

LOW COST CAPTURE USING MULTI-BODY DYNAMICS

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Abstract

Within the last decades, low-energy transfers and capture problem have been studied extensively to understand the dynamical structure of complex multi-body problem with a view to make the space missions much cheaper. Especially in this area, dynamical features of Circular Restricted 3-Body Problem (CR3BP), which is perfectly fitted to the many binary systems in our solar system, is used to determine advantageous and promising low-cost transfer trajectories. However, the capture concept around smaller body, which is highly influenced by gravity force of bigger body in CR3BP, is still being one of the most challenging problems, because of highly chaotic dynamical nature around there. Previously, capture problem have been studied through mapping methods to gather insight into its chaotic dynamics. Although these type of mapping methods can be only adequate for the visualization of planar solution space and provide restricted dynamical insight into planar-capture, they complicate the visualization of the 3-Dimension solution space, and for this reason, making general capture dynamics more difficult to comprehend. However, in this research, a novel map method is designed, which reduces the order of 3-dimension CR3BP, condensing quantities of information into a 2-dimensional image. Thus the 3D capture problem can be better visualized and comprehended. Using this map, any type of low-cost planetary and orbital capture trajectories, including difficult and sensitive permanent captures, which provide completely zero-cost capture solutions, can be found. Therefore, space missions can be not only much cheaper, but also can be more flexible and adaptable to various conditions.

Key words: Circular restricted 3-body problem, Mapping, Lagrange points, Differential Corrector, Linstead-Poincare Method, Capture dynamics, Rotation 2 body problem

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Abbreviations

2BP	2 Body Problem
2D	Two Dimension
3D	Three Dimension
CR3BP	Circular Restricted 3-Body Problem
DC	Differential Corrector method
ESA	European Space Agency
IMT	Invariant Manifold Tube
GNC	Guidance, Navigation, Control
LET	Low Energy transfer
LEO	Low Earth Orbit
LP	Linstead-Poincare perturbation method
WSB	Week Stability boundaries
ZVS	Zero Velocity Surface

Guide of Appendix CD

Matlab Codes (Please put all m files in same folder)

CR3BP_plotter.m	: Given initial conditions, mass ratio, Jacobi constant, it plots sample trajectory in CR3BP equation motion, with zero-velocity surface and equilibrium points, in both rotating and inertia axis.
IMT_capture.m	: A sample plot of L_1 Halo unstable invariant manifold tube (IMT), which is applied a motion map to find both low cost capture and low-cost trajectory change, please see section 4.2.2.
IMTs_cr3bp.m	: An exemplary plot of all stable and unstable IMT's of L_1 and L_2 Halo orbits in Earth-Moon binary system, with 3D-zero velocity surfaces and equilibrium points.
Lagpon.m	: It plots equilibrium points with a given mass ratio.
planet.m	: It plots a simple globe for modelling a planet or a moon with normalised radius and position, to see CR3BP's trajectories with their volume masses relationships.
pot_fun.m	: It plots potential surface $U(x,y)$, see section 3.1.1, with a given mass ratio
zvs.m	: It plots either 2 or 3-dimensional zero velocity surfaces.

Videos (Also see vimeo.com/user2328498/videos)

L1 unstable manifold tube.wmv	: It shows motions of L_1 Halo unstable manifold tube
IMT_MM_variousDV.wmv	: It shows Motion Map for IMT in various velocity decrements, please see section 4.2.2.
MM(0-360deg)_C_3.16.wmv	: It shows Plane Motion Maps in various plane angles for Earth-Moon system, and $C = 3.16$.
MM(X-Z)_various_C.wmv	: It shows X-Z Motion Map in various C (Energy) values.
MM(X-Z)_various_C_withT.wmv	: It shows X-Z Motion Map in colour scaling the final times, in various C (Energy) values.
MM(X-Z)_to_MM(R-I)_various_C.wmv	: It shows X-Z Motion Map in reflection to radius-inclination plane, in various C (Energy) values.
MM(Y-Z)_various_C.wmv	: It shows Y-Z Motion Map in various C (Energy) values.
MM(Y-Z)_to_MM(R-I)_various_C.wmv	: It shows Y-Z Motion Map in reflection to radius-inclination plane, in various C (Energy) values.

I. INTRODUCTION

1.1. Motivation and Overview

Trajectory optimization always becomes main problem in the space mission. Because the most fuel consuming and expensive part is departure (launching) and arrival (stabilization around target planet) part in the space mission, the spacecraft is reduced its payload to decrease the mission cost. The fuel tank of the spacecraft is one of the critical reductions of its payload. It is meant that if the trajectory of mission can be done less fuel, payload of launching part can be reduced and whole mission can be done cheaply, or life time of spacecraft can be increased because of economic usage of fuel. In short, usage of fuel economically makes space mission much cheaper, and also fail-proof for against some emergency case, which is about getting safe parking orbits in fuel shortage.

For decreasing the fuel usage, natural forces can be effectively used. For example, while driving a car, using downhill part of field can provide reduced usage of fuel. Or while sailing a ship, the wind and ocean stream help for free travelling. In the same way, natural effects in space can be useful for trajectory optimization in space mission. There are two main natural forces in space environment, these are; solar radiation pressure for solar sailing and gravitation force. In this research, trajectory optimization is studied using gravitation force of celestial bodies and their dynamics properties. Because every celestial body has an activity-sphere boundaries in its around, while a spacecraft approaching the body, it is slowed down or speed up relative to entering to this sphere influence.

Previously, lunar and interplanetary transfer were made with patch conic (2-body) solution [1-5]. The patched conic method has been used for using joint gravity to our advantage. In this method, this sphere of influence is treated as a point (or strict boundaries) and match Kepler arcs, so spacecraft can “flyby” on a planet. However, in the method of patched conic, mission designers assume that a spacecraft is influenced only by the gravitational field of the planet when it is within the sphere of influence, and is influenced only by the gravity of sun when it is outside any planet’s sphere of influence. The spacecraft is thus always in a 2-body orbit with respect to either the sun or a planet, and we switch points of view when an activity-sphere (that is sphere of influence) boundary is crossed. Actually, this is the best approximation, if distances of celestial bodies are very, very large. But this is not good an approximation for planet-moon system (such as Earth-Moon, or Jupiter-Europa), because gravity of the planet (such as Earth) is not really negligible within the Moon’s sphere of influence, and patch conics method is inappropriate. In this case, we must treat this sphere of influence as a region, where all gravity of bodies effect simultaneously and getting less or high in respect distance of spacecraft. Therefore CR3BP is much more valid for this dynamical system.

In this way, “Low energy transfer” which can be also called “Ballistic Capture” technique is used [21,22] for considering all gravity effects in capture mission. In the transfer between planet and moon or equilibrium point and primaries (planet or moon), spacecraft can be captured and accelerate or de-accelerate by “effect of both bodies simultaneously” with so small manoeuvre for stabilizing. This process provides much less fuel usage than Hohmann Transfer, and also it is operationally safer. But the Capture transfer is much more complicated and sensitive than Hohmann Transfer by the means of dynamical properties. Previous studies have showed that Capture can be quite chaotic, and after the capture, trajectory is needed to be stabilized with huge velocity decrement for preventing the escape.

In space mission history, ballistic capture transfer was firstly used in rescue mission of Japanese spacecraft Hiten by E. Belbruno and J. Miller [21]. The cost of Earth to Moon transfer could be reduced using the gravitational effect of celestial bodies with small manoeuvre in midcourse point, which is found from Poincare map with intersecting Manifolds of Sun-Earth and Earth-Moon. And cost of mission transfer could be about 25% less than classic Hohmann Transfer with extending trajectory time and launching from highly elliptic parking orbits or earth [21,22]. After this mission, the studies about “low energy transfer” and “capture dynamics” have gain a speed.

Previous studies and applications, such as Hiten, have been focus to determination of capture trajectories that are close to planet in planar-position space [21,22 and 33,35] and then it is applied huge velocity increment for stabilizing the trajectory in low-altitude circular orbits. But this way of capture is needed too much velocity increment [16,21,22], so we can't say that it is low-cost capture. Also because of chaotic dynamics in multi-body, some capture trajectories stay bounded in less time span, and some others in long time span, even some of them stay bounded in permanently without using any velocity increment. Because there could be so much trajectory types, a general and systematic mapping approach is needed to locate low-cost capture trajectories which also includes a complete study of out of plane trajectories for examine whole dynamics. If a convenient map can be constructed in a convenient configuration, we can detect all type of ballistic capture trajectories, with none or allowing for small corrective burns. Therefore we need a way of mapping the possibilities and the final bound trajectories for mission planning.

1.2 Statement of Problem

Because of the chaotic nature of multi-body dynamics, and high dimension of 3D-CR3BP (3 position and 3 velocity, total six dimensional phase space), to find desired capture trajectories are becoming too complicated with using traditional solvers, such as Linearisation or some kin of other mapping methods, such as Poincare map. Therefore it is needed to design a new map method for both reduced this 6 dimensional system to 2 dimension and get clear visualization in 3D-CR3BP in a convenient configuration. So then, any desired capture or bounded trajectories can be found no matter how highly chaotic it is. A “new type of mapping” can present clear solutions and make mission quite flexible.

1.3 Aims and Objectives

The aims of this research are listed here:

- Find advantageous “key points” that gives ‘initial conditions’ for low-cost and zero-cost capture trajectories.
- *Determining likelihood of successful low-cost capture in highly chaotic region.*
- Determine the periodic orbits and capture trajectories which can give desired altitude for low-cost parking orbits or trajectories.
- *Find periodic orbits around secondary and map them*
- Mapping from initial conditions to outcomes.
- Determine the velocity decrement as a function of location.
- Identify a set of interesting missions that could benefit from this approach.
- Apply to missions.

In order to achieve the aims given, the following objectives are given:

- Design a new mapping method for finding these “key points” and put different projection on it for more flexible mission design.
- Analyse some key trajectories in the map that gives advantageous parking orbits or trajectory routes.

1.4 Organisation of Work

This thesis investigates low-cost and zero-cost capture missions in CR3BP. In this chapter we have introduced the drives and objectives of the research and also extracted the novelty in section 1.1 that has resulted from this study. The main body of the research is divided into the remaining chapters and appendices, as described next.

Chapter 2 presents literature review about capture mission (low energy transfer) and some previous works about trajectory computations in multi-body dynamics. The computation methods are highlighted with example of missions.

In chapter 3, dynamics of Circular Restricted 3-Body Problem (CR3BP) is explained, as the most suitable model for the most binary systems in our Solar system. Firstly, the derivation of equation of motion is presented, and then finding zero velocity surface and equilibrium points are shown. Secondly, computations of motion methods are introduced for both analytic and numeric ways. Linearized equation motions around equilibrium points, and some mission features are presented. Then, computations of some useful periodic orbits are presented using both differential corrector (numeric) method and Linstead-Poincare (perturbation) method.

And finally computations of “invariant manifold tubes” are introduced as one of the main tool of the capture missions, and then other staff about transit and non-transit trajectories.

In Chapter 4, Motion maps, which are the most important tool of capture mission in this thesis, is introduced. Motion maps actually new mapping design for determination of the motion types in highly chaotic (nonlinear) system in 3D-CR3BP. In this chapter, main computation scheme of motion map explained and some different alternative projections of configurations are shown for possible mission usage. Furthermore some key trajectories are analysed for usage of advantageous mission, such as; “full ballistic capture”, parking orbits with desired attitude and features of collision trajectories.

Chapter 5 presents some interesting mission samples that are done using “motion map”. Firstly, Velocity decrement computations are shown on the map, and then some science and advance transfer orbits, which are found by Motion Map, are show. Also some mission example about full ballistic capture, which provide zero-cost capture, are presented. Finally, the motion mapping method is applied patch CR3BP for more complicated and realistic example.

In Chapter 6 we summarize the research contained in this thesis. This includes a list of Research Achievements and Findings that has been derived from the different chapters. We finish with an assessment of the contribution of this research to the state-of-the-art, and recommended future work to extend the scope of the project.

II. LITERATURE REVIEW

2.1 Introduction

Multi-body dynamics have been one of the most interesting problems in celestial mechanics, it was formulated by Newton firstly and it had been analyzed its mathematical aspects deeply for many years by Lagrange, Poincare, Delaunay, Lindstedt, Newcomb, Sundman and many others. Although two-body problem has an elegant solution, even three-body problem has much complexity with its 18th order system and 10 exact integrals (3 integrals angular momentum, 1 integral energy and 6 integrals motion of mass centre). The general 3-body problem has been studied in special case solutions such as Langrange's equilateral triangle and collinear solution, which is explained in detailly [2,4,6]. Afterwards the general 3-body problem was reduced to relatively more practical form; Circular Restricted 3-body problem (CR3BP), which can be have more information about it in Chapter 3, also [1-6]. In CR3BP, two massive body make circular motion together and so they can fixed in a rotated frame, and third body, whose mass is neglected compared to these two masses, make its motion in this rotated frame. This model actually perfectly fit most of the celestial system in space, because many body in our solar system make nearly circular motion already. So studies about CR3BP have increased, and for obtaining more insight about this model, it has been done great deal of study by Lagrange, Poincare, Hill for years and also many others, too, which are introduces in following sections.

Although CR3BP presents very practical model for celestial mechanics, it has still quite chaotic nature of multi-body dynamics. And as typical in nonlinear dynamic, its complex motions are generally analysed by using 2 main ways;

- The dynamics of its equilibrium points,
- and mapping methods.

Understanding the motion dynamics, including capture problem, and design very smart and economic space missions, these 2 ways have been used. Equilibrium points of CR3BP provide some many advantageous periodic orbits and trajectories, and in section 2.2, great deal of studies about them is shown. Also mapping methods give quite much insight about motions in CR3BP, and section 2.3, some mapping methods, which have been used in CR3BP, is introduced. Finally, in section 2.4, some other studies about searching Periodic Orbits in Multi-Body Dynamics, which can be quite useful in capture problem are shown.

2.2 Dynamics of Equilibrium Points

As it is written, because of the chaotic nature of multi-body dynamics, the complex motion is generally analysed by way of families of periodic orbits associated with equilibrium points. By linearization around these equilibrium (Lagrange) points, families of halo and Lissajous orbits are identified, and many studies have been done about dynamics of Lagrange points to get more advantageous insides in CR3BP. First of all, Farquhar and Dunham, who named halo and Lissajous orbits, did some significant early key works [7]. Then Conley and McGehee firstly described the phase space regions between L_1 and L_2 and the classification of trajectory types, [8,9]. Moreover, there are Llibre, Martinez and Simó's further explorations of the work of Conley and McGehee, and the other significant works about Lagrange points of the Barcelona school including Gómez, Masdemont [10-14]. And lastly, further applications about mission design using Lagrange points was elaborately presented in Koon, Lo, Marsden, and Ross's book [15] and F. Bernelli-Zazzera, F. Topputo, and M. Massari's final report in [16].

In practical usage, there are many good mission examples, which use dynamics of equilibrium points. Such as, they can be very advantageous in missions which need station keeping L_1 of CR3BP has been used for sun examination mission. Because Halo orbits (periodic or Lyapunov orbit) around L_1 (equilibrium point between Sun-Earth) are the best place for examining the sun with its shape that angular deviation from the Sun is required for the communications link [11]. ISEE-3 (1972), SOHO (1986) and GENESIS (2001) were also other examples for halo orbits missions [15]. L_2 equilibrium point of Sun-Earth system is used for positioning of space telescope, such as FIRST and PLANK, please see chapter 6.2 in [15]. Also quasi periodic-Halo orbits are thought for Terrestrial Planet Finder missions, and some studies about formation fly trajectories around equilibrium point and control strategies have been done [3,10,17,18]. Moreover, J. Masdemont made a high-order expansions of invariant manifolds of libration point orbits, as an approach of analytical solution of them, [19] and furthermore all types of periodic and quasi periodic orbits around an L_1 or L_2 equilibrium point have been deeply analysed, please see [13,20].

Furthermore, other than usage in the station keeping missions, equilibrium points of CR3BP has another importance that 'capture transfers' and low energy transfers can be done using asymptotic trajectories of Halo orbits of equilibrium points which are also called invariant manifold tubes of equilibrium points (please see section 3.2.3). Intersection of two manifold tubes, such as in patch CR3BP, spacecraft can be jumped from one tube to another and a 'highway' can be established with connection of these 'tubes chain' [15].

After first ballistic capture mission, Hiten, could be achieved, thanks to Belbruno and Miller [21], these invariant manifold tubes have attracted a lot of attention recently, because they can provide a mechanism for low cost transfer between the primary and secondary bodies and also can be used in capture problems. Therefore in following 2 sections, the first practical capture missions championed by Belbruno [22], and then some other fundamental studies about low-energy transfer would be introduced.

2.2.1 First Practical usage of Ballistic Capture

In 1990, Belbruno and Miller design a transfer trajectory using the regions of the phase space which are chaotic, to rescue a space mission. These chaotic regions have the advantage that neighbouring trajectories diverge, so a satellite coming into the chaotic zone from the primary can make a small adjustment to shift onto a neighbouring orbit that will remain bound by the secondary. It is even possible that no small adjustment may be needed, and this is called *ballistic capture* [22]. The advantage of this approach is that we can achieve capture about the secondary for free, so gain an extra saving in propellant. The disadvantage is that the chaotic region, or weak stability boundary (WSB), is usually further out than the secondary and this significantly increases the transfer time from the primary to the secondary.

According to story, Japan's ISAS Institute launched a pair of small spacecraft to elliptic orbit of Earth [22], in 1990. These were called MUSES-A and MUSES-B. MUSES-B, smaller one, detached to larger one (MUSES-A). Mission was planned that MUSES-B would go to moon with using Hohmann transfer in patch-conic solution of 2-body dynamics. But the communication of MUSES-B broke and ISAS Institute thought that MUSES-A could tried to go to the Moon instead of MUSES-B. But the main problem in here that MUSES-A never designed to go to the moon, and it did not have enough fuel for Hohmann transfer. E. Belbruno and J. Miller found a solution that 'low energy transfer' could be used using gravitation forces of Sun-Earth-Moon for this mission. This transfer is later called 'ballistic capture transfer', figure-1.

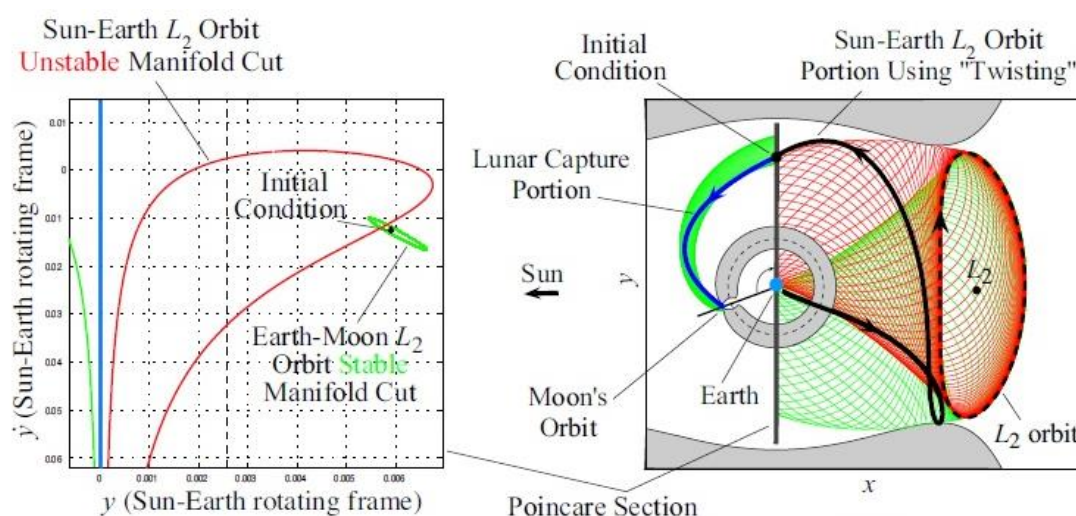


Figure 1 - Low energy Transfer to the Moon, Poincare section for manifold tube transfer (left), Transfer route from Earth to Moon (right) [15].

The mission became a rescue mission, and MUSES-A was renamed as **Hiten**. For determination transfer trajectories, bi-circular patch planar CR3BP model was constructed, this also called restricted four-body problem. In that model, stable and unstable invariant manifold tubes of Sun-Earth Planar- CR3BP were computed numerically (see section 3.2.3) and these tubes were patched with stable invariant tube of Earth-Moon of Planar CR3BP system. While passing between manifold tubes, 'a midcourse manoeuvre' was applied in 'patch point'. The patch points can be determined from intersection of Poincaré section of invariant tubes. But for finding ballistic capture transfer, 'weak stability boundary (WSB)', which is a special map to find

capture trajectories, could be used, see section 2.3.1 and [22]. Finally Hiten could be reached to moon with lunching from highly elliptic earth orbits with $\Delta V = 14\text{m/s}$ and midcourse manoeuvre, which is place to jump Sun-Earth unstable manifold to Earth-Moon stable manifold, is 30m/s . In total, 44m/s velocity increment was done. For Hohmann transfer (patched-conic in 2-body dynamics), it would be 200m/s , please see for more detail in [22].

For more practical comparison, trajectories are assumed to go circular lunar orbit in 100 km altitude for both case (Capture and Hohmann Transfer). Because ‘weak stability boundary’ and so ballistic capture is highly unstable and there is need a final velocity increment around capturer planet for stabilized the orbit. So that, 648m/s velocity increment are added to each case, then total ΔV for Hohmann transfer is 848m/s while it is 678m/s for capture transfer [22]. There is 20% gain about ΔV that this is very significant in space mission. Although flight time is extended much more (about 150 days in capture whereas 5 days in Hohmann transfer), minimization of ΔV is much more significant than flight time in un-manned space mission.

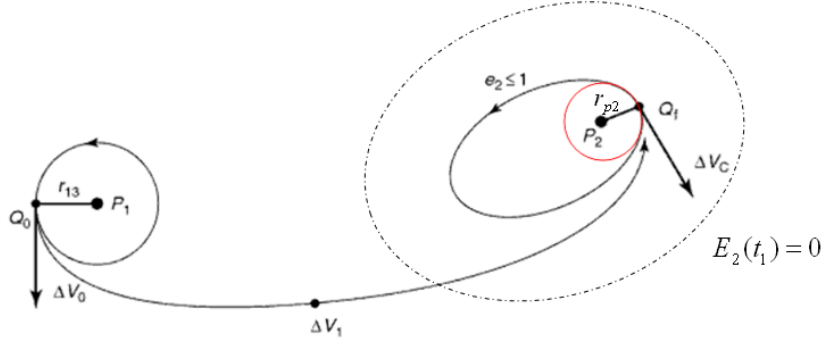


Figure 2 - The capture problem,[22]. Dash line is the capture boundaries where Keplerian energy is equal to the zero relative to the P2. So the capture actualize while the Keplerian energy less then zero.

In figure-2, the capture scheme can be seen more clearly. First of all, a velocity increment – ΔV_0 is applied to jump from initial circular parking orbit to transfer trajectory, which is generally invariant manifold tube. Then another velocity increment (ΔV_1), or midcourse manoeuvre, is done to jump one manifold tube to another one. Finally, the final velocity increment (ΔV_c) is applied to get final circular (or maybe non-circular) parking orbit, red circle in figure-2. The main problem is to minimize all velocity increment in while mission, see equation (2.1).

$$\min\{\Delta V_0 + \Delta V_1 + \Delta V_c\} \quad (2.1)$$

The midcourse manoeuvre (ΔV_1) can already be optimize with Poincare mapping method. Therefore, the main minimizing problem is in initial and final velocity increment. If the trajectory is stabilized to circular parking orbit, the main formulation for velocity increment is as equation 2.2.

$$\Delta V_C = \underbrace{\sqrt{\mu \left(\frac{2}{r_{p2}} - \frac{1}{a_2} \right)}}_{\text{Trajectory}} - \underbrace{\sqrt{\frac{\mu_2}{r_{p2}}}}_{\text{Circular Orbit}} = \sqrt{\frac{\mu_2}{r_{p2}}} (\sqrt{1+e_2} - 1) \quad (2.2)$$

In equation 2.2, the velocity value of capture trajectory, while in its periapsis passage, is subtracted to velocity value of circular parking orbits. As it is seen, if the capture trajectory gets its periapsis passage with highly eccentric ($\sim 0.7 < e < 1$), the velocity increment value becomes huge. But if it passes in nearly circular motion ($e \approx 0$), the velocity increment becomes nearly zero. The passage with zero eccentricity seems quite advantageous, however there is a very important thing that must not be forgotten; highly distribution gravitation effect of external body to target planet. Which means that; higher energy of trajectory, higher distribution effect from external body. In multi body dynamics, low altitude circular orbits are in lower energy, and they are affected less from external body, so that they can be close to Keplerian motion. But high altitude circular orbits are in high energy, and they are affected high from external body, so that their motion becomes much more different from familiar Keplerian motion. The capture trajectories always come to target planet realm with relatively high energy, and even they take a circular passage around target planet, they may not sit in a stable circular parking orbits, directly. It actually means that; equation 2.2 is only valid for high eccentric passage, and stabilization to trajectory in very low energy level. Indeed all this statement can be valid for planar-CR3BP. In 3D-CR3BP, the nature of motion is quite change, as it can be seen in chapter 4.

In addition, this mission and scheme in figure-2 also show three side of the capture process;

- First one is launching part from initial body (Earth). Although the initial velocity increment does not seem too high because of high elliptic initial parking orbit, the initial velocity increment has actually biggest ΔV , if altitudes of initial parking orbits are ~ 200 km LEO (Low Earth Orbits), and ΔV for launching from LEO is about 3000-3200 m/sec [15,16]. This means total ΔV become bigger and so total gain become smaller if initial parking orbit is LEO.
- Second is that captured one manifold tube to another, which is solved easily with Poincaré Map. Although Poincare section may not be so suitable for every situation (see section 3.2.2), jump problem between manifold tubes can be solved relatively easily and low-cost with this method. As it is seen figure-1, the “key point” for optimum transfer can be found with Poincare section (or surface of section).
- The last one is captured by target planet. But Capture around target planet with low cost (without reducing energy too much for getting *suitable* parking orbits), which is the main aim of this thesis, is more complicated problem because highly chaotic nature in there. So for solving this problem, it is done another approach, see chapter 4 and 5.

As it is mentioned before, this Hiten mission pioneer and inspire to other low-energy transfer studies; and cause very various fundamental literature about it.

2.2.2 Low-Energy Transfers

Although Hiten mission caused a mass studies about low energy transfers, using the gravitational forces of celestial bodies was not a new thing. Such as, very previously, planetary flyby technique, also called slingshot maneuver, have been already known, please see [3] section 4.5.3. Also as for another aspect, Bolt and Meiss studied transfer between Earth and Moon's high altitude orbits, and primitively applied of low energy transfers through L_1 aims at guiding the spacecraft from an initial state to a desired final state by using a carefully chosen sequence of small perturbations, [23]. But the real massive and significant studies about transfer using invariant manifold tube are started by the team; Koon, Lo, Marsden, and Ross. They presented very fundamental and detailed information about low-energy transfer using Poincaré surface of section map between invariant manifold tubes [15,24], and also they showed some fundamental resonance orbits, which could be useful in low-energy transfer between primary and secondary region in CR3BP, on the example of Oterma Comet [25]. For further information and detail about their works, also see [26]. Then another intensive study wave about low energy transfer came from by Topputo, Vasile, Bernelli-Zazzera and Ercoli-Finzi, [16, 27-32]. Particularly, their most significant work about low energy transfer is that usage some "transfer arcs" to reach invariant manifold tubes [16], which is briefly explained in below. Furthermore, last but not least, E.Canalias, J. J. Masdemont computed interesting and significant natural transfers between Sun–Earth and Earth–Moon Lissajous libration point orbits in [90,91].

However, Earth-Moon transfer from Earth LEO to Moon (or an invariant manifold tube) is more expensive than from HEO (Highly Elliptic Orbits) and/or GEO (Geostationary Earth Orbit). It is needed $\Delta V \sim 4000 \text{ m/sec}$ from low altitude (167 km) earth orbit to low altitude (100 km) circular moon orbits with Hohmann Transfer, and flight time is 5 days [16,31]. In same altitude departure and arrival orbits, total cost is $\Delta V \sim 3850 \text{ m/sec}$ for WSB (using Sun-Earth-Moon gravity) transfer, and transfer time is ~ 90 days, [31]. Also, in the case when only gravitational force of Earth and Moon must be used, the invariant manifold tube of L_1 Halo orbits is not approached close enough to low altitude of Earth orbit. Therefore, because stable or unstable manifolds of L_1 , whose Halo size about $A_z = 1000\text{-}8000 \text{ km}$, are not approach to any low altitude Earth trajectory [16], it is needed an **intermediate arc** that transfer from Low-Earth orbit to stable manifold of L_1 , [16, 27-32]. This means it is needed two impulsive thrust to reach to Invariant manifold: First thrust (ΔV_1) is from initial orbit (GTO, GEO or LEO) to 'intermediate arc', and second thrust (ΔV_2) is from 'intermediate arc' to invariant manifold. In figure-3, two of Earth-moon transfer methods are shown, using these arcs. Firstly satellite is launched from suitable earth parking orbits with ΔV_1 . In here, LEO would provide less velocity cost than GEO or GTO, because it is needed extra velocity cost to reach them and this increase total cost of mission. The second velocity increment (ΔV_2) is done to get the stable L_1 manifold. Finally last velocity (ΔV_3) increment can be done for stabilizing the trajectory around Moon or just getting the Halo orbits.

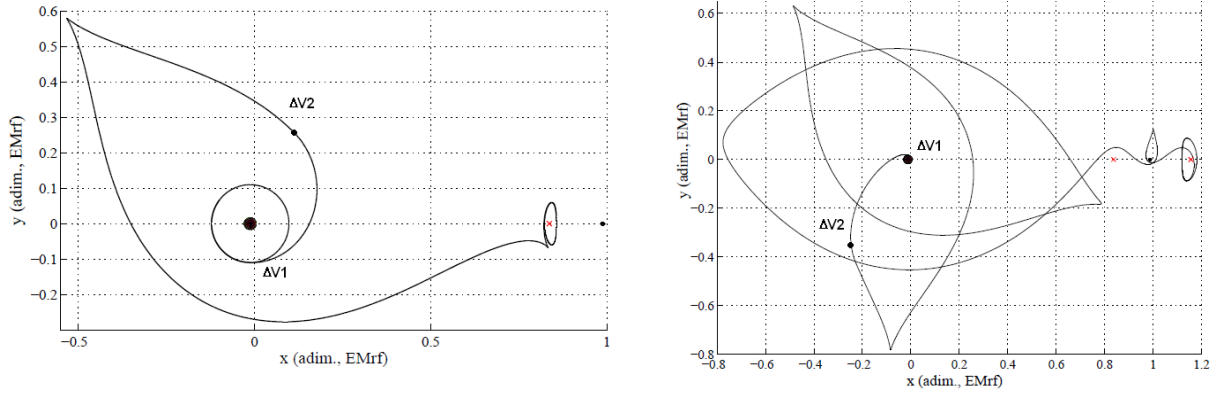


Figure 3 – Earth Moon Transfer with catching Stable L_1 manifold from GEO (Left) and LEO (Right), [16]

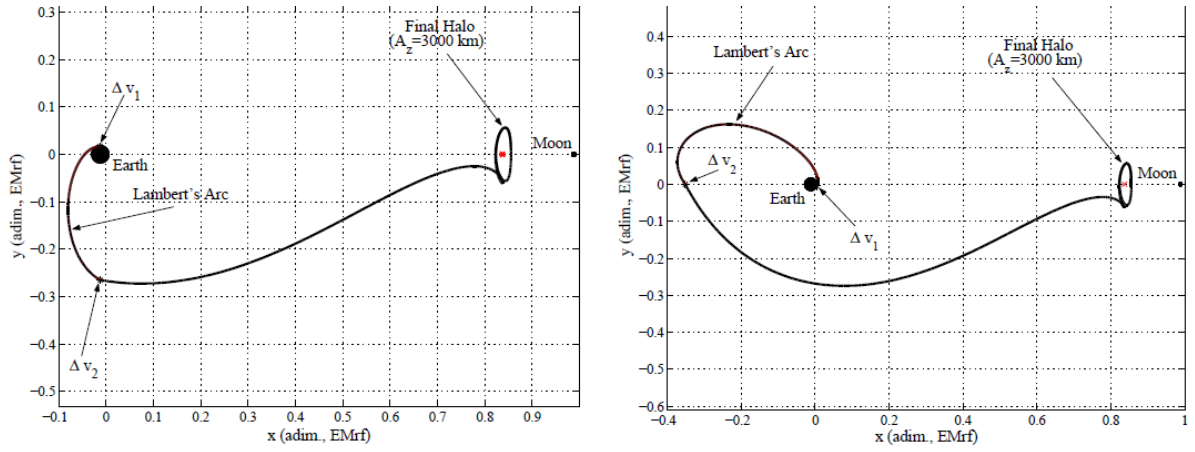


Figure 4 - Earth Moon Transfer with $\Delta V \sim 4144$ m/sec and $\Delta t = 17.2$ days (left), and $\Delta V \sim 3865$ m/sec and $\Delta t = 19.1$ days (right), [16]

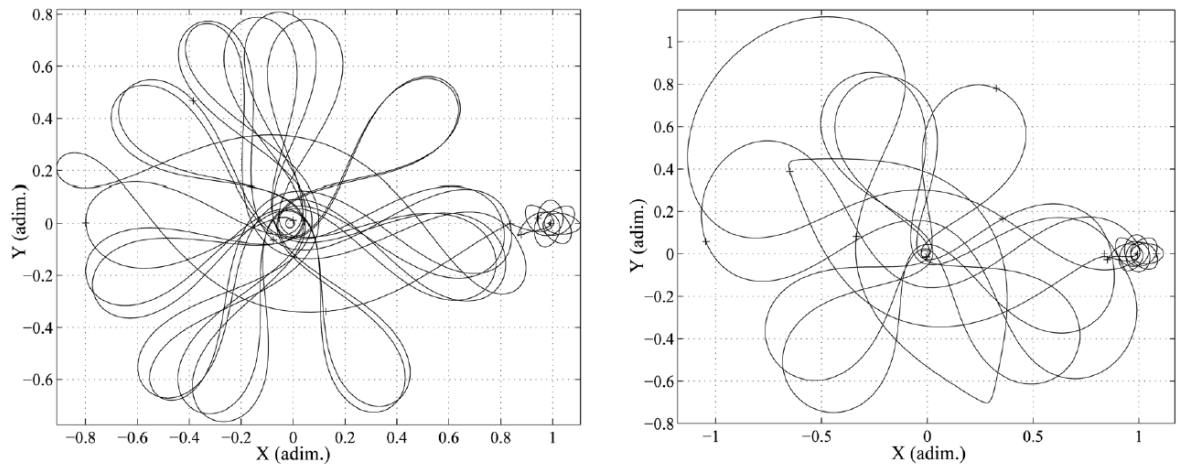


Figure 5 - Earth Moon Transfer with $\Delta V = 3894.9$ m/sec and $\Delta t = 255.5$ days (left), and $\Delta V = 3900$ m/sec and $\Delta t = 193.7$ days (right), [31]

Whole cost is change according to size of halo orbits for this type of transfer, see [16]. In figure-4, total cost and scheme are presented for $A_z=3000$ km halo orbits. As it is mentioned previously, the main problem in this approach is stable manifolds can not approach to low altitude of Earth, so it is needed to make two velocity increment, ΔV_1 and ΔV_2 , and this makes increase the total ΔV . Further more, there would be needed another velocity decrement for stabilizing the trajectory around moon. This ΔV_3 would be about ~ 50 m/sec and much more for high altitude circular moon orbits (with closing both L_1 and L_2 gates of zero velocity surfaces), or about ~ 750 m/sec for low altitude circular moon orbits.

In figure-5, another application is shown. Earth-Moon transfer is done with directly launch from LEO and applying “advance arc transfer” along the trajectory [31]. Even total ΔV is reduced little than previous application, Δt is increased too much.

As a result, total ΔV could be reduced (about %2-4) so little relative to the Hohmann transfer, but Transfer time (Δt) increase too much, if it is used a type of transfer as in figure-5. In Low-thrust transfer, transfer time increase at least more than 10 times relative to Hohman transfer, while total ΔV reduce only % ~ 3 . However transfer types as in figure-3 and 4 can be much more suitable to reduce both ΔV and Δt reasonably. In section 5.4, some applications for reducing both total ΔV and Δt like this are shown using motion map, chapter 4.

2.3 Mapping Studies in CR3BP

The problem with the invariant manifolds is that when they approach to secondary body, their motion become quite chaotic around secondary region, and a very huge corrective burns, which makes mission expensive, is needed to stabilize the trajectory safely. However, a suitable map, which is already widely used in Chaotic and Non-linear dynamics, can solve this problem very practically.

As can be seen from the huge literature review, one of the most practical approaches for studying these chaotic regions is to exploit some mapping methods. And the most widely used approach is to use the Poincaré surface of section which provides a 2D slice through the 4 dimensional phase space (x, y, v_x, v_y) of the planar-CR3BP, although the real problem is in 6 dimensional phase space (x, y, z, v_x, v_y, v_z) of the 3D-CR3BP. But the solution of capture problem get much easier if the study is restricted to the 4D phase space of the planar problem, because Poincaré plots cannot be in good visualization in 6D phase space of the 3D-CR3BP. So to avoid too much complexity in 3D-CR3BP, studies of ballistic capture in the planar case have been done very widely, such as [15, 24,25 and 33,34]. Although classical Poincaré mapping method (Surface of section) gives quite practical approaches to understand and use some chaotic regions of CR3BP, it is insufficient and unpractical in some conditions, such as chaotic capture around secondary body of CR3BP, especially in 3D space. Therefore, some other map techniques and approaches have been derived by many researchers.

2.3.1 Weak Stability Boundaries

The first map approach for capture problem was developed and improved by Belburuno. Name of this map is WSB (weak stability boundaries), which is explained very detail in his book [22] and Belburuno and Topputo's journals [33-36].

Because the manifold tubes transfer could be done relatively easily, then the works focus to capture process near target body. For solving of this problem, Belburuno developed the idea of "Weak stability Boundaries", which present the "key points" around the target planet, in phase space, where spacecraft could ballistic capture without use any propellant in short term, [22,35].

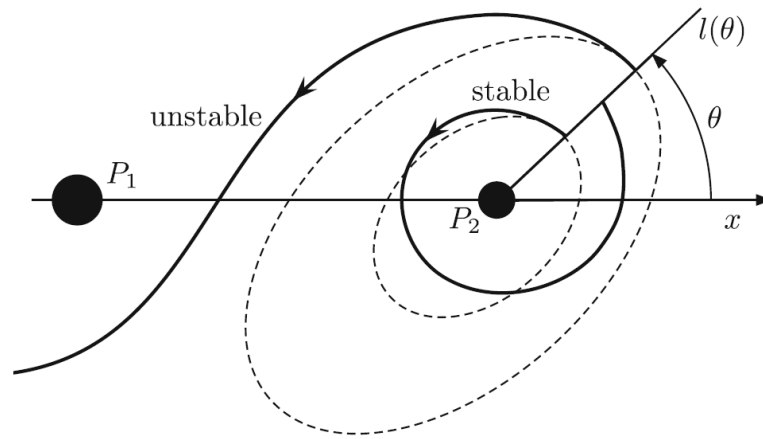


Figure 6 - Example of stable and unstable trajectories by Belbruno, around target planet (P2), [35].

As it is seen in figure-6, the stability and instability definition is done around target planet, and the initial conditions sets like this, [35];

- i. The trajectory start its motion on a radial segments departing from P2 and makes an angle (θ) relative to P1-P2 line. The trajectory is supposed to start at the periapsis of an osculating ellipse around P2.
- ii. The initial velocity of the trajectory is perpendicular to the radial segments. And initial Keplerian elements (θ, e) set as fix.
- iii. The motion is called n-stable if the trajectory makes "n" complete turns around P2, while its Keplerian energy is always negative. Otherwise it is called n-unstable, figure-6.

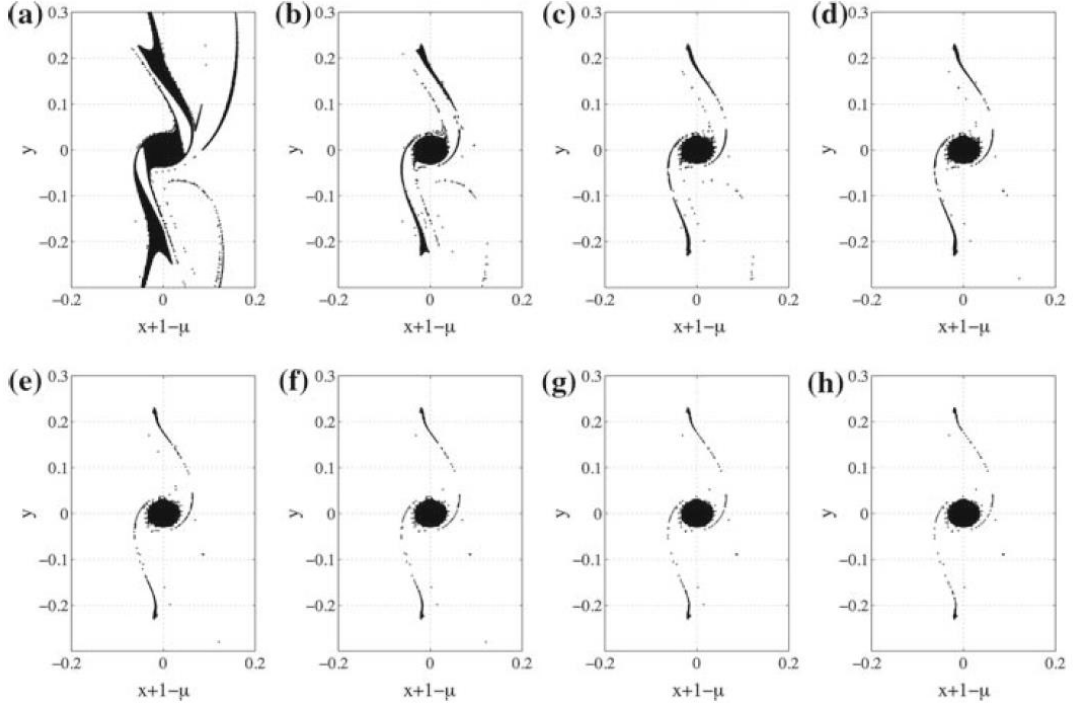


Figure 7 - Collection of W_n , $n=\{1,2,3,4,5,6,7,8\}$. a) $W_1(0,0)$, b) $W_2(0,0)$, c) $W_3(0,0)$, d) $W_4(0,0)$, e) $W_5(0,0)$, f) $W_6(0,0)$, g) $W_7(0,0)$, h) $W_8(0,0)$, [35].

Making numerical computation for large number of orbits with these initial conditions, a set of stable motion can be plotted as $W_n(0,e)$. So, trajectory define stable after makes ‘n’ complete turns, but trajectory can be escape after the ‘n’ turn, so it is needed to stabilized in circular parking orbits. In practically, the capture trajectories are stabilised while their periapsis passage, so that θ is taken as zero. And the final parking orbits are generally circular (generally intended to low altitude circular orbits), so eccentricity (e) is taken zero, too. Then, stable set of Weak stability boundaries are plotted as $W_n(0,0)$. If value of “n” is increased, the stable set is shrinking, of course, figure-7.

Also note that, F. García, G. Gómez, in [37], made another clarify the algorithmic definition of the weak stability boundary in the framework of the planar Restricted Three Body Problem. They highlighted that the precise definition of WSB has not be gotten, at least in the usual mathematical sense of the word, although there is a large number of papers dealing with WSB transfers going from the Earth to the Moon. They showed that Belburuno’s algorithmic definition in [22] is not accurate in some points; Such as the requirements on the initial conditions. So using the algorithmic definition of the WSB previously mentioned, they compute these boundary with for different values of the eccentricity e and using the two different determinations of the velocity. And in the end the gave numerical evidence that these regions seem to be bounded by the stable manifolds of the central objects around L_1 and L_2 at the same energy level; for further detail see [37]. After a year, Belburuno then made more clear definition of WSB in [34]; visualize the weak stability boundary for the first time and showed the results are valid for a large range of the Jacobi constant by using the energy method.

However, no matter how it is defined, the main weakness of this approach is that all the computation and study is done in Planar-CR3BP, 4-dimension phase space (x, y, v_x, v_y) , and also in a relatively restricted time and done for only capture trajectories; so that, it is not reflected whole dynamical features around target planet. Although Planar-CR3BP reduces variables in phase space dimension (and also complexity), the complication is increased much more because of the rotation effect in x-y plane. If the ballistic capture is studied in 3D-CR3BP, we can know all dynamical features, especially it can be seen what happened to the motion if rotation effect reduced in third dimension. Another thing that the “weak stability boundaries”, that define in above, gives too sensitive result and present short term capture that it probably needs some propellant usage for stabilizing the trajectory.

2.3.2 Trajectory Evolutions in the Multi-Body Dynamics

After Belburuno’s WSB approach, some other map approaches with trajectory evaluating have been studies many others. Anderson, Lo and then Parker [38,39] have developed another mapping technique to determine advantageous trajectories from Earth orbit to Lunar orbit in the 3D Sun-Earth-Moon ephemeris model. Especially, Anderson and Parker’s approach is similar to the one in the Hiten mission route [22]. Adding the effects of the Sun and the Earth-Moon eccentricity to the CR3BP, they produced more solutions, which gives more options for Earth-Moon transfer. This work presents some interesting possibilities for transfer with times up to 200 days. A systematic study of the phase space, however, and a study of close lunar capture has not been attempted.

Also, Ross and Scheeres [40] have used Picard’s method as an approximation of a Poincaré return map, a family of two-dimensional symplectic twist maps to approximate a spacecraft’s motion on the manifold tube in the planar case. This approach provides a good approximation to the capture dynamics in the 4D phase space. Furthermore, Paskowitz and Scheeres [41] have studied the dynamics of temporary capture in the 3D Hill three-body problem with application to orbits about Jupiter’s satellite Europa. Using a periapsis Poincaré map for finding these temporarily captured trajectories, the authors considered capture into long-term stable frozen orbits with low-cost (~ 50 to 500 m/s). They found that higher inclination orbits are better for low-cost capture, which is a result that will be verified this paper, too.

Also Howall, Davis and Haapala [42-45] have generated a perapsis initial conditions map, which is similar to Paskowitz and Scheeres’s one, but not in terms of Keplerian elements, to determine capture trajectories for both short and long term, see figure-8 for short term version. In this study, initial condition map of Sun-Saturn system is plotted with 6 revolution time, and all trajectories are classified as 4 main category; escape through L1, escape through L2, impact, and capture. The capture in here actually is temporary capture trajectory, which is captured for only 6 revolution time, then it is more likely escape.

But this map has some weakness to get real insight of dynamics of capture. First of all, the capture trajectories in figure-5 will be actually escape trajectories, after 6 revolutions. So this map only shows escape, capture (but then escape again, because it is temporary capture) and impact. However there are actually 4 main possible motion in around neck region in CR3BP; escape

thought L1, escape thought L2, impact, and **bounded** (All periodic, quasi-periodic, and chaotically bounded trajectories), see chapter 4. To determine bounded trajectories are quite important to provide low-cost capture, which is explained in detail in chapter 4 and 5.

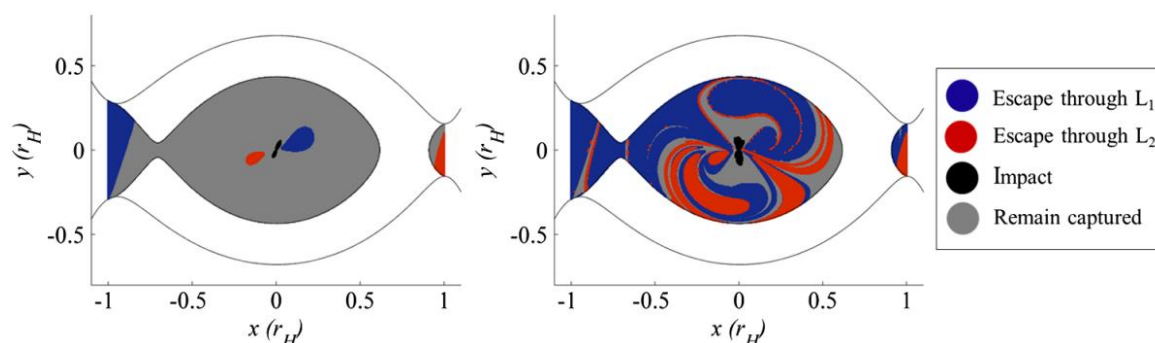


Figure 8 - Initial condition map for one revolution (left) and 6 revolutions (right) for Sun-Saturn system, $C=3.017304659624$, [42].

Also another weakness, although this map presents a nice solution in the planar case, when the z-component to the trajectory's position and velocity are added, the map becomes more complicated for the visualization of the solution space [43]. But the real capture problem is in 3D space, so if a comprehensive way to analysis of 3D-capture trajectories can be found, then the capture problem can be solved the more general way, that suits any mission design.

2.4 Periodic Orbits in Multi-Body Dynamics

Another phenomenon about in multi-body dynamics is the periodic orbits. Periodic orbits, which can be around both primary and secondary bodies, are so important because capture trajectories can be stabilized to these periodic orbits as parking orbits. But, because of its complexity, these periodic orbits, which are in the effect of external gravitational effects, is not as simple as in classical two body Keplerian problem, see figure-9. In the two body problem, motions are always elliptic and its Keplerian elements $\{a, e, i, \Omega, \omega, \theta\}$ are all fixed, it means constant. But in multi-body dynamics, Keplerian elements always osculate, not fixed, and the motion become quite complex that it is hard, even impossible, to get any exact accurate analytical expression and/or definition.

However, orbits, especially periodic orbits in the Restricted Three Body Problem (RTBP) have been studied and documented by many researchers. Such as Broucke [46,47], Henon [48-50], Poincare [51] and Szebehely [2]. For many other studies specifically related to three-dimensional orbits, see Jeffreys [52,53], Goudas [54,55] Zagouras [56,57], Markellos [58] and Lara [59-61]. Especially, A variety of techniques have been introduced to identify both specific solutions and continuous families of solutions, also some of orbit results by Lara and Russel [62-65]. In addition, Russell and Brinckerhoff [66] and Paskowitz and Scheeres [67] made detailed analysis of special type of science orbits.

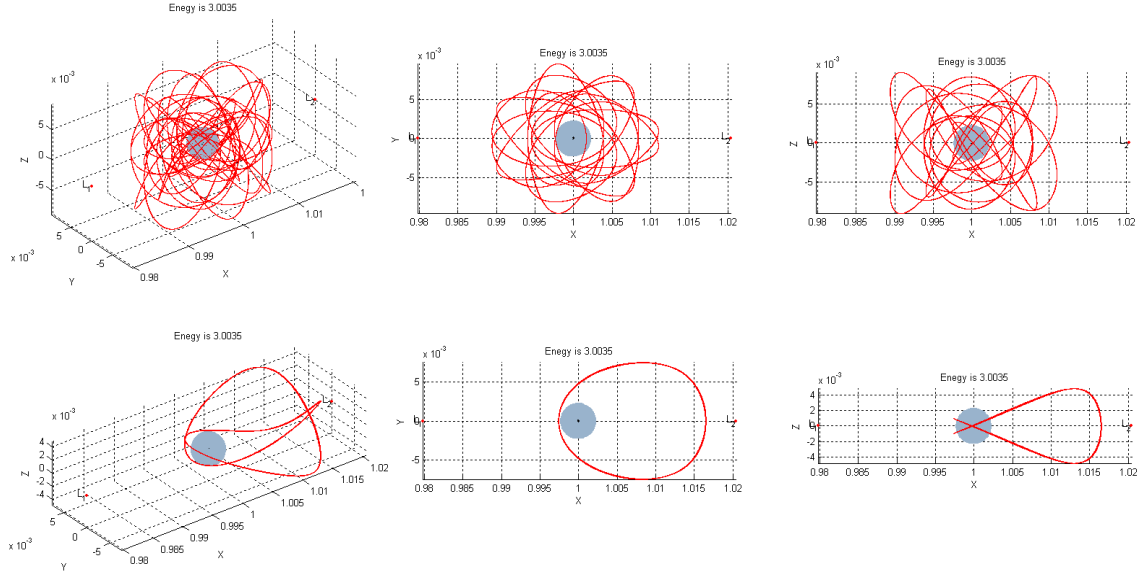


Figure 9 - Two sample periodic orbits around Europa in Jupiter-Europa of CR3BP. Minimum altitude of above orbit is 16.8 km, minimum altitude of below orbit is 206 km, data from [66].

On the other hand, a numerical method known as *differential correction (DC)* (or Newton method), which is explained in section 3.2.2 and for further informations are in [11,14,15], gives an alternative approach to finding periodic orbits. In this approach, initial guess is incorporated in an iterative process aimed at producing initial conditions belonging to a periodic orbit. For example, the orbit is integrated forwards for half a period and then checked to see if the velocity vector has reversed sign but has equal magnitude. If so, then in the second half of the orbit, the trajectory will close. If not, a differential correction is applied to the initial conditions which will bring the final velocity vector closer to the desired state. The iteration then moves closer to the periodic orbit. It is ideally suited for finding symmetric and axi-symmetric periodic orbits, but can be also used to determine other type of trajectories using different procedure explained just above. The method has been also exploited by R. P. Russell [68] to search for periodic orbits in multi-body systems. In this approach a 4-dimensional grid of initial conditions is searched systematically. Two of these dimensions are the energy (Jacobi Constant) and the period of the orbit. From these initial conditions the differential correction method is employed for locating all the periodic orbits by iteration. Over 600,000 periodic orbits (including all planar, 3-dimensional, direct and retrograde orbits) were found taking about 1 week of computation time. The application was looking for periodic orbits around Europa in CR3B model for the Jupiter-Europa system [68].

However, because the differential corrector (or Newton method) needs a first guess to producing initial conditions belonging to a periodic orbit, it is needed to have a good idea where the periodic orbits are likely to be. Otherwise, we have to apply a kind of grid searches or methods as in [68]. Fortunately, another kind of grid search in this research (Motion Maps in Chapter 4) can be also incorporated as initial guess with the differential corrector, quite practically and easily. Therefore, a more simple and fast way of surveying phase space for periodic orbits might be a very useful tool when considering mission analysis to moons of the planets in the Solar System.

Furthermore, in R. P. Russell's study, large amount of periodic orbits are highly-inclined *direct* which are in the same direction with rotation axis in CR3BP, same as result of the "motion map" in chapter 4. This is because highly inclined orbits are not affected so much from rotation effect of 2 massive bodies in CR3BP. This is actually quite a good result, because the capture trajectories are all in direct direction, and to put them to a direct periodic orbit can be quite easy and cheap.

In this thesis, we also consider a method for finding highly inclined and high energy periodic orbits with using motion map (chapter 4). These are particularly interesting as highly inclined orbits are less affected by the orbital rotation of the two masses, and so more periodic families of orbits are expected to see, and also because spacecraft on approach to a secondary will have high energy and insertion into these periodic orbits is of particular usefulness. Such orbits can be used as parking orbits, navigation orbits around the secondary body, intermediate transfer orbits or for scientific measurements of the environment of the secondary. The interesting catalogue about of these orbits can be seen in [68].

Furthermore, using some perturbation method, such as *Linstead-Poincare* (LP) perturbation method (section 3.2.2.3), analytical expression of unstable periodic orbits around equilibrium points can be obtained. Because of calculation burden, this method quite daunted, but once the results are get, the control cost is reduced less than previous simple calculation, [3,11-15].

In summary, R. P. Russell [68] found huge bunch of periodic orbits, which can be useful for capture mission, using DC method, but it took too long time and, their projection is not clear enough that Periodic orbits don't comprehend well. DC and LP method have been associated for finding periodic orbits in effect of multi-body gravity for years. But their too much computation burden and long calculation time makes methods daunted. But using the Motion Map in chapter 4 as initial guess for DC, periodic orbits can be easily determined quite simple and without much calculation time, please see chapter 4 and 5.

2.5 Conclusion

As a conclusion, Capture (Low-Energy) Transfer and using advantageous of multi-body dynamics have gained great interest for last decades. A lot of applications and works are done for planetary transfer low-cost capture and periodic orbits, which are linked with capture trajectories. Further information, interested reader may look at these links [69-73].

Although many studies about this subject, capture in 3D-CR3BP have not studied comprehensively enough, yet. Because it have been thought 3D-CR3BP cause too much complexity to the visualization of the solution space, and they mainly studied capture problem in planar-CR3BP to avoid this complexity. However defining a suitable initial condition map for reducing the system dimension 6D to 2D, then capture problem and its trajectories can be visualized and analysed more comprehensively in 3D solution space, please see chapter 4 and 5.

III. CIRCULAR RESTRICTED 3-BODY PROBLEM

For centuries, a lot of mathematicians have tried to solve and develop some restricted solutions about many-body problem in celestial mechanics, but the general analytical closed solution could not be obtained. Only 2-body problem could be solved and obtained an elegant analytical solution as Keplerian form that one body is centred as origin. For 3-body problem, there are some stationary and restricted applications, such as triangular and collinear stationary solution or Circular Restricted 3-Body Problem (CR3BP). Among of them, only CR3BP can present the most comprehensive application for our solar system. Majority of binary system, Sun-Planet or Planet-Moon, make circular motion together and CR3BP would be the best model.

3.1 Equation of Motion

In the CR3BP, M_1 and M_2 are assumed very massive objects compared to the m_3 . So, gravitational effect of m_3 to the the third body can be neglected; $M_1 \geq M_2 \gg m_3$. The first two masses make circular Keplerian Motion together about their mutual centre of mass. Therefore, the masses M_1 and M_2 affect the motion of m_3 , without in return being affect by m_3 themselves, [1-6], see figure-10.

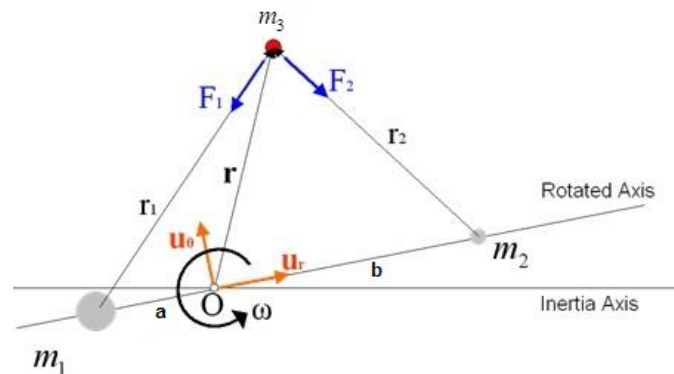


Figure 10- Circular restricted 3-body problem. Rotating axis is rotated in angular velocity relative to inertia axis.

Position vector and angular velocity vector of the rotated frame are respectively, ($\mu_1=GM_1$ and $\mu_2=GM_2$, G is gravitational constant);

$$\vec{r} = r_x \hat{u}_r + r_y \hat{u}_\theta + r_z \hat{u}_z \quad (3.1)$$

$$\vec{\omega} = \dot{\theta} \hat{u}_z = \omega \hat{u}_z, \quad \omega^2 = \frac{\mu_1 + \mu_2}{r_{12}^3} \quad (3.2)$$

r_{12} is the distance between M_1 and M_2 . The inertial acceleration vector of $\vec{r}$ is, [74];

$$\ddot{\vec{r}} = (\ddot{r}_x - 2\dot{r}_y \dot{\theta} - r_x \dot{\theta}^2 - r_y \ddot{\theta}) \hat{u}_r + (\ddot{r}_y + 2\dot{r}_x \dot{\theta} - r_y \dot{\theta}^2 - r_x \ddot{\theta}) \hat{u}_\theta + (\ddot{r}_z) \hat{u}_z \quad (3.2)$$

Because M_1 and M_2 make circular Keplerian 2-body motion, $\omega=d\theta/dt$ is constant and derivation of ω is zero;

$$\ddot{\vec{r}} = (\ddot{r}_x - 2\dot{r}_y \dot{\theta} - r_x \dot{\theta}^2) \hat{u}_r + (\ddot{r}_y + 2\dot{r}_x \dot{\theta} - r_y \dot{\theta}^2) \hat{u}_\theta + (\ddot{r}_z) \hat{u}_z \quad (3.3)$$

And gravitation forces on m_3 is;

$$F = F_1 + F_2 = - \begin{pmatrix} \frac{\mu_2(r_x + a)}{r_1^3} + \frac{\mu_1(r_x - b)}{r_2^3} \\ \frac{\mu_2 r_y}{r_1^3} + \frac{\mu_1 r_y}{r_2^3} \\ \frac{\mu_2 r_z}{r_1^3} + \frac{\mu_1 r_z}{r_2^3} \end{pmatrix} \quad (3.4)$$

Combining (3.3) and (3.4), the equation motion of m_3 can be written;

$$\begin{aligned} \ddot{r}_x - 2\dot{r}_y \omega - r_x \omega^2 + \frac{\mu_2(r_x + a)}{r_1^3} + \frac{\mu_1(r_x - b)}{r_2^3} &= 0 \\ \ddot{r}_y + 2\dot{r}_x \omega - r_y \omega^2 + \frac{\mu_2 r_y}{r_1^3} + \frac{\mu_1 r_y}{r_2^3} &= 0 \\ \ddot{r}_z + \frac{\mu_2 r_z}{r_1^3} + \frac{\mu_1 r_z}{r_2^3} &= 0 \end{aligned} \quad \begin{aligned} r_1 &= \sqrt{(x+a)^2 + y^2 + z^2} \\ r_2 &= \sqrt{(x-b)^2 + y^2 + z^2} \end{aligned} \quad (3.5)$$

In equation 3.5, there are 5 parameters that are ω , μ_1 , μ_2 , a and b (a is distance between M_1 and origin 'O', and b is distance between M_1 and origin 'O'). Equation 3.5 can be arranged to non-dimensional form and equations motion can be written for one parameter, for simplify;

$$\begin{aligned}
\omega^2 = \frac{\mu_1 + \mu_2}{r_{12}^3} = 1 & \quad \mu_2 = \mu = \frac{m_2}{m_1 + m_2} & \quad r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2} \\
\mu_1 + \mu_2 = r_{12} = 1 & \quad \mu_1 = 1 - \mu & \quad r_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}
\end{aligned} \tag{3.6}$$

Time variable can be non-dimensional form as $\tau = \omega t$ (ω as mean motion; $\omega = 2\pi/T$, T is period of binary system) and transformations of variables between the dimensional ($\mathbf{r}_{xyz}, \mathbf{v}_{xyz}$) and non-dimensional (x, y, z, v_x, v_y, v_z) form are below;

$$x, y, z = \frac{r_{x,y,z}}{r_{12}} \quad \text{and} \quad \dot{x}, \dot{y}, \dot{z} = \frac{\dot{r}_{x,y,z}}{\omega r_{12}} \tag{3.7}$$

So that, non-dimensional equation of motion is:

$$\begin{aligned}
\ddot{x} - 2\dot{y} &= U_x = x - (1 - \mu) \frac{x + \mu}{r_1^3} - \mu \frac{x - 1 + \mu}{r_2^3} \\
\ddot{y} + 2\dot{x} &= U_y = y - (1 - \mu) \frac{y}{r_1^3} - \mu \frac{y}{r_2^3} \\
\ddot{z} &= U_z = -(1 - \mu) \frac{z}{r_1^3} - \mu \frac{z}{r_2^3}
\end{aligned} \tag{3.8}$$

And its Potential function;

$$U(x, y, z) = \frac{x^2 + y^2}{2} + \frac{\mu_1}{r_1} + \frac{\mu_2}{r_2} + \frac{\mu_1 \mu_2}{2} \tag{3.9}$$

In here, r_1 and r_2 are the distance from m_3 to M_1 and M_2 . Bigger mass ($\mu_1 = 1 - \mu$) is primary mass and smaller mass ($\mu_2 = \mu$) is secondary mass.

In classical CR3BP, because there are 2 massive body (M_1 and M_2), therefore spacecraft can be captured either of them in that dynamical system. But in real space, one of the bodies is generally much more massive than the other one, and bigger body highly disturb much the realm of the smaller (secondary) body. So then, capture mission would become more challenging in smaller body region, which this thesis actually focus.

3.1.1 Zero Velocity Surface

For finding possible motion area of third body (m_3), the potential function in equation 3.9 is used. The particle (m_3) is actually moving in a field which is also called effective potential [15], and the potential function gives this effective potential surface, which is shown in figure-11-left. A horizontal cut of this plot of the effective potential gives Hill's region (figure 11-right), where the third body can only move in there at a given energy level C (Jacobi constant). Also merging the potential function with kinetic energy $V^2 = v_x^2 + v_y^2 + v_z^2$, Jacobi integral function, which is conservation of energy, can be obtained;

$$V^2 = 2U - C \quad (3.10)$$

Thus, using eq.3.10, at a given energy level C (Jacobi constant), and in zero velocity $V=0$, Hill's region can be shown as $C - 2U > 0$. And for the case $C - 2U=0$, Zero velocity curves (in 2D plot or surfaces in 3D plot) can be plotted in given C , as all shown in figure-11. In this figure, there are 3 main realms; namely the inner region, the neck region and the outer region that are connected by the neck region. For other values of the energy (C), one or more of these realms may be prohibited due to conservation of energy; that is, the necks may close off.

Also note that; please see Matlab code, [zvs.m](#) and [pot_fun.m](#), in appendix CD, to see plotting codes of figure-11.

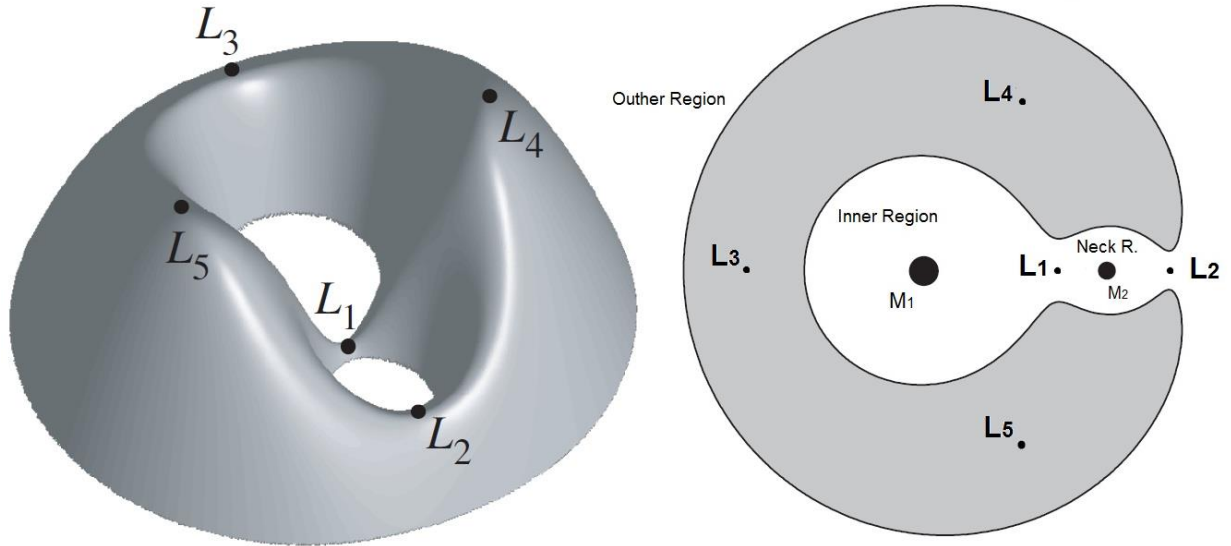


Figure 11 - potential surface $U(x,y)$ left, and Zero velocity curves (right) and equilibrium points [15]

3.1.2 Equilibrium Points

In this system, there are 5 equilibrium points where a satellite can remain indefinitely, stationary in the rotating frame. The locations of these equilibria, or motionless points, or Lagrange points, can be computed from eq. 3.8 by putting the velocity and acceleration components to zero. These equilibria can be divided into two groups: the triangular points, which are stable and marked as L_4 and L_5 in figure-11, and the collinear points, which are unstable and marked as L_1 , L_2 and L_3 in figure-11. Among of them, L_1 and L_2 are the most important ones for this research, because they are such a gate position that connect neck region to either inner realm or outer realm. Therefore, in the neck region there are 4 possible trajectories: ones that escape (or capture in reverse) the region either through L_1 or L_2 they may crash to the secondary body or they remain in a periodic motion. Also please see Matlab code, [lagpon.m](#), in appendix CD, to plot equilibrium points.

3.1.3. Transformation Between Frames

Because equation of motion is relative to rotation axis, trajectories can be transferred to inertia axis. As it is shown in detailly [15,75], transformation equations can be set as;

$$X^{in} = R(X^{ro} - d) \quad (3.11)$$

Where R is rotation matrix and d is a vector that shifts the origin to the one of the body, so trajectory can be plotted with centred either body;

$$R = \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ -\sin t & -\cos t & \cos t & -\sin t \\ \cos t & -\sin t & \sin t & \cos t \end{bmatrix}, \quad d = \begin{bmatrix} \underbrace{-\mu}_{M_1}, \underbrace{1-\mu}_{M_2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.12)$$

After that, trajectory is transferred to dimensional form or another CR3BP system. Such as it may be needed to transfer between 2 CR3BP, as in patch-CR3BP. Let's have 2 different CR3BP system, called "A" and "B". In here, L_A or L_B are the distance between the M_1 and M_2 ; for different CR3BP system. T_A and T_B are the periods of binary system (M_1 and M_2), which makes circular motion together about their mutual centre of mass. So transformation coefficient can be computed as below;

$$L_{AB} = \frac{L_A}{L_B} \quad T_{AB} = \frac{T_A}{T_B} \quad (3.13)$$

Also note that, if the transformation is done only to dimensional form (such as, let non-dimensional form is “A” and dimensional for is B), so L_B and T_B are taken 1 and 2π , respectively, in equation 3.13.

Then using these scale constants, transformation from inertia axis of ‘A’ to inertia axis of ‘B’ is done. In equation 3.14, position, velocity and time transformation shown in compact form.

$$\begin{aligned} X_B^{in,pos} &= L_{AB} X_A^{in,pos} \\ X_B^{in,vel} &= \frac{L_{AB}}{T_{AB}} X_A^{in,vel} \\ t_B &= T_{AB} t_A \end{aligned} \quad (3.14)$$

Then, using inverse of R, trajectories can be transferred to rotation frame of ‘B’. Also orientation of the trajectory, which determines the positions relative to between 2 binary system, can be changed according to the mission design:

$$X_B^{ro,O} = O X_B^{ro} \quad (3.15)$$

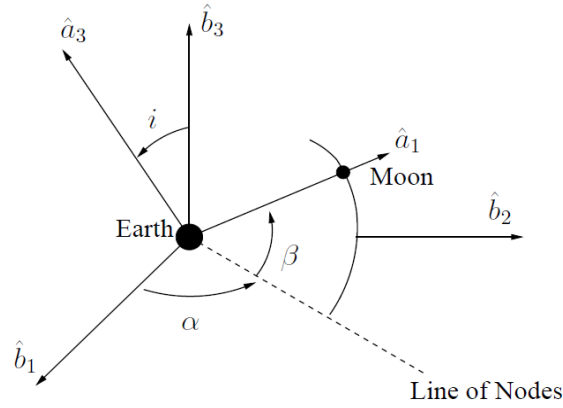


Figure 12 - Orientation between Sun Earth [b1, b2, b3] and Earth-Moon [a1, a2, a3], [75].

Where ‘O’ is the orientation matrix showed figure-12;

$$O = \underbrace{\begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C\alpha} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}}_{Ci} \underbrace{\begin{bmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C\beta} \quad (3.16)$$

In short, while trajectories transfer to one CR3BP system to another ($A \rightarrow B$), first it is transferred to **A** rotation axis to **A** inertia axis, eq. 3.11. Then it is transferred to **A** inertia axis to **B** inertia axis with eq. 3.14, and finally transformation is done from **B** inertia axis to **B** rotation axis, using reverse of eq. 3.11. If there is an orientation differences between **A** and **B**, then orientation transformation is done with setting orientation angles, figure-12.

Note that; You may also see Matlab code, [CR3BP_plotter.m](#), in appendix CD, to plot trajectories both in rotation axis (with zero velocity surface and equilibrium points) and inertia axis to examine CR3BP dynamics in both axis.

3.2 Motion in CR3BP

Because CR3BP is non-linear dynamics, there is no general, easy analytical solution. But, some particular solutions such as linearization or perturbation method can give restricted analytical solution, and some numerical methods, such as differential corrector and Poincare map, can also useful for finding advantageous trajectory and periodic orbits.

3.2.1 Linearization

CR3BP is the simplest model for multi-body dynamics, but it is still have chaotic nature. Equation motion of CR3BP (eq. 3.8) is nonlinear and there is no analytical solution. So that, for gain an insight about motion and simplify the equations, nonlinear system is analysed and linearsed around fix points, such as equilibrium points (figure-11). Because interplanetary capture transfer around secondary realm is studied in many others studies and many times, equation of motion is mostly linearsed in unstable equilibrium points around neck region (section 3.3).

As it is also showed in [15], linearization procedure is shown in below, which would give us equation motion of flow of equation;

$$\begin{aligned}\dot{\bar{x}} &= f(\bar{x}) \longrightarrow \delta\bar{x}(t) = \phi(\bar{x} + \delta\bar{x}) - \phi(\bar{x}) \\ \delta\bar{x}(t) &= \frac{\partial\phi(\bar{x})}{\partial\bar{x}} \delta\bar{x} + h.o.t\end{aligned}\tag{3.17}$$

From this Taylor expansion, the linearsed equation motion and its solution is below:

$$\begin{aligned}\delta\dot{\bar{x}} &= \underbrace{\frac{\partial f(\bar{x}_r)}{\partial\bar{x}}}_A \delta\bar{x} + h.o.t, \quad \bar{x}_r = L_1, L_2, L_3, L_4, L_5 \\ \delta\bar{x}(t) &= e^{At} \delta\bar{x}_0\end{aligned}\tag{3.18}$$

In here, $\bar{x}_r$ can be any coordinate of equilibrium points (see section 3.1.2), and equation of motion (3.18) gives results in the centre of this equilibrium point. Then, matrix **A** can be calculated according to these equilibrium points and stability and dynamics features of equilibrium can be understand better [3,11,15].

Motion around the triangular point: If matrix $\mathbf{A}$ is computed according to triangular equilibrium points L_4 or L_5 , linearized equation gives the local linear motion in reference of these points. Also eigen values of matrix $\mathbf{A}$ gives the stability features of equilibrium point. All Eigen values of matrix $\mathbf{A}$, which is computed using L_4 or L_5 points, are purely imaginary. This means that, motion in the vicinity of L_4 and L_5 points, is stable and most likely to be periodic or quasi-periodic motion, according to a critical mass parameter. Such as Trojan asteroids, which are at the Sun–Jupiter L_4 and L_5 , are naturally stay on these point, stationary. These points can be used for very low-cost station keeping mission. For further information about these points, please see [2,11,13].

Motion around the collinear point: If matrix $\mathbf{A}$ is computed according to co-linear equilibrium points L_1 , L_2 and L_3 , linearized equation gives the local linear motion in reference of these points. Eigen values of the matrix $\mathbf{A}$ have both real ($\lambda_{1,2}=\pm a$) and imaginary values ($\lambda_{3,4}=\pm ib=\pm\omega_{xy}$ and $\lambda_{5,6}=\pm\omega_z$). ω_{xy} is the linearized frequency in x-y plane and ω_z is for z-direction. These frequencies are only valid for vicinity around equilibrium points. If the trajectory is gone far away from equilibrium point, linearised equation motion and frequency would not be valid any more. So that, linearsed equation motion is generally used for periodic solution (which trajectory must stay always around there), but it can be also used for non-periodic solution for gaining insight and understanding characteristic of transfer trajectories.

If the solution of eq. (3.18) is evaluated, then we can get these general form in eq. (3.19), for more details, please see [3,15,19];

$$\begin{aligned}\delta x(t) &= \alpha_1 e^{at} + \alpha_2 e^{-at} + \alpha_3 \cos(bt) + \alpha_4 \sin(bt) \\ \delta y(t) &= k\alpha_1 e^{at} + k\alpha_2 e^{-at} + k\alpha_3 \cos(bt) - k\alpha_4 \sin(bt) \\ \delta z(t) &= \beta_1 \cos(\omega_z t) + \beta_2 \sin(\omega_z t)\end{aligned}\tag{3.19}$$

As you can see exponential part are only in planar component (δ_x , δ_y), and other components with related periodic terms. The *real* Eigen values cause the unstable motion, but *imaginary* Eigen values provide periodic motion. It can be said that colinear equilibrium points are saddles. Therefore, using different combination of the exponential coefficients in eq. (3.19), motion flow around colinear equilibrium points can be classified 4 type, as it is seen in figure-13;

- If $\alpha_1 = \alpha_2 = 0$, periodic orbits is obtained as black line in figure-13,
- If $\alpha_1 \alpha_2 = 0$, asymptotic orbits is obtained as green line in figure-13,
- If $\alpha_1 \alpha_2 < 0$, transits orbits is obtained as red line in figure-13,
- If $\alpha_1 \alpha_2 > 0$, non-transits orbits is obtained as blue line in figure-13,

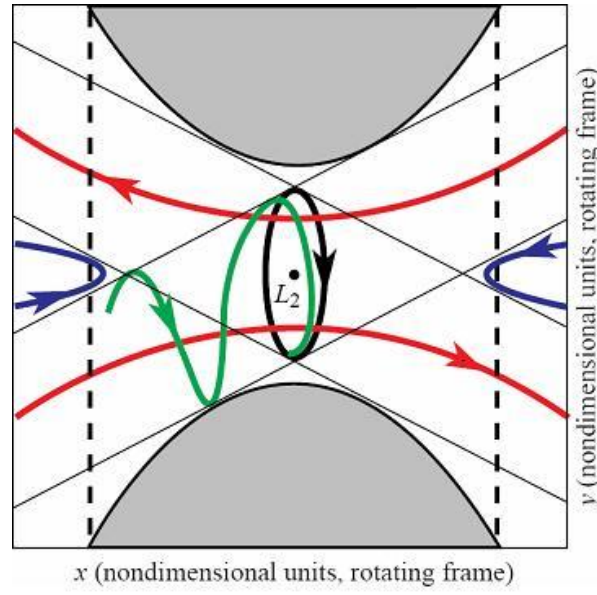


Figure 13 - Flow Around colinear equilibrium point, [15]. Black line is halo orbit, green line is asymptotic orbit, red line is transit orbits, blue line is non transit orbits.

Periodic orbits (halo (lyapunov) orbits and quasi-periodic orbits) around colinear equilibrium points can be used station keeping mission as triangular points. But, because of unstable effect, these periodic orbits are too sensitive and mission cost is much more expensive than triangular equilibrium points, and extra control operation is need. For proper control of mission around these points, periodic orbits must be kept accurately. But Linearized equation can not provide accurate reference trajectories in high amplitude, because of being far from equilibrium point.

Also note that, there are 2 main method for computed periodic orbit accurately around equilibrium points, one is Differential Correction (DC) as numeric computation [such as see 15] and Linstead-Poincare (LP) perturbation method as analytic approach, [3,11,15]. Using both these method, two reference orbits derived for station keeping mission around co-linear equilibrium point and control process can be more accurately.

Furthermore, asymptotic and transit trajectories of L_1 and L_2 Liapounov Periodic Orbits (LPO) are so important for interplanetary capture transfer (which are shown in section 4.2.2 and 4.2.3). L_1 and L_2 are such a gate points that trajectories can pass to one planet region to another, see in figure-11. And also, non-transit trajectories, which they always stay in one realm, can be useful for some parking orbits or station keeping.

As it is seen, equilibrium points and LPO in CR3BP present a lot of advantageous and mission opportunities, such as capture transfer and station keeping missions. But Dynamics of these equilibrium points must be known well to make good use of them.

3.2.2 Periodic Orbits

Although motions in multi-body systems are most likely to be chaotic, unpredictable and unclosed, there are also a lot of group and various types of periodic orbits at various specific states in those systems. For instance, using differential corrector method, over 600,000 periodic orbits could be found around Europa while in gravitation effect of Jupiter (Jupiter-Europa binary system of CR3BP), [68]. But of course, periodic orbits appear as continuous families in Hamiltonian systems and there must be much more periodic orbits. These periodic orbits can be classified with many ways; such as around primary, secondary body and equilibrium points, also they can be stable and unstable.

Furthermore, because the CR3BP is studied in rotation frame, so that trajectories can also be classified against to rotation or not. In CR3BP, there are two types axis; inertia and rotated axis, figure-10. While inertia axis is fixed, rotated axis is rotated with certain angular velocity (ω), relative to inertia axis. If the motion of object around primaries (both bigger and smaller mass) is in the same direction with rotated axis, it is called '*Direct* motion or orbit'. If the motion is inversed relative to rotated axis, it is called '*Retrograde* motion or orbit'. Retrograde motion around primaries is much more stable than direct motion. Because centrifugal effect is reduced in retrograde motion, object (spacecraft or comet) can be staying more stable orbit. Retrograde orbits are generally "stable periodic motion" around primary (for both bigger and second mass) mass even in high energy level and they can be determined easily with Poincare section method. But the direct orbits are not as stable as retrograde orbits because that they are quite chaotic; they tend to make motion around bigger mass and tend to escape from the gravitational effect of smaller mass. Although they make too chaotic and unstable motion, they are quite suitable for transfer (capture/escape) trajectories. For instance "direct stable periodic orbits" can be cheap parking orbit around secondary mass for transfer trajectories. But the problem is that *direct stable periodic orbits around secondary mass* have been thought too hard to find in high energy level. This idea is partly true in Planar-CR3BP actually. But as it is seen in chapter 4, these "direct stable periodic orbits" are generally inhabited in high inclination.

So the problem is to find and determine these periodic orbits for advantageous mission design. Because of non-linear dynamical system, determination methods of these periodic orbits can be different to region to region and dynamical behaviour to dynamical behaviour. For example, it can be used Poincare mapping method around primaries (both bigger and smaller bodies), while DC and LP methods are used around equilibrium points. Which numeric or perturbation methods should be used, are depend on behaviour of dynamical systems. Even in the same problem (as in CR3BP), It should be used different methods in different regions.

3.2.2.1 Mapping Methods:

Mapping methods are actually quite popular and also practical method for analysing dynamical character of the non-linear systems, especially in chaotic systems. Because motion estimations are too complicated and generally impossible using analytical approximation for chaotic systems, the numerical approximations are best options for getting insight of these systems. There are lots of kinds mapping method, but in celestial dynamics, the Poincare map (surface of section) has been generally the most convenient option for analysing motion around gravitational bodies. Also, In CR3BP, Poincare map method gives quite much insight about resonance motion of bigger bodies.

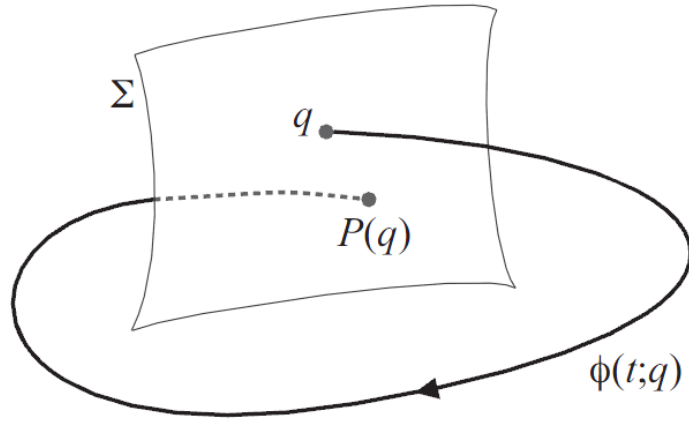


Figure 14 - Poincare section [15].

In the Poincare map method, after huge amount of trajectories integrated in quite enough time, point data is picked from chosen surface, as in figure-14, $\phi(t;q)$ is the trajectory, [5,15];

$$P(q) \equiv \phi(T_{\Sigma}(q) : q)$$

The point's data in the chosen surface present the Poincare map. And the formations of these points give us about dynamical insight of the non-linear system. Such as, if the points form some circles which lie one inside the other, so that motion is the quasi-periodic motion, and middle of the circle gives exact periodic orbits or resonance orbits. If the points spread around so chaotically, this means that motion in that area would be chaotic as also. So that, In the Poincare map, the picture is generally as circles inside to another (which may be called islands) and chaotically spread points (which may be called either sea or ocean). This picture with islands and oceans, presents an initial condition map. For example, if the initial condition is chosen from the island, so that we can obtain a periodic or quasi-periodic orbits, as in figure-15.

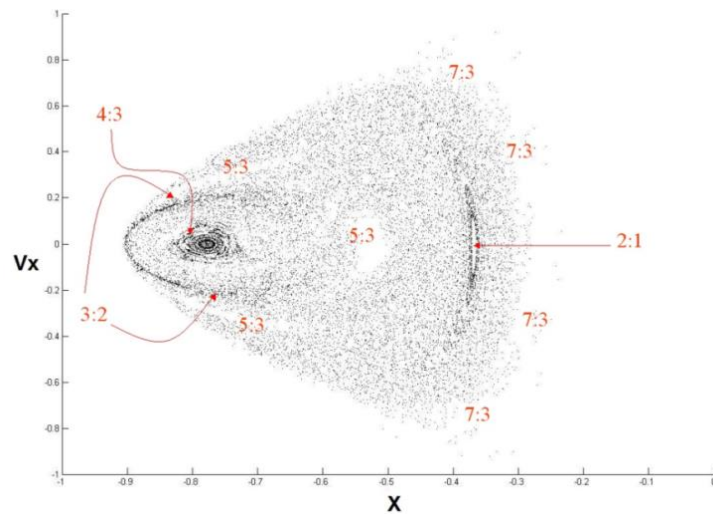


Figure 15 - Poincare map. $(x, dx/dt)$, [74]

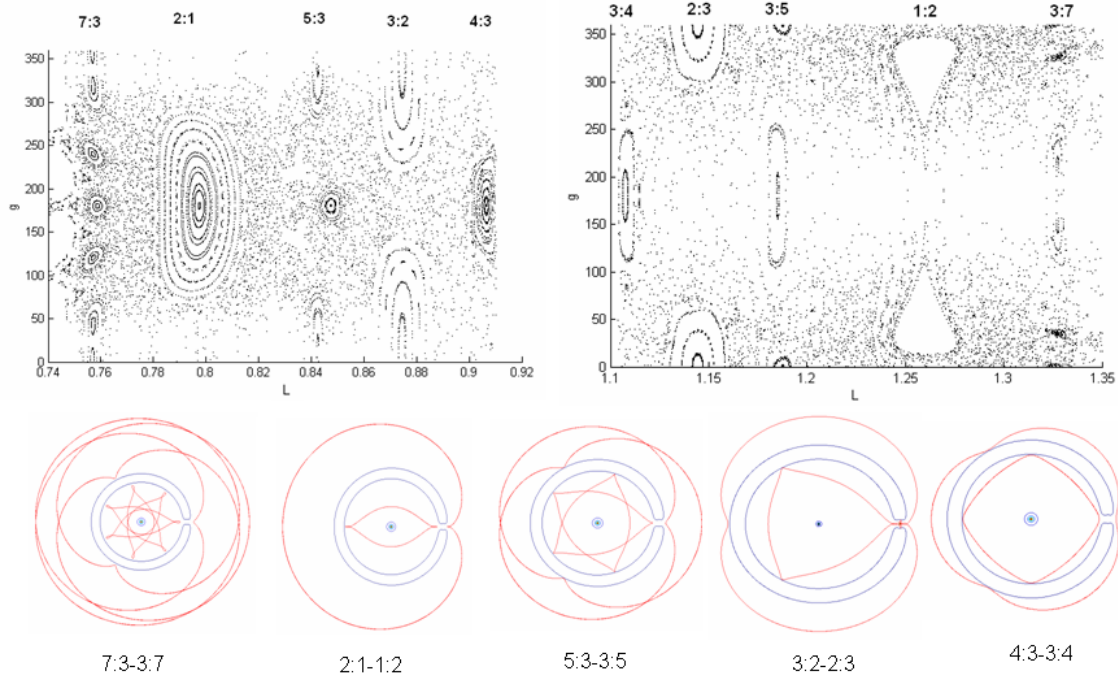


Figure 16 - Poincaré map in reflection of Delaunay variable and resonance motion in Sun-Jupiter system. [25,74].

In the figure-15, the chosen surface is (x, dx) while $y=0$. So in 4 dimensional phase space (x, y, v_x, v_y) , we already know 3 value of them, and The forth one (dy) can be found from using conservation of energy (Jacobi Integral: $V^2=2U-C$). As it is seen, the Poincaré map method is more convenient in 2-D (planar) space with 4 dimensional phase space (x, y, v_x, v_y) . The red numbers on map shows some resonance value (rate) of periodic orbits, which's appeared on Poincaré Map in figure-16, which the surface of section is transfer to Delaunay variables [25], to see more clear the island and oceans of motion flow. And every resonance value of Sun-Jupiter system is shown in rotational axis, for more clear insight.

Also note that; in 3-D space, dimension of phase space increase to 6 dimension (x, y, z, v_x, v_y, v_z) and so the surface of section must be 4-dimension. 4-D surface of section is not so comfortable for analysing the motion, and also the variable number which increase the complexity of solution, is too much. Therefore, it is needed to develop a new mapping method for 3-D space, see Chapter 4.

3.2.2.2 Differential Corrector Method:

This method also called both 'shooting method' [76] and Newton Method [11,77]. In a chaotic system, small differences in initial conditions cause big differences to final state, figure-17. If the relationship between final and initial differences is computed, desired trajectory can be found using this relationship. For this, displacement of actual and desired trajectories is computed [11,13,15,78];

$$\underbrace{\dot{\bar{x}} = f(\bar{x})}_{\text{natural dynamics}} \quad (3.20)$$

$$\delta\bar{x}(t + \delta t) = \phi(t + \delta t, \bar{x}_0 + \delta\bar{x}_0) - \phi(t, \bar{x}_0) \quad (3.21)$$

if the equation is expanded to Taylor series,

$$\delta\bar{x}(t + \delta t) = \underbrace{\frac{\partial \phi(t; \bar{x}_0)}{\partial \bar{x}}}_{STM} \delta\bar{x}_0 + \underbrace{\frac{\partial \phi(t; \bar{x}_f)}{\partial t}}_{\text{vector field}} \delta t = \Phi \delta\bar{x}_0 + \dot{\bar{x}} \delta t \quad (3.22)$$

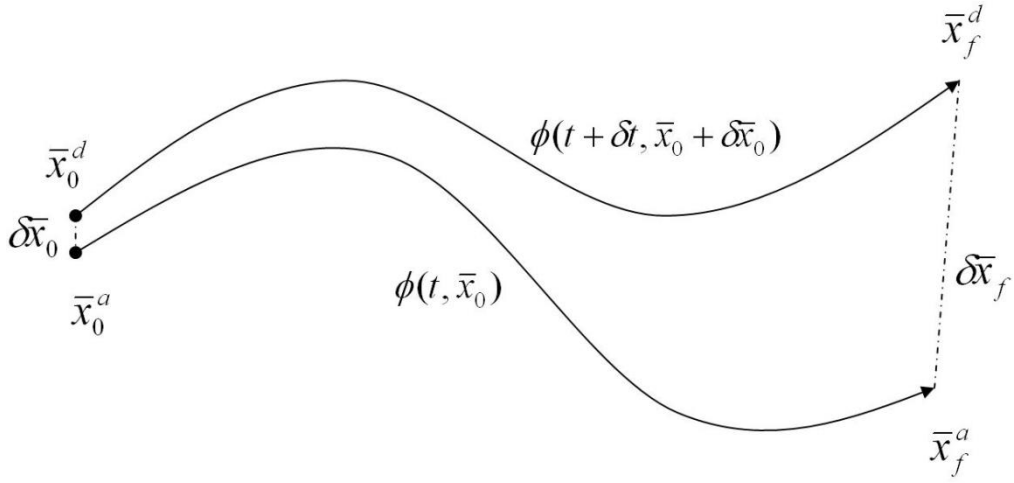


Figure 17 - Trajectories

In eq. 3.22, $\bar{x}(t) \rightarrow \phi(t; \bar{x}_0)$ is the flow map of dynamical system, Φ is state transition matrix and $\dot{\bar{x}}$ is vector field. This equation is numerically derived with equation motion of system. At the end, the final relationship is formed:

$$\delta\bar{x}_f = \Phi \delta\bar{x}_0 + \dot{\bar{x}} \delta t \quad (3.23)$$

For computation of *State Transition Matrix* (STM), [11,13,15,78], the equation of motions and flow map of dynamical system is obtained, see eq.3.24.

$$\underbrace{\dot{\bar{x}} = f(\bar{x})}_{\text{natural dynamics}} \rightarrow \underbrace{\bar{x}(t) = \phi(t; \bar{x}_0)}_{\text{flow}} \quad (3.24)$$

From this, the variation equation is written as eq 3.25;

$$\delta\dot{\bar{x}}(t) = \underbrace{Df(\bar{x}(t))}_{A(t)} \delta\bar{x}(t) \quad (3.25)$$

The solution of variation equation is as equation 3.26;

$$\delta\bar{x}(t) = \Phi(t, t_0) \delta\bar{x}_0 \quad (3.26)$$

In here, Φ is called state transition matrix and gives the linear relationship between initial and final displacements. With combining equation 3.24 and equation 3.26, the equation of motion for state transition matrix can be obtained as in eq. 3.27;

$$\begin{aligned} \frac{d\phi(t, \bar{x}_0)}{dt} &= f(\phi(t, \bar{x}_0)) \\ \frac{d}{dt} \frac{\partial \phi(t, \bar{x}_0)}{\partial \bar{x}_0} &= Df(\phi) \frac{\partial \phi(t, \bar{x}_0)}{\partial \bar{x}_0} \end{aligned} \quad (3.27)$$

So then, the state transition matrix can be compute numerically along the trajectory, as this;

$$\begin{aligned} \dot{\Phi}(t, t_0) &= A(t)\Phi(t, t_0) \\ \dot{\bar{x}} &= f(\bar{x}) \end{aligned} \quad (3.28)$$

From the last equation, solving 42 ODE, final value of STM can be calculated, as initial condition is $\Phi_0 = I_n$ and $\bar{x}_0$. After that, from equation 3.23, and making some restriction in initial conditions [11,13-15,78], the correction equations can be found, as this: restrictions of initial conditions for desired initial and final condition are that for passing x-z plane;

$$\begin{aligned} \bar{x}_0^d &= (x_0, 0, z_0, 0, \dot{y}_0, 0) \\ \bar{x}_f^d &= (x_f, 0, z_f, 0, \dot{y}_f, 0) \end{aligned} \quad (3.29)$$

Then from restriction in initial conditions, the equation (3.23) can be written as the correction equations:

$$\begin{bmatrix} \delta x_f \\ \delta y_f \\ \delta z_f \\ \delta \ddot{x}_f \\ \delta \dot{y}_f \\ \delta \ddot{z}_f \end{bmatrix} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} & \Phi_{16} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} & \Phi_{24} & \Phi_{25} & \Phi_{26} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & \Phi_{34} & \Phi_{35} & \Phi_{36} \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & \Phi_{45} & \Phi_{46} \\ \Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} & \Phi_{56} \\ \Phi_{61} & \Phi_{62} & \Phi_{63} & \Phi_{64} & \Phi_{65} & \Phi_{66} \end{bmatrix} \cdot \begin{bmatrix} \delta \bar{x}_0 \\ \delta \dot{y}_0 \\ \delta \ddot{z}_0 \\ \delta \dot{y}_0 \\ \delta \ddot{z}_0 \end{bmatrix} + \begin{bmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \delta T \quad (3.30)$$

Keep constant

Eq. 3.23 gives desired initial condition with iteration of suitable initial guess. But it has 6 equation and 7 unknowns, so that one of the variable should be fixed. So keeping δx_0 constant, the columns and rows are eliminated as in eq. (3.30) and (3.31). Than final correction equations are:

$$\underbrace{\begin{bmatrix} y_f \\ \dot{x}_f \\ \dot{z}_f \end{bmatrix} = \begin{bmatrix} \Phi_{23} & \Phi_{25} & \dot{y} \\ \Phi_{43} & \Phi_{45} & \ddot{x} \\ \Phi_{63} & \Phi_{65} & \ddot{z} \end{bmatrix} \cdot \begin{bmatrix} \delta z_0 \\ \delta y_0 \\ \delta t \end{bmatrix}}_{\text{Correction Equations}} = 0 \quad (3.31)$$

Because there is no analytical solution of CR3BP, all computation must be done numerically. Differential corrector (DC) method is work properly for only suitable initial guess, because chaotic systems are too sensitive to initial conditions. In figure-18, examples of initial guesses are shown as black trajectory in position space, for getting desired red periodic Halo orbit.

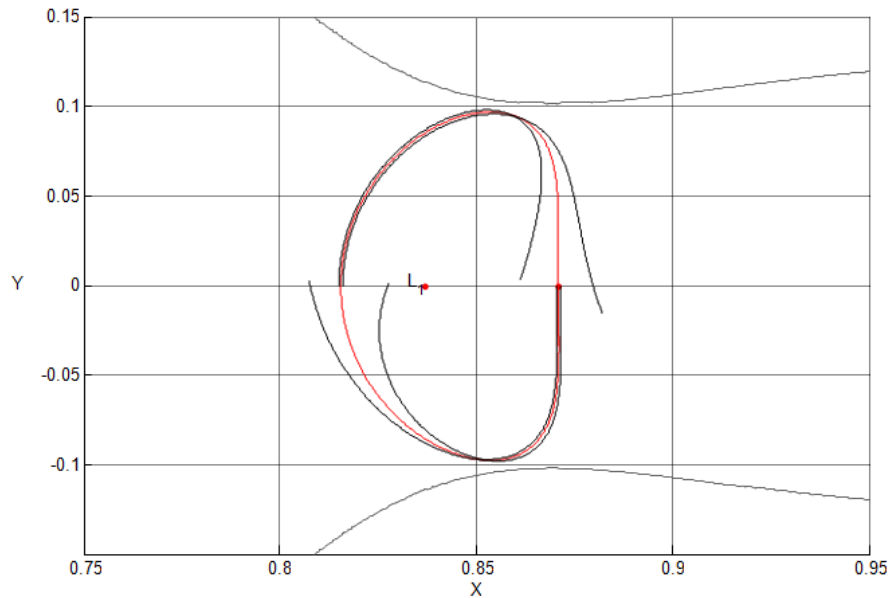


Figure 18 - Ideal initial guess for DC as black lines, Halo orbits as red line.

After the suitable initial condition is chosen, the whole scheme of the iteration of the method is done as in figure-19.

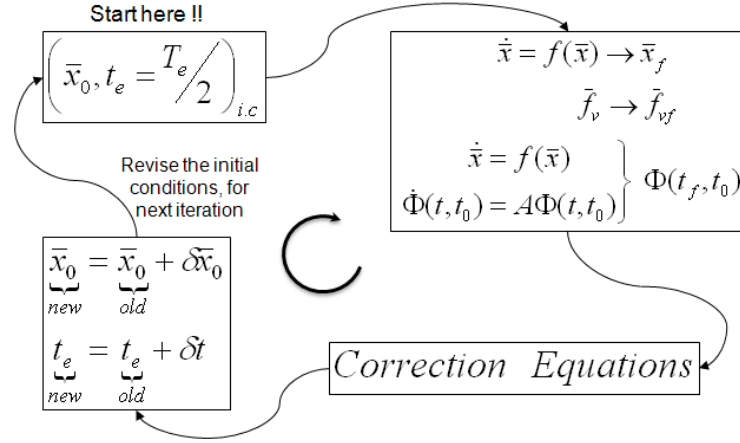


Figure 19 - 3-dimension Halo orbits in different energy are computed by differential corrector (DC) method can be seen. Correction equations are equation 3.31.

In here, T_e is the period of target periodic orbits. Making about 5 iterations, the initial conditions for halo orbits can be found right away. And making so tiny changes in initial guess, the family of halo orbits can be calculated, as in figure-20. But initial guess of this method too sensitive, so should be chosen carefully.

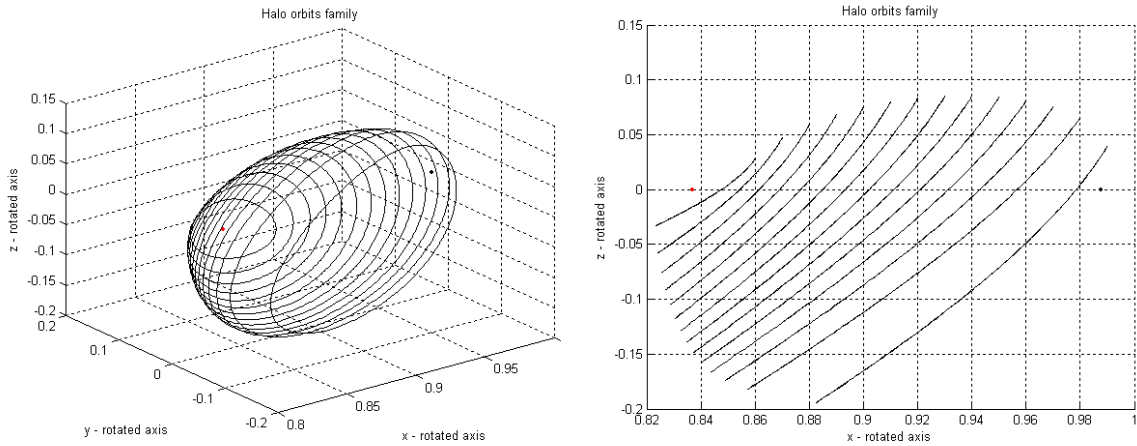


Figure 20 - Halo orbits in different energies, also called Halo orbits Family.

3.2.2.3 Linstead-Poincare perturbation method:

The most critical part in *Differential Correction* (DC) method is to find suitable initial guess, as we say previous section. Because DC method is too sensitive about initial guess, Good initial guess for DC can be found using some Mapping method (such as Poincare) or perturbation method for obtaining analytical approach. For this, *Linstead-Poincare* Perturbation Method (LP) can be used. Also this method gives opportunity to see and compute the motion of periodic and quasi-periodic orbits, analytically.

The main idea is non-linearity change the frequency of linearized system. But adjustment of frequencies provides to avoid secular terms and approximate periodic solutions can be obtained, please see more in [13,15,79-86]. Firstly, nonlinear terms of equation motion expand the Legendre polynomial and added to linear equation of motion. First order solution is taken as solution of linearized equation's solution and this solution added second order equation motion of expansion. So that, n^{th} order solution is used for finding $(n+1)^{\text{th}}$ order solution. While solving the high order expansion, secular terms are eliminated using the frequency expansion, as below;

$$\begin{aligned} \tau &= \omega t & \tau_1 &= \omega_{xy} \tau + \phi_1 \\ \omega &= 1 + \sum_{n \geq 1} \omega_n & \tau_2 &= \omega_z \tau + \phi_2 \end{aligned} \quad (3.32)$$

At the end, all solutions of orders are summed and analytical approach can be obtained;

$$\bar{x}(\tau) = \sum_{n \geq 1}^{\infty} \bar{x}_n \quad (3.33)$$

For analytical approach of halo orbit, frequencies are equalized, see below;

$$\begin{aligned} x(t) &= x_s + \sum_{n=1}^{\infty} a_n \cos(n \tau_1) \\ y(t) &= \sum_{n=1}^{\infty} b_n \sin(n \tau_1) \\ z(t) &= z_s + \sum_{n=1}^{\infty} c_n \cos(n \tau_1) \end{aligned} \quad (3.34)$$

For analytical approach of quasi-periodic solution, such as Lissajous orbit and Quasi-periodic Halo orbit in figure-21, formulation can be generalized as equation 3.35, where m , n , a , b and c (for both eq. 3.34 and 3.35) are coefficients, which can be found after too much daunting computations ;

$$\begin{aligned} x(t) &= \sum_{m,n=1}^{\infty} a_{mn} \cos(n \tau_1 + m \tau_2) \\ y(t) &= \sum_{m,n=1}^{\infty} b_{mn} \sin(n \tau_1 + m \tau_2) \\ z(t) &= \sum_{m,n=1}^{\infty} c_{mn} \cos(n \tau_1 + m \tau_2) \end{aligned} \quad (3.35)$$

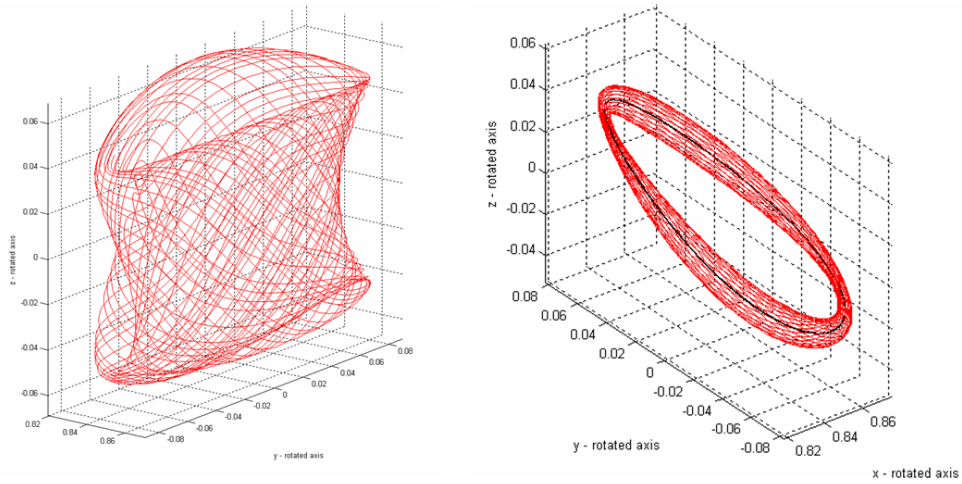


Figure 21 - Lissajous orbit (Left) and Quasi-periodic Halo orbit (Right)

To approach nonlinear solution as accurate as possible, we should expand higher order, of course. This method is rather too daunting and hard about expansions of orders, but using third or fifth expansion solution we can get initial guess for DC method also we can use this solutions as a reference trajectory together with numeric solution. Also this method is used for obtaining realistic periodic solution around equilibrium point, because frequency adjustment is done for fix points. LP method provides a frequency constant for high altitude of periodic solution around equilibrium points with realistic position curves. For more information and insight about LP method is gotten from [13-15,19,79-86].

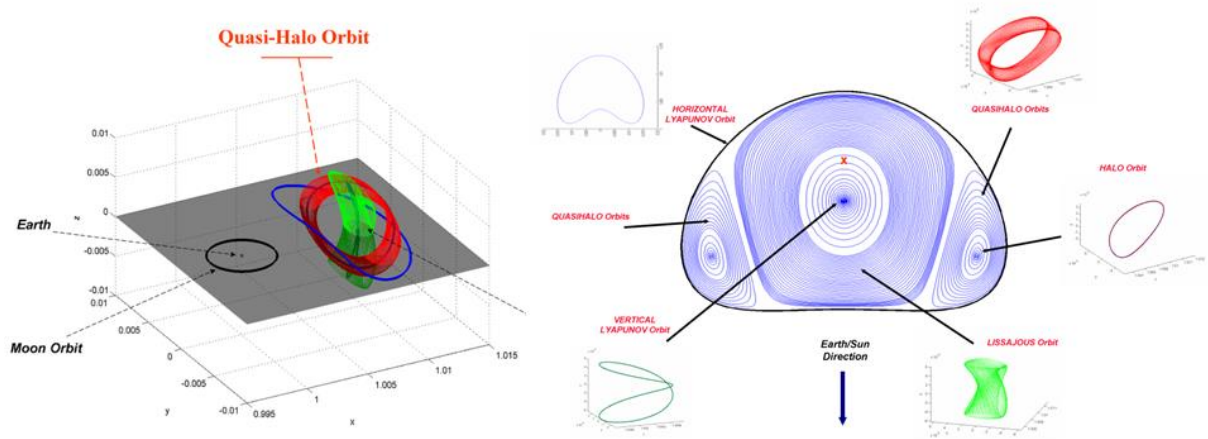


Figure 22 - All periodic and quasi periodic orbits around L2 using multiply Poincare section methods, [13,20].

Lastly, there are other methods for finding unstable periodic orbits in long period computation, such as; '*centre manifold reduction*' or '*multiply Poincare section*' method, [15,20]. The figure-22, the unstable periodic orbits around equilibrium points are plotted with multiply Poincare section method. Similar section also can be gotten with centre manifold reduction. For more information and insight about [11-15,20,79-86].

3.2.3 Invariant Manifold Tubes

In the linearization process (chapter 3.2.1), it is seen that there are 4 type orbit types around unstable equilibrium points; Periodic, Asymptotic, transit and non-transit orbits. While the periodic orbits are used for some kind of station keeping mission, the asymptotic and transit of orbits can be used as transfer trajectories from one body to another. Especially, all asymptotic orbits, which are asymptotic to the Halo orbit, form a tube which is called *invariant manifold tube* (IMT), can present lots of advantageous of mission design. IMT is also a boundary that separates the transit and non-transit tubes. Transit orbits are always in IMT and can pass to one region to another. But the non-transit orbits are outside of the IMT and they are always prison in one region. That's why IMT or Asymptotic orbits are called separatrices, that they separate transit and non-transit trajectories, [87-90]. This phenomenon will be very essential in following Chapter, where motion map method is explained (See Chapter 4). Because these separatrices can be used as permanent capture trajectories, which provide zero-cost capture.

But there is another tricky thing about IMT in CR3BP. IMT are depending on periodic orbits around equilibrium points, and they can be computed thanks to them. But there are various Periodic and quasi periodic around there, so which Asymptotic orbits relate which periodic orbits ? This is very easy, while in Planar-CR3BP; there is only one unique periodic orbit (Horizontal Lypunov Orbit) around unstable equilibrium point in each specific energy. So that Asymptotic orbits and so IMT must be related with this periodic orbit. But in 3-dimensional space, there are more than one periodic orbits (Halo, Quasi Halo, Lissajous Orbit and Vertical Lypunov Orbit) in each specific energy, and all these periodic and quasi-periodic orbits have their own asymptotic orbits and so IMT. But in pratically, 3D-IMT is computed with using either 3D Halo or Vertical Lypunov orbits, for easiness. In addition, a transit orbits seems out side of the manifold tubes while in 3-D space, this means that this transit orbits actually in another IMT of another periodic orbits. This phenomenon is also better explained in [89-91].

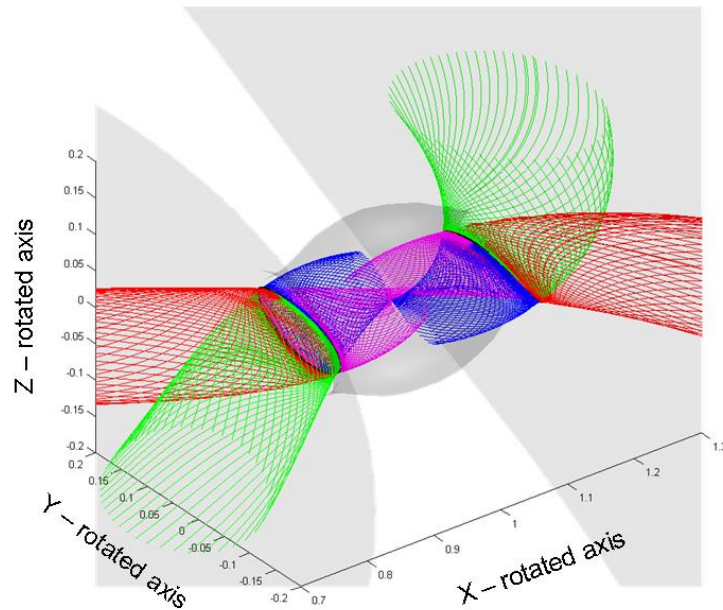


Figure 23 - 3D Manifold tubes of L1 and L2. Green trajectories are stable manifold of outside, red trajectories are unstable manifold of outside, blue trajectories are stable manifold of inside, and purple trajectories are unstable manifold of inside.

As it is seen in figure-13, asymptotic orbits are asymptotic to periodic orbits (such as halo orbits in here). If all asymptotic orbits of periodic orbits are plotted, a manifold tube is appeared, see figure-23.

In figure-23, hot colour orbits (red and pink) are unstable manifold of equilibrium points which are departure from equilibrium point, and cool colour orbits (green and blue) are stable orbits of equilibrium points which are arrival to equilibrium point.

These orbits are important for equilibrium point mission (exp; Plank, Soho, Genesis, [11,15]) and capture transfer missions (exp; Hiten [21,22]). They provide to way to get halo to planet or moon, also capture trajectories are always inside the manifold tubes, and their intersection on Poincare surface of section provides optimum jumping point from one manifold to another. These orbits are used for 'Low Energy Transfer'; they can be reduced mission cost, and have great potential for any flexible and rescue missions. Same as periodic orbits, there are two main way to compute these orbits, numeric method and LP method for analytic approach.

3.2.3.1 Numeric Computation of Asymptotic orbits

Because, asymptotic orbits are asymptotic to periodic orbits, trajectory data of periodic orbits must be known for computation of these orbits. Simply, disturbing the all trajectory data of periodic orbits, initial condition of asymptotic orbits can be gotten easily, see below;

$$\bar{x}_0^a = \bar{x}_0^h \mp \epsilon y \quad (3.36)$$

$\bar{x}_0^h$ is the initial condition of periodic orbit and y is normalized stable or unstable Eigen vectors of Monodromy matrix, which is STM of periodic orbit after one period, $M=\Phi(T)$, such as see, [13-15].

For more simply and practical, the disturbance can be made only on X components. Minus (-) provides the unstable IMT, which move away from equilibrium point (EP) to Minus direction of EP, whereas plus (+) provides the another unstable manifold, which move away from equilibrium point (EP) to plus direction of EP. With sum $\bar{x}_0^h$ and $\pm\epsilon$, the initial condition of a manifold, $\bar{x}_0^a$, can be obtained simply. Moreover, if these initial conditions are integrated in forward time, unstable manifold (Departing from Halo Orbit) is gotten, while if they are integrated in backward time, stable manifold (Approaching to Halo Orbit) is gotten.

After computation, manifold tubes can be intersected, suitable initial condition can be picked using Poincare surface of section. With this method, the best point (patch point) for jumping to another manifold can be determined, [15,25]. For more detail and visual data about Poincare surface of section of IMT, please see section 3.2.3.3.

You may also see Matlab code, [IMTs_cr3bp.m](#), in appendix CD, to plot L_1 and L_2 Halo orbits' IMTs for Earth-Moon binary system. Also there is a video in appendix CD for a example motion of IMT; [L1 unstable manifold tube.wmv](#).

3.2.3.2 Analytical Approach of Asymptotic orbits

LP method of equation of CR3BP can be also arranged for obtaining an analytical approach of asymptotic solution. Analytical approach of asymptotic orbits can be useful for computation of capture orbits analytically and it can give an insight about chaotic capture trajectories and regions.

As it is showed in section 3.2.2.3., the starter solution is taken as periodic solution of linearsed equation motion of CR3BP. For asymptotic orbits, starter solution can be taken asymptotic solution of linearsed equation, too;

$$\begin{aligned}x_1 &= -A \cos(\tau_1) \pm \varepsilon_1 e^{\sigma t} \\y_1 &= B \sin(\tau_1) \mp \varepsilon_2 e^{\sigma t} \\z_1 &= C \cos(\tau_1)\end{aligned}\tag{3.37}$$

With this solution and applying LP method, analytical approach of asymptotic orbits is derived;

$$\begin{aligned}x(t) &= x_h + \sum_{m,n=1} e^{m\sigma t} (a_{1mn} \cos(n\tau_1) + a_{2mn} \sin(n\tau_1)) + \sum_{n=1} a_{3n} e^{n\sigma t} \\y(t) &= y_h + \sum_{m,n=1} e^{m\sigma t} (b_{1mn} \cos(n\tau_1) + b_{2mn} \sin(n\tau_1)) + \sum_{n=1} b_{3n} e^{n\sigma t} \\z(t) &= z_h + \underbrace{\sum_{m,n=1} e^{m\sigma t} (c_{1mn} \cos(n\tau_1) + c_{2mn} \sin(n\tau_1))}_{\text{Shape of tube}} + \underbrace{\sum_{n=1} c_{3n} e^{n\sigma t}}_{\text{Path of tube}}\end{aligned}\tag{3.38}$$

x_h, y_h, z_h are analytical approach of halo orbit, which is given equation 3.34 and a, b, c and m, n are the parameters, which can be found after too daunted calculation procedure, depend on the mass ratio (μ), dimension of Halo orbits and position of equilibrium points. Much more detailed computation methodologies of eq. (3.38) can be seen in G Gómez, W S Koon, M W Lo, J E Marsden, J Masdemont and S D Ross's study, in [19]. In their work, they show two methodologies for computing the stable and unstable manifolds of libration point orbits in series expansions; One is based on the Lindstedt–Poincaré method, and the other is based on a normal form of the Hamiltonian equations of motion.

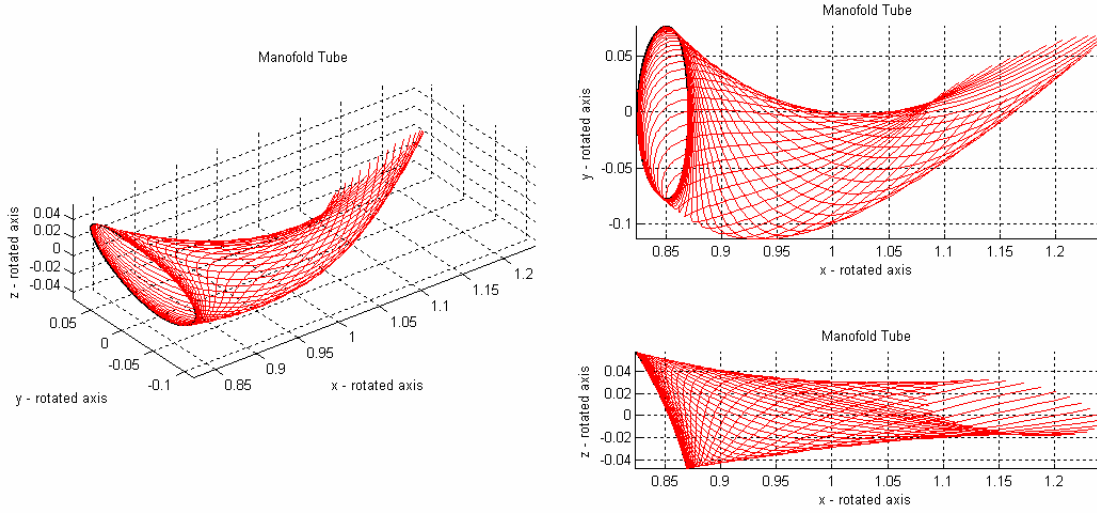


Figure 24 - Sample plot for analytical approach of asymptotic orbits.

In figure-24, sample plot of eq. 3.38 can be seen with random coefficients, which can be given their sample values in below, for demonstration;

$$\begin{aligned}
 IMT = [Halo\ Orbit + Exponential\ Part] = & \underbrace{\begin{bmatrix} x_0 + x_1 \cos(\omega t + \phi) + x_2 \cos 2(\omega t + \phi) \\ y_0 + y_1 \sin(\omega t + \phi) + y_2 \sin 2(\omega t + \phi) \\ z_0 + y_1 \cos(\omega t + \phi) + y_2 \cos 2(\omega t + \phi) \end{bmatrix}}_{Halo\ Orbit} + \dots \\
 & \dots + \underbrace{\begin{bmatrix} x_{s1} e^{ct} \cos(\omega t + \phi) + x_{s2} e^{ct} \sin(\omega t + \phi) \\ y_{s1} e^{ct} \cos(\omega t + \phi) + y_{s2} e^{ct} \sin(\omega t + \phi) \\ z_{s1} e^{ct} \cos(\omega t + \phi) + z_{s2} e^{ct} \sin(\omega t + \phi) \end{bmatrix}}_{Manifold\ Part} + \begin{bmatrix} x_{p1} e^{ct} + x_{p2} e^{2ct} \\ y_{p1} e^{ct} + y_{p2} e^{2ct} \\ 0 \end{bmatrix} \quad (3.39)
 \end{aligned}$$

Where $\omega = 2.086453455$, $c = 2.0152105515$, ϕ is phase difference, and other coefficients;

Xo=L1+0.01215	X1=-0.0228	X2=-0.00215	Xs1= -(2e-6)	Xs2= (1e-6)	Xp1= -1e-5	Xp2= -1e-10
Yo=0	Y1= 0.0773	Y2= 0.001	Ys1= -(1e-6)	Ys2= -(2e-6)	Yp1= -9e-6	Yp2= 4e-10
Zo= 0.0058	Z1= 0.052	Z2=-0.0011	Zs1= -(2e-6)	Zs2= (1e-6)	-	-

Same as periodic orbits approach, this approach could be valid higher order solution with correct coefficients. However computation of frequency is too tricky that frequency would change along the trajectory, especially while moving away from equilibrium point and approaching to the any primary bodies. Besides exponential terms of starter equations increase the computation load, so high order expansion become much more daunting. But if Lindstedt–Poincare’ method can be complemented with other methodology, shown in [19], Invariant manifold Tubes (and so Low energy transfer) are easily implemented with a high degree of accuracy.

3.2.3.3 Mission Design with Invariant Manifolds

In [15 and 89], Mission designs with invariant manifolds are already explained very clearly. The mission design can be done quite suitable using surface of section of invariant manifolds, as in figure-25.

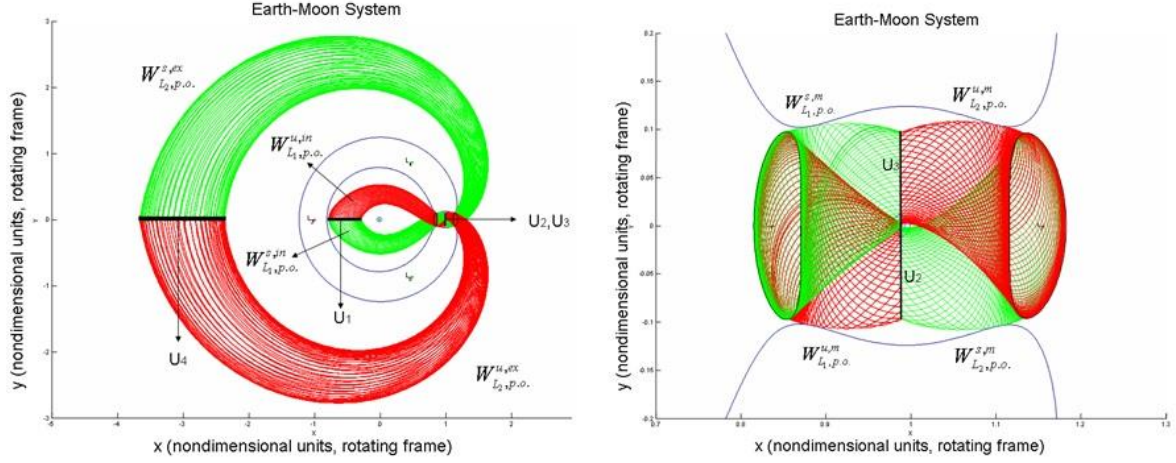


Figure 25 - Flow of stable (green) and unstable (red) invariant manifold tubes around bodies.

In the figure-25, it is show that 4 convenient surface suggestion (U1, U2, U3, U4) which can be used to jump to one invariant manifold to another, [15]. As in explained “mapping method”, Poincare surface of section method can be applied in these, and the convenient initial condition (also called “key point”) for jumping point one tube to another can be found. All we need, to intersect both stable and unstable manifold tubes in a suitable surface (U1, U2, U3, U4 or any others), which is shown in following pages detaily.

Also other than these surface suggestion (U1, U2, U3, U4), there can be many various surface cut according to mission design, of course. Such as Surface cut can be taken as angled, say 45 degree relative to x axis (M1-M2 line), so more convenient “jumping points” can be found with these new perspective, if these surface suggestion (U1, U2, U3, U4) can’t give nice result we desire. Indeed, Poincare surface of section can be used quite flexible, if it is applied conveniently.

In figure 26, three-dimension invariant manifolds tubes are shown with three-dimension zero velocity surface. With this perspective, ‘tube’ shape of invariant manifolds can be seen really literally. But in fact, the usage of “tube” term is for definition of all phase space, that tubes actually lie down in 6 dimensional phase space in figure-26, and also for figure-27, they are lie down in 4 dimensional phase space while in Planar-CR3BP.

Also note that; Because three-dimension invariant manifolds are not so convenient for Poincare surface of section and don’t give clear visual in solution space, as in explained in section 3.2.2.1, the mission design and “key point” determination is mostly done in Planar-CR3BP for simplifying the problem in previous studies. Therefore 2D-IMT is mostly used in surface of section for low-energy transfer and capture problem, see figure-27-31, although the real problem is on 3D position space.

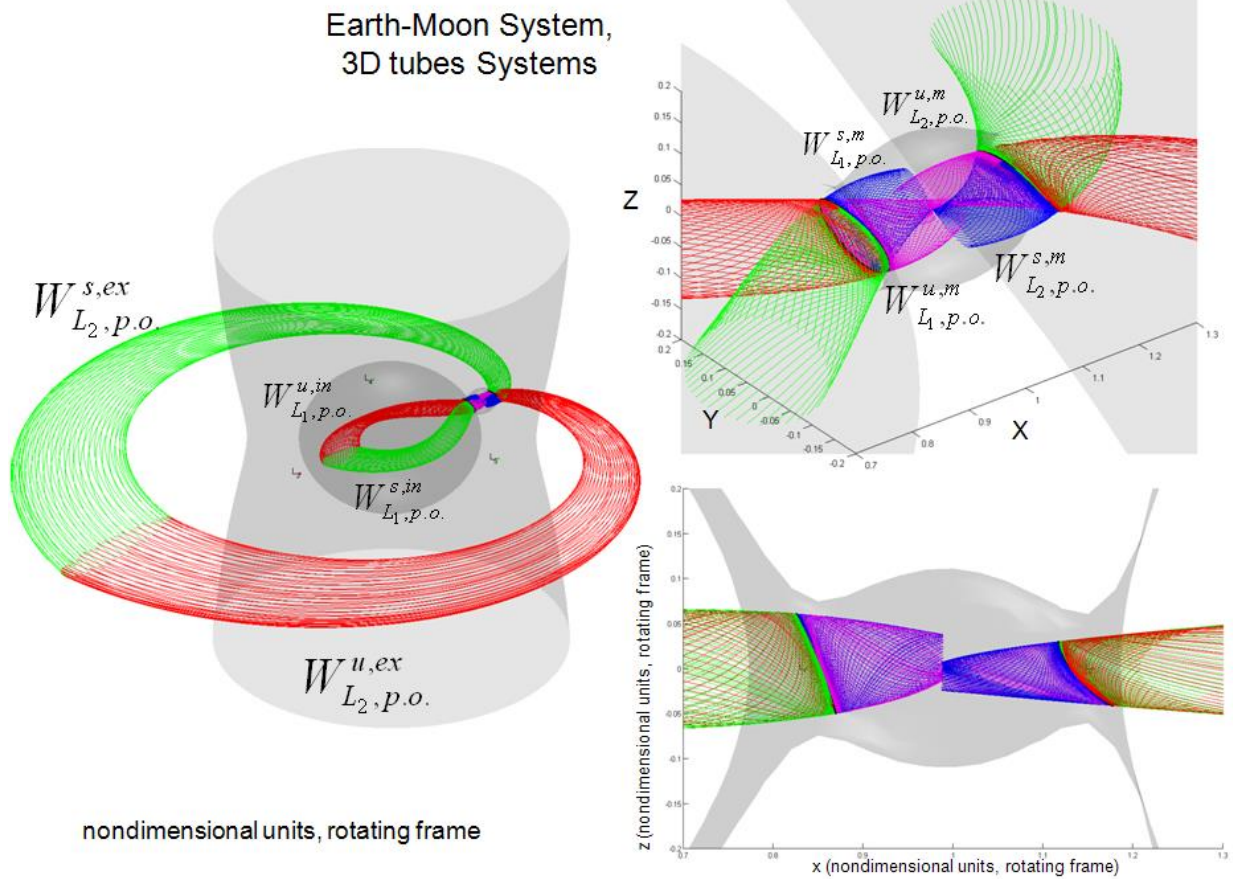


Figure 26 - 3 dimension perspective of stable and unstable manifold tubes. Green trajectories are stable manifold of outside, red trajectories are unstable manifold of outside, blue trajectories are stable manifold of inside, and purple trajectories are unstable.

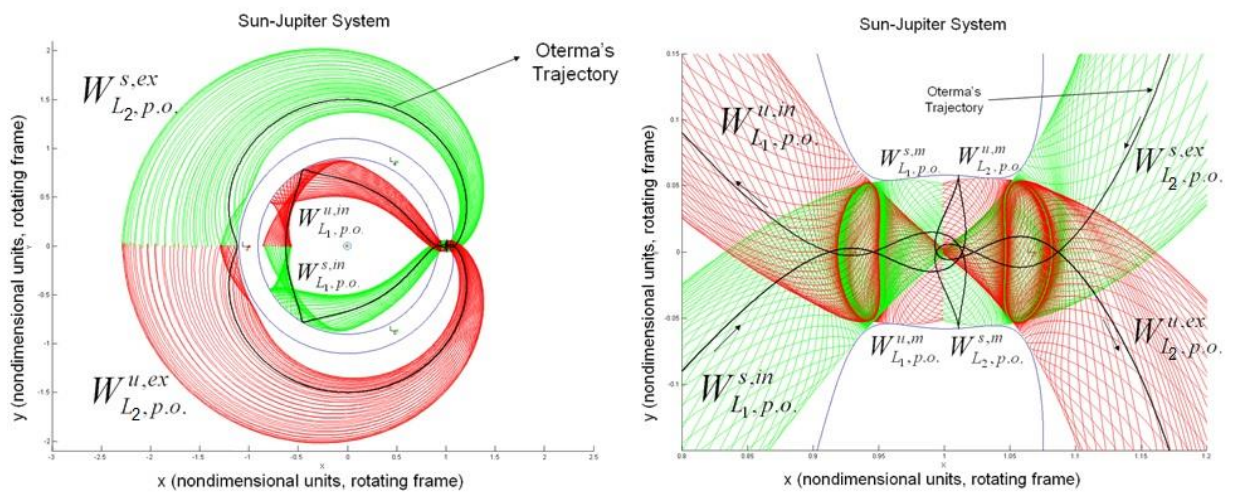


Figure 27 - Stable (green) and unstable (red) manifold tubes.

In here, as an example, 2D IMT is plotted in figure-27. The convenience with [15], the notations of manifold tubes are assigned same as that book. It means that $W_{L_1,p.o.}^{s,m}$ is stable manifold tube of L_1 in secondary body realm. “p.o” means that manifold tube is derived from periodic orbit of equilibrium point. $W_{L_2,p.o.}^{u,m}$ is unstable manifold tube of L_2 in around secondary body. And the meaning of “ex” and “in” is that manifold arrive or departure from *external* or *internal* region according to boundaries of zero velocity surface.

The black trajectory in figure-27 is a good example of transit trajectories, which is such a simulation of Oterma Comet showed in [25]. Also it is kind of a heteroclinic connection that trajectory make a passage from both L_1 and L_2 equilibrium points. In [25], Oterma comet trajectory is analysed well that “gravitational flow” of CR3BP can be so advantageous for space mission.

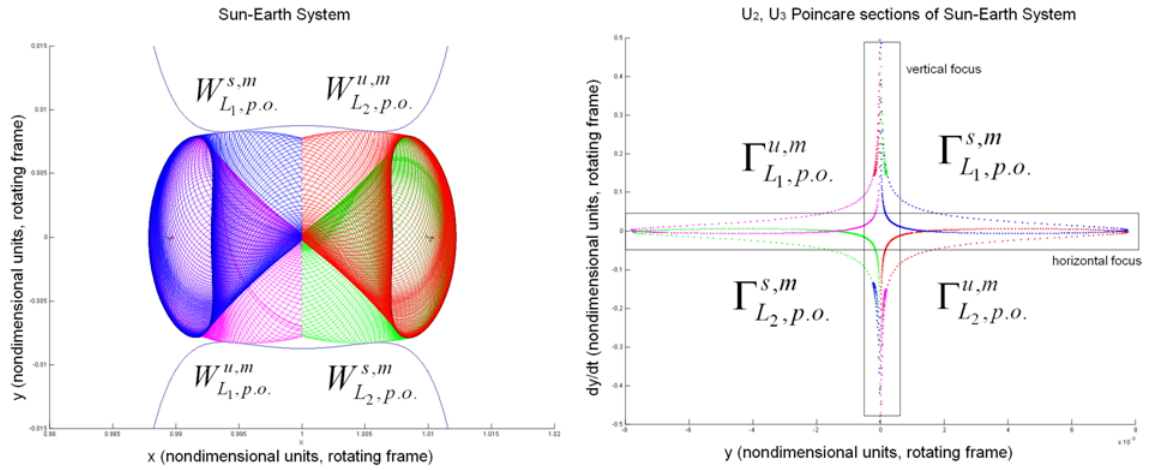


Figure 28 - Neck region (secondary body region) manifold tubes and their Poincare section. Stable manifold tube of L_1 (blue), unstable manifold tube of L_1 (purple), stable manifold tube of L_2 (green), unstable manifold tube of L_2 (red).

In figure-28, Poincare section is shown in standard U_2, U_3 section, see figure-25. With matching with same colour, manifold tubes project the Poincare plane, $W \rightarrow \Gamma$. The section cut is done in $x=1-\mu$, and Poincare map project in (y, v_y) . So the three variable of 4 dimension (x, y, v_x, v_y) is known. The forth variable can be computed by Jacobi integral, eq. 3.10. Therefore, the convenient “key points” can be chosen in Poincare map (y, v_y) directly.

For much more insight, the Poincare map in figure-28 can be focused in horizontally and vertically as in figure-29. In horizontal focus, initial conditions for heteroclinic cycle, which is a periodic orbit that cover both L_1 and L_2 equilibrium points, can be obtained. In vertical focus, the sling and break effect can be seen.

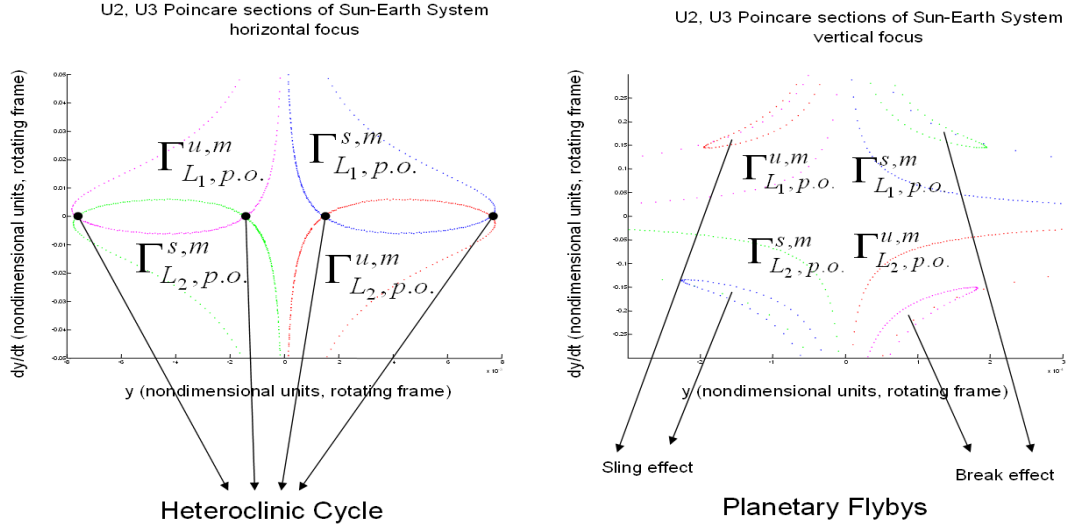


Figure 29 - Horizontal and vertical focus of Poincare section in figure 3.19.

And in figure-30, the trajectory of Oterma comet is shown with flow of the stable and unstable manifold tubes of Sun-Jupiter system. From this example, the reflection of trajectory of Oterma comet can be plotted to Poincare section, for showing the manifold passage clearly, see figure-31. So, same process can be applied as in previous example but for different section; the Poincare sections are done on classical U1,U4 section (as it is defined in figure-25), $y=0$, and Poincare maps are plotted in (x, v_x) , and variable v_y can be computed using Jacobi integral, eq. 3.10. As it is seen in figure-31, the trajectory of Oterma is reflected on Poincare section as black dot, these black dots are exactly on intersection part of stable and unstable manifold reflection. Therefore, it is sure that for jumping one manifold to another, the “key points” must be chosen on the intersection of the manifold reflections.

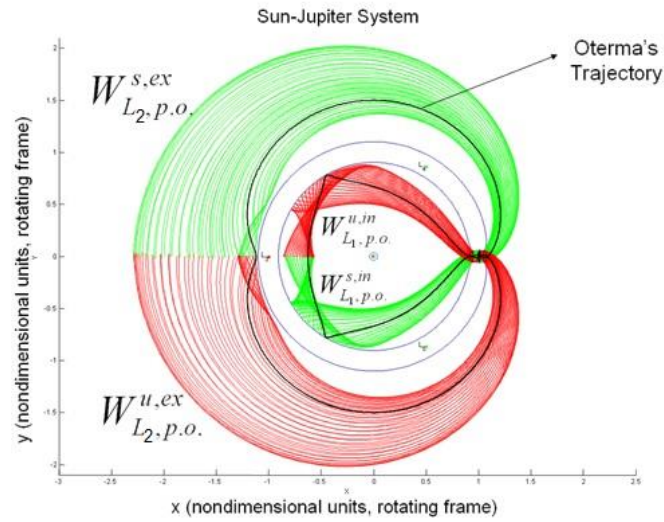


Figure 30 - Oterma trajectory in the flow of the stable and unstable manifold tubes of Sun-Jupiter system.

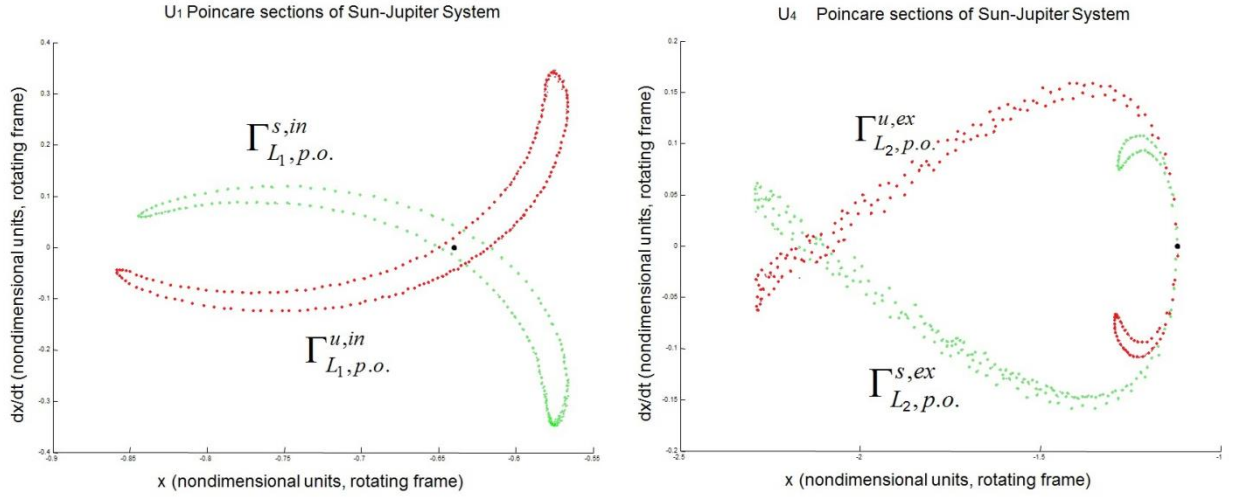


Figure 31 - Poincare section of Sun-Jupiter stable and unstable manifold tubes and Oterma comet projection as black dot. Internal section (left), external section (right).

As it is showed, the manifold intersection is done for only in one-CR3BP. In addition, the same process can be applied the manifold intersection for different CR3BP system. Combining the two different CR3BP is called patch-CR3BP, and this is actually different aspect for restricted 4-body problem. Transformation between 2 system can be done using the equation in section 3.13. As it is seen in figure-32, there are 2 main formations for patch-CR3BP, one is bi-circular model such as Sun-Earth-Moon model in figure-33, and the other one is co-linear model, such as Jupiter-Europa-Ganymede in figure-34.

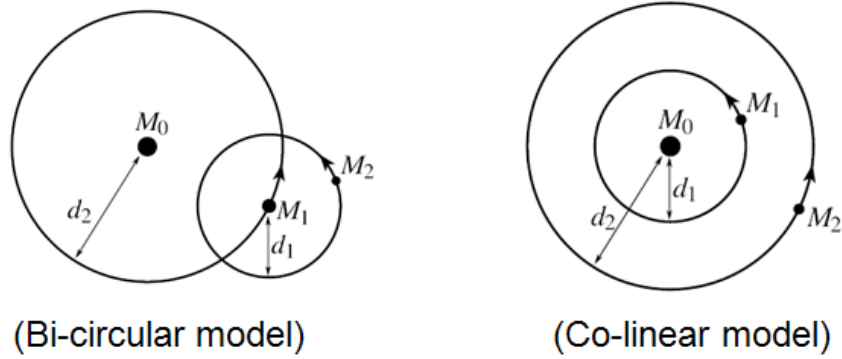


Figure 32 - Two different types for restricted 4 body problem or patch-CR3BP, [15].

For both systems, in figure-33 and 34, transformation process is needed to reflect the trajectory in one binary system to another, which is formulised in section 3.1.3 detaily. For the proper mission design, a trajectory, which is computed in effect of the one binary system (A), must be transferred to other rotation frame of binary system (B). Then rest of the trajectory must be computed in effect of this binary system (B). The important point in here; place (or jumping point) that transferred from (A) to (B) must be determined carefully that sphere of gravitation of

bodies from other binary system shouldn't affect the trajectory computation in other binary system. To provide this, two invariant manifold tubes from different binary system (A and B) are intersected relatively far away from secondary bodies of binary systems. After that, "jumping points" of manifolds of two different systems are found using Poincare map, section 5.4.

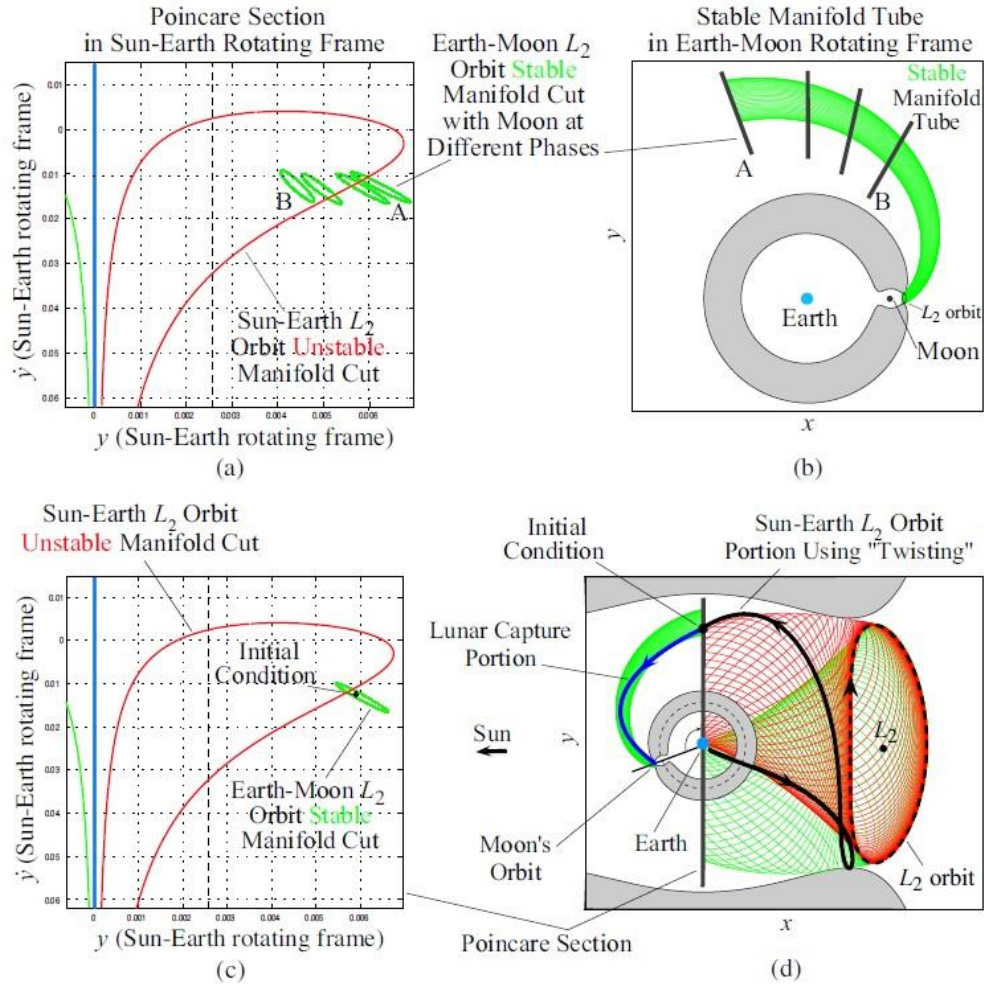


Figure 33 - bi-circular model for Sun-Earth-Moon model and Poincare sections. [15]

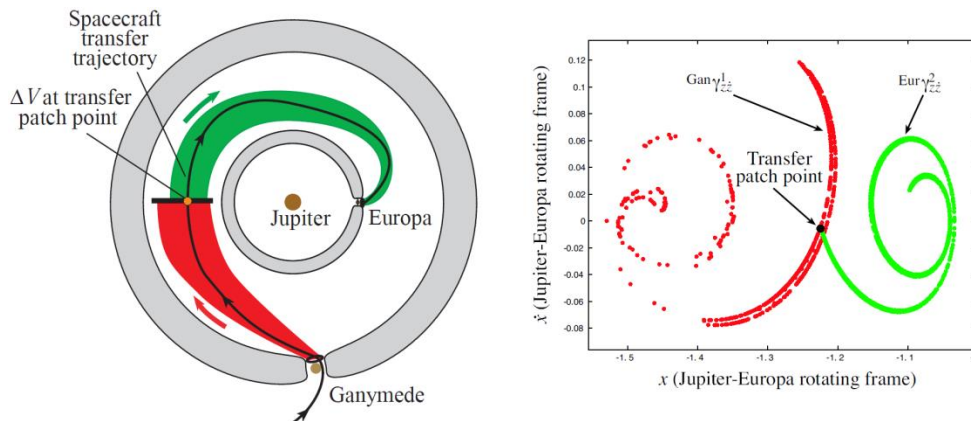


Figure 34 - co-linear model for Jupiter-Europa-Ganymede. [15]

For more accurate study, forth (Sun or Moon) body effect can be directly added to CR3BP as in [15, chapter-5], and also further information, for interested readers, can be seen in [89, chap.-5].

3.2.4 Transit and Non-Transit Trajectories

If the trajectories are projected to phase plane of equilibrium point, it is seen that, asymptotic orbits behave as separatrix; they separate transit and non-transit orbits; which this separatrix is explained very deatily in [15,88-89]. Transit orbits can pass from one realm to another inside the asymptotic tube (Figure-13), while non-transit orbits are stay in same realm. Both transit and non-transit trajectories have their own advantageous feature, or course. Such as transit trajectories can be used transfer for travailing one region to another region in the boundaries of zero velocity surfaces. Also they can be used for capture/escape trajectories as well. However the non-transit trajectories are always stay in same region and they never leave their region, so they can be used station keeping orbits or some kind of science orbits.

In addition, as we mention in section 3.2.3, intersection of these transit and non-transit in flow space gives as a seperatrice trajectories, which can be very useful in zero cost, also called permanent capture. In the permanent capture, trajectories are unbounded (so transit) in backward time, while it is bounded (non-tansit) in forward time. And, reverse of this phenomenon can be also called as permanent escape. As it is going to be seen in following chapters, we may either use this permanent capture/escape phenomenon in converging to exact intersection of transit and non-transit, or develop a leap (or jump) method from transit to non-transit, and reverse.

IV. MOTION MAP

Within the last decade, maps have been used extensively in a wide range of scientific and engineering problems to understand the dynamical structure of complex systems. This stems from the fact that a map reduces the order of the problem, condensing quantities of information into a lower-dimensional image. Because of this advantage, mapping methods are widely used in non-linear dynamics, systems with chaotic behaviour, and so celestial mechanics. Also, because there is not any certain closed solution for multi-body dynamics, as in classical 2-body dynamics, maps help us to gather insight into dynamical behaviours of over dimensional systems, such as in CR3BP. Previously, the solutions, especially manifold intersection (see section 3.2.3.3) and periodic solutions could be found practically using some basics mapping methods. As shown in section 3.2.2.1, Poincaré Map is generally used for determination of periodic solutions in celestial mechanics, multi-body dynamics and so CR3BP. But Poincare map (or surface of sections) may not work for every non-linear dynamical system; it may even not work for a certain realm of a dynamical system while it can work for another part of it. For example, Poincare map method can work well to find periodic orbits around bigger mass and retrograde direction of smaller mass, but it does not work with direct periodic orbits around smaller mass in high energies. Moreover, Poincaré Map also does not give a “nice” visualization for 3D position space to determine 3D periodic orbits. Although gravitational effect of bigger mass so influence to smaller mass realm, especially in highly energy, and it seems as if no any stable direct periodic orbits in there, but there are so many, which are on 3D position space. In previously, all significant studies in CR3BP, including capture dynamics, mission design, searching periodic orbits, were mainly done in planar case, because Poincare map works more convenient in two dimensional case, otherwise too much variables are appeared and four dimensional surface of section in 3D-CR3BP must be done [87].

However, the trajectories have been stabilized around the target planet (smaller mass in CR3BP) using either retrograde orbits or stable-sets of direct trajectories, which are planar and needed high-cost velocity decrement to sit in a parking orbits around M_2 [13,15,20]. Although retrograde orbits seem more safe, stabilize the trajectories in high energy level with direct course can be more advantageous for both low-cost capture and escape. Because CR3BP has its own dynamical flow as a result of circular (or elliptical) rotation of two massive bodies, it can be very useful to utilise this flow in low-energy transfer. Therefore, the main problem is thus; how can we benefit from this dynamical behaviour? Although 3D-CR3BP has its complexity, if a convenient “map” can be constructed, low-cost capture trajectories can be determined more conveniently. 3D-CR3BP can offer much more insight and prove more advantageous, and, if a suitable map can be constructed in secondary realm, we can easily determine any type of trajectories, especially both low cost capture and escape trajectories around target planet, which may be Moon, Planets and other planet’s Moons ...etc.

4.1 Motion Map Definitions

Motion map is a map method that, in a certain or enough (either backward or forward) integration time, defined outcomes (so motions) of a (any) certain trajectory groups are reflected to their initial condition plane, to see the insight of their chaotic dynamical behaviour. So according to this definition, a Motion Map approach is introduced to systematically study the 3-D capture problem around a secondary body. The research focuses on the case where the energy level is sufficiently high that both the L_1 and L_2 equilibrium points's gates are open, mostly in relatively equal size; such as $C=3.16$ for Earth-Moon Binary case, mass ratio (μ); 0.01215. This enables a spacecraft to enter the region around the secondary either from the region around the primary or externally. The mapping procedure needs to distinguish the enough long term outcome of possible trajectories at these energies. Once motion is confined within the region around the smaller body, there are obviously 4 main types of outcomes for the motion around neck (M_2) region:

- Periodic and bound motion,
- Collision with the secondary body,
- Escape/capture through the L_1 opening in the zero velocity surface
- and escape/capture through the L_2 opening,

as similarly defined in [38,39], but different in definition of bounded and configurations. In fact, these outcomes may change depend on different region and map configuration, of course. Such as, if we want to plot a Motion Map around L_4 or L_5 equilibrium points, we must define different outcomes. But, in this research, we focus around M_2 region for finding low-cost capture around there, so the outcomes are defined as above. Also as mentioned before, the energy of approach must be sufficiently high for the L_1 gate to be opened and possibly the 2 L gate as well. Consequently, trajectories can be free to transfer any direction to L_1 or L_2 , and mission design can be made more flexible. The central question this research addresses is how to jump from unbounded (L_1 and L_2 escape trajectories) to bounded (periodic motion) trajectories at low DV cost, or execute the reverse jump. But first of all, let's construct the map.

The CR3BP has a phase space which is 6 dimensional and to survey the full space would be prohibitively costly. Most approaches to explore non-linear dynamical systems rely on using a 2 dimensional cut through a space. This was first introduced by Poincaré and known as a Poincaré surface of section, (see section 3.2.2.1). This research shall refer to such plane as the Motion Map and it is defined and varied a little bit differently than classic Poincaré map. For M_2 region, this map defines a set of initial conditions for the long-term integration of trajectories and their long term behaviour can be classified into the four categories assign above. In this research, we shall display the regions of differing long term behaviour by different colours on the map.

The computation of the motion map can be described systematically as follows:

- Define a grid of initial conditions in the 2-D plane of the Motion Map in phase space.
- Choose a time frame for integration that covers the lifetime of the mission.
- Integrate forwards (or backwards) in time over this selected time span or until it collides with the secondary body or escapes to large distance compared to the L_1 and L_2 points.
- Classify the trajectories according to the possible categories, such as for M_2 region:
 - Collision,
 - Bounded,
 - Escape/capture through L_1 ,
 - or Escape/capture through L_2 .

In figure-35, a quick example of motion map with **x-z plane** configuration is shown. To take advantage of the symmetric features in CR3BP, x-z plane configuration is used; therefore backward and forward integration would give similar result because of symmetry and this save us extra computation load.

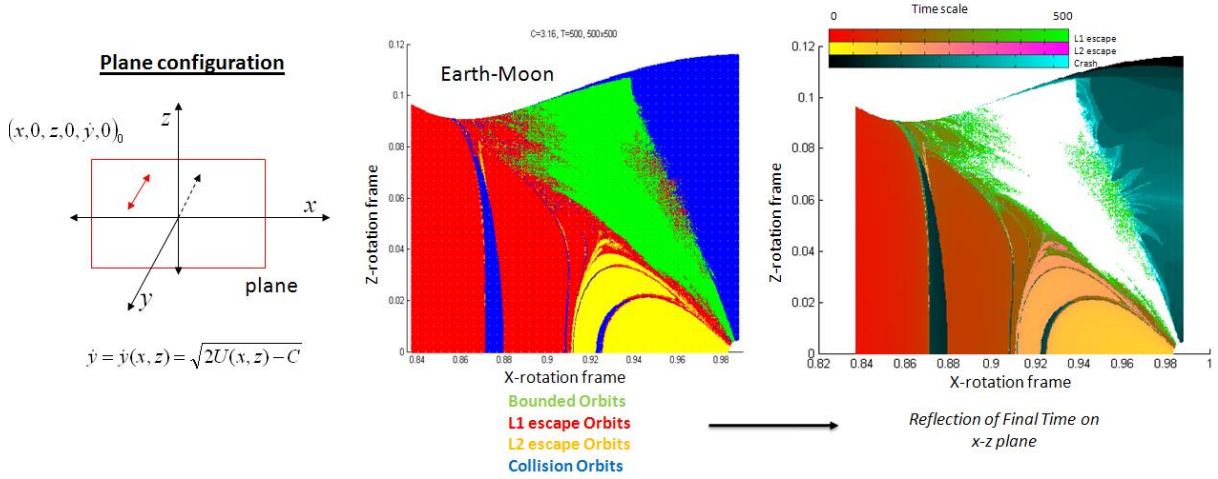


Figure 35 - x-z plane configuration (left), Motion map (Earth-Moon) after integration of grid (middle), Motion map with reflection final time span using colour degrade method (right).

After the integration of the grid of configuration, and assigning the colours, the shape of map is given as in figure-35 (middle);

- Bounded (green), if it stays in secondary region in all along the integration time.
- Escape (or capture, if in backward integration) L_1 (red), it escape secondary region from L_1 gate.
- Escape (or capture, if in backward integration) L_2 (yellow), it escape secondary region from L_2 gate.
- Crash into the Moon (blue), if it falls to the secondary body in the integration time.

Integration is done for $C=3.16$ Jacobi constant value, that means both L_1 and L_2 gate of zero velocity surface are open, and the trajectories can easily escape from both sides. As it is seen already that there is not any green region (even not green points) in $z=0$ case, in this energy. But, even in this high energy level, all bounded trajectories band together in diagonal side (in a certain inclination) of secondary, that, in overview of the map, we can see a solid area for bounded region even in relatively high energy. This illustrates how useful it is to analyse the CR3BP in 3-dimensional case.

For further insight, the third dimension is also assigned as final time of collision and unbounded trajectories, and reflected to the map using colour degrade method, figure-35 (right). Therefore, we can see which trajectories escape (or capture in backward time integration) in short time or long time in 2-D map, too.

As can be expected, there are some elegant tips for plotting and using the motion map accurately, fast and also relatively practically. First of all, we need to choose the right Configuration (see section 4.2), such as x-z plane in figure-35. Then the right integration time and method must be chosen for fast and accurate results (see section 4.3). Also, after obtaining the plot, we should be careful about “fuzzy boundaries” phenomenon (see section 4.4). All these tips are explained in their titles belows.

Also note that, as it is seen this map method is quite different than familiar classical Poincaré section. In Poincaré section, all trajectories are integrated initially, then a surface of section is taken in the phase space to collect the point data of these trajectories on the surface. Then this point data in surface of section can be used for an initial condition map for finding periodic and quasi-periodic orbits. However, in this map method, an initial condition plane is “defined as gird” firstly, then all trajectories using these initial conditions on that plane are integrated in desired time-span, then outcomes of all trajectories are classified (according to problem), then this classification is reflected to this initial condition plane, and finally getting a map.

4.2 Configurations

Because the 3D-CR3BP has a 6 dimensional phase space, there are many alternatives for restricting these variables to the 2D mapping plane. Motion is confined by the conservation of the Jacobi integral which reduces the system dimension to 5, but this still leaves plenty of scope. A most simple and common choice may be to use a plane aligned with two of the coordinate directions, as in Planar-CR3BP. This makes for a simple interpretation of the resulting map. For this reason previous studies, exp. [15, 16, 22,33], have mainly focused upon in-plane motion. In this thesis, however, we argue that out-of-plane orbits can provide a lower cost capture as there is less angular momentum associated with such trajectories. So configurations are set as out of plane.

As it is seen in figure-35, it is needed to initial condition set for plotting the motion map. Because 3D-CR3BP has various variable and trajectory type, there would be a lot of kind initial condition configuration set. This configuration sets can be constructed with restriction in phase space as in figure-35 (left), or it can be projected along a set of trajectory, such as on invariant manifold tubes, or it can be projected along any random trajectory. Furthermore, configuration set can be reflected another projection, after the obtained the map (see section 4.7), too.

Therefore, in this section, some useful configuration sets are introduced and their plotting maps are shown with their feature. Respectively, plane and line configurations, invariant manifold tube projections, random trajectory configurations and angular momentum- inclination projection are introduced.

4.2.1. Plane and Line Configurations

As it mentioned just above, because 3D-CR3BP has 6 dimension phase space (x,y,z,v_x,v_y,v_z) , its dynamics analysis is too complicated for examinations. Even using Jacobi integral for reducing system dimension, there would be still 5 dimensions phase space. Therefore the phase space can be restricted using some convenient configuration for reducing the system dimension from 5 to 2. In figure-36, it is given 2 simple examples for the restriction method, as initial examples.

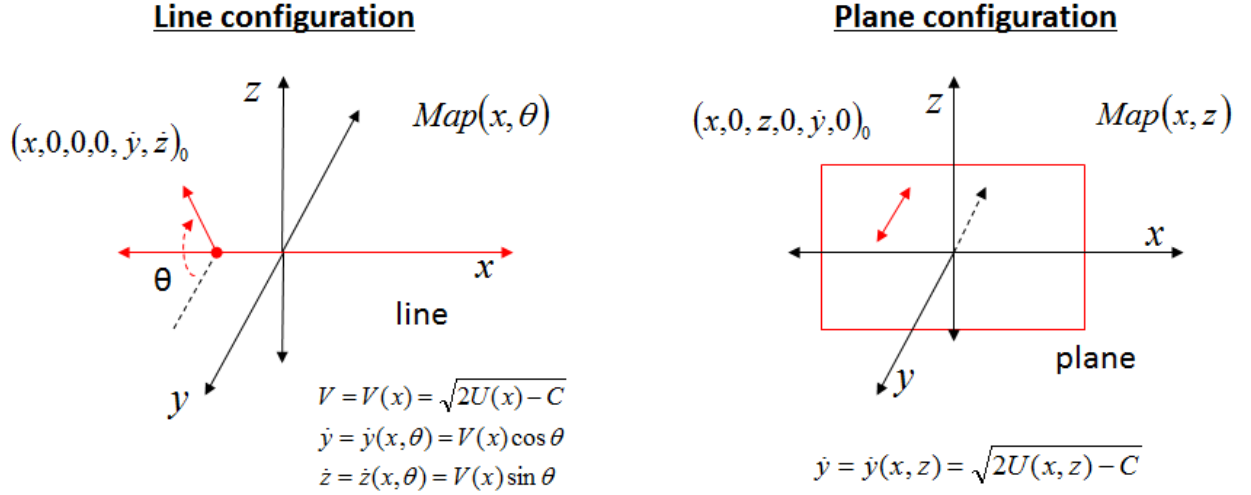
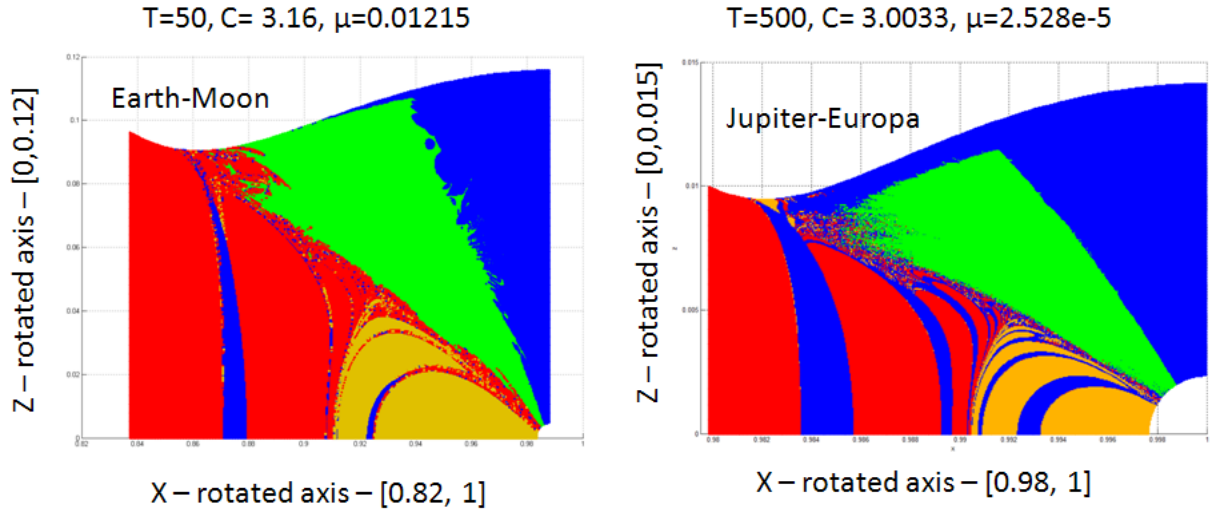


Figure 36 - Line and Plane configuration for initial condition sets

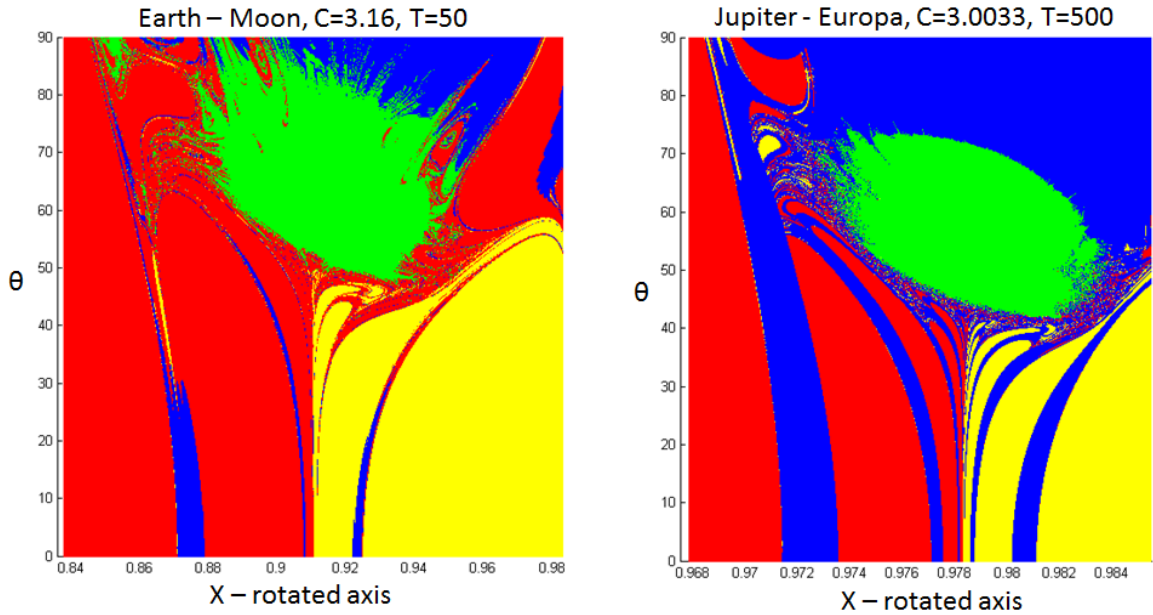
In line configuration (figure-36, left), the line is chosen in position space and the velocity vector is assigned as perpendicular to this line, and its angle (θ) is varied. Jacobi constant is set as constant and Jacobi integral reduce the system configuration as it is seen in figure-36, left. Finally, the initial condition set dimension is reduced 2 dimension as (x, θ) .

In plane configuration (figure-36, right), the plane is chosen in position space and the velocity vector is assigned as perpendicular to this plane. In this example x-z plane or $y=0$ plane is chosen. The advantage of this choice is that the motion is symmetric in CR3BP, so backward and forward integration would be the same, and this halves the amount of computation required. Then Jacobi constant is used to determine the magnitude of the velocity at this point, and reduce the system configuration as it is seen in figure-36, right. Finally, the initial condition set dimension is reduced 2 dimension, such as (x, z) or maybe (y, z) and any other.

Therefore, as it is shown in in figure-37, Maps in two type of configurations are showed. While upper plots (a) are computed using with plane configuration, below plots (b) are computed with line configuration. For all maps, velocity vector directions are kept in Direct (same direction with rotation axis). Upon this plane a grid is defined. This study takes the range of values of “x” to lie between the L1 point and the centre of the Moon, while the z values range from zero (the orbital plane) to the zero velocity curve for the given Jacobi constant in figure-37 (a), and θ is range between 0° to 90° in figure-37 (b). For all map, the Motion Map is plotted for the grid using 500 x 500 cells for both the Earth-Moon system and the Jupiter-Europa system, for comparing two different Binary systems, which have different mass ratios. Also all map only computed for positive side of z-axis, because z-axis always symmetric in CR3BP. Also note that, since the Jacobi constant (C) is fixed, then there is a zero velocity surface that limits the available region of this plane for figure-37- a. And for plots in figure-37- b, secondary body (M2) locate below-right corner.



a) Plane Configuration examples



b) Line Configuration examples

Figure 37 - a)Plane and b) Line motion map for Earth-Moon (left) and Jupiter-Europa (right). Bounded orbits are green, L1 escape trajectories are red, L2 escape trajectories are yellow, and collision trajectories are blue.

For upper plots (figure-37- a, plane configuration), the integration time of map (or the lifetime of the mission) has been fixed to a dimensionless time of 50 units in the Earth-Moon system, corresponding to 217.4189 days, and the velocity vector to be in the forward direction in the orbital rotation direction. For the Jupiter- Europa system the integration time has been set to 500 units as this corresponds to a similar mission lifetime of 282.5939 days.

Similar for below plots, but in different configuration (figure-37- b, line configuration), the integration time of map (or the lifetime of the mission) has been fixed to a dimensionless time of 50 units in the Earth-Moon system, corresponding to 217.4189 days, and the velocity vector is angled as in figure-36-left and it's directed in the forward direction with the same direction

orbital rotation. For the Jupiter- Europa system the integration time has been set to 500 units as this corresponds to a similar mission lifetime of 282.5939 days.

In all these plots the long term behaviour has been colour-coded: the bound trajectories over these periods are shown in green; the trajectories that collide with the secondary (assumed spherical) are shown in blue, and the escape trajectories are red (escape through L_1 gate) and yellow (escape through L_2 gate). Note that those trajectories that escape or collide have only been integrated until the outcome has been determined, which is usually much shorter duration than the 200 plus days chosen. This total integration time is only critical for the green region and in determining the nature of the boundary of this region, which is shall refer to as the “fuzzy boundaries” in the discussions to follow, see section 4.4.

For both configuration and these systems the Jacobi constant (C) have been chosen to set the L_1 and L_2 gates to be open (in the sense of the restriction of the zero velocity surface) with approximately the same opening size, which $C=3.16$ in case of Earth-Moon Binary System, and $C=3.0033$ for Jupiter-Europa system. In other words the height along the z axis of the neck of the zero velocity curve close to L_1 is the same fraction with L_2 side. So, trajectories are in the energy that they may have relatively same chance for both escape (or capture either L_1 or L_2) and bounded around Moon, also see figure-40.

The existence of the green region demonstrates that there is a stationary area over a range of inclinations for long term bound motion around the secondary body, even when the L_1 and L_2 gates are wide open. This region is similar to that result in [68] where many periodic orbits were found at relatively high inclinations. The reason for this is that the gravitational effect of the primary decreases with inclination. At very high inclinations the region becomes blue indicating these trajectories collide with the secondary. This is because these trajectories are becoming increasingly radial with respect to the secondary.

In additions, as can be seen in figure-37, both case of these maps are topologically similar, even though the mass ratio and integration time-span ($T=50$, $T=500$) are very different in the two binary system. For the escape regions, there is a series of narrow blue regions in between them, which we shall refer to as “collision waves”. These waves shrink in accordance with the relative radius of the planet: the bigger the relative radius is the wider the “collision waves” will be.

Furthermore, the integration time also affect the boundaries of green (bounded) region. In the short integration time, the boundaries of green (bounded) region are quiet “clear-cut”. But in the long integration time, the green region is started to shrink and its boundaries become “fuzzy”, and “permanent capture” trajectories, which are bounded in forward time but unbounded in backward time, can be converged. Actually, chaoticity in “fuzzy areas (or boundaries)” and “collision waves” are related to transversal (or multiply) intersections of the invariant manifolds. As we already mention in section 3.2.3, the invariant manifolds, which’s also called asymptotic orbits of periodic orbit around L_1 or L_2 equilibrium points, are a separatrixes that separate transit (which are red or yellow regions in Motion maps) and non-transit (Green or blue (especially polar blue) region in Motion maps. And so, separation of these boundaries, which appear highly fuzzy or chaotic depend on integration time, can be the invariant manifolds of a Liapounov Periodic Orbit of either L_1 or L_2 , which has crossed the plane many times. For more information about multiply manifold section, please see [87,89-94], and more detailed discussion about these fuzzy boundaries are done in section 4.3 and 4.4, and collision waves in section 4.5.1.

Moreover, as it is shown in figure-37, line configuration in figure-36-left (and also many other configurations) might be also chosen for optionally and getting more detailed dynamical insight. But for just focusing this map method better on one example, Plane configuration (in Earth-Moon case) is used in general scope of this thesis; because Map, reflected directly to position space, would be more apprehensible. And also to get more clear and appealing map in position space for sensing dynamics easily, the plane configuration is being mainly analysed for better observation the accommodation with zero velocity surfaces. So from now on, as an initial example, the x-z plane is chosen in the region of target planet (around smaller body in CR3BP), and so we can also get result of backward and forward time integration at the same time using this configuration, because of symmetry in CR3BP.

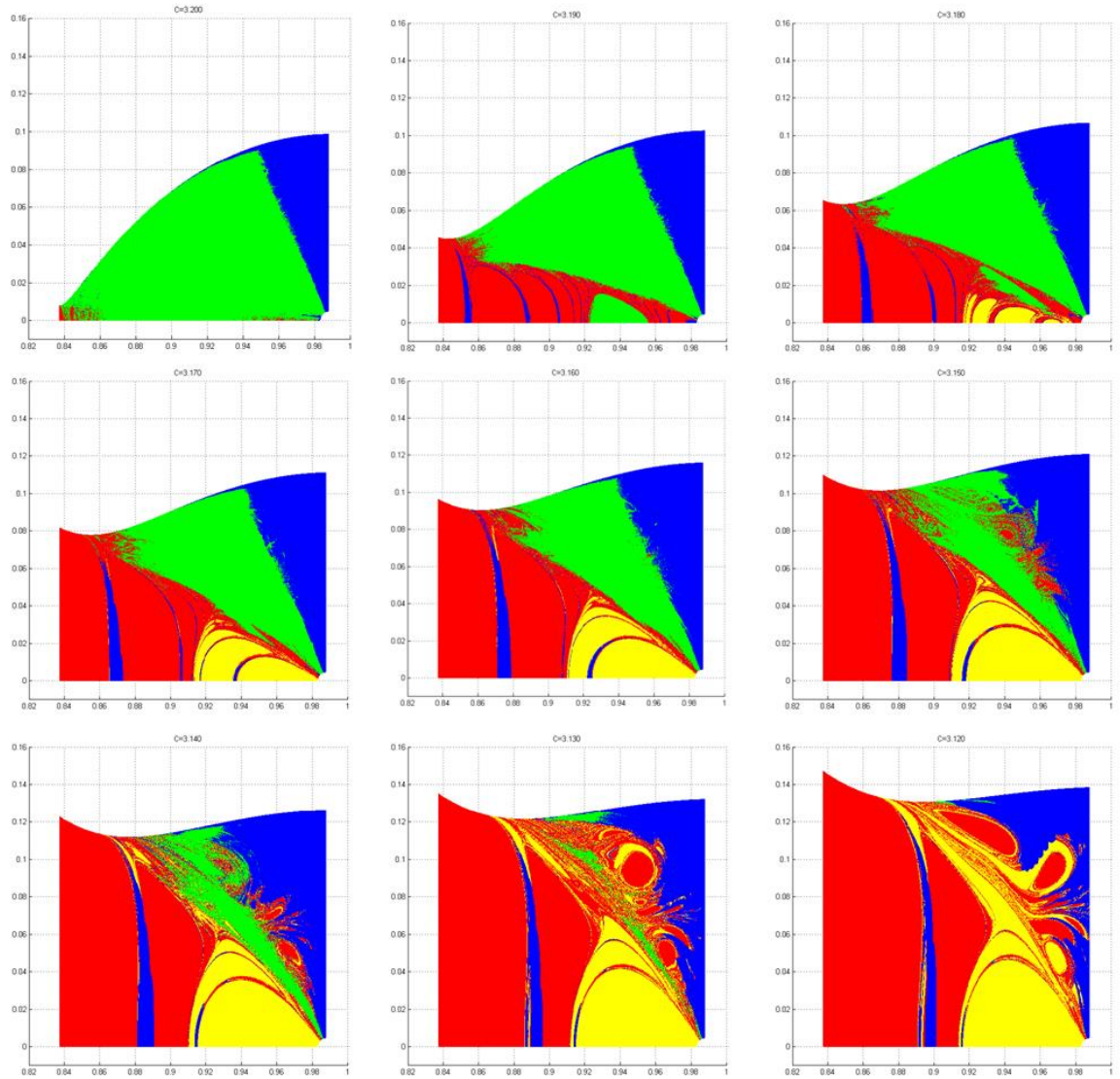


Figure 38 - Plane motion maps in different Jacobi constant for Earth-Moon Bounded orbits are green, L_1 escape trajectories are red, L_2 escape trajectories are yellow, and collision trajectories are blue. $C=3.2$, $C=3.19$, $C=3.18$, $C=3.17$, $C=3.16$, $C=3.15$, $C=3.14$, $C=3.13$ and $C=3.12$, respectively. Please see MM(X-Z)_various_C.wmv in appendix CD to see its video.

Also another note that; the escapes trajectories (red and yellow areas in figure-37 and 38) in these maps are became capture in backward integration, and can be actually defined and called “temporary capture” trajectories, which makes bounded motion in a restricted time (turning several times around secondary body, after it is captured), then escape from either L_1 or L_2 gate. And the bounded trajectories (green areas on Map) are the permanent bounded orbits that can give such a permanent capture orbits, which can be achieved with a so tiny velocity decrement (so energy change). Therefore, the motion map should be also plotted with different Energy (So C , Jacobi Constant) values to see what happened when velocity magnitude so energy change in there, see in figure-38.

In figure-38, motion maps are plotted in different Jacobi constants, for Earth-Moon binary system, in $T=100$. As it is seen, while both L_1 and L_2 gate is nearly close ($C=3.2$), the bounded region (green) has bigger area. But when the Jacobi constant is started to decreased (or energy is increased), the area of bounded region gain an inclination and shrink. And in $C=3.12$ (in highest energy), the green region is disappeared. Therefore spacecraft can be easily sit in high energy parking orbits for $C>3.12$ case. Also using the different energy plots in figure-38, we may jump from one region to another only making a tiny velocity (or energy change). This jumping process is explained in section 5.1, and this type of capture can be called artificial permanent capture.

In addition, to show L_1 and L_2 gate size relationship, figure-39 and figure-40 shows zero-velocity surface in various energy levels and reflection, as x - z and y - z , respectively. Because mass ratio of Earth-Moon binary system is relatively bigger ($\mu=0.01215$) potential effect of L_1 and L_2 and so their gate size of zero velocity surfaces change asymmetrically with the change of Energy. Such as in a specific energy level ($C=3.2$ or $C=3.19$), L_1 gate can be open, while L_2 gate is close. Also when energy is increased more and more, sizes of both gates are changing differently; size of L_2 gate enlarges faster than size of L_1 with increasing the energy level. This potential differences between L_1 and L_2 (see figure 11) makes region sizes of Red and Yellow are differently in various energy levels in figure-38.

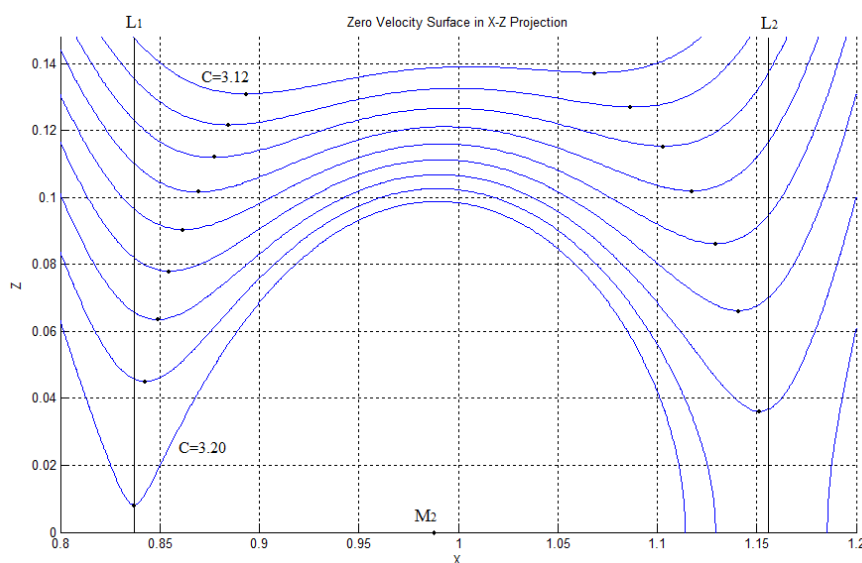


Figure 39 - Zero velocity surfaces in X-Z projection for $C=3.2$, $C=3.19$, $C=3.18$, $C=3.17$, $C=3.16$, $C=3.15$, $C=3.14$, $C=3.13$, $C=3.12$, respectively. Black dots on curves show L_1 and L_2 gates's edges. And place of L_1 and L_2 are extended to Z direction as black lines, to show place relative to black dots.

To able to see change of gates sizes insightfully, zero velocity surfaces (ZVS) can be also plotted in Y-Z projection in figure-39. In these plots, all ZVS are plotted with same energies as in figure-38, and their “X-axis” intersection values are pointed in figure-39 as “black dots”; Black dots near to L_1 equilibrium point are edge of L_1 gates in specific energy level (in this example $C=3.2$ to 3.12), and so; Black dots near to L_2 equilibrium point are edge of L_2 gates in specific energy level, too. So, using the X values of black dots in figure-39, ZVS can be plotted Y-Z projection and edge size of both L_1 and L_2 gate sizes can be seen and compared much better in figure-40.

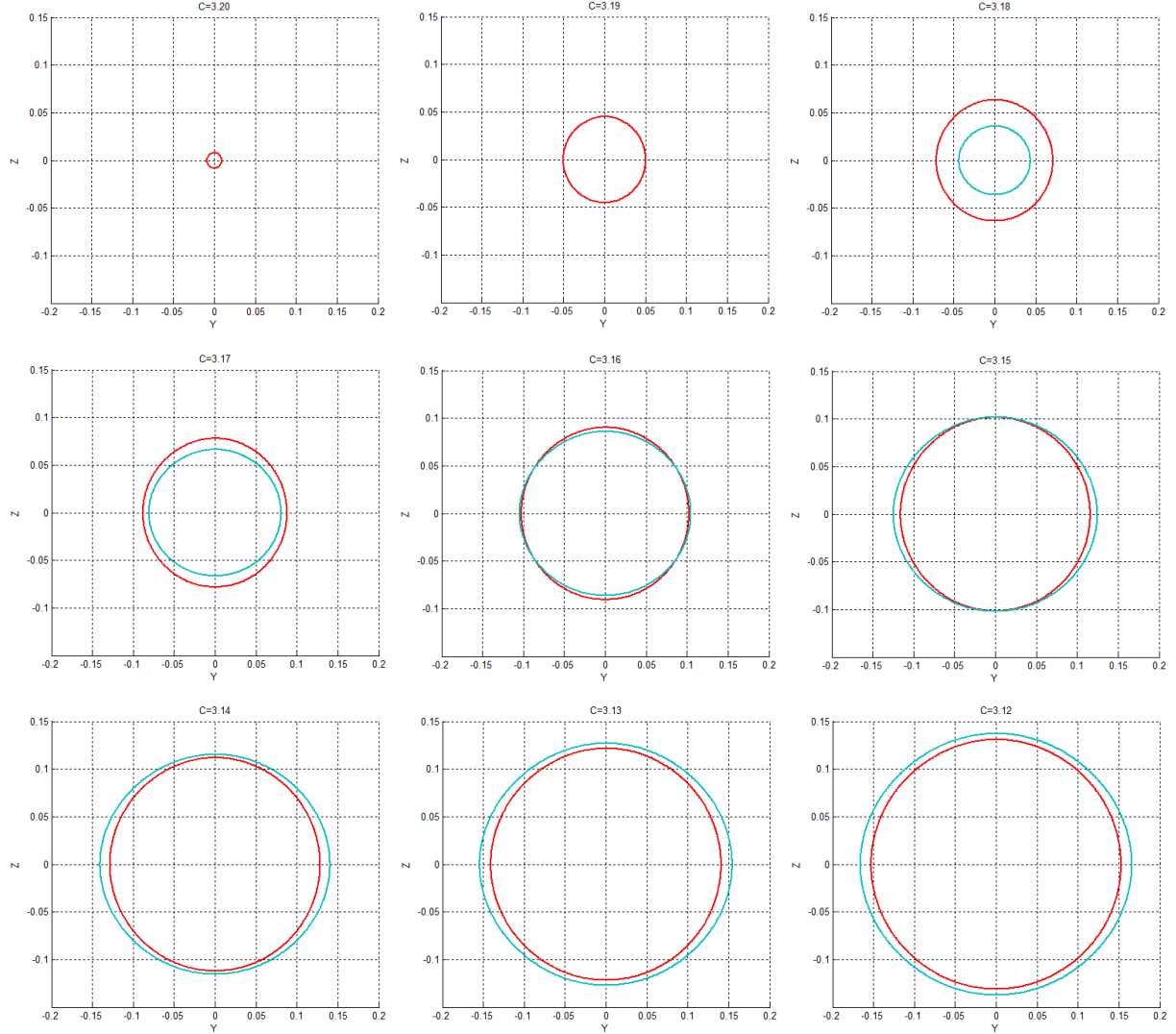


Figure 40 - Zero velocity surfaces in Y-Z projection for $C=3.2$, $C=3.19$, $C=3.18$, $C=3.17$, $C=3.16$, $C=3.15$, $C=3.14$, $C=3.13$, $C=3.12$, respectively. Values of the X-sections are according to black dots in figure 39. Red curves are show intersection taken from L_1 gate side, and Turquoise curves are show intersection taken from L_2 gate side.

In figure-40, Red curves are show (Y-Z) ZVS intersection taken from L_1 gate side, and Turquoise curves are show intersection taken from L_2 gate side. As it is seen, only L_1 gates are open in specific energy range ($C=3.2$ -3.19), and if energy is kept increase, then L_2 gate is open and start to enlarge, and after a specific energy level ($C \sim 3.16$ for Earth-Moon binary system), L_2 gate size become bigger than L_1 gate. That's why we initially choose $C=3.16$ for plotting

motion map at first, see figure-37. $C=3.16$ is such a frontier which both gates' sizes are relatively same size. The motion regions (red, yellow, blue and green) in figure-37 can be better grasped with associating figure-40.

In figure-41, the plane motion map is plotted in another plane, which is for y-z plane, $x=1-\mu$, for gaining much more insight about dynamics of secondary region. This plane is exactly perpendicular to x-z plane, which are plotted as previous motion maps, figure-37-a. The integration in figure-41 is done for both forward time and direct direction. Left side is for plot for negative “y” and right side is for positive “y”. The Jacobi Constant again is set as for both L_1 and L_2 gate is open same size respect to zero velocity surface, $C=3.16$.

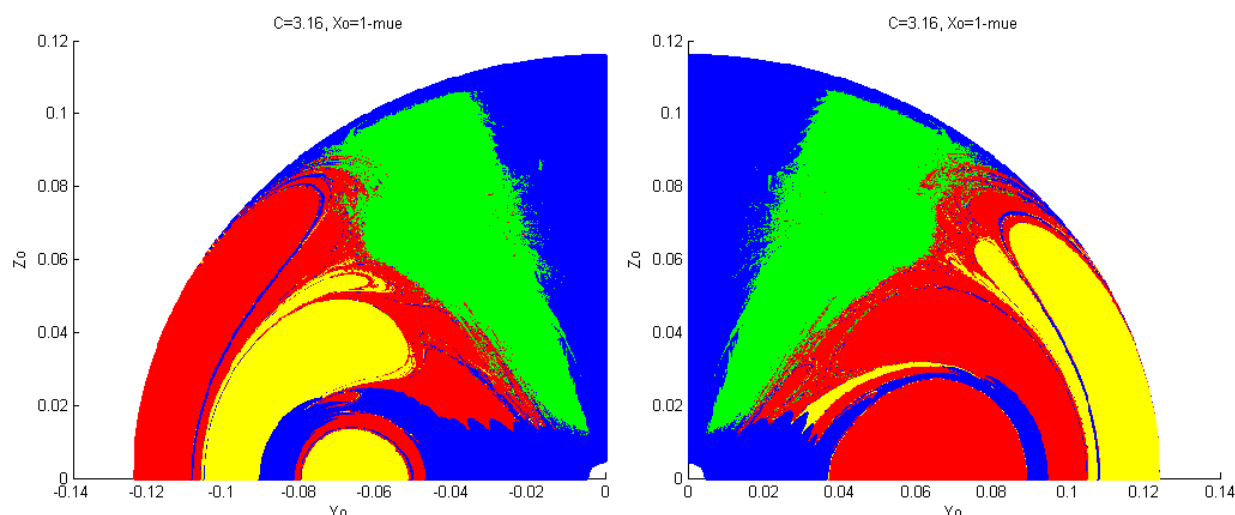


Figure 41 - y-z plane motion maps (Earth Moon), initial conditions are $(x=1-\mu, y, z, dx/dt, 0, 0)$, Collision, bounded, L_1 and L_2 escape trajectories are blue, green red, yellow, respectively. Please see MM(Y-Z)_various_C.wmv in appendix CD to see its video in various C values.

As it is seen in figure-41, x-z motion map (figure-37 or 38) and y-z motion map have some similarities and differences. Firstly, there is a solid polar collision region for both plane maps. Also there is a solid bounded region in diagonal direction. But this bounded region is not able to reach to low altitude of secondary body; however, it can reach to secondary body in x-z plane map. Another difference can be observed in the escape regions. Although both plot in figure-41 are plotted symmetrically, the escape regions are appear too different each other, because L_1 and L_2 equilibrium points are in potentially different level, see figure 11.

There is also another interesting feature about y-z plane motion map in figure-41; it is not symmetric as in X-Z plane but it is opposite-symmetric according to backward or forward time integration. This means, the forward integration of left side map is “symmetrically(mirror) equal” to backward integration of right side plane. So if the right side plane, in figure-41, is integrated to backward time, the result is appeared exactly symmetric to plot in left side in figure-41. Therefore, we don’t need to compute extra map to comprehend both backward and forward time Map for full insight. As it is mentioned before, we can get rid of extra computations in X-Z Map, which gives backward and forward time Map at ones so. Similar with this, Y-Z Map is just plotted forward time for both side as in figure-41, and then overlaid the symmetrical copy of one side to another, we can get full connection of forward and backward integration on y-z motion map, as shown in figure-42.

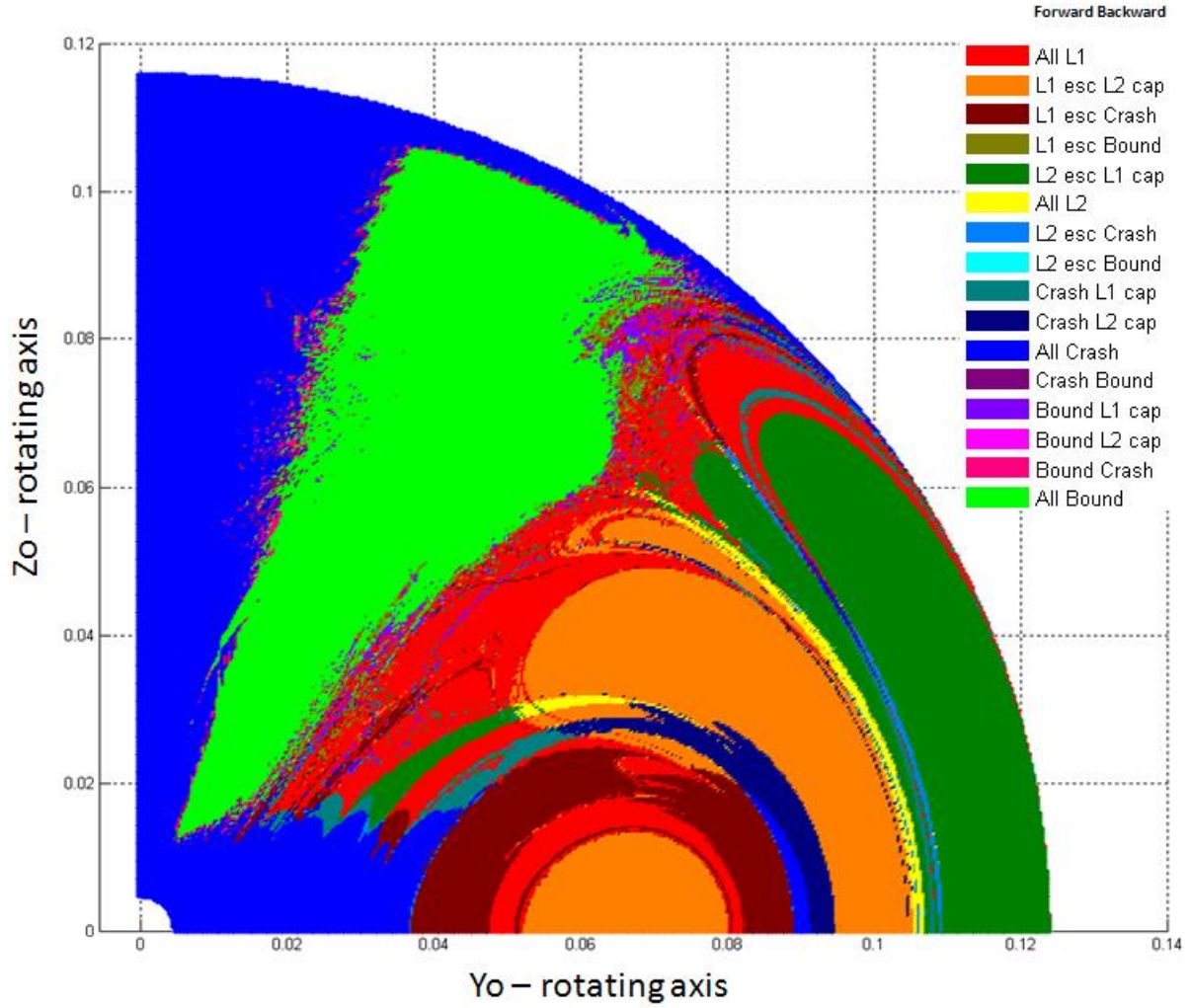


Figure 42 - Connection of forward and backward integration on y-z motion map. $C=3.16$.

So then, the backward and forward integration can be connected in one map to see all motion type; such as; some *capture-bounded* or L_1 *capture*- L_2 *escape* connections. In figure-42, the connection of the backward and forward integrations is shown in one map. In both forward and backward, there are 4 main motions: L_1 , L_2 escape/capture, bounded and collision trajectories. Combination of all this motion results 16 type of motion, figure-42. Such as red means that motion is escape forward and capture backward integration from both L_1 . Orange means motion escape to L_1 in forward, but captured from L_2 in backward integration. And 'bound' to escape or reverse regions can be useful for permanent capture, also see section 5.3.

In figure-43, the whole plane of X-Z motion map is shown. Left and right side of this plot integrated differently to forward time for direct direction. As we mentioned before, because x-z plane, or $y=0$ plane, is symmetric feature in CR3BP, the backward and forward time integration gives same result for this plane. As because, $z=0$ also symmetric in CR3BP, up and down side also symmetrically equal to each other. The symmetry is only broken in left and right side. Even the Jacobi constant is set as both L_1 and L_2 equilibrium points gate are opened in same size, L_1 escape (red) region cover much more area than L_2 escape (yellow) region, in whole map.

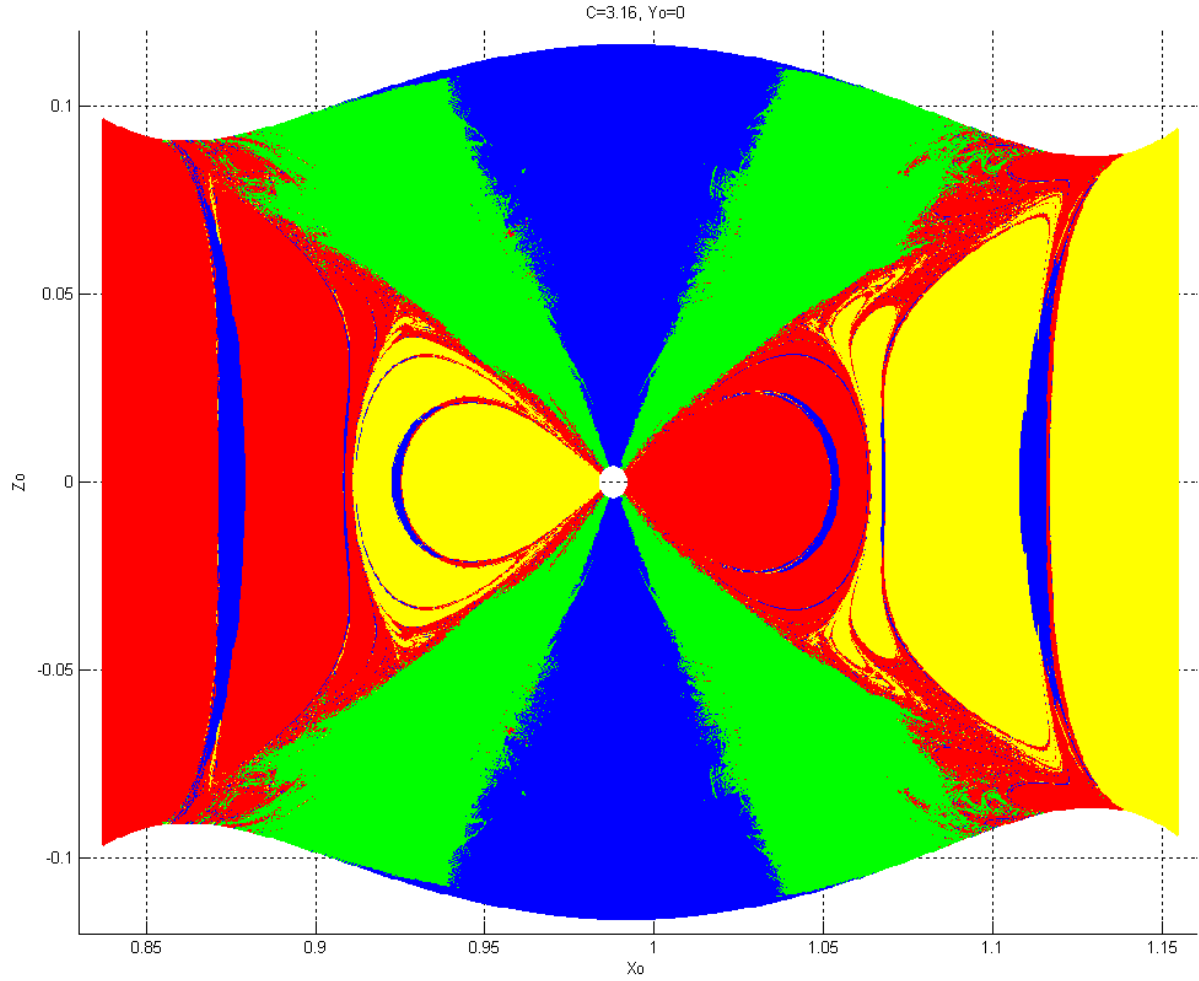


Figure 43 - Whole plane of x-z motion map, Collision, bounded, L_1 escape and L_2 escape trajectories are blue, green red, yellow, respectively.

To show this differences more dramatically, percentages of areas are shown in below table;

Table 4.1 – the percentages of areas of Y-Z Plane Map

	Left side of figure-43	Right side of figure-43
Blue Reg.	% 22.43	% 21.94
Green Reg.	% 26.15	% 26.96
Red Reg.	% 39.87	% 21.48
Yellow Reg.	% 11.54	% 29.61

This is because, as said before, the potential levels of L_1 and L_2 equilibrium points are not same, so the trajectory tends to escape from L_1 gate, rather than L_2 gate, even both gate are open equally in $C=3.16$, please see figure-40. Insomuch that, even the Jacobi constant, and energy level, is varied in different value, red and yellow area would appear in different shape, and again red area would be little bigger than the yellow area.

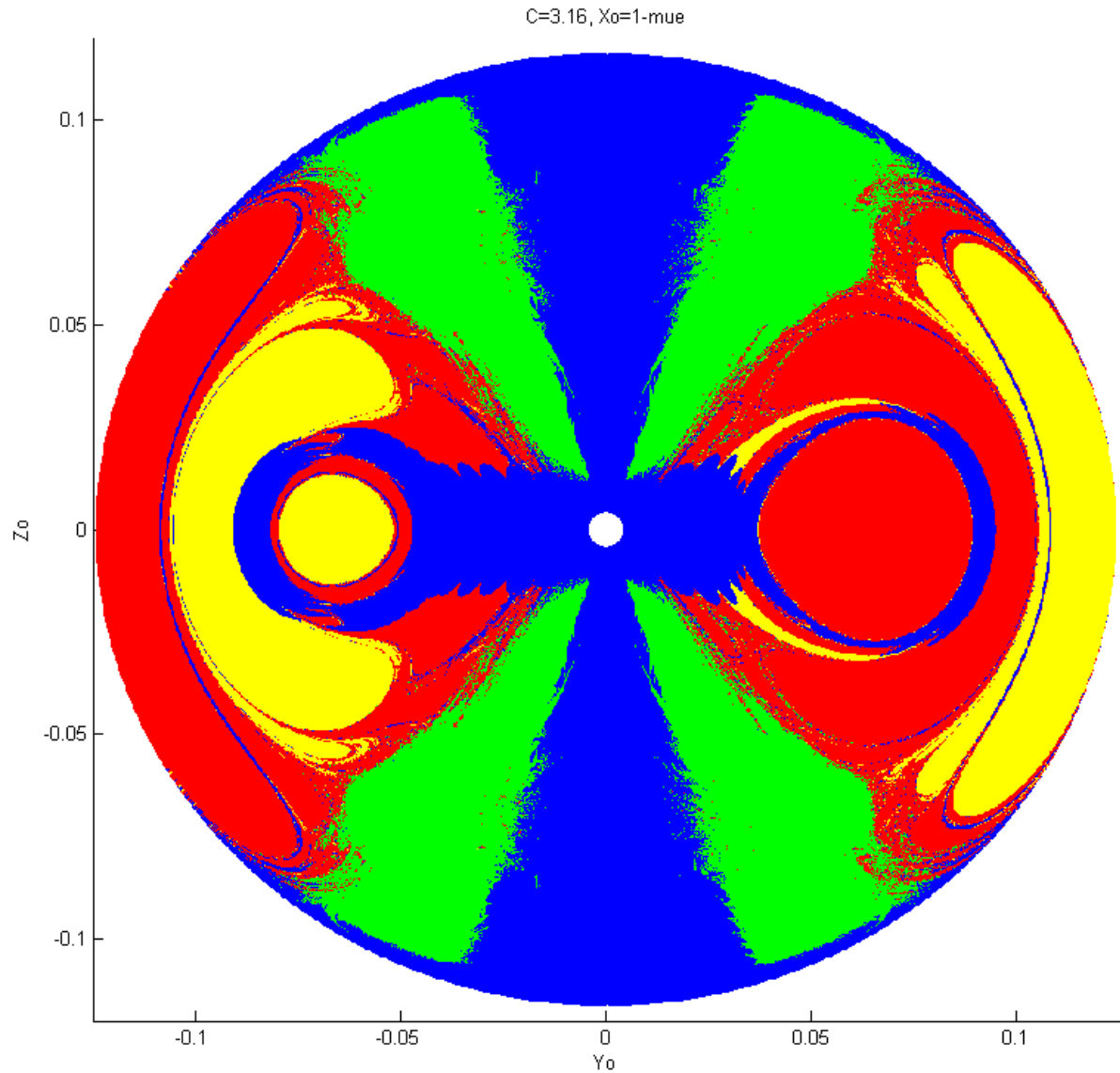


Figure 44 - Whole plane of y-z motion map

In figure-44, the whole plane of Y-Z motion map is shown in this time. Left and right side of this plot integrated differently to forward time for direct direction. Because y-z plane, or $x=1-\mu$ plane, is not symmetric in CR3BP, the backward and forward time integration does not give same result for this plane, but the backward integration of left side gives same result with the forward integration of right side, as mentioned just above. Also because, $z=0$ also symmetric in CR3BP, up and down side also symmetrically equal to each other.

Same as previous X-Z whole map, the percentages of areas are shown in below table;

Table 4.2 – The percentages of areas of Y-Z Plane Map

	Left side of figure-44	Right side of figure-44
Blue Reg.	% 31.63	% 28.82
Green Reg.	% 24.16	% 24.29
Red Reg.	% 28.03	% 32.27
Yellow Reg.	% 16.16	% 14.60

The configuration and initial features of Plane motion map are explained in this section. For further information please see integration process (section 4.3), fuzzy boundaries (section 4.4), bounded orbits (section 4.5). Also the mission application of the plane motion maps is presented in chapter 5. Please also see [MM\(0-360deg\) C 3.16.wmv](#) in appendix CD to see change of various plane angles in $C=3.16$ energy value for Earth-Moon case.

4.2.2. Map of Manifold Tube

As it is seen, the plane motion map is the simplest and easiest to visualize approach. However, there are also many, even maybe numerous, ways to choose or define a map plane according to specific types of mission. For example, in the CR3BP, the invariant manifold tubes are the most commonly used trajectories in space mission designs. Therefore, the “motion map” can be constructed around any invariant manifold tubes (or any other group of orbit we desire or chose) themselves for getting more practical mission designs.

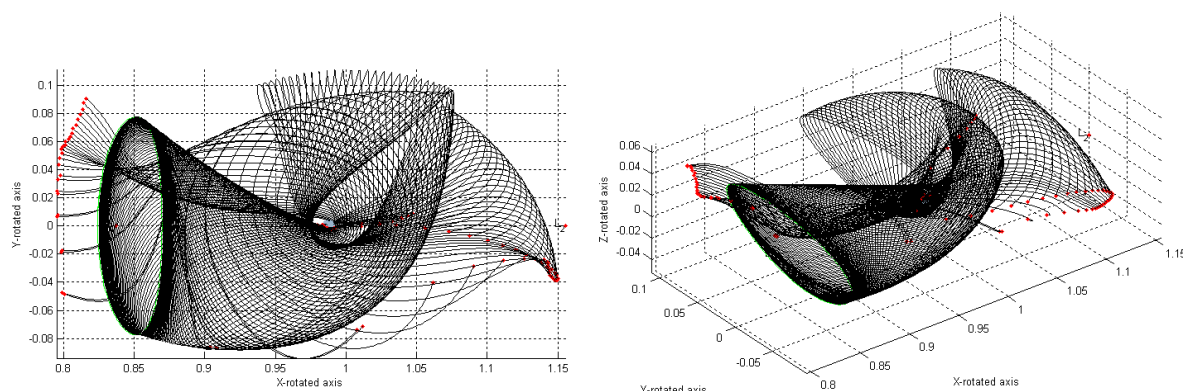


Figure 45 - Unstable invariant manifold tube of L_1 , $C=3.16$. Please see Matlab code, `IMT_capture.m`, in appendix CD, to see the plot code.

In figure-45, a wild unstable invariant manifold tube (IMT) of L_1 is shown, which might be actually any IMT of any equilibrium point. As explained in detail section 3.2.3.1, a manifold tube is obtained by disturbing a halo orbit around L_1 : the initial conditions of each trajectory that forms the tube is obtained by making a tiny disturbance to a halo orbit, at a particular phase. So the halo orbit is subdivided into a series of discrete points and each point is denoted by a trajectory number which is also called “manifold number” in this thesis for easiness. This will compose one strand of the manifold tube. In the example shown in figure-45, the halo orbit is divided into ~ 200 points, and so the tube is made up of ~ 200 strands in that plot. Every trajectory is assigned as a different number, as the manifold number, which is plotted along one axis of the motion map. To form the manifold tube each trajectory is integrated for 425 hours or until they collide with the Moon’s surface, or escape. So the other axis denotes the location along that trajectory. The trajectory is conveniently parameterised by time and a suitable unit is the integration ‘time step’ or ‘step size’. At any one of these time steps we can increase the velocity by a spontaneous ΔV , which causes a change in Jacobi energy.

In figure-46, we illustrate this change in trajectory by a ΔV manoeuvre. The position is fixed but the velocity is changed while remaining tangential to the trajectory causing the Jacobi energy to change from C_1 to C_2 . So every pieces of trajectory, small dots in figure, can be disturbance one by one with constant velocity change (ΔV). Position is already fixed, but magnitude of velocity is changed as tangential to trajectory, see eq. 4.1 and figure-46.

$$\begin{aligned} V_1 &= \sqrt{\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2} = \sqrt{2U(x_1, y_1, z_1) - C_1} \\ V_2 &= \sqrt{\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2} = \sqrt{2U(x_1, y_1, z_1) - C_2} \end{aligned} \quad (4.1)$$

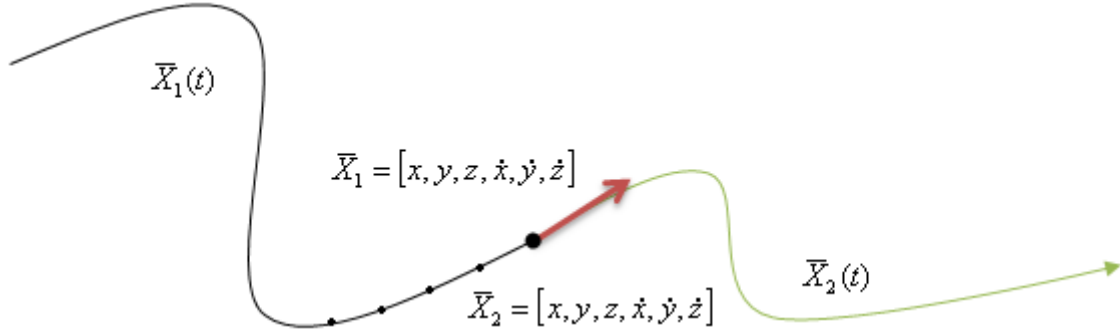


Figure 46 - Scheme for jumping one trajectory for another. Position is set fixed, velocity vector is set tangential to the trajectory but its magnitude is rescaled.

To maintain direction, the other components of the velocity vector can be calculated using a simple scaling method. The velocity increment (ΔV) is added to the final velocity and the scaling ratio k can be found in below;

$$\frac{V_1}{V_2} = \frac{V_1}{V_1 \mp \Delta V} = \frac{\dot{x}_1}{\dot{x}_2} = \frac{\dot{y}_1}{\dot{y}_2} = \frac{\dot{z}_1}{\dot{z}_2} = k \quad (4.2)$$

The relationship between the velocity and Jacobi constant changes is given by eq. 4.3 ;

$$\Delta V = \sqrt{2U(x, y, z) - C_1} - \sqrt{2U(x, y, z) - (C_1 + \Delta C)} \quad (4.3)$$

The motion map then plots “time step” against manifold parameter. For each step a fixed (constant) ΔV is applied at that time step on that trajectory. The new trajectory is then integrated forwards in time for a fixed time, 1.19 years in this example, and then classified as bounded (green), escape through L_1 (red), escape through L_2 (yellow), crash into the Moon (blue), as usual.

In figure-47, the motion map for invariant manifold tube is shown for zero velocity increment, and right side of figure show the projection on Halo orbits of L_1 equilibrium points of Earth-Moon binary system. As it is seen, the map in left side of figure-47 is actually a cylinder, because halo orbit is a periodic orbit. Also, for Halo orbits in figure-47-right, it makes about ~ 12 day for one period. This means that, 100^{th} trajectory comes from 6^{th} days, half period of halo orbits.

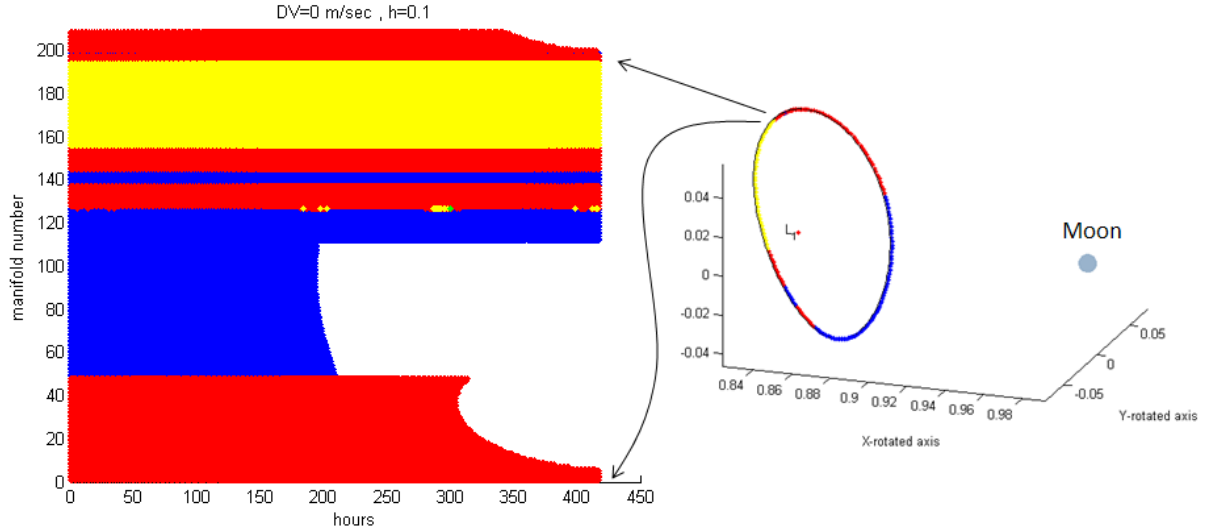


Figure 47 - Motion map for manifold tube for “zero” velocity increment. Collision trajectories are blue, L_1 escape trajectories are red, and L_2 escape trajectories are yellow. $T=100$

Also another note that the initial point depends on the way the manifold is parametrized (the epsilon in eq. (3.36)), whose value may vary depend of manifold. Also as mentioned just before, the Halo Orbit is divided from average ~ 200 points, which would use as initial conditions of Asymptotic orbits (see section 3.2.3). Therefore there would be average ~ 200 Asymptotic orbits, which are also called “manifold number” in this thesis (for easiness) and altogether form Invariant manifold tube in figure-45. But for more sensitive computation, number of points, that divide the Halo Orbit, can be increased, of course.

As there is no velocity increment in figure-47, the “final result” of each trajectory remains the same all the way along the trajectory. In fact all boundaries must be absolutely horizontal in theoretically, but there may be some anomaly in practical computation because of numeric errors in this step size. Such as the only anomaly observed is between the 125th and 130th manifold numbers, because these trajectories approach too close to the singularity of the secondary body, which cause some numerical errors in integration, so the trajectory integration become very sensitive in very vicinity of secondary body.

Thus because there no applying velocity change in figure-47, so there is no green region. This means that there are no trajectories that unassisted will remain in the vicinity of the Moon over the time period considered. But in figure-48, the motion map for this manifold tube is shown applying different velocity decrements from 10 to 40 m/sec. Even at the smallest velocity change some small regions of green stable motion appear. Although they seem very small and therefore may be hard to catch, the timescale is much longer out at the point. To traverse the small green regions takes about 30 minutes, that would be the window in which the maneuver must take place. For 20 m/sec, the maneuver window has increased to ~ 10 hours, for 30m/sec it becomes ~ 30 hours, and for 40 m/sec, the manoeuvre time can be extended to more than ~ 50 hours. It is seen that even for very small changes in velocity in the appropriate window, stable long term motion around the Moon can be achieved. Also note that, same anomalies as in figure-47 (left), may occur some boundaries of colour region, where trajectories close to secondary too much. Thus, we should always keep in mind that those boundaries may not be so reliable.

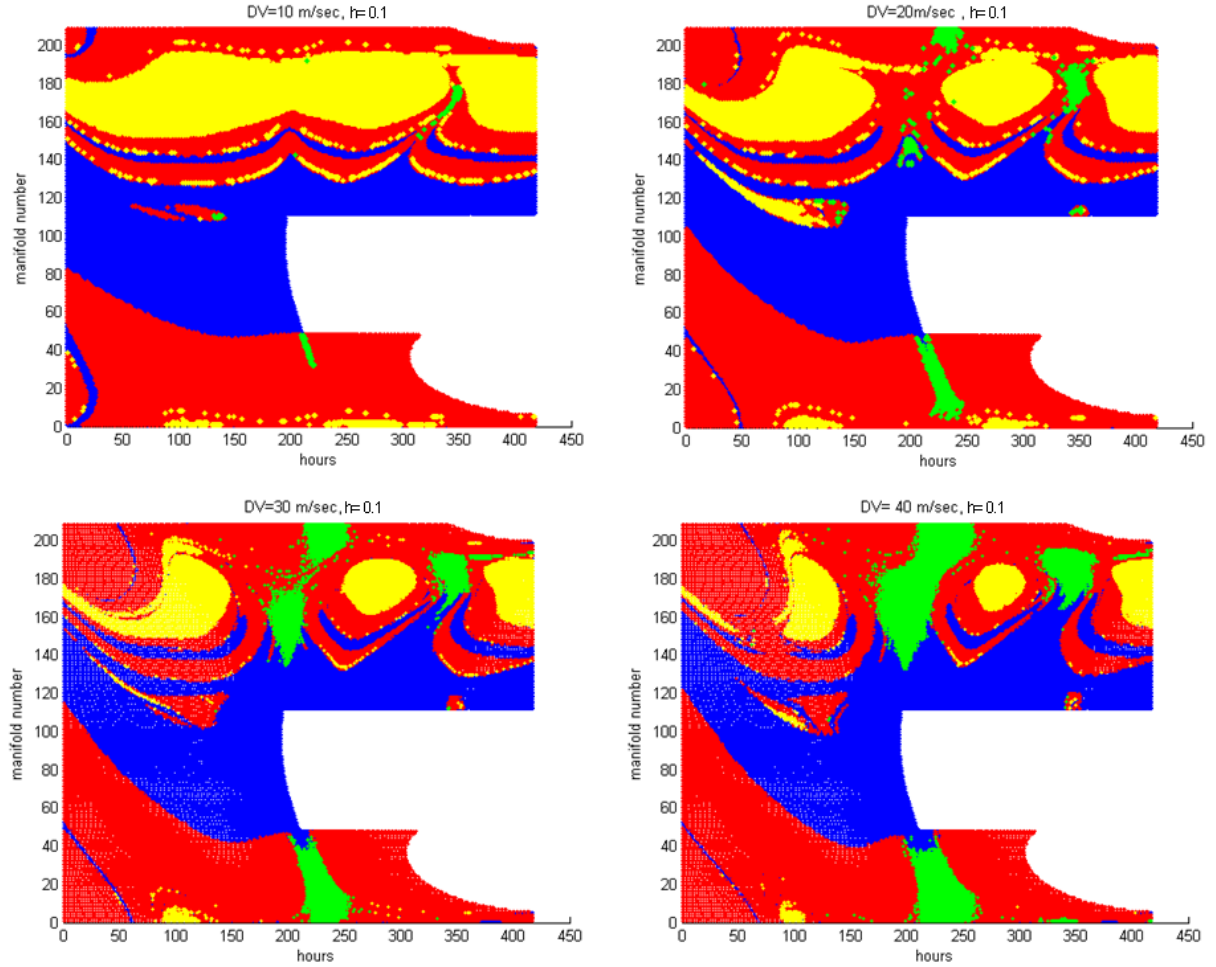


Figure 48 - Motion map of manifold tube in different velocity increment.
Collision trajectories are blue, L_1 escape trajectories are red, L_2 escape trajectories are yellow, bounded trajectories are green. $T=100$. Please see [IMT_MM_variousDV.wmv](#) in appendix CD to see change of various velocity increment.

Another interesting result from these figures is that the green regions are located in the close vicinity of the secondary body. This means that the trajectories can be stabilized by applying the ΔV at periapsis passage. This is demonstrated in figure-49, which shows that green region in figure-48 is reflected on the manifold tube as green dots, and so these green dots show the places to make velocity decrements. Such as for $\Delta V = 0$ m/sec, there is no any green dots, of course. But for $\Delta V = 10$ m/sec, some small “green dots” area appear in very vicinity of secondary body. For $\Delta V = 20$ m/sec, this area of green dots expand, and finally it cover very huge areas for $\Delta V = 30$ m/sec.

Furthermore, according to figure-48, $\sim 150^{\text{th}}$ and $\sim 180^{\text{th}}$ ‘manifold numbers’ are quite convenient trajectories for catching the low-cost stabilizing trajectories, because ‘4 main types of motion’ can be approached for these area. In the middle of the mission, the task purpose might be changed, and spacecraft can be directed to different directions all with low cost. As an example, for the 180^{th} manifold number, with $\Delta V = 10$ m/sec in the 370^{th} hour, or $\Delta V = 20$ m/sec in the 200^{th} hour, can change the trajectory destination from L_2 escape to L_1 escape, see figure-48. Or, considering the 150^{th} manifold number, a $\Delta V = 10$ m/sec in the 200^{th} hour the trajectory destination changes from L_1 escape to collision with the Moon, for example. Also, this type of map can applied to any group of orbit as in this example.

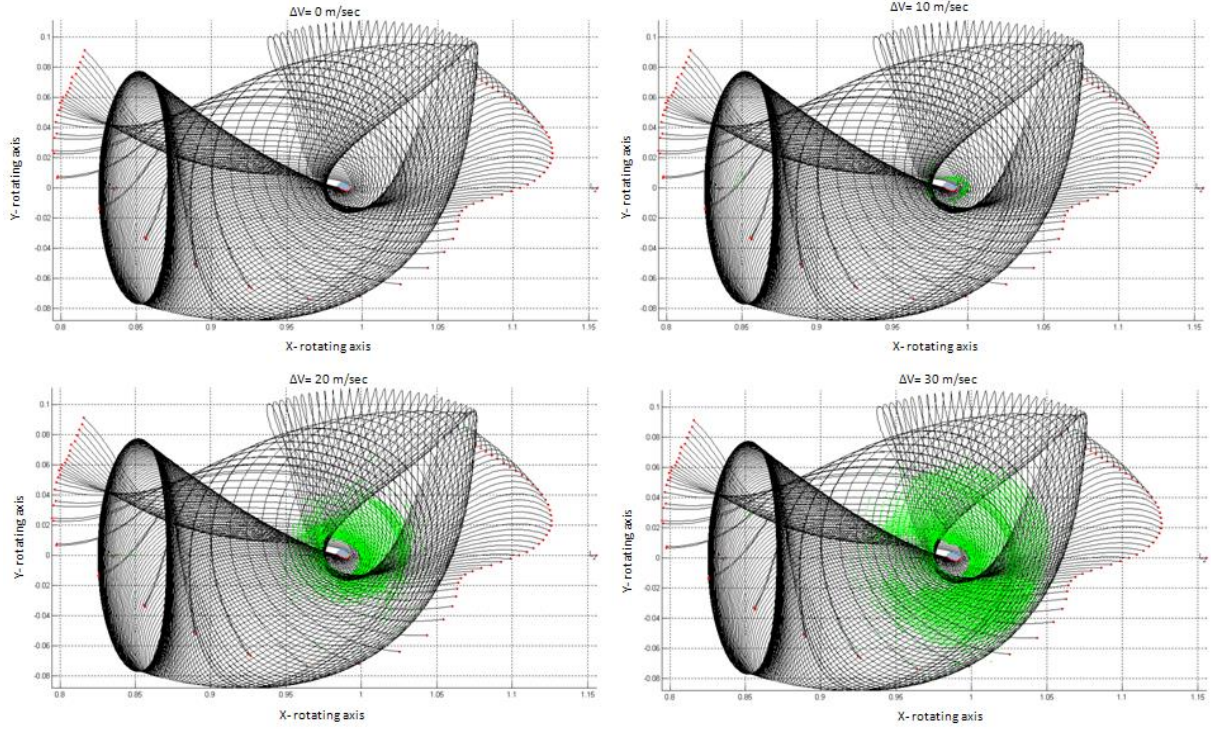


Figure 49 - Manoeuvre coordinates in position space with change of velocity increment

4.2.3. A Map Configuration on a Random Trajectory

In previous two section (section 4.2.1 and 4.2.2), the motion map is plotted for plane configuration and as projection of invariant manifold tube. But there are much more type of trajectories, which are highly chaotic and/or can not be included in any trajectory group as shown in previously. However, the motion map can be also designed with along single trajectory, assigning one axis as velocity change, to see capture possibilities more clearly. So, it is needed to design a new type of map configuration that can be applied any random trajectory and any random mission purpose, for more flexibility.

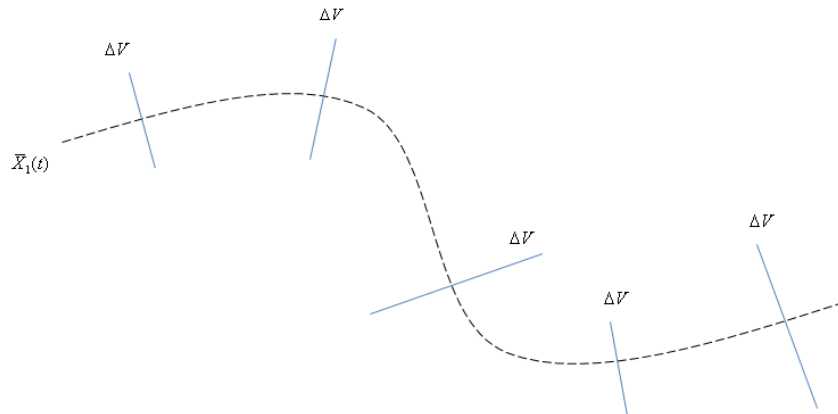


Figure 50 - Scheme for random trajectory map, one axis is velocity increment, another is trajectory along.

So for a wild trajectory, another motion map can be considered where the ΔV that is applied at a given time step is varied. In figure-50, a simple scheme is shown for a random trajectory map. As shown in previous section, in figure-46, one axis is assigned as the “time step” along the trajectory. And in every “time step” of integration, the velocity change is applied in specific range (such as from 0 m/sec to 50 m/sec for low-cost capture), and so the other axis of the map becomes the velocity change (ΔV). Then the trajectories are integrated with these new initial conditions until a desired time (or crash to M_2 , or escape either gates), and the results are classified again with 4 main motions; bounded, collision, L_1 escape, L_2 escape. For demonstrating in the examples, the 150th and 180th manifold numbers are chosen from figure-48, because these trajectories can present all types of motion (bounded, collision, L_1 escape, L_2 escape) by changing ΔV .

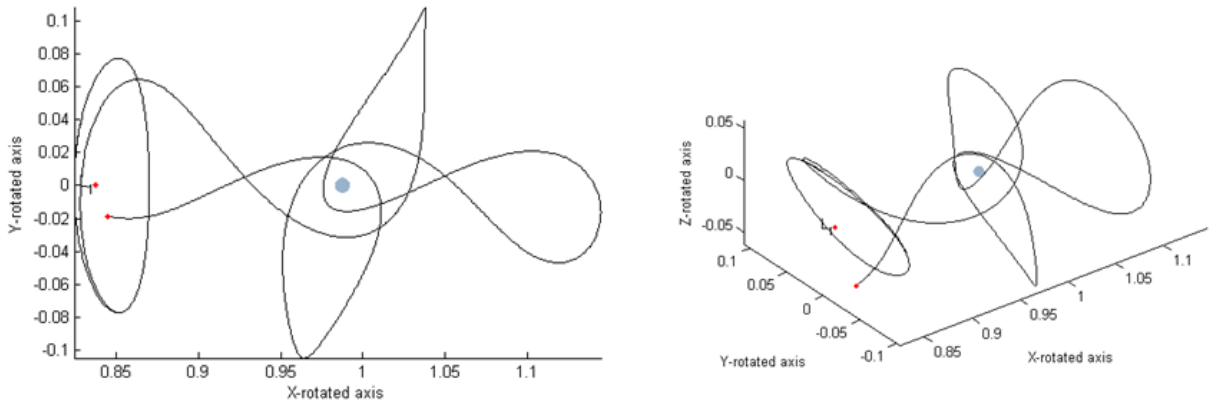


Figure 51 - 150th trajectory escape to L_1 from invariant manifold tube.
 $C=3.16$, in projection in x-y (left), and in x-y-z (3D) (right).

In figure-51, as an example; 150th trajectory is chosen from invariant manifold tube (so 150th manifold number), which was plotted in figure-45. As it is seen the trajectory makes four periapsis passages around the secondary body (Moon), then after the ~681 hours (28.37 days), it escapes from L_1 , if there is no any velocity change.

In figure-52, the motion map for 150th manifold number, which is seen figure-51, is shown. The vertical axis is velocity change (ΔV) and horizontal axis is “step number” for upper plot and time (hours) for below plot. If there is no any velocity decrement (ΔV) on 150th manifold number, it will escape thought to L_1 gate. But if some velocity change (ΔV) is applied up to 50m/sec in chosen time (hour) using this map, its route can be change to different direction. Such as, looking the map (figure-52-below), if it is applied 30m/sec ΔV in 100th hours on 150th manifold, then it escapes thought L_2 gate instead of L_1 . Or if it is applied 20m/sec ΔV in 200th hours on 150th manifold, then it becomes bounded. So looking the motion map in figure-52, which amount of ΔV must be applied in which hour can be seen very clearly for low-cost capture. Thus one can easily change the trajectory direction to desired direction in very low-cost looking this map. Also seeing the green region in this motion map, bounded trajectories can be easily founded; using the Velocity change axis (ΔV) of the motion map, any low-cost trajectories into bound orbit around a secondary mass with a very small velocity change can be found very simply.

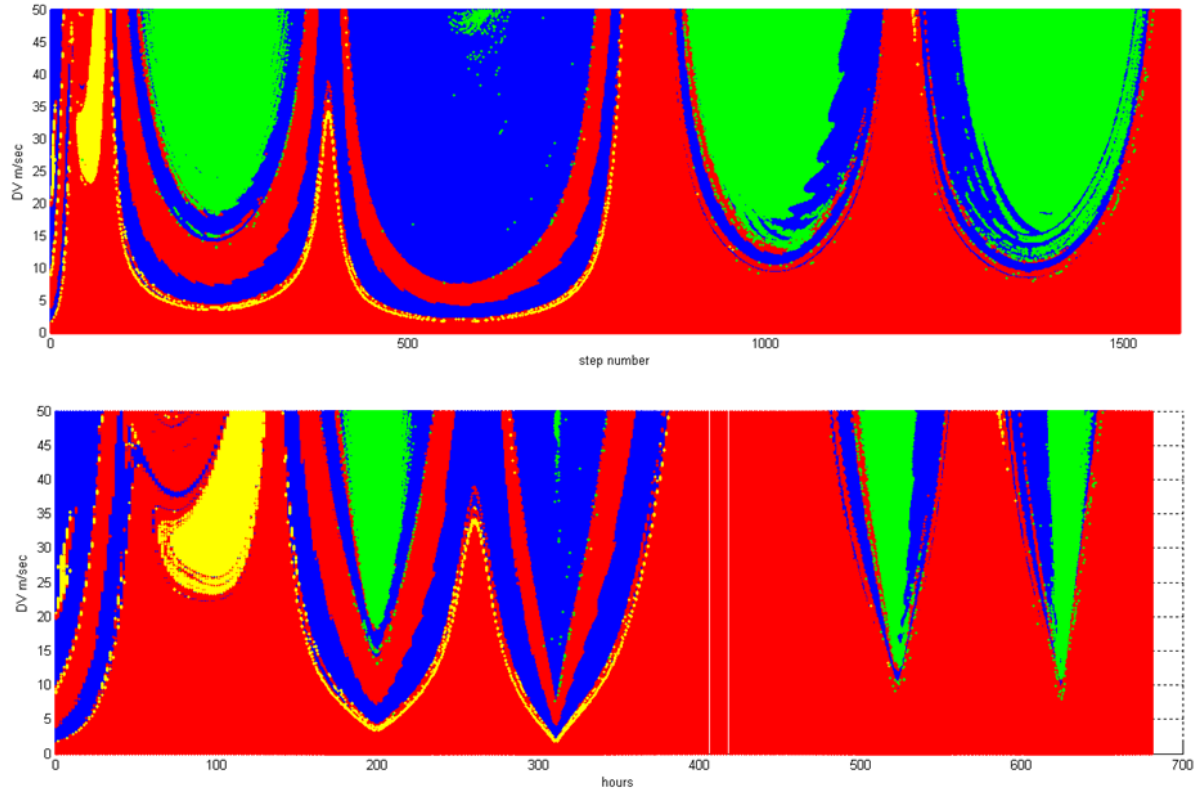


Figure 52 - The random trajectory map for step number (upper) and time step or hours (below), Collision trajectories are blue, L1 escape trajectories are red, L2 escape trajectories are yellow, bounded trajectories are green. $T=100$

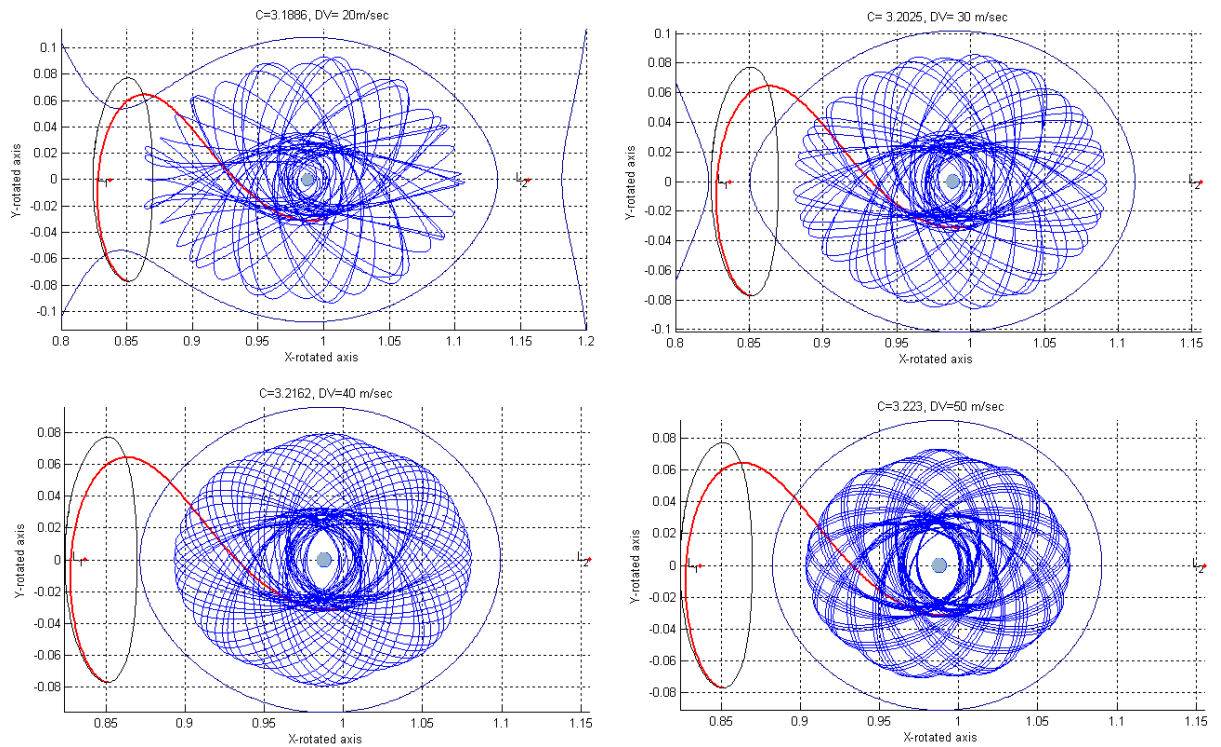


Figure 53 - Bounded trajectory after velocity change, in first “cup”, 200th hours. Minimum altitude 216 km, 1214 km, 2286 km and 3722 km, respectively.

Also as it is seen in figure-52, all four periapsis passages can be seen more clearly; they each have a distinctive “cup” shape. Such as, the green region of the first “cup” reaches about $\Delta V \approx 20$ m/sec as minimum velocity increment for stabilizing the trajectory. Therefore, while traveling the 150th trajectory of Invariant manifold, 20 m/sec velocity change in 200th hours is quite enough to stabilize the trajectory around secondary body, according to map in figure-52. Also, although the second “cup” is not suitable for getting into a parking orbit, the third and fourth “cups” again present opportunities for smaller velocity increments of $\Delta V \approx 15$ m/sec for stabilizing the trajectory.

The motion map plotted in “time step (hours)” (figure-52-below) actually gives much more insight. From this, we can see that in the first “cup”, the trajectory can be stabilized in a window of about ~ 1 hour manoeuvre time and $\Delta V \approx 19$ m/sec, while for $\Delta V \approx 20$ m/sec, the manoeuvre time increases to ~ 8 hours. Also the manoeuvre times are ~ 20 hours for $\Delta V \approx 30$ m/sec, ~ 25 hours for $\Delta V \approx 40$ m/sec, and finally ~ 30 hours for $\Delta V \approx 50$ m/sec. The same analysis can be done for the third and the fourth “cups” as well. Such as, figure-53 shows the final stabilised trajectories using the 200th hour to manoeuvre and with a 20 m/sec, 30 m/sec, 40 m/sec and 50 m/sec, delta-V respectively.

Also note that the minimum ΔV , needed to achieve closes the zero velocity surface (ZVS) and provides permanent capture, is $\Delta V = 28.50$ m/sec for this periapsis passage. So that, for a chosen Altitude-position (in figure-53) below band of “28.50” value in the map can be very useful to achieve low cost manoeuvres without need to close the ZVS (so, both L1 and L2 gates).

As we also mentioned before, this map can be also used for changing the direction of trajectory to any desired direction in the middle of mission. For example, obviously with $\Delta V = 0$ m/sec, the trajectory escapes from L1. And a transfer from L1 to L2 can be made by making a $\Delta V \approx 24$ m/sec velocity change in the 100th hour, or a $\Delta V \approx 5$ m/sec velocity change in ~ 200 th hour or a $\Delta V \approx 3$ m/sec velocity change in ~ 310 th hour. The other combinations can be processed using the same method.

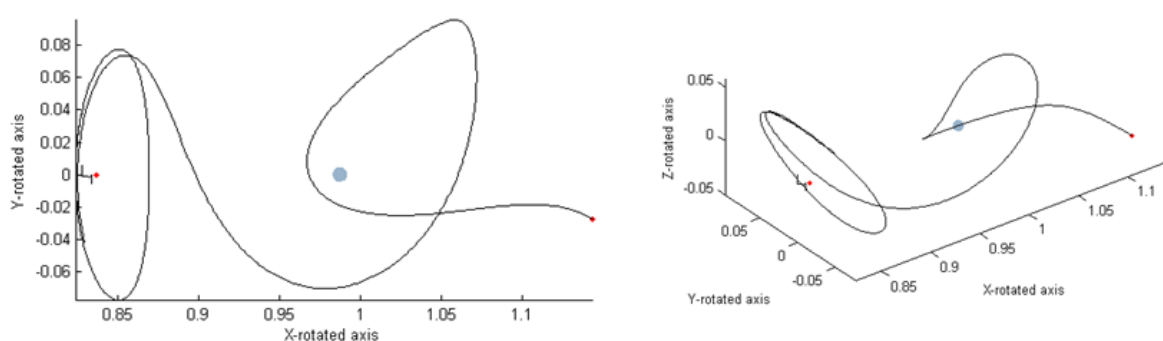


Figure 54 - 180th trajectory escape to L2 from invariant manifold tube.

Another example can also be applied to the trajectory in figure-54, which comes again from the invariant manifold tube in figure-45, 180th manifold number. As can be seen the trajectory makes two periapsis passages around the Moon, then after 417 hours (17.37 days), it escapes from L2.

Applying the same process that is explained in the previous example, the motion map for this trajectory can be plotted as in figure-55. Because there are 2 periapsis passages, there are 2 main “green cups” for stabilizing the trajectory in parking orbits.

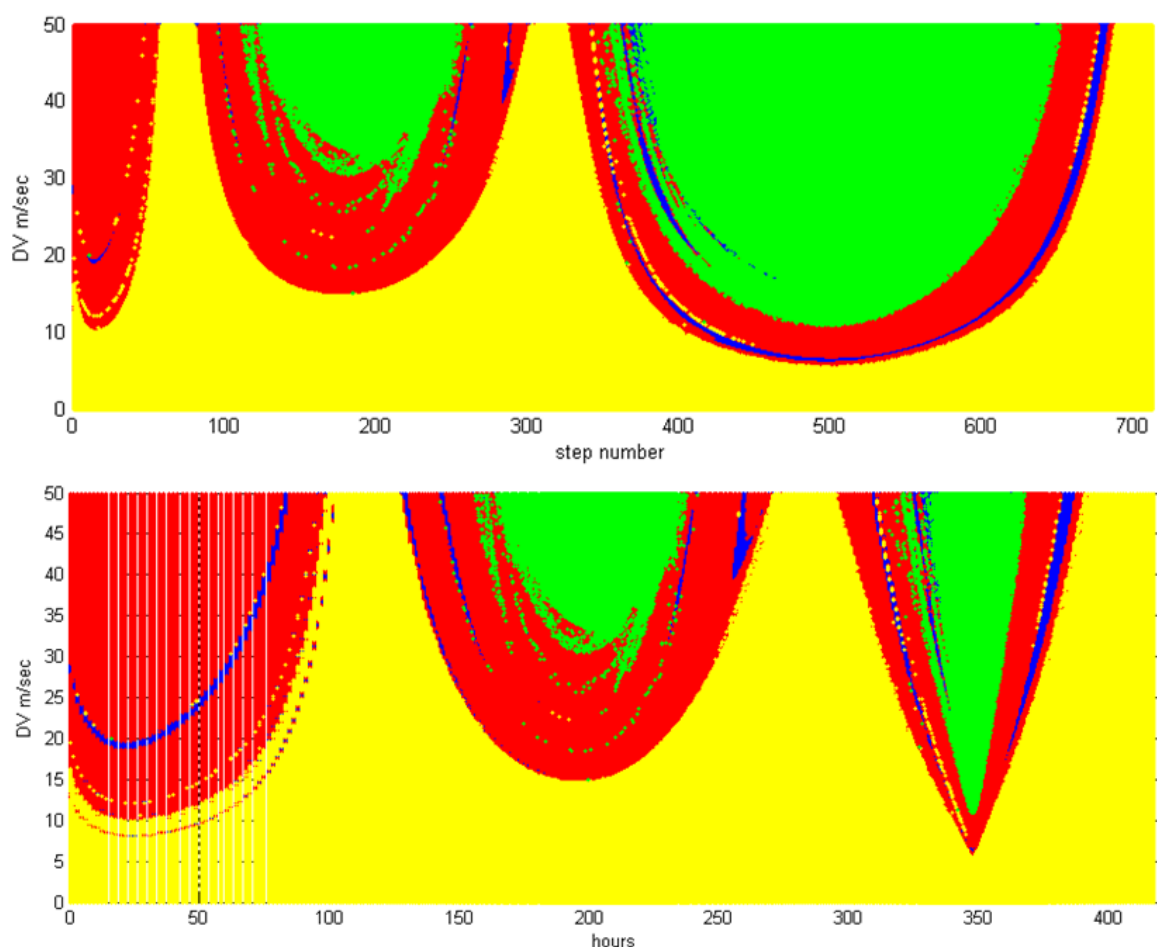


Figure 55 - The random trajectory map for step number (upper) and time step or hours (below), Collision trajectories are blue, L1 escape trajectories are red, L2 escape trajectories are yellow, bounded trajectories are green. $T=100$

As can be seen in figure-55, there are 2 main “cup” with green. The second “cup” presents a much smaller velocity change than the first “cup” for stabilizing the trajectory. While the first “cup” requires $\Delta V \sim 30$ m/sec for minimum velocity cost, the second “cup” gives $\Delta V \sim 13$ m/sec. As in the previous example, the smaller the velocity cost means comes with a reduced manoeuvre window. The manoeuvre time is ~ 4 hours for $\Delta V \sim 13$ m/sec and ~ 6 hours for $\Delta V \sim 15$ m/sec in the second “cup”. The bounded trajectories after the velocity increment is shown in figure-56, for 13 m/sec, 23 m/sec, 33 m/sec and 43 m/sec velocity cost, respectively.

Similar as previous example; the minimum ΔV , needed to achieve closes the ZVS and provides permanent capture, is $\Delta V = 21.828$ m/sec. So that, for a chosen Altitude-position (in figure 56) below band of “21.828” value in the map in figure-55 can be very useful to achieve low cost manoeuvres without need to close the ZVS (so, both L1 and L2 gates).

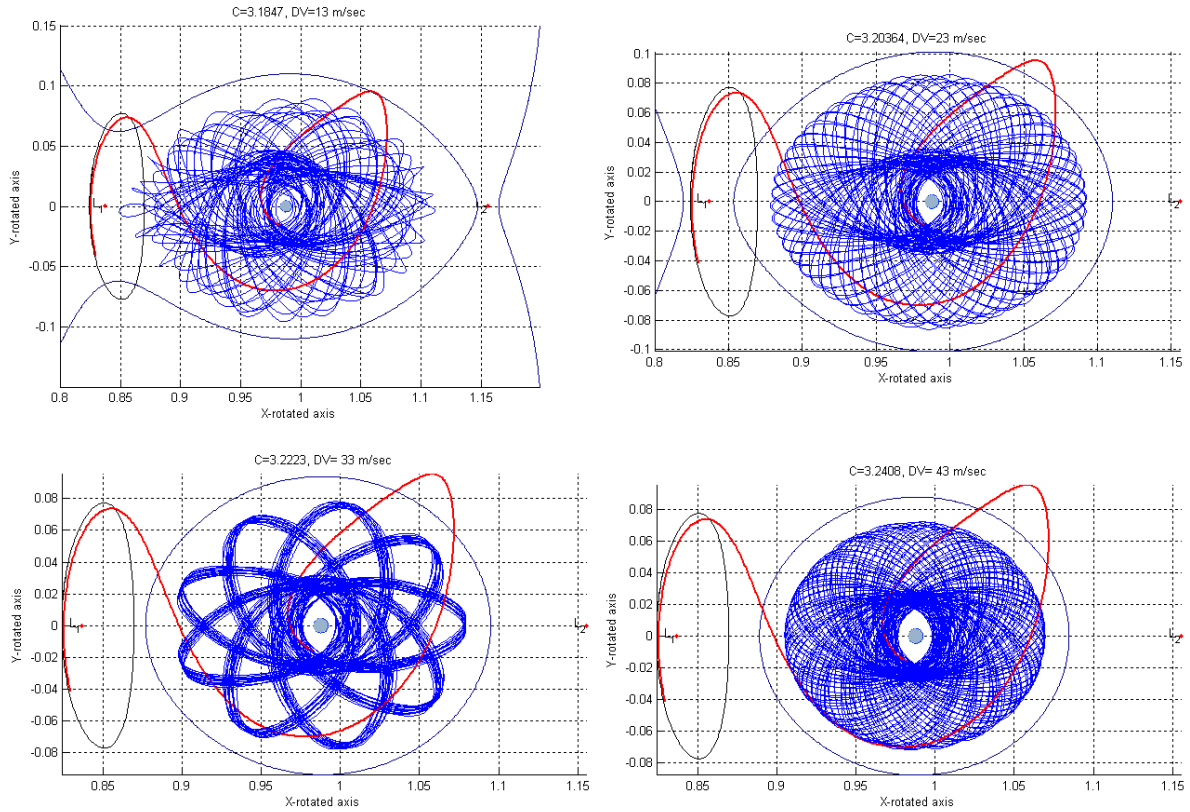


Figure 56 - Bounded trajectory after velocity change, in second “cup”, ~350th hours. Minimum altitude 948 km, 1320 km, 2120 km and 2541 km, respectively.

Again similar with just previous example, the direction of the trajectory escape to L2 (figure-54) can be changed in the middle of the flytime as it is explained previously. For $\Delta V \sim 10 \text{ m/sec}$ in the 25th hour, $\Delta V \sim 15 \text{ m/sec}$ in the 200th hour and $\Delta V \sim 6 \text{ m/sec}$ in the 350th hour, the trajectory can be directed to L₁. Also the trajectories can be crashed to the secondary body (moon) determining the suitable manoeuvre time and velocity cost in these type of maps.

Also last note that, upper these explorations correspond for only to point number 150th and 180th, which could be also any random trajectory around M2 region. “Map configuration Method for a random trajectory” should be only applied for a specific chosen or certain trajectories, and while in the middle of space mission; such as either to change the trajectory direction or capture for low-cost. Because while spacecraft is in the real space mission, there would be only “one trajectory”, which the spacecraft is on, and it may be hard to adapt this trajectory on any Map of trajectory group as shown section 4.2.1 and 4.2.2. So this map technique can be simply applied to directly spacecraft’s current trajectory for determining more flexible task options that could be achieved with very low-cost, than trying to fit to any other map restrictly. But in case of massive and/or generic explorations, this map technique would not be practical in side of the amount of information computed and stored (it is obviously impossible to compute all single random trajectory in CR3BP).

So, Random trajectory map method should be applied to only certain trajectory, which is used in current in mission. And Plane (or line) configuration (in section 4.2.1) and Map of Manifold Method (in section 4.2.1) should be used in phase of space mission design, for choosing suitable trajectory in a mission design. Also for massive and/or generic explorations, plane and line

configurations (and any other similar configurations) can be varied more, and possible adaptive map options for any wild trajectories can be reproduced. Furthermore, Random trajectory map can be also used collaborating with other relatively generic map methods. Such as, in the beginning of a mission design, a suitable trajectory, which may be used for a capture mission plan, is chosen or picked from a genel map (such as from X-Z Map), then Random trajectory map method is applied on to this chosen trajectory to determine the cases the spacecraft can do with low cost on this trajectory.

4.3 Integration of Motion Map

The efficient and accurate computation of the trajectories in the Motion Map is of great importance due to the large numbers of trajectories required, especially some of which pass close by the secondary. The integration method used in this research and found of best performance was a geometric integration scheme based upon a splitting method. The splitting method uses the fact that the Hamiltonian that describes the dynamics is the sum of a kinetic energy part that only depends upon the momenta (velocities) and a potential part that only depends upon the coordinates. Propagation is performed in a leapfrog scheme where each of these two parts is propagated forwards in time as if they were the entire Hamiltonian for the system. The advantage of the scheme is that it avoids any secular growth in the Jacobi constant along the integrated trajectory and preserves the phase space structure.

The implementation of the integrator a Time Transformed Leapfrog (TTL) scheme is employed as detailed in Mikkola and Aarseth (2002), [95]. This is combined with a sixth order symplectic leapfrog integrator. The TTL scheme requires a little extra implementation effort, but offers computational and accuracy benefits. Geometric and symplectic integration methods are well documented in the literature, please see [96,97].

According to this integration method, the escape and collision trajectories are computed for shorter lengths of time and so are more accurate. Numerical errors are expected to be largest in the green region of Motion Map where the integration continues until the most final time. In this research, some tests have been run with very long integrations of these trajectories and compared them with a classical Runge-Kutta integrator. As a result, for the Earth-Moon system, a reasonable agreement could be found between the two integrators for up to 1.5 – 2 years or about 100 in dimensionless time; which is gives same result for both integration method. The integrations were also repeated for some single trajectories and found no qualitative differences. The symplectic integrator uses a transformation of the time and stepsize. The compromise to provide sufficient accuracy and speed dictated a stepsize can be maximum $h=1$, which means for the Earth-Moon system that the average timestep is ~ 0.07 days (1.68 hour) with adjustment.

In figure-57, an example trajectory chosen from the “Fuzzy boundary” region (between red and green areas of the Motion Map) is plotted. Three different trajectories computed using three different stepsizes are shown: for $h=1$, 0.1 and 0.01 (corresponding to 1.68 hours, 10.08 minutes and 60.48 second respectively). The initial conditions for this trajectory were: $X=0.95$, $Z=0.03$ in the X-Z Motion Map (figure-38) which is plotted for $C=3.16$. Because this is an escape trajectory, the spacecraft escapes after about 21 days. This shows there is very little difference between these computed trajectories, and it don't effect much the outcome for the Motion Map. This shows that for the purposes of computing the Motion Map, a stepsize of $h=1$ is entirely adequate.

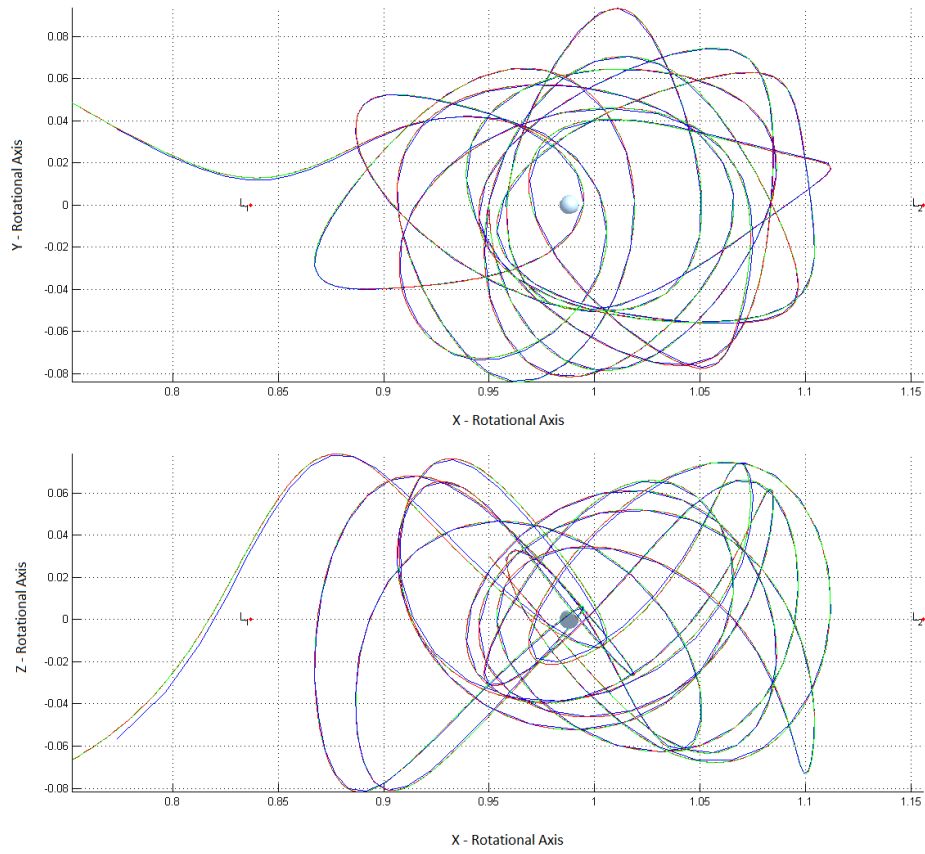


Figure 57 - An escape trajectory seen in two projections computed using three different stepsizes. Blue is $h=1$, red is $h=0.1$, green is $h=0.01$ in symplectic integrator

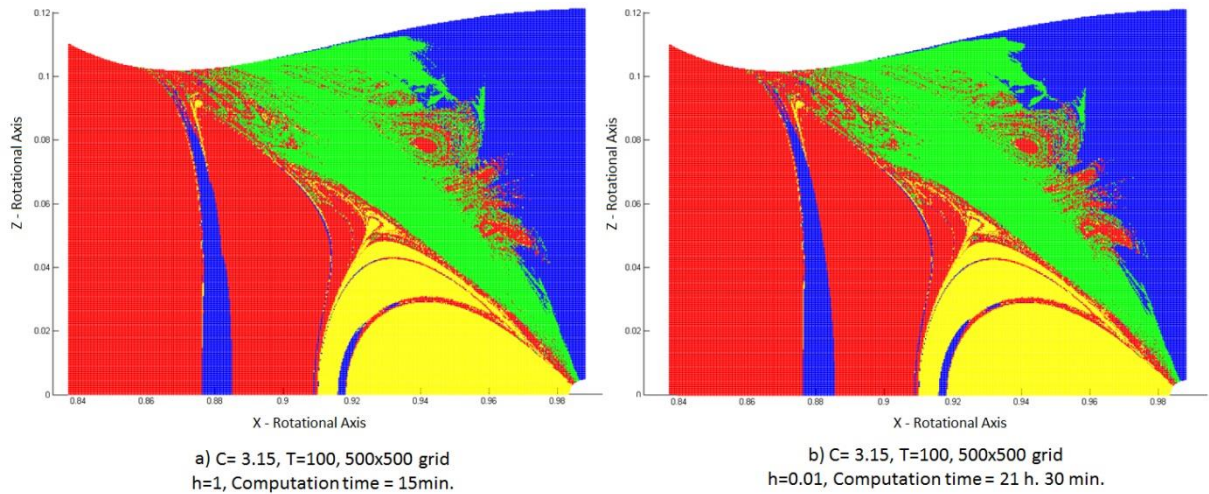


Figure 58 - Two computations of the Motion map using the same conditions but with $h=1$ on left and $h=0.01$ on the right.

In figure-58, the complete Motion map is presented using the stepsizes of $h=1$ and $h=0.01$. This shows that there is no appreciable difference between them so that a stepsize of $h=1$ is really adequate. To compute the Motion Map with $h=0.01$ took 21 hour 30 min on a desktop computer with Intel(R) Core(TM) i7 CPU M 640 @ 2.80 GHz processor while for $h=1$ the map was computed in only ~ 15 min with same computer. Therefore we don't have to waste our time to compute the map in "over accuracy", because there are too tiny differences only around boundaries of them. Also note that, the computational time does vary with Jacobi constant; because for high C (smaller energy) there are more bounded trajectories, and so more trajectories that must be computed to final time. Whereas there are less bounded trajectories in high energies, and most of trajectories are either escape/capture or crash much before than the final computation time. Such as; the computation time using $h=1$ varied from ~ 3 min. for $C=3.12$ to ~ 30 min. for $C=3.20$. So in short, integrations can be done quite fast and accurate with this integrator at stepsize of $h=1$.

4.4 Fuzzy boundaries on Motion Map

However, the motion map is computed numerically, and therefore long integration times can cause the build up of some numerical error, especially in some particular regions called "fuzzy boundaries" surrounding the green region. These regions become highly chaotic and scattering with increasing the integration time. Actually, this "fuzziness or scattered effect" in boundaries can be addressed also in terms of the role of manifolds, which separates transit and non-transit trajectories, see section 3.2.3. Many times transverse intersections of manifold transversal again govern these phenomena in a long time span, and so capture time span of trajectories can be longer and converge naturally permanent capture phenomena in the boundaries between Green (bound, so non-transit) and Red+Yellow (Escape/capture, so transit). Related studies about multiply manifold intersections can be detail find in [88,89].

In fact, Term of Fuzzy boundaries has been already used by Belbruno for defining weak stability boundary [22] to find some temporary capture around secondary region. But in this research, fuzzy boundaries are literally fuzzy area where very very long term Capture trajectories (Green area sides of Red and Yellow areas in figure-37) that converge periodic orbits (Green areas in figure-37), occur. Fuzzy boundaries actually the most important phenomenon in any motion maps in this research: They can appear in any type of motion map. In fact, they cannot be only useful for promising many advantageous missions, but also they can be too dangerous and risky, if Satellites are left on their own around there, because of sensitive dynamics in here. But GNC (Guidance, Navigation, Control) maneuvers, which will be very small, can provide a cheap and robust control in these boundaries. Moreover chaoticity in there is good and can be used advantageously, since the control can shadow many different purpose trajectories at low cost. In addition, the fuzzy boundaries are quite straight cut in short term motion map computation, but their "fuzziness" are increased in long term motion map computation. So, because fuzziness of motion map depends on integration time, it is needed to determine a way for certain integration time span.

In short, if the total integration time is increased, trajectories that were near the boundary side of green region (where we call fuzzy region and boundaries) move into any other regions; the blue region (collision) or the red (escape L1) or yellow region (escape L2). And if we know the features about this phenomenon, we can use it for zero-cost capture for a restricted time span.

4.4.1. Effect of the Integration time

Therefore, the integration time is very important in motion map, because it affects the boundaries of the green (bounded) region and so fuzzy boundaries. For short integration times, the boundaries of this region are “clear-cut”; but for longer integration times, the green region starts to shrink and the boundaries become “fuzzy”.

Since very long term integration (longer than 434 days) of the whole map takes some computation time, a smaller zoomed in region of the motion map shall be plotted to illustrate the effect of increasing integration time. In figure-59, some examples of the motion map in restricted areas for various integration times are presented.

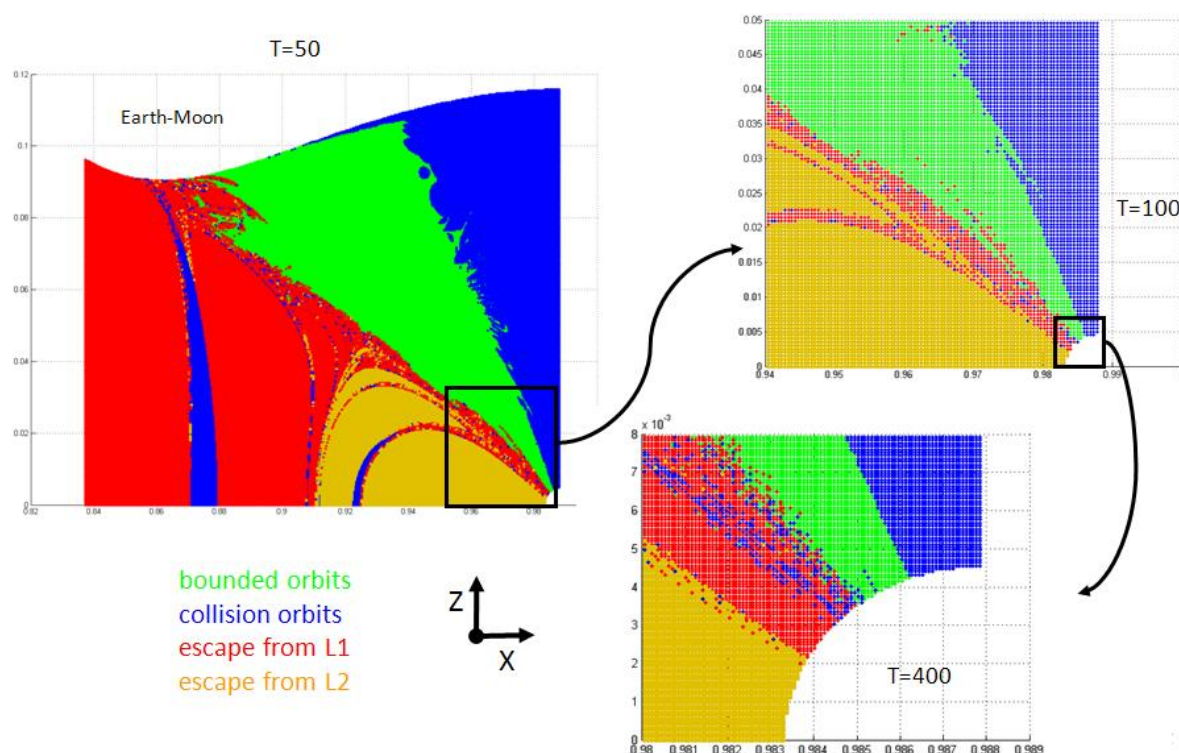


Figure 59 - Closing the fuzzy boundaries in the vicinity of secondary body.
 $T=50$ (217 days), $T=100$ (434 days), $T=400$ (1736 days or 4.75 years)

The first plot is calculated for $T=50$ (217 days). The second one is calculated in a restricted area with $T=100$ (434 days). The third plot is calculated for $T=400$ (4.74 years, more than suggested integration limit. It is exceeded to see fuzziness in boundaries) mission lifetime, and restricted to a very close vicinity to the moon. Therefore, if relatively short space mission which takes less than 217 days is designed, the first plot can be used. If the space mission lifetime is increased, so integration time increase, then computation of whole motion map takes too long time. Thus, for long integration time, the motion map can be computed in a restricted area instead of whole, as show in the second and third plots, so map computation time can be reduced. Note that, the third plot is calculated for $T=400$ (4.74 years) may not be so accurate in boundaries of green (bounded) and red (L_1 escape), of course, but the other region is quite accurate because of their end time is relatively too short. Third plot computed too long to show how “fuzzy boundaries” (which is explained in next section) appear when the integration time is increased.

To see more clearly how the boundaries of the green zone are affected by integration time, the third plot in figure-59 can be compared its twin which has been plotted for $T=50$, see figure-60. This plot shows the boundaries between the green and blue zones and between the red and yellow zones have nearly no differ (Because trajectories in red, yellow and blue regions mostly end off before $T \leq 30$), but the boundary between the red and green zones are strongly affected. This is because trajectories with the longest capture time are situated between the red and green zones. As the integration timespan is increased, the green region shrinks from the boundary with the red region, showing that the bounded trajectories eventually escape the Moon through the L_1 gate, sooner or later.

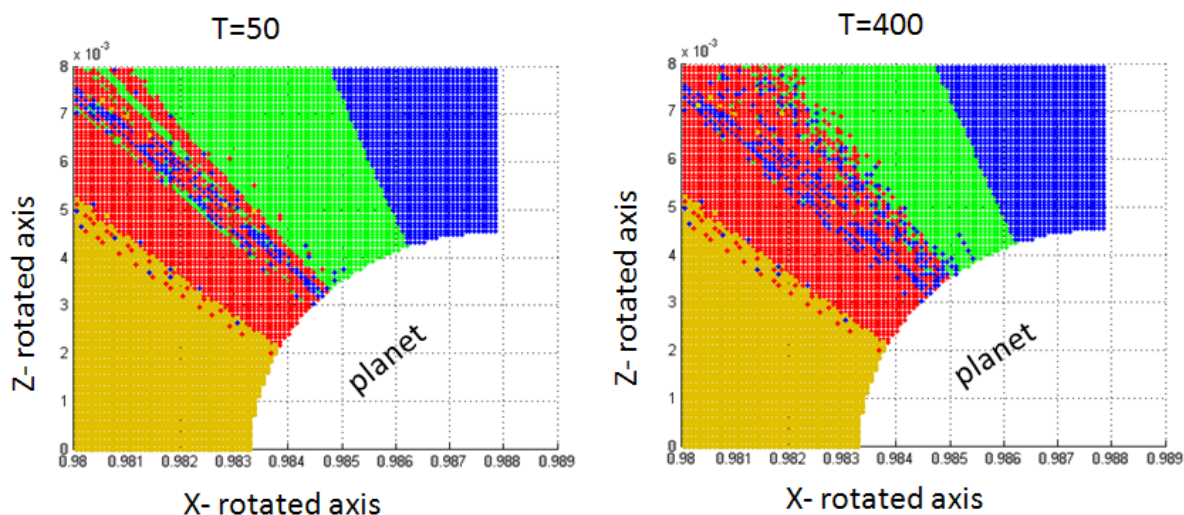


Figure 60 - Effect of Integration time in vicinity of secondary body (Earth-Moon system) $T=50$ (217 days) and $T=400$ (1736 days or 4.75 years), $C=3.16$.

Figure-61 also shows that this is a global result. Here the whole motion map for $T=500$ (5.94 years, more than suggested integration limit again (see section 4.3). It is exceeded to see fuzziness in boundaries) is plotted to see how “fuzzy boundaries” are effected in whole picture, while in the right plot is for $T=100$ (434 days) which is in right integration time. The bounded trajectories near the boundary of the green region start to escape over the longer timespan causing the green region to shrink from some sides. But also note that, as we can see that these shrinkages are not same ratio at every parts of green region; while boundary sides of green region are highly affected, this shrinkages can not reach to relatively middle parts of Green region. It means, even we could integrate the map infinite time, the shrinkages in green region would reach a limit and present robust Green part(s). So the integration time actually only effect certain parts called the fuzzy boundaries between the Green and Red+Yellow regions.

These plot, in figure-60 and 61, illustrate also how the mission lifetime is an important parameter in the mission plan for where in the motion map to target. And also which parts of green (bounded) region and also for other region are relatively unreliable. To see these unreliable parts, it is needed to simply detect differences between left (integrated in 434 days) and right (integrated in 5,94 years) plot in figure-61. The other stationary parts are relatively reliable that they don't have too big change in different integration time, as it is seen.

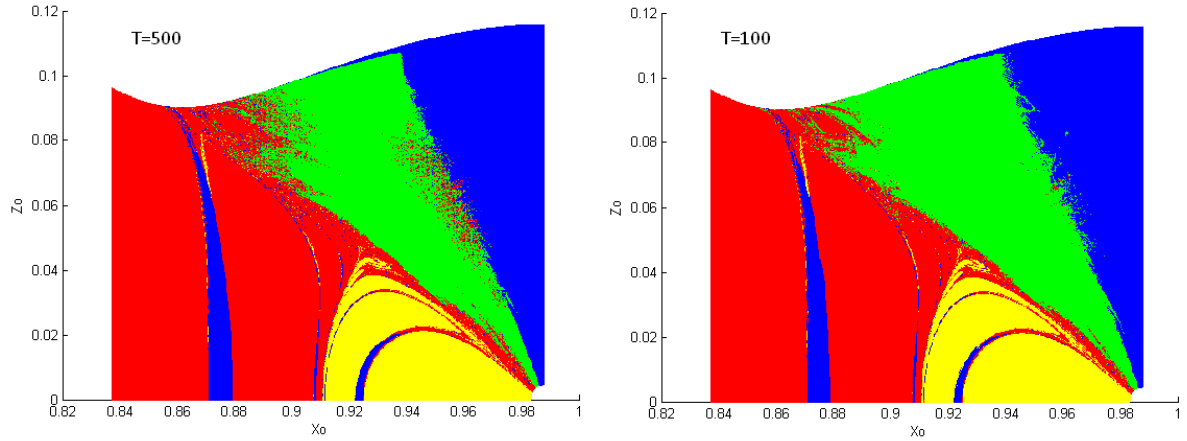


Figure 4.61 - Effect of Integration time in whole map, Bounded orbits are green, L1 escape trajectories are red, L2 escape trajectories are yellow, and collision trajectories are blue. $C=3.16$

In short, as it is mentioned and seen in figure-60 and 61, the increment of integration time span is only effected on specific regions in the motion map, and other regions remain the same. These regions can be simply called “fuzzy boundaries”, because they become fuzzy with high integration time. And because the time span of capture trajectories in there are quite long, “fuzzy boundaries” can be used for zero-cost capture.

Another perspective for fuzzy boundaries can be seen in figure-62, which is an example of fuzzy boundaries for Jupiter-Europa binary system. Left side plot show the x-z plane map with 3 main region intersections; Bounded, L_1 escape and L_2 escape. The blue regions in red and yellow region are called “crash waves”. The right side of figure-62 presents the third dimension extension as “final time” for L_1 , L_2 escape and crash region. The “waves” are seen much more clearly in that perspective. The escape time is increase approaching to boundaries of red and yellow. And the blue region creates some “step waves”. Also the final times go to the limit while approaching to green region.

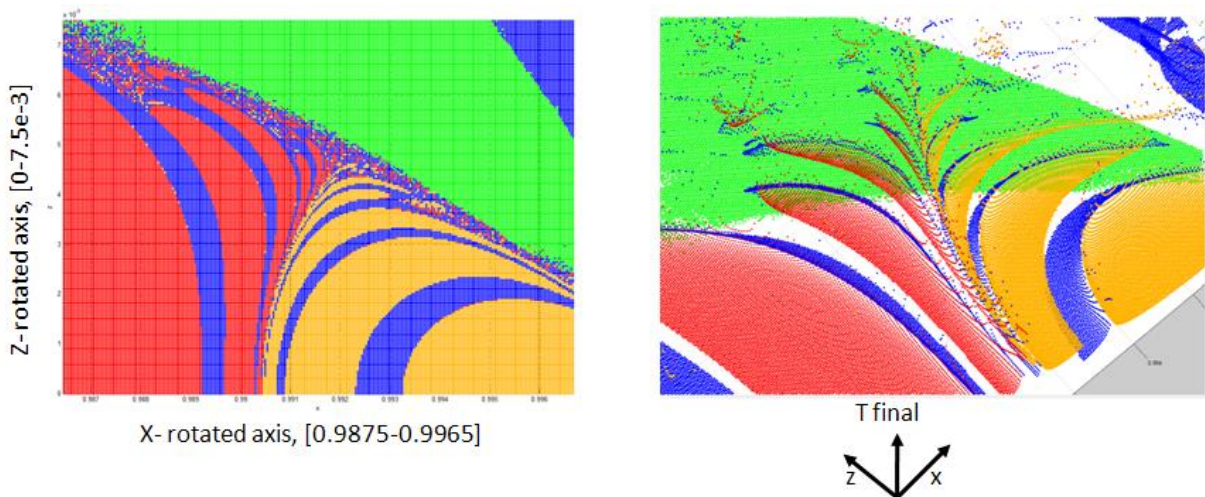


Figure 62 - Fuzzy boundaries for Jupiter- Europa binary system Collision trajectories are blue, L_1 escape trajectories are red, L_2 escape trajectories are yellow, bounded trajectories are green.

4.4.2. Zero Cost Capture

Although the “fuzzy boundaries” are so cloudy and sensitive depending on the integration feature, it may provide a map for zero cost capture and can be still used for practical space mission design using various ways, which are show in following section. In figure-63 we plot an expanded region of this boundary to see the level of complexity within this area of the motion map.

For optionally, we may just avoid complexity of the fuzzy boundaries directly. Such as, the initial condition should be chosen far enough from fuzzy boundaries. So trajectories can be obtained more accurately without any effect of numerical errors. And while going from one region to another, some energy (ΔV) change is made. This can be called “jumping process” which is jumping from one region to another, without touching the boundaries, especially fuzzy boundaries. This process does not give zero-cost capture but it can be simply provided with quite less velocity increment, which is explained in section 5.1.

On the other hand, we may increase the grid resolution in motion map same as in figure-63. With this process, the “fuzzy areas” can be seen more clearly. Also the “small or tiny” region is getting bigger, see figure-63. Therefore initial conditions can be chosen more clearly and so may be applying some GNC maneuvers, which will be very small, can provide a cheap and robust control around there.

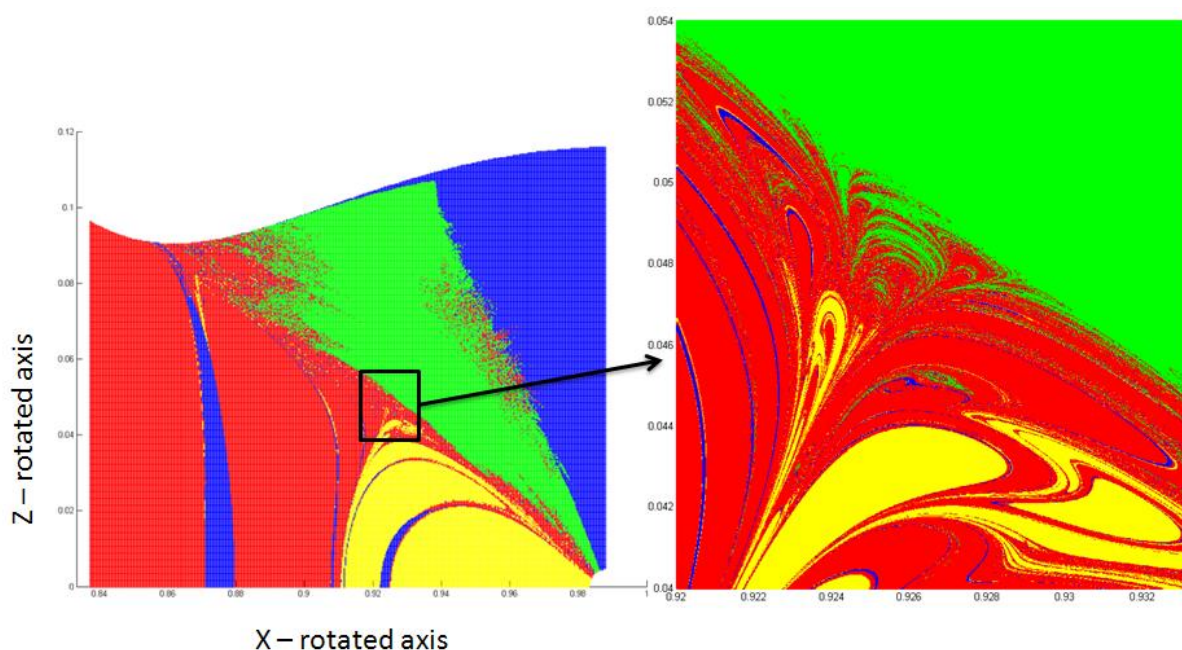


Figure 63 - Focus on fuzzy boundaries for Earth- Moon system.

Furthermore, since in X-Z Motion maps, the forward and backward integrations are symmetric, then in backwards time, trajectories that drift from green to red regions correspond to trajectories entering the region of the secondary from the primary and remaining in orbit about the secondary for a long period of time before escaping again to the primary. Such a trajectory, in the definition used here, is bounded in a restricted time. This is another example of ballistic capture, which is bounded in months or even years, as no propellant is required to keep the

spacecraft in orbit about the secondary after arriving from effective infinity. Therefore if a spacecraft has a mission, which must remain around secondary body only about restricted time, then this type of capture, which can be also called temporary ballistic capture with zero-cost, can be used.

Also note that Belbruno mentioned a permanent capture trajectory in his book [22] as one that is bounded in forward time but unbounded in backward time, or reverse. This type of permanent capture can be either converged or achieved from the Fuzzy boundaries of any type of Motion Map; Plane (X-Z and Y-Z), Line and any other configurations. For trajectories that converge to permanent ballistic capture may be required as small GNC manoeuvres can stabilise the trajectory, see section 5.1 and 5.3. In practice, space missions have a limited life time, and so zero-cost capture can be achieved over the lifetime of the mission. Subsequent escape from the region could also be considered advantageous in clearing regions of space debris. So that, the chaotic region can, however, show where trajectories of arbitrarily long capture can be found. In other words, the motion maps provide such a “cloud” for long term capture in its fuzzy boundaries. So then the capture mission can be done without any velocity cost, but in specific time span.

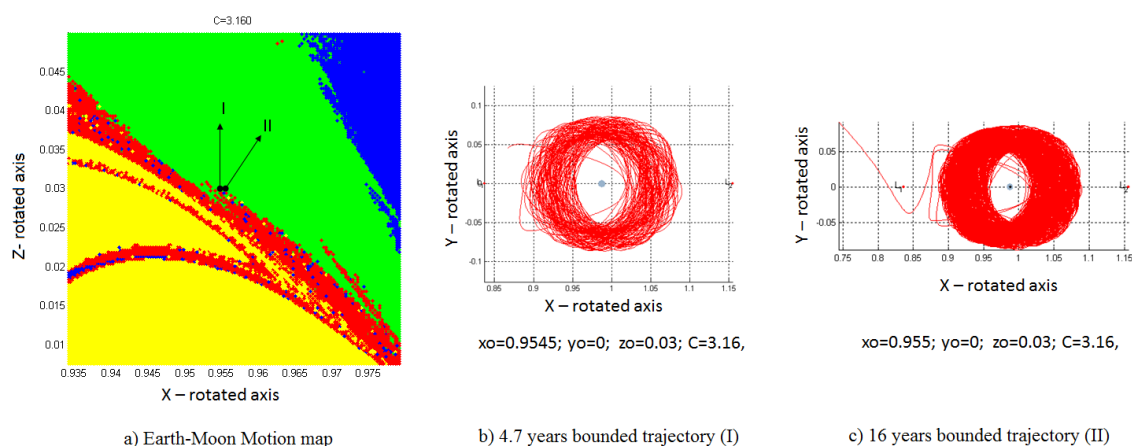


Figure 64 - Chosen the initial conditions for long term capture on fuzzy boundaries. Collision trajectories are blue, L1 escape trajectories are red, L2 escape trajectories are yellow, bounded trajectories are green.

In figure-63, an example of this way is shown as an example, considering two neighbouring trajectories in the region of this fuzzy boundary. They are marked as I and II in figure (a) having initial conditions of $[x_0=0.9545, z_0=0.03]$ for trajectory I and $[x_0=0.9550, z_0=0.03]$ for trajectory II. This corresponds to a physical separation of approximately 125 km. In figure (b) the first of these trajectories is plotted, which is a 3D trajectory, projected onto the orbital (x,y) plane. In figure (c) the second of these trajectories is plotted. Both of these trajectories have very similar appearance in this projection and they also have very similar minimum approach distances to the Moon. The first remains bound around the Moon for 4.6 years while the second is bound for 16 years, with zero-cost. This shows that some characteristics of these trajectories – such as the minimum approach distance, varies smoothly over this region, while the duration of capture is much more sensitive. In addition not that, these 2 trajectories actually can be manifold (see section 3.2.3) of corresponding Lyapunov orbit, because they are in separation between transit (escape or capture; red+yellow region) and non-transit (Bounded; Green region).

Although these ‘so long captures’ are quite advantageous and zero-cost, the numeric errors make the result inaccurate in these high integration times. It means the result (bounded time span) can be change depend on integration step size, and also other disturbances in “real” space. Therefore “the bounded time” span should be chosen much less for accurate trajectory computation. Such as in Earth-Moon system, it is observed that capture trajectory is not change in different step size if the capture time span is restricted less than ~ 1.5 years. But this time can be change relative to energy level and binary system, of course, so before use the trajectories that comes from fuzzy boundaries; it must be tested in different step size and integrator for seeing real accuracy.

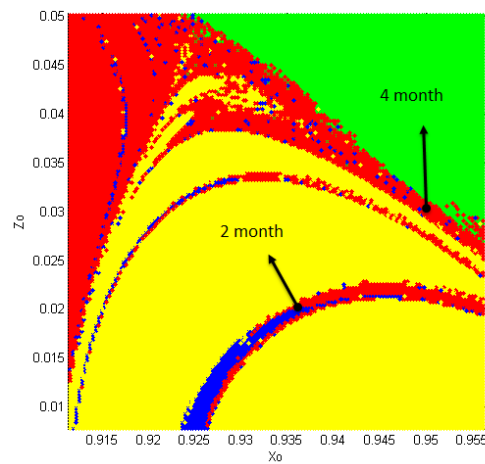
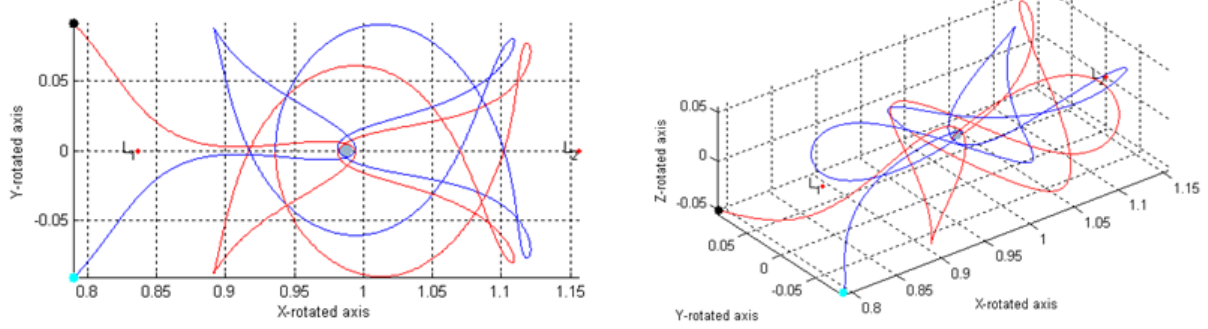


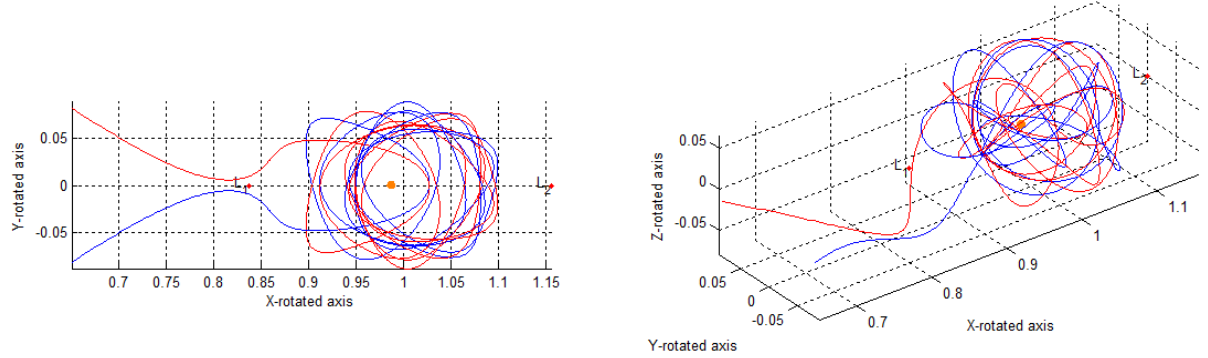
Figure 65 - Chosen the initial conditions for short term capture on fuzzy boundaries of crash waves. Collision trajectories are blue, L_1 and L_2 escape trajectories are red and yellow, bounded trajectories are green.

For getting relatively less long term capture or short term capture, initial conditions must be chosen far away from fuzzy boundaries. In figure-65, the initial condition is chosen from “fuzzy boundaries of crash waves”, so then trajectory also can close to low altitude of target planet. The “final time” is relatively short in that area, so the trajectories can be much more accurate. The initial condition is picked as $[x_0=0.9363, z_0=0.0204]$, and the integration is done for $T=7$ (~ 1 month) in forward time and another integration is done in backward time. As a result, the trajectory is bounded 2 month around Moon, and then it returns where it comes from (so Earth region).

In figure-66, the trajectory is integrated in forward and backward time, with the initial conditions in figure-65. As it is seen in figure-66-a, spacecraft is coming from Earth region with capture trajectory (blue), make about ~ 2 month highly elliptic bounded motion (Blue+red), then it returns to Earth region (red). Because the trajectory is highly elliptic, its minimum altitude can be quite low, that is ~ 420 km. In figure-66-b, same as previous but with making more circular motion for ~ 4 month capture, and in 11,750 km min. altitude. These examples can be useful for average $\sim 1-4$ moon capture, and all it is done with “zero cost”.



a) 2 month capture capture, $[x_0=0.9363, y_0=0, z_0=0.0204]$, $C=3.16$



b) 4 month capture capture, $[x_0=0.9505, y_0=0, z_0=0.03]$, $C=3.16$

Figure 66 - Short time Temporary Capture trajectories.

4.5 Periodic and Bounded Orbits

In previous sections, the bounded trajectories of green region in X-Z Motion Maps have been seen. These bounded trajectories are either highly elliptic and mostly restricted with zero velocity surfaces, (both L_1 and L_2 gates are closed, so they cannot escape) or quite circular and makes robust motions. Even both L_1 and L_2 gate open, trajectories in green region can stay bounded “safely”. For any energy level of motion map, there is a fixed “resonance area” in diagonal side of secondary region, where is in green region, see figure-38. It means that for almost every various value of C (except in very high energies), X-Z Motion map has a constant green region in diagonal direction, which gives all bounded and periodic orbits. So this region can be called as a kind of "resonance area" that trajectories seem always bounded between specific inclinations and majority of stable periodic orbits are inside of this region.

These bounded trajectories provide great promising advantages for parking orbits in high energy level. However the main problem is determination of shapes of these bounded trajectories. In some mission, parking orbits must be low altitude, or nearly circular in high altitude. This is all depended on mission purpose, of course.

In X-Z motion maps, orbits which have too small an inclination either escape in a flyby of the secondary or crash into it, while those with too high an inclination have too little angular momentum and fall into the secondary. This can be seen in map figures in previous sections

(such as figure-37 and 38), where the high inclination orbits are at large z in the blue region and the low inclination orbits are at small z . These trajectories are predominantly red or yellow, indicating escape, with bands of blue between them indicating collision with the secondary. In the region of intermediate inclinations there are regions of stable bounded motion. In fact, the motion maps can be used to find large sets of stable and unstable orbits, both periodic and bounded orbits which remain orbiting around the secondary body for long periods of time.

4.5.1 Unstable Periodic Orbits

In CR3BP, either stable or unstable orbits can be usually found with Differential corrector method. But because Differential Corrector is required too sensitive initial guess, some grid search for finding these proper initial guess can take too much time. But Motion maps can provide these initial guess for Differential Corrector, very fast and accurate. Such as, if X-Z Motion map (or any other configuration Map) is plotted in 3D (see figure-67-left), as X-Z-Tf (Tf is Final time that escape and collision orbits ends), much more insight about trajectories features can be gathered to finding whereabouts the potential advantageous unstable-periodic orbits.

As it is seen in figure-67-right, the “final time” can be reflected on motion map with colour degrade method. In figure, the collision regions is degraded black to light blue, as lightest blue means longest crash time. Same as L_1 escape region is degraded red to green, and for L_2 yellow to purple. Also bounded region is left as blank (white) because its final time already equal to integration time. In addition note that, numbers on regions, which are separated with “collision waves”, shows number of revolutions before it escapes. As it is seen that, number of revolutions increase while closing to bounded region, and when it is in very vicinity their number of revolutions increase exponentially. The same thing can be seen in polar crash area; while closing to “circular belt”, the number of revolutions before they crash increase.

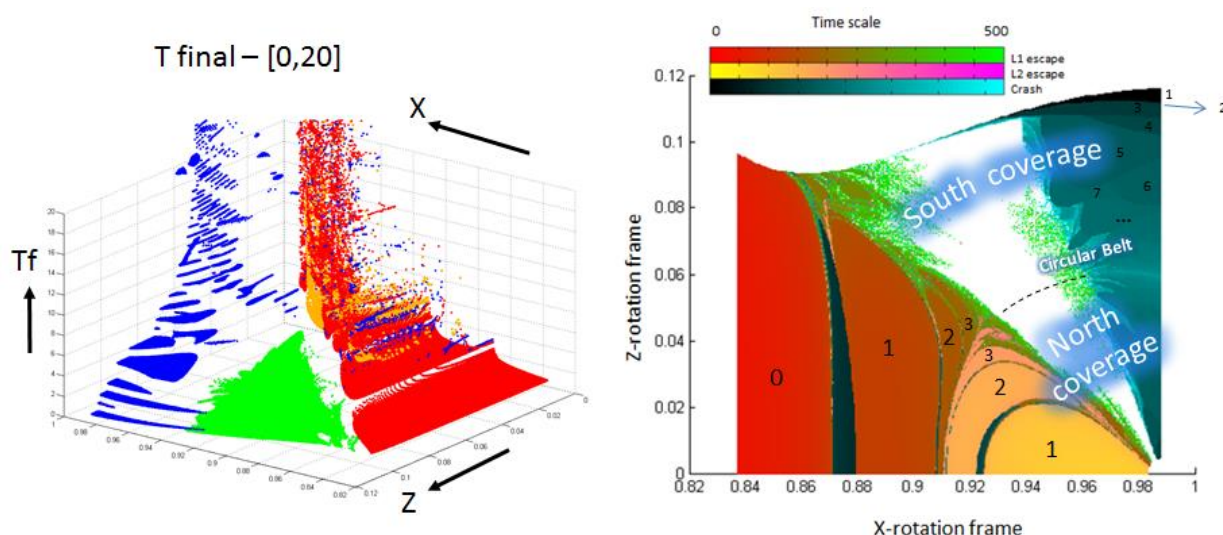


Figure 67 - Reflection of final time on motion map using third dimension (left) [0-20], [0,87 days], and colour degrade method (right), for Earth-Moon system. $C=3.16$, $T=500$. Also, please see MM(X-Z)_various_C_withT.wmv in appendix CD, to see right figure in various C (energy) level.

As I mentioned previous paragraph, the escape and collision time exponentially increase in near the boundaries of green region, figure-67-left. Because the collision and escape time is increase to boundaries of bounded (Green) region, the boundaries in there become quite fuzzy, especially in long integration time. But in the escape and collision region, there are some peak points, which are on place called “collision wave”, and can be show some unstable periodic orbits. It means, the motion map not only shows the stable set of bounded orbits as in green region in figure-67, but also initial condition of “unstable” periodic orbits can be determined with “collision waves” in this map. In addition note that; these “collision waves” are related with transversal manifold intersections either between 2 different Lyapunov Orbits (similar to Heretoclinic connection) or same Lyapunov Orbit (similar to Homoclinic connection), which these type connections are detail studied in [87-94].

The example of how to find these unstable periodic orbits on “collision waves” is shown in figure-68, a plot of the outcome after $T=10$ for a slice through the motion map (Fig.-68) at $z=0$ is considered. Integration of each trajectory is stopped as soon as a collision has occurred (shown as blue) or the spacecraft escapes through the Lagrange points (red or yellow). This figure-68 shows that in between the boundaries where the outcomes change there are 3 main rapid and narrow increase in the final time; see the peaks in figure-68. There are periodic orbits at this transition point, which are unstable. In theoretically, if this map could be computed perfectly (in the absence of numerical noise and with infinite small mesh) then the final times of peak points would increase to infinite. But in practically, final times of these peak points are ended in a finite, because of numeric restrictions.

In figure-68, 3 peak points in the “final time”, which have been labeled as I, II and III, which their initial conditions are;

- I. $C=3.16, x_0 = 0.8708331538104, y_0=0, z_0=0, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- II. $C=3.16, x_0 = 0.9105414412190, y_0=0, z_0=0, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- III. $C=3.16, x_0 = 0.9250645094850, y_0=0, z_0=0, v_{x0}=0, v_{y0}=V, v_{z0}=0$

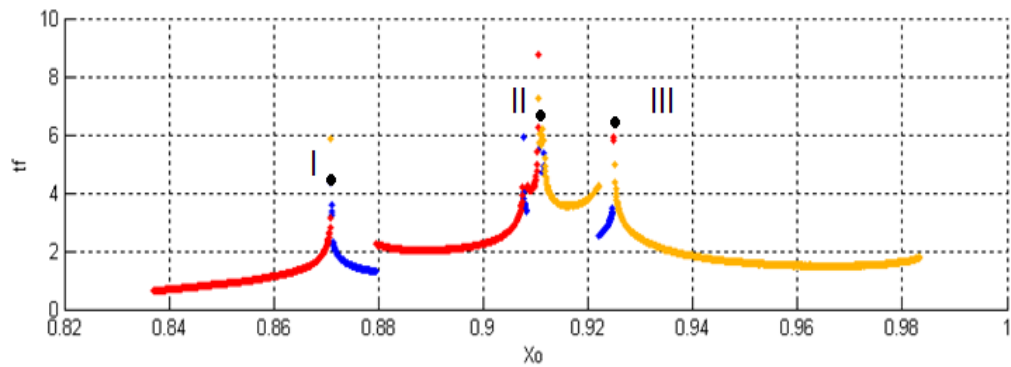


Figure 68 - $z=0$ ($y=0$ already) slice on motion map (Earth Moon) in x-position, final time.

V-velocity is computed from Jacobi Integral. We may either pick the rough initial conditions of these periodic orbits in figure-67 or 68, or the information of whereabouts of the periodic orbits in figure-68 can be used as initial guess for differential corrector (DC) method. In short, this practical map grid (Fig.-68), very accurate initial guess can be provided for DC method, in a very short time, about 2 min or less.

Peaks I and III are associated with a collision inside the escape region, while peak II exists in the boundary between two different escape outcomes (either through L_1 or L_2), which has also a thin collision wave band. These peak points are all planar, unstable, periodic orbits; and are plotted in figure-69. Also other points are just ordinary escape or crash trajectories, which are bounded relatively short time span, and not periodic orbits.

The first peak (shown in red) is the familiar halo orbit around L_1 . It is associated with the blue band in red region to the left of figure-68. The other peak II and III are associated with the other blue band boundary between the red and yellow regions and in the yellow region, respectively. The peak II in figure-68 (for instance) is probably related with a nearby homoclinic or heteroclinic trajectory, while the third one is as if “quasi-homoclinic cycle”, in which the trajectory makes a passage around only one equilibrium point (in this case L_2).

This homoclinic cycle term with Heteroclinic term are better explained by E.Canalias, Josep J. Masdemont in [88,89];

By matching one orbit from the stable manifold with another one from the unstable one, a zero cost trajectory is obtained which asymptotically approaches a Lyapunov orbit both in forward and in backward time. If the manifolds belong to the same orbit, the connections are homoclinic. On the contrary, if they come from different orbits, they are called heteroclinic connections. In our case, as there is a unique periodic orbit around each libration point for any fixed energy level, the classification in homoclinic and heteroclinic can also be regarded as trajectories involving only one L_i (homoclinic) or both of them (heteroclinic).

Approaching with these definition; the green trajectory (figure-69) also makes a flow which close an equilibrium point (L_2). And similarly, another trajectory may make a motion or cycle which close to both equilibrium point (L_1 , L_2). These motion may not be exact homoclinic and heteroclinic motion explained in [88,89], but they may approach them, because they are on “collision waves” which related the manifold intersections of Lyapunov Orbits. And these type trajectories may be called quasi-homoclinic and quasi-heteroclinic motion, for easiness.

So, since these unstable periodic orbits exist in the boundaries between regions, and they are actually such a manifold intersection, the motion map can be used to locate where those unstable periodic orbits and also both potential homoclinic and heteroclinic transitions. A more refined search can be used if such highly unstable periodic orbits are of interest.

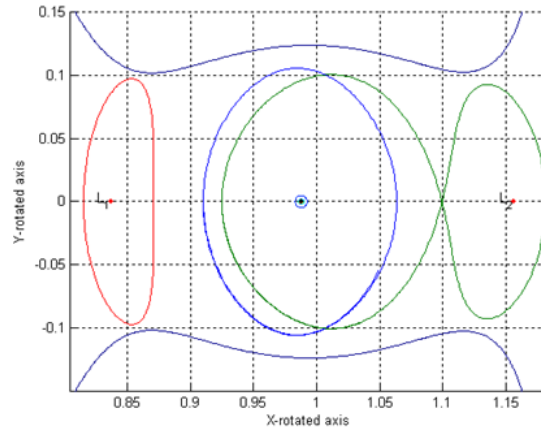


Figure 69 - Unstable periodic orbits from initial conditions of figure 4.5 (Earth Moon). Red is I, blue is II and green is III in figure 68.

Same as previous horizontal slice cut of figure-68, another slice cut of fig.-67 can be taken vertically for finding other periodic orbits in vertical direction, see figure 70. This another separate cut considered through the motion map where's fixed in $x = 1 - \mu$ and vary z . As it is seen in figure-70, maximum final time reach maximum 109 days ($T = 25$ in dimensionless time), which trajectories in this cut are only bounded up to 109 days, before they collide.

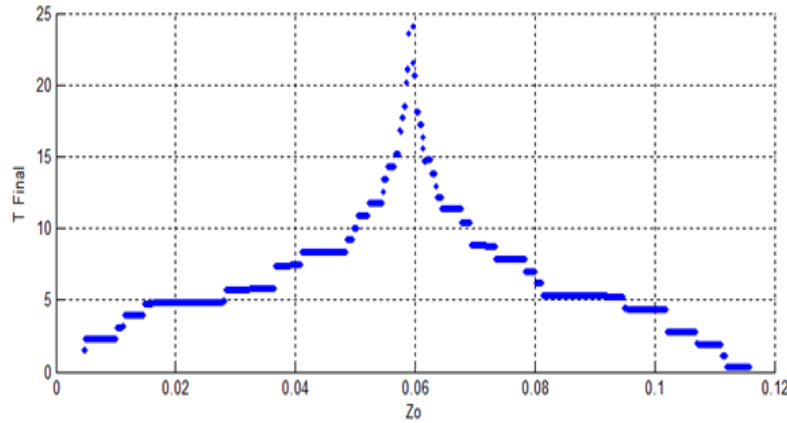


Figure 70 - A slice cut for $x=1-\mu$ on motion map, in z -position, final time.

This cut has been chosen to investigate the light blue changes to final time in figure-67-right. Although there will be no boundaries between regions as in fig. 70, there is one peak point around $z \sim 0.06$, that can clearly be seen in the middle. This shows again there is an unstable periodic orbit, whose exact initial conditions are $C=3.16$, $x_0=(1-\mu)$, $y_0=0$, $z_0=0.059268$, $v_{x0}=0$, $v_{y0}=V$, $v_{z0}=0$, V is again computed from Jacobi Integral. Same as with previous example, this map can also be used to get idea where about the periodic orbits and provide good initial guess to differential corrector method. In fig. 70, the plot resolution is kept high for demonstration, but even with less resolution (so less integration), where about the periodic orbits can be seen, and collaborate with DC method.

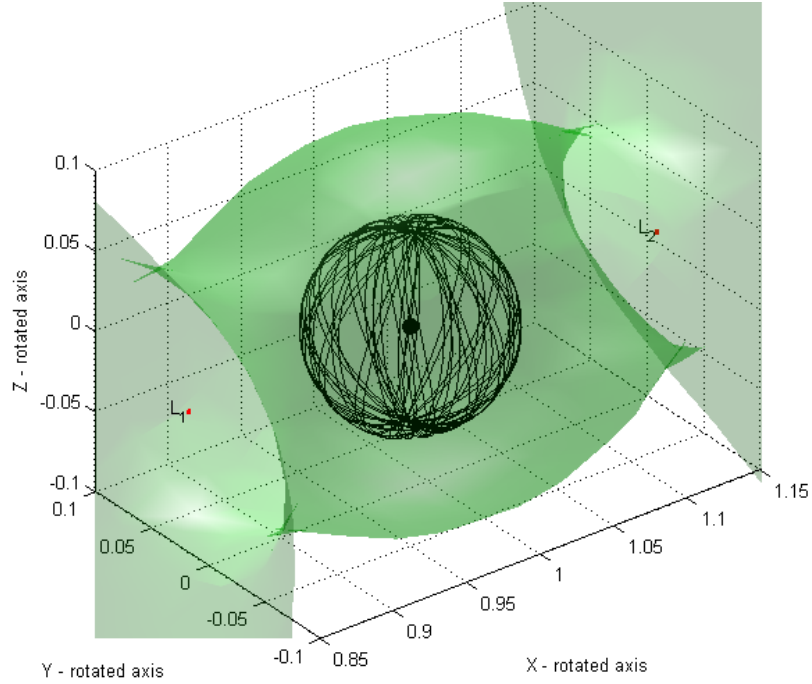


Figure 71 - Polar nearly-circular unstable orbit.

Also it is shown in figure-67-right, light blue arc in collision region present some special polar orbits which are unstable. And one of these orbits has been plotted in figure-71, thanks to data in fig.-70. It is seen that these periodic orbits are nearly circular polar orbits that process around the z axis. Such orbits would be well suited for investigating the polar regions of the Moon, and they are also demonstrated in some studies as well, such as [62]. The fact that there is a band of these pale blue orbits in figure-67-right is a family of unstable periodic orbits, which will be crash to secondary body sooner or later. These orbits are different from the highly unstable halo type periodic orbits, in that these nearly circular polar orbits remain stable for relatively long periods of time before becoming unstable and colliding with the Moon. Because of this feature, station keeping costs might be much cheaper than employing Halo type orbits for spacecraft station keeping. Therefore, all of these trajectories can be used as station keeping or some kind of advance transfer trajectories, depend on the mission purposes.

As it is seen, with taking various slice from the motion map as figure-67, initial conditions of various unstable periodic orbit can be determined for various space mission types, either looking the red and yellow regions boundaries or solid blue areas with final time-axis map as shown in figure-70. Also as we mentioned before, this type of periodic orbit investigation can be used in collaboration with Differential corrector (DC); so Motion Map can be uses a kind of grid search map method, which provides a suitable and practical way to find an initial guess used in DC. For further alternative space mission design about unstable periodic orbit can be found in Chapter 5, see section 5.1 and 5.2.

4.5.2 Stable Bounded Orbits

In this section it is tried to map out the nature of the stable bounded orbits as they appear in the motion map as green (see figure-37 or 38), to show how trajectory design can lead to a focused search for the periodic orbits of interest. Although unstable periodic orbits need corrective control to maintain a spacecraft in them, the stable bounded orbits require no station keeping and are therefore more useful for long term- cheap (nearly zero cost) missions such as for navigation purposes.

In figure-72 (b), the basic schema of motion map (figure-72-a) is shown. It is seen that there is a band of near circular orbits (Circular Belt in Fig.-72). It has been discussed the unstable part of this band (where it crosses the blue region) but it extends into the green region where these orbits become stable. This family of orbits then matches on to the unstable orbits in the region between the red and yellow escape trajectories.

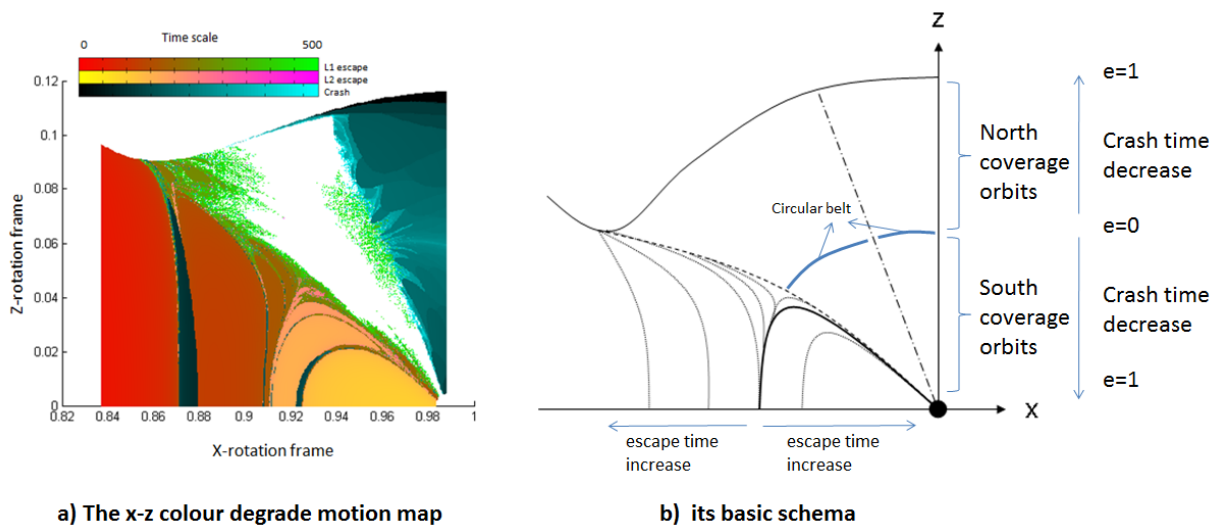


Figure 72 - Motion map Analysis and Orbits orders

As it is moved away from this band an increasing amount of eccentricity in the orbit is had, with the ones at higher inclination having pericentre around the northern hemisphere of the secondary, while the lower z has pericentre in the southern hemisphere. Since these eccentric orbits pass closer to the secondary, unsurprisingly the time before they crash into the lunar surface decreases. Moving away from this circular band causes the time to increase, whether it is to escape the region or crash into the lunar surface.

Also note that; because trajectory (so satellites on it) spend most of the time near apogee and pass very fast in perigee in high elliptic trajectories, whose apogee close north side, are called north coverage, (and so whose apogee close south side, are called south coverage). If geometry of orbits are considered like proximity to planet; then South and North coverage might be the other way around.

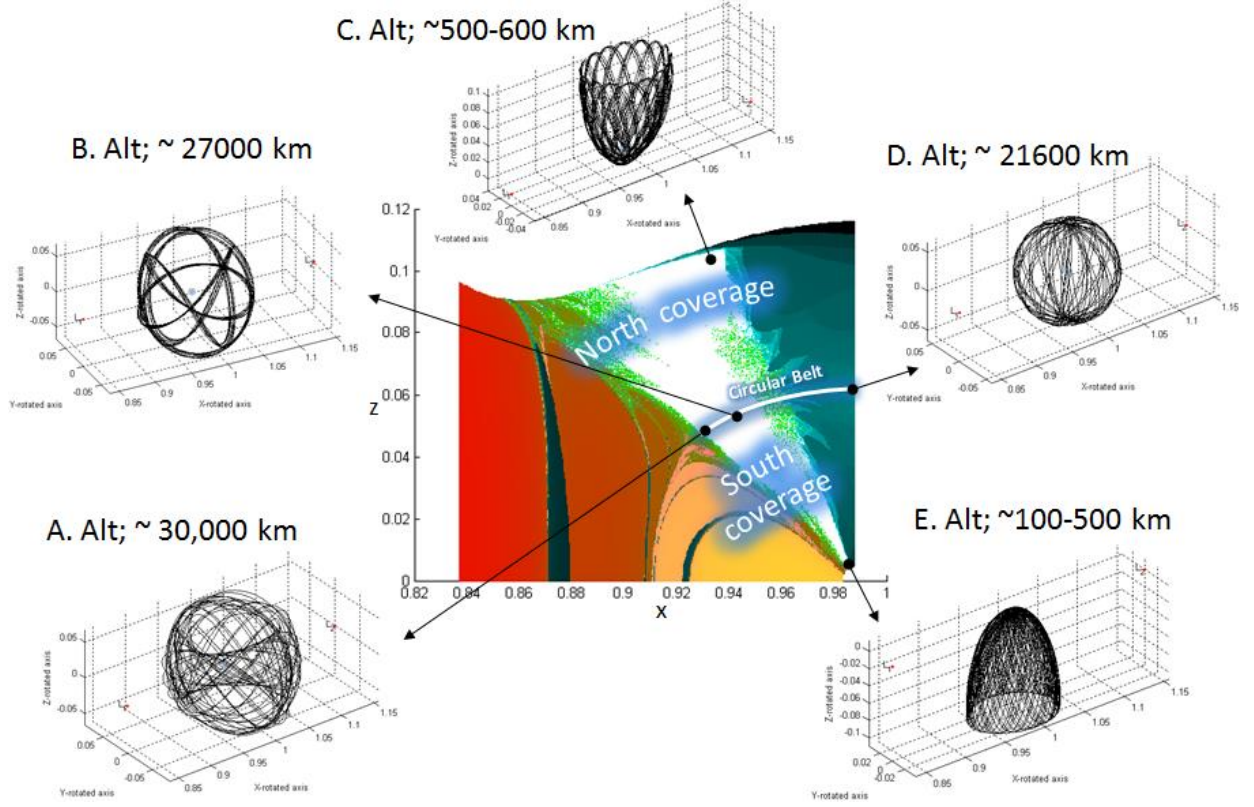


Figure 73 - Samples of parking orbits and their initial conditions in plane motion map, Earth Moon, $C=3.16$.

In figure-73, samples of periodic parking orbits and where they are located in the motion map are shown. For just simple demonstrations of the main orbit types, the initial conditions are picked randomly from various region; 3 of them from circular belt (A, B, D) and others north (E) and south (C) covering regions. Their detailed initial conditions are;

- A. $C=3.16, x_0 = 0.9300, y_0=0, z_0=0.05, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- B. $C=3.16, x_0 = 0.9441, y_0=0, z_0=0.0525, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- C. $C=3.16, x_0 = 0.9315, y_0=0, z_0=0.1029, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- D. $C=3.16, x_0 = (1-\text{mue}), y_0=0, z_0= 0.059268, v_{x0}=0, v_{y0}=V, v_{z0}=0.$
- E. $C=3.16, x_0 = 0.9861, y_0=0, z_0=0.0045, v_{x0}=0, v_{y0}=V, v_{z0}=0$

V is computed from Jacobi Integral.

Trajectories A, B and D taken along the “circular belt”, make nearly circular periodic motion in high altitude. Trajectories A and B are stable (they are taken from the green region, fig.-73), while trajectory D is unstable (taken from the blue region, fig.-73). Trajectories C and E are taken away from the circular belt and show the orbits are both quite elliptic. Trajectory C approaches closest to the Moon in the southern hemisphere while orbit E provides coverage of the northern hemisphere. Many studies about these orbits have also been done previously, such as [60,62-65], and we can find them very fast and easily with these motion maps.

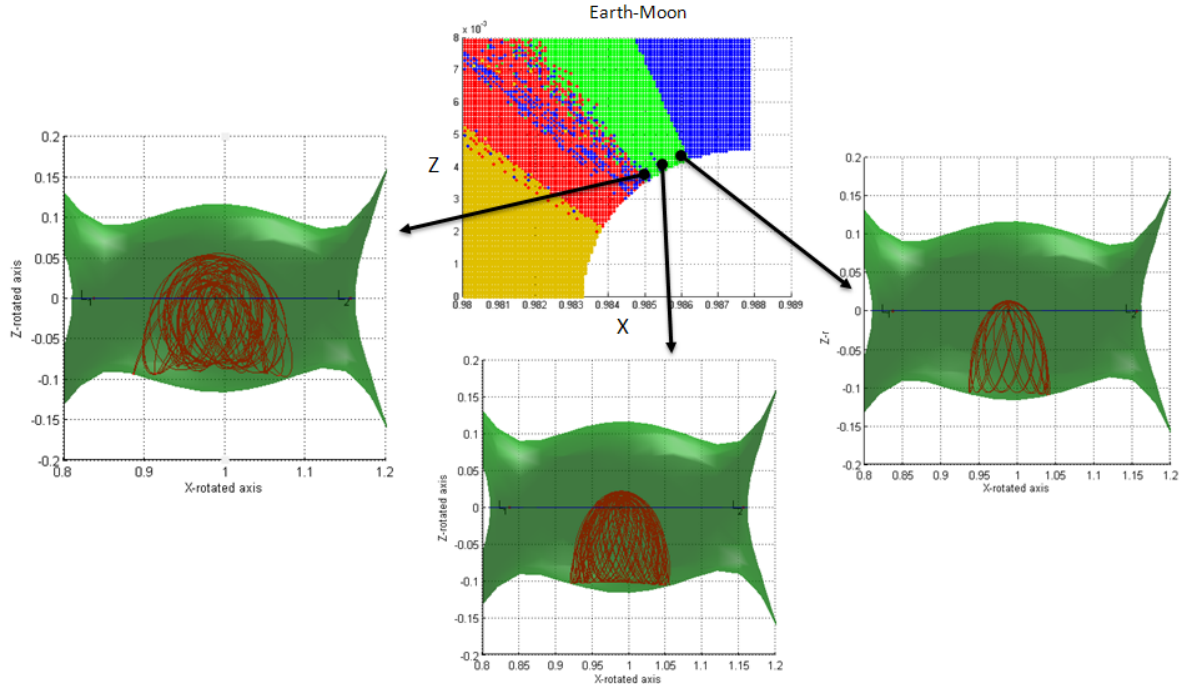


Figure 74 - Samples of south covering trajectories respect to fuzzy boundaries. Collision trajectories are blue, L_1 and L_2 escape trajectories are red and yellow, bounded trajectories are green.

In figure-74, the shapes of the south coverage trajectories is shown in relation to their zero velocity surfaces ($C=3.16$). They are chosen with various distances from the Fuzzy (red-green region boundaries) boundaries of the green bounded region. The trajectory on the left is taken from very close to a fuzzy boundary and its motion is highly elliptic and irregular. The trajectory in the middle is further away from the escape regions and looks far more regular. The third trajectory is on the boundary with the collision region and makes a simple Lissajous type motion. This shows that bounded motions in the green area of the motion map have many sets of periodic, quasi-periodic and chaotic bounded trajectories ... etc in its realm. And shapes of bounded orbits vary depending on distance to boundaries. Therefore, the Green region in the motion map must be analysed more to identify quasi-periodic orbits from chaotic bounded trajectories.

Figure-75 shows two plots of the green bounded region of the motion map, to identify these orbits. Figure-75-a shows two computations of this region: one long-term integration ($T=500$, 5.9567 years) shown in green and the other a short-term integration ($T=50$, 217.4189 days) shown in red. The red region extends underneath the green region, but the colour difference brings out how the boundary of the bounded region changes as the integration time is changed. It means that when integration time is increased in the motion map, the Green (bounded) region of the motion map shrinks; some regions shrink more, some less, as mentioned in section 4.4.1. Highlighted on this figure are two regions, marked by black circles, where the change (shrink) is most pronounced. In these regions, the trajectories will be more chaotic, either escaping by swing by or colliding with the secondary.

To get a feel for the shapes of these orbits, the osculating eccentricity of the periodic orbit is computed over the short timescale ($T=50$) with relatively less grid resolution for speed (so it takes less time for computation), and then a standard deviation of eccentricity is computed. This gives a measure of how much variation in altitude the orbit undergoes. Then a surface plot of

the rms (root mean square) eccentricity is plotted simply in Matlab, right in figure-75-b. Small values (shown in blue) indicate a rather smooth harmonic variation of eccentricity. The regions highlighted by the black circle on the left are also similar on the right and here the rms eccentricity is much larger.

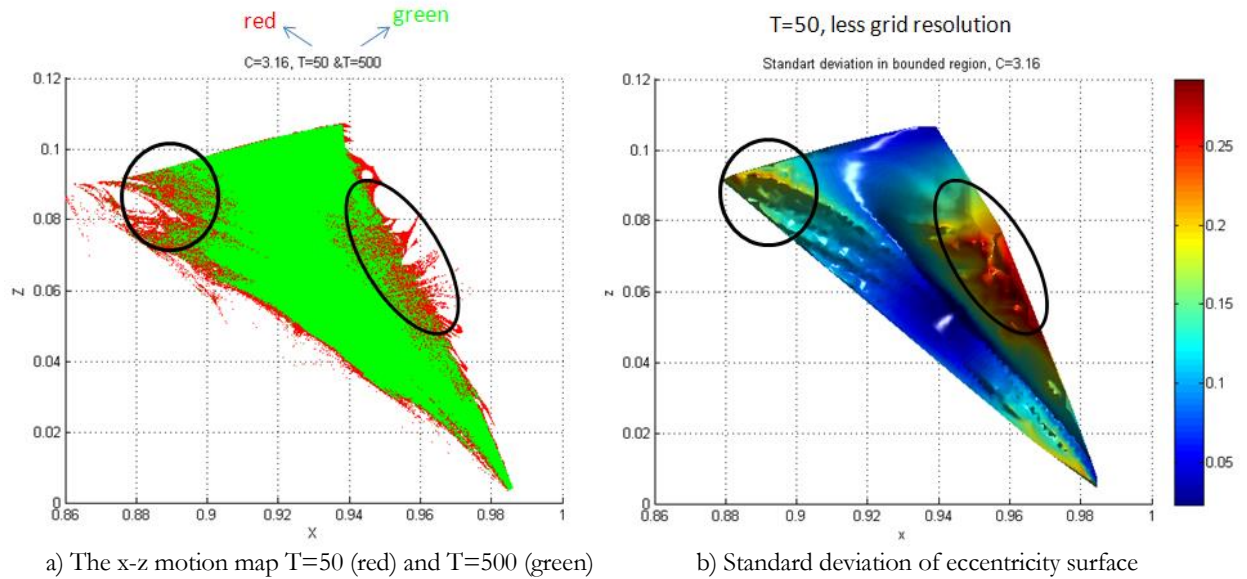


Figure 75 - Stability analysis in bounded region, with effect of long and short term integration in bounded region (left), standard deviation of eccentricity surface in bounded region (right).

As match with together; wide shrink areas in fig.-75-a and Larger eccentricity oscillation areas in fig.-75-b are very parallel that chaotic bounded motions can be also easily determined with the rms (root mean square) eccentricity method. The blue region of this plot is a valley where the motion is more regular and more stable. This provides a region of interest for mission designers.

4.6 Incorporating the Effects of Lunar Shape

The Moon is the second densest satellite in the Solar System after Io and having a mean density of 3,346.4 kg/m³, the Moon is a differentiated body, being composed of a geochemically distinct crust, mantle, and core, and its flattening is 0.00125, [wiki/moon], which may affect the orbit motions.

Fortunately, there are already many studies about effect of Lunar shape, such as [98,99], and for highly elliptic periodic orbits (either south or north coverage), the effects of the non-spherical shape of the secondary may be important for Motion map. To understand whether oblates effect of Moon effect on Motion map, or not, this section will focus upon the effects of the non-spherical shape of the secondary and the small eccentricity of the lunar orbit about the Earth upon the bound periodic orbits.

As seen in figure-72-73, there are two main types of stable direct periodic orbits around the secondary body: nearly circular and highly elliptic. Models for the lunar gravity and the effects of eccentricity in the CR3BP are well known. An analysis of them can be found in [100]. In order to model these disturbances the periodic orbits need to be described in a Moon centred inertial frame. Figure-76 shows the difference in one of the elliptic periodic orbits when computed in the CR3B model (blue) and when the disturbances are added (red).

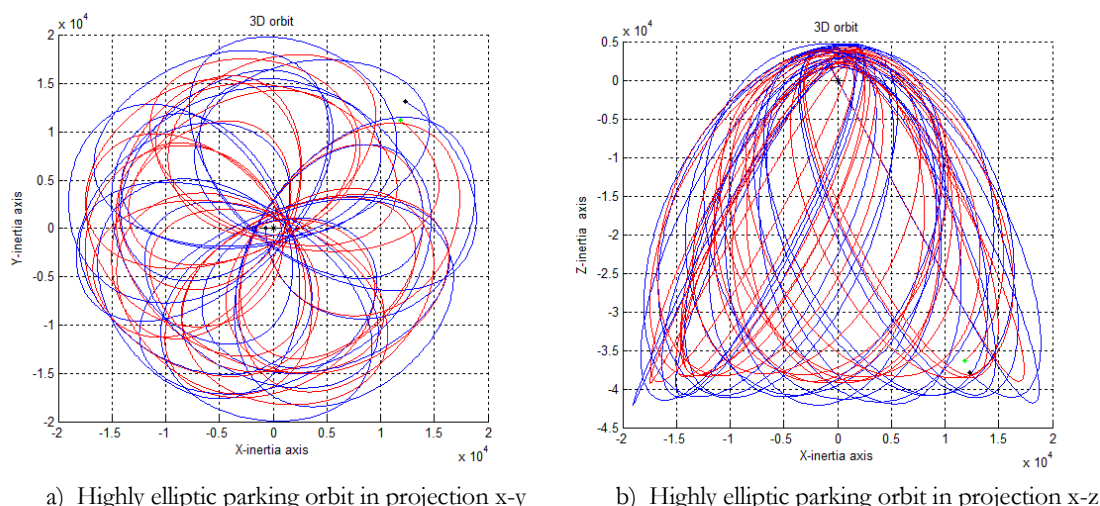
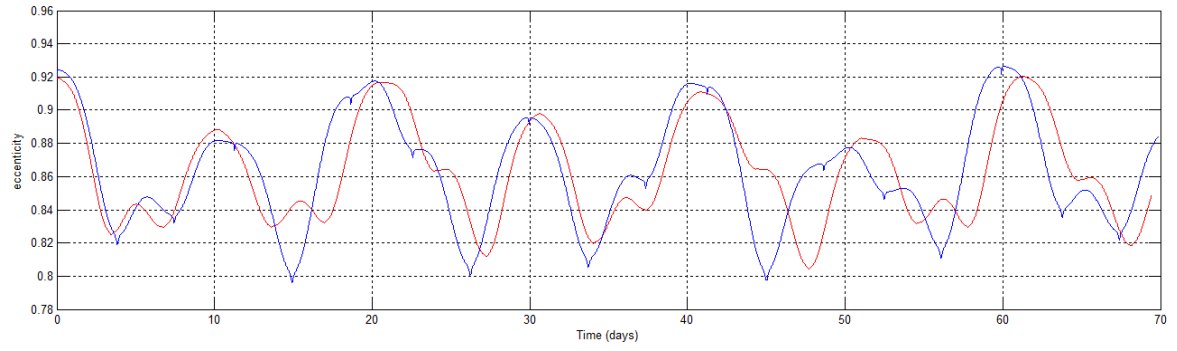


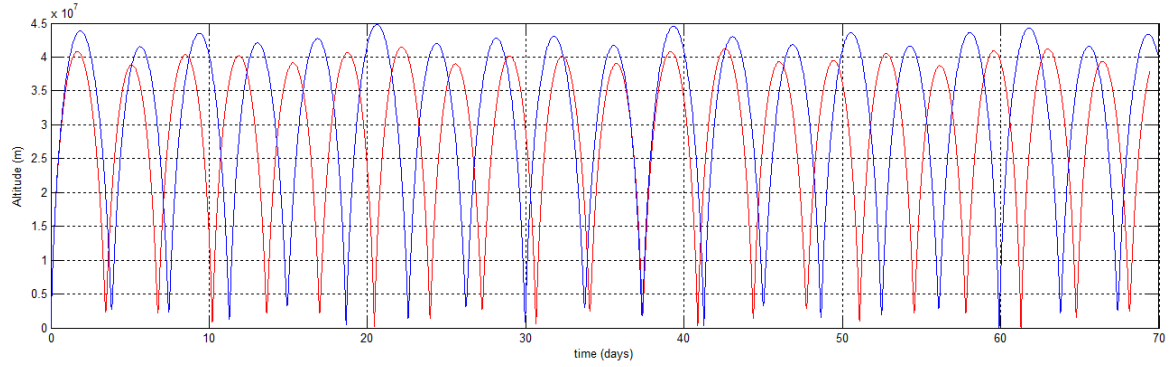
Figure 76 - Highly elliptic periodic motion around Moon (orange) in dimensional (km) inertial axes. The blue trajectory shows the trajectory in the rotating frame of the CR3BP while the red trajectory is computed by effect of moon oblates in. Trajectories are evaluated to 70 days.

This figure illustrates that these disturbances, while they modify the orbits, do not change them in any significant way. The most significant effect of the lunar oblateness is its disturbance to the altitude of apoapsis passage.

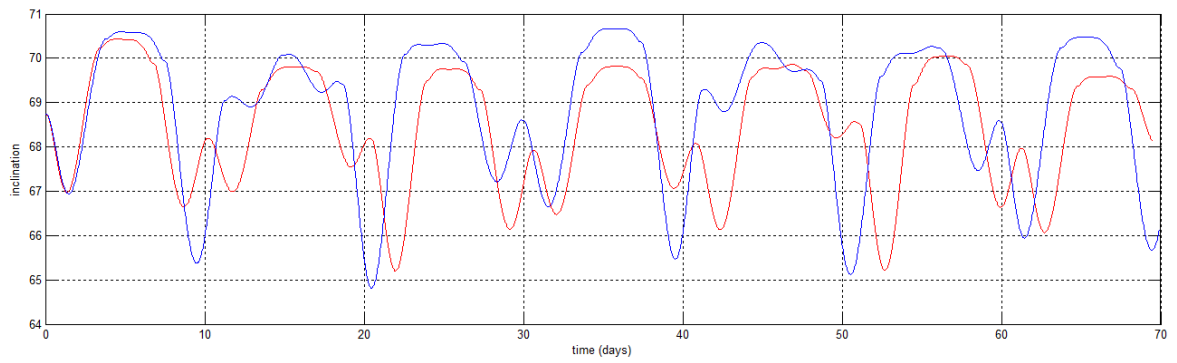
In figure-77, a detailed analysis in terms of osculating Keplerian elements for the two trajectories shown. As it is seen the motion remains bounded in both conditions and the effects of these disturbances can be considered once an orbit has been selected for a mission when a refined orbit will be required. In the case of the computation of motion map, the effects of lunar oblateness can be ignorable in equation. Because the most important thing in Motion map is final outcome of trajectories (Escape, crash, bounded), and of course, this depend on the time length, but Moon oblates effect is not enough to change the final outcome of trajectories in this time span of example. So Moon Oblates effect can be neglected for simplicity. But, of course, if it is studied propagation of one particular trajectory for specific mission, then Moon Oblates effect must be considered.



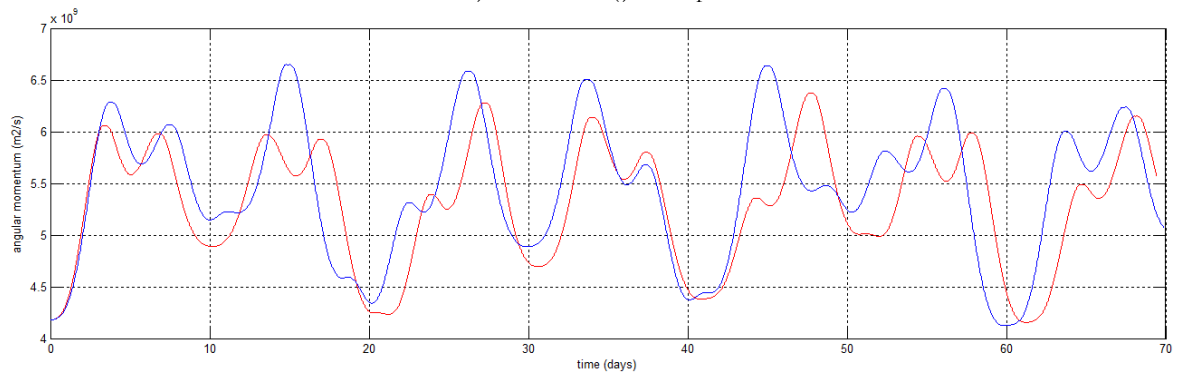
a) Eccentricity-Time plot



b) Altitude-Time plot



c) – Inclination (i) -Time plot



d) Angular momentum – time plot

Figure 77 - Blue line is for the trajectory plotted in CR3BP, red line is the trajectory under effect of Moon Oblates effect and Earth-Moon elliptic motion.

4.7 Keplerian Projection

As it is seen from right plot of figure-72, Keplerian elements may give a clue for dynamics features of secondary body region. Even looking to standard deviation of eccentricity $e(t)$ can give good information whether the trajectory stay its periodic motion or not, see fig.-75.

Therefore, it is showed some transformation that can be useful for understand and gaining an insight about Keplerian motion around primaries. First, trajectories, that are computed by using equation motion of CR3BP, are transform from rotated to inertia axis and then transform to Keplerian elements. Transforming the trajectories to inertia axis and Keplerian elements, it can be understood more insight about Keplerian approximation around capturer planet.

For this, transforming between rotated and inertia axis, orthogonal matrix is used, equation;

$$R(t) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.4)$$

Using this matrix, transformation is done between the axis,

$$\begin{aligned} \bar{x}_{ro} &= R(t)\bar{x}_{in} \\ \bar{x}_{in} &= R^{-1}(t)\bar{x}_{ro} \end{aligned} \quad (4.5)$$

Position vector can be written relative to M_1 or m_2 , origin of inertia axis can be shifted either of bodies:

$$\begin{aligned} \vec{r}_1 &= (x + \mu, y, z) \\ \vec{r}_2 &= (x - 1 + \mu, y, z) \end{aligned} \quad (4.6)$$

So that, after axis transformation, Kepler transformation can be done respect to M_1 or m_2 . And transformations inertia to Keplerian elements are, [5,75];

$$\begin{aligned} R &= \sqrt{x^2 + y^2 + z^2} \\ V &= \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \\ \bar{H} &= (\underbrace{y\dot{z} - z\dot{y}}_{h_x}, \underbrace{z\dot{x} - x\dot{z}}_{h_y}, \underbrace{x\dot{y} - y\dot{x}}_{h_z}) \end{aligned} \quad (4.7)$$

Kepler variables can be seen from figure-75;

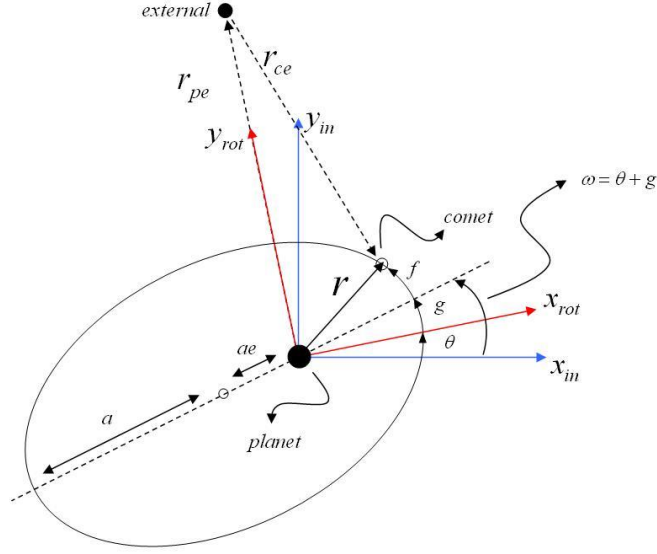


Figure 78 - Osculated Two Body Motion, [5,25].

Radius (R), velocity (V) and angular momentum (H) can be calculated in inertia axis, eq 4.7.

$$a(t) = \left(\frac{2}{R} - \frac{V^2}{\mu} \right)^{-1}, \quad e(t) = \sqrt{1 - \frac{h^2}{a(t)\mu}} \quad \text{and} \quad i = \cos^{-1} \left(\frac{h_z}{h} \right) \quad (4.8)$$

Semi major axis (a), eccentricity (e) and inclination (i) are computed from eq. 4.8.

$$\begin{aligned} \sin \Omega &= \frac{\mp h_x}{h \sin i} & \cos \Omega &= \frac{\mp h_y}{h \sin i} \\ \sin \omega &= e^{-1} \left(-\dot{x}h - \frac{y}{R} \right) & \cos \omega &= e^{-1} \left(\dot{y}h - \frac{x}{R} \right) \\ \sin f &= \frac{h}{\mu e} V & \cos f &= e^{-1} \left(\frac{h^2}{R\mu} - 1 \right) \end{aligned} \quad (4.9)$$

Longitude of ascending node (Ω), argument of perigee (ω) and true anomaly (f) are computed from eq. 4.9.

$$E = -\frac{\mu}{2a(t)} \quad (4.10)$$

And the Keplerian energy is computed from eq. 4.10. In non-dimensional equation of CR3BP, $d\theta/dt=1$. So then, the Keplerian analysis can be done using these transformations.

In figure-79, the x-z map is reflected to (inclination- radius of the secondary) plane. For reminding again, the Jacobi constant is set as both L_1 and L_2 gate open same size, $C=3.16$. Because of this Jacobi constant value (or energy level), Boundaries of red and yellow region is quite straight and perpendicular to the horizontal axis. The other interesting thing is that blue

and green region boundaries are quite parallel to the horizontal axis, and the bounded region is restricted with “specific inclination” that there are no bounded trajectories below then this “specific inclination” for this energy level. In other words, trajectories are sit in stable periodic motion while it in high inclination. To understanding this phenomenon, a simple analysis can be done using rotation 2-body dynamics.

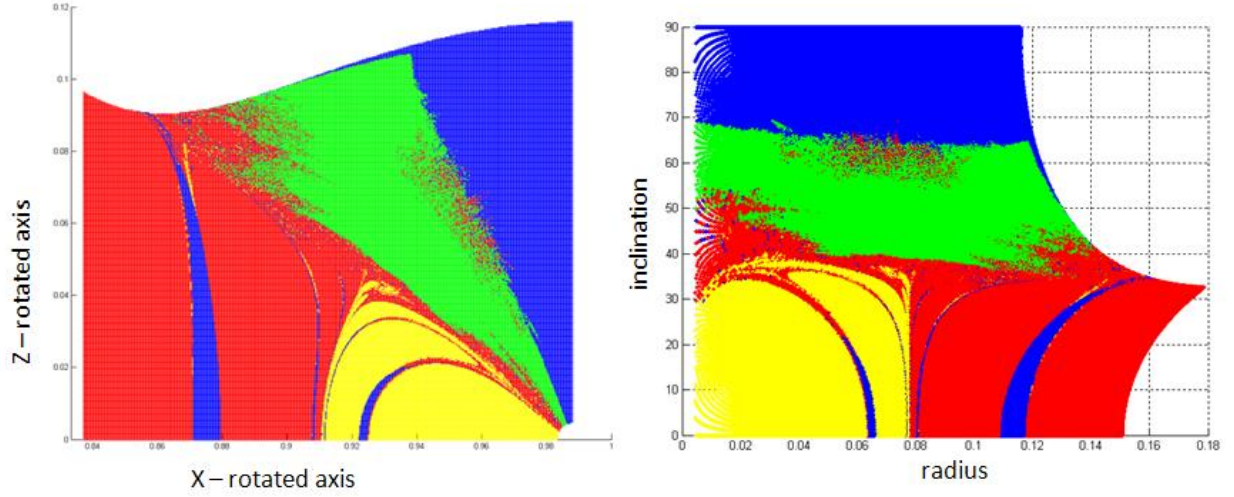


Figure 79 - The x-z plane motion map and projection of (inclination- the radius of the secondary) plane, $C=3.16$. Collision trajectories are blue, L_1 and L_2 escape trajectories are red and yellow, bounded trajectories are green. Please also see MM(X-Z)_to_MM(R-I)_various_C.wmv in appendix CD.

For this, two-body motion can be studied rotated axis for matching CR3BP. This process is quite easy that one of mass parameter (μ_1 or μ_2) is eliminated from equation motion of CR3BP and primary mass is taken as $\mu=1$ in non-dimensional coordinates. But while patching R2BP with CR3BP, primary mass can take any value between $0 < \mu^* \leq 1$ (μ^* can be either bigger or smaller mass, but it would be smaller mass for this work) and angular velocity of rotation axis can equal to 1 for matching angular velocity of CR3BP. So that, the equations motion of rotation 2BP (R2BP) is arranged as;

$$\begin{aligned}\ddot{x} - 2\dot{y} &= U_x = x - \mu^* \frac{x}{r_1^3} \\ \ddot{y} + 2\dot{x} &= U_y = y - \mu^* \frac{y}{r_1^3} \\ \ddot{z} &= U_z = -\mu^* \frac{z}{r_1^3}\end{aligned}\tag{4.11}$$

And Potential function;

$$\bar{U}(x, y) = \frac{x^2 + y^2}{2} + \frac{\mu^*}{r}\tag{4.12}$$

Where,

$$0 < \mu^* \leq 1 \quad r = \sqrt{x^2 + y^2 + z^2}\tag{4.13}$$

The Jacobi integral can give energy equation, same as eq. 3.10. For equality of angular velocity with rotated axis of binary system, unit circle can take value to μ^* ;

$$r_{uc} = \mu^{*1/3} \quad (4.14)$$

r_{uc} is the radius of unit circle. The unit circle is a kind of reference circle that shows the group of the fix points (equilibrium points). In that circle, object is motionless relative to primary mass in rotated axis, in other words rotation frame frequency and orbital frequency of unit circle is same. Energy value in unit circle is;

$$C_{uc} = 3\mu^{*2/3} \quad (4.15)$$

In the condition of $\mu^*=1-\mu$ (for primary body), potential surfaces of R2BP and CR3BP overlapped closely relative to primary body. But for $\mu^*=\mu$, huge energy differences occur and this makes motion more chaotic around secondary body, as it is mention previously.

Resonance motion can be arrange making simple transformation inertia to rotated axis, so for elliptic case ($0 \leq e < 1$), energy can be computed as function of resonance (R), eccentricity (e) and μ^* ;

$$\begin{aligned} a &= r_{uc} R^{2/3}, \quad r_0 = a(1-e) \\ \bar{C} &= 2\sqrt{\mu^* a(1-e^2)} + \frac{\mu^*}{a} \end{aligned} \quad (4.16)$$

For parabolic case ($e=1$), energy can be computed;

$$\bar{C} = 2r_0 \sqrt{\frac{2\mu^*}{r_0}} = 2\sqrt{2r_0\mu^*} \quad (4.17)$$

In 2BP, everything is relatively simpler then multi-body dynamics. If eccentricity less then 1 ($e<1$), the trajectories are bounded, if it is bigger then 1 ($1\leq e$), so trajectories unbounded (parabolic or hyperbolic). So then, using eccentricity equation, eq. 4.8, and combining $a(t)$, and angular momentum ($h^2=r^2V^2$), the new eccentricity function is look like;

$$e = \left| 1 - \frac{r_0 V_0^2}{\mu^*} \right| \quad (4.18)$$

Where r_0 , V_0 and so $\dot{y}_0$ are in eq. 4.19;

$$\begin{aligned}
r_0 &= \sqrt{x_0^2 + z_0^2} \\
V_0^2 &= (x_0 + \dot{y}_0)^2 \\
\dot{y}_0^2 &= x_0^2 + 2\frac{\mu}{r_0} - C
\end{aligned} \tag{4.19}$$

x_0 and z_0 are variable for x-z plane counter plot of eccentricity. Jacobi constant (C) can be defined using eq. 4.16 or 4.17. But for the escape trajectory analysis, the equation 4.17 must be used. Setting $\mu^*=1-\mu$ (normalized mass for secondary body, for Earth-Moon case $\mu^*=0.01215$), and $r_0=0.102$ (in no dimensional form), “C” can be computed $C=0.1$. This means that;

- Trajectories, with $r_0 < 0.102$, $C=0.1$ initial conditions, makes elliptic motion,
- Trajectories, with $r_0 = 0.102$, $C=0.1$ initial conditions, makes parabolic motion,
- Trajectories, with $r_0 > 0.102$, $C=0.1$ initial conditions, makes hyperbolic motion.

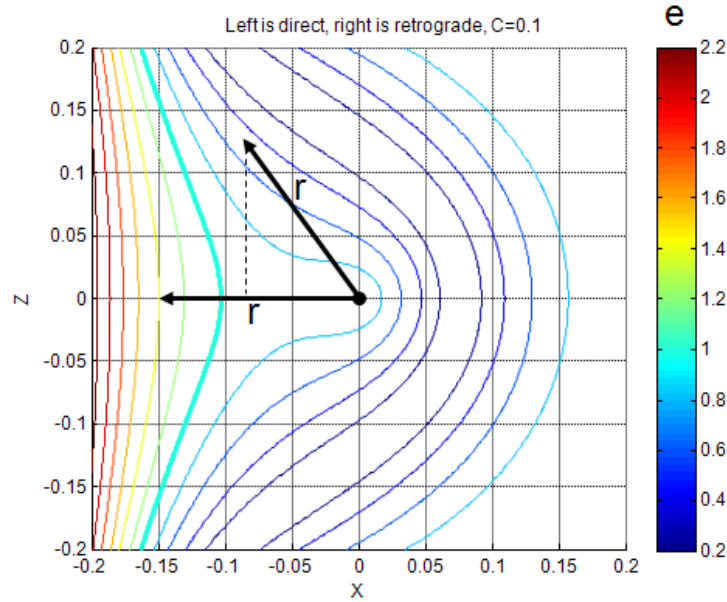


Figure 80 - Contour plot of eccentricity, $e(x,z)$, in x-z plane for rotation 2BP.
The line for $e=1$ is plotted tick.

To see this in suitable phase space, the equation 4.18 is plotted on figure-80. Although this plot for x-z plane, actually it gives same result for all plane that perpendicular to $z=0$ plane, because of no any external body effect in 2BP. From this plot, it is seen that previous “bullets” for motion definitions are only valid in planar case. In 3-dimension case, the dynamics of motion change and “parabolic boundaries” in $e(x,z)$ counter map is bending while the inclination is increased. In other words, planar trajectories makes parabolic motion in $r_0=0.102$, $C=0.1$ initial conditions, but non-planar trajectories can stay bounded with same initial conditions (but in different inclination, of course). With this plot, stability effect of inclination can be seen more clearly. This is because; there is no any centrifugal effect on z direction, as in x-y plane. As it is seen in eq. 3.8 and 4.11, the z component of equations look like “spring” equation, and this direction tend to make harmonic osculation relative to planar axis. So, trajectories with high inclination is more likely to be stable than planar trajectories.

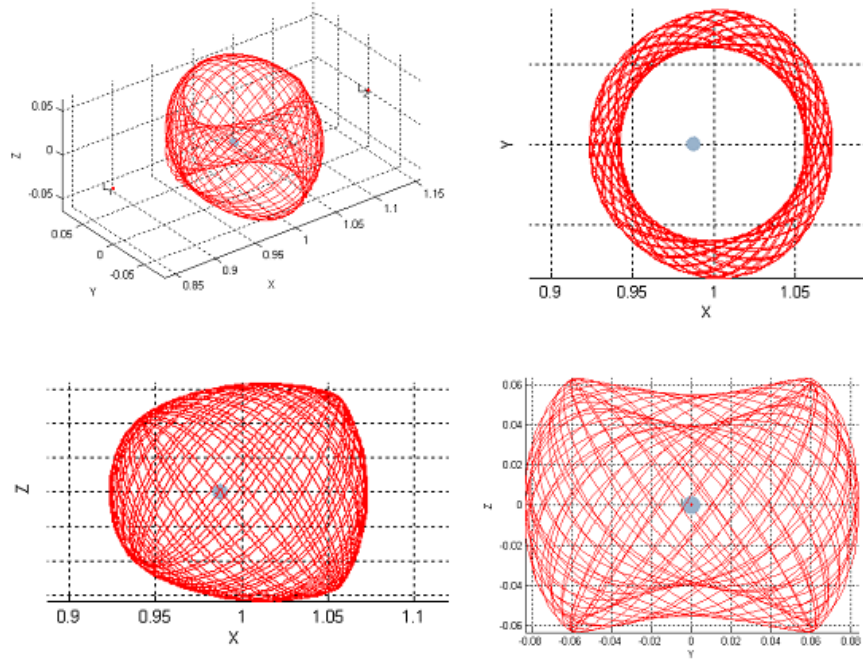


Figure 81 - A stable periodic orbits around Moon (Capturer planet).
 $x_0=0.94421$, $y_0=0$, $z_0=0.04$, $C=3.16$ in non-dimensional rotated axis.

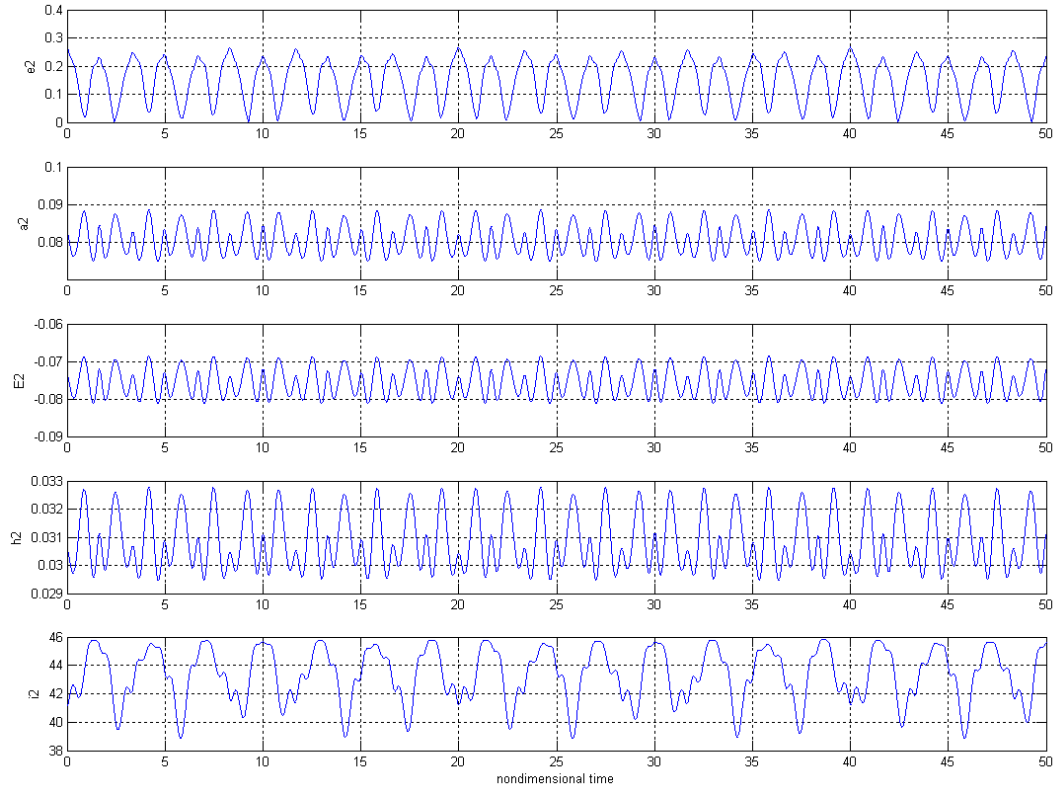


Figure 82 - osculation of Keplerian elements of trajectory in fig. 81.
 Eccentricity, semi-major axis, Keplerian energy, angular momentum and inclination, respectively.

In figure-81 and 82, an example for stable periodic orbits in high inclination, and its Keplerian elements values with along the time are shown. Initial conditions of this stable periodic orbit are chosen “circular belt area” of green region of figure-73. The Keplerian elements functions, see in eq. 4.7-4.10, are in harmonic oscillations.

Although there are many higher energy stable periodic motion around secondary body (see in figure-37 or 38), they are not as so fit classical keplerian motion as it is seen figure-82. However, there are some periodic oscillations in keplerian element, and so other parameter, that these orbits still are useful for any type of reference, parking and science orbits.

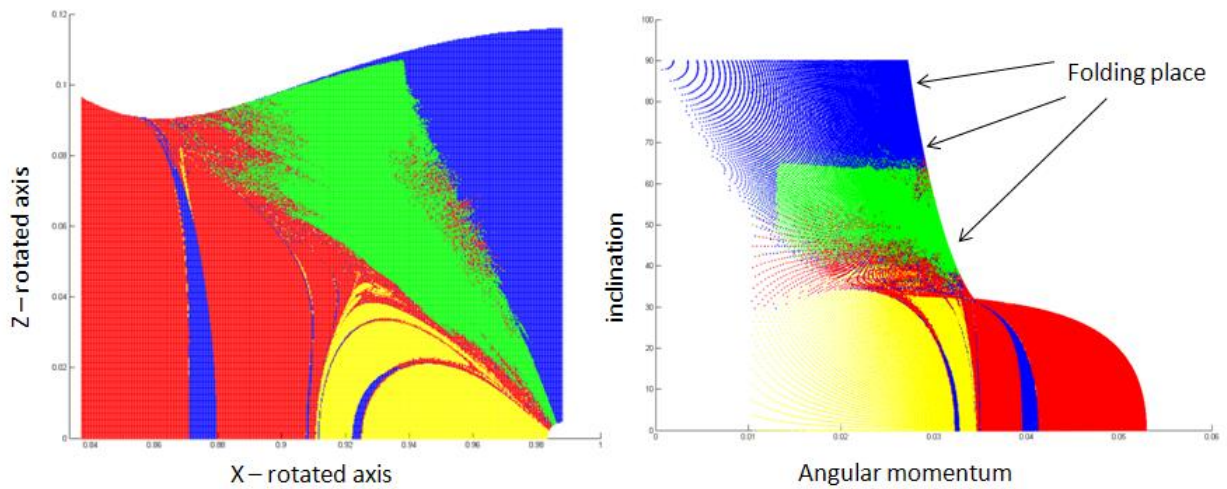


Figure 83 - The x-z plane motion map and projection of inclination-Angular momentum plane, $C=3.16$. Collision trajectories are blue, L1 and L2 escape trajectories are red and yellow, bounded trajectories are green.

In figure-83, the x-z motion map is reflected to the angular momentum (h) - inclination (i) plane, for gaining much more insight. Same as figure-79-right, the boundaries of region are quite straight relative to vertical and horizontal axis (red and yellow boundaries perpendicular to the horizontal axis and boundaries of green and blue region are parallel to the horizontal axis). But the differences, there is also “folding place” in h-i map. This “folding place” is such an interesting place that is quite fit with “circular belt” in figure-72. This mean that maximum points of angular momentum $H(z)$ function gives the place of circular belts. Therefore, the characteristic of motion can be estimated using h-i plot of initial condition.

In figure-84, the initial conditions reflection is focused to close vicinity of secondary body, which transferred from figure-60, to angular momentum-inclination plane. In that focusing area, there is no folding area, so the plot can be seen much more clearly. The boundaries of all region quite parallel to the horizontal axis, and with change of integration time ($T=50$, $T=500$) the fuzzy boundaries and shrink of green region can be seen much more clearly.

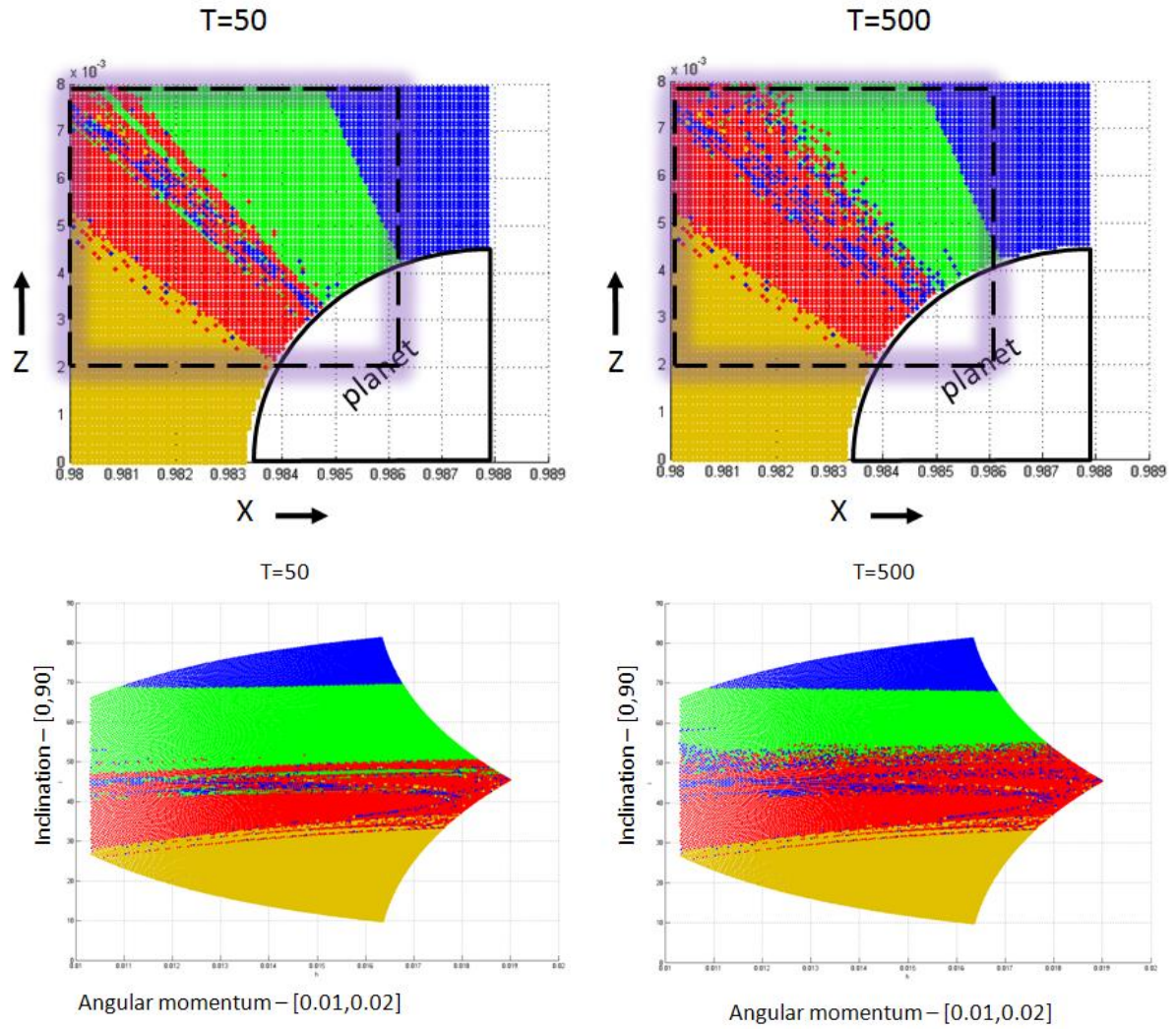


Figure 84 - Angular momentum – inclination plot with focusing close vicinity of secondary body, which are tranfered from dashed part in upper figures. Left side map is integrated T=50, and right one T=500. Collision trajectories are blue, L₁ and L₂ escape trajectories are red and yellow, bounded trajectories are green.

V. MISSION DESIGNS and APPLICATIONS

In Chapter 4, the motion map design process and its types are introduced. As it has been showed, the motion maps are so useful for determining dynamics and trajectories around secondary body in CR3BP, that especially they present great advantages for finding long-term bounded, stable and unstable periodic orbits in ‘high energy levels’ for very low-cost capture, or even ‘zero-cost capture’ using ‘the fuzzy boundaries’. In short, the motion maps provide great flexibility for space mission design.

In this chapter, some mission design and application are introduced with using the motion maps. First of all, computation of velocity increment is shown in section 5.1, then useful science orbits and full ballistic capture missions are applied in section 5.2 and 5.3. Finally Earth to moon transfer approaches are shown using motion maps in 3 different scenarios.

5.1 Cost Computations

In previous chapter, three main types of motion map have been introduced; Plane (x-z or y-z) motion map, motion map of manifold tube and motion map on a random trajectory. The manifold tube and trajectory along maps have already velocity increment ΔV axis. So, looking these maps, the cost computation can be done directly, see section 4.2.2 and 4.2.3.

However ‘plane motion maps’ don’t show ΔV directly. Because they are plotted in specific Jacobi constant (C), and their axis are lie only on the plane, the cost computation can be done combination of various maps with various Jacobi constant. Therefore, plane maps must be plotted for all Jacobi constant values (every energy level). This is already done in previous chapter; the plane (x-z) motion map is plotted from $C=3.2$ (both L_1 and L_2 gates are closed) to $C=3.1$ (very high energy level) as in figure-38. After ~ 100 plots between $C=3.2$ - 3.1 , such a 3 dimensional Motion map, whose axis are X, Z and C (Jacobi Constant), can be obtained. Then simply jumping one map to another, velocity cost can be computed. But the trick is to find right place for jumping, of course.

For this, the boundaries between the regions, especially the fuzzy boundaries of the green region is focused for low-cost jump. To ensure the post burn trajectory is bounded to the Moon, we consider the jump in energy to leapfrog the fuzzy boundary. A manoeuvre does not alter the position in the motion map, assuming it is impulsive, but just shifts the energy. In this way the width of the fuzzy boundary determines the change in energy required to safely jump from the red region (inbound from the Earth) to the green region (capture).

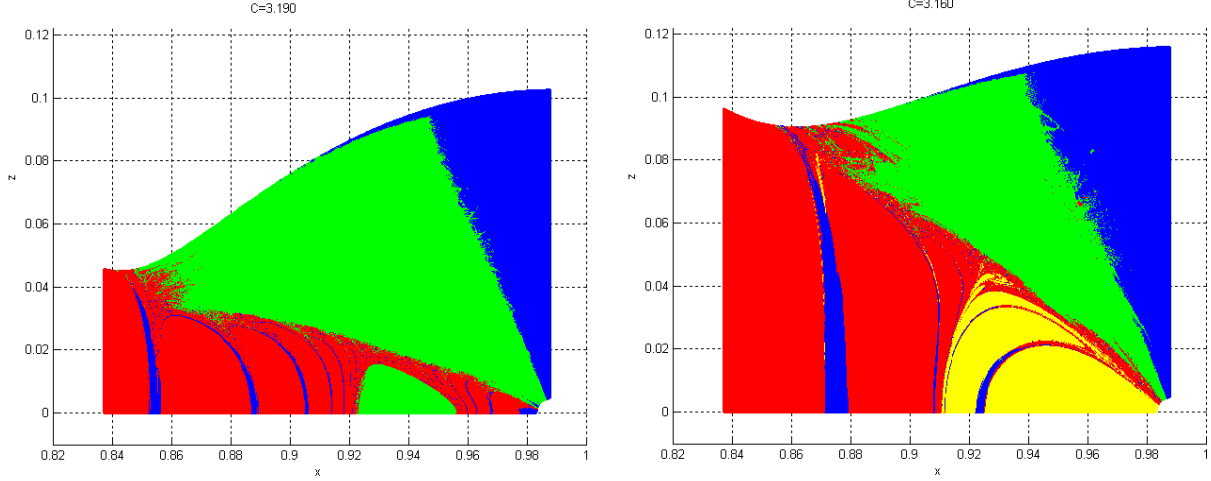


Figure 85 - Plane (x,z) motion map for C=3.19 (left), and for C=3.16 (right).
Collision trajectories are blue, L₁ and L₂ escape trajectories are red and yellow, bounded trajectories are green.

For example, in figure-85, two random motion maps in two different Jacobi constant are shown. In C=3.19, L₂ gate is closed, while L₁ is open. The below boundaries of green region is in $\sim 23^\circ$ inclination in C=3.19. But for C=3.16, inclination of below boundaries is $\sim 40^\circ$. This ‘inclination’ can be also called ‘specific inclination’ which is also give the rough place for ‘fuzzy boundaries’. This means that if the spacecraft is orbiting in upper than 40° inclination with very high energy, say any C value that is $C < 3.16$, then C=3.16-X-Z motion map can be used for low-cost capture. But if it is orbiting in upper than 23° inclination with very high energy, say again any C value that is $C < 3.19$, then C=3.19-X-Z motion map can be used for low-cost capture. Of course, this inclination is not so in straight cut and strictly, but it can give well approximates.

To compute the velocity cost between the maps, we use the expression for the Jacobi integral as in eq. 5.1. The potential function $U(x,z)$ ($y=0$) is fixed by the choice of initial conditions on the motion map. The velocity increment between the Jacobi constant (C_1 and C_2) is then:

$$\Delta V(x, z) = \sqrt{2U(x, z) - C_1} - \sqrt{2U(x, z) - C_2} \quad (5.1)$$

We show a contour plot of these velocity changes (eq. 5.1) over the x-z plane in figure-86. A contour plot of the velocity changes over the x-z plane, where the Jacobi constant is to be decreased from 3.19 to 3.16. Ideally the most advantageous place for the velocity change is in the vicinity of the secondary body, of course, as this exploits the orbital velocity to obtain the greatest shift in energy for the smallest increment of thrust. Minimum velocity decrement in there for bounding the trajectory can be $\Delta V \sim 8$ m/sec while jumping C=3.16 to C=3.19. But even at high altitude, such as a nearly-circular orbit at 21,100 km (initial conditions $x_0 = 0.93$, $z_0 = 0.04$), only 40 m/s velocity decrement is required.

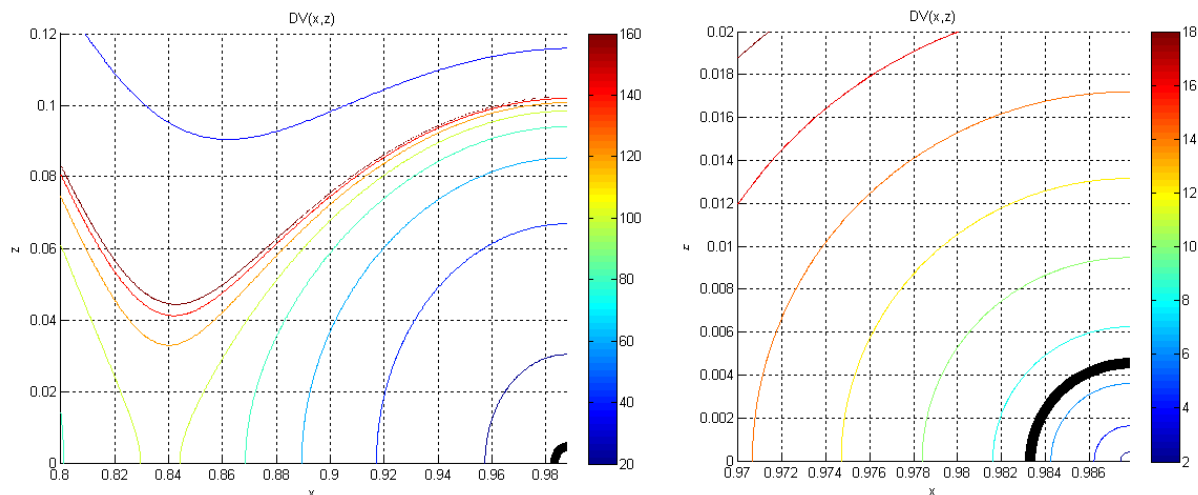


Figure 86 - $\Delta V(x,z)$ contour plot in secondary region for Earth-Moon . Left bar shows velocity increment values (m/sec) that matches contour. Black tick curve is boundaries of Moon. Left side plot is focused in low altitude.

As another example, the minimum velocity decrement for bounding a trajectory (same initial conditions $x_0 = 0.93$, $z_0 = 0.04$) by decreasing the Jacobi constant from $C_1 = 3.16$ to $C_2 = 3.17$ is about 13 m/s over this area of the Motion Map. The key point to emphasize is this means trajectories can become bound without imprisoning them in the zero velocity curves. For a comparison, if the initial Jacobi constant was fixed at the value at the L_1 point ($C=3.2005$, so both L_1 and L_2 gates are closed), then a 56 m/s velocity decrement would be required to bound the orbit to the Moon. The figure of 13 m/s represents a significant saving in propellant required at this late phase of the transfer. Such transfers may be of particular relevance in rescue missions where propellant has been lost for some unforeseen circumstance.

The motion map can be combined with other transfer strategies such as Hohman Transfer, WSB or a 'transfer arc', as described in section 2, [15-16]. Such as ; Using a 'transfer arc', which is showed in [16,29,30], spacecraft can be increased from low Earth orbit to one of transit trajectories, whose initial conditions is given in red region (L_1 escape or capture) in motion map. In this way, if the trajectory arrives at the target planet with a suitable inclination, it can be bounded for less than 20 m/sec. By reducing the final velocity increment in capture can provide much more flexibility in mission and fuel savage can be conserved in long time space mission, such as Voyagers.

For another example, in figure-87 and figure-88, jumping process to one map to another is shown, again. First of all, a suitable parking orbit is chosen in suitable energy level. For this example, a low-altitude (~ 500 km) with high eccentricity periodic orbits is chosen in motion map whose energy is $C=3.185$. Then the desired energy level that is wanted to be jump, is chosen, see in figure-88. Black dots in every plot are show same coordinates in middle plot in figure-87. In figure-88, 'coordinates' are in red (L_1 escape) region in only $C=3.17$, for the other energy level they are in yellow (L_2 escape) region. Therefore, if the spacecraft is wanted to escape to L_1 , the Jacobi constant must be upper than the $C=3.17$. If it is wanted to escape to L_2 , C must be below than the $C=3.17$.

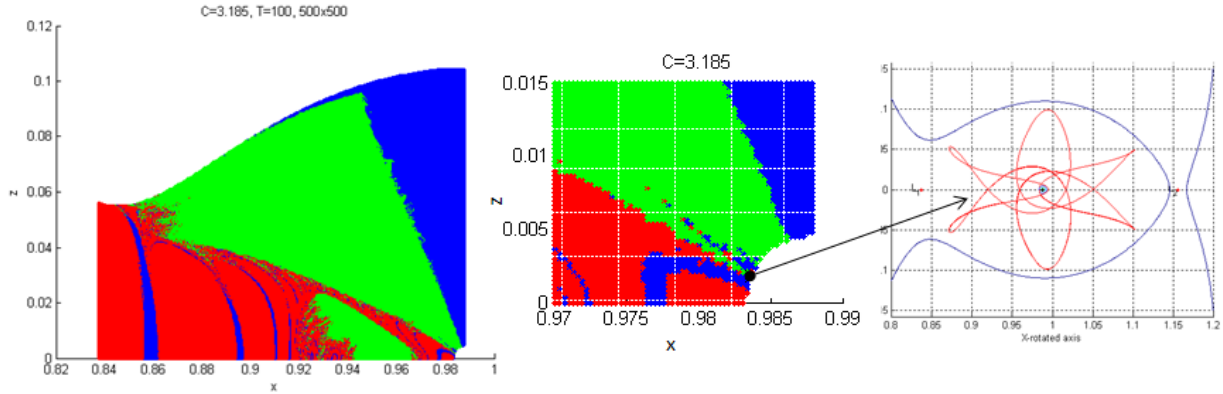


Figure 87 - a stable periodic orbits that chosen low altitude in plane motion map. Chosen coordinates $x=0.9833$, $z=0.0018$, $C=3.185$. Collision trajectories are blue, L_1 and L_2 escape trajectories are red and yellow, bounded trajectories are green.

Table 5.1 – Velocity increment for escaping from Earth-Moon L_2 , $C=3.185$

ΔV (m/sec)	C_f
5.8637	3.16
8.2049	3.15
10.5436	3.14
12.8799	3.13
15.2138	3.12

As an example, the mission is jumping from parking orbits then escape to L_2 gates. So all Jacobi constants, below than $C=3.17$ can provide this duty. Then using eq. 5.1 and ‘fixed coordinates’ ($x=0.9833$, $z=0.0018$), the velocity increment can be computed for different energy level, see table-5.1. Although smallest ΔV seems the most advantageous, the other energy level may be also useful some other missions, see section 5.4.3.

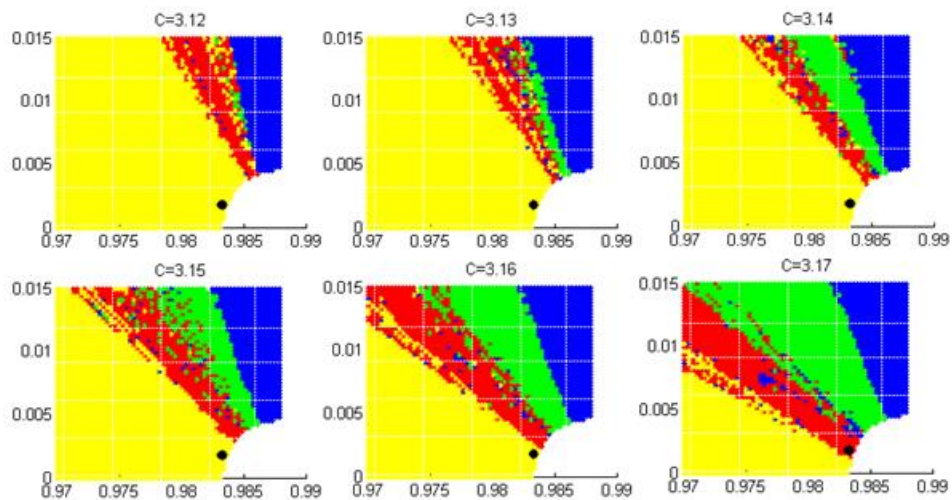


Figure 88 - The motion maps in focus on close vicinity of Moon with various Jacobi constant. Specific inclination is $\sim 60^\circ$ for $C=3.12$, $\sim 55^\circ$ for $C=3.13$, $\sim 50^\circ$ for $C=3.14$, $\sim 45^\circ$ for $C=3.15$, $\sim 40^\circ$ for $C=3.16$, $\sim 35^\circ$ for $C=3.17$, Collision trajectories are blue, L_1 and L_2 escape trajectories are red and yellow respectively, bounded trajectories are green.

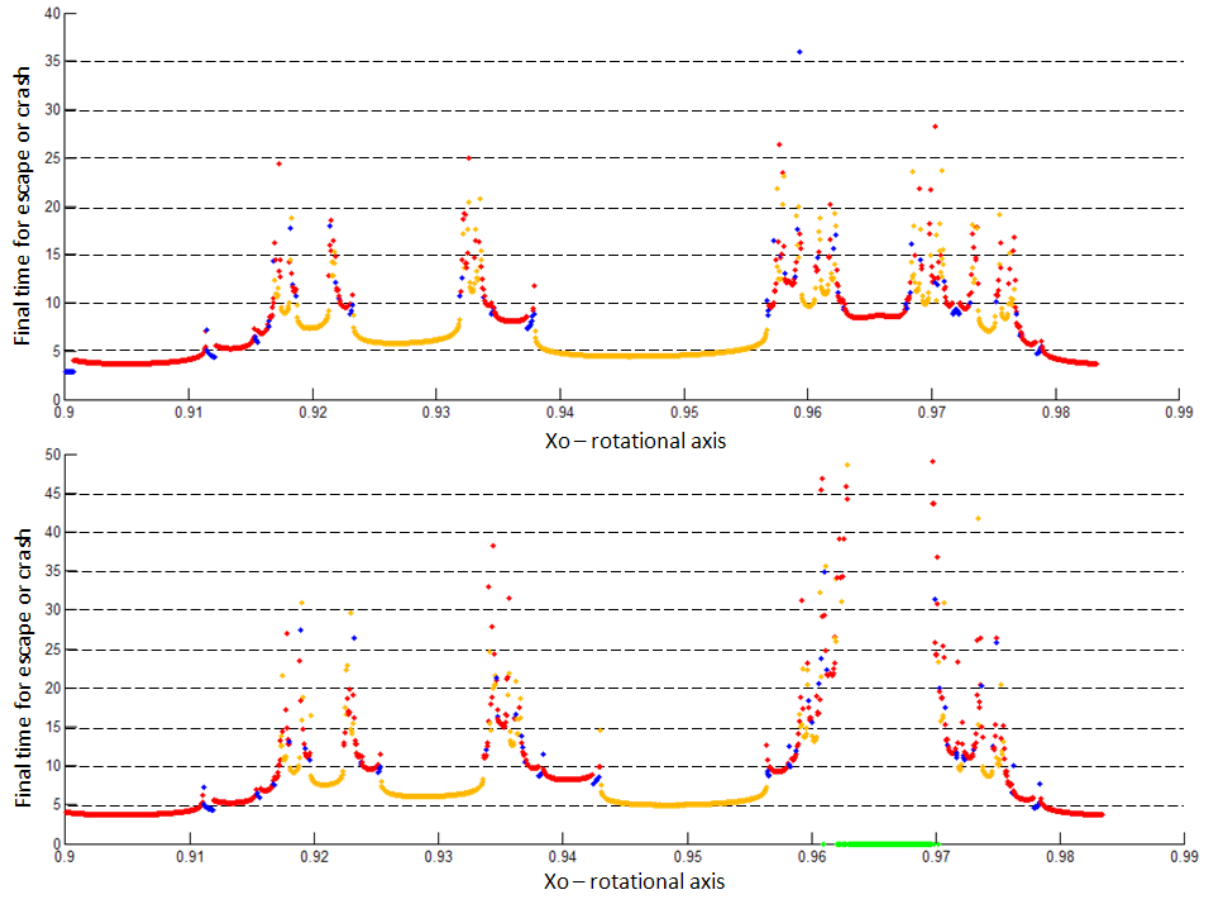


Figure 89 - Two $z=0$ slice for $C=3.181$ (upper) and $C=3.1819$ (below) motion map.

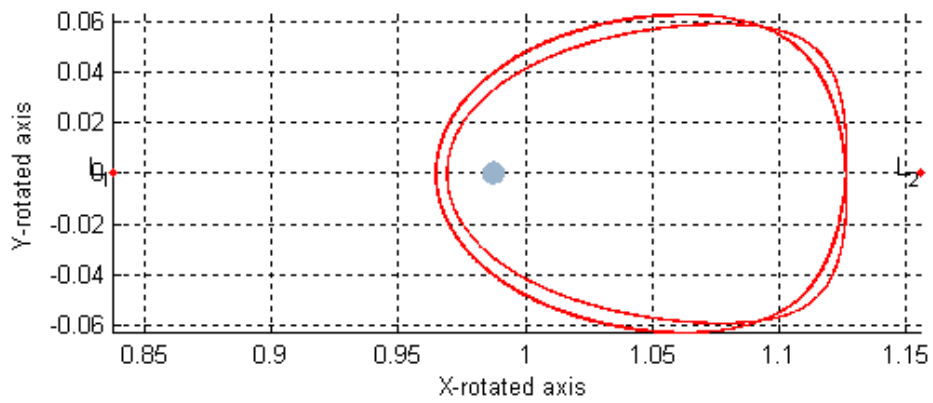


Figure 90 - Stable periodic orbits in $x_0=0.965$, (and $y_0=0, z_0=0$), $C=3.1819$, minimum altitude is 5426 km.

Another example about jumping one map to another is also shown in figure-89. Taking a slice in $z=0$, the motion maps can be reflected x_0-T_f (final time) plane, so planar stable/unstable periodic orbits can be reach. For this example, spacecraft jump from $C=3.181$ to $C=3.1819$ (or reverse for escape). Even this small change in Jacobi constant, the motion characteristic can be change huge in this energy level. Let's the initial condition on x axis is $x_0=0.965$ (and $y_0=0$, $z_0=0$), and velocity vector always perpendicular to the x - z plane. Using the equation 5.1, the velocity increment for jumping is computed as $\Delta V=0.5\text{m/sec}$. So, with $\Delta V=0.5\text{m/sec}$ increment, trajectory can be bounded as in figure-90. Or while spacecraft is in parking orbit as in figure-90, it can be escaped from L_1 for only $\Delta V=0.5\text{ m/sec}$.

In figure-90, the planar stable periodic orbits that picked from figure-89 is shown. Similar examples can be varied, and after finding desired initial or final trajectories on the Motion map for mission, the low-cost computation can be applied easily using equation 5.1. So, another problem is that finding those 'desired trajectories or parking orbits'.

5.2 Science Orbits and Advance transfer trajectories

One of the problems in multi body dynamics is to determining periodic orbits (either stable or unstable). Because of highly disruptive effect of external body, trajectories around secondary body are generally unstable in high energy. But there are also plenty of stable periodic orbits as they are seen in the motion maps, see chapter 4. Of course, depend on the mission; spacecraft can sit either unstable periodic orbits with continuous control process, or stable periodic orbits around secondary body with low-cost, please see section 4.5.

Because there are so many type periodic orbits around secondary body, so there can be so many mission type and application with them. Any stable and unstable periodic orbits can be already determined with various ways, such as: the most commonly used differential corrector method, which is mentioned in section 3.2.2.2. Although differential corrector gives precise initial conditions for periodic orbits, over sensitivity of the initial guess in this method is a main disadvantageous. But the motion map can be used such as grid search method for finding right and proper initial guess for Differential corrector.

Therefore, knowing this features, motion map can be conveniently used for determining approximate place of periodic and bounded orbits. For unstable periodic orbits, it may need more precise data, which can be provided from DC, whereas approximate place can be quite sufficient for stable bounded orbits.

As in section 4.5, stable region in plane motion map is already analysed. In diagonal direction on Motion map, bounded (green) region presents two main type orbits in specific energy level (while both L_1 and L_2 gates are opened); one is semi hemisphere covering orbits, and the other one is high altitude nearly circular orbits, see figure-73. An example for Semi hemisphere covering orbits can be seen in figure-91. These types of orbits can be good for polar probe, such as polar ice water probe or magnetosphere research.

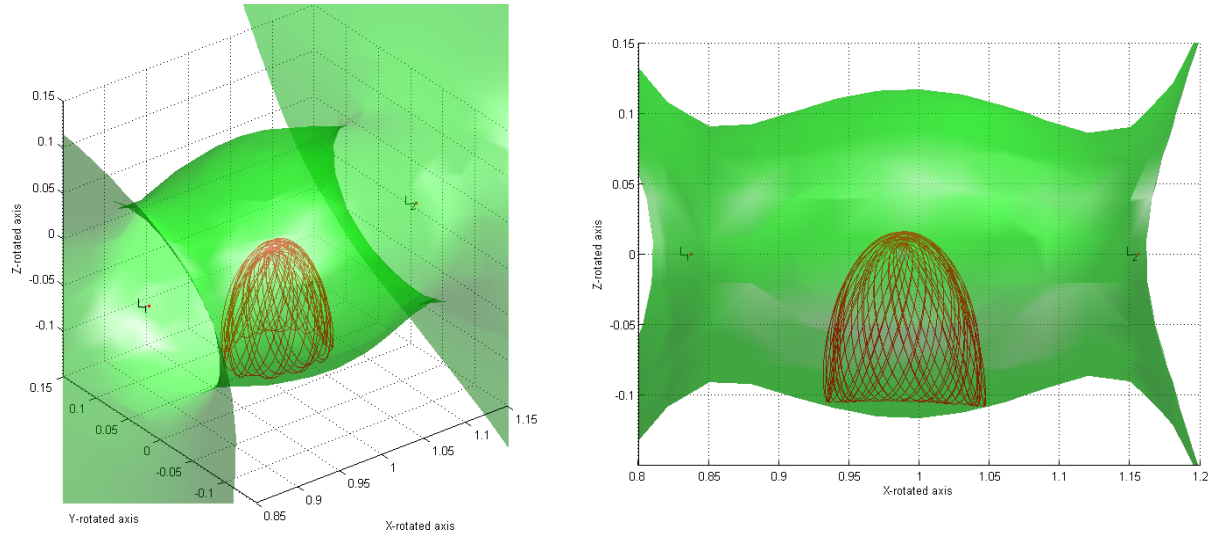


Figure 91 - South semi hemisphere covering around moon for Earth-Moon system. Initial conditions $x_0=0.9858$, $y_0=0$, $z_0=0.0045$, $C=3.16$, velocity vector perpendicular to x-z plane, direct orbits.

In figure-92, it is shown an example for high altitude nearly circular orbits. These types of orbits can be used as safe high altitude parking orbits or as rover trajectory.

These 2 bounded orbits types in figure-91 and 92 can be directly picked from green region of (X-Z) motion map. These type of orbits are quite stable that they don't need any extra control process to keep them bounded. But for unstable periodic orbits, some extra process can be needed, of course.

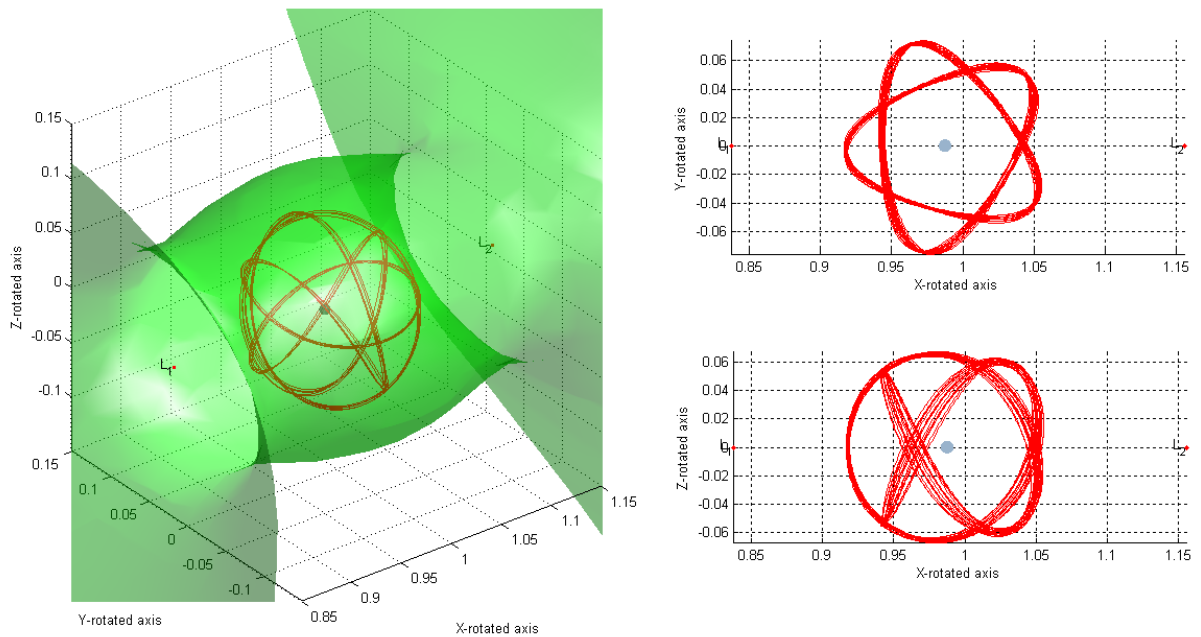


Figure 92 - High altitude nearly circular orbits, for Earth-Moon system. Initial conditions $x_0=0.9456$, $y_0=0$, $z_0=0.051$, $C=3.16$, velocity vector perpendicular to x-z plane, direct orbits.

As it is seen already and many times, the plane (x-z) motion maps in almost every energy level (except too high energies) can present plenty of stable periodic orbits for any mission, that the unstable periodic orbits can be also used in some promising mission trajectory. As it is shown in previous section 4.5.1, unstable trajectories can be used temporary/permanent parking orbits or as advance transfer trajectories. So, unstable orbits in secondary region can be classified in three main types;

- Unstable periodic orbits around secondary body,
- Unstable periodic orbits around co-linear equilibrium points,
- Unstable periodic orbits that makes homoclinic and heteroclinic cycles all around secondary body and co-linear equilibrium points.

Remind again, unstable periodic orbits around secondary body can be already determined differential corrector method. But because it is so sensitive about initial guess, and grid search for initial guess may take too long time, such as in [68]. Fortunately the motion maps can provide quite fast grid search that approximate place of desired periodic orbits can be find relatively short time (such as one map can be computed average ~ 15 min -3 hours, depend on its energy and step size) with great accuracy.

We have already shown that unstable periodic orbits can be also found in the boundaries between red and yellow regions, and from “crash waves” of the motion map, see sec. 4.4. Therefore using various energy level motion map, any unstable periodic between boundaries can be easily located. Such as, in figure-93 left, the motion map for Jacobi constant $C=3.171614$ is plotted. The region has been expanded close to the Moon where there are a number of boundaries between different outcomes are seen. In each of these boundaries there exists an unstable periodic orbit that will pass close to the surface. The periodic orbits in figure-93 right, is picked from the only “crash wave” with zero inclination, $z=0$. Using this location as an initial guess for Differential corrector process, a periodic orbit, which orbits with $z_0=0$, $x_0=0.982938161562$, $C=3.171614$, and passes within an altitude of around 150 km, is plotted.

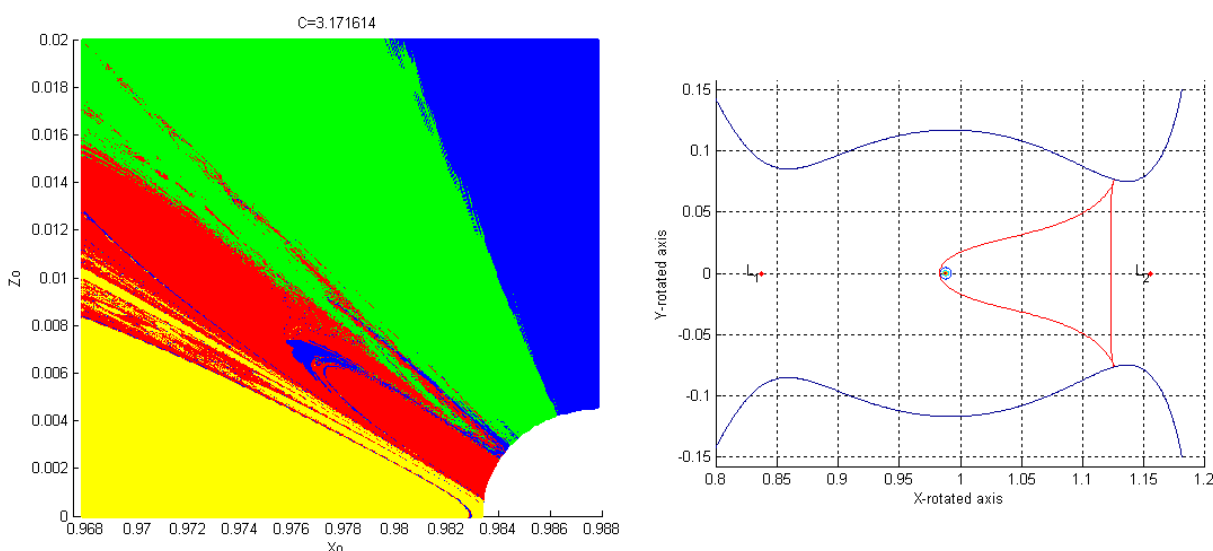


Figure 93 - Planar unstable periodic orbits, minimum altitude 150 km. map
 $T=100$. Initial conditions are picked from first waves in yellow region.
 $x_0=0.982938161562$, $z_0=0$.

As another example, in figure-94 initial conditions are chosen from between the boundaries of the red and yellow regions with $z=0$ and Jacobi constant $C=3.174$, than corrected in DC method. This is a more complex orbit that spends much of its time in elliptical orbit around the Moon before plunging in and passing within 264 km of the surface.

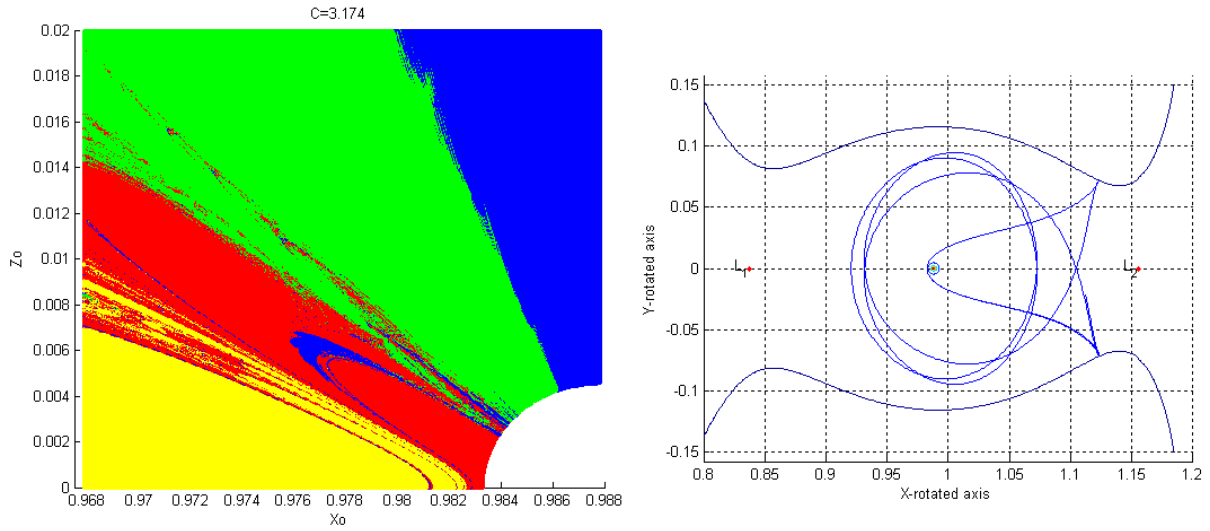
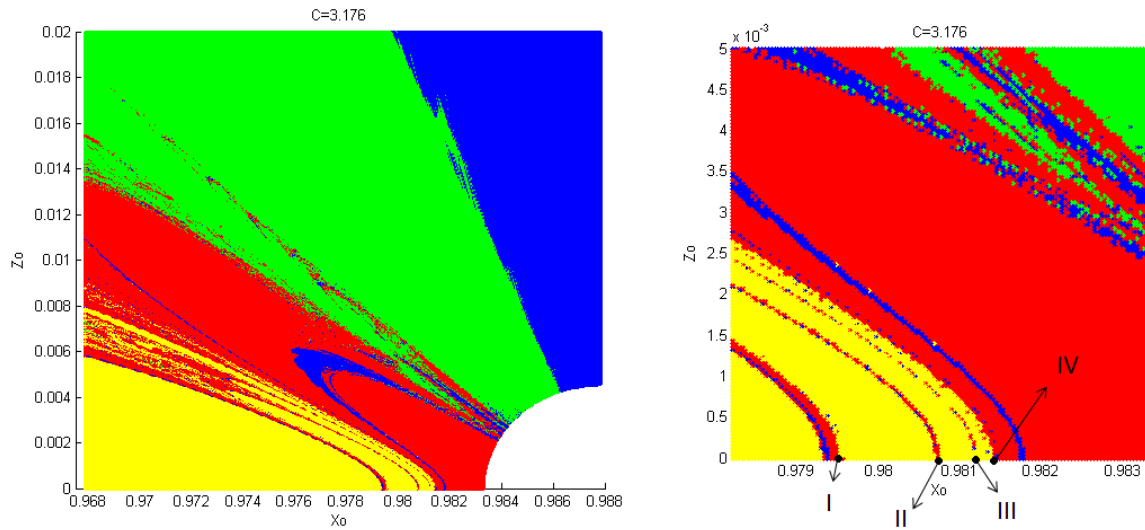


Figure 94 - Unstable periodic orbits, minimum altitude 264 km. Initial conditions are picked from boundaries between red and yellow region in $y=0$. $x_0=0.98264$, $z_0=0$.



a) Motion Map for $C=3.177$

b) Chosen initial conditions

Figure 95 - Plane $(x-z)$ motion map and focus in waves in yellow region. Every wave presents initial conditions of unstable periodic orbits.

By varying the Jacobi constant the minimum altitude to the secondary body can be adjusted and incorporated other useful features of the periodic orbit to exploit. Such as for more interesting example, the periodic orbits close to the secondary are investigated in figures-95 and 96 where a Jacobi constant $C=3.176$ have been chosen as an example. In figure-95 (a) the motion map is shown in this region and in figure-95 (b) four periodic orbits have been selected from their “waves” along the line $z=0$. In figure-96 these four periodic orbits each having a different minimum altitude are plotted, with corrected DC method. It is seen that they are making such a periodic motion with increasing numbers of elliptic turns around the secondary. The Points have been labeled as I, II , III and IV, which their initial conditions are;

- I. $C=3.176, x_0 = 0.979584223146, y_0=0, z_0=0, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- II. $C=3.176, x_0 = 0.980785884663, y_0=0, z_0=0, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- III. $C=3.176, x_0 = 0.981224840000, y_0=0, z_0=0, v_{x0}=0, v_{y0}=V, v_{z0}=0$
- IV. $C=3.176, x_0 = 0.981400680985, y_0=0, z_0=0, v_{x0}=0, v_{y0}=V, v_{z0}=0$

V is computed from Jacobi Integral and their shapes can be seen in fig-96. Same again in previous examples, using the data in fig.- 95, which present good initial guess for Differential corrector (DC) method, so these periodic orbits can be plotted more accurately with DC.

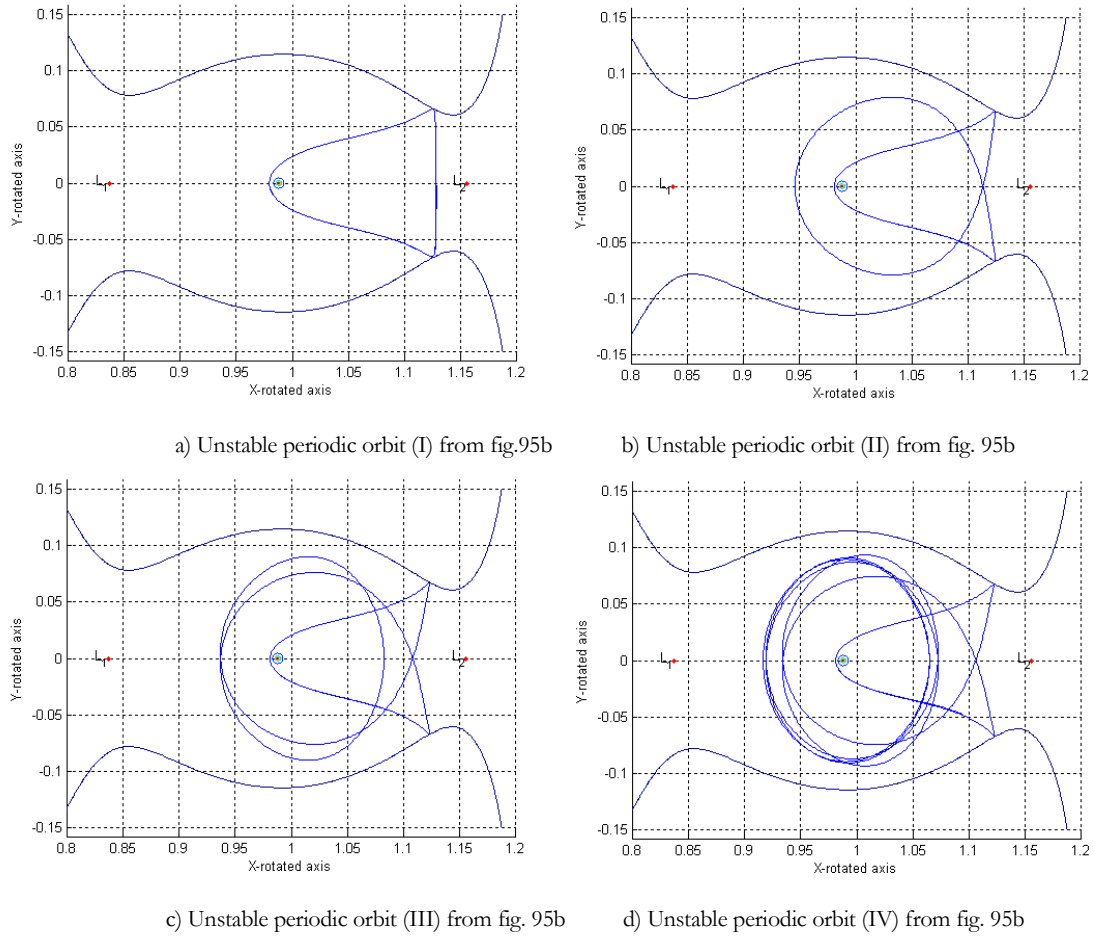


Figure 96 - Minimum altitudes are 1439 km for (I), 977.4 km for (II), 808.65 km for (III) and 741.1 km for (IV), respectively.

As it is seen, if the initial conditions on motion maps are chosen small vicinity of secondary body, low-altitude unstable periodic orbits can be caught. These types of orbits can be used as temporary parking orbits in specific time span, or permanent parking orbits with control process. The ‘waves’ as in figure-95, can occur in any plane motion map and they provide very various science orbits.

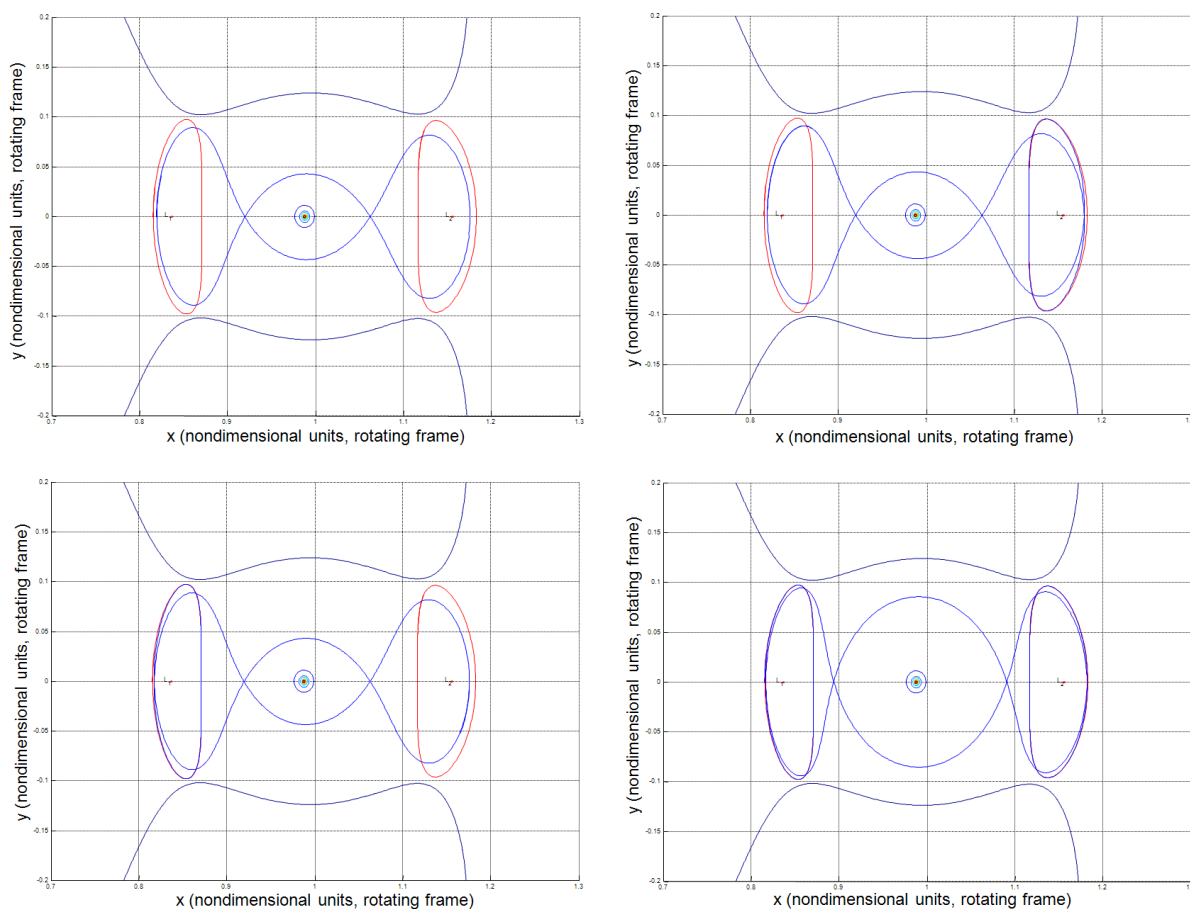


Figure 97 - Unstable periodic orbits, heteroclinic cycles. $C=3.16$.

Furthermore, in figure-97, it is show 4 periodic cycles types, which look like “quasi-heteroclinic orbits”, are shown as another interesting orbit options. The first one (upper left) makes half round around in both L_1 and L_2 equilibrium points. The second (upper-right) and third (below left) ones make half round in one equilibrium points and full round in another equilibrium points, but they make their motion symmetric each other. Such as, the second one makes half round around L_1 and full round around L_2 , while the third one makes half round around L_2 and full round around L_1 . The last one (below right) makes full round around both L_1 and L_2 equilibrium points. As it is seen, heteroclinic cycles provide great advantageous to transfer one halo orbits to another, and in fact, they are such as asymptotic orbits of halo orbits.

In figure-98, initial conditions for those quasi-heteroclinic cycles are shown on motion map. As it is seen initial condition of halo orbits and heteroclinic cycles are too close each other. This means that trajectory direction around equilibrium points can be change with small disruptive effect and so very small cost. Therefore L_1 halo to L_2 halo orbits transfer or heteroclinic transfer between the region can be done very low cost. In addition, interested readers may look at [101] for more homoclinic and heteroclinic cycles shapes, which can be probably found using by Motion map.

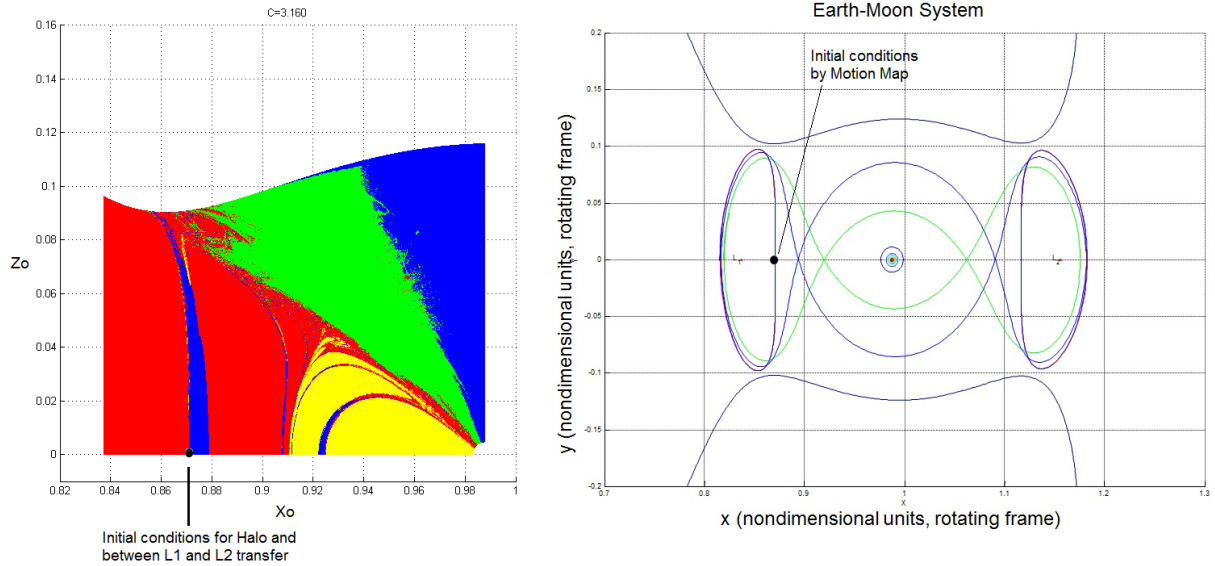


Figure 98 - initial conditions of planar-Lyapunov orbit and heteroclinic cycles on motion map (left), 2 type heteroclinic plots together (right).

5.3 Full Ballistic Capture

As it is introduced in section 4.4, ‘fuzzy boundaries’ (actually all boundaries) are present risky but very advantageous trajectories. They are risky because numeric errors are observed much more in that region, especially too long integration time. But of course satellite are not left on "their own" and some GNC maneuvers, which will be very small, can provide a cheap and robust control in there. So actually they are very advantageous because they present the initial condition of ‘very long-term capture’, which can be close ‘permanent capture’ and capture trajectories can be done so tiny or zero velocity cost. Although full natural ‘permanent capture’, which is mentioned in [22], is really hard to catch in high energy level (such as both L_1 and L_2 gate open) because of some other numeric error or real life disruptive effects, ‘long/short term capture’ can be still used safely for either with tiny controls or in a ‘specific’ time span.

Such as; unstable periodic orbits, which are shown previous section, can be used short term capture missions. In figure-99, two short terms capture trajectory are shown. Left side is captured for ~ 78 days from both sum of forward and backward integration time, and trajectory capture then escape from L_1 . Right side is captured for ~ 68 days from both sum of forward and backward integration time, and trajectory capture then escape from L_2 .

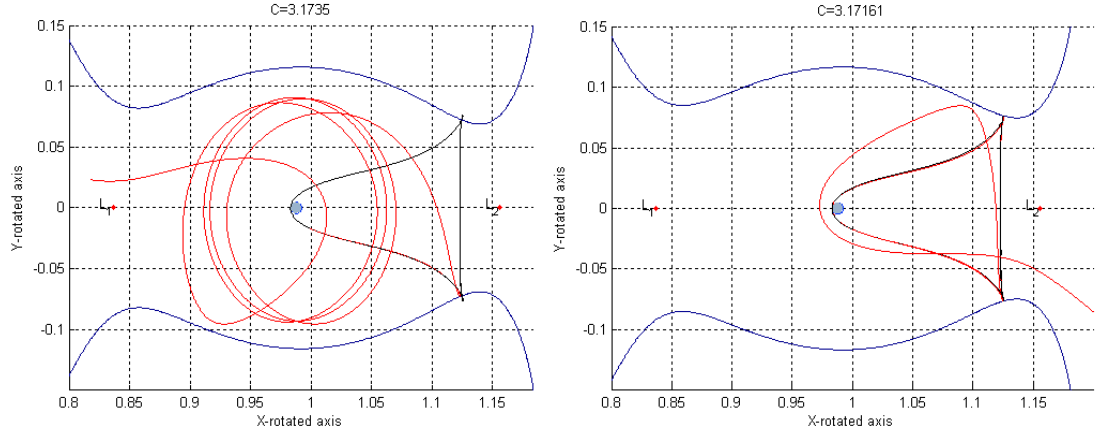


Figure 99 - Short term capture: L_1 escape for ~ 78 days (left), L_2 escape for ~ 68 days (left).

These trajectories in figure-99 are captured then escape without any velocity increment, but if the spacecraft is wanted to stabilised on trajectory, which is shown in black orbit ($C=3.171614$), it is needed to very tiny velocity increment, which can be computed using e.q. 5.1. The initial Jacobi constants may be $C_i=3.1735$ and $C_i=3.1761$ (see figure-99), and final Jacobi constant $C_f=3.171614$. Therefore $\Delta V=0.44323$ m/sec for left plot, $\Delta V=9.4 \cdot 10^{-4}$ m/sec for right plot. Then it is needed control process for station keeping, of course.

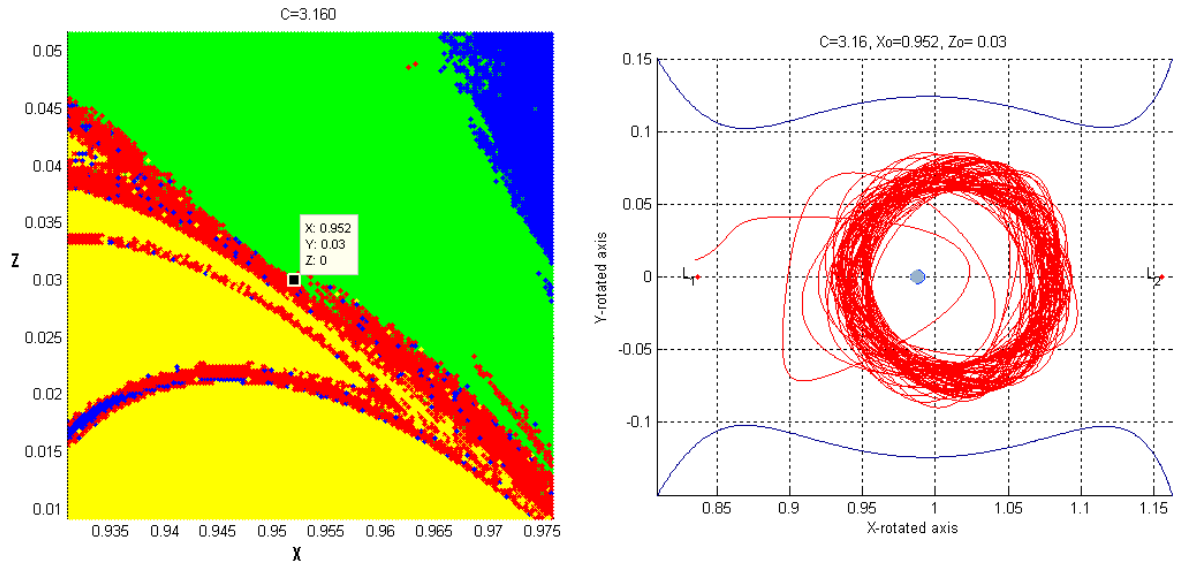


Figure 100 - Moon long term capture using the fuzzy boundaries, $T=\sim 2.9$ years for both sun of backward and forward integration, minimum altitude is 12.110 km.

The long term capture can also obtained using x-z plane motion map. But ‘too long integration time’ cause numeric errors, so there must be limit for time span, which is explained detaily in section 4.3, or it must be used some control process and GNC maneuvers. Such as, in figure-

100, a long term capture example is given. Trajectory escape after making ~ 1.46 years periodic motion around moon, if we don't apply any velocity increment. Same condition is valid in backward integration, it means capture. So in total, trajectory come to moon region, makes ~ 2.9 years nearly circular motion, then it escape to place where it came from, and all these happen in zero-cost. These types of long term capture trajectories makes nearly circular motion and their altitude are relatively too high.

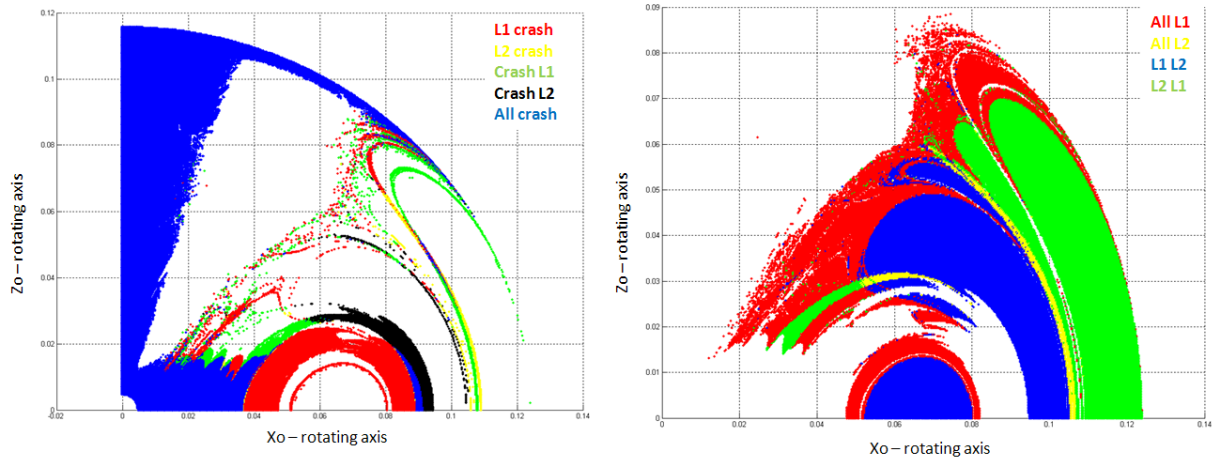


Figure 101 - Apart regions from figure 42. Left side is crash orbits, right side is L_1 and L_2 escape and capture orbits, in $C=3.16$, Earth-Moon system.

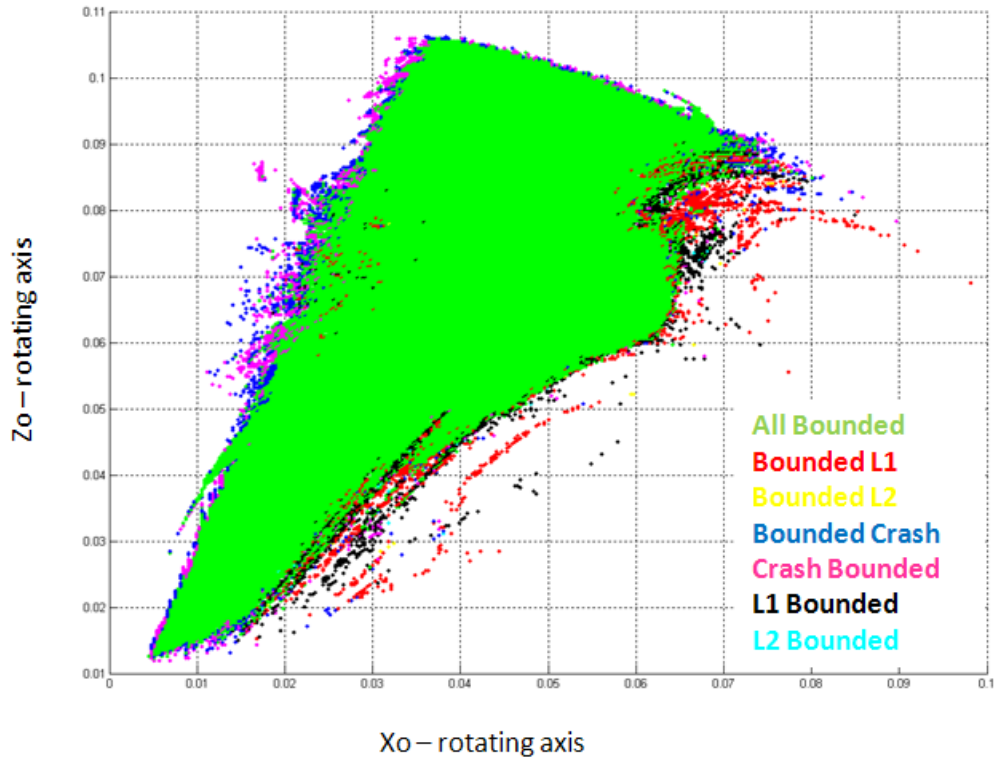


Figure 102 - Apart regions from figure 42. Bounded orbits and other combinations. $C=3.16$, Earth-Moon system.

As it is said in section 4.2.1, x-z plane (or $y=0$ plane) give symmetric result in backward and forward integration, because $y=0$ plane (or x-axis) is in CR3BP, but there is no symmetry in $x=1-\mu$ plane, so backward and forward integration in y-z plane is become different, as it is seen in figure-41. In figure-101, some apart regions from figure-42 is shown. Left plot show crash combination in backward and forward time. Such as; blue region is crash for both backward and forward time, red region is crash in backward, but it escape L_1 for forward integration. The right plot also show escape and capture combinations, such as: blue region is escape L_1 for backward and capture L_2 for forward integration, and the green region is reverse.

In figure-102, apart bounded region is shown. As it is seen, for 'permanent capture', red (forward is bounded, backward is L_1 escape), yellow (forward is bounded, backward is L_2 escape), black (forward is L_1 escape, backward is bounded) and cyan (forward is L_2 escape, backward is bounded) region would be useful. However, the yellow and cyan region is not have so big regions, this means that 'permanent capture' from L_2 gate is quite rare in $C=3.16$ Jacobi constant.

In fact, the "scattered boundary" in figure-102 are actually our fuzzy boundaries, which are discussed in section 4.4. Same as X-Z Plane map show previous sections, these boundaries can be addressed also in terms of the role of manifolds, which make transversal intersections, as showed in figure-103.

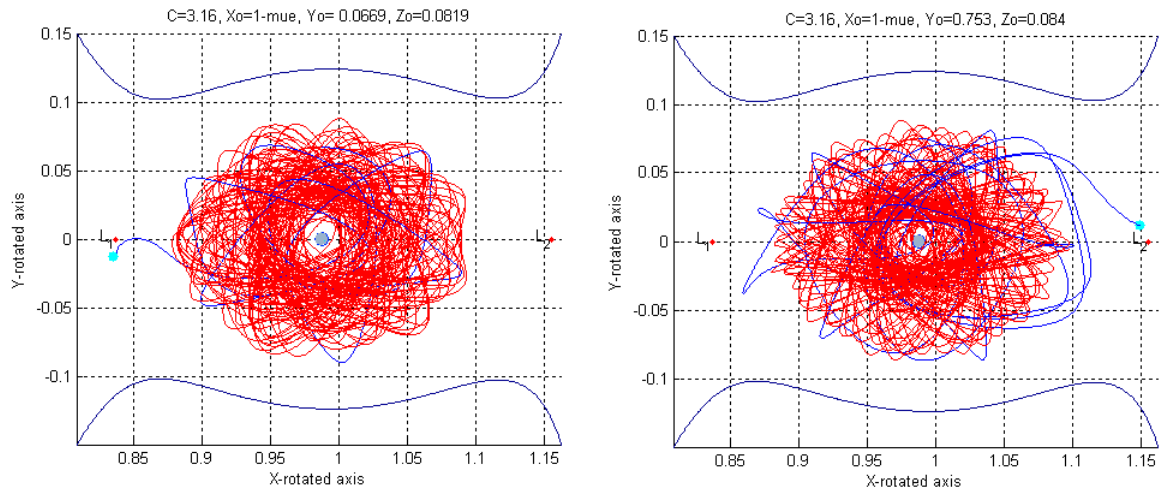


Figure 103 - Permanent capture trajectories from figure 102, Left plot escape L_1 (min. alt. 2446 km), right plot escape to L_2 (min. alt. 1159 km). $C=3.16$, Earth-Moon system.

In figure-103, two sample trajectories for full natural permanent capture, whose initial conditions are taken from fuzzy (or scattered) boundaries in figure-102, is shown. In the left side plot, trajectory is bounded for forward time, but it is captured from L_1 after ~ 43 days in backward time integration. And minimum altitude for this trajectory is 2446 km. In the right side plot, trajectory is bounded for forward time, and it is captured from L_2 after ~ 91 days in backward time integration. And minimum altitude for this trajectory is 1160 km.

5.4 Earth to Moon Transfer

In this section, three scenarios for transfer between the Earth and the Moon shall be considered with using the motion maps:

- I. We use the stable manifold of periodic orbits around the L_1 point to transfer from the Earth to this equilibrium point and then transfers to the unstable manifold to reach the Moon, see section 5.4.1,
- II. We use a Hohmann type transfer from LEO to a location on the motion map and then adjusts the energy to reach a long term bound orbit of the Moon, see section 5.4.2,
- III. and we use the Earth-Sun equilibrium points to reach the Earth-Moon equilibrium L_1 point for ballistic capture such as used by the Hiten spacecraft [21,22].

These approaches are just presented for giving some ideas how the motion maps can be useful and advantageous for mission designs. As it is seen in following applications, the basic strength of motion map is to provide very low-cost final thrust for parking the spacecraft in stable parking orbits, and so prevent fuel waste. Although this fuel gain seems too negligible in relative to total mission cost, it can be very significant under certain circumstances, such as; rescue missions, some life-extension of spacecraft, and/or some other emergency situations.

5.4.1. First Scenario; Earth - Moon Transfer

For the first approach we shall consider the satellite transfer from a high altitude circular orbit around the Earth, as similar in [23]. Designing by backward, as this example, $\sim 20,000$ km altitude the Moon parking orbit is picked as $x_0 = 0.93$, $z_0 = 0.04$ on $C=3.17$ Motion map (velocity vector is perpendicular to the X-Z plane), and L_1 escape (forward time) or capture (backward time) trajectory is set in the same initial conditions but different C value, which is $C=3.16$, and so Motion map, see figure-104. Using the eq. 5.1, the velocity decrement for bounding a trajectory (same initial conditions $x_0 = 0.93$, $z_0 = 0.04$) by decreasing the Jacobi constant from $C_1 = 3.16$ to $C_2 = 3.17$ is about $\Delta V_2 = 13$ m/s over this area of the Motion Map. The key point to emphasize is this means trajectories can become bound without imprisoning them in the zero velocity curves. For a comparison, if the initial Jacobi constant was fixed at the value at the L_1 point ($C=3.2005$), then a 56 m/s velocity decrement would be required to bound the orbit to the Moon. The figure of 13 m/s represents a significant saving in propellant required at this late phase of the transfer. Such transfers may be particularly relevance in rescue missions where propellant has been lost for some unforeseen circumstance.

So in figure-104, $C=3.16$ and $C=3.17$ motion maps and these trajectories picked on $x_0 = 0.93$, $z_0=0.04$ are shown. The upper panes show the locations on the respective motion maps and the lower panes show the actual trajectories. On the below-left, the escape trajectory has been projected onto the (x-y) plane to get an understanding of the motion with respect to the Earth-Moon orbital frame. On the below-right the 3D bounded trajectory around the Moon is shown.

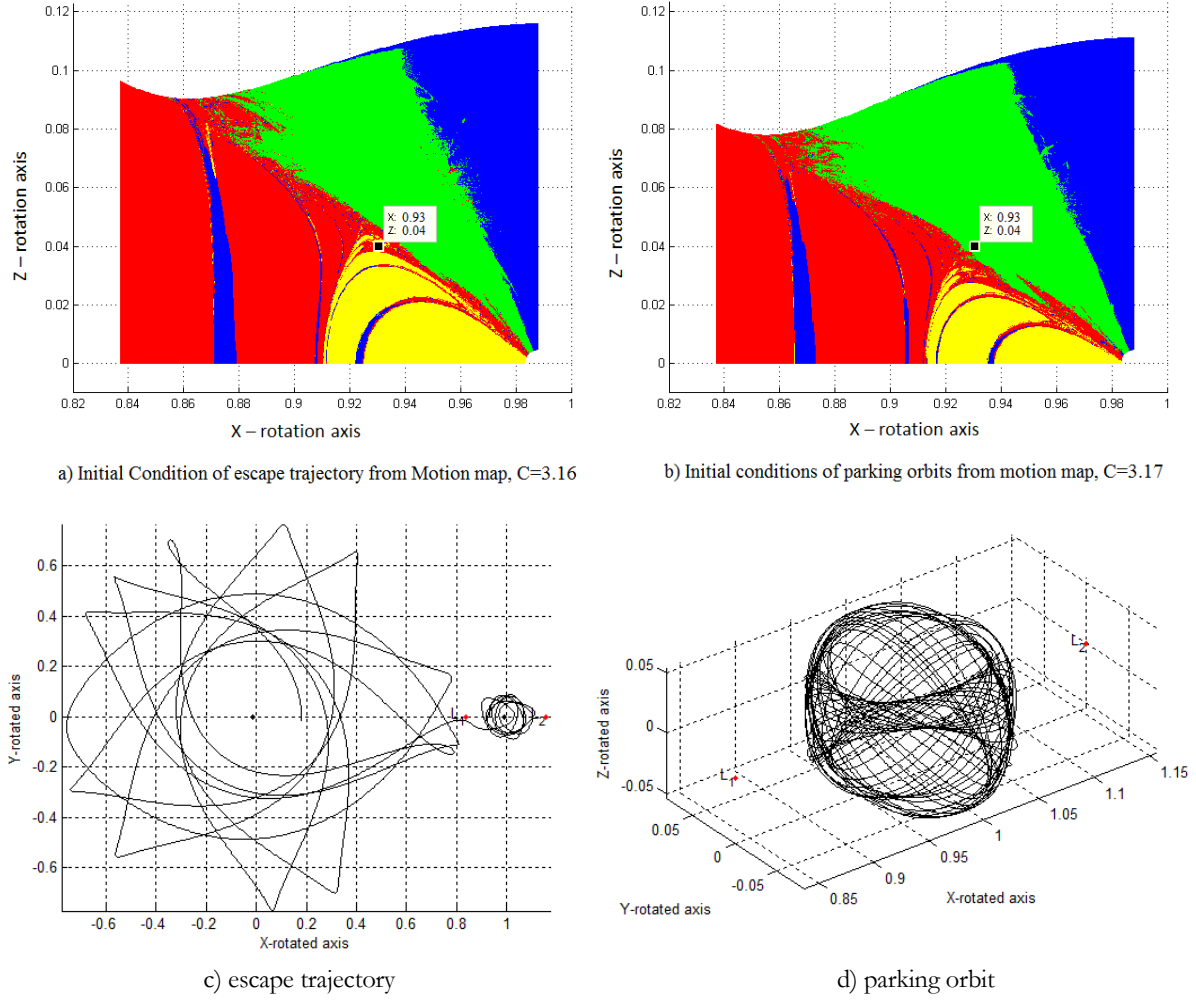


Figure 104 - Earth region to Moon region transfer using motion map. Upper Motion maps show chosen initial conditions in x-z plane, and below plots show the trajectory from these initial conditions. Below left is for $C=3.16$ and below right is for $C=3.17$.

After parking and escape (or capture in backward time) trajectories are chosen, the backward-integration is done until the trajectory (which is in figure-104 left-right) approaches in desired altitude, say, within 63,102 km altitude from Earth after 159 days. A circular trajectory of this radius is in $C=6.1$ energy. Therefore if a spacecraft orbits around the Earth in a circular trajectory of this radius, from $C_1=6.1$ (That is energy of high altitude Earth circular orbit) to $C_2=3.16$, a $\Delta V_1 \approx 621.5$ m/sec would be required to bring the spacecraft to the correct point on the Motion Map.

So using eq. 5.1; initial conditions (i.c.) is taken from position when L_1 capture trajectory (black orbit in figure-105-left) is tangent to high altitude Earth orbit (red orbit in figure-105-left). Also, by increasing the initial altitude of the Earth orbit, such as 118,980 km, the transfer time can be reduced to 7 days and the total ΔV can be depend on initial parking orbit; such as highly elliptic or circular as in show in fig.105 right. As for $C_1=6.136$ highly elliptic initial parking orbit with 233km minimum altitude (red orbit in fig.105 right), it is needed $\Delta V_1 \approx 1543.6$ m/sec (from

$C_1=6.136$ to $C_2=3.16$). Also for $C_1=6.136$ high altitude nearly-circular initial parking orbit, it is needed $\Delta V_1 \approx 316,072$ m/sec (from $C_1=4.14$ to $C_2=3.16$); blue orbit in fig.105 right.

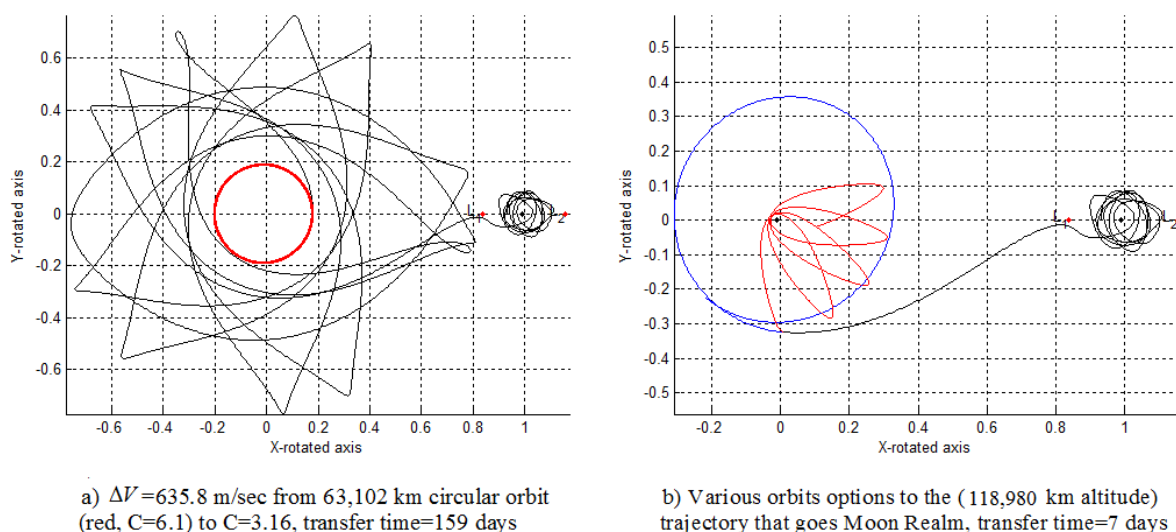


Figure 105 - Comparison of transfer trajectories from high altitude Earth orbits (blue orbits) to Moon.

As it is seen in figure-105, there are various ways to get Earth to Moon transfer; it means once we can catch any “tail” of any transfer trajectories of L1’s IMT (such as black trajectories shown in figures-104 and 105), then we can apply second thrust from $C_1 = 3.16$ to $C_2 = 3.17$ is about $\Delta V_2 = 13$ m/s. And all these transit trajectories from secondary (Moon) region to primary (Earth) region are actually in side of the invariant manifold tube, as explained in section 3.2.3. Unfortunately, stable and unstable manifold tubes of L1 halo orbits are not reach low-altitude of Earth [16], so a dual propulsion system (ΔV_1 and ΔV_2) should be used to exploit this aspect of the natural dynamics for Earth-Moon transfers, in this example. Further similar studies about this type of transfers can be found in [16]. Particularly, merge with intermediate arcs from LEO to HEO with Motion map, much more flexible mission design can be studied.

5.4.2. Second Scenario; Earth - Moon Transfer

The second approach is illustrated simply in figure-106, where we start from a much lower altitude parking orbit, LEO (Low Earth Orbit), and then shot to the motion map (on vicinity Moon) perpendicularly. This can be also thought as a merge of Hohman transfer and Motion map, because beginning of the travel start with Hohman and end with motion map.

The first ΔV_1 transfers the satellite out close to the Moon, then reaching the plane of the motion map. A manoeuvre at this time inserts the satellite into its final parking orbit. The first velocity increment is done from LEO (Low Earth Orbit) and is about 3160 m/sec [16]. Then after reaching the moon region, second velocity increment is done in the vicinity of Moon while trajectory intersects the x-z (motion map plane) perpendicularly. To provide this perpendicular insertion to motion map, backward integration technique is used again. It means, a desired initial conditions of parking orbit is chosen on motion map first, then jumping suitable energy level, this trajectory in this initial conditions is integrated backward time until it reaches to LEO.

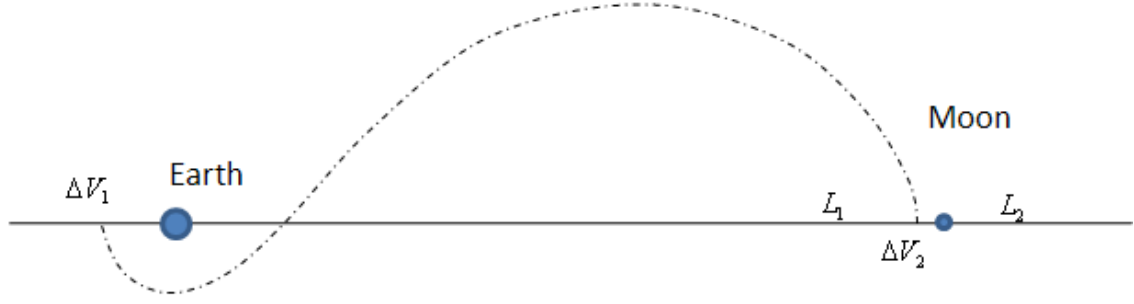


Figure 106 - Earth to Moon transfer scheme. (Earth- moon rotation frame)

The main trick in here, neither “direct” nor “retrograde” trajectories are never close the LEO, unless they aren’t in very high energy level, which about $C < \sim 2.5$. Also, within “direct” and “retrograde” trajectories, only “retrograde” ones can reach to LEO conveniently very short-time, whereas “direct” ones can hardly reach after very long travel. So as the first version, “retrograde” motion map is examined

For initial analyse of “retrograde” motion map, contours of the ΔV required to change the energy from the inbound high energy ($C \sim 2.3$), to the target energy of $C = 3.16$ is shown in figure-107. As similar in previous analyses, the velocity increment is much lower in the vicinity of the Moon, for this transfer the velocity increment, ΔV_2 can be about ~ 200 -400 m/sec.

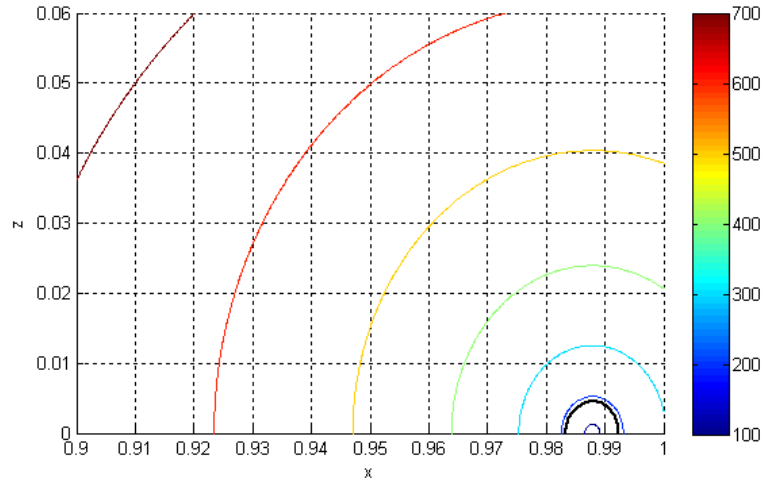


Figure 107 - $\Delta V(x,z)$ contour plot in secondary region for Earth-Moon, jumping $C_1 = 2.3$ to $C_2 = 3.16$, or reverse. Left bar shows velocity increment values (m/sec) that matches contour. Black tick curve is boundaries of Moon. Left side plot is focused in low altitude.

We next consider the parking orbit around the Moon. As it is mentioned in chapter 3, retrograde trajectories are already more stable than direct ones. And because these trajectories reach the Moon with very high energy, retrograde trajectories can be more advantageous to stabilise them in safe. Therefore, to finding suitable low cost parking orbit, we plot the motion map for retrograde trajectories for x-z plane, as in figure-108.

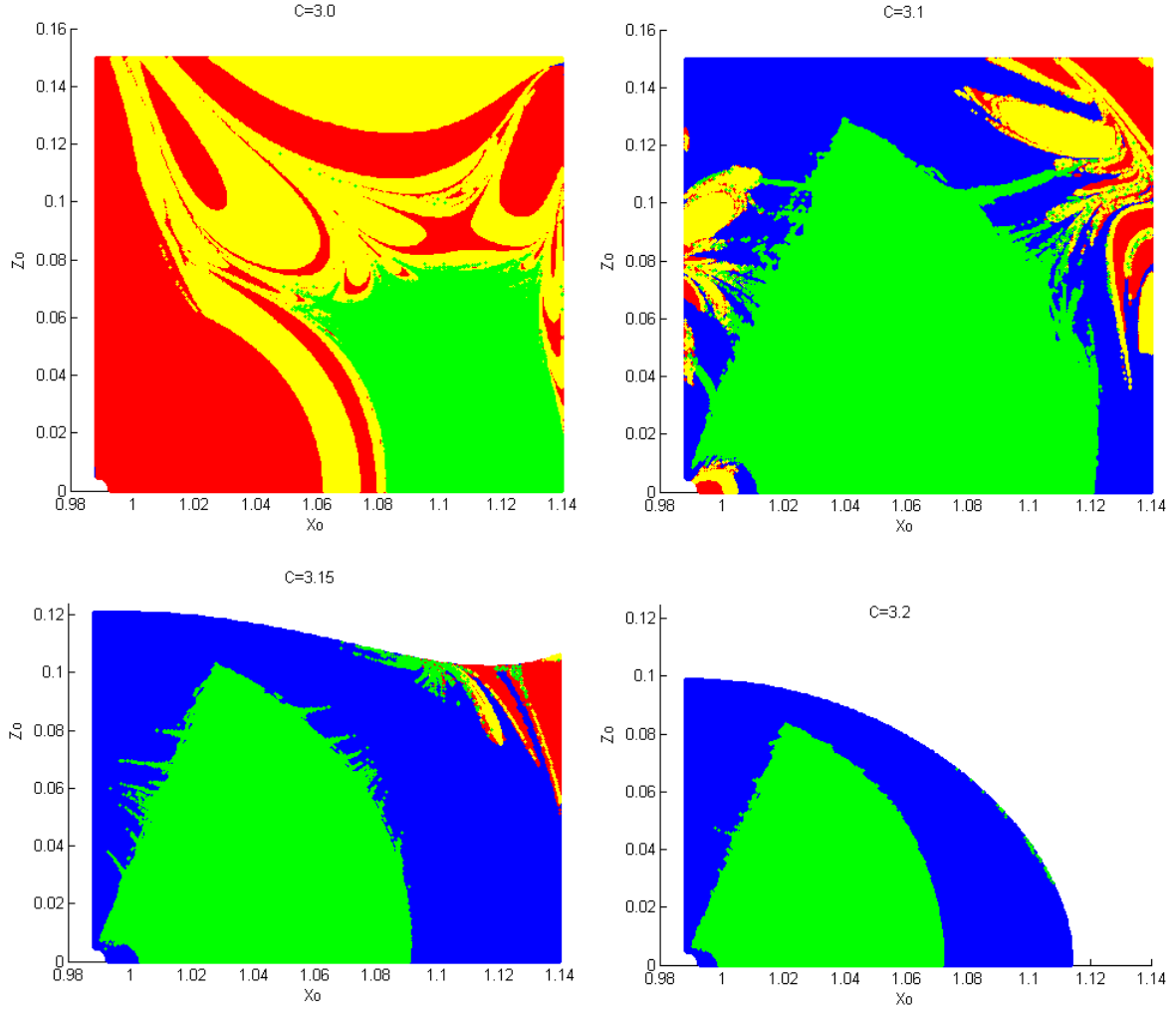


Figure 108 - Motion map for retrograde trajectories. Blue is crash, Green is bounded, red is escape L_1 and yellow is escape to L_2 .

Also very tricky note we mustn't choose high inclination and low altitude initial conditions (i.c.) in the maps (for either retrograde or direct), because in that position, the moon's gravitation catapults the trajectory out towards the Earth with very high inclination oscillations. In backward time such trajectories would not reach low altitude Earth orbit in "reasonable" time. Only **high inclination - high altitude** trajectories and **low inclination- low and high** altitude trajectories can reach low altitude orbits around the Earth in relatively short times, with energies $C < 2.5$. But, of course, **low inclination- low altitude** initial conditions would be preferred to reduce the velocity increment at the motion map. But if we are considering for the direct motion map, the bounded region is always at high inclination. High altitude i.c. choice would present very high ΔV and low altitude i.c. doesn't reach close to the Earth, in reasonable time, which this version will be shown following examples, also.

So the best option for now, therefore, is to use the retrograde map to reduce both **the transfer time and the total cost**. From figure-108 we see there is no green region in the close vicinity of the Moon, unlike the direct motion map (figure-38). But in figure-109, a retrograde parking orbit, which is as close as possible to Moon, is chosen from the motion map for $C=3.2$, as an example. The initial conditions are chosen near the boundary between the blue and green regions at low altitude and zero inclination.

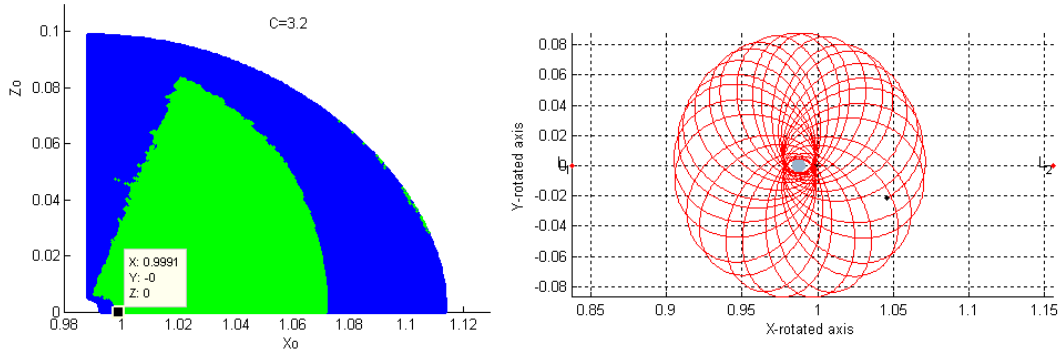


Figure 109 - retrograde parking orbits from motion map. Min. altitude is 71 km, $e \sim 0.7$

After that, initial conditions are integrated backwards in time, and the Jacobi constant ($C=2.052$) is set for reaching 180km altitude of Earth, Figure-110. So in forward time; The ΔV_1 for insertion from LEO into the transfer trajectory (as first impulsive thrust of Hohmann Transfer) was about 3160 m/sec [16], while the ΔV_2 for insertion into a retrograde parking orbit around the Moon is ~ 374 m/sec, using eq.5.1 for jumping from $C=2.052$ to $C=3.2$. The total $\Sigma \Delta V = 3534$ m/sec for a 3.34 day transfer, see Fig.-110.

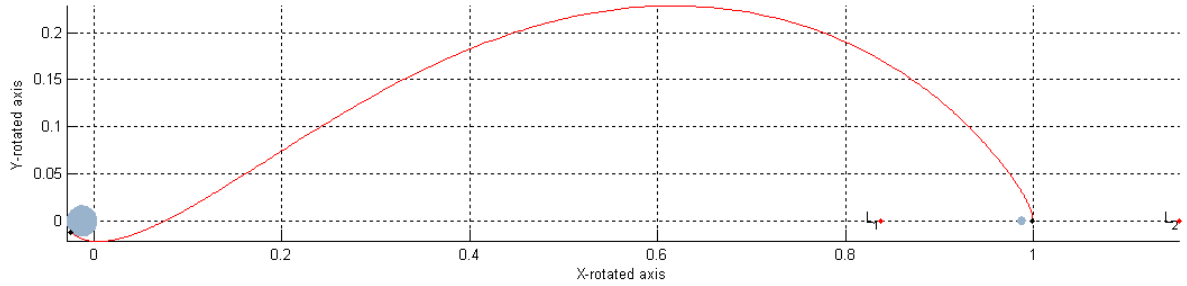


Figure 110 - 1st approach: Short Transfer trajectory for retrograde orbit. $C=2.052$. Transfer time 3.34 days.

If trajectory is kept to integrated in different Jacobi constant (such as $C=2.27$), then another transfer solution can be also derived. Using same initial condition in figure-109, trajectory is integrated backward time in $C=2.27$, and after 30 days, trajectory is reach to 180 km altitude of Earth. ΔV_2 for getting retrograde parking orbits around moon is ~ 309 m/sec for this case. In total $\Sigma \Delta V = 3469$ m/sec for 30 days, see figure-111.

As it is seen, transfers in figure-110 and 111, is such a merge of Hohmann transfer and 'Ballistic capture' process with using motion map. In the beginning of the transfer (From LEO Earth), it is started as Hohmann method, then final low-cost ΔV (In Moon region) is applied using Motion map, figure-109. Using this merge process, many transfer trajectory set with low ΔV and ΔT (transfer) can be obtained. And most significantly low energy transfer doesn't have to be too long time, for reducing ΔV a little.

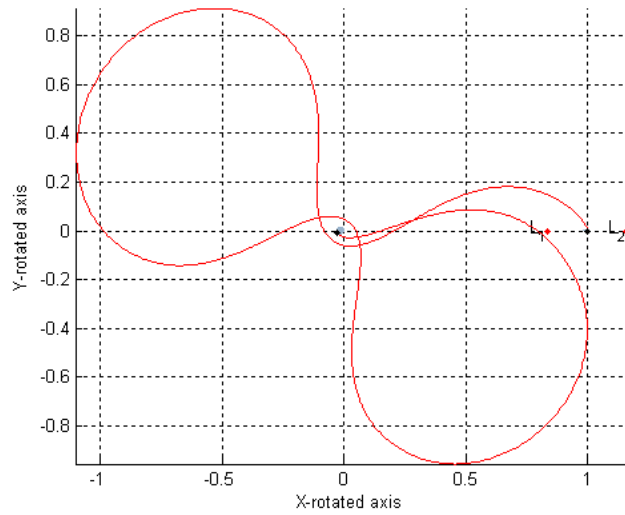


Figure 111 - 2nd approach: Long transfer trajectory for retrograde orbit, $C=2.27$. Transfer time 30 days.

As another example for merging process is applied in figure-112 for high altitude final lunar orbit, using the “direct” motion map. The application can be demonstrated from low Earth orbit (200 km) to high altitude, direct lunar orbit (12,690 km). When the trajectory intersects the x-z plane, a $\Delta V=564\text{m/sec}$ is applied to jump from $C=2.14$ to $C=3.175$, with a total $\Sigma\Delta V=3715\text{ m/sec}$ and a transfer time of 5.67 days.

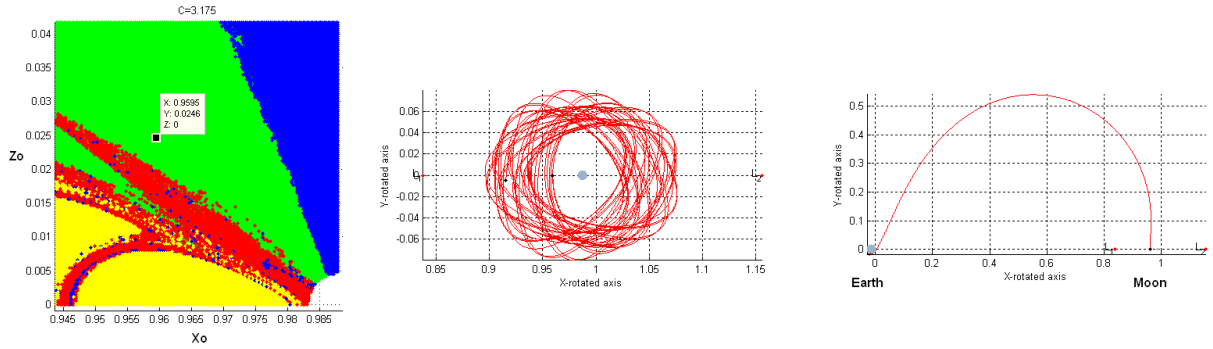


Figure 112 - 3rd approach: Chosen initial conditions from Motion map and transfer trajectory which is computed backward time using these initial conditions for $C=2.14$. Transfer time = 5.67 days.

For transfers where the transfer time is not of importance, then *the high inclination and low altitude* direct trajectories can be also used for reaching LEO. For example, using the initial conditions in the motion map for a 600 km altitude lunar elliptic parking orbit, as shown in figure-113, and integrating backwards in time in $C=2.119$, the trajectory eventually reaches 165 km altitude LEO after 152 days, figure-114. The velocity jump at the motion map (from $C=2.119$ to $C=3.16$) is $\Delta V_2= 217.9\text{ m/sec}$ and $\Delta V_1\sim 3160\text{ m/sec}$ (from LEO to red transfer trajectory in figure-114). So in total, whole mission can cost about $\Sigma\Delta V=3378\text{ m/sec}$ for 152 days.

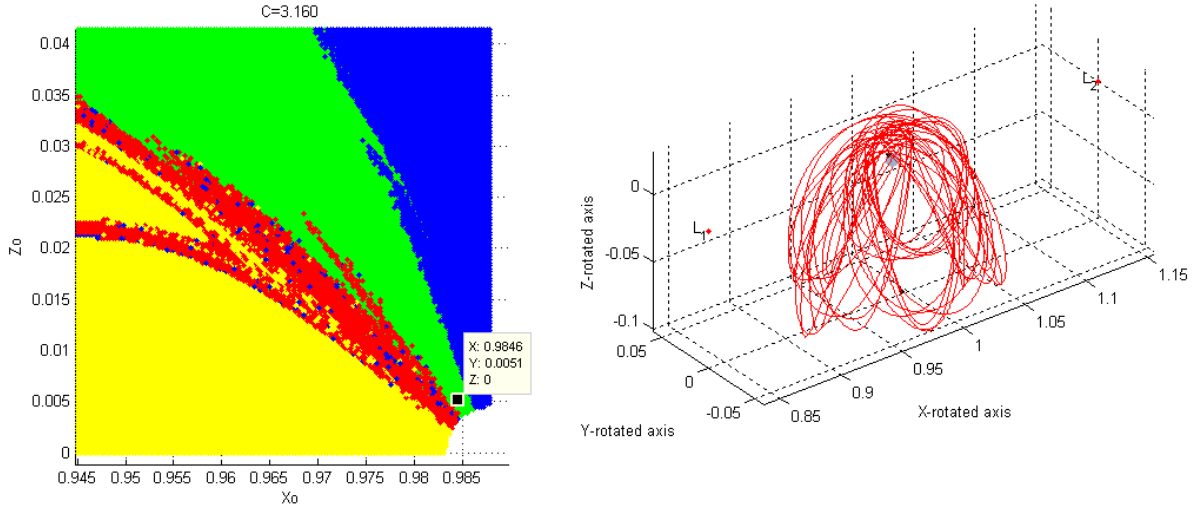


Figure 113 - Chosen initial conditions from Motion map and transfer trajectory which is computed backward time using these initial conditions for $C=2.119$. Transfer time = 200 days.

These examples can be varied for different transfer times and energies (various Jacobi constants), and we can find a family of solutions for various combinations. Although such transfers do not provide much total ΔV saving from Earth-LEO to Moon, they can be quite useful in some critical final ΔV thrusts with done very low fuel. Also note that, if Earth-Moon transfer is done from Earth-GTO (or any Earth's highly elliptic parking orbits, such as shown in figure-105-b red trajectory) to any Moon's stable orbits, whose example's actually given in section 5.4.1, then transfer mission cost can be reduced about one third of traditional Hohman transfer's cost.

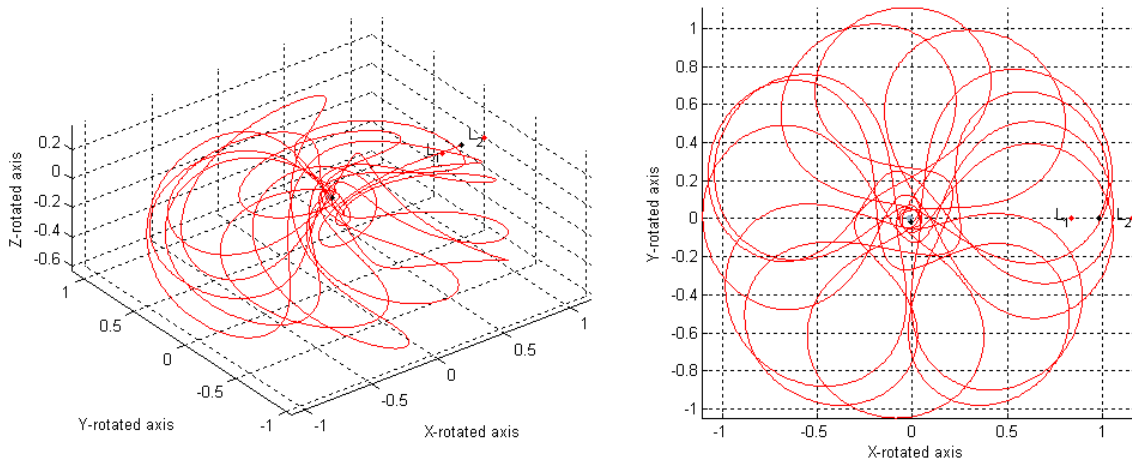


Figure 114 - 4th approach: very long transfer trajectory from 140 km LEO to Moon. $C=2.119$. Transfer time = 200 days.

However, in short for this section now; if Earth-LEO to Moon transfer is considered, so that, the initial conditions on motion map must be chosen in either low (or non) inclination or high altitude-high inclination, for obtaining both less Δt and total ΔV . For this, the approach in figure-110 gives best solution in all scenarios and examples in the term of optimum Δt and ΔV . As in figure-103 and 104, the transfer time is dramatically increased for only about $\sim 4.4\%$ ΔV

gain. Therefore, evaluation of trajectory may not be so necessary if we think about both scope of total Δt and ΔV ; but of course, it is all depend on mission design.

5.4.3. 3rd Scenario, Earth - Moon Transfer

The third way to reach the Moon from the Earth is using the combined gravitational effect of the Sun, Earth and Moon, as similar in Hiten [15,21,22]. This type of complex mission design can be conveniently done by integrating the trajectories backwards in time starting from the lunar parking orbit. So first of all, we start from the motion map of the Earth-Moon system computed for a suitable energy level (see figure-87). Once the parking orbit is chosen, its energy must be close enough to the boundary between the green (bounded) and escape regions to reduce the velocity impulse needed.

For this example, a low-altitude (~ 500 km) high eccentricity periodic orbit is chosen in the motion map with energy $C=3.185$, as in figure-87. And, in figure 88, we've showed the motion map for a range of energies between $C=3.12$ and $C=3.17$ indicating this trajectory. In this mission example, trajectory must approach the Moon through the L_2 gate and the change in Jacobi constant required is sufficiently small (see figures-88) that the velocity change required is much smaller relative to previous works.

For each energy we can use eq. 5.1 at the 'fixed coordinates' ($x=0.9833$, $z=0.0018$) to compute the velocity increment required (see table 5.1). In this example we choose $C=3.13$ with a ΔV of 13 m/s. With this choice we can then compute the trajectory backward in time until it leaves the Moon's gravitational influence.

After that, trajectory must be transferred, and The Sun-Earth and Earth-Moon binary system should be patched, as shown in figure-32,33. For this, it is needed to do some transformation between the binary system as given in section 3.1.3. Let's the Earth-Moon binary system is assign as 'A' and Sun-Earth binary system is assign as 'B'. So trajectory is transferred from A to B; $A \rightarrow B$.

Using the transformation equation in section 3.1.3, Earth-Moon to Sun-Earth transformation is done. At the end, Earth-Moon trajectory relative to the Sun-Earth $\{X_B^{ro,O}\}$ is obtained with including orientation Matrix (O), in the case of orientation differences between A and B (see figure-12; such as there is $i \sim 5^\circ$ inclination between Sun-Earth and Earth-Moon binary system), see in eq. 5.2.

$$X_B^{ro,O}(0,t) = OX_B^{ro} \rightarrow X_B^{ro,O}(\text{final}) \rightarrow C_{SE} \quad (5.2)$$

So then, final data of this trajectory is become initial condition for trajectory that will computed Sun-Earth system. And also using this initial condition, the Jacobi constant for Sun-Earth system can be computed, too. So then while jumping one trajectory to another, the velocity increment may be applied or even it can be set **zero increment** in 'midcourse manoeuvre', such as in [75], and as it is done this example. So then, let's start the first step of transfer.

The first step of the transfer will be the transfer from Moon to Midcourse point, as backward time of course. In figure-115, using the positional condition ($x=0.9833$, $z=0.0018$) in figure-87, a trajectory is integrated to backward from Moon to Midcourse point. The integration can be cut after trajectory pass 45° from Earth-Moon axis, this is averagely enough time for avoiding of gravitational effect of Moon (sphere of influence of Moon) in a rough computation. Same computation can be also done for other energy level that shown in figure-88, for comparison. And, **for zero-cost**, ‘midcourse’ manoeuvre time can be set any time that enough far from moon effect.

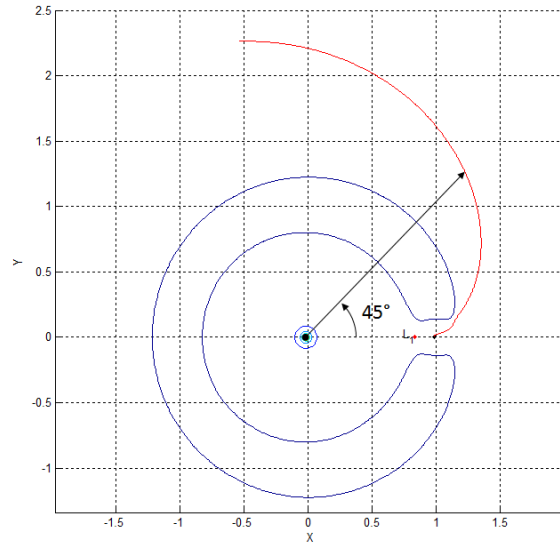


Figure 115 - The first step of the Earth-Moon (or reverse) transfer. Trajectory is integrated to backward.

For this example, to patch the orbit to the Earth-Sun system we adopt the bi-circular model shown in figure-32,33. In this model the eccentricity of both the Earth's orbit about the Sun and the Moon's orbit about the Earth are neglected. To perform the transfer we need to transfer the orbit from the co-rotating Earth-Moon system to the co-rotating Earth-Sun frame. This is achieved by an intermediate transformation to an inertial frame. The computation is complicated by the changing orientation of the Earth-Moon system relative to the Earth-Sun system which is defined by a pair of angles (α, β) [75], see figure-116. So then, the only thing that is needed is to set suitable orientation angle (α, β) , see figure-116) and different energy level (figure-88) are integrated in backward.

Then, it is needed to define the mission aim. The second step of the transfer will be 2 main types. For Sun-Earth-Moon bi-circular model, the mission aim can be target as these;

- Earth to moon, or reverse,
- Sun-Earth L1 or L2 halo orbits to Moon, or reverse.

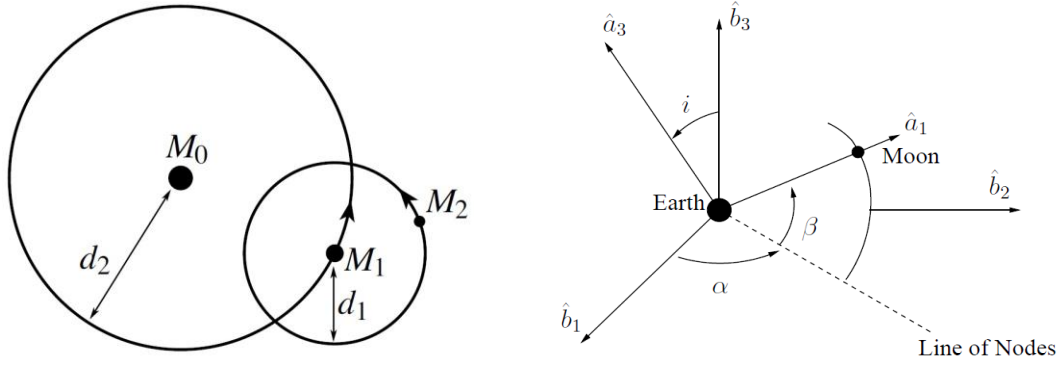


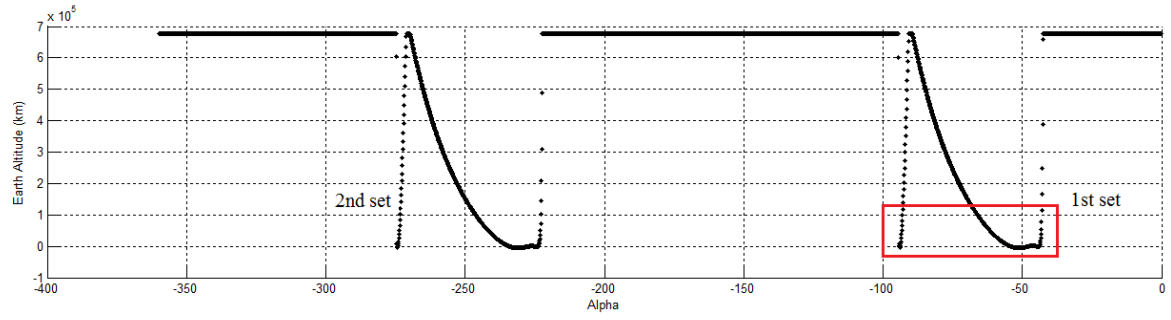
Figure 116 - Bi-circular configuration for patch-CR3BP, such as; Sun (as M_0), Earth (as M_1) and Moon (as M_2), [15,75], left. Orientation angles between Sun-Earth CR3BP $[b_1, b_2, b_3]$ and Earth-Moon CR3BP $[a_1, a_2, a_3]$, $i(\text{inclination})=5.2^\circ$ for Sun-Earth-Moon system, right.

The desired route can be found by varying the orientation angles (α, β) that vary with epoch, see figure-116. The inclination between the Earth-Moon and Sun-Earth orbital planes is always $\approx 5.2^\circ$. And also, we can reduce the search space by fixing one of the remaining angles (α, β) and varying the other.

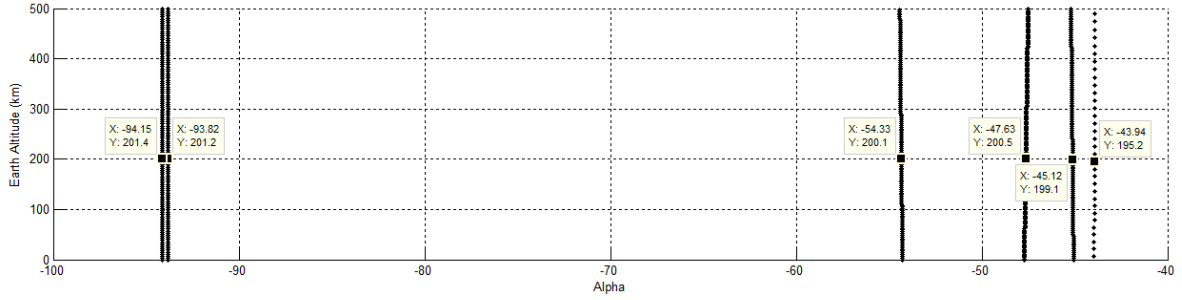
To manoeuvre between two trajectories in the two systems, the common approach is to manoeuvre at the intersection of manifold tubes associated with unstable Lagrange points in the two systems [15]. Such intersections are searched for using Poincare surfaces of section, but this requires the problem to be reducible to Planar-CR3BP. This is usually achieved by ignoring the differences between the orbital planes (around 5.2° in the case of the Earth-Moon system). In our approach the out of plane motion is important and this simplification will not serve. Then, when it leaves from the sphere of influence of the Moon quite enough (as shown in figure-115; in this example, 15.21 days are taken for $C=3.13$), the transformations are done from Earth-Moon to Sun-Earth system by setting a suitable orientation for the desired route and then integrating backwards in the Sun-Earth system until we reach LEO, without applying any ΔV in midcourse.

For finding this suitable orientation angle, the angles in figure-116 should be varied from 0 to 360 degrees. We have chosen to fix β at 0 degrees and vary α , for this example. By integrating back in time we find the options for closest approach to Earth. We plot this as a function of α and Earth's altitude in figure-117.

Figure-117 shows the closest approach to the Earth as a function of the angle α . Angle for an initial parking orbit at 200 km altitude from the Earth, we see that there are 6 possible orientation angles for reaching 200 km Earth altitude, and so 6 epochs for launch. The whole plot repeats when we add 180 degrees to α , so there are two possibilities for each transfer scenario. In figure-118, we plot the 6 different transfers from this parking orbit. We see from figure-117-b these possibilities are grouped into two groups: $\alpha=(-43.94^\circ, -45.12^\circ, -47.63^\circ$ and $-54.33^\circ)$ in plots I to IV, and the other $\alpha=(-93.82^\circ$ and $-94.15^\circ)$ in plots V and VI. The two groups can be distinguished in figure-118 where transfer moves either through the Earth-Sun L_1 gate or the L_2 gate. The transfer times for the first group are (120.3, 106.4, 90.0, 73.8) days respectively while for the second group, the transfers are longer at (154.6 and 164.5) days respectively.



a) Variation over the entire range of Alpha (α)



b) 1st set (red box) is shown enlarged region of the plot, where initial conditions were chosen from

Figure 117 - Closest approach to Earth as a function of orientation angle α with $i \sim 5.2^\circ$, and $\beta = 0$ degrees.

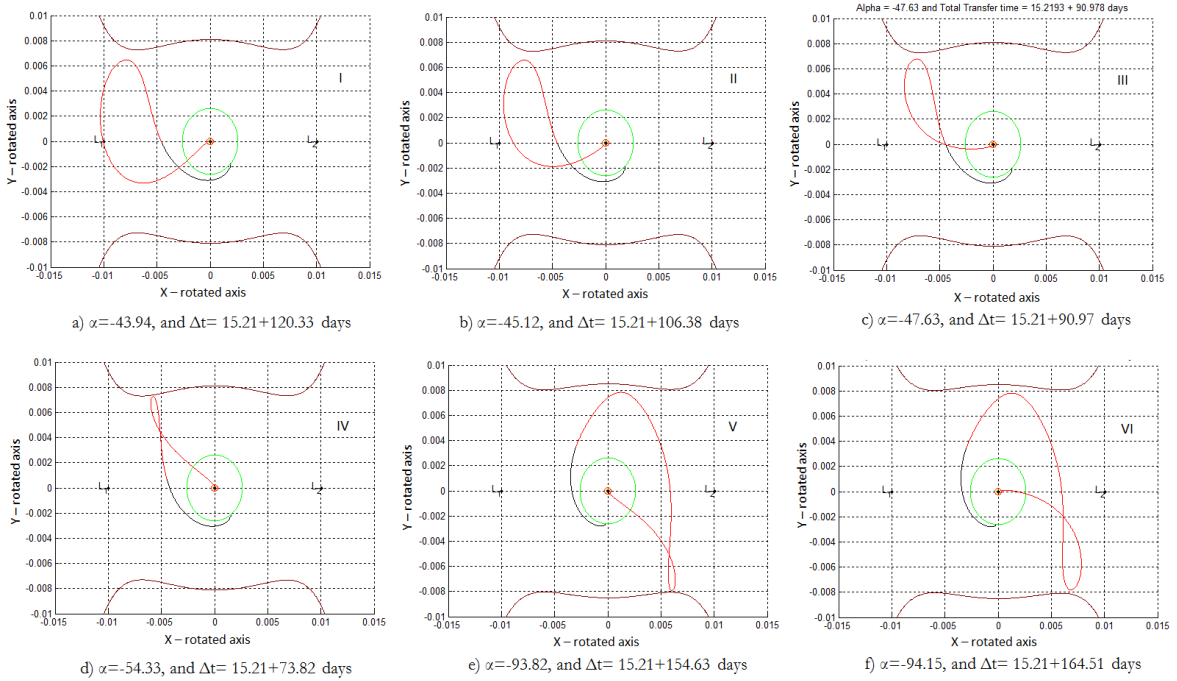


Figure 118 - The 6 different transfer trajectories from a LEO 200 km parking orbit as a function of the different choices of orientation angle α . This plot is in the Earth-Sun frame with the Moon's orbit shown in green. As before the trajectory starts at Earth along the red path and transfers to lunar capture along the black path.

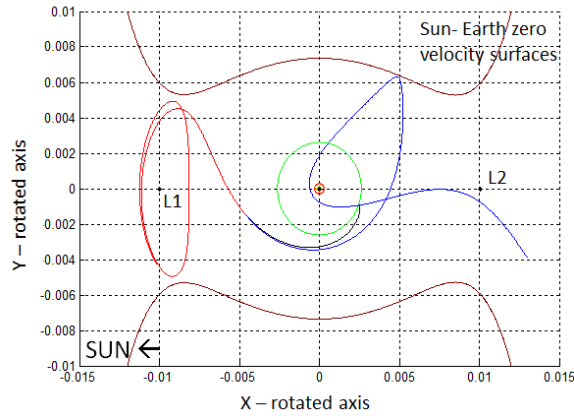
The plots in figure-118 are all for the first set in figure-117 (between -43 to -95 degree). The Hiten mission [15,21,22] used trajectory V, but we can see from this figure that shorter transfer times can be found using trajectory-IV ($\alpha = -54.33$) from LEO altitude of ~ 200 km. If the second set were chosen between -200° and -300° , then all the transfer orientations would be shifted ~ 180 degree because of symmetry between L_1 and L_2 of Sun-Earth, but the transfer times do not change significantly.

An alternative way to transfer from the Sun-Earth system onto the Earth-Moon motion map is to exploit the Lissajous orbits associated with L_1 . In this case, an appropriate choice of the Jacobi constant in the Earth-Moon system is important as this directly determines the size of the Sun-Earth L_1 halo orbit. Table-2 shows the different Jacobi constants of the transfer trajectories in the Earth-Moon system (C_{EM}) and their velocity increment (ΔV) from the parking orbits at these energies. The sizes of the Lissajous orbit are also determined to show how they vary with chosen energy and also their Jacobi constant C_{SE} in Sun-Earth system, after making transformation from EM to SE. So, these transfer trajectories are integrated backwards in time for about ~ 17 days, to move away enough from the gravitational influence of the Moon. Then in a suitable midcourse point, trajectories are transferred and so jump from EM to SE without making any velocity change (as same as in previous example and [75]), and their C_{SE} value in Sun-Earth system and size of Lissajous orbit they aim are computed. And finally all are shown in one table in below;

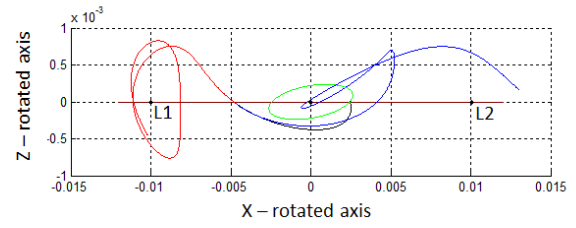
Table 2– Velocity increment and Size of Sun-Earth L1 Lissajous orbits for different Jacobi constants, with zero velocity increment in midcourse.

C_{EM}	ΔV	C_{SE}	$A_y/2$: Size of Lissajous orbit
3.16	5.8637 m/sec	3.000413229	$1,909 \times 10^3$ km
3.15	8.2049 m/sec	3.000542607	$1,520 \times 10^3$ km
3.14	10.5435 m/sec	3.000673082	$1,138 \times 10^3$ km
3.13	12.8799 m/sec	3.000757118	868.7×10^3 km
3.12	15.2138 m/sec	3.000807625	674.9×10^3 km

For the case shown in figure-119, the transfer uses $C_{EM} = 3.12$ case and $C_{SE} = 3.000807625$. Capture by the Moon happens after 17.4 days, the whole transfer taking 75.5 days, and total velocity increment is only about ~ 15.2138 m/sec from Sun-Earth L1 Lissajous orbit to Moon parking orbit. As it is shown that the transfer between Moon and Sun-Earth L_1 or L_2 points can be achieved in very low-cost and in a reasonable time for an un-manned spacecraft.



a) Sun - Earth L1 orbit to Moon Transfer in x,y projection



b) Sun - Earth L1 orbit to Moon Transfer in x,z projection

Figure 119 - Sun-Earth L1 halo orbit to Moon, transfer time is 75.5 days. As before the red trajectory in the Sun-Earth system and would continue along the blue paths were the Moon's gravitational influence be neglected. The black curve shows the Earth-Moon trajectory and the green curve is the Moon's orbit about the Earth in the rotating Sun-Earth frame.

VI. CONCLUSION

In this chapter we summarize the findings of the research presented in this thesis. This is compared with the initial aims and objectives of the research, and the contributions to the state of the art are listed. We finish with a selection of concepts for future work that would extend these concepts.

6.1 Summary and Conclusion

Chapters 1 and 2 define the problem which this research addresses, along with the relevant literature. The concept of capture dynamics and high energy periodic orbits in multi-body dynamics are introduced, the problem is defined. The main problem is to find the ‘key points’ therein the phase space that provides low cost or even zero cost for transfer and capture trajectories. For this, multi-body dynamics must be analysed in 3-dimensional (so full phase space) and different mapping perspective than the previous works.

In chapter 3, Circular Restricted 3-Body Problem (CR3BP) and the trajectory computations methods are introduced as multi-body dynamics system. In this research 3D-CR3BP modes is used, because majority of the binary system in our solar system is fit this model. In section 3.1, the equation of motions and energy integral that is Jacobi integral is shown. Then zero velocity curves, that show the place for particle motion in different energy level, and equilibrium points, that are stationary points and so possible place for station keeping, are introduced. In the section 3.2, the motion type and their computation process are shown. Using the linearised equation around co-linear equilibrium points, 4 type of trajectory around there can be obtained; periodic orbits (halo or lissajous), asymptotic orbits that asymptotic to periodic orbits, transit and non-transit trajectories, see section 3.2.1. Periodic orbits around equilibrium points are used for station keeping missions. Asymptotic orbits, which are asymptotic to periodic orbits around equilibrium points, are used for low-energy transfer trajectories under the name of invariant manifold tubes. Also invariant manifold tubes (IMT) are such a boundaries that separate the transit (potential capture or escape) and non-transit (potential parking) trajectories. This feature around equilibrium points and IMT are very significant that many space mission design in multi-body systems could have been done thanks to this dynamics features, which is display in Motion Maps, too. Also in section 3.2.2, it is introduced 3 different ways to find periodic orbits around equilibrium and bodies. First one is Mapping method, which is mostly used Poincare section but it is not give suitable clear solution in 3D space. Second one is differential corrector Method, which is very sensitive in initial state. And

third one is Linstead-Poincare Perturbation Method, which has too many calculation load, and so daunting. Amongst of them, generally Poincare map method is the most popular and practical one to find both periodic orbits and the transfer points between invariant manifold tube, as it is shown in section 3.2.3. In section 3.2.4, some main concepts about transit and non-transit trajectories are explained.

In chapter 4, the motion map method, which eliminate some weakness in classic Poincare Mapping method, is introduced as new concept for determining the trajectories in CR3BP. Using this map, 'key points', which are the aim of this research for finding the low-cost trajectories, can be found very easily even in 3D solution space, where classic Poincare and other mapping methods are complicated in visualisation of it. In section 4.1, the process about plotting the motion map is introduced. For this, a 2-dimension initial condition grid must be constructed firstly. After that, using these initial conditions, trajectory is integrated forward or backward time, according to mission design. Then the results are classified, again according to the problem, and reflect to the plane to see the Map. In this research, problem is to determine the high energy trajectories around secondary (smaller) body in CR3BP, that both L_1 and L_2 gates are open. So according to this, 4 main type of trajectories can be exist; periodic orbits around secondary body, collusion trajectories that crash to planet and trajectories that escape/capture from L_1 or L_2 . After that classification, it is assigned different colour for different type of trajectories, such as; periodic orbits are green, collusion trajectories are blue, L_1 escape trajectories red and L_2 escape trajectories are yellow. Then the assigned colours of trajectories are reflected to initial conditions grid. Finally, a colourful Motion Map is obtained. The regions in this map show the trajectory results of the initial conditions region. In this method, the most important thing is to define the initial condition grid. Because 3-dimension CR3BP is studied in this research, the variables are too much, so it is needed to restricted initial condition to reduced to grid dimension to 2-dimension, for obtain nice clear visualization of the solution space. In section 4.2, it is introduced 3 main configurations method for initial conditions grid; plane (x-z or y-z) or alternatively line, manifold tube and trajectory along. All of these configurations are very useful in their usage area. Plane configuration shows solid regions clearly for all type of trajectories, especially for bounded trajectories. Such as, bounded region are always in a specific inclination and diagonal direction of secondary body, even in relatively high energy level. This is useful information that stable periodic orbits can be caught in high inclination in high energy level, very low-cost with this map. Manifold tube map would be so useful for mission from co linear equilibrium points to secondary planet, or reverse. Trajectory along motion map (see section 4.2.3) can be also so useful that the motion map can be applied for any random trajectory and it can be determined which part of trajectory presents lowest cost. Last two type configuration also particularly practical that, velocity increment values can be directly shown on those maps, see section 4.2.2 and 4.2.3.

In section 4.3 and 4.4, the most important parts about map are introduced. Because these motion maps are computed numerically, it is possible to occur some numeric errors. So, to get the map fast and accurate, the integration method, time step and integration time span must be chosen carefully, see section 4.3. Also these numeric errors are mostly seen the boundaries that get after long term integration; where it is in boundaries of bounded (green region). So the bounded region's boundaries are the most risky place if spacecraft left in there on "their own", but also the most advantageous place if we apply some tiny manoeuvres, which can provide a very cheap and robust control. In addition, the long term computation cause to shrink the bounded region and boundaries are started to become scattering form, so that boundaries of bounded region is called fuzzy boundaries, because they become really fuzzy

after a certain integration time. Moreover, all these fuzzy parts and also collision waves, which may show unstable periodic orbits that approach homoclinic and heteroclinic cycles, appear in Motion Map are highly related to transversal intersections of the invariant manifolds. As we mentioned in sections 3.2.1 and 3.2.3, there are always asymptotic trajectories called IMT between the transit (potential capture or escape) and non-transit (potential parking) trajectories. So the fuzzy boundaries which are the regions between bounded (green) and unbounded (escape or capture) on motion map must give these asymptotic trajectories, which asymptote any complicated periodic orbits (such as lissajous) around equilibrium points.

In section 4.5, stable and unstable periodic orbits in Motion Map are analysed, because they provide possible ‘parking orbits’ that can be get with very low cost capture process, so features and orbits shapes it gives must be carefully analysed. First, unstable periodic orbits, which are on the place of “collision waves” are presented. Then bounded region, where stable periodic orbits crowd in there, are analysed relative to sub-region. Such as trajectories from circular belt are nearly circular stable periodic orbits, while trajectories far away circular belt give north or south coverage trajectories. So after these analyses, it is showed that capture trajectories can be stabilised to any desired parking orbits very easily and in low cost, thanks motion map.

In section 4.6, Motion map is analysed the effect of Lunar oblates shapes, to see whether oblates effect change the outcome of the Map, or not. It is seen that although oblates effect is very important on single orbits perturbation, it is not enough to change the out-come of the trajectory in particular time, so on the map. And in section 4.7, Keplerian approximation and some analysis are done for about motion map. Firstly, the parabolic boundaries of rotation 2-body problem are showed in counter plot, which display similar result with Motion Map in the place of parabolic boundaries. Then the plane (x-z) motion map is reflected to angular momentum-inclination plane, so the relationship between motion map regions and Keplerian element are seen clearly.

In chapter 5, the mission design and some application about motion map are given. First of all, a cost computation method is introduced for plane motion map in section 5.1. Simply, the velocity increment is computed between different energy level maps, while position is fixes in grid plane. Using this process, velocity decrement can be computed less then $\Delta V \sim 20$ m/sec, but in generally it is about $\Delta V \sim 10$ m/sec for almost any capture trajectory. In section 5.2, Some science orbits, advance transfer trajectories and their usage are introduced and applied on some examples. In section 5.3, Zero-cost capture which can be also called full ballistic capture trajectories are shown. For this type of capture, trajectories are either left their own in a specific time span or applied some tiny velocity decrement to keep them stable, as mention in section 5.1. In section 5.4, motion map is applied in three different transfer scenarios in Earth to Moon case; In first scenario, Transfer between high altitude (circular or elliptic) Earth orbits to high altitude Moon orbits is done using Motion Map. In second scenario, The motion map is partly merged with Hohmann transfer method. And in third scenario, Motion map is used to find final stabilisation parking orbits in bi-circular patch-CR3BP transfer, as same as in Hiten mission. But in this last example, the problem is designed more flexible and more comprehensive. The inclination difference is added between Sun-Earth and Earth-Moon plane. So mission is designed in 3-dimension space and more realistic. And mission aims are expanded flexibly, as transfer may be done from Sun Earth L1 halo orbit to Moon to Earth to moon. In this transfer, total (with midcourse) cost is maximum $\Delta V \sim 15$ m/sec depend on the energy jump and size of halo orbits.

6.2 Research Achievements and Findings

In chapters 1 and 2 the problem statements, previous works and research aim are introduced. In chapter 3, the environment, where the problem is set, is introduced. And in chapter 4 and 5, the research aim is approached with special map designing method, and various accomplishments are done using this map design. Research achievements and findings in this thesis are listed below;

- Multi-body Models as CR3BP enables us to design propellant-effective trajectories to Moon or other planets.
- We present a novel method to visualise the set of solutions to 3D-CR3BP, so;
 - Our method enables 3D capture problem to be visualized clearly.
 - Any type of low or zero-cost (Permanent Capture) planetary and orbital captures can be found in 3D-CR3BP case, using this map.
- Using this method space missions can be designed more cost-effective, flexible, adaptable to any conditions;
 - So spacecraft can be design more smaller and lower mass,
 - Or normal spacecraft can make longer space travel than expected,
 - Or a spacecraft can be rescued from any unpredicted case with low-cost.

In addition to above, there are also some sub-achievements, findings and conclusions;

- Stable and unstable Periodic orbits, which can give desired altitude for low-cost parking orbits or trajectories, are determined and mapped.
- This motion map can be plotted in different configuration of initial conditions grid. Therefore ‘motion maps’ can be applied and used any mission or flexible tasks.
- Plane (x-z or y-z) motion map is show that there are a solids special area for stable periodic orbits in a specific inclination and high energy level. This area can be useful for parking orbits in high energy level.
- The motion map can be also useful to presenting for initial guess for differential corrector method; particularly boundaries of “collision waves” present unstable periodic orbits which can be corrected with differential corrector method.
- Manifold tube motion map can be useful to get stable periodic orbits from halo orbits to secondary body (or reverse) with low-cost. Also it provides to reach other type of trajectories, such as collision or other side escape, with low-cost, too.
- Similar as shown in Manifold tube motion map example, the motion map can be also applied to any other type of manifold tube (such as vertical or horizontal Lyapunov orbit of L_1 or L_2 ...etc) and group of orbit (we desire); such as it can be applied to any piece of Lissajous Orbit of either L_1 or L_2 , applying same scheme shown in section 4.2.2.
- Trajectory along motion map provides chance to change the mission purposes in the middle of the task. So that missions, especially rescue missions, can be applied more flexible using this type of map.
- Fuzzy boundaries in motion map, which are the boundaries of bounded (green) region, provide great opportunity for zero-cost (or very tiny cost) capture.
- Fuzzy boundaries and collision waves are related to transversal intersections of the invariant manifolds; so using them we may design a space mission, which cover both or either L_1 and/or L_2 equilibrium points, with very low-cost.

- Using asymmetric features, The y-z plane motion map can be used for ‘permanent capture’ trajectories. So that capture mission can be achieved absolutely zero-cost.
- The maximum low cost velocity increment that computed with the motion map is $\Delta V \sim 50$ m/sec. But for generally, average velocity increment is done $\Delta V \sim 10-15$ m/sec or less.
- A set of interesting missions that could benefit from this approach, are identified and applied successfully with using motion maps.

In addition, there is an appendix CD that present and show the Videos of Motion Maps and Matlab codes of CR3BP algorithms and trajectories which can be useful to get clear insight of explanation in this research.

6.3 Future Work

This research is determined ‘key points’ for low-cost capture and transfer trajectories are determined using the motion map method. Further work about this research, motion map can be configured in many more different projections of initial condition grid and with various energies. Also permanent capture trajectories and long/short term capture trajectories (for zero-cost mission) can be analysed more deeply. Furthermore it can be derived much more mission type using the motion map.

Lastly, for following studies and videos of different projections of Motion Map with various energies, you may look <https://vimeo.com/user2328498>.

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