lambda_p}\frac{1}{1-\delta}(\pi_t^p)^2 \right) \end{aligned}$$

where $U_t = \int_0^1 U_{jt} dj$.