# An Integer Programming Approach to Layer Planning in Communication Networks

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*Aysu'ya...*

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# Abstract

In this thesis, we introduce the Partitioning-Hub Location-Routing problem (PHLRP), which can be classified as a variant of the hub location problem. PHLRP consists of partitioning a network into sub-networks, locating at least one hub in each subnetwork and routing the traffic within the network such that all inter-subnetwork traffic is routed through the hubs and all intra-subnetwork traffic stays within the sub-networks all the way from the source to the destination. Obviously, besides the hub location component, PHLRP also involves a graph partitioning component and a routing component. PHLRP finds applications in the strategic planning or deployment of the Intermediate System-Intermediate System (ISIS) Internet Protocol networks and the Less-than-truck load freight distribution systems.

First, we introduce three IP formulations for solving PHLRP. The hub-location component and the graph partitioning components of PHLRP are modeled in the same way in all three formulations. More precisely, the hub-location component is represented by the $p$-median variables and constraints; and the graph partitioning component is represented by the size-constrained graph partitioning variables and constraints. The formulations differ from each other in the way the peculiar routing requirements of PHLRP are modeled.

We then carry out analytical and empirical comparisons of the three IP formulations. Our thorough analysis reveals that one of the formulations is provably the tightest of the three formulations. We also show analytically that the LP relaxations of the other two formulations do not dominate each other. On the other hand, our empirical comparison in a standard branch-and-cut framework that is provided by CPLEX shows that not the tightest but the most compact of the three formulations yield the best performance in terms of solution time.

From this point on, based on the insight gained from detailed analysis of the formulations, we focus our attention on a common sub-problem of the three

formulations: the so-called *size-constrained graph partitioning problem*. We carry out a detailed polyhedral analysis of this problem. The main benefit from this polyhedral analysis is that the facets we identify for the size-constrained graph partitioning problem constitute strong valid inequalities for PHLRP.

And finally, we wrap up our efforts for solving PHLRP. Namely, we present the results of our computational experiments, in which we employ some facets of the size-constrained graph partitioning polytope in a branch-and-cut algorithm for solving PHLRP. Our experiments show that our approach brings significant improvements to the solution time of PHLRP when compared with the default branch-and-cut solver of XPress.

# Contents

# Introduction

We live in a world of networks. Networks play a major role in our daily lives. We benefit from innumerable services that are provided through networks. For example, all transportation systems operate over networks (e.g., rail networks, highway networks, etc.). All communication systems, from the most primitive postal communication systems to the most advanced ones, run on networks. Likewise, utilities (e.g., water, electricity) are distributed over networks.

Despite apparent discrepancies in the services they provide, such 'network-driven' systems indeed bear significant similarities. This is mainly due to the fact that these systems essentially fulfill the same function: transfer of items (e.g., people, goods, data packages, etc.) between different locations. For all of these systems, the underlying network constitutes a very expensive asset that needs to be managed with utmost care in order to support both present and future requirements of the customers.

On the one hand, for providers of such services, efficient daily planning of operations (i.e., realizing the regular, daily transfers through the network in a cost-effective manner) constitutes a pre-requisite for profitability of the operations. On the other hand, the service providers face the challenge of preparing their networks for the future (i.e., long-term network planning). That is, they need to plan carefully and thoroughly for investments in facilities and resources so that their networks can support long-term projections and forecasts of product offerings, technological advancements and trends in customer demand.

Network planning decisions, activities and policies in such systems are generally investigated in three classes:

1. *Strategic* (long-term) planning involves long-term macro plans, which require large capital investments over long periods of time. Decisions at this level generally address design and evolution of the network.

2. *Tactical* (medium-term) planning adapts the strategic plans to daily op-

erations. At this level, efficient allocation and utilization of the available resources at hand is the main goal.

3. *Operational* (short-term) planning involves daily service operations with respect to the service level constraints set at strategic or tactical level planning. Examples of planning activities at this level are scheduling, maintenance, crew and vehicle routing.

The planning activities carried out at these three levels differ considerably from each other in scope, objectives, constraints and complexity. This prevents formulation of a unique, all-in-one optimization model which encapsulates all planning decisions involved at these three levels.

In this thesis our focus is on long-term planning. We propose and attempt to solve a network optimization model which could prove to be a very useful tool for long-term planning in certain network driven systems. Namely, we focus on two specific, seemingly disparate long-term network planning problems:

1. implementation planning for the Internet Protocol networks employing the Intermediate System - Intermediate System (ISIS) routing protocol;

2. long-term planning of Less-Than-Truckload Freight Distribution (LTLFD) systems.

Both ISIS routing protocol and LTLFD systems involve *'hub location'* components. *Hubs* are special nodes, which facilitate aggregation and de-aggregation of communication at certain points of the network. They serve as concentration points, where smaller items from non-hub nodes are packed into big lots, which can be processed or transferred much more cost-effectively than the smaller items. They also de-aggregate big lots that they receive into smaller individual items, which can be transferred to their final destination.

Use of hubs induce separate *layers* in flow of communication throughout a network. Transfer of individual items and aggregate lots have generally quite different characteristics and need to be handled by means of different, separate devices, vehicles or processes. For this reason, the hubs in a network divide it into two separate sub-networks (so called, *layers*), one of which is dedicated to transfer of individual items and the other is designed for handling aggregate lots only. Hubs lie at the contact points of these layers.

Location of hubs in a network and deciding on division of the network into two layers fall into the scope of long-term planning. Because locating hubs in a network involve large investments with long-term perspectives.

Below, we give some background on practical aspects of the ISIS routing protocol and the LTLFD systems. We briefly introduce the long-term network planning problems encountered in these two *network driven* systems. Finally, we highlight the common features of these planning problems and conclude that one optimization model can indeed be used to address both problems.

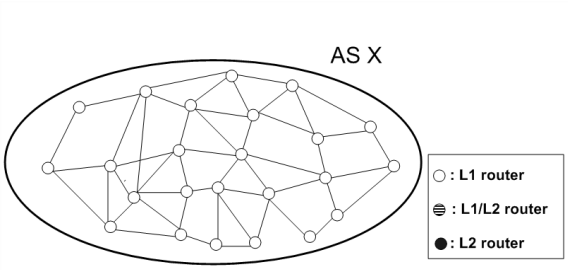
## The ISIS routing protocol

The Internet is a network of networks. In other words, the Internet is composed of many networks that are owned and administered by separate authorities and "continuously" connected to each other. These private networks that make up the Internet are called *Autonomous Systems* (AS). More specifically, an AS is a private network or group of networks which is administered by a single authority. It is necessary to maintain "continuous" communication between and within the ASs to ensure full and reliable communication throughout the Internet.

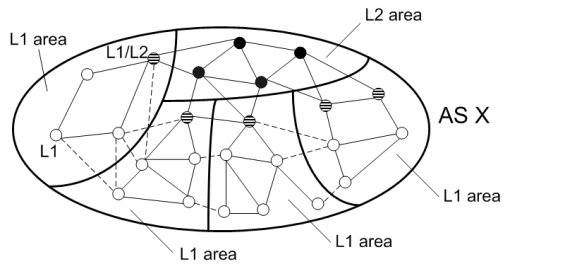
*Routing* is defined as the process of determining the most efficient and effective route (i.e., path) for sending packets towards their destinations in the Internet. Efficiency and effectiveness of a path can be defined with respect to different characteristics, such as cost, security, reliability, optimal utilization of certain network resources, etc. *Routing protocols* are special sets of rules that govern the routing activities in the Internet. A routing protocol may put emphasis on one or more of the mentioned characteristics while determining the paths of communication between the hosts. Routing is carried out by special devices called *routers*.

Routing protocols are divided into two classes: *Interior Gateway Protocols* (IGP) and *Exterior Gateway Protocols* (EGP). IGPs govern the routing activities *within* the ASs and EGPs govern routing activities *between* the ASs. Each AS is free to choose and run its own IGP for its inside communication, however, all of the ASs have to comply with one common EGP. Namely, if two hosts are located in different ASs, then the path of communication between them is collectively determined by the IGPs used in the ASs and the standard EGP used in the Internet; if they are in the same AS, the IGP that is used in that AS determines the path over which the communication takes place.

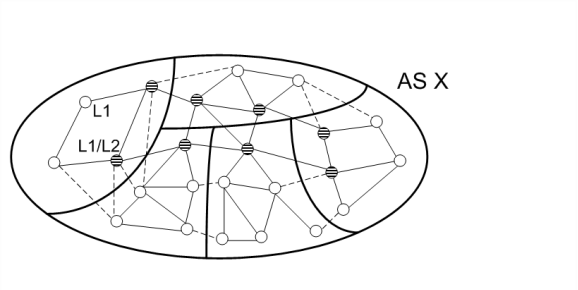
In this thesis, we are interested in a specific IGP called Intermediate System - Intermediate System (ISIS). That is, in this thesis we are only interested in intra-AS communications, and *not* in inter-AS communications.



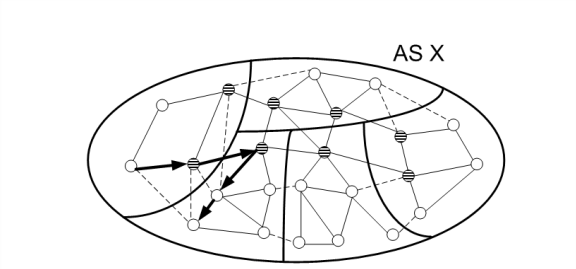

(a) An example Autonomous System.



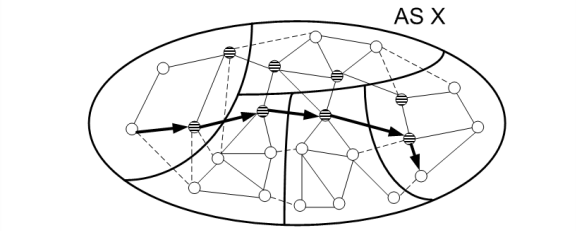

(b) An example ISIS implementation on the example AS. The AS is partitioned into areas and one area is reserved as an L2 area. Note that, inter-area arcs between local nodes (shown dashed) can not be used in this partitioning.




(c) An example implementation without any L2 routers. This implementation does not involve a special L2 area. The backbone is composed of L1/L2 routers.




(d) An example inter-area routing if no L2 area is employed in the implementation.




(e) Another example inter-area routing if no L2 area is employed in the implementation.

Figure 1: Implementation of the ISIS protocol.

Routing protocols are deployed on the routers. For this reason, in this thesis, we will not be dealing with the host computers explicitly. All host computers are assumed to be located in networks that are directly connected to some routers of the AS under consideration. Namely, we assume that each host is reachable by a router. So, in the context of this thesis, we represent an AS with a graph $G = (V, E)$, where $V$ represents the set of routers in the AS, and $E$ stands for the set of links (i.e., edges) between the routers.

In ISIS, the set of all routers in the AS under consideration is divided into subsets, which are called *areas*. Two routers in an area with traffic demand in-between must be able to reach each other over a path that is completely contained within the area. That is, areas induce subgraphs of $G$ in which nodes with traffic demand in-between are connected to each other.

In ISIS, each router in an AS belongs to one of three types: *L1*, *L2* or *L1/L2*. All of the areas but one consist mainly of the L1 routers. The exceptional area consists only of the L2 routers. The areas that consist mainly of the L1 routers are called *L1 areas*, and the area that is composed of the L2 routers is called the *L2 area*.

Two adjacent L1 routers in an L1 area can exchange data packets with each other over the link that lies in-between. On the other hand, an L1 router can not transfer (can not send or receive) packets over an adjacent link if the link is an inter-area link (i.e., if the link traverses the area border). L1 routers belonging to separate L1 areas communicate with each other via the L2 area. In this sense, the L1 areas are *stub areas*, that is, an L1 area can be the source or the sink for an inter-area traffic flow, but it can not be an intermediate area in such a flow. However, the L2 area acts as a *transit area*, that is, this area is responsible for transferring the traffic between the L1 areas. In other words, the L2 area is a backbone area that connects the L1 areas to each other.

Each L1 area has to have special routers that make possible the communication between the L1 routers and the L2 area. L1/L2 routers are responsible for carrying out this liaison task, that is, they facilitate the communication between the L1 routers and the L2 area. Indeed, an L1/L2 router is capable of exchanging data packets over a link with any other router in the network regardless of the type and the area it belongs to.

For this reason, L1/L2 routers act like the gateways of the L1 areas. Note that, two adjacent L1/L2 routers in different areas can exchange data packets with each other over the link that lies in-between. Hence, even if no L2 area exists in the AS, the subgraph induced by the L1/L2 routers can as well be used as the backbone of communication throughout the network. From here on

in this thesis, we will use the terms *L1/L2 router* and *gateway* interchangeably.

If an L1 router in an L1 area wants to send packets to an L1 router in another L1 area, it first forwards them to a gateway of its area (i.e., an L1/L2 router located within its area). This gateway router then passes these packets onto the L2 area, in which the packets will be forwarded towards a gateway (i.e. an L1/L2 router) of the the sink area. Finally, gateway router of the sink area receives the packets from the L2 area and sends them towards the router they are destined to. If no L2 area exists in the AS, once leaving the source area, the packets would be sent towards the sink area over the L1/L2 routers. Clearly, in this case, the subgraph induced by the L1/L2 routers (i.e., the backbone) has to be connected. Figures 1(a)-1(c) display two implementation examples of the protocol, one with an L2 area and one without. Figures 1(d) and 1(e) display two inter-area routing examples when no L2 area is employed.

For the sake of simplicity, in this thesis we ignore L2 routers and the L2 area. In other words, we adopt the deployment approach demonstrated in Figure 1(c). This assumption is not restrictive at all, because, as we have noted above, L1/L2 routers can seamlessly fulfill the tasks carried out by L2 routers. This deployment approach is also preferred generally in practice due to the flexibility it provides in operations.

In ISIS, the L1 routers are conservative in their selection of gateways to communicate with outside of their areas. While communicating with outside of its area, an L1 router always uses the same gateway. In general, the gateway router that is chosen by an L1 router is the closest L1/L2 router in the area. In our Integer Programming (IP) formulations, we allow the routers to pick gateways other than the closest one as well. We assign each L1 router to one gateway router, and force them to communicate with outside of their areas always through the gateways they are assigned to. Clearly, it is pointless to assign L1/L2 routers to gateways other than themselves (i.e., each L1/L2 router is assigned to itself for communicating with outside of its area).

This introduction to ISIS protocol is sufficient for our purposes in this thesis. Naturally, our introduction simplifies certain aspects of the protocol. We consider a simplified implementation here. We refer the interested reader to [34] for a detailed treatment of ISIS.

Based on this background, in this thesis we assume that we are given a graph $G = (V, E)$, which stands for the AS under consideration (recall that $V$ represents the set of routers in the AS, and $E$ stands for the set of links between the routers). We also assume that traffic demand forecasts (i.e., estimates of communication volumes) for every pair of routers in $V$ is available. We assume

further that cost of sending one unit of traffic over every link in $E$ is known. Then, the ISIS implementation planning problem simply consists of

- dividing $G$ (i.e., the AS under consideration) into areas,
- deciding which routers in $V$ to be configured as L1 routers and which to be configured as L1/L2 routers,
- while minimizing the overall cost of forecast communication between the routers.

Note that, the aim of the ISIS implementation planning problem is not to design a network. Rather, the aim is to determine partitions and accordingly locate hubs on a *given* network so that the communication can be realized at minimal cost.

# LTLFD systems

In LTLFD systems, the aim is to collect freight from dispatchers and deliver them to the addressees. In general practice, LTLFD systems provide *door-to-door* service in three-tier networks. At the first tier are the *pick-up-and-delivery (PUD) offices* (for an illustration, see Figure 2). These PUD offices act as the point-of-contact with the customers; and collect from the customers the freight to be transported. An LTLFD system typically has several PUD offices in a city, which are located at convenient locations for the customers. The PUD offices then deliver the freight they collect to the the so-called *end-of-line terminals* (shortly, EOL terminals). EOL terminals are generally large warehouses where the freight collected from PUD offices is classified and consolidated into larger shipments. In an LTLFD system, typically every city is associated with an EOL terminal, and an EOL terminal serves several small cities or a single large city. The EOL terminals constitute the second tier of the three-tier network. At the second tier, these consolidated shipments are forwarded towards *breakbulk hub terminals*, which constitute the third tier of the network (the so-called *breakbulk backbone*). Breakbulk hub terminals are large facilities which classify and consolidate further the shipments they collect. They are located at regional centers with possibly national/international rail, air and/or sea connections to other breakbulk hubs. A breakbulk hub terminal consolidates further the shipments it collects from several EOL terminals and sends the resulting large shipments to other breakbulk hubs that they are destined to. At

the destination end, the shipments are de-consolidated into smaller shipments; and freights are in turn moved to second and first tiers of the network to be delivered to their addressees (by means of the PUD offices).

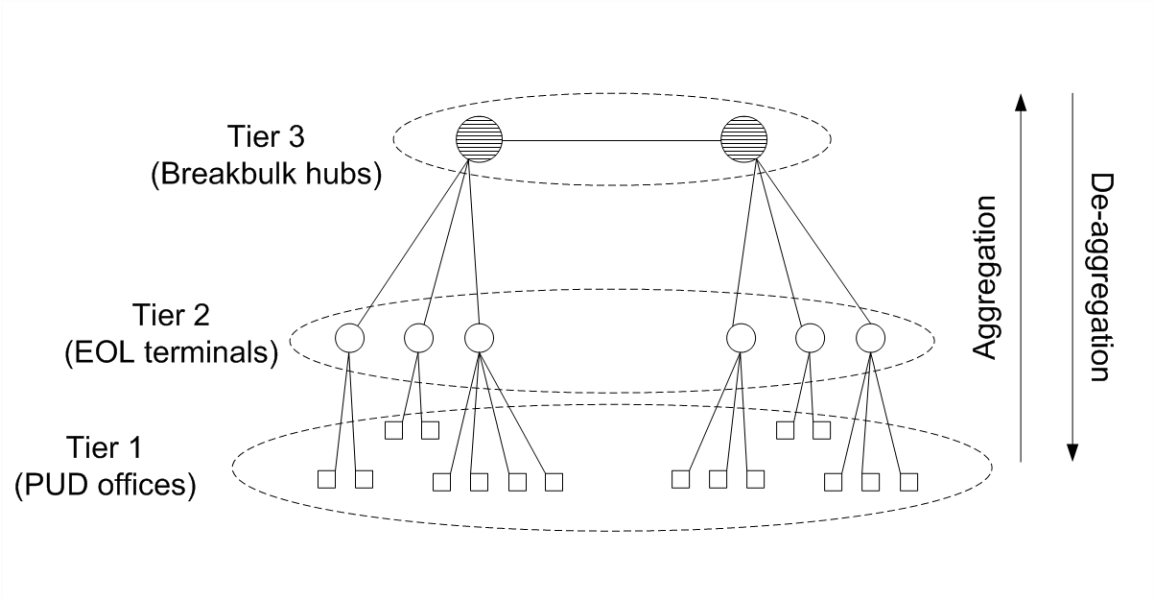



Figure 2: An illustration of the three-tier network structure in LTLFD systems.

In LTLFD systems, the second and third tiers generally serve two purposes: 1. exploitation of economies of scale by means of facilities and vehicles with large capacities, 2. facilitating special services with dedicated vehicles and facilities (e.g., faster transfer with aeroplanes, fresh goods transfer using special vehicles over long distances, etc.). Depending on the nature and volume of transfers and distance in-between, a variety of vehicles (e.g., aircrafts, trains, tractors) can be used to facilitate connections at these tiers.

Generally, in practice direct transfers at the first tier (i.e., direct door-to-door transfers) are not allowed (or are negligible). That is, every package has to pass through at least one EOL terminal. However, direct transfers are generally not forbidden at the second tier (i.e., between EOL terminals). In other words some consolidated shipments can be directly forwarded towards the destination at the second tier without any need for further consolidation at the third tier. In fact, these direct transfers at the second tier bring important advantages, like

- efficiency in second tier services, and,
- reducing the burden on the breakbulk hubs and the breakbulk backbone.

In practice, LTLFD networks are often divided into geographical or administrative regions. Division into regions is generally necessitated by natural or country borders. Regions also provide several important advantages to the service providers:

- Regional operations can be subcontracted.
- Different modes of transportation can be used for inter-regional and intra-regional traffic (e.g., ground transportation for intra-region traffic and air transportation for inter-region traffic).
- Different pricing and service level requirements can be applied for intra-regional and inter-regional traffic (e.g., different delivery times and prices for inter-regional and intra-regional freight traffic).
- Different pricing and service level requirements can be applied to different parts of the network (e.g., highly specialized services for highly industrialized regions, for regions with high density of important clients, etc.).

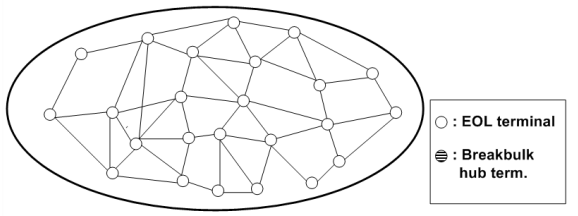
Generally, every region has a breakbulk hub, and all inter-regional traffic flows through the breakbulk hubs. Intra-regional traffic is sent directly from the source EOL terminals to the destination EOL terminals without any need to use breakbulk hub facilities.

In this thesis, we assume that only intra-regional transfers among EOL terminals are allowed. All inter-regional traffic has to pass through the breakbulk hubs.

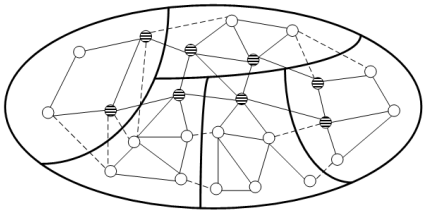
As we confine ourselves to strategic planning issues, this brief introduction to LTLFD networks is sufficient for our purposes. For details on strategic planning freight transportation networks, we refer the interested reader to the excellent surveys by Crainic [8] and Campbell [4].

Based on this background, in this thesis we do not take into consideration the PUD offices as they play a more operational role in the distribution process. We assume that we are given a graph $G = (V, E)$ (see Figure 3 for an example), where $V$ stands for locations of the EOL terminals in the LTLFD

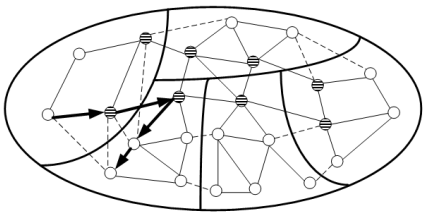
network under consideration, and $E$ stands for the links between the EOL terminals (i.e., a combination of the road network, the rail network and air/sea connections depending on what means of transportation are available to the service provider). In other words, we assume that a preliminary study of surveys, forecasts, etc. is already carried out and locations of EOL terminals in the network are already determined. We also assume that traffic demand forecasts (i.e., estimates of volumes to be transferred) are available for every pair of EOL terminals. We assume further that cost of sending one unit of traffic over every link in $E$ is known.



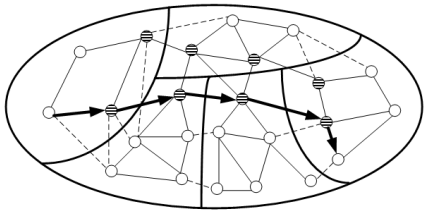

(a) An example LTLFD network. The vertices of the network represent the EOL terminals.



(b) The network is partitioned into regions. Some of the EOL terminals are designated as breakbulk hub terminals. Breakbulk hubs are located such that every region contains at least one breakbulk hub terminal. Note that, inter-region arcs between local nodes (shown dashed) can not be used in this partitioning.



(c) An example inter-regional routing. Routing in LTLFD networks is very similar to ISIS routing.



(d) Another example inter-regional routing.

Figure 3: An example LTLFD network.

Then, the LTLFD long-term planning problem that we deal with consists

of, roughly speaking, dividing the network into regions, determining at least one EOL terminal in each region as a breakbulk hub, and routing the traffic within the network such that inter-regional traffic flows through the breakbulk hubs and intra-regional traffic stays within the regions all the way from the source to the destination.

Note that, like the ISIS implementation problem, the LTLFD long-term planning problem's aim is not to design a network. The aim is to determine regions and to designate some EOL terminals as breakbulk hubs on a *given* transportation network so that the transfers can be carried out at minimal cost.

## Common features of the problems

In this section, we identify the common features of the two problems under consideration.

One can immediately note from these brief introductions to LTLFD networks and ISIS protocol networks that there are considerable similarities between LTLFD networks and ISIS protocol networks. In fact, one can establish close correspondence between

- PUD offices in LTLFD networks and host computers in ISIS networks;
- EOL terminals in LTLFD networks and L1 routers in ISIS networks;
- breakbulk hub terminals in LTLFD networks and L1/L2 routers in ISIS networks;
- regions in LTLFD networks and areas in ISIS protocol networks.

In this thesis, we define our scope as developing a network optimization model that can be used for strategic planning purposes in ISIS protocol networks and LTLFD networks. As strategic planning by nature involves long-term, macro plans, we do not take into consideration explicitly host computers in ISIS networks and PUD offices in LTLFD networks. We assume that every host computer is reachable by an L1 router, and every PUD office is connected to an EOL terminal. In other words, we assume that the traffic demand associated with EOL terminals in LTLFD networks and L1 routers in ISIS protocol networks is an aggregate of the demand associated with the corresponding PUD offices and host computers, respectively.

In both problems, the underlying network is required to be divided into subnetworks (i.e., areas in ISIS networks, and, regions in LTLFD networks). Every subnetwork has to have at least one hub (i.e., L1/L2 router or a breakbulk hub terminal). Routing follows similar principles in both systems: intra-area or intra-regional traffic is routed directly from the source to the destination; inter-area and inter-regional traffic has to be routed through the backbone (i.e., the hubs).

This close correspondence between these two planning problems permits us to develop one single optimization model to address both. In Chapter 1, we describe the problems and the model in a rather technical and precise manner, and in Chapter 2 we introduce three IP formulations to solve the problems.

# Chapter 1

# Problem Definition and notation

We are given a network (a digraph) $D = (V, A)$ and a companion graph $G = (V, E)$. Assume that the arc set $A$ is symmetric (i.e., $(j, i) \in A$ for all $(i, j) \in A$) and for each pair of arcs $(i, j), (j, i)$ in $A$, there exists an edge $e = \{i, j\} \in E$ with $i < j$. We use $\{i, j\}$ and $\{j, i\}$ interchangeably to refer to $e$.

Let $(V_1, V_2, \ldots, V_k)$ be a partition of $V$ into subsets, i.e., $V_1, V_2, \ldots, V_k$ are disjoint subsets of $V$ such that $V = \cup_l V_l$. We assume that $F_L \leq |V_l| \leq F_U$ for $l = 1, 2, \ldots, k$, where $F_L$ and $F_U$ are some lower and upper bound values on the sizes of subsets. Let $D_l = (V_l, A_l)$, $l = 1, 2, \ldots, k$, be subnetworks of $D$ induced by $V_1, V_2, \ldots, V_k$, respectively (see Figures 1(c) and 3(b)). These subnetworks represent the areas in ISIS networks and the regions in LTLFD networks. For the sake of simplicity, in the rest of this chapter, we consistently use the ISIS term *'area'* to refer to these sub-networks.

The use of the parameters $F_L$ and $F_U$ provides a control over the sizes and the number of areas we obtain in the partition. That is, we employ these parameters to protect ourselves against extreme cases in the solution, like very large areas or a very large number of small areas. In fact, partitioning the network into areas of manageable size provides a protection against stability and convergence issues in ISIS (see Chapter 3 of [34] for a detailed treatment of ISIS stability and convergence issues). Capacity or economies-of-scale concerns in freight distribution networks would also justify the use of such bounds on the area sizes.

We designate a subset $H$ of nodes in $V$ as *hub* nodes. We assume $|H| \leq Y$, that is, $Y$ is an upper bound on the number of hubs we can locate in the network. We call the nodes in $V - H$ the *local* nodes (see Figures 1(c) and 3(b)). We call $D_H = (H, A_H)$, the subnetwork of $D$ induced by $H$, the *backbone*.

Obviously, the hubs (i.e., the nodes in $H$) represent the L1/L2 routers in ISIS networks and the breakbulk hub terminals in LTLFD networks. Similarly, the local nodes (i.e., nodes in $V - H$) stand for the L1 routers in ISIS networks and the EOL terminals in LTLFD networks.

Let the set $T$ contain all source-sink pairs which have traffic in-between. For each pair $(u, v) \in T$, we call node $u$ the *source* and node $v$ the *sink*. Let $d_{u,v}$ be the amount of required traffic flow between source-sink pair $(u, v) \in T$. A non-negative cost $c_{i,j}^{u,v}$ is incurred when one unit of flow associated with the pair $(u, v) \in T$ passes through the arc $(i, j) \in A$. The total flow on an arc $(i, j)$ can not exceed a capacity value $C_{i,j}$.

We assume that routing of the traffic associated to $(u, v)$ pairs in $T$ satisfies the following assumptions:

(A1) Inter-area traffic flows can be sent through an arc $(i, j)$ only if $(i, j)$ is on the backbone (i.e., $(i, j) \in A_H$).

(A2) Intra-area traffic is realized over paths which entirely lie within the areas.

(A3) Local nodes always use the same hub to send or receive inter-area flows.

The first assumption implies that two local nodes in different areas can not exchange traffic flows over an arc linking them directly. Local nodes in different areas can send flow to each other only through the hubs located within their areas. We illustrate this in Figures 1(c) and 3(b) by displaying in dashed lines the inter-area arcs which are adjacent to local nodes (i.e., the dashed arcs in these figures are not used at all for transfers). It is worth noting that one can also infer the following from assumptions (A1) and (A2):

- An area can exchange flows with the other areas only if it possesses a hub node (i.e., an area without a hub would be isolated from the rest of the network).
- Hubs of areas with traffic demand in-between must be connected to each other by a path in the backbone.
- If the source and the destination of a traffic flow are in different areas, then the flow is first directed towards the hub within the source area; subsequently, it is sent over the backbone towards the hub of the destination area; finally, it is directed from this hub towards the destination node.

- If the source and the destination are in the same area, then the flow is simply sent over a path that is entirely contained within the area.

Figures 1(d), 1(e), 3(c) and 3(d) display two examples of inter-area flows. Note that in Figures 1(d) and 3(c), the source and the destination are in different areas and although there exists an arc linking them (shown dashed), the flow is sent through the hubs.

Assumption (A3) must be taken into account only if there are more than one hubs in an area. To see this, suppose that area $D_l$ contains at least two hubs, i.e., $|V_l \cap H| \geq 2$, and pick a local node $u \in V_l - H$ in $D_l$. Assumption (A3) implies that $u$ is assigned to one of the hubs in $D_l$, say $i \in V_l \cap H$, and always uses that hub $i$ to send or receive inter-area flows.

Routing in ISIS naturally complies with (A3) because the nodes (which represent *routers*) know only the topology of their areas, i.e., they are blind beyond the borders of their areas. Hence, while sending out inter-area flows, a source router checks if the destination is in the same area as itself. If not, it just sends the flow to the closest hub, no matter what the destination is. In other words, while sending out inter-area flows, routers always minimize the local path within their areas, rather than the global path which extends outside the area towards the destination.

Assumption (A3) has a significant justification in the LTLFD applications, too. In fact, aggregating the inter-area traffic corresponding to a node at a single hub is meaningful only if the fixed cost of dedicating a vehicle to a path is high. For instance, given a local node $u \in V_l - H$ in an area $D_l$, if we split between $i_1, i_2 \in V_l \cap H$ the inter-area traffic associated to $u$, then we would have to dedicate one vehicle to each of the paths from $u$ to $i_1$ and $i_2$. This might prove costly if the associated fixed cost is rather high, e.g., in rail networks. We do not explicitly account for fixed costs in this paper; however, assumption (A3) allows us to take into consideration, albeit implicitly, this relevant fixed cost of dedicating a vehicle to a line.

Against this background, it is worth noting that the IP formulations we present in the next chapter provide the flexibility to impose or relax assumption (A3) in the solution. In fact, in Section 2.6 we discuss the necessary modifications that need to be made in the formulations when we relax assumption (A3).

Having set the grounds, we are now ready to give an informal definition of our problem. Given the network $D = (V, A)$, it consists of:

1. partitioning $D$ into areas $D_1, D_2, \ldots, D_k$ of size at least $F_L$ and at most

$F_U$ (i.e., partitioning $V$ into subsets $V_1, V_2, \ldots, V_k$ such that $F_L \leq |V_l| \leq F_U$ for $l = 1, 2, \ldots, k$),

2. determining $H$ such that $|H| \leq Y$ (i.e., the hub nodes),

3. assigning every local router to a hub within its area (assumption (A3)),

4. routing the traffic for every $(u, v) \in T$ in accordance with the assumptions (A1), (A2), and capacity restrictions over the arcs,

5. with the objective of minimizing the total cost of routing.

In the sequel, we refer to this problem as *the Partitioning-Hub-Location-Routing Problem* ($PHLRP$).

PHLRP has a graph (or network) partitioning component, a hub location component and a routing component. We formulate the graph partitioning component (i.e., partitioning $D$ into $D_1, D_2, \ldots, D_k$ of size at least $F_L$ and at most $F_U$) as a *size constrained graph partitioning problem*, which is investigated in Chapter 4. We model the routing component using multi-commodity arc flow variables and constraints (i.e., flow balance constraints, arc-capacity constraints).

The hub location component of PHLRP differs from the hub location problems explored in the literature. According to a recent survey by Alumur and Kara [1], hub location problems in the literature are generally based on three prevalent assumptions:

(B1) The hub network is complete with a link between every hub pair.

(B2) Cost of transfer between the hubs is less than the cost of transfer between non-hub nodes and hub nodes (due to economies of scale).

(B3) No direct transfer (between two non-hub nodes) is allowed.

On the other hand, in PHLRP we completely relax assumption (B1) and adopt a weaker version of assumption (B3). Assumption (B2) can be accommodated in PHLRP.

Assumption (B1) does not hold in the application context we consider. In fact, nodes in transportation networks are generally connected to each other over paths that traverse several intermediate nodes. Similarly, in large Internet protocol networks (e.g. country wide large networks) setting up a physical link

between every hub, especially between the ones that are far apart, might prove very costly.

In logistics networks, relaxing (B3) (i.e., permitting transfers between non-hub nodes) might decrease transfer costs significantly (for instance, see Lie et al. [20], Lin et al. [18]). Direct transfers between local routers are allowed in ISIS protocol, too.

Assumptions (A1) and (A2) account for a weaker version of (B3) and impose that local transfers within the areas can be carried out directly (i.e., not necessarily through hubs). They are not restrictive at all for ISIS deployment applications and LTLFD applications. On the contrary, they are necessary to ensure aggregation of inter-area flows on the backbone and to protect the backbone from the burden of intra-area flows.

We conclude this section by introducing more notation that we use throughout the thesis. Let $Q \subseteq V$ and $S \subseteq V$. We denote by $A(Q)$ the set of all arcs in $A$ whose both end-nodes are in $Q$, i.e., $A(Q) = \{(i,j) \in A : i,j \in Q\}$. We denote by $A(Q,S)$ the set of arcs from $Q$ to $S$, i.e., $A(Q,S) = \{(i,j) \in A : i \in Q, j \in S\}$. We denote by $A^+(Q)$ and $A^-(Q)$ the set of outgoing arcs from $Q$ and incoming arcs into $Q$, respectively, i.e., $A^+(Q) = A(Q, V-Q)$ and $A^-(Q) = A(V-Q, Q)$.

Similarly, we denote by $E(Q)$ the set of all edges in $E$ whose both end-nodes lie in $Q$, i.e., $E(Q) = \{\{i,j\} \in E : i,j \in Q\}$; we denote by $E(Q,S)$ the set of edges in $E$ whose one end-node lies in $Q$ and the other in $S$, i.e., $E(Q,S) = \{\{i,j\} \in E : i \in Q, j \in S \text{ or vice versa}\}$; and, we denote by $\delta(Q)$ the set of edges whose only one end-node lies in $Q$, i.e., $\delta(Q) = E(Q, V-Q)$.

For any $x \in \mathbb{R}^{|E|}$, let $x(F) = \sum_{e \in F} x_e$ for $F \subseteq E$. For ease of notation, a set with only one element is represented with the element itself in mathematical expressions: namely, for instance, $Q - u$ stands for $Q - \{u\}$.

We denote by $K_n = (V, E_n)$ the complete graph defined on a vertex set $V$ with $n$ vertices. Throughout, $E_n$ stands for the complete set of edges among the $n$ vertices of $V$.

We denote by $N \in \mathbb{R}^{|V| \times |A|}$ the vertex-arc incidence matrix of $D$, i.e.,

$$N_{u,(i,j)} = \begin{cases} 1, & \text{if } (i,j) \in A^+(\{u\}), \\ -1, & \text{if } (i,j) \in A^-(\{u\}), \\ 0, & \text{otherwise,} \end{cases} \qquad \forall u \in V, (i,j) \in A.$$

$Nf$ is the product of the matrix $N$ by a flow vector $f \in \mathbb{R}^{|A|}$; and, $(Nf)_i$ stands for the $i^{\text{th}}$ component of the vector $Nf$, i.e., $(Nf)_i$ is equal to the net flow for vertex $i$. We denote $\sum_{i \in S} (Nf)_i$ by $(Nf)_S$.

# Chapter 2

# Integer Programming Formulations for solving PHLRP

In this chapter, we introduce the three IP formulations we propose for solving PHLRP. First, we present their common variables and common constraints.

## 2.1 Common features

There are two sets of common variables used in the formulations:

- The partitioning variables $w_e \in \{0,1\}$ for all $e \in E_n$. For $e = \{u, v\}$,

$$w_e = \begin{cases} 1, & \text{if } u \text{ and } v \text{ are in the same area;} \\ 0, & \text{otherwise.} \end{cases}$$

  Note that throughout the thesis we use $w_e$, $w_{u,v}$ and $w_{v,u}$ interchangeably for $e = \{u, v\}$.

- The hub-location and allocation (assignment) variables $x_{u,v}$ for all $u, v \in V$:

$$x_{u,v} = \begin{cases} 1, & \text{if node } u \text{ is assigned to the hub } v \text{ for exchanging flows} \\ & \text{with outside of its area;} \\ 0, & \text{otherwise.} \end{cases}$$

Setting $x_{u,u}$ to 1 or 0 implies designating node $u$ as a hub or a local node, respectively.

Note that, this definition of hub-location and allocation (assignment) variables complies with assumption (A3) of PHLRP.

We can also relax assumption (A3) from PHLRP in such a way that a local node does not have to use the same hub for all of its inter-area communications (i.e., it can use different hubs depending on the sender / receiver of the packages). The IP formulations can be modified to accommodate such a variation of the problem (see Section 2.6). In fact, in this setting we would not have to assign the local nodes to hubs and hence we would not need anymore the allocation variables ($x_{u,v}$ for $u \neq v$) in the Integer Programming formulations. On the other hand, we would always need the location variables and associated constraints in our formulations, no matter if assumption (A3) is taken into consideration or not.

In the sequel, we develop the formulations assuming that assumption (A3) is in effect.

The common constraints of the three formulations are as follows:

$$w_{u,v} + w_{u,t} - w_{v,t} \leq 1 \qquad \forall u, v, t \in V : u, v, t \text{ different}, \tag{2.1a}$$

$$\sum_{v \in V-u} w_{u,v} \leq F_U - 1 \qquad \forall u \in V, \tag{2.1b}$$

$$\sum_{v \in V-u} w_{u,v} \geq F_L - 1 \qquad \forall u \in V, \tag{2.1c}$$

$$\sum_{v \in V} x_{u,v} = 1 \qquad \forall u \in V, \tag{2.1d}$$

$$x_{u,v} \leq x_{v,v} \qquad \forall u, v \in V : u \neq v, \tag{2.1e}$$

$$\sum_{u \in V} x_{u,u} \leq Y, \tag{2.1f}$$

$$x_{u,v} + x_{v,u} \leq w_{u,v} \qquad \forall u, v \in V : u < v. \tag{2.1g}$$

Valid partitioning of the network into areas is ensured by means of the constraints (2.1a), the so-called triangle inequalities. These constraints, first used by Grötschel and Wakabayashi [14] for the clique partitioning problem, guarantee that if two edges of a triangle lie *within* an area, the third edge lies within that same area, too. Constraints (2.1b) and (2.1c) impose upper and lower bounds, $F_U$ and $F_L$ respectively, on the sizes of the areas. Constraints (2.1d) ensure that each node is allocated to a hub to communicate with nodes outside its area. Note that when $x_{u,u} = 1$ for a node $u$, then $u$ cannot be allocated

to a node other than itself (i.e., $x_{u,v} = 0$ for $v \in V \setminus \{u\}$); otherwise, when $x_{u,u} = 0$, $u$ is necessarily allocated to a hub. Constraints (2.1e) prevent a local node from being assigned to a local node. Constraint (2.1f) imposes an upper bound $Y$ on the number of hubs that can be located in the network. Finally, constraints (2.1g) ensure that each local node is assigned to a hub that lies within the same area as itself.

One can appreciate hardness of PHLRP by observing the well-known structures in constraints (2.1a)-(2.1g). The constraints (2.1a) define the clique partitioning problem, which is $\mathcal{NP}$-Hard (see [33]). The constraints (2.1d)-(2.1f) are the $p$-median constraints, a widely known $\mathcal{NP}$-Hard problem in the discrete location literature (Kariv and Hakimi [17]).

In fact, when we introduce Formulation 1 in Section 2.2 (page 25), we show that PHLRP is $\mathcal{NP}$-Hard as the clique partitioning problem is a special case of it.

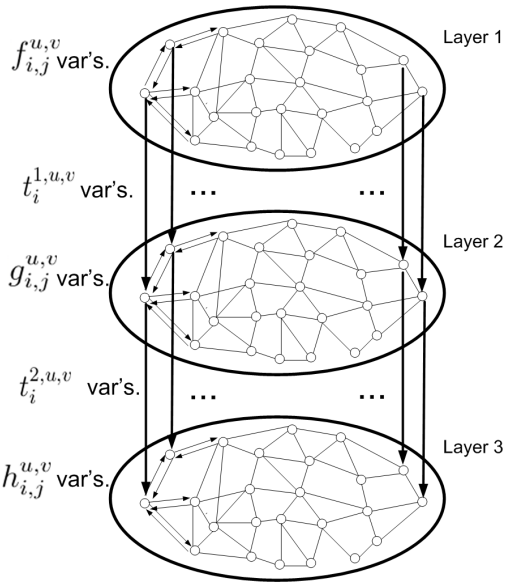
The graph partitioning and location components of PHLRP are represented by means of these common variables and constraints (2.1a)-(2.1g) in our Integer Programming formulations. The formulations differ from each other in the ideas they employ for incorporating the routing of the intra-area and inter-area flows.

The following sections describe in detail the characteristic features of the three IP formulations.

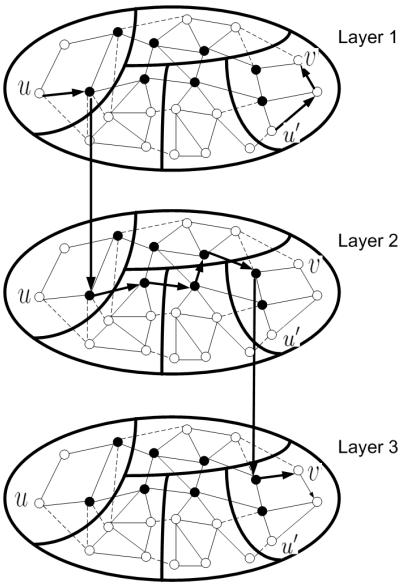
## 2.2 Formulation 1

Formulation 1 is based on the idea of replicating the digraph $D = (V, A)$ three times (see Figure 2.1(a)). We refer to these replicates as *layers*. The sets of nodes and the arcs in the three layers are identical. In addition, we define cost-free and uncapacitated inter-layer arcs between identical nodes of the subsequent layers. Namely, there is an inter-layer arc from node $i$ of Layer 1 (respectively, Layer 2) to node $i$ of Layer 2 (respectively, Layer 3) for each $i$ in $V$. Note that, unlike the arcs within the layers, the inter-layer arcs are all in one direction. It is only possible to send flow from Layer 1 to Layer 2 and from Layer 2 to Layer 3, but not in the opposite direction.

In this formulation, we impose that if the source and the sink nodes, say $(u, v)$, lie in the same area, then the flow starts at $u$ and ends at $v$ of Layer 1.




(a) The flow variables in Formulation 1. Flows in Layers 1, 2 and 3 are represented by the arc-flow variables $f_{i,j}^{u,v}$, $g_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$, respectively. Inter-layer flows are represented by $t_{i,j}^{1,u,v}$ and $t_{i,j}^{2,u,v}$ variables.




(b) Examples of intra-area and inter-area flows in Formulation 1. Inter-area flow from $u$ to $v$ starts from Layer 1 and ends at Layer 3. Intra-area flow from $u'$ to $v$ stays within Layer 1.

Figure 2.1: Formulation 1 employs three replicates (i.e., Layers 1, 2 and 3) of the network $D$. Inter-layer arcs are used to connect Layer 1 to Layer 2 and Layer 2 to Layer 3. For the sake of simplicity in illustration, only some of the intra-layer and inter-layer arcs are displayed in the figures. Each of the undirected links within the layers stands for two arcs in opposite directions.

Otherwise, if $u$ and $v$ lie in different areas, then the flow starts at the source node $u$ in Layer 1, passes through Layer 2 and ends at the sink node $v$ of Layer 3 (see Figure 2.1(b)).

We also impose that Layer 2 hosts backbone flows. That is, in Layer 2 we permit flows only through arcs belonging to $A_H$. On the other hand, in Layers 1 and 3 we permit flows only through arcs that lie *within* areas. Moreover, we permit inter-layer flows only through the inter-layer arcs corresponding to hubs.

## Variables

Besides the variables introduced in Section 2.1, Formulation 1 employs five sets of flow variables (see Figure 2.1(a)):

- $f_{i,j}^{u,v}$, $g_{i,j}^{u,v}$, $h_{i,j}^{u,v} \in \{0,1\}$, for all $(u,v) \in T$ and $(i,j) \in A$, to represent the flows in Layers 1,2 and 3, respectively;

- $t_i^{1,u,v}$, $t_i^{2,u,v} \in \{0,1\}$, for all $(u,v) \in T$ and $i \in V$, to represent the inter-layer flows from Layer 1 to Layer 2 and from Layer 2 to Layer 3, respectively.

Specifically, we define $f_{i,j}^{u,v}$ as:

$$f_{i,j}^{u,v} = \begin{cases} 1, & \text{if the flow from } u \text{ to } v \text{ passes through the arc } (i,j) \text{ in Layer 1;} \\ 0, & \text{otherwise.} \end{cases}$$

This definition also applies to $g_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$, if 'Layer 1' is replaced by the appropriate layer label. Similarly, we define $t_i^{1,u,v}$ as:

$$t_i^{1,u,v} = \begin{cases} 1, & \text{if the flow from } u \text{ to } v \text{ passes from Layer 1 to Layer 2 over node } i \\ & \text{(i.e., through the inter-layer arc } (i,i) \text{ between the two layers);} \\ 0, & \text{otherwise.} \end{cases}$$

This definition also applies to $t_i^{2,u,v}$ if the layer labels are changed accordingly.

Formulation 1 is as follows:

$$\min \sum_{(u,v)\in T} \sum_{(i,j)\in A} c_{i,j}^{u,v} d_{u,v} \left(f_{i,j}^{u,v} + g_{i,j}^{u,v} + h_{i,j}^{u,v}\right), \tag{2.2a}$$

$$\text{subject to} \qquad \text{constraints } (2.1a) - (2.1g),$$

$$\sum_{(i,j)\in A^+(\{i\})} g_{i,j}^{u,v} \leq x_{i,i} \qquad \forall (u,v) \in T, \forall i \in V, \tag{2.2b}$$

$$f_{i,j}^{u,v} + f_{j,i}^{u,v} + h_{i,j}^{u,v} + h_{j,i}^{u,v} \leq w_{i,j} \qquad \forall (u,v) \in T, \forall (i,j) \in E, \tag{2.2c}$$

$$\sum_{(u,v)\in T} d_{u,v} \left(f_{i,j}^{u,v} + g_{i,j}^{u,v} + h_{i,j}^{u,v}\right) \leq C_{i,j} \qquad \forall (i,j) \in A, \tag{2.2d}$$

$$(Nf^{u,v})_i + t_i^{1,u,v} = \begin{cases} 1 & \text{if } i = u \\ 0 & \text{if } i \in V - \{u,v\} \\ -w_{u,v} & \text{if } i = v \end{cases} \quad \forall (u,v) \in T, \tag{2.2e}$$

$$(Ng^{u,v})_i + t_i^{2,u,v} - t_i^{1,u,v} = 0, \qquad \forall i \in V, \forall (u,v) \in T, \tag{2.2f}$$

$$(Nh^{u,v})_i - t_i^{2,u,v} = \begin{cases} w_{u,v} - 1 & \text{if } i = v \\ 0 & \text{if } i \in V - \{v\} \end{cases} \quad \forall (u,v) \in T, \tag{2.2g}$$

$$t_i^{1,u,v} \leq x_{u,i} \qquad \forall (u,v) \in T, \forall i \in V, \tag{2.2h}$$

$$t_i^{2,u,v} \leq x_{v,i} \qquad \forall (u,v) \in T, \forall i \in V. \tag{2.2i}$$

The objective function (2.2a) stands for the total cost of routing the traffic within the network. Constraints (2.2b) guarantee that the backbone is composed only of hubs. Specifically, in (2.2b), if a node is not a hub then it does not receive any flow from the other nodes in Layer 2. Constraints (2.2c) ensure that the flows in Layer 1 and Layer 3 do not cut across area borders. Constraints (2.2d) stand for the capacity restrictions over the arcs.

Constraints (2.2e)-(2.2g) are the flow balance equations of Layers 1-3, respectively. These constraints ensure that for each source-sink pair $(u, v) \in T$:

- $u$ in Layer 1 sends out one unit of flow, and
- that one unit of flow is absorbed by
  - $v$ in Layer 1 if $w_{u,v} = 1$ (i.e., if $u$ and $v$ are in the same area),
  - $v$ in Layer 3 if $w_{u,v} = 0$ (i.e., if $u$ and $v$ are in different areas).

Constraints (2.2h) and (2.2i) ensure that an inter-area flow from a source $u$ to a sink $v$ gets onto the backbone on the hub node to which the source is allocated to and gets off the backbone on the hub node to which the sink is allocated to. The flow from $u$ to $v$ is allowed to pass from Layer 1 to Layer

2 over node $i$ only if $i$ is a hub and the source $u$ is allocated to $i$ (i.e., only if $x_{u,i} = 1$). Similarly, the flow is allowed to pass from Layer 2 to Layer 3 over node $i$ only if $i$ is a hub and the sink $v$ is allocated to $i$ (i.e., $x_{v,i} = 1$).

It is worth noting that this formulation complies with the assumptions (A1)-(A3). In fact, constraints (2.2b) and (2.2c) represent assumption (A1); constraints (2.2c) and (2.2e) represent assumption (A2); finally, constraints (2.1d) stand for assumption (A3).

Note further that assumption (B2) of general hub location problems (i.e., transfers on the backbone at reduced unit flow cost), can also be accommodated by replacing the cost coefficient of $g_{i,j}^{u,v}$ variables in the objective function.

### 2.2.1 $\mathcal{NP}$-Hardness of PHLRP

Before concluding this section, we would like to exploit the 3-layer representation of Formulation 1 in order to prove $\mathcal{NP}$-hardness of PHLRP. We show that the size-constrained clique partitioning problem, which we investigate in Chapter 4, is a special case of PHLRP.

**Proposition 2.2.1.** *PHLRP is $\mathcal{NP}$-hard.*

*Proof. Clique partitioning problem* consists of clustering nodes of a complete graph into subsets such that total weight of the edges within the subsets is minimized. Its Integer Programming formulation is given by constraints (2.1a) and the following objective function:

$$\min \sum_{e \in E_n} \tilde{c}_e w_e. \tag{2.3}$$

This problem is shown to be $\mathcal{NP}$-hard by Wakabayashi [33]. Now consider the *size-constrained clique partitioning problem* which imposes upper and lower bounds on the sizes of the subsets. Its Integer Programming formulation consists of constraints (2.1a)-(2.1c) and objective function (2.3). We show in Chapter 4 that size-constrained clique partitioning problem is a generalization of the clique partitioning problem. Hence, it is also $\mathcal{NP}$-hard.

We show that, under the following assumptions, solving PHLRP is in fact equivalent to solving the size-constrained clique partitioning problem:

1. Graph $G$ itself is a complete graph.

2. $Y$ in constraint (2.1f) is equal to $n$.

3. All the arcs in Layers 1 and 3 are defined to be cost-free. The arc costs in Layer 2 are defined as follows:

$$c_{i,j}^{u,v} = \begin{cases} \tilde{c}_{u,v} & \text{if } (i,j) = (u,v) \\ M & \text{otherwise} \end{cases}$$

   where $M$ is a sufficiently large real number.

4. $T = \{(u,v) : u,v \in V; u < v\}$.

5. $d_{u,v} = 1$ for all $(u,v) \in T$.

When $Y$ is equal to $n$, we would ideally designate all the nodes in the network as hubs (note that adding a hub to the network would *never* increase the total routing costs), i.e., $x_{i,i} = 1$ for all $i \in V$. Note also that, due to assumptions 1-3 above, all intra-area flows (i.e., flows for which $w_{u,v} = 1$ for all $(u,v) \in T$) would be cost-free and all inter-area flows (i.e., flows for which $w_{u,v} = 0$ for all $(u,v) \in T$) would incur a cost that is equal to $\tilde{c}_{u,v}$. Hence, the objective function (2.2a) would be equivalent to (2.3).

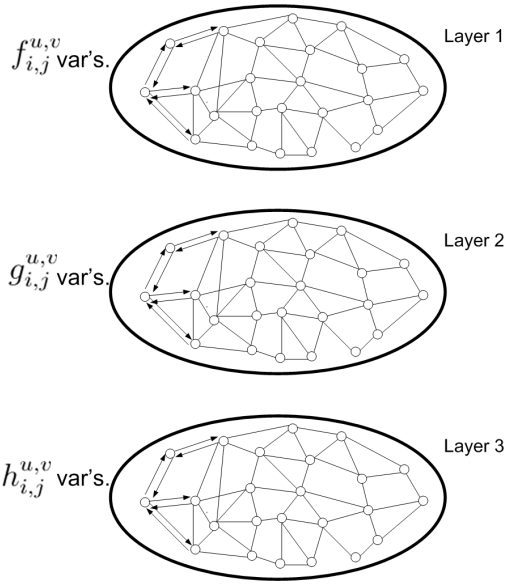
This shows that the size-constrained clique partitioning problem is a special case of PHLRP, which proves $\mathcal{NP}$-hardness of PHLRP. □

## 2.3 Formulation 2

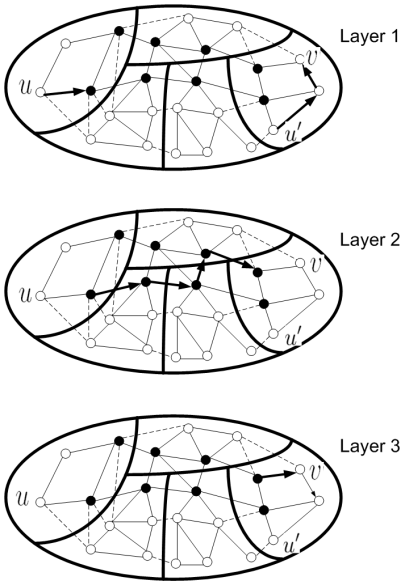
Formulation 2 is also based on a three-layer replication of the network $D$. But in contrast with Formulation 1, no inter-layer arcs and inter-layer variables are employed in Formulation 2. Hence, only three sets of flow variables are used: $f_{i,j}^{u,v}$, $g_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$. These flow variables are defined exactly like their counterparts in Formulation 1 (see Figure 2.2(a)).

The main idea behind Formulation 2 is mimicking (reformulating) the multi-commodity flow constraints of Formulation 1 (2.2e)-(2.2i) without using the inter-layer variables $t_i^{1,u,v}$ and $t_i^{2,u,v}$.

On the other hand, just like in Formulation 1, we impose in this formulation too that Layer 2 accommodates the backbone. That is, in Layer 2 we permit flows only through arcs belonging to $A_H$, in Layers 1 and 3 we permit flows only through arcs that lie *within* areas. Hence, we employ the constraints (2.2b)-(2.2d) in Formulation 2 as well.




(a) The flow variables in Formulation 2. Flows in Layers 1, 2 and 3 are represented by the arc-flow variables $f_{i,j}^{u,v}$, $g_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$, respectively. There are no inter-layer arcs in Formulation 2.




(b) Examples of intra-area and inter-area flows in Formulation 2. The inter-area flow from $u$ to $v$ consists of three portions: 1. the flow within the source area, 2. the flow on the backbone, and, 3. the flow within the sink area. Layers 1, 2 and 3 respectively host these three portions of the flow from $u$ to $v$. Intra-area flow from $u'$ to $v$ stays within Layer 1.

Figure 2.2: Formulation 2 employs three replicates (i.e., Layers 1, 2 and 3) of the network $D$. For the sake of simplicity in illustration, only some of the intra-layer arcs are displayed. Each of the undirected links within the layers stands for two arcs in opposite directions.

Obviously, since the objective function (2.2a) of Formulation 1 is defined over the variables $f_{i,j}^{u,v}$, $g_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$, we can use this objective function in Formulation 2 as well.

We now explain how we reformulate in Formulation 2 the flow balance equalities without using inter-layer variables between Layers 1, 2 and 3. Pick a source-sink pair $(u, v) \in T$. If $u$ and $v$ lie in the same area (i.e., if $w_{u,v} = 1$), we would like to have the flow from $u$ to $v$ stay within Layer 1. More precisely, when $w_{u,v} = 1$ we require that the following constraints be in effect:

$$(Nf^{u,v})_i = \begin{cases} 1, & \text{if } i = u, \\ 0, & \text{if } i \in V - \{u, v\}, \\ -1, & \text{if } i = v, \end{cases} \tag{2.4a}$$

$$(Ng^{u,v})_i = 0 \quad \forall i \in V, \tag{2.4b}$$

$$(Nh^{u,v})_i = 0 \quad \forall i \in V. \tag{2.4c}$$

Now suppose that $u$ and $v$ lie in different areas (i.e., $w_{u,v} = 0$). In this case, we would like to have the flow from $u$ to $v$ split into the three layers. We impose that

- the portion of the flow within the source area (i.e., from $u$ to its gateway) lies in Layer 1;
- the portion of the flow on the backbone (i.e., from $u$'s gateway to $v$'s gateway) lies in Layer 2;
- the portion of the flow within the sink area (i.e., from $v$'s gateway to $v$) lies in Layer 3.

In fact, these requirements imply that we would like to have the following equalities in effect when $w_{u,v} = 0$:

$$(Nf^{u,v})_i = \begin{cases} 1 - x_{u,i}, & \text{if } i = u, \\ -x_{u,i}, & \text{if } i \in V - \{u\}, \end{cases} \tag{2.5a}$$

$$(Ng^{u,v})_i = x_{u,i} - x_{v,i} \quad \forall i \in V, \tag{2.5b}$$

$$(Nh^{u,v})_i = \begin{cases} x_{v,i}, & \text{if } i \in V - \{v\}, \\ x_{v,i} - 1, & \text{if } i = v. \end{cases} \tag{2.5c}$$

When $u$ itself is a gateway (i.e., if $x_{u,u} = 1$) in (2.5a) we would have $f_{i,j}^{u,v} = 0$ for all $(i, j) \in A$ (due to constraints (2.1d)). Otherwise, in Layer 1 the flow sent by $u$ would be absorbed by the gateway that $u$ is assigned to (i.e., node $i$ for which $x_{u,i} = 1$). Likewise, when $v$ itself is a gateway, in (2.5c) we would have $h_{i,j}^{u,v} = 0$ for all $(i, j) \in A$. Otherwise, in Layer 3 $v$ absorbs the flow that is sent out by its gateway.

In Formulation 2, which we present below, we aim at having (2.4) in effect when $w_{u,v} = 1$ and having (2.5) in effect when $w_{u,v} = 0$ for all $(u, v) \in T$.

$$\min \sum_{(u,v)\in T} \sum_{(i,j)\in A} c_{i,j}^{u,v} d_{u,v} (f_{i,j}^{u,v} + g_{i,j}^{u,v} + h_{i,j}^{u,v}) \tag{2.6a}$$

subject to the const. $(2.1a) - (2.1g)$ and (2.2b) – (2.2d),

$$(Nf^{u,v})_i \geq \begin{cases} 1 - x_{u,i} & \text{if } i = u \\ -x_{u,i} & \text{if } i \in V - \{u, v\} \\ -x_{u,i} - w_{u,v} & \text{if } i = v \end{cases} \qquad \forall (u, v) \in T, \tag{2.6b}$$

$$(Nf^{u,v})_i \leq \begin{cases} 1 & \text{if } i = u \\ 0 & \text{if } i \in V - \{u, v\} \\ -w_{u,v} & \text{if } i = v \end{cases} \qquad \forall (u, v) \in T, \tag{2.6c}$$

$$(Ng^{u,v})_i \leq x_{u,i} - x_{v,i} + w_{u,v} \qquad \forall (u, v) \in T, \forall i \in V, \tag{2.6d}$$

$$(Ng^{u,v})_i \geq x_{u,i} - x_{v,i} - w_{u,v} \qquad \forall (u, v) \in T, \forall i \in V, \tag{2.6e}$$

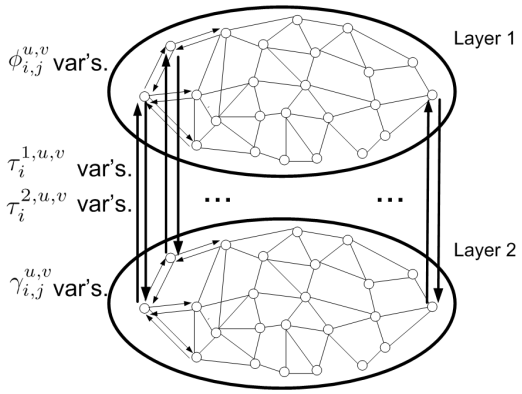
$$(Nh^{u,v})_i \leq \begin{cases} x_{v,i} & \text{if } i \in V - \{v\} \\ x_{v,i} - 1 + w_{u,v} & \text{if } i = v \end{cases} \qquad \forall (u, v) \in T. \tag{2.6f}$$

In fact, for every $(u, v) \in T$, when $w_{u,v} = 1$ equalities (2.4a) are in effect because constraints (2.6c) dominate (2.6b). When $w_{u,v} = 0$, equalities (2.5a) are in effect since constraints (2.6c) become redundant.

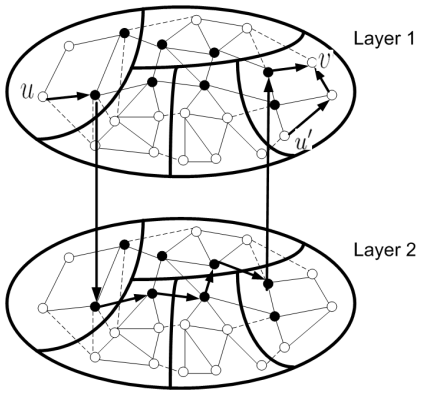
Likewise, when $w_{u,v} = 1$ equalities (2.4b) and (2.4c) are in effect. Because, constraints (2.6d), (2.6e) and (2.6f) become redundant and hence, the flow variables $g_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$ take on 0 value in the optimal solution since the coefficients $c_{i,j}^{u,v}$ are nonnegative in the objective function (2.6a). On the other hand, when $w_{u,v} = 0$, equalities (2.6d), (2.6e) and (2.6f) are equivalent to (2.5b) and (2.5c).

Formulation 2, too, complies with the assumptions (A1)-(A3). Like in Formulation 1, constraints (2.2b) and (2.2c) ensure that assumption (A1) is satisfied; and constraints (2.1d) ensure that assumption (A3) is satisfied. Now, we show that optimal solutions of Formulation 2 always satisfy assumption (A2). Pick an optimal solution $\tilde{\boldsymbol{z}} = (\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})$. Let $T'$ denote the set of all source-sink pairs $(u, v) \in T$ that lie within the same area with respect to $\tilde{\boldsymbol{z}}$ (i.e., $T' = \{(u, v) \in T : \tilde{w}_{u,v} = 1\}$). First, note that as arc costs are non-negative, we have $\tilde{g}_{i,j}^{u,v} = \tilde{h}_{i,j}^{u,v} = 0$ for all $(u, v) \in T'$ and $(i, j) \in A$ (due to constraints (2.6d), (2.6e) and (2.6f)). Next, note that the constraints corresponding to $(u, v) \in T'$ in (2.6c) would always be binding for $\tilde{\boldsymbol{z}}$. Then, (2.2c) would imply that assumption (A2) holds for all $(u, v) \in T'$.

As a final remark, note also that assumption (B2) of general hub location problems can be accommodated in Formulation 2 as well, by replacing the cost coefficient of $g_{i,j}^{u,v}$ variables in the objective function.




(a) The flow variables in Formulation 3. Flows in Layers 1 and 2 are represented by the arc-flow variables $\phi_{i,j}^{u,v}$ and $\gamma_{i,j}^{u,v}$, respectively. Inter-layer flows from Layer 1 to Layer 2 are represented by the variables $\tau_{i,j}^{1,u,v}$; and inter-layer flows from Layer 2 to Layer 1 are represented by the variables $\tau_{i,j}^{2,u,v}$.




(b) Examples of intra-area and inter-area flows in Formulation 3. The inter-area flow from $u$ to $v$ consists of three portions: 1. the flow within the source area, 2. the flow on the backbone, and, 3. the flow within the sink area. Layers 1 hosts the first and the third portions of the flow from $u$ to $v$. Layer 2 hosts the second portion. The intra-area flow from $u'$ to $v$ stays within Layer 1 all the way from the source to the destination.

Figure 2.3: Formulation 3 employs two replicates (i.e., Layers 1 and 2) of the network $D$. For the sake of simplicity in illustration, only some of the intra-layer and inter-layer arcs are displayed. Each of the undirected links within the layers stands for two arcs in opposite directions.

## 2.4 Formulation 3

In Formulation 3, we employ only two replicates of $D$ (see Figure 2.3(a)). We define costless and uncapacitated inter-layer arcs in both directions between identical nodes of the two layers. Namely, there is an inter-layer arc from node $i$ of Layer 1 to node $i$ of Layer 2 and vice versa.

Like in the first two formulations, we impose that if a source-sink pair $(u, v)$ lies in the same area, then the flow starts at $u$ and ends at $v$ of Layer 1. Otherwise, if $u$ and $v$ lie in different areas, then the flow starts at $u$ of Layer 1, and jumps to Layer 2 over the hub that $u$ is allocated to. It continues within Layer 2 until it reaches the hub that $v$ is allocated to, and *returns* over this hub to Layer 1, where it is sent towards $v$. (see Figure 2.3(b)).

We also impose, like in the first two formulations, that Layer 2 hosts backbone flows. That is, in Layer 2 we permit flows only through arcs belonging to $A_H$. On the other hand, in Layer 1 we permit flows only through arcs that lie *within* areas. Moreover, we permit inter-layer flows only through the inter-layer arcs corresponding to hubs.

Note that this formulation is closely related to Formulation 1. Layers 1 and 3 of Formulation 1 merge into a single layer, namely Layer 1, in this formulation. Layer 2 hosts in both formulations the backbone and has the same characteristics. The inter-layer arcs from Layer 1 to Layer 2 in both formulations are equivalent to each other; and the inter-layer arcs from Layer 2 to Layer 3 in Formulation 1 are equivalent to the inter-layer arcs from Layer 2 to Layer 1 in Formulation 3.

### Variables

Formulation 3 employs four sets of flow variables (see Figure 2.1(a)):

- $\phi_{i,j}^{u,v}$, $\gamma_{i,j}^{u,v}$ for all $(u, v) \in T$ and $(i, j) \in A$, to represent the flows in Layers 1 and 2, respectively;
- $\tau_i^{1,u,v}$, $\tau_i^{2,u,v} \in \{0, 1\}$, for all $(u, v) \in T$ and $i \in V$, to represent the inter-layer flows from Layer 1 to Layer 2 and from Layer 2 to Layer 1, respectively.

In particular, we define $\phi_{i,j}^{u,v}$ as

$$\phi_{i,j}^{u,v} = \begin{cases} 1, & \text{if the flow from } u \text{ to } v \text{ passes through the arc } (i,j) \text{ in Layer 1;} \\ 0, & \text{otherwise.} \end{cases}$$

The definition also applies to $\gamma_{i,j}^{u,v}$ if 'Layer 1' is replaced by 'Layer 2' above. Similarly, we define $\tau_i^{1,u,v}$ as:

$$\tau_i^{1,u,v} = \begin{cases} 1, & \text{if the flow from } u \text{ to } v \text{ passes from Layer 1 to Layer 2 over vertex } i \\ & \text{(i.e., through arc } (i,i) \text{ between the two layers);} \\ 0, & \text{otherwise.} \end{cases}$$

This definition also applies to $\tau_i^{2,u,v}$ if the layer labels are changed accordingly.

Obviously, there is a strong relation between the variables of Formulations 1 and 3. Namely,

$$\begin{aligned} \phi_{i,j}^{u,v} &= f_{i,j}^{u,v} + h_{i,j}^{u,v} && \forall (u,v) \in T, \forall (i,j) \in A, \\ \gamma_{i,j}^{u,v} &= g_{i,j}^{u,v} && \forall (u,v) \in T, \forall (i,j) \in A, \\ \tau_i^{1,u,v} &= t_i^{1,u,v} && \forall (u,v) \in T, \forall (i,j) \in A, \\ \tau_i^{2,u,v} &= t_i^{2,u,v} && \forall (u,v) \in T, \forall (i,j) \in A. \end{aligned}$$

Formulation 3 is as follows:

$$\min \sum_{(u,v)\in T} \sum_{(i,j)\in A} c_{i,j}^{u,v} d_{u,v} (\phi_{i,j}^{u,v} + \gamma_{i,j}^{u,v}) \tag{2.7a}$$

$$\text{subject to} \quad \text{the cons. } (2.1a) - (2.1g),$$

$$\sum_{\substack{j\in V: \\ (i,j)\in A}} \gamma_{i,j}^{u,v} \leq x_{i,i} \qquad \forall (u,v) \in T, \forall i \in V, \tag{2.7b}$$

$$\phi_{i,j}^{u,v} + \phi_{j,i}^{u,v} \leq w_{i,j} \qquad \forall (u,v) \in T, \forall (i,j) \in E, \tag{2.7c}$$

$$\sum_{(u,v)\in T} d_{u,v} (\phi_{i,j}^{u,v} + \gamma_{i,j}^{u,v}) \leq C_{i,j} \qquad \forall (i,j) \in A, \tag{2.7d}$$

$$(N\phi^{u,v})_i + \tau_i^{1,u,v} - \tau_i^{2,u,v} = \begin{cases} 1 & \text{if } i = u \\ 0 & \text{if } i \in V - \{u,v\} \\ -1 & \text{if } i = v \end{cases} \qquad \forall (u,v) \in T, \tag{2.7e}$$

$$(N\gamma^{u,v})_i - \tau_i^{1,u,v} + \tau_i^{2,u,v} = 0 \qquad \forall (u,v) \in T, \tag{2.7f}$$

$$\sum_{i\in V} \tau_i^{1,u,v} = 1 - w_{u,v} \qquad \forall (u,v) \in T, \tag{2.7g}$$

$$\tau_i^{1,u,v} \leq x_{u,i} \qquad \forall (u,v) \in T, \forall i \in V, \tag{2.7h}$$

$$\tau_i^{2,u,v} \leq x_{v,i} \qquad \forall (u,v) \in T, \forall i \in V. \tag{2.7i}$$

The objective function (2.7a), which is equivalent to (2.2a), represents the total cost of routing within the network. The constraints (2.7b)-(2.7d) are equivalent to (2.2b)-(2.2d). Constraints (2.7e) and (2.7f) are flow balance equations for Layers 1 and 2, respectively. Constraints (2.7g) ensure that the flow stays within Layer 1, if the source-sink pair $(u, v) \in T$ lie within the same area. (i.e., if $w_{u,v} = 1$). Constraints (2.7h) allow the flow to get on the backbone (i.e., to pass to Layer 2) only through the hub the source is allocated to (i.e., through $i \in V$ for which $x_{u,i} = 1$). Constraints (2.7i) fulfill the symmetrical requirement, that is, the flow is forced to get off the backbone on the hub that the sink is allocated to.

Formulation 3 complies with the assumptions (A1)-(A3), too. Constraints (2.7b) and (2.7c) represent assumption (A1); constraints (2.7c) and (2.7e) represent assumption (A2); finally, constraints (2.1d) stand for assumption (A3).

Like the first two formulations, Formulation 3 can also accommodate assumption (B2) of general hub location problems by replacing the cost coefficient of $\gamma_{i,j}^{u,v}$ variables in the objective function.

In the next two sections, we investigate two relaxations to PHLRP and their impacts on the three formulations we have introduced. In Section 2.5, we consider relaxation of capacity restrictions on the arcs. We show that under this assumption we can relax the integrality restrictions on some of the variables in the formulations . In Section 2.6 we consider relaxing assumption (A3) from PHLRP. We show that more compact versions of Formulations 1 and 3 can be obtained when this assumption is relaxed from the definition of PHLRP.

## 2.5 Relaxing the capacity constraints

In this section, we investigate the impact of relaxing the capacity constraints from the formulations (i.e., (2.2d) in Formulations 1 and 2, (2.7d) in Formulation 3). In particular, we show in Propositions 2.5.1-2.5.3 that, in absence of capacity constraints, we can relax the integrality restrictions on the flow variables and the allocation variables ($x_{u,v}$, $u \neq v$). In other words, if capacity restrictions are ignored, only a small portion of variables in the formulations (i.e., only variables $w_{u,v}$ and $x_{u,u}$) need to be binary.

For the sake of simplicity, in this section we refer to the un-capacitated

versions of the formulations as Formulation 1R, 2R and 3R, respectively.

**Proposition 2.5.1.** *Let* $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{t^1}, \boldsymbol{t^2})$ *and let* $\mu$ *denote the number of variables in Formulation 1R (i.e.,* $\mu = n^2 + \binom{n}{2} + 3|T||A| + 2n|T|$*). Further, let* $\mathcal{A}$ *denote the convex hull of feasible solutions of Formulation 1R, i.e.,*

$$\begin{aligned}
\mathcal{A} = conv\{\boldsymbol{z} \in \mathbb{R}^{\mu} | \, \boldsymbol{z} \text{ satisfies all constraints of Formulation 1 except (2.2d)},\\
w_e \in \{0,1\} \qquad & \forall e \in E_n,\\
x_{u,v} \in \{0,1\} \qquad & \forall u, v \in V,\\
f_{i,j}^{u,v}, g_{i,j}^{u,v}, h_{i,j}^{u,v} \in \{0,1\} \qquad & \forall (u,v) \in T, (i,j) \in A,\\
t_i^{1,u,v}, t_i^{2,u,v} \in \{0,1\} \qquad & \forall (u,v) \in T, i \in V\};
\end{aligned}$$

*and let* $\mathcal{B}$ *stand for the convex hull of feasible solutions of Formulation 1R when the integrality restrictions on the assignment variables and the flow variables are relaxed, i.e.,*

$$\begin{aligned}
\mathcal{B} = conv\{\boldsymbol{z} \in \mathbb{R}^{\mu} | \, \boldsymbol{z} \text{ satisfies all constraints of Formulation 1 except (2.2d)},\\
w_e \in \{0,1\} \qquad & \forall e \in E_n, && (2.8a)\\
x_{u,u} \in \{0,1\} \qquad & \forall u \in V, && (2.8b)\\
x_{u,v} \in [0,1] \qquad & \forall u, v \in V : u \neq v, && (2.8c)\\
f_{i,j}^{u,v}, g_{i,j}^{u,v}, h_{i,j}^{u,v} \in [0,1] \qquad & \forall (u,v) \in T, (i,j) \in A, && (2.8d)\\
t_i^{1,u,v}, t_i^{2,u,v} \in [0,1] \qquad & \forall (u,v) \in T, i \in V\}. && (2.8e)
\end{aligned}$$

*Then,* $\mathcal{A} = \mathcal{B}$.

*Proof.* Let $\tilde{\boldsymbol{z}} = \left(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{t}}^1, \tilde{\boldsymbol{t}}^2\right)$ be an extreme point of $\mathcal{B}$. We show that $\tilde{\boldsymbol{z}}$ is fully integral and hence it is an extreme point of $\mathcal{A}$, too.

Let $\tilde{T}^0 = \{(u,v) \in T : \tilde{w}_{u,v} = 0\}$ and $\tilde{T}^1 = \{(u,v) \in T : \tilde{w}_{u,v} = 1\}$. For $(u,v) \in \tilde{T}^1$, (2.8a) and the flow balance constraints (2.2e)-(2.2g) jointly ensure

that[1]

$$\begin{aligned} \tilde{f}_{i,j}^{u,v} &\in \{0,1\} \qquad \forall (i,j) \in A \\ \tilde{g}_{i,j}^{u,v} &= 0 \qquad \forall (i,j) \in A \\ \tilde{h}_{i,j}^{u,v} &= 0 \qquad \forall (i,j) \in A. \end{aligned}$$

Now, consider a source-sink pair $(u^*, v^*) \in \tilde{T}^0$. By constraints (2.2e) and (2.2g), we have

$$\sum_{i \in V} \tilde{t}_i^{1,u^*,v^*} = \sum_{i \in V} \tilde{t}_i^{2,u^*,v^*} = 1.$$

Then, constraints (2.1d), (2.2h) and (2.2i) imply

$$\tilde{t}_i^{1,u^*,v^*} = \tilde{x}_{u^*,i} \qquad \forall i \in V \tag{2.9a}$$

$$\tilde{t}_i^{2,u^*,v^*} = \tilde{x}_{v^*,i} \qquad \forall i \in V. \tag{2.9b}$$

Suppose that there exist $i_1, i_2, j_1, j_2 \in V$ such that $0 < \tilde{x}_{u^*,i_1}, \tilde{x}_{u^*,i_2} < 1$ and $0 < \tilde{x}_{v^*,j_1}, \tilde{x}_{v^*,j_2} < 1$. Then, the flow variables in $\tilde{\boldsymbol{z}}$ can be interpreted as in Figure 2.4, where $P^1, P^2, \ldots, P^6$ represent paths with non-zero flows in $\tilde{\boldsymbol{f}}$, $\tilde{\boldsymbol{g}}$ and $\tilde{\boldsymbol{h}}$. By (2.9), as $\tilde{x}_{u^*,i_1}, \tilde{x}_{u^*,i_2}, \tilde{x}_{v^*,j_1}$ and $\tilde{x}_{v^*,j_2}$ are fractional, the following variables also take on fractional values:

- $\tilde{t}_{i_1}^{1,u^*,v^*}, \tilde{t}_{i_2}^{1,u^*,v^*}, \tilde{t}_{j_1}^{2,u^*,v^*}$ and $\tilde{t}_{j_2}^{2,u^*,v^*}$,
- $\tilde{f}_{i,j}^{u^*,v^*}$, $\tilde{g}_{i,j}^{u^*,v^*}$ and $\tilde{h}_{i,j}^{u^*,v^*}$ variables for arcs $(i,j)$ which lie on $P^1, \ldots, P^6$.

Consider the following perturbations:

- $\tilde{x}_{u^*,i_1}$ to $\tilde{x}_{u^*,i_1} + \epsilon$,
- $\tilde{x}_{u^*,i_2}$ to $\tilde{x}_{u^*,i_2} - \epsilon$,

[1]Here, constraints (2.2e)-(2.2g) might also imply non-zero flows along cycles in Layers 1, 2 and 3. In this presentation, we ignore such cycle flows for the following two reasons:

1. If such cycles carry unit flows, then $\tilde{\boldsymbol{z}}$ would be fully integral.
2. If they carry fractional flows, then this would contradict extremeness of $\tilde{\boldsymbol{z}}$. It would be possible to perturb $\tilde{\boldsymbol{z}}$ with very small positive and negative flows along these cycles to obtain two new feasible solutions. Naturally, $\tilde{\boldsymbol{z}}$ can be written as a convex combination of these two new solutions.

- $\tilde{x}_{v^*,j_1}$ to $\tilde{x}_{v^*,j_1} + \epsilon$,
- $\tilde{x}_{v^*,j_2}$ to $\tilde{x}_{v^*,j_2} - \epsilon$

as well as the ones displayed for the flow variables in Figure 2.4. Due to (2.8a) and (2.8b), the new values of the perturbed variables do not violate any constraints in Formulation 1R. We call this new feasible solution obtained from $\tilde{\boldsymbol{z}}$ by means of these perturbations $\tilde{\boldsymbol{z}}^1$. Similarly, by reversing the signs of the $\epsilon$-perturbations, we obtain yet another feasible solution and call it $\tilde{\boldsymbol{z}}^2$. Naturally, we have $\tilde{\boldsymbol{z}} = \frac{1}{2}\tilde{\boldsymbol{z}}^1 + \frac{1}{2}\tilde{\boldsymbol{z}}^2$, which contradicts extremeness of $\tilde{\boldsymbol{z}}$.

It is important to note that the $\epsilon$-perturbations applied to the flow variables to obtain $\tilde{\boldsymbol{z}}^1$ and $\tilde{\boldsymbol{z}}^2$ must be applied to all fractional flow variables for all $(u, v) \in \tilde{T}^0$. □

Before moving on to the analogous results on Formulations 2R and 3R, we would like to elaborate a little bit more on the constraint matrix of Formulation 1R. One might suspect that, in Formulation 1R, when the partitioning variables $w_{u,v}$ ($\{u, v\} \in E_n$) and the location variables $x_{u,u}$ ($u \in V$) are treated as constants, coefficients of the rest of the variables in constraints (2.1d), (2.1e), (2.1g), (2.2b), (2.2c) and (2.2e)-(2.2i) constitute a Totally Unimodular (TU) matrix. In such a setting, the only variables we would have in Formulation 1R are the flow variables ($\boldsymbol{f}, \boldsymbol{g}$, $\boldsymbol{h}$, $\boldsymbol{t^1}$ and $\boldsymbol{t^2}$) and the assignment variables ($x_{u,v}$ where $u \neq v$). The problem would reduce to finding optimal

- assignment of local nodes to hubs, and
- flows between source-sink pairs

in a network where a valid partitioning is done and at least one hub is located in each area. If the mentioned coefficient matrix was TU, the result of Proposition 2.5.1 would follow directly from Total Unimodularity as $w_{u,v}$ and $x_{u,u}$ variables would appear in the formulation only as constant integers.

However, we show below that this matrix is not TU. For this, we refer to the following characterization of TU matrices by Camion:

**Definition 2.5.1.** *A matrix is Eulerian if the sum of the elements in each row and column is even.*

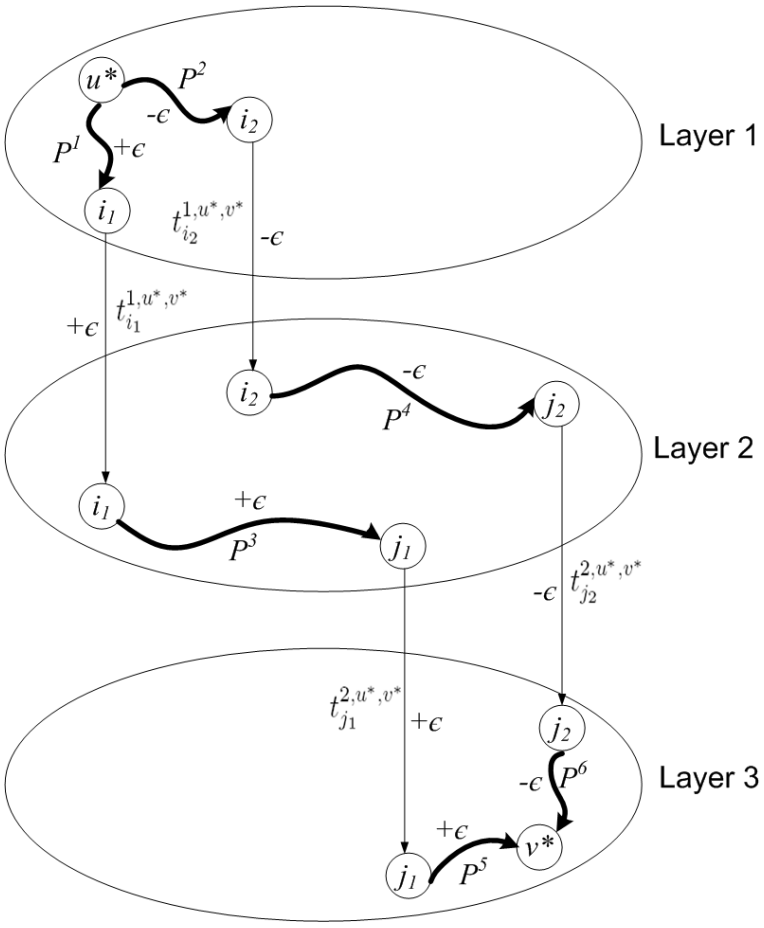



Figure 2.4: Perturbation used to obtain the feasible solution $\tilde{z}^1$ in the proof Proposition 2.5.1. Dark, curved arrows in the figure represent paths.

**Theorem 2.5.1** (Camion [3])**.** *A (-1,0,+1)-matrix A is TU if and only if the sum of elements in each Eulerian square submatrix is a multiple of 4.*

Consider three source-sink pairs $(u, v_1), (u, v_2), (u, v_3) \in T$ and three nodes $i_1, i_2, i_3 \in V - \{u\}$. Based on these pairs and nodes, we construct the following Eulerian submatrix of the coefficient matrix under consideration. The column labels and row labels of the matrix display the corresponding variables and constraints of Formulation 1R, respectively. To give an example, the first row of the matrix represents the constraint corresponding to $(u, v_1) \in T$ and $i_1 \in V - \{u\}$ in (2.2e). See Figure 2.5 for an illustration of the flow variables

used to construct this Eulerian submatrix.

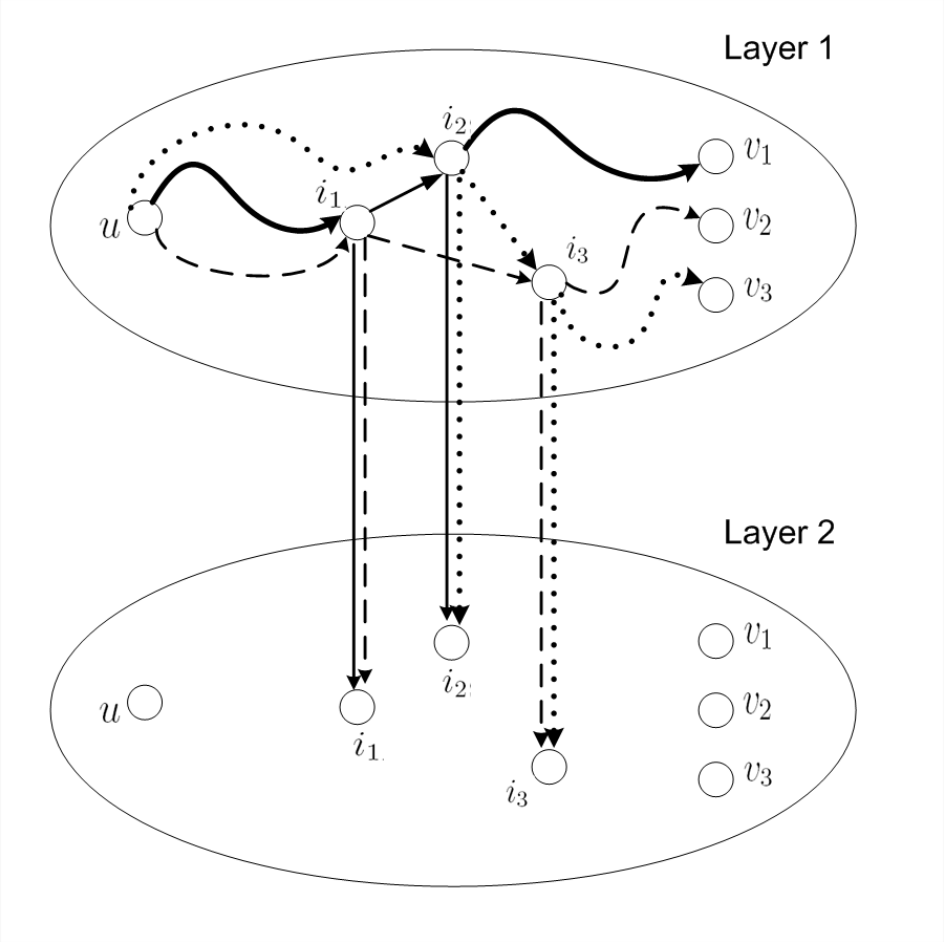



Figure 2.5: An illustration of the flow variables that are used to construct the Eulerian submatrix on page 39. These variables appear in constraints (2.2e) and (2.2h) for $(u, v_1), (u, v_2), (u, v_3) \in T$ and $i_1, i_2, i_3 \in V - \{u\}$. Each of the source-sink pairs is depicted by a different line-style, i.e., $(u, v_1)$ by solid, $(u, v_2)$ by dashed and $(u, v_3)$ by dotted. The curved arcs stand for the flow on paths between the corresponding nodes in Layer 1. The straight arcs within Layer 1 represent the flow variables $f_{i_\cdot,i_\cdot}^{u,v_\cdot}$ and the straight arcs between the layers represent the flow variables $t_{i_\cdot}^{1,u,v_\cdot}$.

| | $f^{u,v_1}_{i_1,i_2}$ | $f^{u,v_2}_{i_1,i_3}$ | $f^{u,v_3}_{i_2,i_3}$ | $t^{1,u,v_1}_{i_1}$ | $t^{1,u,v_1}_{i_2}$ | $t^{1,u,v_2}_{i_1}$ | $t^{1,u,v_2}_{i_3}$ | $t^{1,u,v_3}_{i_2}$ | $t^{1,u,v_3}_{i_3}$ | $x_{u,i_1}$ | $x_{u,i_2}$ | $x_{u,i_3}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (2.2e), $(u,v_1),i_1$ | +1 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (2.2e), $(u,v_1),i_2$ | −1 | 0 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| (2.2h), $(u,v_1),i_1$ | 0 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | 0 | −1 | 0 | 0 |
| (2.2h), $(u,v_1),i_2$ | 0 | 0 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | 0 | −1 | 0 |
| (2.2e), $(u,v_2),i_1$ | 0 | +1 | 0 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | 0 | 0 |
| (2.2e), $(u,v_2),i_3$ | 0 | −1 | 0 | 0 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | 0 |
| (2.2h), $(u,v_2),i_1$ | 0 | 0 | 0 | 0 | 0 | +1 | 0 | 0 | 0 | −1 | 0 | 0 |
| (2.2h), $(u,v_2),i_3$ | 0 | 0 | 0 | 0 | 0 | 0 | +1 | 0 | 0 | 0 | 0 | −1 |
| (2.2e), $(u,v_3),i_2$ | 0 | 0 | +1 | 0 | 0 | 0 | 0 | +1 | 0 | 0 | 0 | 0 |
| (2.2e), $(u,v_3),i_3$ | 0 | 0 | −1 | 0 | 0 | 0 | 0 | 0 | +1 | 0 | 0 | 0 |
| (2.2h), $(u,v_3),i_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | +1 | 0 | 0 | −1 | 0 |
| (2.2h), $(u,v_3),i_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | +1 | 0 | 0 | −1 |

The sum of the elements of this Eulerian submatrix is equal to 6, which shows, by Camion's theorem, that the coefficient matrix we are considering is not TU.

This proof shows that the coefficient matrix under consideration is *in general* not TU. While constructing the Eulerian submatrix, we assume existence of three distinct source-sink pairs $(u,v_1),(u,v_2)$ and $(u,v_3)$. Likewise, we assume existence of three distinct nodes $i_1,i_2,i_3 \in V-\{u\}$ and three arcs $(i_1,i_2),(i_2,i_3),(i_1,i_3) \in A$.

Of course, some special instances of PHLRP *might* lead to TU coefficient matrices. In this thesis, we do not dive into the tedious task of identifying these special cases and leave it as a future research area.

It is also worth noting that this result does not contradict with Proposition 2.5.1. Proposition 2.5.1 shows that the solution would be integral for the specific right-hand-side vector that is being used in Formulation 1R. But, as the coefficient matrix is not TU, one can in general find an integral right-hand-side vector for which the solution would not be integral.

Now, we continue with the analogous impact of relaxation of the arc-capacity constraints for Formulations 2R and 3R.

**Proposition 2.5.2.** *Let* $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{w},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ *and let* $\mu$ *denote the number of variables in Formulation 2R (i.e.,* $\mu = n^2+\binom{n}{2}+3|T||A|$*). Further, let* $\mathcal{A}$

*denote the convex hull of feasible solutions of Formulation 2R, i.e.,*

$$\begin{aligned}\mathcal{A} = conv\{\boldsymbol{z} \in \mathbb{R}^{\mu} | \boldsymbol{z} \text{ satisfies all constraints of Formulation 2 except } (2.2\text{d}),\\ w_e \in \{0,1\} \qquad & \forall e \in E_n,\\ x_{u,v} \in \{0,1\} \qquad & \forall u,v \in V,\\ f_{i,j}^{u,v}, g_{i,j}^{u,v}, h_{i,j}^{u,v} \in \{0,1\} \qquad & \forall (u,v) \in T, (i,j) \in A\};\end{aligned}$$

*and let $\mathcal{B}$ stand for the convex hull of feasible solutions of Formulation 2R when the integrality restrictions on the assignment variables and the flow variables are relaxed, i.e.,*

$$\begin{aligned}\mathcal{B} = conv\{\boldsymbol{z} \in \mathbb{R}^{\mu} | \boldsymbol{z} \text{ satisfies all constraints of Formulation 2 except } (2.2\text{d}),\\ w_e \in \{0,1\} \qquad & \forall e \in E_n,\\ x_{u,u} \in \{0,1\} \qquad & \forall u \in V,\\ x_{u,v} \in [0,1] \qquad & \forall u,v \in V : u \neq v,\\ f_{i,j}^{u,v}, g_{i,j}^{u,v}, h_{i,j}^{u,v} \in [0,1] \qquad & \forall (u,v) \in T, (i,j) \in A\}.\end{aligned}$$

The proof of this proposition is very similar to the proof of Proposition 2.5.1.

**Proposition 2.5.3.** *Let $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{w}, \boldsymbol{\phi}, \boldsymbol{\gamma}, \boldsymbol{\tau^1}, \boldsymbol{\tau^2})$ and let $\mu$ denote the number of variables in Formulation 3R (i.e., $\mu = n^2 + \binom{n}{2} + 2|T||A| + 2n|T|$). Further, let $\mathcal{P}$ denote the convex hull of feasible solutions of Formulation 3R, i.e.,*

$$\begin{aligned}\mathcal{A} = conv\{\boldsymbol{z} \in \mathbb{R}^{\mu} | \boldsymbol{z} \text{ satisfies } (2.1a) - (2.1g) \text{ and } (2.7e) - (2.7c)\\ w_e \in \{0,1\} \qquad & \forall e \in E_n\\ x_{u,v} \in \{0,1\} \qquad & \forall u,v \in V\\ \phi_{i,j}^{u,v}, \gamma_{i,j}^{u,v} \in \{0,1\} \qquad & \forall (u,v) \in T, (i,j) \in A\\ \tau_i^{1,u,v}, \tau_i^{2,u,v} \in \{0,1\} \qquad & \forall (u,v) \in T, i \in V\};\end{aligned}$$

*and let $\mathcal{B}$ stand for the convex hull of feasible solutions of Formulation 3R when the integrality restrictions on the assignment variables and the flow variables*

*are relaxed, i.e.,*

$$\begin{aligned}\mathcal{B} = conv\,\{\boldsymbol{z} \in \mathbb{R}^{\mu} |\, \boldsymbol{z} \text{ satisfies } (2.1a)-(2.1g) \text{ and } (2.7e)-(2.7c)\\ w_e \in \{0,1\} \qquad & \forall e \in E_n\\ x_{u,u} \in \{0,1\} \qquad & \forall u \in V\\ x_{u,v} \in [0,1] \qquad & \forall u,v \in V : u \neq v\\ \phi_{i,j}^{u,v}, \gamma_{i,j}^{u,v} \in [0,1] \qquad & \forall (u,v) \in T, (i,j) \in A\\ \tau_i^{1,u,v}, \tau_i^{2,u,v} \in [0,1] \qquad & \forall (u,v) \in T, i \in V\}\,.\end{aligned}$$

*Then,* $\mathcal{A} = \mathcal{B}$.

The proof of this proposition is very similar to the proof of Proposition 2.5.1.

## 2.6 Relaxing assumption (A3)

In this section, we investigate the impact of relaxing (A3) from PHLRP. Namely, we consider relaxing the restriction that all the inter-area traffic that is sent out or received by a local node be transferred through the same hub. We assume that a local node has the flexibility to use any hub within its area for its inter-area communication. Note that, under this assumption we do not need to explicitly assign the local nodes to the hubs.

In this setting, the only set of flow-balance equalities that would be affected in Formulation 1 are (2.2h) and (2.2i). Because, inter-area flow between a source-sink pair $(u,v)$ is no more restricted to pass through *specific hubs* which $u$ and $v$ are assigned to. On the contrary, the flow is now allowed to pass through any hub within the source and the sink areas. Hence, the term $x_{u,i}$ in the right-hand-sides of constraints (2.2h) can be replaced by $x_{i,i}$, for all $i \in V$, i.e.,

$$t_i^{1,u,v} \leq x_{i,i} \qquad \forall (u,v) \in T, \forall i \in V. \tag{2.10a}$$

Similarly, we can replace (2.2i) by

$$t_i^{2,u,v} \leq x_{i,i} \qquad \forall (u,v) \in T, \forall i \in V. \tag{2.10b}$$

But then, constraints (2.1d), (2.1e), (2.1g) and the allocation variables $x_{u,v}$, $u \neq v$, become redundant and should be removed from Formulation 1.

In Formulation 2, however, we do not have inter-layer variables and we explicitly make use of the assignment variables $x_{u,i}$ and $x_{v,i}$ in the flow-balance inequalities (2.6c)-(2.6f). More precisely, for any source-sink pair $(u, v)$ such that $w_{u,v} = 0$, we make sure that:

- The portion in Layer 1 of the flow from $u$ to $v$ starts at $u$ and ends at node $i$ for which $x_{u,i} = 1$ (constraint (2.6b)).
- The portion in Layer 2 of the flow starts at $i$ and ends at $j$ for which $x_{v,j} = 1$ (constraints (2.6d) and (2.6e)).
- The portion in Layer 3 of the flow starts at $j$ and ends at $v$ (constraint (2.6f)).

Obviously, when assumption (A3) is relaxed in definition of PHLRP and hence there is no explicit assignment of local nodes to hubs, the flow-balance inequalities (2.6b)-(2.6f) would not be valid anymore for PHLRP. Hence, Formulation 2 can not be used for solving PHLRP when assumption (A3) is relaxed.

For Formulation 3, we can argue in the same manner as we have done for Formulation 1 and conclude that constraints (2.7h) and (2.7i) can be replaced by

$$\tau_i^{1,u,v} \leq x_{i,i} \qquad \forall (u,v) \in T, \forall i \in V, \tag{2.11a}$$

$$\tau_i^{2,u,v} \leq x_{i,i} \qquad \forall (u,v) \in T, \forall i \in V. \tag{2.11b}$$

Naturally, this implies that we can remove constraints (2.1d), (2.1e) and (2.1g) and the allocation variables from the formulation.

We highlight these results in the following lemmas.

**Proposition 2.6.1.** *Suppose that assumption (A3) is relaxed in definition of PHLRP. Then a modified version of Formulation 1, which consists of the following constraints, can be used for solving PHLRP:* (2.1a)-(2.1c), (2.1f), (2.2b)-(2.2g), (2.10a) *and* (2.10b).

**Proposition 2.6.2.** *Suppose that assumption (A3) is relaxed in definition of PHLRP. Then a modified version of Formulation 3, which consists of the following constraints, can be used for solving PHLRP:* (2.1a)-(2.1c), (2.1f), (2.7b)-(2.7g) (2.11a) *and* (2.11b).

Note that, the versions of Formulations 1 and 3 that are described in these propositions do not employ the assignment variables $x_{u,v}$ $(u \neq v)$. But they contain the hub-location variables $x_{u,u}$ for all $u \in V$.

## 2.7 Relaxing arc-capacity constraints and assumption (A3) jointly

Finally in this section, we consider the joint impact of relaxing the capacity constraints and assumption (A3) from PHLRP. In fact, in this case we would benefit from all the relaxation / reductions that are presented in Sections 2.5 and 2.6. Namely, in this setting we can relax the integrality restrictions on the flow variables and the assignment variables ($x_{u,v}$ for $u, v \in V : u \neq v$) in the compact versions of Formulations 1 and 3 which are described in Propositions 2.6.1 and 2.6.2..

The following propositions are direct consequences of Propositions 2.5.1, 2.5.3, 2.6.1 and 2.6.2.

**Proposition 2.7.1.** *If capacity restrictions are ignored and assumption (A3) is relaxed, Formulation 1 reduces to the following:*

$$
\begin{aligned}
&\textit{min} && (2.2a) \\
&\textit{s. to} && (2.1a) - (2.1c), (2.1f), (2.2b), (2.2c), (2.2e) - (2.2g), (2.10a) \textit{ and } (2.10b), \\
w_e &\in \{0,1\} && \forall e \in E_n, \\
x_{u,u} &\in \{0,1\} && \forall u \in V, \\
f_{i,j}^{u,v}, g_{i,j}^{u,v}, h_{i,j}^{u,v} &\in [0,1] && \forall (u,v) \in T, (i,j) \in A, \\
t_i^{1,u,v}, t_i^{2,u,v} &\in [0,1] && \forall (u,v) \in T, i \in V.
\end{aligned}
$$

**Proposition 2.7.2.** *If capacity restrictions are ignored assumption (A3) is relaxed, Formulation 3 reduces to the following:*

$$
\begin{aligned}
&\textit{min} && (2.7a) \\
&\textit{s. to} && (2.1a) - (2.1c), (2.1f), (2.7b), (2.7c), (2.7e), (2.7f), (2.11a) \textit{ and } (2.11b), \\
w_e &\in \{0,1\} && \forall e \in E_n, \\
x_{u,u} &\in \{0,1\} && \forall u \in V, \\
\phi_{i,j}^{u,v}, \gamma_{i,j}^{u,v} &\in [0,1] && \forall (u,v) \in T, (i,j) \in A, \\
\tau_i^{1,u,v}, \tau_i^{2,u,v} &\in [0,1] && \forall (u,v) \in T, i \in V.
\end{aligned}
$$

# Chapter 3

# Analytical and empirical comparison of the IP formulations

In this chapter we present analytical and empirical comparison of Formulations 1, 2 and 3. In Section 3.1 we attempt to provide pairwise comparisons of the formulations' LP relaxation strengths. Section 3.2 is devoted to a comparative analysis of their computational performances in solving random instances of PHLRP.

## 3.1 Analytical comparisons

In this section we attempt to identify how Formulations 1, 2 and 3 compare with each other in terms of LP relaxation tightness. More precisely, in Sections 3.1.1 and 3.1.2, we show that Formulation 1 is stronger than the other two, and hence the strongest of the three. And, in Section 3.1.3, we compare Formulations 2 and 3 with each other, and show that none is provably stronger than the other.

### 3.1.1 Comparison of Formulations 1 and 2

Note that, the definitions of the flow variables $f_{i,j}^{u,v}$, $g_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$ are exactly the same in these two formulations. This means that all the variables of Formulation 2 are also present in Formulation 1. Differently, Formulation 1 retains two extra sets of variables, $t_i^{1,u,v}$ and $t_i^{2,u,v}$, which are not used at all in Formu-

lation 2. Hence, we first project out $t_i^{1,u,v}$ and $t_i^{2,u,v}$ variables from the linear relaxation of Formulation 1; then, we compare this projected formulation with the linear relaxation of Formulation 2, to decide which of Formulations 1 and 2 is stronger.

### Projecting out $t_i^{1,u,v}$ and $t_i^{2,u,v}$ from Formulation 1

We project out these variables using Fourier-Motzkin elimination (for a detailed treatment of this method, see [21]). We start by projecting out the $t_i^{2,u,v}$ variables. The relevant constraints that involve the $t_i^{2,u,v}$ variables in the linear relaxation of Formulation 1 are (2.2f), (2.2g), (2.2i) and the non-negativity constraints. We rearrange these constraints below:

$$
\begin{array}{lllll}
(1) & t_i^{2,u,v} = -(Ng^{u,v})_i + t_i^{1,u,v} & \forall (u,v) \in T, \forall i \in V & \text{(2.2f)} \\
(2) & t_i^{2,u,v} = (Nh^{u,v})_i - d_i & \forall (u,v) \in T, \forall i \in V & \text{(2.2g)} \\
(3) & t_i^{2,u,v} \leq x_{v,i} & \forall (u,v) \in T, \forall i \in V & \text{(2.2i)} \\
(4) & t_i^{2,u,v} \geq 0 & \forall (u,v) \in T, \forall i \in V & \text{(Non-neg. constraints)}
\end{array}
$$

where

$$
d_i = \begin{cases} 0, & \text{if } i \in V - \{v\}, \\ w_{u,v} - 1 & \text{if } i = v. \end{cases}
$$

We eliminate the $t_i^{2,u,v}$ variables and obtain the following inequalities:

$$
\begin{array}{lrll}
(5) & t_i^{1,u,v} = (Ng^{u,v} + Nh^{u,v})_i - d_i, & \forall (u,v) \in T, \forall i \in V & \text{(from (1) and (2))} \\
(6) & t_i^{1,u,v} \leq (Ng^{u,v})_i + x_{v,i}, & \forall (u,v) \in T, \forall i \in V & \text{(from (1) and (3))} \\
(7) & t_i^{1,u,v} \geq (Ng^{u,v})_i, & \forall (u,v) \in T, \forall i \in V & \text{(from (1) and (4))} \\
(8) & (Nh^{u,v})_i \leq x_{v,i} + d_i, & \forall (u,v) \in T, \forall i \in V & \text{(from (2) and (3))} \\
(9) & (Nh^{u,v})_i \geq d_i, & \forall (u,v) \in T, \forall i \in V & \text{(from (2) and (4))} \\
(10) & x_{v,i} \geq 0. & \forall (u,v) \in T, \forall i \in V & \text{(from (3) and (4))}
\end{array}
$$

Clearly, the inequalities (10) are redundant. We now project out the $t_i^{1,u,v}$ variables. Apart from (5), (6) and (7) above, the relevant constraints in the linear relaxation of Formulation 1 that involve the $t_i^{1,u,v}$ variables are (2.2e), (2.2h) and the non-negativity constraints. We now rearrange these constraints

below:

$$t_i^{1,u,v} = c_i - (Nf^{u,v})_i \qquad \forall (u,v) \in T, \forall i \in V \qquad \text{(2.2e)} \tag{11}$$

$$t_i^{1,u,v} \leq x_{u,i} \qquad \forall (u,v) \in T, \forall i \in V \qquad \text{(2.2h)} \tag{12}$$

$$t_i^{1,u,v} \geq 0 \qquad \forall (u,v) \in T, \forall i \in V \qquad \text{(Non-neg. constraints)} \tag{13}$$

where

$$c_i = \begin{cases} 1, & \text{if } i = u, \\ 0, & \text{if } i \in V - \{u, v\}, \\ -w_{u,v}, & \text{if } i = v. \end{cases}$$

Fourier-Motzkin elimination for the $t_i^{1,u,v}$ variables yield the following inequalities:

$$(Nh^{u,v})_i \leq x_{v,i} + d_i \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (5) and (6))} \tag{14}$$

$$(Nh^{u,v})_i \geq d_i \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (5) and (7))} \tag{15}$$

$$(Nf^{u,v} + Ng^{u,v} + Nh^{u,v})_i = c_i + d_i \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (5) and (11))} \tag{16}$$

$$(Ng^{u,v} + Nh^{u,v})_i \leq x_{u,i} + d_i \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (5) and (12))} \tag{17}$$

$$(Ng^{u,v} + Nh^{u,v})_i \geq d_i \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (5) and (13))} \tag{18}$$

$$x_{v,i} \geq 0 \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (6) and (7))} \tag{19}$$

$$(Nf^{u,v} + Ng^{u,v})_i \geq c_i - x_{v,i} \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (6) and (11))} \tag{20}$$

$$(Ng^{u,v})_i \geq -x_{v,i} \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (6) and (13))} \tag{21}$$

$$(Nf^{u,v} + Ng^{u,v})_i \leq c_i \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (7) and (11))} \tag{22}$$

$$(Ng^{u,v})_i \leq x_{u,i} \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (7) and (12))} \tag{23}$$

$$(Nf^{u,v})_i \geq c_i - x_{u,i} \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (11) and (12))} \tag{24}$$

$$(Nf^{u,v})_i \leq c_i \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (11) and (13))} \tag{25}$$

$$x_{u,i} \geq 0 \qquad \forall (u,v) \in T, \forall i \in V \quad \text{(from (12) and (13))} \tag{26}$$

Obviously, (19) and (26) are redundant. The inequalities (8) and (9) are equivalent to (14) and (15). The inequalities (17) are dominated by (16) and (24); the inequalities (18) are dominated by (16) and (25); the inequalities (20) are dominated by (14) and (16); the inequalities (22) are dominated by (15) and (16); the inequalities (23) are dominated by (15), (16) and (24); and the inequalities (21) are dominated by (14), (16) and (23). Hence, the projected model, which we call Formulation 1P in the sequel, is determined by the inequalities (14)-(16), (24) and (25).

More formally, the projected model Formulation 1P can be stated as follows:

$$\min \sum_{(u,v)\in T} \sum_{(i,j)\in A} c_{i,j}^{u,v} d_{u,v} (f_{i,j}^{u,v} + g_{i,j}^{u,v} + h_{i,j}^{u,v})$$

subject to the constraints (2.1a)-(2.1g), (2.2b), (2.2c),

$$(Nf^{u,v})_i \geq \begin{cases} 1 - x_{u,u} & \text{if } i = u \\ -x_{u,i} & \text{if } i \in V - \{u,v\} \\ -x_{u,v} - w_{u,v} & \text{if } i = v \end{cases} \qquad \forall (u,v) \in T \qquad (3.1a)$$

$$(Nf^{u,v})_i \leq \begin{cases} 1 & \text{if } i = u \\ 0 & \text{if } i \in V - \{u,v\} \\ -w_{u,v} & \text{if } i = v \end{cases} \qquad \forall (u,v) \in T \qquad (3.1b)$$

$$(Nh^{u,v})_i \geq \begin{cases} w_{u,v} - 1 & \text{if } i = v \\ 0 & \text{if } i \in V - \{v\} \end{cases} \qquad \forall (u,v) \in T \qquad (3.1c)$$

$$(Nh^{u,v})_i \leq \begin{cases} x_{v,v} + w_{u,v} - 1 & \text{if } i = v \\ x_{v,i} & \text{if } i \in V - \{v\} \end{cases} \qquad \forall (u,v) \in T \qquad (3.1d)$$

$$(Nf^{u,v} + Ng^{u,v} + Nh^{u,v})_i = \begin{cases} 1, & \text{if } i = u, \\ 0, & \text{if } i \in V - \{u,v\}, \\ -1, & \text{if } i = v. \end{cases} \qquad \forall (u,v) \in T \qquad (3.1e)$$

$$x_{u,v} \geq 0 \qquad \forall u,v \in V \qquad (3.1f)$$

$$w_{u,v} \geq 0 \qquad \forall \{u,v\} \in E_n \qquad (3.1g)$$

$$f_{i,j}^{u,v}, g_{i,j}^{u,v}, h_{i,j}^{u,v} \geq 0 \qquad \forall (u,v) \in T, (i,j) \in A \qquad (3.1h)$$

### Comparison of Formulations 1P and 2

In this section, we show that Formulation 2 is implied by Formulation 1P. Note that, constraints (2.1a)-(2.1g), (2.2b), (2.2c), (2.6b), (2.6c) and (2.6f) in Formulation 2 are all present in Formulation 1P, too.

We show that the constraints (2.6d) and (2.6e) are dominated. Consider a constraint in (2.6d) corresponding to an arbitrary source-sink pair $(u,v) \in T$ and an arbitrary node $i \in V$. From (3.1a) and (3.1d), using (2.1d), we can obtain

$$(Nh^{u,v})_{V-\{i\}} - (Nf^{u,v})_i \leq \begin{cases} x_{u,i} - x_{v,i} + w_{u,v} - 1, & \text{if } i = u, \\ x_{u,i} - x_{v,i} + w_{u,v}, & \text{if } i \in V - \{u,v\}, \\ x_{u,i} - x_{v,i} + w_{u,v} + 1, & \text{if } i = v; \end{cases}$$

$$(Nf^{u,v})_{V-\{i\}} - (Nh^{u,v})_i \geq \begin{cases} x_{u,i} - x_{v,i} - w_{u,v} - 1, & \text{if } i = u, \\ x_{u,i} - x_{v,i} - w_{u,v}, & \text{if } i \in V - \{u,v\}, \\ x_{u,i} - x_{v,i} - w_{u,v} + 1, & \text{if } i = v. \end{cases}$$

Since $(Nh^{u,v})_{V-\{i\}} = -(Nh^{u,v})_i$ and $(Nf^{u,v})_{V-\{i\}} = -(Nf^{u,v})_i$, adding the corresponding constraint of (3.1e) to these inequalities yields the result.

We now present a solution which is feasible for Formulation 2 but not for Formulation 1P. Consider the graph with six nodes in Figure 3.1. Let $n = 6$, $F_L = 2$, $F_U = 4$ and $Y = 3$. For the sake of simplicity assume that $T = \{(5, 6)\}$.

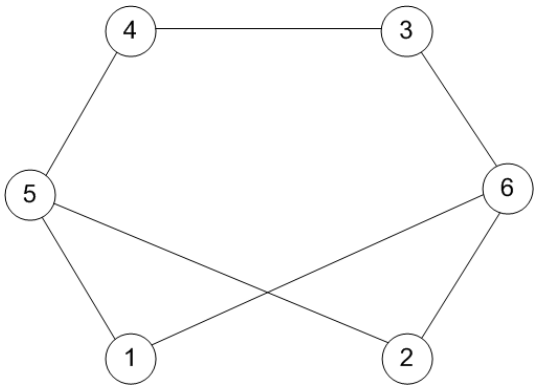



Figure 3.1: An example graph.

Now, consider the following solution $\tilde{\boldsymbol{z}} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})$:

$$\tilde{\boldsymbol{x}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.50 & 0.50 & 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0.50 & 0.50 & 0 \\ 0.25 & 0 & 0 & 0.25 & 0.50 & 0 \\ 0 & 0.25 & 0.25 & 0 & 0 & 0.50 \end{bmatrix} \end{array} ;$$

$$\tilde{w}_{h,l} = 0.5 \qquad \forall \{h, l\} \in E_6;$$

$$\tilde{f}_{i,j}^{5,6} = \begin{cases} 0.50, & \text{if } (i,j) = (5,2), \\ 0.50, & \text{if } (i,j) = (2,6), \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{g}_{i,j}^{5,6} = 0 \qquad \forall (i,j) \in A;$$

$$\tilde{h}_{i,j}^{5,6} = 0 \qquad \forall (i,j) \in A.$$

This solution is feasible for Formulation 2, but violates (3.1e).

### 3.1.2 Comparison of Formulations 1 and 3

The variables $g_{i,j}^{u,v}$ in Formulation 1 and $\gamma_{i,j}^{u,v}$ in Formulation 3 are equivalent to each other, since both represent the portions on the backbone of inter-area flows. Moreover, since Layers 1 and 3 of Formulation 1 are merged into Layer 1 of Formulation 3, flow variables $\phi_{i,j}^{u,v}$ in Formulation 3 are equivalent to the sum of $f_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$ in Formulation 1, for any $(u,v) \in T$ and $(i,j) \in A$. Furthermore, for any $(u,v) \in T$ and $i \in V$, the definitions of $t_i^{1,u,v}$ and $t_i^{2,u,v}$ in Formulation 1 and $\tau_{i,j}^{1,u,v}$ and $\tau_{i,j}^{2,u,v}$ in Formulation 3 are respectively equivalent to each other.

All these equivalence relations among the variables of these two formulations imply that all constraints but (2.7c), (2.7e) and (2.7g) of Formulation 3 are also present in Formulation 1. Hence, to show that Formulation 1 is stronger than Formulation 3, we only need to prove that constraints (2.7c), (2.7e) and (2.7g) are implied by the constraints of Formulation 1. Due to the equivalence of $\phi_{i,j}^{u,v}$ to the sum of $f_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$, (2.7e) are equal to the sum of (2.2e) and (2.2g). Constraints (2.7g) are implied by the summation of (2.2e) over all $i \in V$. And, constraints (2.7c) are equivalent to (2.2c).

We illustrate strict inclusion of Formulation 1 in Formulation 3 by means of an example over the example graph in Figure 3.1. Let $n = 6$, $F_L = 2$, $F_U = 4$, $N = 3$ and $T = \{(5,6)\}$.

Consider the following solution $\tilde{\boldsymbol{z}} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{f}}, \tilde{\boldsymbol{g}}, \tilde{\boldsymbol{h}})$:

$$\tilde{\boldsymbol{x}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.50 & 0.50 & 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0.50 & 0 & 0 & 0 \\ 0 & 0 & 0.50 & 0.50 & 0 & 0 \\ 0.50 & 0 & 0 & 0.50 & 0 & 0 \\ 0.25 & 0 & 0 & 0.25 & 0.50 & 0 \\ 0.25 & 0 & 0 & 0.25 & 0 & 0.50 \end{bmatrix} \end{array};$$

$$\tilde{w}_{h,l} = 0.5 \qquad \forall \{h,l\} \in E_6$$

$$\tilde{f}_{i,j}^{5,6} = \begin{cases} 0.50, & \text{if } (i,j) = (5,2), \\ 0.50, & \text{if } (i,j) = (2,6), \\ 0.50, & \text{if } (i,j) = (5,1), \\ 0.50, & \text{if } (i,j) = (1,6), \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{g}_{i,j}^{5,6} = 0 \qquad \forall (i,j) \in A,$$

$$\tilde{h}_{i,j}^{5,6} = 0 \qquad \forall (i,j) \in A,$$

$$\tilde{t}_i^{1,5,6} = \tilde{t}_i^{2,5,6} = \begin{cases} 0.25, & \text{if } i = 1, \\ 0.25, & \text{if } i = 4, \\ 0, & \text{otherwise;} \end{cases}$$

This solution is not feasible for Formulation 1 as it violates constraint (2.2g). But, the solution $\left(\tilde{\phi}_{i,j}^{u,v}, \tilde{\gamma}_{i,j}^{u,v}, \tilde{\tau}_{i,j}^{1,u,v}, \tilde{\tau}_{i,j}^{2,u,v}\right)$, with $\tilde{\phi}_{i,j}^{u,v} = \tilde{f}_{i,j}^{u,v} + \tilde{h}_{i,j}^{u,v}$, $\tilde{\gamma}_{i,j}^{u,v} = \tilde{g}_{i,j}^{u,v}$, $\tilde{\tau}_l^{1,u,v} = \tilde{t}_l^{1,u,v}$ and $\tilde{\tau}_l^{2,u,v} = \tilde{t}_l^{2,u,v}$ for all $(u,v) \in T$, $(i,j) \in A$ and $l \in V$, is feasible for Formulation 3.

In this section and the previous, we have proved that Formulation 1 is the strongest of the three formulations introduced.

### 3.1.3 Comparison of Formulations 2 and 3

To be able to gain more analytical insight on the formulations introduced, we compare the polytopes of Formulations 2 and 3 as well. For this, like we do in Sections 3.1.1 and 3.1.2, we translate Formulations 2 and 3 to the same subspace. More precisely, we recast them on the basis of two layers with no inter-area links. In other words, we rewrite them using the variables $w_{u,v}$, $x_{u,v}$, $\phi_{i,j}^{u,v}$ and $\gamma_{i,j}^{u,v}$ only, the former two and the latter two being defined exactly in the same manner as they are defined in Sections 2.1 and 2.4 respectively.

Nevertheless, in this case, this comparison does not lead to firm conclusions about which formulation is stronger than the other. For this reason, we present this lengthy comparison in Section A.1 of the appendix. Namely, we show that neither of Formulations 2 and 3 implies the other. We are also able to provide example instances where there exists a feasible solution for one formulation but not the other.

We conclude analytical comparisons of the formulations by wrapping up our results in Figure 3.2. In the figure, region $F_i$ symbolizes the polytope associated with the LP relaxation of Formulation $i$, $i = 1, 2, 3$. $F_1$ is contained in both $F_2$ and $F_3$ (i.e., Formulation 1 is stronger than both Formulations 2 and 3); and, $F_2$ and $F_3$ intersect (i.e., neither Formulation 2 nor Formulation 3 is provably stronger than the other).

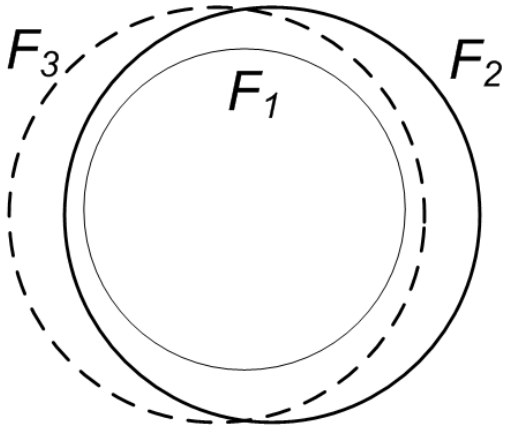



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Figure 3.2: Analytical comparison of Formulations 1, 2 and 3.

## 3.2 Empirical Comparisons of the Formulations

In this section, we compare the performances of the three formulations from a computational efficiency point of view. As the test platform, we use the MIP solver of CPLEX 11 (with default settings) on a machine with Pentium 4, 3GHz with 2GB of memory. Due to hardness of the problem, we ignore the capacity constraints and run our experiments on the reduced forms of the formulations. Namely, we run our experiments on the reduced form of Formulations 1 and 3 as implied by Propositions 2.7.1 and 2.7.2, respectively, and on the reduced form of Formulation 2 as implied by Lemma 2.5.2.

We run the experiments on randomly generated instances, each of which we characterize by

- a graph $G = (V, E)$ and a corresponding network $D = (V, A)$ (where every edge in $E$ is represented in $A$ by two arcs in opposite directions),
- a cost matrix where a random integer cost value between 1 and 20 is assigned to every edge in $E$, and,
- an upper-triangular traffic matrix whose non-zero entries, random integers between 1 and 200, represent the source-sink pairs with traffic demand in-between (i.e., elements of $T$).

We consider four different values for $|V|$: 9,10,12 and 15; four different values for edge density (i.e., $|E|/\binom{n}{2}$): 0.3,0.5,0.7 and 0.9; and five different values for the traffic matrix density (i.e., $|T|/\binom{n}{2}$):0.1, 0.3, 0.5, 0.7 and 1. We generate 20 instances for every possible combination of $|V|$, edge density and traffic matrix density. That is, our test bed consists of 1600 instances (20x4x4x5). We use a time limit of 1 hour per instance in our experiments.

For $F_L$, $F_U$ and $N$ we use the following settings:

- When $|V| = 9$, we set $F_L = F_U = N = 3$;
- when $|V| = 10$, we set $F_L = 3$, $F_U = 4$ and $N = 3$;
- when $|V| = 12$, we set $F_L = 3$, $F_U = 4$ and $N = 4$;
- when $|V| = 15$, we set $F_L = 3$, $F_U = 5$ and $N = 5$.

Tables 3.1, 3.2 and 3.3 display, respectively, average solution time, average LP relaxation gap ratios (percentage) and average branch-and-cut tree sizes for the instances with size 9 and 10. Percentage LP relaxation gap values are computed as follows:

$$\frac{OPT_{IP} - OPT_{LP}}{OPT_{IP}} \times 100, \tag{3.2}$$

where $OPT_{IP}$ and $OPT_{LP}$ represent the optimal objective values of the IP model and its LP relaxation, respectively.

All instances with size 9 and 10 could be solved to optimality by all formulations within the time limit of one hour. Each of the entries in Tables 3.1, 3.2 and 3.3 corresponds to average values for 400 instances. Obviously, performance of Formulation 2 is very poor compared to the other two. Formulation 3 performs slightly better than Formulation 1 in terms of solution time. Although Formulation 1 is theoretically stronger than Formulation 3, the difference in the LP relaxation gap ratios is insignificant. We can observe the impact of the strength of Formulation 1 in the sizes of the branch-and-cut trees generated. But, owing to its compactness, Formulation 3 manages to yield better solution times.

Clearly, Formulation 2 is very weak. Analytically, Formulation 3 is not stronger than Formulation 2, as shown in Section 3.1.3. But, empirically we can definitely conclude that Formulation 2 has the largest LP relaxation gap and the worst performance. Hence, we exclude this formulation from our empirical analysis in the rest of this section.

| $\|V\|$ | Form. 1 | Form. 2 | Form. 3 |
|---|---|---|---|
| 9 | 2.675 | 17.848 | 1.998 |
| 10 | 14.042 | 180.310 | 9.718 |

Table 3.1: Average solution time (seconds) for instances with size 9 and 10.

| $\|V\|$ | Form. 1 | Form. 2 | Form. 3 |
|---|---|---|---|
| 9 | 8.514% | 61.691% | 8.767% |
| 10 | 8.980% | 63.604% | 9.120% |

Table 3.2: Average LP relaxation gap ratios for instances with size 9 and 10.

| $\|V\|$ | Form. 1 | Form. 2 | Form. 3 |
|---|---|---|---|
| 9 | 23.18 | 71.1925 | 25.32 |
| 10 | 119.3225 | 468.385 | 128.6075 |

Table 3.3: Average number of nodes in the branch-and-cut tree for instances with size 9 and 10.

We now compare performances of Formulations 1 and 3 on the larger instances (with sizes 12 and 15). Tables 3.4, 3.5 and 3.6 display for these instances, respectively, average solution times, average LP relaxation gap ratios and average size of the branch-and-cut trees of the formulations. Both of the formulations could solve to optimality within the time limit (of 1 hour) all of the 400 instances with size 12. But, out of 400 instances with size 15, Formulations 1 and 3 failed to solve (within the time limit), 23 and 15, respectively. Hence, the average figures displayed in the last rows ($|V| = 15$) of Tables 3.4, 3.5 and 3.6 are computed over 377 instances for Formulation 1 and over 385 instances for Formulation 3.

Note that, in Table 3.5 LP relaxation gap ratios seem to be lower for instances with size 15. However, such a comparison would be misleading as the figures in the $|V| = 15$ row exclude the hardest instances among the 400 (the ones that could not be solved).

The results follow the same pattern as in Tables 3.1, 3.2 and 3.6. Formulation 3 performs better in solution time. And the difference in LP relaxation gap ratios is not significant.

There are 15 instances with size 15 (out of 400) that both formulations failed to solve within 1 hour. For these instances, the average gap at termination (the ratio of the difference between the best lower bound and best upper bound to the best upper bound) is 6.943% for Formulation 1 and 5.502% for Formulation 3. There are 8 instances with size 15 that could be solved to optimality within the time limit by Formulation 3 but not by Formulation 1. For these 8 instances, gap at termination exhibited by Formulation 1 is on the average 3.283 (with maximum 4.961 and minimum 2.057).

Finally, we would like to say a few words on our computational experience with instances of size 20. Our experiments on 20-node instances is not "complete", that is, due to practical limitations on experimentation time we confine ourselves to instances with edge densities 0.3 and 0.5 only as they are easier to solve. We consider the same five traffic demand density values as above (0.1, 0.3, 0.5, 0.7 and 1). Hence, our experiments on instances of size 20 involve only 200 (20x2x5) instances. Table 3.7 displays the status of the formulations on these instances after one hour of solution time.

Formulation 1, on the average, takes 1023.748 seconds and creates 627.64 branch-and-cut nodes to solve 78 instances that it can solve to optimality. For Formulation 3, average solution time and average number of branch-and-cut nodes for solving 82 instances to optimality is 956.144 seconds and 852.28. The gap in average solution times for these instances is not large.

| $\|V\|$ | Form. 1 | Form. 3 |
|---|---|---|
| 12 | 57.804 | 46.375 |
| 15 | 510.323 | 442.906 |

Table 3.4: Average solution time (seconds) for instances with size 12 and 15.

| $\|V\|$ | Form. 1 | Form. 3 |
|---|---|---|
| 12 | 7.108% | 7.193% |
| 15 | 6.310% | 6.498% |

Table 3.5: Average LP relaxation gap ratios for instances with size 12 and 15.

| $\|V\|$ | Form. 1 | Form. 3 |
|---|---|---|
| 12 | 168.41 | 192.0125 |
| 15 | 568.725 | 667.0175 |

Table 3.6: Average number of nodes in the branch-and-cut tree for instances with size 12 and 15.

| Status | Form. 1 | Form. 3 |
|---|---|---|
| Optimal | 78 | 82 |
| Int. feasible soln. found | 79 | 106 |
| No int. feasible soln. found | 43 | 12 |
| Total | 200 | 200 |

Table 3.7: Status of formulations on instances with size 20 at the end of the time limit (1 hour).

Formulation 3 fails to solve to optimality 118 instances out of 200. But it can find an integer feasible solution for 106 of those instances (86.83% of 118). On the other hand, Formulation 1 can find an integer feasible solution for only 79 of 122 instances that it can not solve to optimality (only 64.75% of 122). Average gap at termination (ratio between the best lower bound and upper bound at termination) is 18.39% (respectively, 17.59%) for 106 (respectively, 79) instances for which Formulation 1 (respectively, Formulation 3) can find an integer feasible solution.

From the empirical evidence presented in this section, we conclude that Formulation 3 is the best of the three formulations introduced in this paper. Formulation 2 performs by far the worst of the three. The difference between the performances of Formulation 1 and 3 are quite similar to each other. But, in general, Formulation 3 yields better results.

## 3.3 Conclusion

This chapter is devoted to thorough comparative analyses of Formulations 1, 2 and 3. In Sections 3.1 and 3.2 we carry out analytical and empirical comparisons of these formulations.

In Section 3.1, we prove that Formulation 1 is stronger than the other two formulations. Polytopes corresponding to Formulations 2 and 3 intersect (none contains the other).

However, in Section 3.2 we see that LP relaxation gap and solution time associated with Formulation 2 is much higher than the other two formulations. We also see that LP relaxation gap for Formulation 1 is only very slightly lower than that of Formulation 3. Although there is not much difference in the LP relaxation gaps associated with these two formulations, strength of Formulation 1 has a significant impact on the sizes of the branch-and-cut trees created for solving the instances. Formulation 1 explores much smaller branch-and-cut trees than Formulation 3. However, Formulation 3 is more compact and hence it takes shorter time to solve the corresponding LPs in the nodes of the branch-and-cut trees. That is, the usual trade-off between formulation strength and compactness reveals itself in our experiments as well. And compactness of Formulation 3 turns out to be more crucial than strength of Formulation 1 in solving PHLRP.

# Chapter 4

# The size-constrained graph partitioning polytope

In Chapter 2 we conclude that when capacity restrictions on the arcs of $D$ are ignored and assumption (A3) is relaxed, Formulations 1, 2 and 3 can be recast in simpler, more compact forms. More precisely, Lemma 2.5.2 and Propositions 2.7.1 and 2.7.2 show that in that case the integrality restriction on all flow variables and all $x_{u,v}$ with $u \neq v$ can be relaxed. Then, the only binary variables that remain in the formulations are the partitioning variables (i.e., $w_e$ where $e \in E_n$) and the location variables (i.e., $x_{u,u}$ where $u \in V$).

In this chapter we attempt to exploit this feature. We present a thorough polyhedral analysis of the convex hull of $w \in \{0,1\}^{\binom{n}{2}}$ that satisfy (2.1a), (2.1b) and (2.1c). Lemma 2.5.2 and Propositions 2.7.1 and 2.7.2 imply that strong valid inequalities or facets for this polytope constitute strong valid inequalities for Formulations 1, 2 and 3, too.

In fact, constraints (2.1a), (2.1b) and (2.1c) define a generic graph partitioning polytope which has strong links with several well-known graph partitioning polytopes from the literature. For this reason, in this chapter we analyze this polytope within the context of the graph partitioning literature.

Owing to this treatment, this chapter is somewhat disconnected from the other chapters of this thesis. In this chapter[1], PHLRP does not occupy a central location. Our focus is rather on the graph partitioning polytope, which is defined by (2.1a), (2.1b) and (2.1c).

[1]An abridged version of this chapter is published in December 2010: "Size-constrained graph partitioning polytopes", *Discrete Mathematics*, 310(24), 3473–3493.

## 4.1 Introduction

The problem of partitioning a set of items into clusters to achieve an objective (e.g., to obtain homogeneous clusters and extract valuable information on each of the clusters, to divide the items into clusters that have minimal interaction in-between, etc.) is encountered in a wide range of disciplines such as marketing, economics, biology, psychology, politics, etc. Owing to this wide spectrum of applications, the clustering problem has been studied extensively in Operations Research.

Three classes of clustering problems can be differentiated in Operations Research applications. In the first class of problems, the items to be clustered are articulated by numerical attributes. Generally, the items are represented by vectors in a real space which correspond to a set of attribute measurements. This class of problems find applications in many areas like pattern recognition (see [10] for a general treatment) and marketing (for example, [28]), among others.

In the second class of problems, there are no attributes defined at all and only certain pairwise relations between the items are provided. For example in some telecommunications networks, the nodes are clustered on the basis of the communication level between pairs of nodes. Generally, the aim in such problems is to cluster the items (routers, users, customers, etc.) so as to minimize the communication between the clusters (for example, see [19], [27]), or minimize total communication cost at all within the whole network (for example, see [32]). Other application areas of this class of clustering problems are compiler design, parallel computing and electronic circuit design (for example, see [11], [12], [16]).

In the third class, the data relevant to clustering involve attributes that are defined categorically. Namely, attributes do not assume numerical values and are evaluated in categories (for example, see [25]). For example, if the objects under consideration are people, hair color is an example of an attribute which is best stated categorically (brown, black, blond, etc.); for consumers, the attitude against a product or brand can be more meaningfully articulated categorically (positive, negative, neutral); in some applications, it might be more meaningful to state the size categorically as “large”, “medium” or “small” rather than some numerical measures. Fortunately, this class of problems can also be analyzed in the second class of clustering problems above owing to some qualitative data analysis techniques which convert such type of data into pairwise relations between the items. An example of such a conversion is given

by Grötschel and Wakabayashi ([13]).

Generally in Operations Research literature, the clustering problems in the second and third classes above are classified as *graph partitioning problems*. In this scheme, the items to be clustered are represented by the nodes of a graph, and pairwise relations among them are represented by edge weights. The aim is to find the 'best' clustering, which optimizes a certain objective function, of the nodes of the graph into subsets.

In practical clustering applications, the subsets to be obtained might be subject to various constraints such as restrictions on the number of subsets, restrictions on the sizes of the subsets, requirement that certain items be or not be in the same subset, etc. In this chapter, we confine ourselves to clustering problems with restrictions on the subset sizes. Namely, the clustering problems that are addressed here involve parameters $F_U$ and $F_L$, which constrain the sizes of the subsets to be obtained from above and below, respectively. For instance, in a specific clustering application, the subsets we obtain might represent voting districts for a certain elective procedure, or distribution regions belonging to a distribution network. It might not be possible to support voting districts or distribution regions larger than a pre-specified size $F_U$ because some resources at disposal (like human resources for the elective procedure, or vehicles for the distribution network) might not be adequate in quantity or in quality. In addition, reasons like economies of scale or need for balanced partitions, where the sizes of the subsets obtained are 'almost' equal to each other, might give rise to minimum size requirements as well on the subsets. For instance, we might want to impose the restriction that the subsets be larger than a pre-specified size, $F_L$, in order to make the voting districts large enough to allow meaningful statistical inferences, or, in order to make the distribution regions large enough to benefit from economies of scale, or, in order to obtain almost equally sized subsets that will allow applying similar administrative procedures.

Besides bounds $F_L$ and $F_U$ on the subset sizes, we assume as well that the data relevant to clustering is provided in the form of pairwise relations among the items. This locates our problem in the second and third class of clustering problems described above. Following the prevalent practice in the literature, we formulate it as a graph partitioning problem. The formal definition and an integer programming formulation of the problem are given in Section 4.3. But before, in Section 4.2, we present the notation that is used in this chapter.

## 4.2 Notation

We represent the set of items to be clustered by the set $V = \{1, 2, \ldots, n\}$. Consider a graph $G = (V, E)$ on $V$. Each edge $e \in E$ is also represented by means of its adjacent nodes $u$ and $v$: the terms $\{u, v\}$ and $\{v, u\}$ are used interchangeably to refer to the edge $e$.

A *partition*, $\pi$, is a collection of non-empty subsets $N_1, N_2, \ldots, N_k$ of $V$ such that $N_i \cap N_j = \emptyset$ for all $i \neq j$, and $\bigcup_{\forall i} N_i = V$. A *size constrained partition* (shortly, an *sc-partition*) is a partition with $F_L \leq |N_i| \leq F_U$ for all $i = 1, \ldots, k$. We associate a characteristic vector $w^\pi \in \{0, 1\}^{|E|}$ with each partition $\pi$, where $w^\pi_{u,v} = 1$ if $u, v \in N_i$ for some $i \in \{1, 2, \ldots, k\}$, and 0 otherwise. We use the representations $\pi$ and $(N_1, N_2, \ldots, N_k)$, and its characteristic vector $w^\pi$ interchangeably to refer to a partition. We use the term *subclique* to refer to the subsets (i.e., $N_i$'s) in the partitions.

Throughout, $conv(X)$ and $aff(X)$ denotes the convex hull and the affine hull of a set $X \subset \mathbb{R}^{|E|}$, respectively. The equality set $M(P)$ of a polytope $P$ is a maximal set of linearly independent hyperplanes that contain $P$. The dimension of $P$ is denoted by $dim(P)$.

For convenience, we also recite the relevant parts of the notation that was introduced in Chapter 1. Let $Q \subseteq V$ and $S \subseteq V$. We denote the set of all edges whose both endnodes are in $Q$ by $E(Q)$, i.e., $E(Q) = \{\{u, v\} \in E | u, v \in Q\}$. The set of edges with one endnode in $Q$ and the other endnode in $S$ are denoted by $E(Q, S)$, i.e., $E(Q, S) = \{\{u, v\} \in E | u \in Q, v \in S\}$. We denote by $\delta(Q)$ the set of edges whose only one end-node lies in $Q$, i.e., $\delta(Q) = E(Q, V - Q)$.

For any $x \in \mathbb{R}^{|E|}$, let $x(F) = \sum_{e \in F} x_e$ for $F \subseteq E$. For ease of notation, a set with only one element is represented with the element itself in mathematical expressions: namely, for instance, $Q - u$ stands for $Q - \{u\}$.

## 4.3 Problem definition and formulation

The problem we address in this paper and its companion consists of finding the sc-partition $\pi$ that optimizes

$$\sum_{e \in E} c_e w^\pi_e.$$

This is a quite common objective function for most of the graph partitioning problems in the literature. Some articles interpret the minimization of this

objective function as clustering of the most similar items together when the edge weights $c_e$'s represent a measure of dissimilarity (like distance) between the items (see [13], [14], [23] and [22] among others). Some others interpret the maximization of it as division of a network or a compiler into clusters which have minimal interaction in-between (see [11], [12], [16] and [19] among others).

This objective function, which is a sum of the weights of a set of edges, allows us to assume that $G = (V, E)$ is a complete graph. Because, if it is not, we can always convert it to one by adding the missing edges with weights 0. Solving the problem on this complete graph would be completely equivalent to solving it on $G$. Hence, in the rest of this chapter, we assume that $G = (V, E)$ is a complete graph defined on $n$ nodes. That is, in the rest of this chapter, we assume that $E = E_n$ and $G = K_n$ (note that this assumption holds in this chapter only).

The integer programming formulation we propose for our problem is as follows:

$$\text{minimize or maximize} \quad \sum_{e \in E} c_e w_e$$

$$\text{subject to}$$

$$\begin{aligned}
w_{u,v} + w_{u,t} - w_{v,t} &\leq 1 \qquad \forall u, v, t \in V : u, v, t \text{ different} && \text{(4.1a)}\\
w(\delta(u)) &\geq F_L - 1 \qquad \forall u \in V && \text{(4.1b)}\\
w(\delta(u)) &\leq F_U - 1 \qquad \forall u \in V && \text{(4.1c)}\\
w_{u,v} &\in \{0, 1\} \qquad \forall \{u, v\} \in E && \text{(4.1d)}
\end{aligned}$$

where the variable $w_{u,v}$ takes value 1 if nodes $u$ and $v$ are in the same subclique, and 0 otherwise. Constraints (4.1a), called triangle inequalities, ensure that if any two edges of a triangle linking three nodes in $V$ are contained in one subclique (i.e., all three of the nodes are packed in the same subclique), then the third edge of the triangle is also contained in that subclique. Constraints (4.1b) and (4.1c) stand for the size restrictions on the subcliques. It is easy to see that the set of characteristic vectors of sc-partitions of $G$ coincide exactly with set of feasible solutions of formulation (4.1).

This paper and its sequel are devoted to the analysis of the structure of the polytope

$$\mathcal{P}^{lu}(n, F_L, F_U) = conv\{w^\pi \in \mathbb{R}^{|E|} | w^\pi \text{ feasible for (4.1) (i.e., } \pi \text{ is an sc-partition)}\}.$$

Obviously, this polytope is uniquely determined by the three parameters $n$, $F_L$ and $F_U$.

Formulation (4.1) and $\mathcal{P}^{lu}(n, F_L, F_U)$ generalize the following several graph partitioning problems from the literature and their corresponding polytopes.

- For $F_L = 1$ and $F_U = n$ (i.e., when (4.1b) and (4.1c) become redundant), this formulation turns into the so-called *clique partitioning problem* which was introduced by Grötschel and Wakabayashi ([13],[14]; see also [2], [26]). This problem addresses the most basic clustering problem where there is no restriction on the number of subsets or the sizes of the subsets (i.e., for a feasible solution $(N_1, \ldots, N_k)$ of the clique partition problem, $1 \leq k \leq n$, and $1 \leq |N_i| \leq n$ for all $i = 1, \ldots, k$). For further use, we define

  $$\mathcal{P}(n) = conv\{w^\pi \in \mathbb{R}^{|E|} | w^\pi \text{ satisfies (4.1a) and (4.1d) (i.e., } \pi \text{ is a partition)}\}.$$

- When $F_U = n$ (i.e., when the upper bound on the subclique sizes is removed), we get the so-called *clique partitioning problem with minimum clique size requirement* of Ji and Mitchell [15]. Let the corresponding polytope be denoted as $\mathcal{P}^l(n, F_L)$, i.e.,

  $$\mathcal{P}^l(n, F_L) = conv\{w^\pi \in \mathbb{R}^{|E|} | w^\pi \text{ satisfies (4.1a), (4.1b) and (4.1d)}\}.$$

- When $F_L = 1$ (i.e., the lower bound is removed), we end up with the so-called *simple graph partitioning problem* of Sørensen ([30] and [29]). We denote the corresponding polytope as

  $$\mathcal{P}^u(n, F_U) = conv\{w^\pi \in \mathbb{R}^{|E|} | w^\pi \text{ satisfies (4.1a), (4.1c) and (4.1d)}\}.$$

- The equipartition problem consists of dividing a graph $G$ into two subgraphs with sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. Let $\mathcal{P}^{equi}(n)$ denote the equi-partition polytope, that is, the convex hull of characteristic vectors $w^\pi$ such that $\pi = (N_1, N_2)$, $|N_1| = \lfloor \frac{n}{2} \rfloor$ and $|N_2| = \lceil \frac{n}{2} \rceil$. Conforti et al. ([6], [7]) provide a detailed analysis of $\mathcal{P}^{equi}(n)$.

- The $k$-way equipartition problem ($k \geq 3$) consists of dividing a graph $G$ with ($n$ mod $k$) $= 0$ into $k$ equally sized subgraphs. And, similarly, let $\mathcal{P}^{k-way}(n, k)$ denote the $k$-way equi-partition polytope, which is the convex hull of $w^\pi$ such that $\pi = (N_1, \ldots, N_k)$ and $|N_i| = F = \frac{n}{k}$ for $i = 1, \ldots, k$. For a detailed treatment of $\mathcal{P}^{k-way}(n, k)$, see [23] and [22] by Mitchell.

In this chapter, we attribute special importance to the equi-partition polytope $\mathcal{P}^{equi}(n)$ and the $k$-way equi-partition polytope $\mathcal{P}^{k-way}(n,k)$. Because, in the sequel, we present several results, which are new to the literature, on $\mathcal{P}^{equi}(n)$ and $\mathcal{P}^{k-way}(n,k)$. In Section 4.4, we state conditions on $n$, $F_L$ and $F_U$ that make $\mathcal{P}^{lu}(n,F_L,F_U)$ equivalent to $\mathcal{P}^{equi}(n)$ or $\mathcal{P}^{k-way}(n,k)$.

For ease of notation, throughout we shortly use $\mathcal{P}^{lu}$ to refer to $\mathcal{P}^{lu}(n,F_L,F_U)$ unless otherwise stated. In the next section, we start with the analysis of $\mathcal{P}^{lu}$ by proving feasibility conditions and dimension.

## 4.4 Non-emptiness and dimension of $\mathcal{P}^{lu}$

Not all $(n,F_L,F_U)$ triples yield a nonempty $\mathcal{P}^{lu}$. For example, for $n=31$, $F_L=8$ and $F_U=10$ it is not possible to find an sc-partition, i.e., $\mathcal{P}^{lu}(31,8,10)=\emptyset$. Lemma 4.4.1 states the necessary and sufficient condition $(n,F_L,F_U)$ should satisfy for $\mathcal{P}^{lu}$ to be non-empty. We assume in this section that $F_L \geq 2$.

**Lemma 4.4.1.** *$\mathcal{P}^{lu} \neq \emptyset$ if and only if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq \left\lceil \frac{n}{F_U} \right\rceil$.*

*Proof.* Consider an sc-partition $\pi = (N_1,\ldots,N_k)$. By definition of an sc-partition (see Section 4.2), $F_L \leq |N_i| \leq F_U$ for all $i=1,2,\ldots,k$. That is, $w^\pi \in \mathcal{P}^{lu}$ if and only if $kF_L \leq n \leq kF_U$, which holds if and only if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq k \geq \left\lceil \frac{n}{F_U} \right\rceil$. □

Note that, non-emptiness of $\mathcal{P}^{lu}$ is dependent only on the values of $n$, $F_L$ and $F_U$, and not on the set $V$ itself. This implies that non-emptiness of $\mathcal{P}^{lu}$ is preserved through permutations of $V$. One can also infer from the proof of this lemma that an sc-partition has at least $\left\lceil \frac{n}{F_U} \right\rceil$ and at most $\left\lfloor \frac{n}{F_L} \right\rfloor$ subcliques.

We now pass on to determining the dimension of $\mathcal{P}^{lu}$. When $F_U = F_L$, the problem reduces to the $k$-way equipartition problem whose dimension is shown to be $\binom{n}{2} - n$ by Mitchell [22]. Hence, assume in the rest of this section that $F_U \geq F_L + 1$.

We now give two technical lemmas that will be used in establishing the dimension of $\mathcal{P}^{lu}$ and in some of the facetness proofs in the following sections.

**Lemma 4.4.2.** *Let $P$ denote a face of $\mathcal{P}^{lu} \neq \emptyset$. Let $\pi = (N_1,N_2,\ldots,N_k)$ be an sc-partition such that $w^\pi \in P$. Let $\alpha^T w = \alpha_0$ ($\alpha \neq 0$) be a hyperplane in*

$\mathbb{R}^{\binom{n}{2}}$ *and suppose* $P \subset \{w \in \mathbb{R}^{\binom{n}{2}} | \alpha^T w = \alpha_0\}$. *Let* $\{i,j\} \subseteq N_1$ *and* $h \in N_2$. *Then* $\alpha_{i,h} = \alpha_{j,h}$ *provided that the following are satisfied for* $i$, $j$ *and* $h$*:*

- $w^{\pi^1} \in P$ *where* $\pi^1 = (N_1^*, N_2^*, \ldots, N_k^*)$, $N_1^* = N_1 - i$, $N_2^* = N_2 \cup i$, $N_l^* = N_l \; \forall l \in \{3, \ldots, k\}$*;*
- $w^{\pi^2} \in P$ *where* $\pi^2 = (N_1^*, N_2^*, \ldots, N_k^*)$, $N_1^* = N_1 - j$, $N_2^* = N_2 \cup j$, $N_l^* = N_l \; \forall l \in \{3, \ldots, k\}$*;*
- $w^{\pi^3} \in P$ *where* $\pi^3 = (N_1^*, N_2^*, \ldots, N_k^*)$, $N_1^* = (N_1 - i) \cup h$, $N_2^* = (N_2 - h) \cup i$, $N_l^* = N_l \; \forall l \in \{3, \ldots, k\}$*;*
- $w^{\pi^4} \in P$ *where* $\pi^4 = (N_1^*, N_2^*, \ldots, N_k^*)$, $N_1^* = (N_1 - j) \cup h$, $N_2^* = (N_2 - h) \cup j$, $N_l^* = N_l \; \forall l \in \{3, \ldots, k\}$.

*Proof.* Using the fact that $\alpha^T w^\pi = \alpha_0$ and $\alpha^T w^{\pi^1} = \alpha_0$, we obtain

$$\alpha_{i,j} + \alpha(i, N_1 - \{i,j\}) = \alpha_{i,h} + \alpha(i, N_2 - h). \tag{4.2}$$

From $\alpha^T w^\pi = \alpha_0$ and $\alpha^T w^{\pi^2} = \alpha_0$ we have

$$\alpha_{i,j} + \alpha(j, N_1 - \{i,j\}) = \alpha_{j,h} + \alpha(j, N_2 - h). \tag{4.3}$$

Similarly, from $\alpha^T w^\pi = \alpha_0$ and $\alpha^T w^{\pi^3} = \alpha_0$ we have

$$\alpha_{i,j} + \alpha(i, N_1 - \{i,j\}) + \alpha(h, N_2 - h) = \alpha_{j,h} + \alpha(h, N_1 - \{i,j\}) + \alpha(i, N_2 - h), \tag{4.4}$$

and from $\alpha^T w^\pi = \alpha_0$ and $\alpha^T w^{\pi^4} = \alpha_0$ we have

$$\alpha_{i,j} + \alpha(j, N_1 - \{i,j\}) + \alpha(h, N_2 - h) = \alpha_{i,h} + \alpha(h, N_1 - \{i,j\}) + \alpha(j, N_2 - h). \tag{4.5}$$

From equations (4.2) and (4.4) we obtain

$$\alpha_{i,h} + \alpha(h, N_2 - h) = \alpha_{j,h} + \alpha(h, N_1 - \{i,j\}). \tag{4.6}$$

From equations (4.3) and (4.5) we obtain

$$\alpha_{j,h} + \alpha(h, N_2 - h) = \alpha_{i,h} + \alpha(h, N_1 - \{i,j\}). \tag{4.7}$$

Equations (4.6) and (4.7) imply $\alpha_{i,h} = \alpha_{j,h}$. □

**Lemma 4.4.3.** *Let $\alpha^T w = \alpha_0$ ($\alpha \neq 0$) be a hyperplane in $\mathbb{R}^{\binom{n}{2}}$. Suppose that $\mathcal{P}^{lu} \subset \{w \in \mathbb{R}^{\binom{n}{2}} | \alpha^T w = \alpha_0\}$. If there exists an integer $k \geq 2$ such that $kF_L < n < kF_U$, then $\alpha_{e'} = \alpha_{e''}$ for all $e', e'' \in E$.*

*Proof.* The inequalities $kF_L < n < kF_U$ imply that there exists an sc-partition $\pi = (N_1, \ldots, N_k)$ such that $|N_1| > F_L$ and $|N_2| < F_U$. Since such an sc-partition would be feasible for all different permutations of $V$, setting $P = \mathcal{P}^{lu}$ in Lemma 4.4.2 yields the result. □

The following two theorems determine the dimension of $\mathcal{P}^{lu}$. The first one proves the dimension when $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil$, i.e., when all sc-partitions in $\mathcal{P}^{lu}$ have the same number of subcliques. The second one proves the dimension when $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lceil \frac{n}{F_U} \right\rceil$, namely, when the number of subcliques in the sc-partitions can take on several values.

**Theorem 4.4.1.** *Suppose $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil \geq 2$.*

*(i) If $(n \bmod F_L) = 0$ or $(n \bmod F_U) = 0$, then $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$.*

*(ii) Suppose $F_U - F_L = 1$. If $(n \bmod F_L) \neq 0$ and $(n \bmod F_U) \neq 0$, then $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$.*

*(iii) Suppose $F_U - F_L \geq 2$.*

*(a) If $(n \bmod F_L) = 1$ or $(n \bmod F_U) = F_U - 1$, then $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$.*

*(b) If $(n \bmod F_L) > 1$ and $0 < (n \bmod F_U) < F_U - 1$, then $dim(\mathcal{P}^{lu}) = \binom{n}{2}$ (i.e., it is full-dimensional).*

*Proof.* All sc-partitions have $k = \left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil$ subcliques. Note that when $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 1$, $\mathcal{P}^{lu}$ reduces to a single point, in which case the dimension is equal to 0.

(i) If $(n \bmod F_L) = 0$ or $(n \bmod F_U) = 0$, then in any sc-partition all subcliques have size $F_L$ or $F_U$, respectively. This means, $\mathcal{P}^{lu}$ reduces to the $k$-way equipartition polytope whose dimension is equal to $\binom{n}{2} - n$ by [22].

(ii) We have $kF_L < n < kF_U$. By Lemma 4.4.3, the only hyperplane that contains $\mathcal{P}^{lu}$ is

$$w(E) = (n \bmod F_L)\binom{F_U}{2} + [k - (n \bmod F_L)]\binom{F_L}{2}.$$

(iii) (a) When $(n \bmod F_L) = 1$, again we have $kF_L < n < kF_U$ and hence by Lemma 4.4.3, the only hyperplane that contains $\mathcal{P}^{lu}$ is

$$w(E) = \binom{F_L + 1}{2} + (k-1)\binom{F_L}{2}.$$

A symmetric reasoning can be applied when $(n \bmod F_U) = F_U - 1$.

(b) Arguing similarly, we infer that the only hyperplane that contains $\mathcal{P}^{lu}$ is in the form $w(E) = C$ where $C$ is a constant. But, when $(n \bmod F_L) > 1$ and $0 < (n \bmod F_U) < F_U - 1$, there always exist sc-partitions $\pi_1$ and $\pi_2$ with $w^{\pi_1}(E) \neq w^{\pi_2}(E)$. For example, suppose $(n \bmod F_L) = 2$. Consider an sc-partition $\pi_1 = (N_1, \ldots, N_k)$ where $|N_1| = |N_2| = F_L + 1$ and $|N_i| = F_L$ for $i = 3, \ldots, k$. We now construct another sc-partition $\pi_2$ by moving one node from $N_2$ to $N_1$, i.e., $\pi_2 = (\tilde{N}_1, \ldots, \tilde{N}_k)$ where $|\tilde{N}_1| = F_L + 2$, $|\tilde{N}_i| = F_L$ for $i = 2, \ldots, k$. Clearly, $w^{\pi_2}(E) = w^{\pi_1}(E) + 1$.

□

**Theorem 4.4.2.** *Suppose that* $\left\lfloor \frac{n}{F_L} \right\rfloor \geq \left\lceil \frac{n}{F_U} \right\rceil + 1$.

*(i) Suppose that* $\left\lceil \frac{n}{F_U} \right\rceil = 1$ *and* $\left\lfloor \frac{n}{F_L} \right\rfloor = 2$.

*(a) If* $(n \bmod F_L) = 0$, *then* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$.

*(b) If* $(n \bmod F_L) \neq 0$, *then* $dim(\mathcal{P}^{lu}) = \binom{n}{2}$ *(i.e., it is full-dimensional).*

*(ii) Suppose that* $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$ *and* $\left\lceil \frac{n}{F_U} \right\rceil \geq 2$.

*(a) If* $(n \bmod F_L) = 0$ *and* $(n \bmod F_U) = 0$, *then* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$.

*(b) If* $(n \bmod F_L) \neq 0$ *or* $(n \bmod F_U) \neq 0$, *then* $dim(\mathcal{P}^{lu}) = \binom{n}{2}$ *(i.e., it is full-dimensional).*

*(iii) If* $\left\lfloor \frac{n}{F_L} \right\rfloor \geq \left\lceil \frac{n}{F_U} \right\rceil + 2$, *then* $dim(\mathcal{P}^{lu}) = \binom{n}{2}$ *(i.e., it is full-dimensional).*

*Proof.* We distinguish two cases:
*Case 1:* $(n \bmod F_L) \neq 0$,
*Case 2:* $(n \bmod F_L) = 0$.

*Case 1:* Since $(n \bmod F_L) \neq 0$, we have $F_L \left\lfloor \frac{n}{F_L} \right\rfloor < n$. And, $n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil$ implies $n < F_U \left\lfloor \frac{n}{F_L} \right\rfloor$ since $\left\lfloor \frac{n}{F_L} \right\rfloor \geq \left\lceil \frac{n}{F_U} \right\rceil + 1$. Then, by Lemma 4.4.3, the only hyperplane that contains $\mathcal{P}^{lu}$ is $w(E) = C$ for some constant $C$.

Now, consider an sc-partition $\pi_1$ with $\left\lfloor \frac{n}{F_L} \right\rfloor$ subcliques. Since $\left\lfloor \frac{n}{F_L} \right\rfloor \geq \left\lceil \frac{n}{F_U} \right\rceil + 1$, it is possible to construct another sc-partition $\pi_2$ with $\left\lfloor \frac{n}{F_L} \right\rfloor - 1$ subcliques by distributing the nodes of the smallest subclique of $\pi_1$ over the others. It is easy to see that $w^{\pi_1}(E) < w^{\pi_2}(E)$, because while moving nodes from the smallest subclique of $\pi_1$ to the others, we are actually increasing the number of edges contained within the subcliques. Then, $\mathcal{P}^{lu}$ is full-dimensional. This proves parts $(i-b)$, $(ii-b)$ and $(iii)$ of the theorem for *Case 1* (*Case 1* does not apply to parts $(i-a)$ and $(ii-a)$).

*Case 2:* On the one hand, since $(n \bmod F_L) = 0$, we have $\left\lfloor \frac{n}{F_L} \right\rfloor = \frac{n}{F_L}$, which implies $F_L \left( \frac{n}{F_L} - 1 \right) < n$. On the other hand, we have

$$n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil \leq F_U \left( \frac{n}{F_L} - 1 \right).$$

Then, we can apply Lemma 4.4.3 if $n < F_U \left\lceil \frac{n}{F_U} \right\rceil$ (i.e., if $(n \bmod F_U) \neq 0$) or $\left\lceil \frac{n}{F_U} \right\rceil < \frac{n}{F_L} - 1$. But, we can always create sc-partitions $\pi_1$ and $\pi_2$ like we do in the proof of *Case 1*, and existence of two such sc-partitions contradicts Lemma 4.4.3. This proves parts $(ii-b)$ and $(iii)$ of the theorem for this case. Now remains parts $(i-a)$ and $(ii-a)$. (*Case 2* does not apply to part $(i-b)$).

*Part $(i-a)$:* We have, $\mathcal{P}^{lu} = conv(A_1 \cup \{w^{\pi^*}\})$ where

$$A_1 = \{w \in \{0,1\}^{|E|} | w \text{ satisfies } (4.1a) \text{ and } w(\delta(u)) = F_L - 1 \; \forall u \in V\}$$

and $\pi^*$ is the sc-partition with only one subclique, i.e., $w_e^{\pi^*} = 1$ for all $e \in E$. The convex hull of $A_1$, $conv(A_1)$, is the polytope of a $k$-way equipartition problem whose dimension is shown to be $\binom{n}{2} - n$ by Mitchell [22]. It is easy to verify that

$$aff(A_1) = \{w \in \mathbb{R}^{|E|} | w(\delta(u)) = F_L - 1 \; \forall u \in V\}.$$

Clearly, $w^{\pi^*} \notin aff(A_1)$ and so, dimension of $\mathcal{P}^{lu}$ is equal to $\binom{n}{2} - (n-1)$.

*Part $(ii-a)$:* We have $\mathcal{P}^{lu} = conv(A_1 \cup A_2)$ where

$$A_1 = \{w \in \{0,1\}^{|E|} | w \text{ satisfies } (4.1a) \text{ and } w(\delta(u)) = F_L - 1 \; \forall u \in V\},$$

and

$$A_2 = \{w \in \{0,1\}^{|E|} | w \text{ satisfies } (4.1a) \text{ and } w(\delta(u)) = F_U - 1 \; \forall u \in V\}.$$

Both of $A_1$ and $A_2$ are the feasible sets of k-way equipartition problems. It is easy to verify that

$$aff(A_1) = \{w \in \mathbb{R}^{|E|} | w(\delta(u)) = F_L - 1 \; \forall u \in V\}$$

and

$$aff(A_2) = \{w \in \mathbb{R}^{|E|} | w(\delta(u)) = F_U - 1 \; \forall u \in V\}.$$

We know that both of $aff(A_1)$ and $aff(A_2)$ have dimension $d = \binom{n}{2} - n$. The fact that they are parallel affine sets directly implies that $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. □

Now, we can state the conditions that make $\mathcal{P}^{lu}$ equivalent to $\mathcal{P}^{equi}(n)$ or $\mathcal{P}^{k-way}(n,k)$.

$$\mathcal{P}^{lu} = \begin{cases} \mathcal{P}^{equi}(n) & \text{when } n \text{ is odd, } \dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1 \text{ and } \left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2; \\ \mathcal{P}^{equi}(n) & \text{when } n \text{ is even, } \dim(\mathcal{P}^{lu}) = \binom{n}{2} - n \text{ and } \left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2; \\ \mathcal{P}^{k-way}(n,k), & \text{when } \dim(\mathcal{P}^{lu}) = \binom{n}{2} - n. \end{cases}$$

We summarize the results of Theorems 4.4.1 and 4.4.2 in Tables 4.1 and 4.2, respectively. While Table 4.1 includes all the results related with Theorem 4.4.1, Table 4.2 only displays results corresponding to parts $(i-a)$, $(i-b)$, $(ii-a)$ and $(ii-b)$ of Theorem 4.4.2 (i.e., the parts corresponding to the case $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$).

## 4.5 More on $\mathcal{P}^{lu}$

In this section, we deepen the analysis of the polytope $\mathcal{P}^{lu}$. First, we give a characterization of the full-dimensional $\mathcal{P}^{lu}$'s. Secondly, we give expressions for the equality sets of $\mathcal{P}^{lu}$, $M(\mathcal{P}^{lu})$.

We assume in this section that $F_L \geq 2$. Moreover, note that when $\left\lceil \frac{n}{F_U} \right\rceil = 1$, $w^\pi = 1 \in \mathcal{P}^{lu}$ (i.e., not partitioning the set $V$ and packing all items in one subclique is feasible). Hence, in the rest of this chapter we avoid this trivial solution and assume that $n > F_U$.

| | $(n \bmod F_L) = 0$ | $(n \bmod F_L) = 1$ | $1 < (n \bmod F_L)$ |
|---|---|---|---|
| $(n \bmod F_U) = 0$ | $\binom{n}{2} - n$ | $\binom{n}{2} - n$ | $\binom{n}{2} - n$ |
| $(n \bmod F_U) = F_U - 1$ | | $\binom{n}{2} - 1$ | $\binom{n}{2} - 1$ |
| $0 < (n \bmod F_U) < F_U - 1$ | | | $\binom{n}{2} - 1$ or $\binom{n}{2}$ |

Table 4.1: Dimension of $\mathcal{P}^{lu}$ when $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil$ (Theorem 4.4.1). Whenever $0 < (n \bmod F_U) < F_U - 1$ and $1 < (n \bmod F_L)$, dimension depends on the value of $F_U - F_L$: if $F_U - F_L \geq 2$, $\mathcal{P}^{lu}$ is full-dimensional; and if $F_U - F_L = 1$, it has dimension $\binom{n}{2} - 1$.

| | $(n \bmod F_L) = 0$ | $(n \bmod F_L) \neq 0$ | |
|---|---|---|---|
| | $\binom{n}{2} - (n-1)$ | $\binom{n}{2}$ | When $\left\lceil \frac{n}{F_U} \right\rceil = 1$ |
| $(n \bmod F_U) = 0$ | $\binom{n}{2} - (n-1)$ | $\binom{n}{2}$ | When $\left\lceil \frac{n}{F_U} \right\rceil \geq 2$ |
| $(n \bmod F_U) \neq 0$ | $\binom{n}{2}$ | | |

Table 4.2: Dimension of $\mathcal{P}^{lu}$ when $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$ (parts $(i-a)$, $(i-b)$, $(ii-a)$ and $(ii-b)$ of Theorem 4.4.2). Note that, whenever $\left\lfloor \frac{n}{F_L} \right\rfloor \geq \left\lceil \frac{n}{F_U} \right\rceil + 2$, $\mathcal{P}^{lu}$ is full-dimensional no matter what the values of $(n \bmod F_L)$, $(n \bmod F_U)$ and $\left\lceil \frac{n}{F_U} \right\rceil$ are (part $(iii)$ of Theorem 4.4.2).

### 4.5.1 Characterization of Full-dimensional $\mathcal{P}^{lu}$

In this section, we establish five categories that full-dimensional $\mathcal{P}^{lu}$'s fall into. This characterization is going to be very helpful in proving the conditions that make several classes of valid inequalities facet-defining from Section 4.6 on. We need the following definitions to be able to prove this result.

**Definition 4.5.1.** *Whenever $F_U - F_L = 1$, an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ $(k \geq 4)$ is called 1-loose if $|\{i : |N_i| = F_L\}| \geq 2$ and $|\{i : |N_i| = F_U\}| \geq 2$.*

**Definition 4.5.2.** *Whenever $F_U - F_L \geq 2$, an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ is called 2-loose if it has at least two subcliques, say $N_i$ and $N_j$, such that $F_L < |N_i| < F_U$ and $F_L < |N_j| < F_U$.*

**Definition 4.5.3.** *Whenever an sc-partition is 1-loose or 2-loose, it is called a loose sc-partition.*

We now present a proposition that characterizes $\mathcal{P}^{lu}$'s with loose sc-partitions.

**Proposition 4.5.1.** *The polytope $\mathcal{P}^{lu}$ contains loose partitions if and only if there exists an integer $k$ for which $kF_L + 1 < n < kF_U - 1$.*

We omit the proof because it is straightforward. Propositions 4.5.2 and 4.5.3 show that existence of 1-loose or 2-loose sc-partitions in $\mathcal{P}^{lu}$ implies that $\mathcal{P}^{lu}$ is full-dimensional.

**Proposition 4.5.2.** *Suppose that $F_U - F_L = 1$ and $\mathcal{P}^{lu}$ has a feasible 1-loose sc-partition. Then, $\mathcal{P}^{lu}$ is full dimensional only if $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lceil \frac{n}{F_U} \right\rceil$.*

*Proof.* From Theorem 4.4.1, we know that $\mathcal{P}^{lu}$ is not full-dimensional if $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil$. So, suppose $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lceil \frac{n}{F_U} \right\rceil$. Existence of a feasible 1-loose sc-partition in $\mathcal{P}^{lu}$ implies that we can employ Lemma 4.4.3. Consider an arbitrary sc-partition $\pi_1$ with $\left\lfloor \frac{n}{F_L} \right\rfloor$ subcliques. We can always find another sc-partition $\pi_2$ with $\left\lfloor \frac{n}{F_L} \right\rfloor - 1$ subcliques since

$$F_L \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) < n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil \leq F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right).$$

But, then $w^{\pi_1}(E) < w^{\pi_2}(E)$. □

**Proposition 4.5.3.** *Suppose that $F_U - F_L \geq 2$. If $\mathcal{P}^{lu}$ has a feasible 2-loose sc-partition, then it is full dimensional.*

*Proof.* Let $\pi_1 = (N_1, N_2, \ldots, N_k)$ be a feasible 2-loose sc-partition in $\mathcal{P}^{lu}$. Then $\mathcal{P}^{lu}$ satisfies the conditions of Lemma 4.4.3. Without loss of generality, assume that $F_L < |N_1| \leq |N_2| < F_U$. Now, we can create another sc-partition $\pi_2$ by shifting a node from $N_1$ to $N_2$. But then, we contradict Lemma 4.4.3 since $w^{\pi_1}(E) < w^{\pi_2}(E)$. □

The converse of these propositions does not hold; for example, when $|V| = 49$, $F_L = 8$ and $F_U = 10$, the sc-partitions have either 5 or 6 subcliques. The ones with 5 subcliques have 4 subcliques with size 10 and one subclique with

size 9; the ones with 6 subcliques have 5 subcliques with size 8 and one subclique with size 9 (i.e., no feasible 2-loose sc-partitions). But, from part $(ii-b)$ of Theorem 4.4.2, the polytope corresponding to this instance is full-dimensional.

The following two theorems, one written for the case $F_U - F_L \geq 2$ and the other for the case $F_U - F_L = 1$, describe the categories that full-dimensional $\mathcal{P}^{lu}$'s fall into.

**Theorem 4.5.1.** *Suppose that $F_U - F_L \geq 2$ and $\mathcal{P}^{lu}$ is full-dimensional. Then, it complies with exactly one of the following:*

**(FD-1)** *It has feasible 2-loose sc-partitions,*

**(FD-2)** *$(n \ mod \ F_L) = 1$, $(n \ mod \ F_U) = 0$ and $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$;*

**(FD-3)** *$(n \ mod \ F_L) = 1$, $(n \ mod \ F_U) = F_U - 1$ and $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$;*

**(FD-4)** *$(n \ mod \ F_L) = 0$, $(n \ mod \ F_U) = F_U - 1$ and $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$.*

*Proof.* First, let $k = \left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil$. According to part $(iii-b)$ of Theorem 4.4.1, we must have $F_U - F_L \geq 2$ and $kF_U - 2 \geq n \geq kF_L + 2$. Then, by Proposition 4.5.1, $\mathcal{P}^{lu}$ contains 2-loose sc-partitions (i.e., **(FD-1)**).

Now, suppose $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lceil \frac{n}{F_U} \right\rceil$. We distinguish three cases: $(n \bmod F_L) > 1$, $(n \bmod F_L) = 1$ and $(n \bmod F_L) = 0$.

*Case 1 $(n \ mod \ F_L) > 1$*: In this case, obviously we have $F_L \left\lfloor \frac{n}{F_L} \right\rfloor + 2 \leq n$. On the other hand, we have

$$n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil \leq F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) = F_U \left\lfloor \frac{n}{F_L} \right\rfloor - F_U \leq F_U \left\lfloor \frac{n}{F_L} \right\rfloor - 2,$$

which by Proposition 4.5.1 implies that $\mathcal{P}^{lu}$ contains 2-loose sc-partitions (i.e., **(FD-1)**).

*Case 2 $(n \ mod \ F_L) = 1$*: We have

$$n = F_L \left\lfloor \frac{n}{F_L} \right\rfloor + 1.$$

Adding $0 > 1 - F_L$ to this equation (note that we assume $F_L \geq 2$) yields

$$n > F_L \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) + 2.$$

We also have, $n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil$.

- First, consider $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lceil \frac{n}{F_U} \right\rceil + 1$. We have

$$F_U \left\lceil \frac{n}{F_U} \right\rceil \leq \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) F_U - F_U \leq \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) F_U - 2,$$

  which proves then that $\mathcal{P}^{lu}$ contains 2-loose sc-partitions by Proposition 4.5.1.

- Secondly, consider $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$. If

$$n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil - 2 = F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - 2,$$

  then $\mathcal{P}^{lu}$ contains 2-loose sc-partitions. On the other hand, if $n = F_U \left\lceil \frac{n}{F_U} \right\rceil - 1$, $\mathcal{P}^{lu}$ complies with **(FD-3)**; and, if $n = F_U \left\lceil \frac{n}{F_U} \right\rceil$, $\mathcal{P}^{lu}$ complies with **(FD-2)**.

*Case 3 ($n$ mod $F_L$) $= 0$*: First, note that we must have $\left\lceil \frac{n}{F_U} \right\rceil \geq 2$, because otherwise, $\mathcal{P}^{lu}$ is not full-dimensional by part $(i-a)$ of Theorem 4.4.2. From ($n$ mod $F_L$) $= 0$ and the assumption that $F_L \geq 2$, we infer $F_L(\frac{n}{F_L} - 1) + 2 \leq n$.

- Consider first $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lceil \frac{n}{F_U} \right\rceil + 1$. From $n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil$, we have

$$n \leq F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - F_U < F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - 2,$$

  which proves existence of 2-loose sc-partitions in $\mathcal{P}^{lu}$.

- Now, consider $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$. If $n \leq F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - 2$, $\mathcal{P}^{lu}$ has loose sc-partitions; if, $n = F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - 1$, $\mathcal{P}^{lu}$ complies with **(FD-4)**. Note that, $\mathcal{P}^{lu}$ is not full-dimensional when $n = F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right)$ (i.e., ($n$ mod $F_U$) $= 0$) by part $(ii-a)$ of Theorem 4.4.2.

□

**Theorem 4.5.2.** *Suppose $F_U - F_L = 1$ and $\mathcal{P}^{lu}$ is full-dimensional. Then it complies with exactly one of* **(FD-2)**, **(FD-3)**, **(FD-4)**, *and,* **(FD-5)***: it has feasible 1-loose sc-partitions and $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lceil \frac{n}{F_U} \right\rceil$.*

We skip the proof because it is very similar to that of Theorem 4.5.1.

In the rest of this chapter, when we say that $\mathcal{P}^{lu}$ complies with **(FD-1)** we implicitly assume that $F_U - F_L \geq 2$. Similarly, when we say that $\mathcal{P}^{lu}$ complies with **(FD-5)**, we implicitly assume that $F_U - F_L = 1$.

Remark: In Theorems 4.5.1 and 4.5.2, we can still show full-dimensionality for **(FD-1)**, **(FD-2)**, **(FD-3)** and **(FD-5)** when $\left\lceil \frac{n}{F_U} \right\rceil = 1$. On the other hand, **(FD-4)** implies full-dimensionality only if $\left\lceil \frac{n}{F_U} \right\rceil \geq 2$.

### 4.5.2 Equality Sets of $\mathcal{P}^{lu}$

In this section we express the equality sets of $\mathcal{P}^{lu}$ when it is not full-dimensional. These characterizations are going to be helpful in proving the conditions under which several valid inequalities are facet defining in Section 4.6.

When $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, the equality set of $\mathcal{P}^{lu}$ consists of a single hyperplane $w(E) = C$ where $C$ is an appropriate constant. When $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$ the equality set consists of the hyperplanes

$$w(\delta(u)) = F - 1 \ \ \forall u \in V,$$

where $F \in \{F_L, F_U\}$. These equality sets can trivially be inferred from the proofs of Theorems 4.4.1 and 4.4.2. However, when $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$, the equality set is not trivial, and is shown in the following proposition.

**Proposition 4.5.4.** *Suppose $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. Then, the equality set of $\mathcal{P}^{lu}$ consists of $n-1$ linearly independent hyperplanes in the form $\sum_{(i,j)\in E} (\lambda_i + \lambda_j) w_{i,j} = 0$ where $\lambda_i \in \mathbb{R}$ for $i \in V$ and $\sum_{i \in V} \lambda_i = 0$.*

*Proof.* If $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$, then feasible solutions in $\mathcal{P}^{lu}$ lie in either of the following parallel affine spaces (by the proof of Theorem 4.4.2):

$$A_1 = \{w \in \mathbb{R}^{\binom{n}{2}} | w(\delta(u)) = F_L - 1 \quad \forall u \in V\};$$
$$A_2 = \{w \in \mathbb{R}^{\binom{n}{2}} | w(\delta(u)) = F_U - 1 \quad \forall u \in V\}.$$

We now assign a coefficient $\lambda_u$ ($u \in V$) to each of the hyperplanes defining $A_1$ and take linear combinations of these hyperplanes to obtain

$$\sum_{u \in V} \lambda_u w(\delta(u)) = \sum_{u \in V} \lambda_u (F_L - 1). \tag{4.8}$$

Now, we search for $\vec{\lambda}$'s for which $A_2$ is contained in the affine space defined by (4.8). All points in $A_2$ satisfy $w(\delta(u)) = F_U - 1$ for all $u \in V$. Inserting this into (4.8) yields

$$\sum_{u \in V} \lambda_u (F_U - 1) = \sum_{u \in V} \lambda_u (F_L - 1).$$

This equality implies $\sum_{u \in V} \lambda_u = 0$. □

## 4.6 Trivial Facets for $\mathcal{P}^{lu}$

Having set the grounds, we now pass on to the facetness proofs. In the rest of this chapter, we assume that $F_L \geq 3$. Recall that, we also assume $n > F_U$ (i.e. sc-partitions in $\mathcal{P}^{lu}$ have at least two subcliques). We start with several technical lemmas.

**Lemma 4.6.1.** *Suppose that $F_U - F_L \geq 2$. Suppose also that $g^T w \leq h$ is a valid inequality for $\mathcal{P}^{lu}$. Let $P = \{w \in \mathcal{P}^{lu} | g^T w = h\}$. Suppose further that there exists $k \in \mathbb{Z}_+$ such that $kF_L + 1 < n < kF_U - 1$ (i.e., there exist 2-loose sc-partitions in $\mathcal{P}^{lu}$). Pick such a 2-loose sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^\pi \in P$, $i \in N_1$, $j \in N_2$, $F_L < |N_1| < F_U$ and $F_L < |N_2| < F_U$ for $i, j \in V$. Then $g_{i,j} = 0$ if the following hold:*

- *$w^{\pi^*} \in P$ where $\pi^* = (N_1^*, \ldots, N_k^*)$, $N_1^* = (N_1 - i) \cup j$, $N_2^* = (N_2 - j) \cup i$ and $N_l^* = N_l$ for all $l \in \{3, \ldots, k\}$.*
- *$w^{\widehat{\pi}} \in P$ where $\widehat{\pi} = (\widehat{N}_1, \ldots, \widehat{N}_k)$, $\widehat{N}_1 = N_1 - i$, $\widehat{N}_2 = N_2 \cup i$ and $\widehat{N}_l = N_l$ for all $l \in \{3, \ldots, k\}$.*
- *$w^{\tilde{\pi}} \in P$ where $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$, $\tilde{N}_1 = N_1 \cup j$, $\tilde{N}_2 = N_2 - j$, $\tilde{N}_l = N_l$ for all $l \in \{3, \ldots, k\}$.*

*Proof.* From the equalities $g^T w^\pi = h$ and $g^T w^{\pi^*} = h$, we obtain

$$g(i, N_1 - i) + g(j, N_2 - j) = g(i, N_2 - j) + g(j, N_1 - i). \tag{4.9}$$

From the equalities $g^T w^\pi = h$ and $g^T w^{\widehat{\pi}} = h$, we obtain

$$g(i, N_1 - i) = g(i, N_2 - j) + g_{i,j}. \tag{4.10}$$

From the equalities $g^T w^\pi = h$ and $g^T w^{\tilde{\pi}} = h$, we get

$$g(j, N_2 - j) = g(j, N_1 - i) + g_{i,j}. \tag{4.11}$$

Now summing the equalities (4.10) and (4.11) and comparing the sum with (4.9), we get $g_{i,j} = 0$. □

**Lemma 4.6.2.** *Suppose that $g^T w \leq h$ is a valid inequality for $\mathcal{P}^{lu}$. Let $P = \{w \in \mathcal{P}^{lu} | g^T w = h\}$. Let $u$, $v$ and $x$ be three nodes in $V$.*

- *Let $\pi = (N_1, N_2, \ldots, N_k)$ be an sc-partition such that $\{u, v\} \subset N_1$, $x \in N_2$ and $|N_1| = |N_2| + 1$;*
- *let $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$ where $\tilde{N}_1 = (N_1 - v) \cup x$, $\tilde{N}_2 = (N_2 - x) \cup v$, $\tilde{N}_l = N_l$ for $l \in \{3, \ldots, k\}$;*
- *let $\pi^* = (N_1^*, \ldots, N_k^*)$ where $N_1^* = (N_2 - x) \cup \{u, v\}$, $N_2^* = (N_1 - \{u, v\}) \cup x$, $N_l^* = N_l$ for $l \in \{3, \ldots, k\}$;*
- *let $\widehat{\pi} = (\widehat{N}_1, \ldots, \widehat{N}_k)$ where $\widehat{N}_1 = (N_1^* - v) \cup x$, $\widehat{N}_2 = (N_2^* - x) \cup v$, $N_l^* = N_l$ for $l = \{3, \ldots, k\}$.*

*If $\{w^\pi, w^{\tilde{\pi}}, w^{\pi^*}, w^{\widehat{\pi}}\} \subset P$ then $g_{u,v} = g_{u,x}$.*

*Proof.* Comparing the equalities, $g^T w^\pi = h$ and $g^T w^{\tilde{\pi}} = h$, we obtain

$$g_{u,v} + g(v, N_1 - \{u, v\}) + g(x, N_2 - x) = g_{u,x} + g(x, N_1 - \{u, v\}) + g(v, N_2 - x),$$

which is equivalent to $g_{u,v} - g_{u,x} = A$ where

$$A = [g(x, N_1 - \{u, v\}) - g(x, N_2 - x)] - [g(v, N_1 - \{u, v\}) - g(v, N_2 - x)].$$

Comparing the equalities $g^T w^{\pi^*} = h$ and $g^T w^{\widehat{\pi}} = h$, we obtain

$$g_{u,v} + g(v, N_1^* - \{u, v\}) + g(x, N_2^* - x) = g_{u,x} + g(x, N_1^* - \{u, v\}) + g(v, N_2^* - x),$$

which is equivalent to $g_{u,v} - g_{u,x} = B$ where

$$B = [g(x, N_1^* - \{u, v\}) - g(x, N_2^* - x)] - [g(v, N_1^* - \{u, v\}) - g(v, N_2^* - x)].$$

Since $N_1^* - \{u, v\} = N_2 - x$ and $N_2^* - x = N_1 - \{u, v\}$, we have $A = -B$. Hence, $g_{u,x} = g_{u,v}$. □

**Lemma 4.6.3.** *Suppose that $g^T w \leq h$ is a valid inequality for $\mathcal{P}^{lu}$. Let $P = \{w \in \mathcal{P}^{lu} | g^T w = h\}$. Let $u$, $v$, $x$ and $y$ be four nodes in $V$.*

- *Let $\pi = (N_1, N_2, \ldots, N_k)$ be an sc-partition such that $\{u, x\} \subset N_1$, $\{v, y\} \subset N_2$;*
- *let $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$ where $\tilde{N}_1 = (N_1 - u) \cup v$, $\tilde{N}_2 = (N_2 - v) \cup u$, $\tilde{N}_l = N_l$ for $l \in \{3, \ldots, k\}$;*
- *let $\pi^* = (N_1^*, \ldots, N_k^*)$ where $N_1^* = (N_1 - x) \cup y$, $N_2^* = (N_2 - y) \cup x$, $N_l^* = N_l$ for $l \in \{3, \ldots, k\}$;*
- *let $\widehat{\pi} = (\widehat{N}_1, \ldots, \widehat{N}_k)$ where $\widehat{N}_1 = (N_1^* - u) \cup v$, $\widehat{N}_2 = (N_2^* - v) \cup u$, $N_l^* = N_l$ for $l = \{3, \ldots, k\}$.*

*If $\{w^\pi, w^{\tilde{\pi}}, w^{\pi^*}, w^{\widehat{\pi}}\} \subset P$ then $g_{u,x} + g_{v,y} = g_{u,y} + g_{v,x}$.*

*Proof.* Comparing the equalities, $g^T w^\pi = h$ and $g^T w^{\tilde{\pi}} = h$ we obtain

$$g_{u,x}+g(u, N_1-\{u,x\})+g_{v,y}+g(v, N_2-\{v,y\}) = g_{u,y}+g(u, N_2-\{v,y\})+g_{v,x}+g(v, N_1-\{u,x\}),$$

which is equivalent to $(g_{u,x} + g_{v,y}) - (g_{u,y} + g_{v,x}) = A$ where

$$A = [g(u, N_2 - \{v, y\}) - g(u, N_1 - \{u, x\})] - [g(v, N_2 - \{v, y\}) - g(v, N_1 - \{u, x\})] .$$

Comparing the equalities $g^T w^{\pi^*} = h$ and $g^T w^{\widehat{\pi}} = h$, we obtain

$$g_{u,y}+g(u, N_1^*-\{u,y\})+g_{v,x}+g(v, N_2^*-\{v,x\}) = g_{u,x}+g(u, N_2^*-\{v,x\})+g_{v,y}+g(v, N_1^*-\{u,y\}),$$

which is equivalent to $(g_{u,x} + g_{v,y}) - (g_{u,y} + g_{v,x}) = B$ where

$$B = [g(u, N_1^* - \{u, y\}) - g(u, N_2^* - \{v, x\})] - [g(v, N_1^* - \{u, y\}) - g(v, N_2^* - \{v, x\})] .$$

Since $N_1^* - \{u, y\} = N_1 - \{u, x\}$ and $N_2^* - \{v, x\} = N_2 - \{v, y\}$, we have $A = -B$. Hence, $g_{u,x} + g_{v,y} = g_{u,y} + g_{v,x}$. □

**Lemma 4.6.4.** *Suppose that $g^T w \leq h$ is a valid inequality for $\mathcal{P}^{lu}$. Let $p \in V$ and $Q \subseteq V$. Suppose that $g(p, \tilde{Q}_1) + \alpha = g(p, \tilde{Q}_2)$ for any disjoint $\tilde{Q}_1, \tilde{Q}_2 \subset Q$ with $|\tilde{Q}_1| = \sigma_1$, $|\tilde{Q}_2| = \sigma_2$ where $\alpha \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{Z}$, $\sigma_1 > 0$ and $\sigma_2 \geq 0$. Then, $g_{p,q_1} = g_{p,q_2}$ for any $q_1, q_2 \in Q$.*

*Proof.* Pick arbitrarily $q_1 \in \tilde{Q}_1$ and $q_2 \in R$ where

$$R = \begin{cases} \tilde{Q}_2, & \text{if } \tilde{Q}_2 \neq \emptyset, \\ Q - \tilde{Q}_1, & \text{if } \tilde{Q}_2 = \emptyset. \end{cases}$$

Consider the sets $\tilde{Q}_1^* = (\tilde{Q}_1 - q_1) \cup q_2$, $R^* = (R - q_2) \cup q_1$. Comparing $g(p, \tilde{Q}_1) + \alpha = g(p, \tilde{Q}_2)$ and $g(p, \tilde{Q}_1^*) + \alpha = g(p, \tilde{R}^*)$ yields the result. □

Note that, when $\tilde{Q}_2 = \emptyset$, it is crucial to have in this lemma that $\tilde{Q}_1 \subsetneq Q$.

In the sequel, we prove the conditions under which several classes of valid inequalities become facet defining for $\mathcal{P}^{lu}$. Each class is investigated in a separate section. Commonly, for each of the valid inequalities, we assume that the face defined by the valid inequality under consideration, say $a^T w \leq b$, is contained in the face defined by some valid inequality $g^T w \leq h$. In our proofs, with reference to Theorem 3.6 in Section I.4.3 of [24], we basically show that $g^T w = h$ is equal to a linear combination of $a^T w = b$ and the hyperplanes in $M(\mathcal{P}^{lu})$.

### 4.6.1 The Non-Negativity Constraints

In this section, we provide the conditions that make the non-negativity constraints facet defining. We denote the face corresponding to a non-negativity constraint $w_e \geq 0$ by $P_e$, i.e., $P_e = \{w \in \mathcal{P}^{lu} | w_e = 0\}$. We assume that there exists a valid inequality $g^T w \leq h$ such that

$$P_e \subseteq \{w \in \mathcal{P}^{lu} | g^T w = h\}.$$

In Theorems 4.6.1, 4.6.2, 4.6.3 and 4.6.4 we prove the conditions under which $P_e$'s are facets, i.e., $g^T w = h$ can be written as a linear combination of $w_e = 0$ and the equalities in $M(\mathcal{P}^{lu})$, if any.

Before moving on to these theorems, we present the following useful proposition.

**Proposition 4.6.1.** *If $\left\lfloor \frac{n}{F_L} \right\rfloor = 2$, the non-negativity constraints $w_e \geq 0$ are not facet defining for $\mathcal{P}^{lu}$.*

*Proof.* Recall that we assume $n > F_U$ (i.e., $\left\lceil \frac{n}{F_U} \right\rceil \geq 2$). So, when $\left\lfloor \frac{n}{F_L} \right\rfloor = 2$, we can conclude that all sc-partitions have 2 subcliques. Then, for each edge $e = \{u, v\}$ we have

$$P_e \subseteq \left\{ w \in \mathcal{P}^{lu} | w_{x,u} + w_{x,v} - w_{u,v} = 1 \ \ \forall x \in V - \{u, v\} \right\}.$$

□

**Theorem 4.6.1.** *Suppose $\mathcal{P}^{lu}$ is full-dimensional. Then $w_e \geq 0$ is facet-defining for $\mathcal{P}^{lu}$ if and only if one of the following hold:*

- $n \geq 3F_L + 1$, *or,*
- $n = 3F_L \leq 2F_U - 2$.

*Proof.* Let $e = \{u, v\}$ for $u, v \in V$. We present the proof separately for **(FD-1)**-**(FD-5)**. Note that, by Proposition 4.6.1 we have $n \geq 3F_L$.

**(FD-1)** : Note that, in this case we assume $F_U - F_L \geq 2$. We analyze two cases separately:

(a) $\mathcal{P}^{lu}$ has 2-loose sc-partitions with at least 3 sub-cliques: Note that, in this case, $n$ is at least $3F_L + 2$. Pick an arbitrary edge $\{i, j\} \in E - \{u, v\}$. Pick a 2-loose sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with at least three subcliques and where $u \in N_1$, $i \in N_1$, $j \in N_2$, $v \in N_3$ ($i$ is not necessarily different from $u$), $F_L < |N_1| < F_U$, $F_L < |N_2| < F_U$. By Lemma 4.6.1 and symmetry, we conclude that $g_{i,j} = 0$ for any $\{i, j\} \in E - \{u, v\}$. Thus, $w_{u,v} \geq 0$ defines a facet of $\mathcal{P}^{lu}$.

(b) All 2-loose sc-partitions in $\mathcal{P}^{lu}$ have at most 2 sub-cliques: Pick two arbitrary nodes $x, y \in V - \{u, v\}$. Pick a 2-loose sc-partition, say $\pi^1 = (N_1^1, N_2^1)$ such that $\{u, x\} \subset N_1^1$ and $\{v, y\} \subset N_2^1$. By Lemma 4.6.1, $g_{x,y} = 0$. By arbitrariness of $x$ and $y$, $g_e = 0$ for all $e \in E(V - \{u, v\})$.

Pick another sc-partition, say $\pi^2 = (N_1^2, N_2^2, \ldots, N_k^2)$ ($k \geq 3$), such that $u \in N_1^2$, $\{v, x\} \subset N_2^2$ and $y \in N_3^2$. Switching $x$ and $y$ in $\pi^2$, we get $g_{v,x} = g_{v,y}$ for all $x, y \in V - \{u, v\}$, which can be rewritten as $g_{v,z} = \alpha \in \mathbb{R}$ for all $z \in V - \{u, v\}$. Similarly, we can also infer $g_{u,z} = \beta \in \mathbb{R}$ for all $z \in V - \{u, v\}$. Now, in $\pi^1$, shifting $x$ from $N_1^1$ to $N_2^1$ shows that $\alpha = \beta$.

Since $\{w^{\pi^1}, w^{\pi^2}\} \subset P_e$, comparing $g^T w^{\pi^1} = h$ and $g^T w^{\pi^2} = h$ yields $\alpha = \beta = 0$, due to the fact that $w^{\pi^2}(\{u, v\}, V - \{u, v\}) < w^{\pi^1}(\{u, v\}, V - \{u, v\}) = n - 2$.

Note that, this proof for **(FD-1)** is applicable even if $3F_L \leq n \leq 3F_L + 1$ and $n \leq 2F_U - 2$.

**(FD-2)** : Proposition 4.6.1 and **(FD-2)** imply that $n \geq 3F_L + 1$.

Pick two arbitrary nodes $x, y \in V - \{u, v\}$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with $k = \left\lfloor \frac{n}{F_L} \right\rfloor \geq 3$, $\{v, x\} \subset N_1$, $y \in N_2$, $u \in N_3$, $|N_1| = F_L + 1$ and $|N_i| = F_L$ for $i = 2, \ldots, k$. Using Lemma 4.6.2, we infer $g_{v,x} = g_{x,y}$. By symmetry, we infer this result for $u$ as well and generalize this result as $g_{e'} = \alpha$ for all $e' \in E - \{u, v\}$.

Now, consider sc-partitions $\pi^1 = (N_1^1, \ldots, N_{k_1}^1)$ with $k_1 = \left\lfloor \frac{n}{F_L} \right\rfloor$, $u \in N_1^1$, $v \in N_2^1$; and, $\pi^2 = (N_1^2, \ldots, N_{k_2}^2)$ with $k_2 = \left\lceil \frac{n}{F_U} \right\rceil = k_1 - 1$, $u \in N_1^2$, $v \in N_2^2$. Clearly, $\{w^{\pi^1}, w^{\pi^2}\} \subset P_e$. Comparing $g^T w^{\pi^1} = h$ and $g^T w^{\pi^2} = h$, we infer that $\alpha = 0$.

**(FD-3)** : The proof for this case is the same as the proof of case $(\mathbf{FD-2})$.

**(FD-4)** : Pick three nodes $x, y, z \in V - \{u, v\}$ arbitrarily. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with $k \geq 2$ such that $\{v, z\} \subset N_1$, $\{u, x, y\} \subset N_2$, $|N_1| = F_U - 1$, $|N_i| = F_U$ for all $i = 2, \ldots, k$. Applying Lemma 4.6.2, we obtain $g_{x,y} = g_{y,z}$, which by arbitrariness of $x$, $y$ and $z$ implies $g_{e'} = \beta$ for all $e' \in E(V - \{u, v\})$ where $\beta \in \mathbb{R}$. Now, shifting $x$ from $N_2$ to $N_1$ in $\pi$ we get $g_{u,x} = g_{v,x}$. Since $x$ is an arbitrary node, we infer $g_{u,x} = g_{v,x} = \alpha_x \in \mathbb{R}$ for all $x \in V - \{u, v\}$. Now, pick another sc-partition $\pi^* = (N_1^*, \ldots, N_{k+1}^*)$ with $|N_i^*| = F_L$ for $i = 1, \ldots, k+1$, $u \in N_1^*$, $\{x, v\} \subset N_2^*$ and $y \in N_3^*$. Now, switching $x$ and $y$ in $\pi^*$, we get $\alpha_x = \alpha_y$ which implies $\alpha_x = \alpha_y = \alpha$ for all $x, y \in V - \{u, v\}$.

- Now, assume that $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 4$ (i.e., $n \geq 4F_L$). Pick an sc-partition $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k)$ where $k \geq 3$, $|\tilde{N}_1| = F_U - 1$ and $|\tilde{N}_i| = F_U$ for $i = 2, \ldots, k$. Suppose $u \in \tilde{N}_1$, $\{x, y\} \subset \tilde{N}_2$ and $v \in \tilde{N}_3$. Shifting $y$ from $\tilde{N}_2$ to $\tilde{N}_1$ we get $\beta(F_U - 1) = \beta(F_U - 2) + \alpha$ which implies $\alpha = \beta$. Hence, we infer that, if $\alpha = \beta \neq 0$, $P_e$ is contained in an hyperplane $g_{u,v} w_{u,v} + \alpha w(E - \{u, v\}) = h$ for some constants $h \in \mathbb{R}$ and $g_{u,v} \in \mathbb{R}$. But, this contradicts the fact that $\{\pi^*, \tilde{\pi}\} \subset P_e$ do not both lie on this hyperplane since $w^{\pi^*}(E - \{u, v\}) < w^{\tilde{\pi}}(E - \{u, v\})$. Then, $\alpha = \beta = 0$.
- Finally, assume that $\left\lfloor \frac{n}{F_L} \right\rfloor = 3$ (i.e., $n = 3F_L = 2F_U - 1$). In this

case, $P_e$ is contained in the hyperplane

$$g_{u,v}w_{u,v}+\alpha w(E(\{u,v\},V-\{u,v\}))+\beta w(E(V-\{u,v\}))=h \quad (4.12)$$

for some constants $g_{u,v} \in \mathbb{R}$ and $h \in \mathbb{R}$. Now, consider two sc-partitions $\pi^1 = (N_1^1, N_2^1, N_3^1)$ and $\pi^2 = (N_1^2, N_2^2)$ implied by **(FD-4)**, where $|N_i^1| = F_L$ for $i = 1, 2, 3$, $|N_1^2| = F_U - 1$, $|N_2^2| = F_U$, $u \in N_1^1 \cap N_1^2$, $v \in N_2^1 \cap N_2^2$. For $\pi^1$, the hyperplane equation (4.12) turns into

$$2(F_L-1)\alpha + \left[2\binom{F_L-1}{2} + \binom{F_L}{2}\right]\beta = h;$$

and, for $\pi^2$ it becomes

$$(2F_U-3)\alpha + \left[\binom{F_U-1}{2} + \binom{F_U-2}{2}\right]\beta = h.$$

$P_e$ is contained in the hyperplane represented by (4.12) obtained by setting

$$\begin{aligned}\alpha &= 2\binom{F_L-1}{2} + \binom{F_L}{2} - \binom{F_U-1}{2} - \binom{F_U-2}{2} \\ &= (F_L-1)\left(\frac{3}{2}F_L-2\right) - (F_U-2)^2,\end{aligned}$$

and $\beta = 2(F_U - F_L) - 1$. Hence, when $\mathcal{P}^{lu}$ complies with **(FD-4)** and $n = 3F_L = 2F_U - 1$, $w_{u,v} \geq 0$ is not facet defining. Indeed, when $n = 3F_L = 2F_U - 1$ it is possible to show that

$$\begin{aligned}\alpha w_{u,v} + \alpha w(E(\{u,v\},V-\{u,v\})) + \beta w(E(V-\{u,v\})) &\geq \\ (F_L-1)\left(\frac{3}{2}F_L-2\right)(2F_U-3) &- 2(F_L-1)(F_U-2)^2\end{aligned}$$

is facet defining if $\beta - \alpha \geq 0$; and,

$$\begin{aligned}\alpha w_{u,v} + \alpha w(E(\{u,v\},V-\{u,v\})) + \beta w(E(V-\{u,v\})) &\leq \\ (F_L-1)\left(\frac{3}{2}F_L-2\right)(2F_U-3) &- 2(F_L-1)(F_U-2)^2\end{aligned}$$

is facet defining if $\beta - \alpha < 0$.

**(FD-5)** : Note that, in this case we assume $F_U - F_L = 1$. The proof for this case is very similar to the proof of case **(FD-2)**.

□

**Theorem 4.6.2.** *Suppose that* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. *Then, the nonnegativity constraints* $w_e \geq 0$ *are facet defining for* $\mathcal{P}^{lu}$ *if and only if* $n \geq 3F_L + 1$.

*Proof.* By Proposition 4.6.1, we consider $n \geq 3F_L$. But, when $n = 3F_L$, $\dim(\mathcal{P}^{lu})$ is never equal to $\binom{n}{2} - 1$. Hence, we have $n \geq 3F_L + 1$. We give the proof for only the case $(n \bmod F_L) = 1$. The proof for $(n \bmod F_U) = F_U - 1$ is very similar.

Let $e = \{u, v\}$ for $u, v \in V$. Pick two arbitrary nodes $x, y \in V - \{u, v\}$. Using $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we can set $g_{x,y} = 0$. Now, pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $|N_1| = F_L + 1$, $|N_i| = F_L$ for $i = 2, \ldots, k$, $\{x, v\} \subset N_1$, $y \in N_2$ and $u \in N_3$. Using Lemma 4.6.2 we obtain $g_{v,x} = g_{x,y}$. By symmetry, we can also infer $g_{u,x} = g_{x,y}$, which generalizes into $g_{e'} = 0$ for all $e' \in E - \{u, v\}$ for we have already set $g_{x,y} = 0$. □

**Theorem 4.6.3.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. *Then the non-negativity constraints* $w_e \geq 0$ *are facet-defining for* $\mathcal{P}^{lu}$ *if and only if* $n \geq 3F_L$.

*Proof.* When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$, we have $n \geq 3F_L$ by Theorem 4.4.2 and the assumption that $n > F_U$.

Let $e = \{u, v\}$ and without loss of generality, assume $u = n$, $v = n - 1$. Let $k = \frac{n}{F_U} = \frac{n}{F_L} - 1$.We know from Proposition 4.5.4 that $M(\mathcal{P}^{lu})$ consists of linearly independent hyperplanes in the form $\sum_{(u,v)\in E}(\lambda_u + \lambda_v)w_{u,v} = 0$ where $\boldsymbol{\lambda} \in \mathbb{R}^n$ and $\sum_{u\in V} \lambda_u = 0$. We choose $n-1$ such hyperplanes $H_1, H_2, \ldots, H_{n-1}$ which are determined by the $\boldsymbol{\lambda}$'s displayed in the rows of the following matrix:

$$\begin{array}{r} \\ H_1 \longrightarrow \\ H_2 \longrightarrow \\ H_3 \longrightarrow \\ \vdots \\ H_{n-1} \longrightarrow \end{array} \begin{array}{c} \begin{array}{cccccc} \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_{n-1} & \lambda_n \end{array} \\ \left[\begin{array}{cccccc} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{array}\right] \end{array} .$$

One can verify that $H_1, H_2, \ldots, H_{n-1}$ are linearly independent. Now, consider the set of edges $S = \{\{1, n\}, \{2, n\}, \ldots, \{n-2, n\}, \{1, n-1\}\}$. The following matrix displays the coefficients of the edges belonging to $S$ in the equations

representing $H_1$ to $H_{n-1}$:

$$\begin{array}{r} \\ H_1 \longrightarrow \\ H_2 \longrightarrow \\ H_3 \longrightarrow \\ H_4 \longrightarrow \\ \vdots \\ H_{n-3} \longrightarrow \\ H_{n-2} \longrightarrow \\ H_{n-1} \longrightarrow \end{array} \begin{array}{c} \begin{array}{cccccccc} \{1,n\} & \{2,n\} & \{3,n\} & \{4,n\} & \dots & \{n-2,n\} & \{1,n-1\} \end{array} \\ \left[\begin{array}{cccccccc} 1 & -1 & 0 & 0 & & 0 & 1 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & -1 & -1 & -1 & \cdots & -1 & 1 \end{array}\right] \end{array}.$$

This matrix is non-singular. Hence, we can set $g_e = 0$ for all $e \in S$ (again, by Theorem 3.6 in Section I.4.3 of [24]). Now, pick four arbitrary nodes $x, y, z, t \in V - \{n-1, n\}$ and an sc-partition $\pi = (N_1, N_2, \dots, N_{k+1})$ where $|N_i| = F_L$ for all $i = 1, 2, \dots, k+1$, $\{n, x\} \subset N_1$, $\{n-1, y\} \subset N_2$ and $\{z, t\} \subset N_3$. Using Lemma 4.6.3 we infer $g_{x,n} + g_{y,n-1} = g_{y,n} + g_{x,n-1}$. Since $g_{x,n} = g_{y,n} = 0$, we have $g_{x,n-1} = g_{y,n-1}$. But we have already set $g_{1,n-1} = 0$, hence, we obtain $g_{q,n-1} = 0$ for all $q \in V - \{n-1, n\}$. Applying Lemma 4.6.3 once more, we can obtain $g_{n-1,y} + g_{z,t} = g_{n-1,z} + g_{y,t}$, which implies $g_{e'} = \alpha \in \mathbb{R}$ for all $e' \in E(V - \{n-1, n\})$.

Finally, construct another sc-partition by distributing the nodes of $N_{k+1}$ over the others and call this new sc-partition $\pi^*$. Clearly, $w^{\pi^*} \in P_e$. Comparing $g^T w^\pi = h$ and $g^T w^{\pi^*} = h$ yields $\alpha = 0$. □

**Theorem 4.6.4.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. *Then the non-negativity constraints* $w_e \geq 0$ *are facet-defining for* $\mathcal{P}^{lu}$ *if and only if* $n \geq 3F_L$.

*Proof.* When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$, $M(\mathcal{P}^{lu})$ consists of the hyperplanes $H_1, H_2, \dots, H_n$, where $H_u$ is defined as

$$w(\delta(u)) = F - 1$$

for $F \in \{F_L, F_U\}$. Let $e = \{u, v\}$, and assume without loss of generality that $v = n-1$ and $u = n$. Let $S = \{\{1, n\}, \{2, n\}, \dots, \{1, n-1\}, \{1, n-2\}\}$. In the following matrix, the coefficients of the edges belonging to $S$ in the equations

representing $H_1$ to $H_n$ are displayed:

$$\begin{array}{r c} & \begin{array}{ccccccc} \{1,n\} & \{2,n\} & \{3,n\} & \dots & \{n-2,n\} & \{1,n-1\} & \{1,n-2\} \end{array} \\ \begin{array}{r} H_1 \rightarrow \\ H_2 \rightarrow \\ H_3 \rightarrow \\ H_4 \rightarrow \\ \vdots \\ H_{n-3} \rightarrow \\ H_{n-2} \rightarrow \\ H_{n-1} \rightarrow \\ H_n \rightarrow \end{array} & \left[\begin{array}{ccccccc} 1 & 0 & 0 & \dots & 0 & 1 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 & 0 \end{array}\right] \end{array} .$$

This matrix is non-singular. With reference to Theorem 3.6 in Section I.4.3 of [24], we use this fact to set $g_{e'} = 0$ for all $e' \in S$.

Now, pick four arbitrary nodes $x, y, z, t \in V - \{n, n-1\}$; and, consider an sc-partition $\pi = (N_1, N_2, \dots, N_k)$ where $k = \frac{n}{F}$, $|N_i| = F$ for all $i = 1, 2, \dots, k$, $\{n, x\} \subset N_1$, $\{n-1, y\} \subset N_2$ and $\{z, t\} \subset N_3$. Applying Lemma 4.6.3 to this sc-partition yields $g_{n,x} + g_{n-1,y} = g_{n,y} + g_{n-1,x}$. But, since we have already set $g_{n,x} = g_{n,y} = 0$, we obtain $g_{n-1,x} = g_{n-1,y}$. But then, since $\{1, n-1\} \in S$ and we have already set $g_{1,n-1} = 0$, this implies that $g_{q,n-1} = 0$ for all $q \in V - \{n-1, n\}$. Now, applying Lemma 4.6.3 once more gives $g_{n-1,y} + g_{z,t} = g_{n-1,t} + g_{y,z}$, which, by $g_{1,n-2} = 0$, directly implies $g_{e'} = 0$ for all $e' \in E(V - \{n-1, n\})$. □

### 4.6.2 Upper Bound Constraints

Let $P$ denote the face defined by a constraint in (4.1c), i.e.,

$$P = \{w \in \mathcal{P}^{lu} | w(\delta(u)) = F_U - 1\} \qquad \text{where } u \in V. \tag{4.13}$$

For some $(n, F_L, F_U)$ triples, it is possible that $P = \emptyset$. For such instances, the right hand side of the constraints (4.1c) has to be strengthened to ensure that $P$ is non-empty. For example, consider the triple $(33, 8, 10)$, which imply $\left\lceil \frac{n}{F_U} \right\rceil = \left\lfloor \frac{n}{F_L} \right\rfloor = 4$. We have $\max_i\{|N_i|\} = 9$ for any sc-partition $\pi = (N_1, N_2, N_3, N_4)$. Clearly, in this example we need to use 9 rather than 10 in the right hand side of (4.1c) to ensure that $P \neq \emptyset$. In other words, we need to be able to find the correct right hand side value for which (4.1c) remain valid and their corresponding faces become non-empty.

Now, we introduce a series of numbers, $\phi_i^U$, where $\phi_1^U - 1$ gives us this correct right hand side value we look for. The other elements of this series (i.e., $\phi_i^U$ for

$i > 1$) will be utilized in Sections 4.9 and 4.12. Consider the definition

$$\phi_i^U = \max_{\substack{\pi=(N_1,\ldots,N_k):\\ w^\pi \in \mathcal{P}^{lu}(R_{i-1},F_L,F_U)}} \left\{ \max_{j=1,\ldots,k} \{|N_j|\} \right\}.$$

where $\phi_0 = F_U$, $R_0 = n$ and $R_i = R_{i-1} - \phi_i^U$. According to this definition, $\phi_1^U$ gives the maximal size the subsets can attain in sc-partitions (i.e., in feasible solutions of $\mathcal{P}^{lu}(n, F_L, F_U)$), $\phi_2^U$ gives the second maximal size the subsets can attain in sc-partitions, and so on. Equivalently we can also say that $\phi_2^U$ is the maximal size the subsets can attain in feasible solutions of $\mathcal{P}^{lu}(n-\phi_1^U, F_L, F_U)$, $\phi_3^U$ is the maximal size the subsets can attain in feasible solutions of $\mathcal{P}^{lu}(n - \phi_1^U - \phi_2^U, F_L, F_U)$, and so on.

The following formula can be used for computing the values of $\phi_i^U$:

$$\phi_i^U = max \left\{ \phi \in \mathbb{Z}_+ \mid F_L \leq \phi \leq F_U, \left\lfloor \frac{R_{i-1} - \phi}{F_L} \right\rfloor \geq \left\lceil \frac{R_{i-1} - \phi}{\phi} \right\rceil \right\}. \quad (4.14)$$

Note that, $R_k = 0$ for $k = \left\lceil \frac{n}{F_U} \right\rceil$, and hence $\phi_i^U$ for $i > \left\lceil \frac{n}{F_U} \right\rceil$ is not defined.

Now, we define $P_u^U$'s as follows:

$$P_u^U = \{w \in \mathcal{P}^{lu} | w(\delta(u)) = \phi_1^U - 1\} \qquad \text{where } u \in V.$$

Due to the definition of $\phi_1^U$, $P_u^U$ is never empty for any $u \in V$.

Assume that there exists a valid inequality $g^T w \leq h$ such that

$$P_u^U \subseteq \{w \in \mathcal{P}^{lu} | g^T w = h\}$$

for a node $u$. In Theorems 4.6.5, 4.6.6 and 4.6.7 we prove the conditions under which $P_u^U$ is a facet, i.e., $g^T w = h$ can be written as a linear combination of $w(\delta(u)) = \phi_1^U - 1$ and the equalities in $M(\mathcal{P}^{lu})$, if any. Beforehand, we present a technical lemma.

**Lemma 4.6.5.** *If* $dim(n - \phi_1^U, F_L, F_U) \neq \binom{n-\phi_1^U}{2} - (n - \phi_1^U)$, $\phi_1^U = F_U$.

*Proof.* Suppose that $\phi_1^U < F_U$. If $\dim(n-\phi_1^U, F_L, F_U) \neq \binom{n-\phi_1^U}{2} - (n-\phi_1^U)$, there exists $\pi = (N_1, \ldots, N_k)$ in $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ such that $|N_l| > F_L$ for some $l \in \{1, \ldots, k\}$. Then, we can construct an sc-partition $\pi^* = (N_1^*, \ldots, N_{k+1}^*)$ in $\mathcal{P}^{lu}(n, F_L, F_U)$ such that $|N_i^*| = |N_i|$ for $i = 1, \ldots, k$ and $|N_{k+1}^*| = \phi_1^U$. But, then we would be able to find another sc-partition $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$ by shifting a node from $N_l^*$ to $N_{k+1}^*$. Then, we would have $|\tilde{N}_{k+1}| = \phi_1^U + 1$, which contradicts the definition of $\phi_1^U$. □

**Theorem 4.6.5.** *Suppose $\mathcal{P}^{lu}$ is full-dimensional and $F_U - F_L \geq 2$. Then, (4.1c) are facet defining for $\mathcal{P}^{lu}(n, F_L, F_U)$ if and only if*

*(i) $dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U)) = \binom{n-F_U}{2} - (n - F_U - 1)$, or,*

*(ii) $dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U)) = \binom{n-F_U}{2}$.*

*Proof.* Due to Lemma 4.6.5, the conditions of this theorem dictate us to consider (4.1c) rather than their strengthened version $w(\delta(u)) \leq \phi_1^U$. We pick a node $u$ and prove that the conditions (i) and (ii) make its corresponding inequality in (4.1c) facet defining.

(i) By Theorem 4.4.2, we have $\frac{n-F_U}{F_L} = \frac{n-F_U}{F_U} + 1$. Let $k = \frac{n-F_U}{F_U}$. Pick four nodes $p, q, x, y \in V - u$ arbitrarily. Pick an sc-partition $\pi = (N_1, \ldots, N_{k+2})$ in $\mathcal{P}^{lu}(n, F_L, F_U)$ such that $|N_1| = F_U$, $|N_i| = F_L$ for $i = 2, \ldots, k+2$, $u \in N_1$, $\{p, x\} \subset N_2$ and $\{q, y\} \subset N_3$. Applying Lemma 4.6.3 gives $g_{p,x} + g_{q,y} = g_{p,y} + g_{q,x}$, which implies $g_{p,z} - g_{q,z} = \alpha \in \mathbb{R}$ for all $z \in V - \{u, p, q\}$. Now, pick another sc-partition $\pi^* = (N_1^*, N_2^*, \ldots, N_{k+2}^*)$ in $\mathcal{P}^{lu}(n, F_L, F_U)$ such that $|N_1^*| = F_U$, $|N_i^*| = F_L$ for $i = 2, \ldots, k+2$, $\{u, q\} \subset N_1^*$ and $p \in N_2^*$. Switching $p$ and $q$ in $\pi^*$ gives

$$g_{q,u} + g(q, N_1^* - \{u, q\}) + g(p, N_2^* - p) = g_{p,u} + g(p, N_1^* - \{u, q\}) + g(q, N_2^* - p). \tag{4.15}$$

Now, construct yet another sc-partition $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_{k+1})$ in $\mathcal{P}^{lu}(n, F_L, F_U)$ by distributing the nodes of $N_{k+2}^*$ in $\pi^*$ over $N_2^*, \ldots, N_{k+1}^*$. By Theorem 4.4.2 we have $|\tilde{N}_i| = F_U$ for $i = 1, \ldots, k+1$. Clearly, we also have $\{u, q\} \subset \tilde{N}_1$ and $p \in \tilde{N}_2$. Switching $p$ and $q$ in $\tilde{\pi}$ gives

$$g_{q,u} + g(q, N_1^* - \{u, q\}) + g(p, (N_2^* - p) \cup \bar{N}_2^*) = g_{p,u} + g(p, N_1^* - \{u, q\}) + g(q, (N_2^* - p) \cup \bar{N}_2^*),$$

where $\bar{N^*}_2 = \tilde{N}_2 - N_2^*$ with $|\bar{N^*}_2| = F_U - F_L$. Comparing this latter equation with (4.15) yields

$$g(p, \bar{N^*}_2) = g(q, \bar{N^*}_2).$$

Plugging $g_{p,z} - g_{q,z} = \alpha$ into this equality implies that $\alpha = 0$, from which we infer $g_{p,z} = g_{q,z}$ for all $z \in V - \{u, p, q\}$. But, since $p$ and $q$ are arbitrary, this immediately gives $g_e = \beta \in \mathbb{R}$ for all $e \in E(V - u)$. Further, comparing $g^T w^{\pi^*} = h$ and $g^T w^{\tilde{\pi}} = h$ we obtain $\beta = 0$. Then, (4.15) directly implies $g_{u,p} = \gamma \in \mathbb{R}$ for all $p \in V - u$.

(ii)
- If $\mathcal{P}^{lu}(n - F_U, F_L, F_U)$ has a 2-loose sc-partition (i.e., it complies with **(FD-1)**), applying Lemma 4.6.1 to such a 2-loose sc-partition and generalizing, one can infer $g_e = 0$ for all $e \in E(V - u)$. Now, consider two arbitrary nodes $x, y \in V - u$ and an sc-partition $\pi = (N_1, \ldots, N_k)$ in $\mathcal{P}^{lu}(n, F_L, F_U)$ such that $\{u, x\} \subset N_1$, $y \in N_2$ and $|N_1| = F_U$. Switching $x$ and $y$ in $\pi$ yields $g_{u,x} = g_{u,y}$. Since $x$ and $y$ are arbitrary, we infer $g_e = \alpha \in \mathbb{R}$ for $e \in \delta(u)$.
- If $(n - F_U, F_L, F_U)$ satisfies **(FD-2)**, pick an sc-partition in $\mathcal{P}^{lu}(n, F_L, F_U)$, $\pi^1 = (N_1^1, N_2^1, \ldots, N_{k+2}^1)$, such that $k = \left\lceil \frac{n-F_U}{F_U} \right\rceil \geq 1$, $u \in N_1^1$, $|N_1^1| = F_U$, $|N_2^1| = F_L + 1$ and $|N_i^1| = F_L$ for $i = 3, \ldots, k + 2$. Applying Lemma 4.6.2 to this sc-partition and generalizing, we infer $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - u)$. Now, pick another sc-partition $\pi^2 = (N_1^2, N_2^2, \ldots, N_{k+1}^2)$ in $\mathcal{P}^{lu}(n, F_L, F_U)$, where $N_1^2 = N_1^1$, $|N_i^2| = F_U$ for $i = 1, 2, \ldots, k + 1$. Comparing $g^T w^{\pi^1} = h$ and $g^T w^{\pi^2} = h$ we infer that $\alpha = 0$ for all $e \in E(V - u)$. Now, pick two arbitrary nodes $x, y \in V - u$ such that, without loss of generality, $x \in N_1^1$ and $y \in N_2^1$. Switching $x$ and $y$ in $\pi^1$, we obtain $g_{u,x} = g_{u,y}$. This implies $g_{u,v} = \beta \in \mathbb{R}$ for all $v \in V - u$ and hence that (4.16) is facet defining. We can use this proof when $(n - F_U, F_L, F_U)$ satisfies **(FD-3)** or **(FD-5)** as well, by choosing $\pi^1$ and $\pi^2$ accordingly.
- If $(n - F_U, F_L, F_U)$ satisfies **(FD-4)**, pick an sc-partition $\pi^1 = (N_1^1, N_2^1, \ldots, N_{k+1}^1)$ in $\mathcal{P}^{lu}(n, F_L, F_U)$ where $k = \left\lceil \frac{n-F_U}{F_U} \right\rceil \geq 2$ (recall from the Remark of Theorem 4.5.1 that **(FD-4)** implies full-dimensionality of $\mathcal{P}^{lu}(n - F_U, F_L, F_U)$ if and only if $\left\lceil \frac{n-F_U}{F_U} \right\rceil \geq 2$), $u \in N_1^1$, $|N_1^1| = F_U$, $|N_2^1| = F_U - 1$, $|N_i^1| = F_U$ for $i = 3, \ldots, k + 1$. Applying Lemma 4.6.2 to this sc-partition and generalizing, we infer that $g_e = \alpha$ for all $e \in E(V - u)$. Now, pick another sc-partition $\pi^2 = (N_1^2, N_2^2, \ldots, N_{k+2}^2)$ in $\mathcal{P}^{lu}(n, F_L, F_U)$ where $N_1^2 = N_1^1$, $|N_i^2| = F_L$ for $i = 2, \ldots, k + 2$. Comparing $g^T w^{\pi^1} = h$ and $g^T w^{\pi^2} = h$ we infer $\alpha = 0$. Now, in a similar manner to the previous paragraph (where facetness for **(FD-2)** is proved), we can show that $g_{u,v} = \beta \in \mathbb{R}$ for all $v \in V - u$.

Now, suppose that $\dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U))$ is either $\binom{n-F_U}{2} - 1$ or $\binom{n-F_U}{2} - (n - F_U)$. Observe from Section 4.5.2 that, $\mathcal{P}^{lu}(n - F_U, F_L, F_U)$ is contained in the hyperplane $w(\bar{E}) = C$ where $C$ is a constant and $\bar{E}$ represents the edge

set of a complete graph with $n - F_U$ nodes. Then,

$$P_u^U \subseteq \{w \in \mathbb{R}^{\binom{n}{2}} | w(E) = C + \binom{F_U}{2}\}.$$

Note that, $w(E) \leq C + \binom{F_U}{2}$ is valid for $\mathcal{P}^{lu}$. □

**Theorem 4.6.6.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. *The inequalities*

$$w(\delta(u)) \leq \phi_1^U - 1 \qquad \forall u \in V \tag{4.16}$$

*are facet defining for* $\mathcal{P}^{lu}$ *if and only if one of the following hold:*

- $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$ *and* $(n \ mod \ F_U) = F_U - 1$.
- $F_U - F_L = 1$, $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$ *and* $(n \ mod \ F_L) > 1$.

*Proof.* If $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, by Theorem 4.4.1 we have $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil$. Theorem 4.4.1 also shows that $\dim(\mathcal{P}^{lu})$ is $\binom{n}{2} - 1$ when $(n \bmod F_L) = 1$ or $(n \bmod F_U) = F_U - 1$. We know furthermore that, when $F_U - F_L = 1$, $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$ holds even if $(n \bmod F_L) > 1$ and $0 < (n \bmod F_U) < F_U - 1$.

When $(n \bmod F_L) = 1$ or $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2$, $P_u^U$ is contained in the faces defined by the upper 2-star inequalities introduced in Section 4.12 of the sequel paper.

Now, we show that (4.16) are facet defining if $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$ and $(n \bmod F_U) = F_U - 1$. Since $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$ we can fix $g_{e^*} = 0$ for some $e^* \in E(V - u)$. Now, pick four arbitrary nodes $v, t, x, y \in V - u$ and an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ $(k \geq 3)$ such that $\{u, y\} \subset N_1$, $\{v, t\} \subset N_2$, $x \in N_3$, $|N_1| = |N_2| = F_U$, $|N_3| = F_U - 1$ and $|N_i| = F_U$ for $i = 4, \ldots, k$. Applying Lemma 4.6.2 to $\pi$, we can infer that $g_{v,t} = g_{v,x}$ for all $v, t, x \in V - u$. Since $v$, $t$ and $x$ are arbitrary and $g_{e^*} = 0$ for some $e^* \in E(V - u)$, we infer $g_e = 0$ for $e \in E(V - u)$. Switching $v$ and $y$ in $\pi$, we get $g_{u,v} = g_{u,y}$, which implies $g_{u,v} = \alpha \in \mathbb{R}$ for all $v \in V - u$.

Now suppose that $F_U - F_L = 1$ and consider an sc-partition $\pi^* = (N_1^*, N_2^*, \ldots, N_k^*)$ $(k \geq 3)$ such that $\{u, y\} \subset N_1^*$, $\{v, t\} \subset N_2^*$, $x \in N_3^*$, $|N_1^*| = |N_2^*| = F_U = F_L + 1$ and $|N_3| = F_U - 1 = F_L$. Proceeding in the same manner as we have done with $\pi$ in the previous paragraph, we can show that (4.16) are facet defining for $\mathcal{P}^{lu}$ when $F_U - F_L = 1$, $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$ and $(n \bmod F_L) > 1$. □

**Theorem 4.6.7.** *When* $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$*, (4.16) are facet defining for* $\mathcal{P}^{lu}$*.*

*Proof.* When $\dim(\mathcal{P}^{lu})=\binom{n}{2}-(n-1)$, $\mathcal{P}^{lu}(n, F_L, F_U) = \mathcal{P}^{lu}(n, F_L, F_L) \cup \mathcal{P}^{lu}(n, F_U, F_U)$. Since $\mathcal{P}^{lu}(n, F_U, F_U) \subseteq P_u^U$, we have $\dim(P_u^U) = \binom{n}{2} - n$. □

### 4.6.3 Lower Bound Constraints

In this section, we express the conditions that make the strengthened version of the lower bound constraints (4.1b) facet defining. The proofs of the theorems (except a part of the proof of Theorem 4.6.9) are not presented because they are very similar to their counterparts in the previous section.

In this section, we introduce the series $\phi_i^L$, which is in a sense the symmetric of $\phi_i^U$ of the previous section. We define $\phi_i^L$ in the following manner:

$$\phi_i^L = \min_{\substack{\pi=(N_1,\ldots,N_k):\\ w^\pi \in \mathcal{P}^{lu}(R_{i-1},F_L,F_U)}} \left\{ \min_{j=1,\ldots,k} \{|N_j|\} \right\}.$$

where $\phi_0 = F_L$, $R_0 = n$ and $R_i = \max\{0, R_{i-1} - \phi_i^L\}$. Similarly, $\phi_1^L$ gives the *minimal* size the subsets can attain in sc-partitions (i.e., in feasible solutions of $\mathcal{P}^{lu}(n, F_L, F_U)$), and, $\phi_2^L$ gives the second *minimal* size the subsets can attain in sc-partitions, and so on. We can equivalently say that $\phi_2^L$ is the minimal size the subsets can attain in feasible solutions of $\mathcal{P}^{lu}(n-\phi_1^L, F_L, F_U)$, $\phi_3^L$ is the minimal size the subsets can attain in feasible solutions of $\mathcal{P}^{lu}(n-\phi_1^L-\phi_2^L, F_L, F_U)$, and so on. Likewise, $\phi_i^L$ for $i > 1$ will be utilized in Section 4.11.

The following formula, which is symmetric of (4.14), can be employed for computing the values of $\phi_i^L$'s:

$$\phi_i^L = min \left\{ \phi \in \mathbb{Z}_+ \mid F_L \leq \phi \leq F_U, \left\lfloor \frac{R_{i-1} - \phi}{\phi} \right\rfloor \geq \left\lceil \frac{R_{i-1} - \phi}{F_U} \right\rceil \right\}. \quad (4.17)$$

Note that, for $k = \left\lfloor \frac{n}{F_L} \right\rfloor$, $R_k = 0$ and hence, $\phi_i^L$ for $i > k$ is not defined.

Now follows the lemma and the theorems that summarize the main results of this section regarding the lower bound constraints.

**Lemma 4.6.6.** *If* $dim(n - \phi_1^L, F_L, F_U) \neq \binom{n-\phi_1^L}{2} - (n - \phi_1^L)$*,* $\phi_1^L = F_L$*.*

**Theorem 4.6.8.** *Suppose* $\mathcal{P}^{lu}$ *is full-dimensional and* $F_U - F_L \geq 2$*. Then, (4.1b) are facet defining for* $\mathcal{P}^{lu}$ *if and only if*

*(i)* $dim(\mathcal{P}^{lu}(n-F_L,F_L,F_U))=\binom{n-F_L}{2}-(n-F_L-1)$, *or,*

*(ii)* $dim(\mathcal{P}^{lu}(n-F_L,F_L,F_U))=\binom{n-F_L}{2}$.

**Theorem 4.6.9.** *Suppose* $dim(\mathcal{P}^{lu})=\binom{n}{2}-1$. *Then,*

$$w(\delta(u))\geq\phi_1^L-1\qquad\forall u\in V \tag{4.18}$$

*are facet defining for* $\mathcal{P}^{lu}$ *if and only if one of the following hold:*

- $\left\lfloor\frac{n}{F_L}\right\rfloor=\left\lceil\frac{n}{F_U}\right\rceil\geq 3$ *and* $(n\ mod\ F_L)=1$.
- $F_U-F_L=1$, $\left\lfloor\frac{n}{F_L}\right\rfloor=\left\lceil\frac{n}{F_U}\right\rceil\geq 3$ *and* $(n\ mod\ F_U)<F_U-1$.

*Proof.* When $\left\lfloor\frac{n}{F_L}\right\rfloor=\left\lceil\frac{n}{F_U}\right\rceil=2$ or $(n \bmod F_U)=F_U-1$, $P_u^L=\{w\in\mathcal{P}^{lu}|w(\delta(u))\geq\phi_1^L-1\}$ is contained in the faces defined by the lower 2-star inequalities in Section 4.11. □

**Theorem 4.6.10.** *Suppose* $\dim(\mathcal{P}^{lu})=\binom{n}{2}-(n-1)$. *Then (4.18) are facet defining for* $\mathcal{P}^{lu}$.

Now, we move on to the non-trivial facets of $\mathcal{P}^{lu}$: the 2-partition inequalities, the lower and upper general clique inequalities, the cycle inequalities and the lower and upper 2-star inequalities. As we had already stated, the equi-partition polytope $\mathcal{P}^{equi}(n)$ and the $k$-way equi-partition polytope $\mathcal{P}^{k-way}(n,k)$ are special cases of $\mathcal{P}^{lu}$. Certain theorems in the sequel prove for the first time in the literature that some of the aforementioned valid inequalities are facet defining for these two polytopes as well. We highlight such results as corollaries to corresponding theorems.

Recall that we assume $F_L\geq 3$ and $n>F_U$ in the rest of this chapter.

## 4.7 2-Partition inequalities

This section is devoted to the 2-partition inequalities, which are defined in the following manner over two disjoint subsets $S$ and $T$ of $V$ (see Figure 4.1 for a graphical representation):

$$w(E(S,T))-w(E(S))-w(E(T))\leq\min(|S|,|T|)\qquad\text{for } S,T\subset V: S\cap T=\emptyset. \tag{4.19}$$

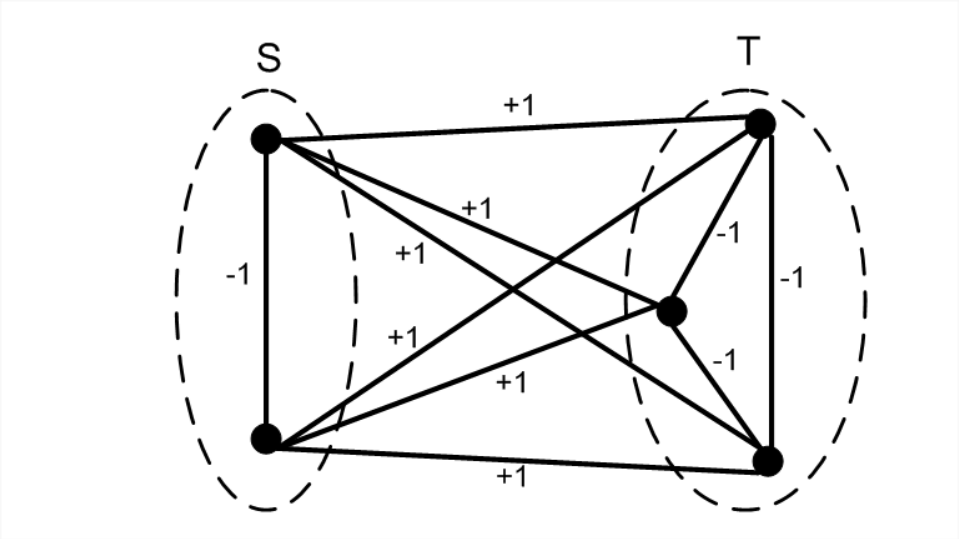


Figure 4.1: Support graph of an example 2-partition inequality with $|S| = 2$ and $|T| = 3$. Numbers on the edges stand for the coefficients of the corresponding partitioning variables.

The 2-partition inequalities are introduced by Grötschel and Wakabayashi [14] for the clique partitioning polytope $\mathcal{P}(n)$ and shown to be facet defining for this polytope if and only if $|S| \neq |T|$. Oosten et al. [26] introduce some generalizations of these inequalities (the weighted 2-partition inequalities), which are also facet defining for $\mathcal{P}(n)$. Ji and Mitchell [15] and Sørensen [29] show that these inequalities are facet defining for $\mathcal{P}^l(n, F_L)$ and $\mathcal{P}^u(n, F_U)$ as well, respectively. These inequalities are also a special case of the clique-web inequalities, which are introduced and proved to be facet defining by Deza, Grötschel and Laurent [9] for the multicut polytope, the complement of $\mathcal{P}(n)$.

One special feature of the 2-partition inequalities is their having the triangle inequalities as special cases: for disjoint $S, T \subset V$ with $|S| = 1$ and $|T| = 2$, (4.19) turns into a triangle inequality.

In this section, we denote by $P_{S,T}$ the face defined by a 2-partition inequality over the sets $S$ and $T$, i.e.,

$$P_{S,T} = \left\{ w \in \mathcal{P}^{lu} | w(E(S,T)) - w(E(S)) - w(E(T)) = \min(|S|, |T|) \right\}.$$

Now, in Theorems 4.7.1-4.7.6 we prove several sufficient conditions that make a 2-partition inequality facet defining for $\mathcal{P}^{lu}$. As before, we assume that there

exists a valid inequality $g^T w \leq h$ such that

$$P_{S,T} \subseteq \{w \in \mathcal{P}^{lu} | g^T w = h\}.$$

We will show that $g^T w = h$ is a linear combination of the 2-partition inequality under consideration and the equalities in $M(\mathcal{P}^{lu})$, if any.

But, before, in Lemma 4.7.10 we give a characterization of the sc-partitions that lie on $P_{S,T}$. This lemma was already proved by Deza, Grötschel and Laurent [9] for the multicut polytope. We just provide an alternative proof here.

**Lemma 4.7.10.** *Consider two disjoint subsets $S, T \subset V$ and the corresponding face $P_{S,T}$. Suppose without loss of generality that $|S| < |T|$. Then, $w^\pi \in P_{S,T}$ for an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ if and only if $0 \leq |N_i \cap T| - |N_i \cap S| \leq 1$ for all $i = 1, 2, \ldots, k$.*

*Proof.* Let $s_i^\pi = |N_i \cap S|$ and $t_i^\pi = |N_i \cap T|$ for all $i = 1, 2, \ldots, k$. Then,

$$w^\pi(E(S,T)) - w^\pi(E(S)) - w^\pi(E(T)) = \sum_{i=1}^{k} \left( s_i^\pi t_i^\pi - \binom{s_i^\pi}{2} - \binom{t_i^\pi}{2} \right).$$

Rearranging we get

$$w^\pi(E(S,T)) - w^\pi(E(S)) - w^\pi(E(T)) = \sum_{i=1}^{k} \left( \frac{(s_i^\pi + t_i^\pi) - (t_i^\pi - s_i^\pi)^2}{2} \right). \quad (4.20)$$

Let $\mu_i^\pi = \min\{s_i^\pi, t_i^\pi\}$ and $\delta_i^\pi = |t_i^\pi - s_i^\pi|$. Then, we can rewrite (4.20) as

$$w^\pi(E(S,T)) - w^\pi(E(S)) - w^\pi(E(T)) = \sum_{i=1}^{k} \left( \mu_i^\pi - \binom{\delta_i^\pi}{2} \right),$$

which attains its maximum value $|S|$ if and only if $\mu_i^\pi = s_i^\pi$ and $0 \leq \delta_i^\pi \leq 1$ for all $i = 1, 2, \ldots, k$. □

**Theorem 4.7.1.** *Suppose that $F_U - F_L \geq 2$. Suppose further that $\mathcal{P}^{lu}$ is full-dimensional and complies with* **(FD-1)**. *Let $S$ and $T$ be two proper and disjoint subsets of $V$ such that $|S| < |T|$ and $|V - (S \cup T)| \geq 2$. Let $\Delta = |T| - |S|$. Pick an integer $k$ such that $kF_L + 1 < n < kF_U - 1$ (note that such a $k$ exists since $\mathcal{P}^{lu}$ complies with* **(FD-1)**). *The 2-partition inequality (4.19) defined over $S$ and $T$ is facet defining for $\mathcal{P}^{lu}$ if*

- $\Delta < k$, *and,*
- *there exist an sc-partition* $\pi = (N_1, N_2, \ldots, N_k)$ *such that* $F_L < |N_i| < F_U$ *for* $i = 1, k$, $F_L \leq |N_i| \leq F_U$ *for* $i = 2, \ldots, k-1$, *and*

$$|S| \leq \left( \left\lfloor \frac{|N_1| - 1}{2} \right\rfloor + \sum_{i=2}^{k-\Delta} \left\lfloor \frac{|N_i|}{2} \right\rfloor + \sum_{i=k-\Delta+1}^{k-1} \left\lfloor \frac{|N_i| - 1}{2} \right\rfloor + \left\lfloor \frac{|N_k| - 2}{2} \right\rfloor \right). \tag{4.21}$$

*Proof.* Let $L = \min\{2, |S|\}$. There exist integer $\mu_i$ $(i = 1, \ldots, k)$ values such that

(i) $\mu_i > 0$ for $i = 1, \ldots, L$ and $\mu_i \geq 0$ for $i = L+1, \ldots, k$,

(ii) $\sum_{i=1}^{k} \mu_i = |S|$,

(iii) $2\mu_1 + 1 \leq |N_1|$,

(iv) $2\mu_i \leq |N_i|$ for $i = 2, \ldots, k - \Delta$,

(v) $2\mu_i + 1 \leq |N_i|$ for $i = k - \Delta + 1, \ldots, k-1$,

(vi) $2\mu_k + 2 \leq |N_k|$.

Pick $\pi$ in such a way that

- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta, \\ \mu_i + 1, & \text{for } i = k - \Delta + 1, \ldots, k. \end{cases}$

By Lemma 4.7.10, $w^\pi \in P_{S,T}$. The conditions on $\mu_1$ and $\mu_k$ imply $N_1 - (S \cup T) \neq \emptyset$ and $N_k - (S \cup T) \neq \emptyset$. Pick arbitrarily $u \in N_1 - (S \cup T)$ and $v \in N_k - (S \cup T)$. From Lemma 4.6.1, we infer $g_{u,v} = 0$, which leads to $g_e = 0$ for all $e \in E(V - (S \cup T))$ due to arbitrariness of $u$ and $v$. We can apply Lemma 4.6.1 also to an arbitrary $s \in N_1 \cap S$ and $v$ to infer $g_{s,v} = 0$, which generalizes into $g_e = 0$ for all $e \in E(S, V - (S \cap T))$.

Moving $u$ from $N_1$ to $N_k$ yields

$$g(u, N_1 \cap T) = g(u, N_k \cap T). \tag{4.22}$$

Due to Lemma 4.6.4, (4.22) implies that $g_{u,t'} = g_{u,t''}$ for all $t', t'' \in T$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for all $e \in E(T, V - (S \cup T))$. Then, (4.22) is equivalent to

$$\mu_1 \alpha = (\mu_k + 1)\alpha. \tag{4.23}$$

Now, obtain another sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ from $\pi$ by shifting an arbitrary $t^* \in N_k \cap T$ from $N_k$ to $N_1$. Shifting $u$ from $N_1^*$ to $N_k^*$, we get

$$g(u, N_1^* \cap T) = g(u, N_k^* \cap T),$$

which is equivalent to $(\mu_1 + 1)\alpha = \mu_k \alpha$. Solving this equation and (4.23) simultaneously yields $\alpha = 0$.

Now, pick an arbitrary $t' \in N_k \cap T$ and shift it from $N_k$ to $N_1$ in $\pi$ to get

$$g(t', N_k \cap (T - t')) + g(t', N_k \cap S) = g(t', N_1 \cap T) + g(t', N_1 \cap S). \tag{4.24}$$

*Claim 1:* Equation (4.24) implies that $g_{e'} = -g_e = \beta \in \mathbb{R}$ for all $e' \in E(T)$ and $e \in E(S, T)$.
*Proof:* When $\mu_1 = 1$ and $\mu_k = 0$, the result is trivial. When $\mu_1 > 1$ or $\mu_k > 0$, condition 1 on $\mu_i$ values allows us to apply Lemma 4.6.4 twice and conclude that $g_{e'} = \beta \in \mathbb{R}$ for all $e' \in E(T)$ and $g_e = \gamma$ for all $e \in E(S, T)$. Then (4.24) can be rewritten as

$$(\mu_k - \mu_1)(\beta + \gamma) = 0. \tag{4.25}$$

If $\mu_k = 0$ or $\mu_k \neq \mu_1$, (4.25) implies $\beta = -\gamma$ since $\mu_1 > 0$. Suppose $\mu_k = \mu_1 > 0$. We obtain a new sc-partition $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$ from $\pi$ in the following way:

- If $|N_1 - (S \cup T)| \geq 2$, pick $\{u, x\} \subset N_1 - (S \cup T)$ arbitrarily. Obtain $\tilde{\pi}$ by switching $\{u, x\}$ in $N_1$ and $\{\tilde{s}, \tilde{t}\}$ in $N_k$, where $\tilde{s} \in N_k \cap S$ and $\tilde{t} \in N_k \cap T$.
- If $|N_1 - (S \cup T)| = 1$ (i.e., $|N_1| = 2\mu_1 + 1$), we have $|N_1| + 1 \leq |N_k|$. This implies that $|N_1| < F_U - 1$ and $|N_k| > F_L + 1$ (by conditions 3 and 6). Obtain $\tilde{\pi}$ by switching $u$ in $N_1$ and $\{\tilde{s}, \tilde{t}\}$ in $N_k$.

In either case, shifting $t'$ from $\tilde{N}_k$ to $\tilde{N}_1$ yields $(\mu_k - \mu_1 - 2)(\beta + \gamma) = 0$. Solving this and (4.25) simultaneously, we get $\beta = -\gamma$. This completes the proof of the claim.

*Claim 2:* $g_e = \beta$ for all $e \in E(S)$.
*Proof:* Shift $s$ from $N_1$ to $N_k$ in $\pi$ to get

$$g(s, N_1 \cap (S - s)) - \mu_1 \beta = g(s, N_k \cap S) - (\mu_k + 1)\beta. \tag{4.26}$$

We investigate two cases:
*Case 1:* $k \geq 3$ and $\mu_k > 0$. If $k \geq 3$ and $\mu_k > 0$, due to Lemma 4.6.4 and the assumption that $\mu_2 > 0$, equation (4.26) implies that $g_{s,s'} = \theta \in \mathbb{R}$ for all $s' \in S - s$. Then (4.26) turns into

$$(\mu_1 - \mu_k - 1)(\theta - \beta) = 0. \tag{4.27}$$

If $\mu_1 \neq \mu_k + 1$, then we infer $\theta = \beta$. Suppose $\mu_1 = \mu_k + 1$. Consider the sc-partition $\tilde{\pi}$ obtained above in the proof of Claim 1. Shifting $s$ from $\tilde{N}_1$ to $\tilde{N}_k$ in $\tilde{\pi}$ gives $(\mu_1 - \mu_k + 1)(\theta - \beta) = 0$. Solving this simultaneously with (4.27) yields $\theta = \beta$.
*Case 2:* $k \geq 3$ and $\mu_k = 0$. We know from condition 1 that $\mu_2 > 0$ if $|S| \geq 2$. We can always obtain a new sc-partition $\hat{\pi} = (\hat{N}_1, \hat{N}_2, \ldots, \hat{N}_k)$ from $\pi$ by switching $\{\hat{s}, \hat{t}\} \subset N_2$ and $\{y, z\} \in N_k - (S \cup T)$, where $\hat{s} \in N_2 \cap S$ and $\hat{t} \in N_2 \cap T$. Now, we can proceed in the same manner as we have done for Case 1 to show that $g_e = -\beta$ for all $e \in E(S)$.

*Case 3:* $k = 2$. Consider the partition $\tilde{\pi}$ obtained as in the proof of Claim 1. Shifting $s$ from $\tilde{N}_1$ to $\tilde{N}_2$ in $\tilde{\pi}$ yields

$$g(s, N_1 \cap (S - s)) + g_{s,\tilde{s}} - (\mu_1 + 1)\beta = g(s, N_2 \cap S) - g_{s,\tilde{s}} - \mu_2\beta. \tag{4.28}$$

Due to symmetry, equations (4.26) and (4.28) imply $g_e = \beta$ for all $e \in E(S)$. This completes the proof of the claim and the theorem. □

When

$$|S| \leq \left( \left\lfloor \frac{F_L - 2}{2} \right\rfloor + \Delta \left\lfloor \frac{F_L - 1}{2} \right\rfloor + (k - \Delta - 1) \left\lfloor \frac{F_L}{2} \right\rfloor \right), \tag{4.29}$$

this theorem proves that $P_{S,T}$ is a facet.

On the other hand, when $|S|$ violates (4.29), it is not a trivial task to check if there exists an sc-partition that satisfies (4.21). For this case, we introduce a very simple polynomial algorithm to check existence of such an sc-partition.

Let $\pi = (N_1, \ldots, N_k)$ be an sc-partition such that

$$F_L < |N_i| < F_U \qquad \text{for } i = 1, k, \tag{4.30}$$
$$F_L \leq |N_i| \leq F_U \qquad \text{for } i = 2, \ldots, k-1. \tag{4.31}$$

Let

$$\tau_i(\pi) = \begin{cases} \left\lfloor \frac{|N_i|-1}{2} \right\rfloor, & \text{for } i = 1 \text{ and } i = k-\Delta+1, \ldots, k-1, \\ \left\lfloor \frac{|N_i|}{2} \right\rfloor, & \text{for } i = 2, \ldots, k-\Delta, \\ \left\lfloor \frac{|N_i|-2}{2} \right\rfloor, & \text{for } i = k. \end{cases} \tag{4.32}$$

If $\sum_{i=1}^{k} \tau_i \geq |S|$ for $\pi$, this means $P_{S,T}$ is a facet.

The aim of our algorithm is to construct an sc-partition $\pi$ that is in accordance with (4.30) and (4.31) and that maximizes $\sum_{i=1}^{k} \tau_i(\pi)$. Clearly, $\tau_i(\pi)$'s depend only on the subclique sizes $|N_1|, \ldots, |N_k|$ of $\pi$. Hence, the aim of our algorithm is essentially to construct valid subclique sizes (i.e., $|N_i|$'s for $i = 1, \ldots, k$) in accordance with (4.30) and (4.31) such that $\sum_{i=1}^{k} \tau_i(\pi)$ is maximized.

Initially in the algorithm, we put $F_L$ nodes in each of the $k$ subcliques of $\pi$ (i.e., we assign $|N_i| = F_L$ for $i = 1, \ldots, k$) and distribute the remaining $r = n - kF_L$ nodes over these subcliques. During the course of this construction, we make use of a concept which we call *the growth potential*. We define the growth potential corresponding to a subclique $i$ as the increase in the value of $\tau_i(\pi)$ induced by unit increase in the value of $|N_i|$. For example, when $|N_1| = 8$, $\tau_1(\pi) = 3$. If we add 1 to $|N_1|$, $\tau_1(\pi)$ increases to 4. Hence, when $|N_1| = 8$ we say that the growth potential of subclique $N_1$ is 1. By definition, the growth potential can only take on 0-1 values. We denote the growth potential of a subclique $N_i$ by $\psi(|N_i|)$, where

$$\psi(|N_i|) = \begin{cases} ((1 + |N_i|) \bmod 2), & \text{for } i = 1 \text{ and } i = k-\Delta+1, \ldots, k-1. \\ (|N_i| \bmod 2), & \text{for } i = 2, \ldots, k-\Delta \text{ and } i = k. \end{cases}$$

Besides, we use the following constants for denoting the maximum allowed subclique sizes as implied by (4.30) and (4.31):

$$\chi_i = \begin{cases} F_U - 1, & \text{for } i = 1, k; \\ F_U, & \text{for } i = 2, \ldots, k-1. \end{cases}$$

The algorithm is based on the following four index sets $\mathcal{A}$, $\mathcal{Y}$, $\mathcal{S}$ and $\mathcal{U}$:

$$\begin{aligned}\mathcal{A} &= \{i : \psi(|N_i|) = 1, |N_i| < \chi_i\},\\ \mathcal{Y} &= \{i : \psi(|N_i|) = 0, |N_i| < \chi_i - 1\},\\ \mathcal{S} &= \{i : \psi(|N_i|) = 0, |N_i| = \chi_i - 1\},\\ \mathcal{U} &= \{i : |N_i| = \chi_i\}.\end{aligned}$$

Note that, no matter what the values of $|N_i|$'s are, we always have

$$\mathcal{A} \cup \mathcal{Y} \cup \mathcal{S} \cup \mathcal{U} = \{1, \ldots, k\}.$$

For a subclique $j \in \mathcal{A}$, increasing $|N_j|$ by one makes $j$ leave $\mathcal{A}$ and move into one of $\mathcal{Y}$, $\mathcal{S}$ or $\mathcal{U}$. If $j \in \mathcal{Y}$ and if we increase $|N_j|$ by one, $j$ leaves $\mathcal{Y}$ and moves into $\mathcal{A}$. Clearly, if $j \in \mathcal{S}$ and we increase $|N_j|$ by one, $j$ shifts into $\mathcal{U}$. And, we can not increase $|N_j|$ by one for a subclique $N_j$ with $j \in \mathcal{U}$.

Among these four sets, $\mathcal{A}$ retains a special feature in that if $j \in \mathcal{A}$ and we increase $|N_j|$ by one, $\tau_j(\pi)$ increases by 1, too. This is not the case for $j \in \mathcal{Y} \cup \mathcal{S} \cup \mathcal{U}$. For this reason, in order to obtain an increase in $\sum_{i=1}^{k} \tau_i$, we need to add nodes to subcliques with indices in $\mathcal{A}$. This is the basic logic we use in the algorithm, which we present below:

**Algorithm:**

Input: $n, k, |S|, \Delta$;

Output: Valid subclique sizes $|N_1|, \ldots, |N_k|$ in accordance with (4.30) and (4.31) and that maximizes $\sum_{i=1}^{k} \tau_i$;

Variable: integer $r$;

Constants: $\chi_i$ for $i = 1, \ldots, k$ as defined above;

Functions: $\psi(|N_i|)$ for $i = 1, \ldots, k$ as defined above;

Initialization: $|N_i| = F_L$ for $i = 1, \ldots, k$; $r = n - kF_L$;

**Step 1:** For all $j \in \mathcal{A}$ do $|N_j| = |N_j| + \min(1, r)$ and $r = r - \min(1, r)$;

**Step 2:** Pick a $j \in \mathcal{Y}$ do $|N_j| = |N_j| + \min(1, r)$ and $r = r - \min(1, r)$;

**Step 3:** If $(\mathcal{A} \cup \mathcal{Y} \neq \emptyset)$ then go to Step 1, else go to Step 4.

**Step 4:** For all $j \in \mathcal{S}$ do $|N_j| = |N_j| + \min(1, r)$ and $r = r - \min(1, r)$;

**Return:** $|N_i|$ values for $i = 1, \ldots, k$.

In Step 1 we add one node to each of the subcliques whose indices belong to $\mathcal{A}$. For each node added, $\sum_{i=1}^{k} \tau_i(\pi)$ increases by 1. Once Step 1 is completed, growth potentials of all the subcliques are 0 (i.e., $\mathcal{A}$ is empty and we can not induce an increase in $\sum_{i=1}^{k} \tau_i(\pi)$ by adding a node to a subclique). At this point, the best action to be taken would be to move a subclique into $\mathcal{A}$. This

is exactly what Step 2 carries out by adding one node to a subclique in $\mathcal{Y}$. In Step 3, we check if there remains any subcliques in $\mathcal{A} \cup \mathcal{Y}$ (i.e., if there remains any subcliques with growth potential equal to 1, or, whose growth potential would be equal to 1 when size is increased by 1). If there are any, we return to Step 1 and continue in the same manner. If there are not any, we can infer that $\{1, \ldots, k\} = \mathcal{S} \cup \mathcal{U}$ and we can put the remaining $r$ nodes we have into the subcliques in $\mathcal{S}$ only. Step 4 carries out this terminating task.

We plug the $|N_i|$'s returned by the algorithm into (4.32) to obtain $\tau_i(\pi)$'s. As we have mentioned above, if $\sum_{i=1}^{k} \tau_i(\pi) \geq |S|$, then we conclude that $P_{S,T}$ is a facet.

Now, we continue with the other theorems of this section.

**Theorem 4.7.2.** *Suppose that $\mathcal{P}^{lu}$ is full-dimensional and complies with* **(FD-2)**. *Let $S$ and $T$ be two proper and disjoint subsets of $V$ such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \left\lfloor \frac{n}{F_L} \right\rfloor$. The 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k - 1$ and*

$$|S| \leq (\Delta + 1) \left\lfloor \frac{F_L - 1}{2} \right\rfloor + (k - \Delta - 2) \left\lfloor \frac{F_L}{2} \right\rfloor . \tag{4.33}$$

*Proof.* Let $L = \min\{2, |S|\}$. By (4.33), there exist integer $\mu_i$ $(i = 1, \ldots, k-1)$ values such that

(i) $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L+1, \ldots, k-1$,

(ii) $\sum_{i=1}^{k-1} \mu_i = |S|$,

(iii) $2\mu_1 + 2 \leq F_L + 1$,

(iv) $2\mu_i \leq F_L$ for $i = 2, \ldots, k - \Delta - 1$,

(v) $2\mu_i + 1 \leq F_L$ for $i = k - \Delta, \ldots, k - 1$.

Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that

- $|N_1| = F_L + 1$ and $|N_i| = F_L$ for $i = 2, \ldots, k$,
- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k-1$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta - 1, \\ \mu_i + 1, & \text{for } i = k - \Delta, \ldots, k - 1. \end{cases}$

By Lemma 4.7.10, $w^\pi \in P_{S,T}$. Conditions on $\mu_1$ imply that $|N_1 - (S \cup T)| \geq 2$. Pick arbitrarily $u, v \in N_1 - (S \cup T)$ and $x \in N_k - (S \cup T)$. We can apply Lemma 4.6.2 to infer $g_{u,v} = g_{u,x}$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - (S \cup T)$.

Moving $u$ from $N_1$ to $N_{k-1}$ in $\pi$ gives

$$\begin{aligned} g(u, N_1 \cap S) \quad + \quad & g(u, N_1 \cap T) + g(u, N_1 - (S \cup T \cup u)) = \\ & g(u, N_{k-1} \cap S) + g(u, N_{k-1} \cap T) + g(u, N_{k-1} - (S \cup T)). \end{aligned} \tag{4.34}$$

Applying Lemma 4.6.4 to this equation, one can infer that $g_{u,t} = \beta$ for all $t \in T$ and $g_{u,s} = \gamma$ for all $s \in S$. This turns (4.34) into

$$(\mu_{k-1} - \mu_1)(\gamma + \beta - 2\alpha) = \alpha - \beta. \tag{4.35}$$

Now, obtain another sc-partition $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k)$ from $\pi$ by switching $v \in N_1 - (S \cup T \cup u)$ and $t \in N_{k-1} \cap T$. Moving $u$ from $\tilde{N}_1$ to $\tilde{N}_{k-1}$ yields

$$(\mu_{k-1} - \mu_1)(\gamma + \beta - 2\alpha) = \beta - \alpha, \tag{4.36}$$

which, together with the former equation, implies

- $\alpha = \beta = \gamma$ if $\mu_1 \neq \mu_{k-1}$, and,
- $\alpha = \beta$ if $\mu_1 = \mu_{k-1}$.

If $\mu_1 = \mu_{k-1}$, move $u$ from $N_1$ to $N_k$ in $\pi$ to obtain $\mu_1(\beta + \gamma) = 2\mu_1\alpha$, which obviously leads to $\gamma = \alpha$.

Next, we obtain a new sc-partition $\pi^* = (N_1^*, N_2^*, \ldots, N_{k-1}^*)$ from $\pi$ by removing subclique $N_k$ by distributing its nodes over the other subcliques. Comparing $g^T w^\pi = h$ with $g^T w^{\pi^*} = h$, we obtain $\alpha = \beta = \gamma = 0$.

Now, shifting $t \in \tilde{N}_1 \cap T$ from $\tilde{N}_1$ to $\tilde{N}_{k-1}$ in $\tilde{\pi}$ yields

$$g(t, \tilde{N}_1 \cap S) + g(t, \tilde{N}_1 \cap (T - t)) = g(t, \tilde{N}_{k-1} \cap S) + g(t, \tilde{N}_{k-1} \cap T). \tag{4.37}$$

Applying Lemma 4.6.4 to this equation and generalizing, we get $g_e = \sigma \in \mathbb{R}$ for all $e \in E(S, T)$ and $g_e = \omega \in \mathbb{R}$ for all $e \in E(T)$. But, then (4.37) turns into $(\mu_1 - \mu_{k-1})(\sigma + \omega) = 0$, which immediately implies that $\sigma = -\omega$ if $\mu_1 \neq \mu_{k-1}$. If $\mu_1 = \mu_{k-1}$, one can shift $t$ from $\tilde{N}_1$ to $\tilde{N}_k$ to get $\mu_1(\sigma + \omega) = 0$, which immediately implies that $\sigma = -\omega$ since $\mu_1 > 0$.

Finally, we show $g_e = -\sigma$ for all $e \in E(S)$. It is always possible to obtain a new sc-partition $\pi' = (N_1', \ldots, N_k')$ by rearranging $\pi$ such that

- $|N_1' \cap S| = 2$ if $|S| = 2$;
- $|N_1' \cap S| \geq 2$ and $|N_2' \cap S| > 0$ if $|S| \geq 3$.

Let $\mu_1' = |N_1' \cap S|$ and $\mu_{k-1}' = |N_{k-1}' \cap S|$. Shifting $s \in N_1' \cap S$ from $N_1'$ to $N_{k-1}'$ in $\pi'$ yields

$$g(s, N_1' \cap (S - s)) + (\mu_1')\sigma = g(s, N_{k-1}' \cap S) + (\mu_{k-1}' + 1)\sigma.$$

If $|S| = 2$ this equation already implies $g_e = -\sigma$ for $e \in E(S)$. If $|S| \geq 3$, one can apply Lemma 4.6.4 to infer that $g_e = -\sigma$ for all $e \in E(S)$. □

**Theorem 4.7.3.** *Suppose that $F_U - F_L \geq 2$. Suppose further that $\mathcal{P}^{lu}$ is full-dimensional and complies with* **(FD-3)** *or* **(FD-4)**. *Let $S$ and $T$ be two proper and disjoint subsets of $V$ such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \left\lceil \frac{n}{F_U} \right\rceil$. The 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k$ and*

$$|S| \leq (k - \Delta) \left\lfloor \frac{F_L}{2} \right\rfloor + \Delta \left\lfloor \frac{F_L - 1}{2} \right\rfloor - 1.$$

*Proof.* The proof is very similar to the proof of Theorem 4.7.2. □

*Remark:*

- When $\mathcal{P}^{lu}$ complies with **(FD-1)**, the triangle inequalities (i.e., $|S| = 1$ and $|T| = 2$) are facet defining if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 2$ (i.e., $n \geq 2F_L + 2$).
- When $\mathcal{P}^{lu}$ complies with **(FD-2)**, the triangle inequalities are facet defining if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 3$ (i.e., $n \geq 3F_L + 1$).
- When $\mathcal{P}^{lu}$ complies with **(FD-3)** or **(FD-4)**, the triangle inequalities are facet defining if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 3$ (i.e., $n \geq 3F_L + 1$ for **(FD-3)** and $n \geq 3F_L$ for **(FD-4)**).

**Theorem 4.7.4.** *Suppose that $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. Let $S$ and $T$ be two proper and disjoint subsets of $V$ such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and*

$$k = \begin{cases} \left\lfloor \frac{n}{F_L} \right\rfloor, & \text{if } (n \bmod F_L) = 1, \\ \\ \left\lceil \frac{n}{F_U} \right\rceil, & \text{if } (n \bmod F_U) = F_U - 1, \end{cases}$$

*The 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k$ and*

$$|S| \leq \begin{cases} \Delta \left\lfloor \frac{F_L-1}{2} \right\rfloor + (k-\Delta-1) \left\lfloor \frac{F_L}{2} \right\rfloor + \left\lfloor \frac{F_L-2}{2} \right\rfloor & \text{if } (n \bmod F_L) = 1, \\ \left\lfloor \frac{F_U-2}{2} \right\rfloor + (k-\Delta-1) \left\lfloor \frac{F_U}{2} \right\rfloor + (\Delta-1) \left\lfloor \frac{F_U-1}{2} \right\rfloor + \left\lfloor \frac{F_U-4}{2} \right\rfloor & \text{if } (n \bmod F_U) = F_U - 1. \end{cases} \tag{4.38}$$

*Proof.* Let $L = \min\{2, |S|\}$. By (4.38), there exist integer $\mu_i$ $(i = 1, \ldots, k)$ values such that

(i) $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L+1, \ldots, k$,

(ii) $\sum_{i=1}^{k} \mu_i = |S|$,

(iii) $2\mu_1 + 2 \leq \begin{cases} F_L + 1, & \text{if } (n \bmod F_L) = 1, \\ F_U, & \text{if } (n \bmod F_U) = F_U - 1; \end{cases}$ ,

(iv) $2\mu_i \leq \begin{cases} F_L, & \text{if } (n \bmod F_L) = 1, \\ F_U, & \text{if } (n \bmod F_U) = F_U - 1; \end{cases}$ for $i = 2, \ldots, k - \Delta$,

(v) $2\mu_i + 1 \leq \begin{cases} F_L, & \text{if } (n \bmod F_L) = 1, \\ F_U, & \text{if } (n \bmod F_U) = F_U - 1; \end{cases}$ for $i = k - \Delta + 1, \ldots, k - 1$,

(vi) $2\mu_k + 2 \leq \begin{cases} F_L, & \text{if } (n \bmod F_L) = 1, \\ F_U - 2, & \text{if } (n \bmod F_U) = F_U - 1. \end{cases}$

Using the fact that $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we assign $g_{e^*} = 0$ for some $e^* \in E(V - (S \cup T))$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with

- $|N_1| = F_L + 1$ and $|N_i| = F_L$ for $i = 2, \ldots, k$ if $(n \bmod F_L) = 1$,
- $|N_i| = F_U$ for $i = 1, \ldots, k-1$ and $|N_k| = F_U - 1$ if $(n \bmod F_U) = F_U - 1$,
- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta, \\ \mu_i + 1, & \text{for } i = k - \Delta + 1, \ldots, k. \end{cases}$

By Lemma 4.7.10, $w^\pi \in P_{S,T}$. Values of $\mu_1$ and $\mu_k$ allow us to apply Lemma 4.6.2 to infer $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - (S \cup T))$. But, then $g_{e^*} = 0$ implies $\alpha = 0$.

Moving $u \in N_1 - (S \cup T)$ from $N_1$ to $N_k$ in $\pi$, and proceeding in the same way as for (4.34) in the proof of Theorem 4.7.2, one can show that $g_e = \beta \in \mathbb{R}$ for all $e \in E(T, V - (S \cup T))$ and $g_e = \gamma \in \mathbb{R}$ for all $e \in E(S, V - (S \cup T))$. Moreover, continuing in the same manner, one can obtain

$$(\mu_k - \mu_1)(\gamma + \beta - 2\alpha) = \alpha - \beta, \tag{4.39}$$
$$(\mu_k - \mu_1)(\gamma + \beta - 2\alpha) = \beta - \alpha, \tag{4.40}$$

which are essentially very similar to (4.35) and (4.36). These two equalities imply $\beta = \alpha = 0$. Now, we obtain a new sc-partition $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$ from $\pi$ by switching $v \in N_k - (S \cup T)$ and $s \in N_1 \cap S$. Moving $u \in \tilde{N}_1 - (S \cup T)$ to $\tilde{N}_k$ in $\tilde{\pi}$ yields $(\mu_k - \mu_1 + 2)\gamma = 0$, which, together with (4.40) implies $\gamma = 0$.

Switching $u \in N_1 - (S \cup T)$ and $t \in N_k \cap T$ in $\pi$ yields

$$g(t, N_1 \cap S) + g(t, N_1 \cap T) = g(t, N_k \cap S) + g(t, N_k \cap (T - t)). \tag{4.41}$$

From Lemma 4.6.4, one can infer that $g_e = \sigma$ for all $e \in E(S, T)$ and $g_e = \omega$ for all $e \in E(T)$. This turns (4.41) into $(\mu_1 - \mu_k)(\sigma + \omega) = 0$. If $\mu_1 \neq \mu_k$, then we have $\sigma = -\omega$. Suppose $\mu_1 = \mu_k$. Due to restrictions on the values of $\mu_1$ and $\mu_k$, when $\mu_1 = \mu_k$, we have $|N_1 - (S \cup T)| \geq 3$. Pick $\{v, x\} \subset N_1 - (S \cup T \cup u)$ and obtain a new sc-partition $\widehat{\pi} = (\widehat{N}_1, \ldots, \widehat{N}_k)$ from $\pi$ by switching $\{v, x\}$ and $\{\widehat{s}, \widehat{t}\} \subset N_k$ where $\widehat{s} \in N_k \cap S$ and $\widehat{t} \in N_k \cap (T - t)$. Switching $u$ and $t$ in $\widehat{\pi}$ yields $(\mu_1 - \mu_k + 2)(\sigma + \omega) = 0$, which leads to $\sigma = -\omega$.

We can show that $g_e = -\sigma$ for all $e \in E(S)$ just like we do in the proof of Theorem 4.7.2. □

According to this theorem, the triangle inequalities are always facet defining for $\mathcal{P}^{lu}$ when its dimension is $\binom{n}{2} - 1$.

Another very important result of Theorem 4.7.4 is that the 2-partition inequalities are facet defining for the equi-partition polytope $\mathcal{P}^{equi}(n)$ when $n$ is odd and certain other conditions are satisfied. We highlight this result, which is new to the literature, by means of the following corollary.

**Corollary 4.7.1.** *Consider the equi-partition polytope $\mathcal{P}^{equi}(n)$ where $n = 2p+1$ (i.e., $n$ is odd). Let $S$ and $T$ be two disjoint subsets of $V$ such that $|T| = |S|+1$ and*

$$|S| \leq \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p-2}{2} \right\rfloor.$$

*Then, the 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{equi}(n)$.*

This corollary shows in particular that the triangle inequalities are facet defining for $\mathcal{P}^{equi}(n)$ when $n$ is odd.

**Theorem 4.7.5.** *Suppose that $F_L \geq 4$. Suppose further that $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. Let $S$ and $T$ be two proper and disjoint subsets of $V$ such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \frac{n}{F_L}$. The 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k - 1$ and*

$$|S| \leq 2 \left\lfloor \frac{F_L - 2}{2} \right\rfloor + (k - \Delta - 2) \left\lfloor \frac{F_L}{2} \right\rfloor + (\Delta - 1) \left\lfloor \frac{F_L - 1}{2} \right\rfloor \tag{4.42}$$

*Proof.* Let $L = \min\{2, |S|\}$. There exist integer $\mu_i$ $(i = 1, \ldots, k-1)$ values such that

(i) $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L+1, \ldots, k-1$,

(ii) $\sum_{i=1}^{k-1} \mu_i = |S|$,

(iii) $2\mu_1 + 2 \leq F_L$,

(iv) $2\mu_i \leq F_L$ for $i = 2, \ldots, k - \Delta - 1$,

(v) $2\mu_i + 1 \leq F_L$ for $i = k - \Delta, \ldots, k - 2$,

(vi) $2\mu_{k-1} + 2 \leq F_L$.

By part $(iii)$ of Theorem 4.4.2, we know $(k-1)F_U = kF_L$ where $k = \frac{n}{F_L} = \frac{n}{F_U} + 1$. Let $\pi = (N_1, N_2, \ldots, N_k)$ be an sc-partition with $|N_i| = F_L$ for $i = 1, 2, \ldots, k$. Suppose the following conditions hold for $\pi$:

- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k-1$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta - 1, \\ \mu_i + 1, & \text{for } i = k - \Delta, \ldots, k - 1. \end{cases}$

By Lemma 4.7.10, $w^\pi \in P_{S,T}$.

Without loss of generality, suppose $n \notin S \cup T$. Consider the set of edges $Q = \{\{1, n\}, \{2, n\}, \ldots, \{n-1, n\}\}$. Recall the hyperplanes $H_1, H_2, \ldots, H_{n-1}$

we considered belonging to $M(\mathcal{P}^{lu})$ in Theorem 4.6.3. The coefficients in the equations representing $H_1, \ldots, H_{n-1}$ of the edges in $Q$ are as follows:

$$\begin{array}{c} \{1,n\} \quad \{2,n\} \quad \{3,n\} \quad \ldots \quad \{n-2,n\} \quad \{n-1,n\} \\ \left[\begin{array}{ccccccc} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ -1 & -1 & -1 & \ldots & -1 & 0 \end{array}\right] \end{array}.$$

This matrix is non-singular. Hence, set $g_e = 0$ for all $e \in Q$.

Suppose that $n \in N_1$ in $\pi$. Pick $u \in N_1 - (S \cup T)$ and $\{x, y\} \subset N_k$ arbitrarily (note that $N_k \cap (S \cup T) = \emptyset$). By Lemma 4.6.3, we have $g_{n,u} + g_{x,y} = g_{n,x} + g_{u,y}$. Since we have already set $g_{n,u} = g_{n,x} = 0$, this implies $g_{x,y} = g_{u,y}$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - (S \cup T \cup n)$.

Now, pick an sc-partition $\pi^1 = (N_1^1, N_2^1, \ldots, N_k^1)$ such that $n \in N_k^1$, $u \in N_1^1 - (S \cup T)$, $|N_i^1 \cap S| = |N_i \cap S|$ and $|N_i^1 \cap T| = |N_i \cap T|$ for $i = 1, 2, \ldots, k$. Switching $n$ and $u$ in $\pi^1$ gives

$$g(u, N_1^1 \cap S) + g(u, N_1^1 \cap T) + \alpha(F_L - 2\mu_1 - 1) = \alpha(F_L - 1). \tag{4.43}$$

Applying Lemma 4.6.4, we infer that $g_e = \beta \in \mathbb{R}$ for all $e \in E(V - (S \cup T), T)$ and $g_e = \gamma \in \mathbb{R}$ for all $e \in E(V - (S \cup T), S)$. Then (4.43) turns into

$$\beta + \gamma = 2\alpha. \tag{4.44}$$

Now, consider the sc-partition $\pi^2 = (N_1^2, \ldots, N_k^2)$ obtained from $\pi^1$ by switching $t \in N_{k-1}^1 \cap T$ and $x \in N_k^1 - (S \cup T)$. Switching $u$ and $n$ in $\pi^2$ yields

$$(F_L - 2\mu_1 - 1)\alpha + \mu_1\beta + \mu_1\gamma = (F_L - 2)\alpha + \beta.$$

Comparing this equation with (4.44) yields $\beta = \gamma = \alpha$.

Now, consider the sc-partition $\pi^*$ obtained from $\pi$ by distributing $N_k$ over the other subsets. Comparing $g^T w^\pi = h$ with $g^T w^{\pi^*} = h$ gives $\alpha = \beta = \gamma = 0$.

Now, switching $t \in N_{k-1} \cap T$ and $u \in N_1 - (S \cup T)$ in $\pi$ gives

$$g(t, N_1 \cap S) + g(t, N_1 \cap T) = g(t, N_{k-1} \cap S) + g(t, N_{k-1} \cap (T - t)). \tag{4.45}$$

This equality and Lemma 4.6.4 leads us to $g_e = \sigma \in \mathbb{R}$ for all $e \in E(S, T)$ and $g_e = \omega \in \mathbb{R}$ for all $e \in E(T)$. Then, (4.45) turns into

$$(\mu_1 - \mu_{k-1})(\sigma + \omega) = 0. \tag{4.46}$$

If $\mu_1 \neq \mu_{k-1}$ then we have $\sigma = -\omega$. Suppose $\mu_1 = \mu_{k-1}$. Then, obtain a new sc-partition $\pi' = (N'_1, \ldots, N'_k)$ by switching $\{x, y\} \subset N_k$ and $\{s', t'\} \subset N_{k-1}$ in $\pi$ where $s' \in N_{k-1} \cap S$ and $t' \in N_{k-1} \cap T$. Switching $t \in N'_{k-1} \cap T$ and $u \in N'_1 - (S \cup T)$ yields $(\mu_1 - \mu_{k-1} + 1)(\sigma + \omega) = 0$, which, together with (4.46) implies $\sigma = -\omega$.

Finally, we show that $g_e = -\sigma$ for all $e \in E(S)$. Pick an sc-partition $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$ where $n \in N_{k-1}$ and

- $|N_1 \cap S| = 2$ if $|S| = 2$,
- $|N_1 \cap S| \geq 2$ and $|N_2 \cap S| > 0$ if $|S| \geq 3$.

Note that, we can always find such an sc-partition $\tilde{\pi}$. Switching $s \in \tilde{N}_1 \cap S$ and $n$ in $\tilde{\pi}$ yields

$$g(s, \tilde{N}_1 \cap (S - s)) + g(s, \tilde{N}_1 \cap T) = g(s, \tilde{N}_{k-1} \cap S) + g(s, \tilde{N}_{k-1} \cap T).$$

Applying Lemma 4.6.4 to this equality yields $g_e = -\sigma$ for all $e \in E(S)$. □

*Remark:* When $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$, the triangle inequalities are facet defining for $\mathcal{P}^{lu}$ if $n \geq 3F_L$.

**Theorem 4.7.6.** *Suppose $F_L \geq 4$. Suppose also that $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. Let $S$ and $T$ be two proper and disjoint subsets of $V$ such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \frac{n}{F}$ where*

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

*The 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k$ and*

$$|S| \leq (k - \Delta - 1) \left\lfloor \frac{F}{2} \right\rfloor + (\Delta - 1) \left\lfloor \frac{F-1}{2} \right\rfloor + \left\lfloor \frac{F-2}{2} \right\rfloor + \left\lfloor \frac{F-3}{2} \right\rfloor. \quad (4.47)$$

*Proof.* Let $L = \min\{2, |S|\}$. There exist integer $\mu_i$ $(i = 1, \ldots, k)$ values such that

(i) $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L+1, \ldots, k$,

(ii) $\sum_{i=1}^{k} \mu_i = |S|$,

(iii) $2\mu_1 + 2 \leq F$,

(iv) $2\mu_i \leq F$ for $i = 2, \ldots, k - \Delta$,

(v) $2\mu_i + 1 \leq F$ for $i = k - \Delta + 1, \ldots, k - 1$,

(vi) $2\mu_k + 3 \leq F$.

Without loss of generality, assume that $\{n, n-1, n-2\} \subset V - (S \cup T)$. Let $Q = \{\{1, n\}, \{2, n\}, \ldots, \{n-1, n\}, \{n-2, n-1\}\}$. When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$, $M(\mathcal{P}^{lu})$ consists of hyperplanes $H_1, \ldots, H_n$ where $H_u$ is defined as

$$w(\delta(u)) = F - 1.$$

The columns of the edges belonging to $Q$ in the equalities corresponding to $H_1, \ldots, H_n$ are as follows:

$$\begin{array}{c} \{1,n\} \quad \{2,n\} \quad \{3,n\} \quad \ldots \quad \{n-2,n\} \quad \{n-1,n\} \quad \{n-2,n-1\} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 0 \end{array}\right] \end{array} .$$

This matrix is non-singular. Hence, we set $g_e = 0$ for all $e \in Q$.

Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ such that

- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta, \\ \mu_i + 1, & \text{for } i = k - \Delta + 1, \ldots, k; \end{cases}$
- $n \in N_1$.

By Lemma 4.7.10, $w^\pi \in P_{S,T}$.

Pick arbitrarily $u \in N_1 - (S \cup T)$ and $\{x, y\} \subset N_k - (S \cup T)$. Lemma 4.6.3 implies that $g_{u,n} + g_{x,y} = g_{u,y} + g_{n,x}$. Since $\{u, n\}$ and $\{n, x\}$ are in $Q$, we infer $g_{x,y} = g_{u,y}$, which generalizes into $g_e = 0$ for all $e \in E(V - (S \cup T \cup n))$ since we have already set $g_{n-2,n-1} = 0$.

Now, switch $x$ and $n$ in $\pi$ to obtain

$$g(x, N_1 \cap S) + g(x, N_1 \cap T) = g(x, N_k \cap S) + g(x, N_k \cap T). \tag{4.48}$$

Lemma 4.6.4 implies that $g_e = \beta \in \mathbb{R}$ for all $e \in E(T, V - (S \cup T))$ and $g_e = \gamma \in \mathbb{R}$ for all $e \in E(S, V - (S \cup T))$. Then, (4.48) turns into

$$(\mu_k - \mu_1 + 1)\beta + (\mu_k - \mu_1)\gamma = 0. \tag{4.49}$$

Now, we obtain a new sc-partition $\pi^1 = (N_1^1, \ldots, N_k^1)$ from $\pi$ by switching $t \in N_k \cap T$ and $u \in N_1 - (S \cup T)$. Switching $x$ and $n$ in $\pi^1$ yields

$$(\mu_k - \mu_1 - 1)\beta + (\mu_k - \mu_1)\gamma = 0.$$

Solving this equation simultaneously with (4.49) gives $\beta = 0$. Now, obtain yet another sc-partition $\pi^2 = (N_1^2, \ldots, N_k^2)$ from $\pi$ by switching $s \in N_1 \cap S$ and $y \in N_k - (S \cup T)$. Switching $x$ and $n$ in $\pi^2$ yields

$$(\mu_k - \mu_1 + 2)\gamma = 0,$$

which, when compared with (4.49), implies $\gamma = 0$.

Switching $t \in N_k \cap T$ and $u \in N_1 - (S \cup T)$ in $\pi$ gives

$$g(t, N_1 \cap S) + g(t, N_1 \cap T) = g(t, N_k \cap S) + g(t, N_k \cap (T - t)). \tag{4.50}$$

Again, applying Lemma 4.6.4, we infer $g_e = \sigma$ for all $e \in E(S, T)$ and $g_e = \omega$ for all $e \in E(T)$. Then, (4.50) can be rewritten as

$$(\mu_k - \mu_1)(\sigma + \omega) = 0. \tag{4.51}$$

If $\mu_k \neq \mu_1$, we have $\sigma = -\omega$. Suppose $\mu_k = \mu_1$. Obtain a new sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ by switching $\{s^*, t^*\} \subset N_1$ and $\{x, y\} \subset N_k - (S \cap T)$, where $s^* \in N_1 \cap S$ and $t^* \in N_1 \cap T$. Switching $t \in N_k^* \cap T$ and $u \in N_1^* - (S \cup T)$ yields

$$(\mu_k - \mu_1 + 2)(\sigma + \omega) = 0,$$

which implies $\omega = -\sigma$. In a very similar manner, one can also show that $g_e = -\sigma$ for all $e \in E(S)$. □

This theorem suggests that, under certain conditions, the 2-partition inequalities and in particular the triangle inequalities are facet defining for the equi-partition polytope $\mathcal{P}^{equi}(n)$ for even $n$, and the $k$-way equi-partition polytope ($k \geq 3$). We highlight these two important results in the following corollaries. To our knowledge, these results as well are new to the literature.

**Corollary 4.7.2.** *Consider the equi-partition polytope $\mathcal{P}^{equi}(n)$ where $n = 2p$ (i.e., $n$ is even). Let $S$ and $T$ be two disjoint and proper subsets of $V$ such that $|T| = |S| + 1$ and*

$$|S| \leq \left\lfloor \frac{p-2}{2} \right\rfloor + \left\lfloor \frac{p-3}{2} \right\rfloor .$$

*Then, the 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{equi}(n)$.*

**Corollary 4.7.3.** *Consider the $k$-way equi-partition polytope $\mathcal{P}^{k-way}(n,k)$ where $k \geq 3$ and $n = kF$. Let $S$ and $T$ be two disjoint and proper subsets of $V$ such that $\Delta = |T| - |S| < k$ and (4.47) is satisfied. Then, the 2-partition inequality (4.19) is facet defining for $\mathcal{P}^{k-way}(n,k)$.*

## 4.8 The lower general clique inequalities

Let $Q$ be, throughout this section, a subset of $V$ such that $|Q| > \left\lfloor \frac{n}{F_L} \right\rfloor$ and $\left(|Q| \bmod \left\lfloor \frac{n}{F_L} \right\rfloor\right) \neq 0$. Likewise, let $k$, $r$, $p$ and $q$ be defined, throughout this section, in the following manner:

- $k = \left\lfloor \frac{n}{F_L} \right\rfloor$;
- $r = n \bmod F_L$, i.e., $n = kF_L + r$,
- $p = \left\lfloor \frac{|Q|}{k} \right\rfloor$ ($p \geq 1$ always due to choice of $|Q|$), and,
- $q = (|Q| \bmod k)$ ($q > 0$ always due to choice of $|Q|$).

Consider the following inequality:

$$w(E(Q)) \geq (k-q)\binom{p}{2} + q\binom{p+1}{2}. \tag{4.52}$$

These inequalities are first introduced by Chopra and Rao [5] for a graph partitioning problem. Chopra and Rao call these inequalities 'the general clique inequalities'. But in this context, we prefer using the term 'the lower general clique inequalities' because in Section 4.9 we introduce another set of valid inequalities, 'the upper general clique inequalities', which have the same left hand side as and the opposite sense of (4.52).

Chopra and Rao [5] introduce these inequalities for the polytope corresponding to the problem of partitioning $G$ into at most $\bar{l}$ subcliques without any size restrictions, i.e., they deal with the following polytope:

$$conv\{w^\pi | \pi = (N_1, \ldots, N_l), l \leq \bar{l}, 1 \leq |N_i| \leq n \text{ for } i = 1, \ldots, l\}.$$

Obviously, when $\bar{l} = k$, $\mathcal{P}^{lu}$ is a subset of this polytope. And, this means that (4.52) is valid for $\mathcal{P}^{lu}$ as well.

Ji and Mitchell [15] prove the conditions for these inequalities to be facet defining for $\mathcal{P}^l(n, F_L)$. Theorems (4.8.1)-(4.8.3) of this section are devoted to proving the conditions that make (4.52) facet defining for $\mathcal{P}^{lu}$. We denote by $P_Q$ the face defined by a lower general clique inequality written for a subset $Q$, i.e.,

$$P_Q = \left\{ w \in \mathcal{P}^{lu} | w(E(Q)) = (k-q)\binom{p}{2} + q\binom{p+1}{2} \right\}.$$

As usual, we assume that there exists a valid inequality $g^T w = h$ such that

$$P_Q \subseteq \{w \in \mathcal{P}^{lu} | g^T w = h\}.$$

We aim to show that $g^T w = h$ can be written as a linear combination of (4.52) and the hyperplane equations in $M(\mathcal{P}^{lu})$, if any. Before moving on to the theorems, we give a simple characterization of $P_Q$ in the following lemma.

**Lemma 4.8.1.** *For any sc-partition $\pi = (N_1, \ldots, N_l)$, $w^\pi \in P_Q$ if and only if*

*(i)* $l = k$,

*(ii)* $|\{i : |N_i \cap Q| = p\}| = k - q$, *and,*

*(iii)* $|\{i : |N_i \cap Q| = p + 1\}| = q$.

*Proof.* Sufficiency is obvious. We prove necessity. Note that any sc-partition $\pi^*$ that minimizes $w^\pi(E(Q))$ necessarily has $k$ subcliques, because, minimizing $w^\pi(E(Q))$ requires that the nodes of $Q$ be dispersed to the maximal number of subcliques.

Secondly, for $\pi^1 = (N_1^1, \ldots, N_k^1)$ with $w^{\pi^1} \in P_Q$, we necessarily have

$$\big||N_i^1 \cap Q| - |N_j^1 \cap Q|\big| \leq 1 \qquad \forall i, j = 1, \ldots, k. \tag{4.53}$$

To see this, suppose that $|N_{i^*}^1 \cap Q| - |N_{j^*}^1 \cap Q| \geq 2$ for some $i^*$ and $j^*$. Now, obtain another sc-partition $\pi^2 = (N_1^2, \ldots, N_k^2)$ from $\pi^1$ by switching $x \in N_{i^*}^1 \cap Q$

and $y \in N^1_{j^*} - Q$. Clearly, $w^{\pi^1}(E(Q)) > w^{\pi^2}(E(Q))$. This contradicts the assumption that $w^{\pi^1}$ minimizes $w^{\pi}(E(Q))$.

The conditions 2 and 3 follow from (4.53) and the fact that any $\pi$ with $w^{\pi} \in P_Q$ has $k$ subcliques. □

**Theorem 4.8.1.** *Suppose that the polytope $\mathcal{P}^{lu}$ is full-dimensional and suppose $F_U - F_L \geq 2$.*

*(i) When $r = 0$, (4.52) is valid for $\mathcal{P}^{lu}$, but not facet-defining.*

*(ii) When $r = 1$, (4.52) is facet-defining for $\mathcal{P}^{lu}$ if and only if $Q = V$.*

*(iii) When $r > 1$, (4.52) is facet-defining for $\mathcal{P}^{lu}$.*

*Proof.* (i) When $r = 0$,

$$P_Q \subseteq \{w \in \mathcal{P}^{lu} | w(\delta(u)) = F_L - 1\}$$

for any $u \in V$.

(ii) When $r = 1$, note that, all sc-partitions with $k$ subcliques in $\mathcal{P}^{lu}$ are composed of $k-1$ subcliques with size $F_L$ and one subclique with size $F_L + 1$.

We first show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(Q)$. Pick arbitrarily three nodes $u, v, x \in Q$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^{\pi} \in P_Q$, $\{u, v\} \subset N_1$, $x \subset N_2$, $|N_1| = F_L + 1$, $|N_i| = F_L$ for $i = 2, \ldots, k$, $|N_1 \cap Q| = p + 1$ and $|N_2 \cap Q| = p$. Applying Lemma 4.6.2 to $\pi$ yields $g_{u,v} = g_{u,x}$. Since $u$, $v$ and $x$ are arbitrary, we have $g_e = \alpha \in \mathbb{R}$ for all $e \in E(Q)$.

Secondly, we show that $g_e = \beta \in \mathbb{R}$ for all $e \in E - E(Q)$. Pick arbitrarily $u \in Q$, $v, x \notin Q$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^{\pi} \in P_Q$, $\{u, v\} \subset N_1$, $x \in N_2$, $|N_1| = F_L + 1$, $|N_i| = F_L$ for $i = 2, \ldots, k$, $|N_1 \cap Q| = p + 1$ and $|N_2 \cap Q| = p$. Using Lemma 4.6.2, one can show $g_{v,x} = g_{v,u} = g_{u,x} = \beta \in \mathbb{R}$. Arbitrariness of $u$, $v$ and $x$ implies $g_e = \beta$ for all $e \in E - E(Q)$.

We infer that

$$P_Q \subseteq \{w \in \mathcal{P}^{lu} | \alpha w(E(Q)) + \beta w(E - E(Q)) = C\}$$

for some constants $\alpha$, $\beta$ and $C$. Then, (4.52) is facet defining for $\mathcal{P}^{lu}$ if and only if $Q = V$.

(iii) When $r > 1$, we know from Theorem 4.5.1 that, all the sc-partitions with $k$ subcliques are 2-loose sc-partitions (note that $F_U - F_L \geq 2$). We first show that $g_e = 0$ for all $e \in E(Q, V-Q)$. Pick two arbitrary nodes $u \in Q$ and $v \notin Q$. Pick a 2-loose sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^\pi \in P_Q$, $F_L < |N_1| < F_U$, $F_L < |N_2| < F_U$, $u \in N_1$, $v \in N_2$, $|N_1 \cap Q| = p+1$ and $|N_2 \cap Q| = p$. Using Lemma 4.6.1, we get $g(u, v) = 0$. Since $u$ and $v$ are picked arbitrarily, we can infer $g_e = 0$ for all $e \in E(Q, V-Q)$.

Now, picking the same sc-partition $\pi$ and two arbitrary nodes $x \in N_1 - Q$ and $y \in N_2 - Q$, and applying Lemma 4.6.1, one can infer that $g(x, y) = 0$, i.e., $g_e = 0$ for all $e \in E(V-Q)$ since $x$ and $y$ are arbitrary.

Finally, pick three nodes $u, v, x \in Q$. Consider the sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^\pi \in P_Q$, $u \in N_1$, $v, x \in N_2$, $F_L < |N_1| < F_U$, $F_L < |N_2| < F_U$, $|N_1 \cap Q| = p$ and $|N_2 \cap Q| = p+1$. Applying Lemma 4.4.2 to $\pi$, $u$, $v$ and $x$ by setting $P$ to $P_Q$, one can infer that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(Q)$.

□

**Theorem 4.8.2.** *If $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, the lower general clique inequality (4.52) is facet defining for $\mathcal{P}^{lu}$.*

*Proof.* Using $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we set $g'_e = 0$ for one $e' \in E - E(Q)$. Proceeding like in the proof of part 2 in Theorem 4.8.1, we show $g_e = \beta$ for all $e \in E - E(Q)$ and $g_e = \alpha$ for all $e \in E(Q)$. But, $g'_e = 0$ implies $\beta = 0$. □

Note that, Theorem 4.8.2 proves as well that (4.52) is facet defining for $\mathcal{P}^{equi}(n)$. This result is already proved by Chopra and Rao [5]. Theorem 4.8.2 proposes just an alternative proof for this result.

**Proposition 4.8.1.** *Suppose $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. The general clique inequality (4.52) is not facet defining for $\mathcal{P}^{lu}$.*

*Proof.* In this case,

$$P_Q \subseteq \{w \in \mathcal{P}^{lu} | w(\delta(u)) = F_L - 1\}$$

for any $u \in V$. □

**Theorem 4.8.3.** *Suppose $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. Let*

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

*The lower general clique inequality (4.52) is facet defining for $\mathcal{P}^{lu}$ if and only if $p < F - 1$.*

*Proof.* First note that, when $p = F - 1$, $n - |Q| = k - q$. In any $\pi$ such that $w^\pi \in \mathcal{P}^{lu}$, the nodes in $V - Q$ would be dispersed among $k - q$ subcliques. Hence,

$$P_Q \subseteq \{w \in \mathcal{P}^{lu} | w_{u,v} = 0\}$$

for all $u, v \in V - Q$. Suppose $p < F - 1$. Without loss of generality, let $Q = \{1, 2, \ldots, |Q|\}$ and $V - Q = \{|Q| + 1, \ldots, n\}$. Consider the set $S = \{\{1, n\}, \{2, n\}, \ldots, \{n - 1, n\}, \{1, n - 1\}\}$. We know that $M(\mathcal{P}^{lu})$ consists of the hyperplanes $H_1, \ldots, H_n$ where $H_u$ is represented by the equation

$$w(\delta(u)) = F - 1.$$

In the following matrix, the coefficients of the edges belonging to $S$ in the equations representing $H_1$ to $H_n$ are displayed:

$$\begin{array}{r} \\ H_1 \to \\ H_2 \to \\ H_3 \to \\ \vdots \\ H_{n-2} \to \\ H_{n-1} \to \\ H_n \to \end{array} \begin{array}{c} \begin{array}{ccccccc} \{1,n\} & \{2,n\} & \{3,n\} & \ldots & \{n-2,n\} & \{n-1,n\} & \{1,n-1\} \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 0 \end{array}\right] \end{array} .$$

This matrix is non-singular. With reference to Theorem 3.6 in Section I.4.3 of [24], we use this fact to set $g_e = 0$ for all $e \in S$.

Now, pick an arbitrary node $u \in Q$ and an sc-partition $\pi = (N_1, \ldots, N_k)$ such that $w^\pi \in P_Q$. Suppose that $u \in N_1$, $n \in N_2$, $|N_1 \cap Q| = p + 1$ and $|N_2 \cap Q| = p$. Switching $u$ and $n$, we get

$$g(u, (N_1 - u) \cap Q) + g(u, (N_1 - u) - Q) = g(u, (N_2 - n) \cap Q) + g(u, (N_2 - n) - Q).$$

Applying Lemma 4.6.4 to this equation yields $g_e = \alpha \in \mathbb{R}$ for $e \in E(Q)$ and $g_e = \beta$ for $e \in E(Q, (V - n) - Q)$. But, then since we have already fixed $g_{1,n-1} = 0$, we immediately have $\beta = 0$.

Now, pick an arbitrary node $x \in (V - n) - Q$. Pick an sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ such that $w^{\pi^*} \in P_Q$, $x \in N_1^*$, $n \in N_2^*$, $|N_1 \cap Q| = p + 1$ and $|N_2 \cap Q| = p$. Switching $x$ and $n$ yields

$$g(x, (N_1^* - x) - Q) = g(x, (N_2^* - n) - Q). \tag{4.54}$$

Applying Lemma 4.6.4 to this equation gives $g_e = \gamma$ for all $e \in E((V-n)-Q)$. Plugging this into (4.54), we infer $\gamma = 0$. □

Theorem 4.8.3 suggests that (4.52) is facet defining for $\mathcal{P}^{k-way}(n,k)$ as well. We highlight this result, which is new to the literature, in the following corollary.

**Corollary 4.8.1.** *The lower general clique inequality (4.52) is facet defining for $\mathcal{P}^{k-way}(n,k)$ if and only if*

*(i)* $(|Q| \bmod k) \neq 0$*, and,*

*(ii)* $|Q| < n - k$.

## 4.9 The Upper General Clique Inequalities

In this section we investigate the upper general clique inequalities, and prove conditions for which these inequalities become facet defining for $\mathcal{P}^{lu}$. A special case of these inequalities were first introduced by Sørensen [31] for $\mathcal{P}^{u}(n, F_U)$. Here we consider a more general version of Sørensen's inequalities.

Let $Q \subseteq V$. Consider the $\phi_i^U$ values computed by means of the formula (4.14) in Section 4.6.2. Let $k_Q$ and $n_Q$ be defined throughout this section as follows:

- $k_Q = \max\{i | \sum_{l=1}^{i} \phi_l^U \leq |Q|\}$, and,
- $n_Q = |Q| - \sum_{l=1}^{k_Q} \phi_l^U$.

Contrary to the lower general clique inequality (4.52), the following upper general clique inequality aims to bound $w(E(Q))$ from above:

$$w(E(Q)) \leq \sum_{l=1}^{k_Q} \binom{\phi_l^U}{2} + \binom{n_Q}{2} \qquad \forall Q \subseteq V. \tag{4.55}$$

Note that in the statement of this inequality, we adopt the convention that $\binom{0}{2} = \binom{1}{2} = 0$.

**Lemma 4.9.1.** *The upper general clique inequality (4.55) is valid for $\mathcal{P}^{lu}$.*

*Proof.* The set $Q$ should be split into the largest subcliques possible in order to maximize $w(E(Q))$. This can be achieved by putting $\phi_1^U$ nodes of $Q$ in one subclique, $\phi_2^U$ nodes in another subclique, and so on. □

Throughout this section, $\mathcal{P}_Q$ stands for the face defined by (4.55). Next, we give a lemma that characterizes the sc-partitions whose characteristic vectors lie on $P_Q$. We skip the proof because it is trivial.

**Lemma 4.9.2.** *For an sc-partition $\pi = (N_1, \ldots, N_k)$, $w^\pi \in P_Q$ if and only if $|N_i| = |N_i \cap Q| = \phi_i^U$ for $i = 1, \ldots, k_Q$ and $|N_i| \geq |N_i \cap Q| = n_Q$ for $i = k_Q + 1$.*

Now, we continue with two lemmas that state some conditions on $Q$ determining if (4.55) is or is not facet defining.

**Lemma 4.9.3.** *Suppose $F_U - F_L \geq 2$ and $Q = V$. The upper general clique inequality (4.55) is facet defining for $\mathcal{P}^{lu}$ if and only if it is full-dimensional and complies with* **(FD-1)**, **(FD-3)** *or* **(FD-4)**.

*Proof.* When

- $\mathcal{P}^{lu}$ is full-dimensional and it complies with **(FD-2)**, or,
- $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$,

we have

$$P_Q \subseteq \{w \in \mathcal{P}^{lu} | w(\delta(u)) = F_U - 1\} \qquad \forall u \in V.$$

When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$,

$$P_Q \subseteq \{w \in \mathcal{P}^{lu} | w(\delta(u)) = F - 1\} \qquad \forall u \in V$$

where

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

And, finally when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, $P_Q = \mathcal{P}^{lu}$.

When $\mathcal{P}^{lu}$ is full-dimensional and it complies with one of **(FD-1)**, **(FD-3)** or **(FD-4)**, there always exist $\phi_{i^*}^U$ and $\phi_{j^*}^U$ such that $\phi_{i^*}^U \neq \phi_{j^*}^U$. When $\phi_{i^*}^U = \phi_{j^*}^U + 1$, using Lemma 4.6.2, one can easily show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E$.

Suppose that $\phi_{i^*}^U \geq \phi_{j^*}^U + 2$. Pick four arbitrary nodes $x, y, z, t$ and an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ where $k = \left\lceil \frac{n}{F_U} \right\rceil$, $\{x, z\} \subset N_{i^*}$, $\{y, t\} \subset N_{j^*}$, $|N_i| = \phi_i^U$ for $i = 1, 2, \ldots, k$. Applying Lemma 4.6.3 to $\pi$ yields

$$g_{x,z} - g_{y,z} = g_{x,t} - g_{y,t} = \alpha \in \mathbb{R} \qquad \forall z, t \in V - \{x, y\}. \tag{4.56}$$

But, switching $x$ and $y$ in $\pi$, we obtain

$$g(x, N_{i^*} - x) + g(y, N_{j^*} - y) = g(x, N_{j^*} - y) + g(y, N_{i^*} - x). \tag{4.57}$$

Plugging (4.56) into (4.57), we get $\alpha = 0$. Since $x, y, z$ and $t$ are arbitrary, we have $g_e = \gamma \in \mathbb{R}$ for all $e \in E$. □

**Lemma 4.9.4.** *Suppose $F_U - F_L \geq 2$. Let $Q$ be a proper subset of $V$. The upper general clique inequality (4.55) is not facet defining for $\mathcal{P}^{lu}$ if*

*(i)* $n_Q = 0$, *or,*

*(ii)* $|Q| \leq \phi_1^U + 1$.

*When $\mathcal{P}^{lu}$ is full-dimensional, (4.55) is not facet defining for $\mathcal{P}^{lu}$ if*

*(iii)* $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ *is not full-dimensional.*

*Proof.* (i) When $n_Q = 0$, $P_Q$ lies in the intersection of the hyperplanes $w_{u,v} = 0$ for all $u \in Q$ and $v \in V - Q$.

(ii) When $|Q| \leq \phi_1^U$, $P_Q$ lies in the intersection of the hyperplanes $w_{u,v} = 1$ for all $u, v \in Q$. When $|Q| = \phi_1^U + 1$, $P_Q$ is contained in the face defined by the cycle inequalities of Section 4.10. More precisely, let $Q = \{q_1, q_2, \ldots, q_{\phi_1^U + 1}\}$; the face defined by the valid inequality $g^T w \leq \phi_1^U - 1$ with

$$g_{u,v} = \begin{cases} 1, & \text{if } u = q_i \text{ and } v = q_{i+1}, \text{ for all } i = 1, 2, \ldots, \phi_1^U + 1, \\ 0, & \text{otherwise,} \end{cases}$$

where all indices are taken modulo $\phi_1^U + 1$, contains $P_Q$.

(iii) Let $\tilde{n} = n - \sum_{i=1}^{k_Q} \phi_i^U$ and $E_{\tilde{n}}$ be the edge set of the complete graph defined on $\tilde{n}$ nodes. When $\dim(\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)) = \binom{\tilde{n}}{2} - \tilde{n}$, $P_Q$ consists of $w^\pi$ such that $\pi = (N_1, \ldots, N_k)$, $|N_1| = |N_1 \cap Q| = \phi_i^U$ for $i = 1, \ldots, k_Q$,

$|N_{k_Q+1} \cap Q| = n_Q$, $|N_i| = F$ for $i = k_Q + 1, \ldots, k$ where $F$ is equal to either $F_L$ or $F_U$. This implies that $P_Q$ is contained in the intersection of the hyperplanes $w(\delta(u)) = F - 1$ for all $u \in V - Q$.

Now, suppose $\dim(\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)) = \binom{\tilde{n}}{2} - (\tilde{n} - 1)$. All points $w^\pi$ in $P_Q$ where $\pi = (N_1, N_2, \ldots, N_k)$ have $|N_i| = |N_i \cap Q| = \phi_i^U$ for $i = 1, \ldots, k_Q$ and $|N_{k_Q+1} \cap Q| = n_Q$. Let $P_1$ consist of points $w^\pi \in P_Q$ such that $|N_i| = F_L$ for $i = k_Q + 1, \ldots, k$; and, let $P_2$ consist of points $w^\pi \in P_Q$ such that $|N_i| = F_U$ for $i = k_Q + 1, \ldots, k$. Since $P_Q = P_1 \cup P_2$, obviously, $P_Q$ is contained in hyperplanes $w(\delta(u)) - w(\delta(v)) = 0$ for all $u, v \in V - Q$, which is in accordance with Proposition 4.5.4.

Finally, suppose $\dim(\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)) = \binom{\tilde{n}}{2} - 1$. Suppose that $\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)$ is contained in an hyperplane $w(E_{\tilde{n}}) = C$ for some constant $C \in \mathbb{R}$. Then $P_Q$ is contained in $w(E) = C + \sum_{i=1}^{k_Q} \binom{\phi_i^U}{2}$, where $E$, as before, denotes the edge set of the complete graph defined on $n$ nodes.

□

The following Theorem 4.9.1 states the conditions that make (4.55) facet defining for a full-dimensional $\mathcal{P}^{lu}$. With reference to Lemma 4.9.4, we assume that $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ is full-dimensional, and hence by Theorem 4.5.1, it falls into one of four categories **(FD-1)**,**(FD-2)**-**(FD-4)**.

**Theorem 4.9.1.** *Suppose $\mathcal{P}^{lu}$ is full-dimensional and $F_U - F_L \geq 2$. Let $Q$ be a subset of $V$ which does not comply with any of the conditions 1-3 of Lemma 4.9.4 (i.e., $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ is full-dimensional, $n_Q > 0$ and $|Q| > \phi_1^U + 1$).*

*(i) When $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-1)**, *(4.55) is facet defining for $\mathcal{P}^{lu}$ if $1 \leq n_Q \leq F_L$.*

*(ii) When $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-2)**, *(4.55) is facet defining for $\mathcal{P}^{lu}$ if and only if*

- *$1 \leq n_Q < F_L$ and $k_Q \leq \left\lceil \frac{n}{F_U} \right\rceil - 1$, or,*
- *$n_Q = F_L$ and $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$.*

*(iii) When $F_U - F_L = 2$ and $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-3)**, *(4.55) is facet defining for $\mathcal{P}^{lu}$ if and only if*

- $1 \leq n_Q < F_L$ *and* $k_Q \leq \left\lceil \frac{n}{F_U} \right\rceil - 1$, *or,*
- $n_Q = F_L$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$, *or,*
- $n_Q = F_L + 1$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 2$.

*(iv) When* $F_U - F_L > 2$ *and* $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ *complies with* **(FD-3)**, *(4.55) is facet defining for* $\mathcal{P}^{lu}$ *if and only if*

- $1 \leq n_Q < F_L$ *and* $k_Q \leq \left\lceil \frac{n}{F_U} \right\rceil - 1$, *or,*
- $F_L \leq n_Q \leq F_L + 1$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$.

*(v) When* $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ *complies with* **(FD-4)**, *(4.55) is facet defining for* $\mathcal{P}^{lu}$ *if and only if* $1 \leq n_Q \leq F_L$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$.

*Proof.* We prove the cases 1 and 2 only. The proofs of parts 3, 4 and 5 are similar.

(i) Pick two arbitrary nodes $u, v \in V - Q$ and an sc-partition $\pi = (N_1, \ldots, N_h)$ where $\left\lceil \frac{n}{F_U} \right\rceil \leq h \leq \left\lfloor \frac{n}{F_L} \right\rfloor$, $|N_i| = |N_i \cap Q| = \phi_i^U$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1} \cap Q| = n_Q$, $F_L < |N_{k_Q+1}| < F_U$, $F_L < |N_{k_Q+2}| < F_U$. Note that such an sc-partition with $h \geq k_Q + 2$ exists since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with **(FD-1)**. Suppose that $u \in N_{k_Q+1}$ and $v \in N_{k_Q+2}$ (there exists a $u \in N_{k_Q+1} - Q$ since we assume $n_Q \leq F_L$). By Lemma 4.6.1, $g_{u,v} = 0$. This means, we have $g_e = 0$ for all $e \in E(V - Q)$.

Now, in $\pi$, shift $v$ from $N_{k_Q+2}$ to $N_{k_Q+1}$ to obtain

$$g(v, N_{k_Q+1} \cap Q) = 0. \tag{4.58}$$

When applied to this equation, Lemma 4.6.4 implies that $g_e = \alpha \in \mathbb{R}$ for $e \in E(Q, V - Q)$. But, then inserting this into (4.58) we immediately infer $g_e = 0$ for $e \in E(Q, V - Q)$.

Now, we will show $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$. We prove separately for two cases: a) $n_Q \geq 2$, b) $n_Q = 1$ and $k_Q \geq 2$ (we do not consider $k_Q = 1$, because then (4.55) would not be facet defining by part 2 of Lemma 4.9.4).

Case a): Pick four nodes $x, y, z, t \in Q$ and suppose that $\{x, z\} \subset N_{k_Q}$ and $\{y, t\} \subset N_{k_Q+1}$ in $\pi$. By Lemma 4.6.3, we have $g_{x,z} - g_{y,z} = g_{x,t} - g_{y,t}$, which implies

$$g_{x,q} - g_{y,q} = \gamma \in \mathbb{R} \qquad \forall q \in Q - \{x, y\} \tag{4.59}$$

by arbitrariness of $z$ and $t$. Switching $x$ and $y$ in $\pi$, we get

$$g(x, (N_{k_Q} \cap Q) - x) + g(y, (N_{k_Q+1} \cap Q) - y) = g(x, (N_{k_Q+1} \cap Q) - y) + g(y, (N_{k_Q} \cap Q) - x). \tag{4.60}$$

Plugging (4.59) into (4.60) yields $\gamma = 0$ since $\phi^U_{k_Q} = |N_{k_Q} \cap Q| \geq \phi^U_{k_Q+1} > |N_{k_Q+1} \cap Q|$ due to definition of $k_Q$. Then, we infer $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$.

Case b): Pick four nodes $x, y, z, t \in Q$ and suppose that $t \in N_1$, $\{x, z\} \subset N_{k_Q}$ and $y \in N_{k_Q+1}$ in $\pi$. Obtain a new sc-partition $\pi^* = (N^*_1, \ldots, N^*_k)$ from $\pi$ by switching $z$ and $t$. Clearly, $w^\pi \in P_Q$ and $w^{\pi^*} \in P_Q$. Switching $x$ and $y$ in $\pi$ yields (4.60) once more. Switching $x$ and $y$ in $\pi^*$ gives

$$g(x, (N^*_{k_Q} \cap Q) - x) + g(y, (N^*_{k_Q+1} \cap Q) - y) = g(x, (N^*_{k_Q+1} \cap Q) - y) + g(y, (N^*_{k_Q} \cap Q) - x).$$

Comparing this equation with (4.60) leads us to (4.59). Now, arguing in a similar manner to the proof of case $(a)$ we can obtain the result.

(ii) *Sufficiency:* Pick an sc-partition $\pi = (N_1, \ldots, N_{k+1})$ where $k = \left\lceil \frac{n}{F_U} \right\rceil$, $|N_i| = |N_i \cap Q| = \phi^U_i$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1}| = F_L + 1$, $|N_i| = F_L$ for $i = k_Q + 2, \ldots, k+1$ and $|N_{k_Q+1} \cap Q| = n_Q$. Clearly, $w^\pi \in P_Q$. Note that such an sc-partition exists since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi^U_i, F_L, F_U)$ complies with **(FD-2)**. We organize the proof in four steps.

(1) We first show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - Q)$. Assuming $n_Q < F_L$, pick three arbitrary nodes $u, v, t \in V - Q$ and suppose $\{u, v\} \subset N_{k_Q+1}$ and $t \in N_{k_Q+2}$. By Lemma 4.6.2, we infer $g_{u,v} = g_{u,t}$. Arbitrariness of these three nodes implies $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - Q)$.

Now, assume $n_Q = F_L$ and $k_Q < k - 1$. This latter implies that the number of subcliques in $\pi$ is strictly larger than $k_Q + 2$ (i.e., $k_Q + 2 < k + 1$). Pick an arbitrary node $u \in V - Q$ and suppose $u \in N_{k_Q+1}$. Shift $u$ from $N_{k_Q+1}$ to $N_{k_Q+2}$ to obtain

$$g(u, N_{k_Q+1} - u) = g(u, N_{k_Q+2}). \tag{4.61}$$

Since $k_Q + 2 < k + 1$, we can apply Lemma 4.6.4 and get $g_{u,x} = g_{u,y}$ for all $x, y \in (V - Q) - u$. Obviously, this result generalizes into $g_e = \alpha \in \mathbb{R}$ for $e \in E(V - Q)$.

(2) Now, we show $g_e = \alpha$ for $e \in E(Q, V - Q)$. Note that, for an arbitrary $u \in V - Q$, the equation (4.61) applies for any value of $n_Q$, no matter if it is smaller than $F_L$ or not. The previous result $g_e = \alpha$ for $e \in E(V - Q)$ turns (4.61) into

$$g(u, N_{k_Q+1} - u) = F_L \alpha. \tag{4.62}$$

Applying Lemma 4.6.4 to this equation gives $g_{u,q_1} = g_{u,q_2}$ for all $q_1, q_2 \in Q$, which implies $g_e = \beta \in \mathbb{R}$ for $e \in E(Q, V - Q)$. But inserting this result into (4.62) gives $\beta = \alpha$.

(3) Next, we show $\alpha = 0$. Distribute the nodes in $N_{k+1}$ over $N_{k_Q+1}, \ldots, N_k$ in $\pi$ (note that, this is possible since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with **(FD-2)**) and name this new sc-partition as $\tilde{\pi}$. Clearly, $w^{\tilde{\pi}} \in P_Q$. Comparing $g^T w^\pi = h$ and $g^T w^{\tilde{\pi}} = h$ yields $\alpha = 0$, because $w^\pi(E(V - Q, V)) < w^{\tilde{\pi}}(E(V - Q, V))$.

(4) Finally we can show that $g_e = \gamma \in \mathbb{R}$ for $e \in E(Q)$ in the same manner as in the proof of part 1.

*Necessity:* We consider three cases and prove separately:

a) $n_Q > F_L + 1$;

b) $n_Q = F_L + 1$;

c) $n_Q = F_L$ and $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 1$.

a) When $n_Q > F_L + 1$, $P_Q$ consists of sc-partitions $\pi = (N_1, N_2, \ldots, N_k)$ with $k = \left\lceil \frac{n}{F_U} \right\rceil$, $|N_i| = F_U$ for $i = 1, 2, \ldots, k$, $|N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, 2, \ldots, k_Q$ and $|N_{k_Q+1} \cap Q| = n_Q$. This means, $P_Q$ lies in the intersection of the hyperplanes $w(\delta(u)) = F_U - 1$ for all $u \in V$.

b) Let $P_1$ be the set of sc-partitions $\pi = (N_1, \ldots, N_{k+1})$ with $k = \left\lceil \frac{n}{F_U} \right\rceil$, $|N_i| = |N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1}| = |N_{k_Q+1} \cap Q| = F_L + 1$ and $|N_i| = F_L$ for $i = k_Q + 2, \ldots, k$. Let $P_2$ be the set of sc-partitions $\pi = (N_1, \ldots, N_k)$ with $|N_i| = F_U$ for $i = 1, \ldots, k$,

$|N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1} \cap Q| = n_Q = F_L + 1$. When $n_Q = F_L + 1$, we have $P_Q = P_1 \cup P_2$. In all the solutions in $P_1$ and $P_2$, all the nodes in $V - Q$ are always in subcliques with the same size. That is, we have $w(\delta(u)) = w(\delta(v))$ for all $u, v \in V - Q$, which shows that (4.55) is not facet defining in this case.

c) All $w^\pi \in P_Q$ with $\pi = (N_1, N_2, \ldots, N_k)$ has $|N_i| = |N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, \ldots, k_Q$ and $|N_{k_Q+1} \cap Q| = n_Q = F_L$. Let $P_1$ consist of sc-partitions in $P_Q$ with $k = k_Q + 2$, one of $|N_{k_Q+1}|$ and $|N_{k_Q+2}|$ is $F_L$ and the other is $F_L + 1$; and, let $P_2$ consist of sc-partitions in $P_Q$ with $k = k_Q + 1$ and $|N_{k_Q+1}| = F_U$. Since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with **(FD-2)**, we have $P_Q = P_1 \cup P_2$. In other words, $|V - Q| = F_L + 1$ and it is ensured that in any sc-partition of $P_Q$, at most one node of $V - Q$ is packed in a different subclique than the others (indeed, in the sc-partitions of $P_2$ it is ensured that all are packed in the same subclique).

Now, let $Q' = V - Q = \{q'_1, q'_2, \ldots, q'_{F_L+1}\}$. Both $P_1$ and $P_2$ (and hence $P_Q$) are contained in the face defined by the 2-chorded cycle inequalities, which were introduced in [14] for $\mathcal{P}(n)$. More precisely, $P_1$ and $P_2$ are contained in an hyperplane $g^T w = 0$ such that

$$g_{u,v} = \begin{cases} 1, & \text{if } u = q'_i \text{ and } v = q'_{i+1}, \text{ for all } i = 1, 2, \ldots, F_L + 1; \\ -1, & \text{if } u = q'_i \text{ and } v = q'_{i+2}, \text{ for all } i = 1, 2, \ldots, F_L + 1; \\ 0, & \text{otherwise,} \end{cases}$$

where all indices are taken modulo $F_L + 1$.

□

Now, we prove necessary and sufficient conditions for (4.55) to be facet defining for $\mathcal{P}^{lu}$ with dimension $\binom{n}{2} - 1$. Note that in this case, $\mathcal{P}_Q = \mathcal{P}^{lu}$ (i.e., (4.55) constitutes the equality set of $\mathcal{P}^{lu}$) when $Q = V$. So, we assume in the following theorem that $Q \subset V$.

**Theorem 4.9.2.** *Suppose* $dim(\mathcal{P}^{lu}) = \frac{n(n-1)}{2} - 1$. *Let* $Q$ *be a subset of* $V$ *for which the conditions 1 and 2 of Lemma 4.9.4 are not satisfied (i.e.* $n_Q > 0$ *and* $|Q| > \phi_1^U + 1$*). The upper general clique inequality (4.55) is not facet defining for* $\mathcal{P}^{lu}$ *when* $(n \bmod F_L) = 1$*; when* $(n \bmod F_U) = F_U - 1$*, it is facet defining for* $\mathcal{P}^{lu}$ *if and only if*

- $0 < n_Q < F_U - 1$ *and* $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 2$, *or,*
- $0 < n_Q \leq F_U - 1$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 2$.

*Proof.* When $(n \bmod F_L) = 1$, $P_Q$ lies in the hyperplanes $w(\delta(u)) = F_L - 1$ for all $u \in V - Q$. When $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 1$, $P_Q$ lies in the hyperplanes $w(\delta(u)) = F_U - 2$ for all $u \in V - Q$. And, in a similar manner to part 2 of Theorem 4.9.1, one can show that (4.55) is not facet defining for $\mathcal{P}^{lu}$ if $n_Q = F_U - 1$ and $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 2$.

When $(n \bmod F_U) = F_U - 1$, we set $g_{e'} = 0$ for one $e' \in E(V - Q)$. Proceeding in a similar manner to part 2 of Theorem 4.9.1, one can show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E - E(Q)$. But then, $g_{e'} = 0$ implies $g_e = 0$ for all $e \in E - E(Q)$. Proceeding like in part 1 of Theorem 4.9.1, one can show $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$. □

Proposition 4.9.1 states that (4.55) is not facet defining at all when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. The following Theorem 4.9.3 states the necessary and sufficient conditions that make (4.55) facet defining for $P^{lu}$ when its dimension is $\binom{n}{2} - n$.

**Proposition 4.9.1.** *The upper general clique inequality (4.55) is not facet defining for* $\mathcal{P}^{lu}$ *when* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$.

*Proof.* In this case, $P_Q$ is contained in the hyperplanes $w(\delta(u)) = F_U - 1$ for all $u \in V$.

□

**Theorem 4.9.3.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. *Let*

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0; \end{cases}$$

*and let* $k = \frac{n}{F}$. *Let* $Q$ *be a subset of* $V$ *which does not comply with conditions 1 and 2 in Lemma 4.9.4 (i.e.,* $n_Q > 0$ *and* $|Q| > F + 1$*). The upper general clique inequality (4.55) is facet defining for* $\mathcal{P}^{lu}$ *if and only if* $Q$ *complies with one of the following:*

*(i)* $n_Q = 1$ *and* $2 \leq k_Q < k - 1$, *or*

*(ii)* $2 \leq n_Q < F - 1$ *and* $1 \leq k_Q < k - 1$, *or*

*(iii)* $n_Q = F - 1$ *and* $1 \leq k_Q < k - 2$.

*Proof.* If $k_Q = k-1$,

$$P_Q = \{w \in \mathcal{P}^{lu} | w_{u,v} = 1\} \qquad \forall u, v \in V - Q.$$

And, when $n_Q = F - 1$ and $k_Q = k - 2$, one can show in a similar manner to the proof of part $(ii)$ of Lemma 4.9.4 that $P_Q$ is contained in the face defined by the cycle inequalities.

Without loss of generality, let $Q = \{1, 2, \ldots, |Q|\}$ and $V - Q = \{|Q| + 1, \ldots, n\}$. Consider the set $S = \{\{1, n\}, \{2, n\}, \ldots, \{n-1, n\}, \{1, n-1\}\}$. Like in the proof of Theorem 4.8.3, one can fix the values of the $g_e$ variables to 0 for all $e \in S$.

Now, we prove the separately for the parts 1, 2 and 3.

(i) Pick $x, y, z \in V - Q$ and $p, q \in Q$ arbitrarily. Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ such that $w^\pi \in P_Q$, $n_Q = 1$, $2 \leq k_Q < k - 1$, $p \in N_{k_Q}$, $\{n, z, q\} \subset N_{k_Q+1}$ and $\{x, y\} \subset N_{k_Q+2}$. Applying Lemma 4.6.3, one can infer that $g_{n,z} + g_{x,y} = g_{n,y} + g_{x,z}$, which leads to $g_{x,y} = g_{x,z}$ since $\{n, z\}, \{n, y\} \in S$. Since, $x$, $y$ and $z$ are arbitrary nodes, we infer that

$$g_e = \alpha \in \mathbb{R} \qquad \forall e \in E(V - Q). \tag{4.63}$$

Switching $n$ and $x$ in $\pi$ yields

$$g(x, N_{k_Q+2} - x) = g(x, q) + g(x, N_{k_Q+1} - \{q, n\}). \tag{4.64}$$

If we plug (4.63) into (4.64) we get $g_{x,q} = \alpha$. Since $x$ and $q$ are arbitrary, this implies $g_e = \alpha$ for all $e \in E(Q, V - Q)$. But then since we have set $g_{1,n-1} = 0$, we get $\alpha = 0$.

Now, switch $p$ and $q$ in $\pi$ to obtain

$$g(p, N_{k_Q} - p) = g(q, N_{k_Q} - p). \tag{4.65}$$

Pick $q_1 \in N_1$ and $q_2 \in N_{k_Q}$ (note that, $k_Q \geq 2$). Obtain a new sc-partition $\tilde{\pi}$ by switching $q_1$ and $q_2$. Switching $p$ and $q$ in $\tilde{\pi}$ yields

$$g(p, N_{k_Q} - \{p, q_2\}) + g_{p,q_1} = g(q, N_{k_Q} - \{p, q_2\}) + g_{q,q_1}.$$

Comparing this equality with (4.65) gives $g_{p,q_1} - g_{q,q_1} = g_{p,q_2} - g_{q,q_2}$, which generalizes into $g_{p,\hat{q}} - g_{q,\hat{q}} = \beta \in \mathbb{R}$ for $\hat{q} \in Q - \{p, q\}$. However, due to (4.65), this implies that $\beta = 0$, which implies further that $g_e = \gamma \in \mathbb{R}$ for $e \in E(Q)$.

(ii) Proceeding in a similar manner to part 1, one can show that $g_e = 0$ for all $e \in E - E(Q)$. Now, we show $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$. Pick $u \in Q$ arbitrarily. Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ such that $w^\pi \in P_Q$, $2 \leq n_Q < F - 1$, $1 \leq k_Q < k - 1$, $u \in N_{k_Q}$ and $n \in N_{k_Q+1}$ arbitrarily. Now, switching $u$ and $n$ yields

$$g(u, N_{k_Q} - u) = g(u, N_{k_Q+1} \cap Q).$$

Applying Lemma 4.6.4 to this equation yields $g_{u,q} = \beta$, which generalizes into $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$ due to arbitrariness of $u$.

(iii) The proof of this part is similar to those of parts 1 and 2.

□

This theorem suggests that the upper general inequality (4.55) is facet defining for the k-way equi-partition polytope $\mathcal{P}^{k-way}(n, k)$. In fact, Mitchell [22] shows that (4.55) is valid for this polytope. Here in Theorem 4.9.3, we advance this result one step further and show for the first time that (4.55) is indeed facet defining for $\mathcal{P}^{k-way}(n, k)$. We highlight this contribution by means of the following corollary:

**Corollary 4.9.1.** *The upper general clique inequality (4.55) is facet defining for $\mathcal{P}^{k-way}(n, k)$ if and only if*

- $F + 2 \leq |Q| \leq n - F - 2$, *and,*
- $(|Q| \ mod \ F) \neq 0$.

## 4.10 The Cycle Inequalities

Let $G_C = (V_C, E_C)$ be a cycle in $G$. The cycle inequality

$$w(E_C) \leq |V_C| - 2 \tag{4.66}$$

is first introduced by Conforti, Rao and Sassano [7] for the equipartition polytope $\mathcal{P}^{equi}(n)$. They show that it is facet defining for $\mathcal{P}^{equi}(n)$ when $n$ is odd and $|V_C| = \left\lceil \frac{n}{2} \right\rceil + 1$. Ferreira *et al.* [11] introduce a generalization of the cycle inequality, the so-called *cycle with ear inequalities* for a node-capacitated graph partitioning polytope. Sørensen [31] show that the cycle with ear inequalities (hence the cycle inequalities) are facet defining for $\mathcal{P}^u(n, F_U)$.

In this section, we generalize this result for $\mathcal{P}^{lu}$. In the rest of this section, let $\phi_1^U$ and $\phi_2^U$ be defined as in (4.14), and all the indices be taken modulo $|V_C|$. And, let $P_C$ denote the face of $\mathcal{P}^{lu}$ defined by (4.66) corresponding to the cycle $G_C$.

We first present a lemma that states the validity condition of (4.66) for $\mathcal{P}^{lu}$. We skip the proof because it is trivial.

**Lemma 4.10.1.** *The cycle inequality (4.66) is valid for $\mathcal{P}^{lu}$ if and only if $|V_C| \geq \phi_1^U + 1$.*

Now follows the theorems that state the conditions of facetness for $\mathcal{P}^{lu}$.

**Theorem 4.10.1.** *Suppose $\mathcal{P}^{lu}$ is full-dimensional and $F_U - F_L \geq 2$. Pick a cycle $G_C = (V_C, E_C)$ with $|V_C| = \phi_1^U + 1$. The cycle inequality (4.66) is facet defining for $\mathcal{P}^{lu}$ if $\phi_2^U \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$ and $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ is full-dimensional.*

*Proof.* Let $V_C = \{u_1, u_2, \ldots, u_{|V_C|}\}$ and

$$E_C = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{|V_C|-1}, u_{|V_C|}\}, \{u_{|V_C|}, u_1\}\}.$$

Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with $|N_1| = \phi_1^U$, $N_1 \cap V_C = \{u_2, u_3, \ldots, u_{\phi_1^U+1}\}$ and $N_2 \cap V_C = \{u_1\}$. Clearly, $w^\pi(E_C) = |V_C| - 2$. If $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ complies with **(FD-1)**, suppose that we have picked $\pi$ such that $F_L < |N_2| < F_U$ and $F_L < |N_3| < F_U$. Using Lemma 4.6.1, we can show that $g_{v,v'} = 0$ for two arbitrary nodes $v, v'$ in $V - V_C$. If $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ complies with **(FD-2)**, **(FD-3)** or **(FD-4)**, suppose that we have picked $\pi$ such that $|N_2| = |N_3| + 1$. Then, we can apply Lemma 4.6.2 so as to show that $g_{v,v'} = \alpha \in \mathbb{R}$ for any two arbitrary nodes $v, v'$ in $V - V_C$. But, with **(FD-2)**, **(FD-3)** and **(FD-4)** it is always possible to construct another sc-partition with one less or one more number of subcliques than $\pi$ (like in step (3) in the proof of part (ii) in Theorem 4.9.1). This implies $\alpha = 0$, namely, $g_e = 0$ for $e \in E(V - V_C)$.

Now, pick an arbitrary $v \in V - V_C$ and $u_i \in V_C$. Furthermore, pick an sc-partition $\pi' = (N_1', \ldots, N_k')$ such that $|N_1'| = \phi_1^U$, $N_1' \cap V_C = \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{\phi_1^U+1}\}$, $N_2' \cap V_C = \{u_i\}$ , $|N_2'| < F_U$, $|N_3'| > F_L$ and $v \in N_3'$. Note that full-dimensionality of $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ ensures existence of such an sc-partition. Shifting $v$ from $N_3'$ to $N_2'$, we get $g_{v,u_i} = 0$, which generalizes into $g_e = 0$ for all $e \in E(V_C, V - V_C)$.

Now, we show $g_{u_i,u_{i+l}} = 0$ for $i = 1, 2, \ldots, |V_C|$ and $l = 2, 3, \ldots, \left\lfloor \frac{|V_C|}{2} \right\rfloor$. Pick an sc-partition $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k)$ with $|\tilde{N}_1| = \phi_1^U$, $\tilde{N}_1 \cap V_C = \{u_{l'+1}, u_{l'+2}, \ldots, u_{|V_C|}\}$,

$|\tilde{N}_2| = \phi_2^U$ and $\tilde{N}_2 \cap V_C = \{u_1, u_2, \ldots, u_{l'}\}$ where $3 \leq l' \leq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$ (this is possible even for $l' = \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$ since $\phi_2^U \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$). Clearly, $w^{\tilde{\pi}}(E_C) = |V_C| - 2$. Since $|V_C| = \phi_1^U + 1$ and $|\tilde{N}_1| = \phi_1^U$, there exists $\{v, v'\} \subset \tilde{N}_1 - V_C$. Switching $v$ and $u_1$ yields

$$g(u_1, \{u_2, \ldots, u_{l'}\}) = g(u_1, \{u_{l'+1}, \ldots, u_{|V_C|}\}). \tag{4.67}$$

Now, consider another sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ obtained from $\tilde{\pi}$ by switching $u_{l'}$ and $v$. Obviously, $w^{\pi^*}(E_C) = |V_C| - 2$. Now, switching $u_1$ and $v'$ in $\pi^*$ yields

$$g(u_1, \{u_2, \ldots, u_{l'}\}) - g_{u_1,u_{l'}} = g(u_1, \{u_{l'+1}, \ldots, u_{|V_C|}\}) + g_{u_1,u_{l'}}.$$

Comparing this with (4.67), we infer $g_{u_1,u_{l'}} = 0$. By symmetry, the same result can be obtained for any $u_i \in V_C$, that is, $g_{u_i,u_{i+l'-1}} = 0$ for any $i = 1, 2, \ldots, |V_C|$. This implies that $g_e = 0$ for all $e \in E(V_C) - E_C$. But, then (4.67) directly gives $g_{u_1,u_2} = g_{u_1,u_{|V_C|}}$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for $e \in C$ by symmetry. □

**Theorem 4.10.2.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. *Pick a cycle* $G_C = (V_C, E_C)$ *with* $|V_C| = \phi_1^U + 1$.

*(i) When* $(n \bmod F_U) = F_U - 1$ *and* $F_U - F_L \geq 2$, *the cycle inequality (4.66) is facet defining for* $\mathcal{P}^{lu}$ *if* $\phi_2^U \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$.

*(ii) When* $(n \bmod F_U) = F_U - 1$ *and* $F_U - F_L = 1$, *the cycle inequality (4.66) is facet defining for* $\mathcal{P}^{lu}$ *if* $F_L \geq 4$ *and* $\phi_2^U \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$.

*(iii) When* $(n \bmod F_L) = 1$, *the cycle inequality (4.66) is facet defining for* $\mathcal{P}^{lu}$ *if* $F_L \geq 4$ *and* $F_L \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$.

*Proof.* We present the proof for case 1 only. The proofs for the other cases are similar. Note that in case 1,

- when $\left\lceil \frac{n}{F_U} \right\rceil = 2$, $\phi_1^U = F_U$; $\phi_2^U = F_U - 1$; and,
- when $\left\lceil \frac{n}{F_U} \right\rceil \geq 3$, $\phi_1^U = \phi_2^U = F_U$.

Let $V_C = \{u_1, u_2, \ldots, u_{|V_C|}\}$ and

$$E_C = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{|V_C|-1}, u_{|V_C|}\}, \{u_{|V_C|}, u_1\}\}.$$

Since $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we can set $g_{e'} = 0$ for one $e' \in E(V - V_C)$. Pick three arbitrary nodes $x, y$ and $z$ in $V - V_C$. Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ with, without loss of generality, $|N_1| = F_U$, $|N_2| = F_U - 1$, $N_1 \cap V_C = \{u_1, u_2, \ldots, u_{F_U-2}\}$, $N_2 \cap V_C = \{u_{F_U-1}, u_{F_U}, u_{F_U+1}\}$, $\{x, y\} \subset N_1$ and $z \in N_2$ (note that, $N_2 - V_C$ is non-empty because $|N_2| = F_U - 1 \geq (F_L + 2) - 1 \geq (3 + 2) - 1$, since we assume $F_L \geq 3$). Using Lemma 4.6.2, we get $g_e = \alpha \in \mathbb{R}$ for $e \in E(V - V_C)$ since $x, y$ and $z$ are arbitrary.

Pick another sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ with $|N_1^*| = F_U$, $|N_2^*| = F_U - 1$, $N_1^* \cap V_C = \{u_1, \ldots, u_{F_U-1}\}$, $N_2^* \cap V_C = \{u_{F_U}, u_{F_U+1}\}$ and $x \in N_1^*$. Shifting $x$ from $N_1^*$ to $N_2^*$ yields

$$g(x, \{u_1, u_2, \ldots, u_{F_U-1}\}) = (F_U - 3)\alpha + g(x, \{u_{F_U}, u_{F_U+1}\}). \tag{4.68}$$

Now, we obtain a new sc-partition $\tilde{\pi}$ by only switching $u_{F_U-1}$ and $u_{F_U+1}$ in $\pi^*$. Shifting $x$ from $\tilde{N}_1$ to $\tilde{N}_2$ in $\tilde{\pi}$ we get

$$g(x, \{u_1, u_2, \ldots, u_{F_U-2}, u_{F_U+1}\}) = (F_U - 3)\alpha + g(x, \{u_{F_U-1}, u_{F_U}\}).$$

Comparing this with (4.68) we infer $g_{x,u_{F_U-1}} = g_{x,u_{F_U+1}}$, which generalizes into $g_{x,u_i} = g_{x,u_{i-2}}$ for $i = 1, 2, \ldots, |V_C|$ since $u_{F_U+1}$ is arbitrary in the sense that we could have placed any two consecutive nodes of $V_C$ into $N_2^*$.

When $|V_C|$ is odd, $g_{x,u_i} = g_{x,u_{i-2}}$ for any $i$ implies that $g_e = \beta \in \mathbb{R}$ for $e \in E(V - V_C, V_C)$ since $x$ is arbitrary. When $|V_C|$ is even, we infer $g_{x,u_i} = \beta_1 \in \mathbb{R}$ if $i$ is odd and $g_{x,u_i} = \beta_2 \in \mathbb{R}$ if $i$ is even. Now, we repeat the same procedure using $\pi$ instead of $\pi^*$. Namely, we obtain a new sc-partition $\pi'$ by switching $u_{F_U-2}$ and $u_{F_U+1}$. We compare the two equations we obtain by shifting $x$ in $\pi$ and $\pi'$ and get $g_{x,u_{F_U-2}} = g_{x,u_{F_U+1}}$, which generalizes into $g_{x,u_i} = g_{x,u_{i-3}}$ since $u_{F_U+1}$ is arbitrary in the sense that we could have placed any three consecutive nodes of $V_C$ into $N_2$. This shows that $\beta_1 = \beta_2$ when $|V_C|$ is even. In other words, for any $|V_C|$, we have shown $g_e = \beta \in \mathbb{R}$ for $e \in E(V - V_C, V_C)$. Inserting this into (4.68) gives $\alpha = \beta$. But, since we have already set $g_{e'} = 0$ for one $e' \in E(V - V_C)$, we infer $\alpha = \beta = 0$.

We can show $g_e = \gamma \in \mathbb{R}$ for $e \in E_C$ and $g_e = 0$ for $E(V_C) - E_C$ just like we do in Theorem 4.10.1. □

**Theorem 4.10.3.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. *Pick a cycle* $G_C = (V_C, E_C)$ *with* $|V_C| = F_U + 1$. *The cycle inequality (4.66) is facet defining for* $\mathcal{P}^{lu}$ *if and only if* $F_U < 2F_L$.

*Proof.* First assume $F_U < 2F_L$. Let $V_C = \{u_1, u_2, \ldots, u_{F_U+1}\}$ and

$$E_C = \{\{u_1, u_2\}, \ldots, \{u_{F_U}, u_{F_U+1}\}, \{u_{F_U+1}, u_1\}\}.$$

Without loss of generality, suppose $n \notin V_C$. We use the fact that $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$ for setting $g_e = 0$ for $e \in S$ where $S = \{\{1, n\}, \{2, n\}, \ldots, \{n-1, n\}\}$, just like we do in the proof of Theorem 4.7.5. Pick two arbitrary nodes $x, y \in V - V_C$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with $|N_i| = F_U$ for $i = 1, 2, \ldots, k$, $N_1 \cap V_C = \{u_1, u_2, \ldots, u_l\}$, $N_2 \cap V_C = \{u_{l+1}, \ldots, u_{F_U+1}\}$, $n \in N_1$ and $\{x, y\} \in N_2$ where $l = \left\lfloor \frac{|V_C|}{2} \right\rfloor$. Switching $x$ and $n$ yields

$$g(x, (N_2 - (V_C \cup x)) + g(x, N_2 \cap V_C) = g(x, N_1 - V_C) + g(x, N_1 \cap V_C). \quad (4.69)$$

Now, switch $y$ and $u_1$ in $\pi$ and call this new sc-partition $\pi^*$. Switching $x$ and $n$ in $\pi^*$ yields

$$\begin{aligned} &g(x, N_2 - (V_C \cup x)) - g_{x,y} + g(x, N_2 \cap V_C) + g_{x,u_1} = \\ &g(x, N_1 - V_C) + g_{x,y} + g(x, N_1 \cap V_C) - g_{x,u_1}. \end{aligned}$$

Comparing this equation with (4.69) we obtain $g_{x,y} = g_{x,u_1}$. We can generalize this result to $g_e = \alpha \in \mathbb{R}$ for $e \in E((V-n) - V_C, V-n)$ since $x$, $y$ and $u_1$ are arbitrary.

Next, pick another sc-partition $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_{k+1})$ where $|\tilde{N}_i| = F_L$ for $i = 1, 2, \ldots, k+1$, $\tilde{N}_1 \cap V_C = \{u_1, u_2, \ldots, u_l\}$, $\tilde{N}_2 \cap V_C = \{u_{l+1}, \ldots, u_{F_U+1}\}$ where $l = \left\lfloor \frac{|V_C|}{2} \right\rfloor$. Note that such an sc-partition exists since we assume $F_U < 2F_L$ and $|V_C| = F_U + 1$. Comparing $g(w^{\pi}) = h$ and $g(w^{\tilde{\pi}}) = h$, we can infer $\alpha = 0$.

We can show $g_e = \gamma \in \mathbb{R}$ for $e \in E_C$, $g_e = 0$ for $E(V_C) - E_C$ like we do in Theorem 4.10.1.

Finally, when $F_U \geq 2F_L$, all the points that satisfy the cycle inequality at equality lie on the hyperplanes $w(\delta(u)) = F_U - 1$ for all $u \in V$. □

**Theorem 4.10.4.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. *Pick a cycle* $G_C = (V_C, E_C)$ *with* $|V_C| = F + 1$ *where*

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

*The cycle inequality (4.66) is facet defining for* $\mathcal{P}^{lu}$.

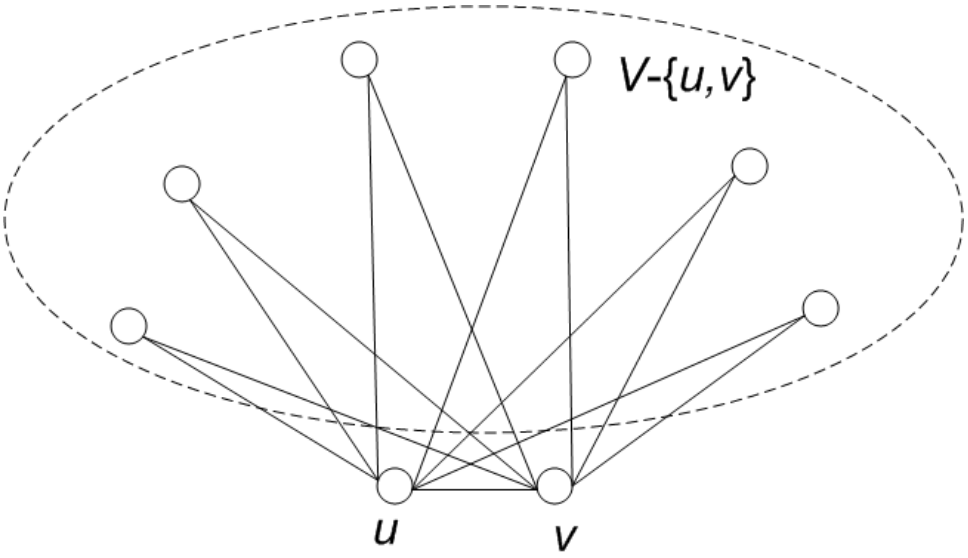


Figure 4.2: Support graph of the 2-star inequalities.

*Proof.* Let $V_C = \{u_1, u_2, \ldots, u_{F_U+1}\}$ and $E_C = \{\{u_1, u_2\}, \ldots, \{u_{F_U}, u_{F_U+1}\}, \{u_{F_U+1}, u_1\}\}$. Suppose, without loss of generality that $\{n-1, n\} \subset V - V_C$. Like we do in the proof Theorem 4.8.3, we set $g_e = 0$ for $e \in S = \{\{1, n\}, \{2, n\}, \ldots, \{n-1, n\}, \{1, n-1\}\}$. Then, like in the proof of Theorem 4.10.3, we can show $g_e = \alpha \in \mathbb{R}$ for $e \in E((V-n) - V_C, V-n)$. But, of course, combining this with $g_{1,n-1} = 0$, we get $g_e = 0$ for $e \in E(V - V_C, V)$. Finally, we can show $g_e = \gamma \in \mathbb{R}$ for $e \in E_C$, $g_e = 0$ for $E(V_C) - E_C$ like we do in Theorem 4.10.1. □

Mitchell [22] shows that the cycle inequality (4.66) is facet defining for $\mathcal{P}^{equi}(n)$ for even $n$ and $\mathcal{P}^{k-way}(n, k)$. Theorem 4.10.4 presents an alternative proof for this result.

## 4.11 The lower 2-star inequalities

In this section, we introduce the following *lower 2-star inequalities* (see 4.2 for a graphical representation of the support graph):

$$(2 + \phi_2^L - \phi_1^L) w_{u,v} + w(\{u, v\}, V - \{u, v\}) \geq \phi_1^L + \phi_2^L - 2 \qquad \forall u, v \in V. \quad (4.70)$$

Let $P_{u,v}$ denote the face defined by the lower 2-star inequality corresponding to the nodes $u$ and $v$, i.e.

$$P_{u,v} = \{w \in \mathcal{P}^{lu} | (2 + \phi_2^L - \phi_1^L)w_{u,v} + w(\{u,v\}, V - \{u,v\}) = \phi_1^L + \phi_2^L - 2\}.$$

The following lemma proves that (4.70) is valid for $\mathcal{P}^{lu}$.

**Lemma 4.11.1.** *The lower 2-star inequality (4.70) is valid for $\mathcal{P}^{lu}$.*

*Proof.* Consider two nodes $u, v \in V$ and the corresponding lower 2-star inequality (4.70). Pick an sc-partition $\pi = (N_1, \ldots, N_k)$. First suppose that $u$ and $v$ are packed in the same subset, say $N_i$, in $\pi$. In this case, the left hand side of (4.70) is equal to

$$(2 + \phi_2^L - \phi_1^L) + (2|N_i| - 4), \tag{4.71}$$

which is always larger than or equal to the right hand side of (4.70). Now, suppose that $u$ and $v$ are packed in different subsets, i.e., say $u \in N_i$ and $v \in N_j$. Then, the left hand side of (4.70) is equal to

$$|N_i| + |N_j| - 2, \tag{4.72}$$

which is always larger than or equal to the right hand side of (4.70). □

The next lemma characterizes $P_{u,v}$.

**Lemma 4.11.2.** *The characteristic vector $w^\pi$ of an sc-partition $\pi = (N_1, \ldots, N_k)$ is contained in $P_{u,v}$ if and only if there exist $i^*$ and $j^*$ such that*

*(i)* $|(N_{i^*} \cup N_{j^*}) \cap \{u,v\}| = 2$,

*(ii)* $|N_{i^*}| = \phi_1^L$ *when* $|N_{i^*} \cap \{u,v\}| = 2$,

*(iii)* $|N_{i^*}| + |N_{j^*}| = \phi_1^L + \phi_2^L$ *when* $|N_{i^*} \cap \{u,v\}| = 1$.

*Proof. Sufficiency:* Case 1. $|N_{i^*} \cap \{u,v\}| = 2$: By condition 1, $|N_{j^*} \cap \{u,v\}| = 0$. As $|N_{i^*}| = \phi_1^L$, (4.71) is equal to the right hand side of (4.70). Note that, in this case $|N_{j^*}|$ can take any value larger than or equal to $\phi_2^L$.
Case 2. $|N_{i^*} \cap \{u,v\}| = 1$: Then, we have $|N_{i^*}| + |N_{j^*}| = \phi_1^L + \phi_2^L$ and (4.72) is equal to the right hand side of (4.70).
*Necessity:* Consider an sc-partition $\pi = (N_1, \ldots, N_k)$. Obviously, there must exist $i^*$ such that $|N_{i^*}| = \phi_1^L$ and $|N_{i^*} \cap \{u,v\}| \geq 1$, because otherwise (4.71)

and (4.72) are never equal to the right hand side of (4.70). In fact, when $|N_{i^*} \cap \{u, v\}| = 2$, (4.71) is equal to the right hand side of (4.70) only if $|N_{i^*}| = \phi_1^L$. When $|N_{i^*} \cap \{u, v\}| = 1$ and there must exist $j^*$ such that $|N_{i^*}| + |N_{j^*}| = \phi_1^L + \phi_2^L$, because otherwise (4.72) would never be equal to the right hand side of (4.70). □

As usual, we start by considering the full-dimensional case.

**Theorem 4.11.1.** *Suppose that $\mathcal{P}^{lu}$ is full-dimensional. Then the lower 2-star inequalities (4.70) define facets of $\mathcal{P}^{lu}$ if and only if one of the following hold:*

*(i)* $\phi_2^L \geq \phi_1^L + 2$;

*(ii)* $\phi_2^L = \phi_1^L + 1$, $F_U - F_L \geq 3$.

*Proof. Sufficiency*: Pick $x, y, z \in V - \{u, v\}$ arbitrarily; and, pick an sc-partition $\pi = (N_1, \ldots, N_k)$ where $\{v, z\} \subset N_1$, $\{u, x, y\} \subseteq N_2$, $|N_i| = \phi_i^L$ for $i = 1, 2, \ldots, k$, and $\phi_2^L > \phi_1^L$. Note that, when $\phi_2^L > \phi_1^L$, by definition of $\phi_i^L$'s, we have $\phi_i^L = F_U$ for $i = 3, \ldots, k \left(= \left\lfloor \frac{n}{F_L} \right\rfloor\right)$. Clearly, $w^\pi \in P_{u,v}$. Obtain a new sc-partition $\pi^* = (N_1^*, N_2^*, \ldots, N_k^*)$ by only switching $y$ and $z$ in $\pi$ (obviously, $w^{\pi^*} \in P_{u,v}$). Now, shift $x$ from $N_2$ to $N_1$ in $\pi$ to obtain

$$g_{x,u} + g_{x,y} + g(x, N_2 - \{u, x, y\}) = g_{x,z} + g_{x,v} + g(x, N_1 - \{v, z\}). \tag{4.73}$$

Shifting $x$ from $N_2^*$ to $N_1^*$ in $\pi^*$ yields

$$g_{x,u} + g_{x,z} + g(x, N_2^* - \{u, x, y\}) = g_{x,y} + g_{x,v} + g(x, N_1^* - \{v, z\}).$$

Comparing this with (4.73) gives $g_{x,y} = g_{x,z}$, which implies that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - \{u, v\})$. Plugging this into (4.73) and rearranging we obtain

$$g_{x,v} - g_{x,u} = (\phi_2^L - \phi_1^L - 1)\alpha. \tag{4.74}$$

Now, consider yet another sc-partition $\tilde{\pi}$ which is obtained by switching $u$ and $v$ in $\pi$. Clearly, we still have $w^{\tilde{\pi}} \in P_{u,v}$. If we proceed in the same manner for $\tilde{\pi}$, we get

$$g_{x,u} - g_{x,v} = (\phi_2^L - \phi_1^L - 1)\alpha. \tag{4.75}$$

<u>*Claim:*</u> Equations (4.74) and (4.75) imply $\alpha = 0$ under conditions 1 and 2.
*Proof:* The result is obvious when $\phi_2^L - \phi_1^L \geq 2$. Suppose $\phi_2^L - \phi_1^L = 1$ and $F_U - F_L \geq 3$. Obtain an sc-partition $\pi' = (N_1', \ldots, N_k')$ by switching $u$ and $z$ in

$\pi$. Obviously $w^{\pi'} \in P_{u,v}$. Note that, definition of $\phi_i^L$'s imply that $\phi_2^L = F_L + 1$ and $\phi_3^L = F_U$. Shifting a node $t \in N_3'$ from $N_3'$ to $N_2'$ yields that $\alpha = 0$ (we can always shift such a node from $N_3'$ to $N_2'$ since we assume $F_U - F_L \geq 3$). This completes the proof of the claim.

From this claim, (4.74) and (4.75), we can directly infer $g_{x,u} = g_{x,v} = \gamma_x \in \mathbb{R}$ for all $x \in V - \{u, v\}$.

Now, consider $\hat{\pi} = (\hat{N}_1, \hat{N}_2, \ldots, \hat{N}_k)$ with $\{u, v, y\} \subseteq \hat{N}_1$, $x \in \hat{N}_2$, $|\hat{N}_1| = \phi_1^L$ and $|\hat{N}_2| = \phi_2^L$. Clearly, $w^{\hat{\pi}} \in P_{u,v}$. Switching $x$ and $y$ in $\hat{\pi}$ yields

$$\gamma_x = \gamma_y, \tag{4.76}$$

which completes the sufficiency part of the proof.

*Necessity*: When $\phi_1^L = \phi_2^L$, a lower 2-star inequality (4.70) is always implied by the corresponding lower bound inequalities (4.18).

When $\phi_2^L = \phi_1^L + 1$ and $F_U - F_L = 2$, all feasible sc-partitions that lie on $P_{u,v}$ have one subclique of size $F_L$, one subclique of size $F_L + 1$ and $k - 2$ subcliques of size $F_U$, where $k = \left\lfloor \frac{n}{F_L} \right\rfloor$. Hence, for all sc-partitions that belong to $P_{u,v}$, the following equality holds:

$$w(E) = \sum_{i=1}^{k} \binom{\phi_i^L}{2} = \binom{F_L}{2} + \binom{F_L + 1}{2} + (k-2)\binom{F_U}{2}.$$

Then, this implies that $P_{u,v}$ lies in the face defined by the valid inequality (it is not difficult to show that this inequality is valid for $\mathcal{P}^{lu}$ when $\phi_2^L = \phi_1^L + 1$ and $F_U - F_L = 2$):

$$w(E) \leq \binom{F_L}{2} + \binom{F_L + 1}{2} + (k-2)\binom{F_U}{2}.$$

Note that, when $\phi_2^L = \phi_1^L + 1$ and $F_U - F_L \leq 1$, $\mathcal{P}^{lu}$ is not full-dimensional. This completes the necessity part of the proof.

□

Now, we prove the facet defining conditions for the lower 2-star inequalities when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. In this case, the lower 2-star inequalities are facet defining for $\mathcal{P}^{lu}$ if and only if the lower bound inequalities (4.18) are not facet defining. This complementary facetness property between these two sets of inequalities is a direct consequence of the following theorem and Theorem 4.6.9.

**Theorem 4.11.2.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. *The lower 2-star inequalities (4.70) are facet defining for* $\mathcal{P}^{lu}$ *if and only if*

- $(n \ mod \ F_L) = 1$ *and* $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2$, *or,*
- $(n \ mod \ F_U) = F_U - 1$.

*Proof.* Using the fact that $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we set $g_{e'} = 0$ for some $e' \in E(V - \{u, v\})$. Proceeding like in the proof of Theorem 4.11.1, we can show $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - \{u, v\})$. But, $g_{e'} = 0$ implies $\alpha = 0$. Now, we can complete the proof proceeding in the same manner as in the proof of Theorem 4.11.1. Note that, when $(n \bmod F_L) = 1$ and $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$, $P_{u,v}$ lies in the corresponding hyperplanes defined by the lower bound inequalities (4.18) as we would have $\phi_1^L = \phi_2^L = F_L$. □

Now, we state that the lower 2-star inequalities are not facet defining when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$.

**Proposition 4.11.1.** *Suppose* $dim(\mathcal{P}^{lu}) \leq \binom{n}{2} - (n-1)$. *The lower 2-star inequalities (4.70) are not facet defining for* $\mathcal{P}^{lu}$.

*Proof.* When, $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$, $P_{u,v}$ lies in the intersection of the hyperplanes $w(\delta(u)) = F_L - 1$. When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$, the lower 2-star inequalities are just the sum of two hyperplanes from $M(\mathcal{P}^{lu})$. □

## 4.12 The upper 2-star inequalities

In this section, we introduce the following *upper 2-star inequalities*:

$$(2-(\phi_1^U-\phi_2^U))w_{u,v}+w(\{u,v\},V-\{u,v\}) \leq \phi_1^U+\phi_2^U-2 \qquad \forall u,v \in V. \ (4.77)$$

The proofs of the following theorems are very similar to their counterparts in Section 4.11, so we present them without proofs. Just like in Section 4.11, when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, the upper 2-star inequalities are facets of $\mathcal{P}^{lu}$ if and only if the upper bound inequalities (4.16) are not facet defining.

**Theorem 4.12.1.** *Suppose that* $\mathcal{P}^{lu}$ *is full-dimensional. Then the upper 2-star inequalities (4.77) define facets of* $\mathcal{P}^{lu}$ *if and only if one of the following hold:*

*(i)* $\phi_1^U \geq \phi_2^U + 2$.

*(ii)* $\phi_1^U = \phi_2^U + 1$, $F_U - F_L \geq 3$.

**Theorem 4.12.2.** *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. *The upper 2-star inequalities (4.77) are facet defining for* $\mathcal{P}^{lu}$ *if and only if*

- $(n \ mod \ F_U) = F_U - 1$ *and* $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2$, *or,*
- $(n \ mod \ F_L) = 1$.

**Proposition 4.12.1.** *Suppose* $dim(\mathcal{P}^{lu}) \leq \binom{n}{2} - (n-1)$. *The upper 2-star inequalities (4.70) are not facet defining for* $\mathcal{P}^{lu}$.

# Chapter 5

# Computational Experimentation

In this chapter[1], we present our computational experience with a simple branch-and-cut algorithm which employs the strong valid inequalities that are highlighted in Chapter 4. Namely, we test the effectiveness in solving PHLRP of the following strong valid inequalities from Chapter 4:

- (4.19) from Section 4.7 (the 2-partition inequalities),
- (4.52) from Section 4.8 (the lower general clique inequalities),
- (4.55) from Section 4.9 (the upper general clique inequalities),
- (4.66) from Section 4.10 (the cycle inequalities).

We exclude the lower and upper 2-star inequalities ((4.70) from Section 4.11 and (4.77) from Section 4.12) from our discussion in this chapter. Because they are facet defining for the size-constrained graph partitioning polytope only under very restrictive conditions and hence are not as strong as the above inequalities for PHLRP.

We run our experiments on Formulation 3 as it emerges as the most efficient of the three in our computational experiments in Section 3.2.

PHLRP is a very difficult problem. For this reason, in our experiments we ignore assumption (A3) and consider only the un-capacitated version of the problem. That is, we exploit the reduction implied by Proposition 2.7.2 and run our experiments on this reduced version of Formulation 3. For the sake

[1]The content of this chapter is going to be published in February 2011: "A branch-and-cut algorithm for the partitioning-hub location-routing problem", *Computers & Operations Research*, 38(12), 539–549.

| Impl. ID | Description |
|---|---|
| 1 | Form. 3C with Xpress proprietary cuts |
| 2 | Form. 3C with 2-partition ineq, i.e., (4.19) |
| 3 | Form. 3C with lower gen. clique ineq.,i.e., (4.52) |
| 4 | Form. 3C with upper gen. clique ineq.,i.e., (4.55) |
| 5 | Form. 3C with cycle ineq., i.e., (4.66) |
| 6 | Form. 3C with all ineq., i.e., (4.19), (4.52), (4.55), (4.66) |
| 7 | Form. 3C with Xpress proprietary cuts and all ineq., i.e., (4.19), (4.52), (4.55), (4.66) |

Table 5.1: List of the seven implementations of Formulation 3C used in the experiments.

of simplicity, in the rest of this chapter we refer to this compact version of Formulation 3 as *Formulation 3C*.

## 5.1 Computational experimentation details

We consider instances having 9, 10, 12, 15 and 20 vertices, respectively. For each value of network size $|V|$ we consider different values for the edge density Ed= $|E|/\binom{|V|}{2}$, namely 0.3, 0.6, and 0.9, respectively. Similarly, we consider three possible values for the demand density Dd=$|T|/\binom{|V|}{2}$ equal to 0.3, 0.6, and 1.0, respectively. For each possible assignment of network size $|V|$, Ed and Dd, we generated 10 random instances of PHLRP, leading to an overall number of 450 instances of the problem.

We implemented Formulation 3C by means of Mosel 2.0 of Xpress-MP, Optimizer version 18, running on a 2.8GHz Intel Core i7 CPU, equipped with 3 GByte RAM and Windows Vista operating system. The Xpress-MP primal heuristic was turned on and the maximum running time was limited to 1 hour per instance.

We implemented seven versions of the Formulation 3C, each one employing a different cut set (see Table 5.1). In the 1$^{\text{st}}$ version we relied only on Xpress proprietary cuts. In the 2$^{\text{nd}}$, 3$^{\text{rd}}$, 4$^{\text{th}}$ and 5$^{\text{th}}$ versions we deactivated Xpress proprietary cuts and used the valid inequalities (4.19), (4.52), (4.55) and (4.66), respectively. In the 6$^{\text{th}}$ version we used valid inequalities (4.19), (4.52), (4.55) and (4.66) jointly while keeping off Xpress proprietary cuts. Finally, in the 7$^{\text{th}}$ version we used Xpress proprietary cuts in combination with valid inequal-

ities (4.19), (4.52), (4.55) and (4.66). We used the following algorithms for separating inequalities (4.19), (4.52), (4.55) and (4.66):

**Separation oracle for the 2-partition inequalities** We used a slightly modified version of the heuristic proposed by [14] and [32] to separate the 2-partition inequalities. Specifically, for every vertex $v \in V$ we consider the set of vertices $W := \{v' \in V \setminus \{v\} \ | w(v, v') > 0\}$. Then, we randomly pick two vertices, say $i$ and $j$, from $W$ and set $T = \{i, j\}$. Subsequently, for every vertex $k$ in $W - T$, we check if $w(k, v) - w(E(\{k\}, T))$ is strictly positive and if so we set $T = T \cup \{k\}$. Finally, we check if $|T| \geq 3$ and $w(E(\{v\}, T) - w(E(T)) > 1$. If both conditions hold, we add the violated inequality $w(E(\{v\}, T) - w(E(T)) \leq 1$. Note that, if $|T| = 2$, the 2-partition inequality reduces to a triangle inequality, which is already present in the formulation.

**Separation oracle for the lower general cliques inequalities** As observed in Section 4.8, inequalities (4.52) are not facet defining for the size-constrained graph partitioning polytope when $|Q| \leq \left\lfloor \frac{|V|}{F_L} \right\rfloor$. This fact can be exploited for separating the lower general clique inequalities. Specifically, for every $t \in \left\{ \left\lfloor \frac{|V|}{F_L} \right\rfloor + 1, \left\lfloor \frac{|V|}{F_L} \right\rfloor + 2, \ldots, |V| \right\}$ we randomly pick a subset $Q$ of $V$ such that $|Q| = t$. We check whether $w(E(Q)) < (k - q)\binom{p}{2} + q\binom{p+1}{2}$ and if so, we add the inequality $w(E(Q)) \geq (k - q)\binom{p}{2} + q\binom{p+1}{2}$. We apply this procedure 100 times to every fractional solution encountered in the branch-and-cut tree.

**Separation oracle for the upper general cliques inequalities** As observed in Section 4.9, inequalities (4.55) are not facet defining for the size-constrained graph partitioning polytope when $|Q| \leq \phi_1^U$ or $|Q| \geq |V| - \phi_{\bar{k}}^U$, where $\bar{k} = \left\lceil \frac{|V|}{F_U} \right\rceil$. This fact can be exploited for separating the upper general clique inequalities. Specifically, for every $t \in \left\{ \phi_1^U + 1, \phi_1^U + 2, \ldots, |V| - \phi_{\bar{k}}^U - 1 \right\}$, we randomly pick a subset $Q$ of $V$ such that $|Q| = t$ and we check whether $w(E(Q)) > \sum_{l=1}^{k_Q} \binom{\phi_l^U}{2} + \binom{n_Q}{2}$. If so we add the inequality $w(E(Q)) \leq \sum_{l=1}^{k_Q} \binom{\phi_l^U}{2} + \binom{n_Q}{2}$. We apply this procedure 100 times to every fractional solution encountered in the branch-and-cut tree. The whole procedure can be accelerated by precomputing the values $\{\phi_i^U\}$, $i = 1, \ldots, \bar{k}$.

**Separation oracle for the cycle inequalities** The cycle inequalities are separated by means of a heuristic algorithm working as follows. For every vertex $v \in V$ we set $V_c := \{v\}$, and we add the remaining vertices to $V_c$ by picking at random $(\phi_1^U - 1)$ vertices in $V \setminus \{v\}$. Subsequently, we check whether $w(E_c) > \phi_1^U - 2$. If so we add the inequality $w(E_c) \leq \phi_1^U - 2$.

### 5.1.1 Discussion

In this section we provide a general overview of the numerical experiments performed on the considered 450 random instances of the PHLRP. As general trend, implementation 7 performed the best, resulting, in average, faster and tighter than any other implementation. Tables 5.2-5.7 show a more detailed analysis of the results. Specifically, Tables 5.2 and 5.3 show the performances of the seven implementations with respect to the solution time. Tables 5.4 and 5.5 show the performances of the implementations with respect to the gap, i.e., the difference between the optimal value found and the value of linear relaxation at root node of the branch-and-cut tree, divided by the optimal value. Finally, Tables 5.6 and 5.7 show the performances of the implementations with respect to the number of nodes explored in the branch-and-cut tree.

Each row of Tables 5.2 and 5.3 shows, for each implementation, the average, the maximum and the minimum solution time required to solve the corresponding 10 random instances. When the running time exceeds the time limit, it appears as around 3600 seconds in the tables. The $13^{\text{th}}$ and $14^{\text{th}}$ columns of Table 5.3 display the best and the worst performing implementations with respect to the average solution times. The columns show that, as size gets larger, implementation 7 performs generally better than the others. On the contrary, the exclusive use of Xpress proprietary cuts (i.e., implementation 1) generally leads to poor performances. This fact clearly shows that the valid inequalities (4.19), (4.52), (4.55) and (4.66) bring significant improvements on the solution time.

Implementation 2 turns out to be generally the worst in terms of solution time. This fact is in contrast with Tables 5.4 and 5.5 which show the ability of implementation 2 to provide tight lower bounds for the problem. This behavior is possibly due to the hardness of separating inequalities (4.19), so that the time spent for the separation is not worth its use. On the other hand, we can observe from the penultimate column of Table 5.3 that implementation 6 performs the best in a considerable number of cases. Implementations 3 and 5 also perform very well and yield quite good solution times when compared with the other

implementations. In absence of Xpress proprietary cuts we would recommend the use of valid inequalities (4.19), (4.52), (4.55) and (4.66) jointly in a branch-and-cut framework.

Tables 5.4 and 5.5 show the effectiveness of the valid inequalities in providing tight lower bounds for the problem. Each row displays, for each implementation, the average, the maximum, and the minimum gap on the corresponding 10 random instances. If for some instance the running time exceeded the time limit, the best so far primal bound is used instead of the optimal value.

We observed that implementation 7 performs generally better by always yielding the best average gaps. When we exclude implementation 7 from the analysis we observed that implementation 6 generally performs the best and globally better than implementation 1 (see the $13^{\text{th}}$ and $14^{\text{th}}$ columns of Table 5.5). This fact confirms again that the joint use of valid inequalities (4.19), (4.52), (4.55) and (4.66) significantly improves over Xpress proprietary cuts in terms of LP relaxations. As implementation 6 employs all the corresponding valid inequalities there is no surprise in seeing that it performs better than implementations 2-5. Interestingly, due to the random nature of our separation algorithms, in some cases implementation 6 is beaten by another one. However, this fact is observed very rarely and does not change the conclusion that implementation 6 is a reliable option to obtain tight LP relaxations for formulation Formulation 3C.

The last two columns of Table 5.5 show a comparative analysis of implementations 2-5. Specifically, the results showed that implementations 2 and 5 performs the best and the worst respectively. Tables 5.4 and 5.5 also highlight that implementation 2 always performs better than implementation 1, and that implementations 3 and 4 perform at least as good as implementation 1 in most of the cases. This fact suggests that valid inequalities (4.19), (4.52) and (4.55) individually constitute sharp cuts for formulation Formulation 3C: they strengthen the formulation more than Xpress proprietary cut strategy does.

Tables 5.6 and 5.7 compare the number of nodes explored in the branch-and-cut tree. Although implementations 1 and 2 appear to explore less nodes, the results do not exhibit a clear pattern. Possibly, this is due to our interference with the branch-and-cut implementation of Xpress Optimizer. Apparently, the Xpress proprietary cuts and search strategy help to reduce the size of the branch-and-cut tree, although this does not automatically implies a quick convergence to the optimum.

| | | | Time (sec.) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1st cut set: Xpress cuts | | | 2nd cut set: Valid Ineq. (4.19) | | | 3rd cut set: Valid Ineq. (4.52) | | | 4th cut set: Valid Ineq. (4.55) | | |
| $\|V\|$ | Ed | Dd | Average | Max | Min | Average | Max | Min | Average | Max | Min | Average | Max | Min |
| 9 | 0.3 | 0.3 | 0.32 | 0.92 | 0.03 | 0.46 | 0.77 | 0.02 | 0.70 | 2.09 | 0.25 | 0.34 | 0.75 | 0.14 |
| 9 | 0.3 | 0.6 | 0.77 | 2.02 | 0.02 | 0.47 | 1.45 | 0.02 | 0.66 | 1.50 | 0.02 | 0.59 | 1.66 | 0.02 |
| 9 | 0.3 | 1.0 | 1.10 | 2.23 | 0.00 | 1.24 | 2.78 | 0.00 | 1.21 | 2.97 | 0.00 | 0.91 | 2.13 | 0.00 |
| 9 | 0.6 | 0.3 | 1.01 | 2.47 | 0.02 | 1.08 | 3.19 | 0.14 | 0.98 | 2.16 | 0.02 | 0.87 | 1.75 | 0.02 |
| 9 | 0.6 | 0.6 | 2.82 | 5.06 | 1.28 | 1.59 | 3.80 | 0.58 | 2.24 | 3.78 | 0.84 | 1.51 | 2.67 | 0.59 |
| 9 | 0.6 | 1.0 | 5.58 | 9.92 | 1.22 | 2.20 | 5.84 | 0.47 | 3.47 | 6.89 | 0.91 | 2.23 | 8.02 | 0.27 |
| 9 | 0.9 | 0.3 | 3.12 | 7.42 | 0.73 | 0.88 | 3.58 | 0.20 | 2.37 | 7.99 | 0.75 | 1.48 | 4.53 | 0.49 |
| 9 | 0.9 | 0.6 | 5.75 | 9.36 | 0.03 | 1.19 | 2.58 | 0.02 | 2.55 | 4.09 | 0.17 | 1.07 | 2.89 | 0.11 |
| 9 | 0.9 | 1.0 | 10.42 | 21.22 | 3.77 | 2.41 | 4.39 | 0.67 | 4.69 | 7.74 | 2.25 | 2.53 | 5.80 | 0.72 |
| 10 | 0.3 | 0.3 | 2.07 | 4.80 | 0.03 | 1.10 | 2.69 | 0.02 | 1.48 | 2.99 | 0.14 | 1.09 | 2.34 | 0.02 |
| 10 | 0.3 | 0.6 | 3.81 | 6.66 | 0.48 | 2.04 | 4.67 | 0.94 | 3.31 | 11.47 | 0.49 | 1.61 | 3.39 | 0.41 |
| 10 | 0.3 | 1.0 | 7.37 | 17.00 | 1.97 | 3.91 | 13.17 | 1.13 | 6.98 | 32.72 | 0.56 | 4.43 | 22.70 | 0.58 |
| 10 | 0.6 | 0.3 | 5.85 | 9.86 | 1.63 | 2.22 | 3.66 | 0.88 | 3.06 | 6.14 | 0.78 | 1.34 | 2.33 | 0.50 |
| 10 | 0.6 | 0.6 | 7.94 | 12.78 | 4.14 | 4.58 | 13.94 | 1.73 | 3.80 | 7.75 | 0.89 | 3.62 | 15.06 | 0.77 |
| 10 | 0.6 | 1.0 | 17.87 | 38.83 | 5.97 | 11.13 | 29.83 | 2.84 | 9.00 | 22.56 | 2.19 | 9.81 | 45.44 | 1.17 |
| 10 | 0.9 | 0.3 | 7.30 | 16.50 | 1.84 | 3.14 | 6.77 | 1.17 | 2.66 | 7.05 | 1.31 | 3.24 | 12.41 | 0.42 |
| 10 | 0.9 | 0.6 | 14.90 | 37.22 | 5.95 | 7.24 | 21.55 | 1.00 | 4.36 | 9.86 | 1.08 | 5.68 | 15.70 | 1.44 |
| 10 | 0.9 | 1.0 | 28.84 | 63.47 | 7.17 | 12.05 | 58.64 | 1.11 | 18.18 | 96.09 | 1.19 | 15.89 | 90.03 | 0.97 |
| 12 | 0.3 | 0.3 | 8.98 | 31.31 | 1.30 | 5.82 | 13.77 | 1.08 | 8.64 | 18.94 | 1.02 | 4.65 | 13.67 | 0.30 |
| 12 | 0.3 | 0.6 | 12.05 | 32.17 | 3.78 | 2.89 | 5.97 | 1.39 | 8.57 | 24.41 | 2.53 | 4.66 | 9.85 | 0.91 |
| 12 | 0.3 | 1.0 | 14.16 | 28.72 | 0.27 | 18.66 | 69.39 | 0.66 | 15.29 | 72.80 | 1.95 | 9.08 | 29.97 | 1.28 |
| 12 | 0.6 | 0.3 | 11.11 | 58.92 | 0.31 | 6.80 | 38.80 | 0.41 | 12.81 | 83.28 | 0.25 | 4.90 | 23.59 | 0.11 |
| 12 | 0.6 | 0.6 | 38.03 | 127.35 | 13.36 | 17.54 | 44.02 | 1.69 | 24.78 | 77.27 | 2.95 | 28.18 | 90.02 | 2.30 |
| 12 | 0.6 | 1.0 | 66.09 | 150.69 | 8.58 | 38.73 | 135.08 | 2.52 | 32.33 | 109.78 | 4.56 | 35.75 | 153.24 | 2.39 |
| 12 | 0.9 | 0.3 | 12.86 | 46.69 | 1.89 | 6.82 | 18.92 | 1.25 | 7.65 | 18.89 | 2.08 | 11.14 | 32.63 | 3.47 |
| 12 | 0.9 | 0.6 | 30.40 | 53.22 | 10.00 | 15.02 | 47.14 | 2.56 | 19.12 | 45.31 | 5.20 | 19.64 | 38.38 | 5.61 |
| 12 | 0.9 | 1.0 | 59.98 | 141.27 | 2.36 | 20.56 | 76.22 | 2.19 | 32.72 | 240.64 | 0.61 | 34.80 | 201.14 | 1.77 |
| 15 | 0.3 | 0.3 | 33.73 | 71.69 | 14.03 | 11.71 | 42.44 | 2.66 | 13.71 | 29.58 | 3.20 | 17.97 | 33.57 | 6.78 |
| 15 | 0.3 | 0.6 | 54.08 | 84.46 | 12.25 | 41.94 | 142.66 | 8.08 | 55.08 | 153.33 | 10.23 | 55.53 | 120.45 | 11.84 |
| 15 | 0.3 | 1.0 | 144.45 | 388.97 | 9.72 | 107.36 | 341.47 | 14.39 | 132.93 | 358.39 | 11.31 | 92.74 | 302.74 | 12.95 |
| 15 | 0.6 | 0.3 | 48.23 | 146.92 | 10.84 | 30.81 | 87.53 | 12.42 | 62.95 | 169.94 | 10.84 | 41.59 | 184.21 | 8.88 |
| 15 | 0.6 | 0.6 | 73.02 | 167.67 | 11.06 | 46.13 | 112.72 | 14.55 | 56.00 | 163.56 | 14.33 | 36.85 | 147.25 | 6.99 |
| 15 | 0.6 | 1.0 | 234.64 | 929.85 | 41.06 | 222.61 | 1019.03 | 27.66 | 187.32 | 917.88 | 27.06 | 136.60 | 784.69 | 11.13 |
| 15 | 0.9 | 0.3 | 122.53 | 373.75 | 21.38 | 87.82 | 287.16 | 20.73 | 127.23 | 304.63 | 13.25 | 60.51 | 212.79 | 8.22 |
| 15 | 0.9 | 0.6 | 508.47 | 3599.76 | 41.36 | 473.41 | 3599.93 | 17.33 | 313.59 | 2018.29 | 14.88 | 340.83 | 2386.35 | 11.33 |
| 15 | 0.9 | 1.0 | 204.51 | 722.67 | 34.22 | 284.03 | 798.24 | 55.97 | 286.92 | 812.31 | 27.88 | 268.03 | 828.57 | 33.42 |
| 20 | 0.3 | 0.3 | 777.55 | 3599.83 | 21.42 | 812.37 | 2444.62 | 17.63 | 933.47 | 3599.06 | 10.95 | 769.55 | 3599.64 | 17.00 |
| 20 | 0.3 | 0.6 | 2698.56 | 3599.86 | 209.22 | 2732.19 | 3599.97 | 238.17 | 2501.49 | 3600.11 | 260.89 | 2672.87 | 3599.94 | 240.11 |
| 20 | 0.3 | 1.0 | 2325.85 | 3599.68 | 165.32 | 2082.02 | 3599.94 | 56.75 | 1727.37 | 3599.91 | 42.75 | 2274.20 | 3599.93 | 112.99 |
| 20 | 0.6 | 0.3 | 1329.04 | 3599.53 | 54.56 | 1159.43 | 3599.58 | 96.24 | 832.77 | 3599.03 | 40.52 | 1204.35 | 3599.64 | 38.22 |
| 20 | 0.6 | 0.6 | 2664.42 | 3600.04 | 102.67 | 2502.63 | 3600.07 | 93.17 | 2394.14 | 3600.11 | 75.17 | 2499.41 | 3599.83 | 74.84 |
| 20 | 0.6 | 1.0 | 3371.46 | 3599.83 | 1320.56 | 3136.07 | 3600.05 | 860.57 | 3362.26 | 3600.54 | 1226.28 | 3364.53 | 3600.03 | 1248.83 |
| 20 | 0.9 | 0.3 | 2583.07 | 3600.05 | 172.50 | 2214.88 | 3600.08 | 166.67 | 2307.30 | 3599.85 | 169.67 | 2085.34 | 3599.74 | 49.17 |
| 20 | 0.9 | 0.6 | 2937.88 | 3599.93 | 715.91 | 2717.52 | 3600.10 | 740.53 | 3316.32 | 3599.99 | 766.41 | 2972.07 | 3599.99 | 407.29 |
| 20 | 0.9 | 1.0 | 3466.21 | 3600.05 | 2265.20 | 3469.87 | 3600.13 | 2301.82 | 3599.76 | 3600.08 | 3599.11 | 3488.88 | 3599.98 | 2501.92 |

Table 5.2: Performances of the seven implementations of Formulation 3C with respect to the solution time (part 1).

| | | | Time (sec.) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $5^{th}$ cut set: Valid Ineq. (4.66) | | | $6^{th}$ cut set: Valid Ineq. (4.19), (4.52), (4.55) and (4.66) | | | $7^{th}$ cut set: Xpress Cuts + V.I.(4.19), (4.52), (4.55) and (4.66) | | | Among $1^{st}$-$7^{th}$ | | Among $2^{nd}$-$6^{th}$ | |
| $\|V\|$ | Ed | Dd | Average | Max | Min | Average | Max | Min | Average | Max | Min | Best | Worst | Best | Worst |
| 9 | 0.3 | 0.3 | 0.16 | 0.30 | 0.02 | 0.42 | 0.83 | 0.22 | 0.29 | 0.80 | 0.03 | 5 | 3 | 5 | 3 |
| 9 | 0.3 | 0.6 | 0.25 | 0.72 | 0.02 | 0.64 | 1.45 | 0.02 | 0.73 | 1.97 | 0.02 | 5 | 1 | 5 | 1 |
| 9 | 0.3 | 1.0 | 0.51 | 1.86 | 0.00 | 0.91 | 2.84 | 0.00 | 1.06 | 2.17 | 0.00 | 5 | 2 | 5 | 2 |
| 9 | 0.6 | 0.3 | 0.64 | 2.06 | 0.16 | 1.02 | 3.59 | 0.02 | 0.82 | 2.22 | 0.02 | 5 | 2 | 5 | 2 |
| 9 | 0.6 | 0.6 | 1.30 | 1.89 | 0.34 | 1.83 | 3.63 | 0.88 | 2.13 | 3.31 | 1.08 | 5 | 1 | 5 | 3 |
| 9 | 0.6 | 1.0 | 2.66 | 4.77 | 1.09 | 3.64 | 9.99 | 0.61 | 3.40 | 5.84 | 0.75 | 2 | 1 | 2 | 6 |
| 9 | 0.9 | 0.3 | 1.48 | 3.44 | 0.58 | 1.58 | 7.61 | 0.22 | 1.49 | 2.95 | 0.19 | 2 | 1 | 2 | 3 |
| 9 | 0.9 | 0.6 | 1.62 | 2.70 | 0.45 | 1.80 | 4.61 | 0.02 | 2.59 | 3.86 | 0.02 | 4 | 1 | 4 | 3 |
| 9 | 0.9 | 1.0 | 5.52 | 11.67 | 0.83 | 4.17 | 15.30 | 0.80 | 4.82 | 7.80 | 1.88 | 2 | 1 | 2 | 5 |
| 10 | 0.3 | 0.3 | 0.73 | 1.31 | 0.41 | 1.22 | 3.24 | 0.03 | 1.06 | 2.27 | 0.02 | 5 | 1 | 5 | 3 |
| 10 | 0.3 | 0.6 | 1.68 | 3.17 | 0.52 | 2.64 | 5.69 | 0.56 | 1.90 | 3.88 | 0.41 | 4 | 1 | 4 | 3 |
| 10 | 0.3 | 1.0 | 3.43 | 15.25 | 0.36 | 5.38 | 29.17 | 0.53 | 3.45 | 6.83 | 1.23 | 5 | 1 | 5 | 3 |
| 10 | 0.6 | 0.3 | 1.64 | 3.05 | 0.61 | 2.57 | 5.45 | 0.56 | 2.11 | 3.55 | 0.70 | 4 | 1 | 4 | 3 |
| 10 | 0.6 | 0.6 | 2.83 | 5.14 | 0.95 | 3.51 | 8.80 | 1.36 | 3.64 | 5.95 | 2.30 | 5 | 1 | 5 | 2 |
| 10 | 0.6 | 1.0 | 7.30 | 13.42 | 2.33 | 16.11 | 57.00 | 2.84 | 7.34 | 13.70 | 2.38 | 5 | 1 | 5 | 6 |
| 10 | 0.9 | 0.3 | 3.01 | 6.00 | 1.36 | 3.69 | 6.88 | 2.00 | 3.08 | 5.30 | 0.73 | 3 | 1 | 3 | 6 |
| 10 | 0.9 | 0.6 | 4.82 | 11.19 | 1.36 | 9.21 | 23.52 | 1.34 | 7.13 | 15.99 | 3.81 | 3 | 1 | 3 | 6 |
| 10 | 0.9 | 1.0 | 9.50 | 26.55 | 1.49 | 18.46 | 99.11 | 1.05 | 10.22 | 18.31 | 3.64 | 5 | 1 | 5 | 6 |
| 12 | 0.3 | 0.3 | 4.96 | 13.28 | 1.05 | 9.49 | 30.17 | 0.33 | 4.38 | 8.39 | 0.66 | 7 | 6 | 4 | 6 |
| 12 | 0.3 | 0.6 | 5.20 | 12.02 | 1.74 | 7.87 | 27.02 | 1.72 | 5.75 | 10.31 | 2.69 | 2 | 1 | 2 | 3 |
| 12 | 0.3 | 1.0 | 10.70 | 46.19 | 1.38 | 11.06 | 32.94 | 0.83 | 7.31 | 19.45 | 0.27 | 7 | 2 | 4 | 2 |
| 12 | 0.6 | 0.3 | 6.09 | 31.64 | 0.63 | 6.59 | 35.80 | 0.16 | 4.75 | 15.88 | 0.14 | 7 | 3 | 4 | 3 |
| 12 | 0.6 | 0.6 | 20.50 | 94.92 | 5.30 | 39.88 | 176.64 | 2.73 | 14.98 | 38.16 | 5.52 | 7 | 6 | 2 | 6 |
| 12 | 0.6 | 1.0 | 46.36 | 210.11 | 5.69 | 53.39 | 227.53 | 2.63 | 21.06 | 39.81 | 5.73 | 7 | 1 | | |
| 12 | 0.9 | 0.3 | 6.93 | 14.97 | 1.17 | 15.10 | 53.22 | 3.49 | 5.63 | 15.83 | 1.67 | 7 | 6 | 3 | 2 |
| 12 | 0.9 | 0.6 | 13.46 | 24.74 | 5.67 | 23.43 | 50.16 | 7.09 | 13.32 | 23.45 | 6.03 | 7 | 1 | 6 | 2 |
| 12 | 0.9 | 1.0 | 25.58 | 135.33 | 3.08 | 40.88 | 233.05 | 1.45 | 20.37 | 67.38 | 0.50 | 7 | 1 | 3 | 2 |
| 15 | 0.3 | 0.3 | 9.88 | 20.19 | 3.23 | 21.76 | 62.50 | 7.84 | 10.46 | 24.88 | 5.41 | 5 | 1 | 6 | 2 |
| 15 | 0.3 | 0.6 | 41.40 | 78.84 | 8.34 | 70.03 | 133.81 | 19.99 | 21.98 | 36.78 | 10.81 | 7 | 6 | 6 | 5 |
| 15 | 0.3 | 1.0 | 85.98 | 233.27 | 10.58 | 116.60 | 339.63 | 12.08 | 51.40 | 121.55 | 7.24 | 7 | 1 | 6 | 2 |
| 15 | 0.6 | 0.3 | 33.19 | 125.38 | 5.24 | 67.22 | 237.17 | 13.78 | 13.50 | 42.16 | 5.53 | 7 | 6 | 3 | 4 |
| 15 | 0.6 | 0.6 | 39.67 | 92.64 | 8.33 | 71.46 | 225.02 | 18.56 | 23.51 | 39.86 | 7.38 | 7 | 1 | 5 | 2 |
| 15 | 0.6 | 1.0 | 228.90 | 1501.73 | 11.08 | 220.51 | 1295.81 | 11.84 | 69.84 | 271.49 | 14.23 | 7 | 1 | 5 | 2 |
| 15 | 0.9 | 0.3 | 70.80 | 199.75 | 12.84 | 77.54 | 296.66 | 8.92 | 34.75 | 85.03 | 4.69 | 7 | 3 | 5 | 4 |
| 15 | 0.9 | 0.6 | 226.52 | 1432.84 | 12.47 | 321.16 | 2252.91 | 9.59 | 142.73 | 951.64 | 14.17 | 7 | 1 | 6 | 2 |
| 15 | 0.9 | 1.0 | 197.03 | 644.44 | 37.45 | 242.47 | 754.94 | 27.75 | 78.47 | 232.17 | 23.45 | 7 | 3 | 6 | 4 |
| 20 | 0.3 | 0.3 | 850.45 | 3599.82 | 12.48 | 920.30 | 3599.72 | 7.05 | 212.77 | 716.43 | 4.64 | 7 | 3 | 5 | 4 |
| 20 | 0.3 | 0.6 | 2645.31 | 3600.04 | 130.50 | 2797.62 | 3599.99 | 427.50 | 2153.49 | 3599.88 | 61.52 | 7 | 6 | 4 | 3 |
| 20 | 0.3 | 1.0 | 1970.84 | 3599.84 | 73.52 | 2119.13 | 3600.12 | 145.16 | 1617.00 | 3599.95 | 48.31 | 7 | 1 | 4 | 2 |
| 20 | 0.6 | 0.3 | 981.69 | 3599.71 | 36.74 | 1490.85 | 3599.30 | 91.33 | 580.50 | 2170.87 | 39.00 | 7 | 6 | 4 | 2 |
| 20 | 0.6 | 0.6 | 2565.72 | 3600.02 | 73.25 | 2570.47 | 3599.91 | 123.61 | 2033.48 | 3599.65 | 32.53 | 7 | 1 | 4 | 2 |
| 20 | 0.6 | 1.0 | 3202.41 | 3600.08 | 1229.05 | 3365.06 | 3599.91 | 1255.95 | 3208.31 | 3599.89 | 357.49 | 2 | 1 | 3 | 2 |
| 20 | 0.9 | 0.3 | 2364.13 | 3599.84 | 98.53 | 2153.84 | 3599.54 | 115.53 | 1592.37 | 3599.37 | 52.99 | 7 | 1 | 5 | 2 |
| 20 | 0.9 | 0.6 | 3160.54 | 3599.99 | 492.68 | 2994.83 | 3599.83 | 307.46 | 2160.30 | 3599.96 | 200.57 | 7 | 3 | 3 | 4 |
| 20 | 0.9 | 1.0 | 3599.63 | 3600.06 | 3599.14 | 3500.59 | 3599.74 | 2610.90 | 3466.21 | 3600.05 | 2265.20 | 1, 7 | 3 | 2 | 4 |

Table 5.3: Performances of the seven implementations of Formulation 3C with respect to the solution time (part 2).

| | | | Gap (%) | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1st cut set: Xpress cuts | | | 2nd cut set: Valid Ineq. (4.19) | | | 3rd cut set: Valid Ineq. (4.52) | | | 4th cut set: Valid Ineq. (4.55) | | |
| $\|V\|$ | Ed | Dd | Average | Max | Min | Average | Max | Min | Average | Max | Min | Average | Max | Min |
| 9 | 0.3 | 0.3 | 7.68 | 21.31 | 0.56 | 7.55 | 21.28 | 2.43 | 7.60 | 20.92 | 3.68 | 7.60 | 20.92 | 3.68 |
| 9 | 0.3 | 0.6 | 3.47 | 13.73 | 0.46 | 3.34 | 13.60 | 5.10 | 3.44 | 13.73 | 5.88 | 3.44 | 13.73 | 5.88 |
| 9 | 0.3 | 1.0 | 7.77 | 16.02 | 0.22 | 7.34 | 15.66 | 0.01 | 7.75 | 16.02 | 0.22 | 7.75 | 16.02 | 0.22 |
| 9 | 0.6 | 0.3 | 5.57 | 16.75 | 0.84 | 5.51 | 16.50 | 0.80 | 5.57 | 16.75 | 0.84 | 5.57 | 16.75 | 0.84 |
| 9 | 0.6 | 0.6 | 12.87 | 24.79 | 3.57 | 12.78 | 24.76 | 3.57 | 12.86 | 24.63 | 3.57 | 12.86 | 24.63 | 3.57 |
| 9 | 0.6 | 1.0 | 12.14 | 26.14 | 2.12 | 11.83 | 25.87 | 2.12 | 12.14 | 26.14 | 2.12 | 12.14 | 26.14 | 2.12 |
| 9 | 0.9 | 0.3 | 9.14 | 16.27 | 1.34 | 9.10 | 16.15 | 1.34 | 9.14 | 16.27 | 1.34 | 9.14 | 16.27 | 1.34 |
| 9 | 0.9 | 0.6 | 9.37 | 16.42 | 5.02 | 9.25 | 16.42 | 4.88 | 9.36 | 16.42 | 5.02 | 9.36 | 16.42 | 5.02 |
| 9 | 0.9 | 1.0 | 10.32 | 17.45 | 4.28 | 10.05 | 17.45 | 4.10 | 10.32 | 17.45 | 4.28 | 10.32 | 17.45 | 4.28 |
| 10 | 0.3 | 0.3 | 7.63 | 23.80 | 2.11 | 7.63 | 23.80 | 0.54 | 7.63 | 23.80 | 2.11 | 7.63 | 23.80 | 2.11 |
| 10 | 0.3 | 0.6 | 11.54 | 31.37 | 0.43 | 11.51 | 31.37 | 0.43 | 11.54 | 31.37 | 0.43 | 11.54 | 31.37 | 0.43 |
| 10 | 0.3 | 1.0 | 11.43 | 21.92 | 3.08 | 11.37 | 21.92 | 3.08 | 11.43 | 21.92 | 3.08 | 11.43 | 21.92 | 3.08 |
| 10 | 0.6 | 0.3 | 7.28 | 14.68 | 3.33 | 7.25 | 14.61 | 3.33 | 7.28 | 14.68 | 3.33 | 7.28 | 14.68 | 3.33 |
| 10 | 0.6 | 0.6 | 8.89 | 20.45 | 3.20 | 8.86 | 20.45 | 3.15 | 8.89 | 20.45 | 3.18 | 8.89 | 20.45 | 3.18 |
| 10 | 0.6 | 1.0 | 13.00 | 18.59 | 7.38 | 12.85 | 18.56 | 7.38 | 12.98 | 18.59 | 7.38 | 12.98 | 18.59 | 7.38 |
| 10 | 0.9 | 0.3 | 8.57 | 15.89 | 0.14 | 8.55 | 15.80 | 0.14 | 8.56 | 15.89 | 0.14 | 8.56 | 15.89 | 0.14 |
| 10 | 0.9 | 0.6 | 10.59 | 18.03 | 3.25 | 10.48 | 17.91 | 3.01 | 10.58 | 18.03 | 3.25 | 10.58 | 18.03 | 3.25 |
| 10 | 0.9 | 1.0 | 11.44 | 19.52 | 1.61 | 11.25 | 19.08 | 1.55 | 11.43 | 19.52 | 1.61 | 11.43 | 19.52 | 1.61 |
| 12 | 0.3 | 0.3 | 7.77 | 14.97 | 0.75 | 7.66 | 14.70 | 0.75 | 7.77 | 14.97 | 0.75 | 7.77 | 14.97 | 0.75 |
| 12 | 0.3 | 0.6 | 10.27 | 21.61 | 4.22 | 10.06 | 21.47 | 4.06 | 10.27 | 21.61 | 4.21 | 10.27 | 21.61 | 4.21 |
| 12 | 0.3 | 1.0 | 9.43 | 19.69 | 3.10 | 9.33 | 19.61 | 3.10 | 9.43 | 19.69 | 0.89 | 9.43 | 19.69 | 0.01 |
| 12 | 0.6 | 0.3 | 6.11 | 15.17 | 0.30 | 6.08 | 14.99 | 0.25 | 6.11 | 15.17 | 0.30 | 6.11 | 15.17 | 0.30 |
| 12 | 0.6 | 0.6 | 10.95 | 19.82 | 3.58 | 10.81 | 19.53 | 3.57 | 10.95 | 19.82 | 3.58 | 10.95 | 19.82 | 3.58 |
| 12 | 0.6 | 1.0 | 9.71 | 15.77 | 2.42 | 9.52 | 15.51 | 2.39 | 9.71 | 15.77 | 2.42 | 9.71 | 15.77 | 2.42 |
| 12 | 0.9 | 0.3 | 7.69 | 15.41 | 1.79 | 7.65 | 15.33 | 1.77 | 7.69 | 15.41 | 1.79 | 7.69 | 15.41 | 1.79 |
| 12 | 0.9 | 0.6 | 9.50 | 16.03 | 2.24 | 9.34 | 15.98 | 2.14 | 9.45 | 16.03 | 1.78 | 9.45 | 16.03 | 1.78 |
| 12 | 0.9 | 1.0 | 8.62 | 16.34 | 0.03 | 8.47 | 16.26 | 0.03 | 8.62 | 16.34 | 0.03 | 8.62 | 16.34 | 0.03 |
| 15 | 0.3 | 0.3 | 4.75 | 9.53 | 1.55 | 4.71 | 9.48 | 1.55 | 4.75 | 9.53 | 1.55 | 4.75 | 9.53 | 1.55 |
| 15 | 0.3 | 0.6 | 7.31 | 10.48 | 2.70 | 7.21 | 10.33 | 2.52 | 7.31 | 10.48 | 2.70 | 7.31 | 10.48 | 2.70 |
| 15 | 0.3 | 1.0 | 9.17 | 13.11 | 3.52 | 9.06 | 13.01 | 3.52 | 9.17 | 13.11 | 3.52 | 9.17 | 13.11 | 3.52 |
| 15 | 0.6 | 0.3 | 5.13 | 10.09 | 2.24 | 5.10 | 9.93 | 2.24 | 5.13 | 10.09 | 2.24 | 5.13 | 10.09 | 2.24 |
| 15 | 0.6 | 0.6 | 5.36 | 8.88 | 1.51 | 5.28 | 8.79 | 1.51 | 5.36 | 8.88 | 1.51 | 5.36 | 8.88 | 1.51 |
| 15 | 0.6 | 1.0 | 7.63 | 13.53 | 2.31 | 7.48 | 13.41 | 2.23 | 7.63 | 13.53 | 2.31 | 7.63 | 13.53 | 2.31 |
| 15 | 0.9 | 0.3 | 6.56 | 9.11 | 0.61 | 6.52 | 9.10 | 0.61 | 6.56 | 9.11 | 0.61 | 6.56 | 9.11 | 0.61 |
| 15 | 0.9 | 0.6 | 8.66 | 13.70 | 2.17 | 8.44 | 13.53 | 2.07 | 8.66 | 13.70 | 2.17 | 8.66 | 13.70 | 2.17 |
| 15 | 0.9 | 1.0 | 7.89 | 11.84 | 3.31 | 7.77 | 11.66 | 3.31 | 7.89 | 11.84 | 3.31 | 7.89 | 11.84 | 3.31 |
| 20 | 0.3 | 0.3 | 8.88 | 13.75 | 0.63 | 8.54 | 13.14 | 0.52 | 9.19 | 15.73 | 0.63 | 9.12 | 15.01 | 0.63 |
| 20 | 0.3 | 0.6 | 15.80 | 24.33 | 7.63 | 14.83 | 20.19 | 7.44 | 14.69 | 19.60 | 7.63 | 15.39 | 21.02 | 7.63 |
| 20 | 0.3 | 1.0 | 11.76 | 18.67 | 5.70 | 10.71 | 15.85 | 5.54 | 10.43 | 16.08 | 5.70 | 11.21 | 19.12 | 5.70 |
| 20 | 0.6 | 0.3 | 10.64 | 16.77 | 5.92 | 10.10 | 15.40 | 5.56 | 10.70 | 17.99 | 5.92 | 10.83 | 19.27 | 5.92 |
| 20 | 0.6 | 0.6 | 15.98 | 29.66 | 3.92 | 13.69 | 26.37 | 3.91 | 14.53 | 26.69 | 3.92 | 14.89 | 26.69 | 3.92 |
| 20 | 0.6 | 1.0 | 18.55 | 32.91 | 6.26 | 16.31 | 23.98 | 6.11 | 17.96 | 28.60 | 6.26 | 17.61 | 26.64 | 6.26 |
| 20 | 0.9 | 0.3 | 14.10 | 23.91 | 4.04 | 12.57 | 20.95 | 3.78 | 13.00 | 22.10 | 4.04 | 12.69 | 21.23 | 4.04 |
| 20 | 0.9 | 0.6 | 13.89 | 21.67 | 7.81 | 13.06 | 20.05 | 7.22 | 14.44 | 22.56 | 7.81 | 14.08 | 21.32 | 7.81 |
| 20 | 0.9 | 1.0 | 21.86 | 37.01 | 6.04 | 18.44 | 27.86 | 5.82 | 18.68 | 32.04 | 7.70 | 17.36 | 30.10 | 6.04 |

Table 5.4: Performances of the seven implementations of Formulation 3C with respect to the gap (part 1).

| | | | Gap (%) | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 5th cut set: Valid Ineq. (4.66) | | | 6th cut set: Valid Ineq. (4.19), (4.52), (4.55) and (4.66) | | | 7th cut set: Xpress Cuts + V.I.(4.19), (4.52), (4.55) and (4.66) | | | Among 1st-6th | | Among 2nd-5th | |
| $\|V\|$ | Ed | Dd | Average | Max | Min | Average | Max | Min | Average | Max | Min | Best | Worst | Best | Worst |
| 9 | 0.3 | 0.3 | 6.60 | 31.35 | 10.57 | 6.81 | 21.19 | 1.58 | 6.28 | 20.53 | 1.37 | 5 | 1 | 5 | 3, 4 |
| 9 | 0.3 | 0.6 | 6.52 | 30.72 | 9.13 | 2.48 | 13.35 | 4.83 | 1.65 | 12.80 | 4.36 | 6 | 5 | 2 | 5 |
| 9 | 0.3 | 1.0 | 6.85 | 33.09 | 10.08 | 7.23 | 15.52 | 0.55 | 6.54 | 15.02 | 0.11 | 5 | 1 | 5 | 3, 4 |
| 9 | 0.6 | 0.3 | 14.13 | 23.35 | 0.84 | 5.13 | 16.26 | 0.98 | 4.64 | 15.43 | 0.56 | 6 | 5 | 2 | 5 |
| 9 | 0.6 | 0.6 | 26.59 | 39.21 | 11.26 | 11.90 | 24.57 | 2.75 | 11.56 | 24.48 | 2.66 | 6 | 5 | 2 | 5 |
| 9 | 0.6 | 1.0 | 19.93 | 34.22 | 15.11 | 11.39 | 25.32 | 1.90 | 11.08 | 24.36 | 1.17 | 6 | 5 | 2 | 5 |
| 9 | 0.9 | 0.3 | 15.94 | 25.68 | 4.64 | 8.54 | 15.60 | 0.64 | 8.51 | 14.93 | 0.78 | 6 | 5 | 2 | 5 |
| 9 | 0.9 | 0.6 | 19.13 | 34.44 | 3.86 | 8.57 | 15.68 | 4.56 | 7.61 | 15.26 | 4.11 | 6 | 5 | 2 | 5 |
| 9 | 0.9 | 1.0 | 20.52 | 35.11 | 5.30 | 9.36 | 16.46 | 3.45 | 9.24 | 16.22 | 3.45 | 6 | 5 | 2 | 5 |
| 10 | 0.3 | 0.3 | 9.32 | 19.94 | 2.11 | 7.57 | 22.95 | 0.64 | 7.55 | 22.48 | 0.39 | 6 | 5 | 3, 4 | 5 |
| 10 | 0.3 | 0.6 | 12.35 | 30.89 | 2.76 | 11.11 | 30.71 | 0.66 | 10.91 | 29.97 | 0.76 | 6 | 5 | 2 | 5 |
| 10 | 0.3 | 1.0 | 11.27 | 21.92 | 2.52 | 11.12 | 21.31 | 2.45 | 11.01 | 20.74 | 1.66 | 6 | 1, 3, 4 | 5 | 3, 4 |
| 10 | 0.6 | 0.3 | 7.19 | 14.48 | 3.33 | 6.58 | 14.07 | 2.90 | 6.11 | 13.27 | 1.98 | 6 | 1, 3, 4 | 5 | 3, 4 |
| 10 | 0.6 | 0.6 | 8.89 | 20.04 | 3.18 | 8.30 | 19.84 | 2.46 | 7.67 | 19.09 | 1.57 | 6 | 1 | 2 | 5 |
| 10 | 0.6 | 1.0 | 14.00 | 18.58 | 7.38 | 12.52 | 18.09 | 7.28 | 11.81 | 17.74 | 6.91 | 6 | 5 | 2 | 5 |
| 10 | 0.9 | 0.3 | 9.09 | 15.89 | 0.14 | 8.05 | 15.12 | 0.54 | 7.40 | 14.42 | 0.39 | 6 | 5 | 2 | 5 |
| 10 | 0.9 | 0.6 | 11.54 | 17.86 | 3.20 | 9.77 | 17.52 | 2.89 | 9.01 | 17.32 | 2.87 | 6 | 5 | 2 | 5 |
| 10 | 0.9 | 1.0 | 11.79 | 19.50 | 0.89 | 10.70 | 18.67 | 0.80 | 9.82 | 18.16 | 0.79 | 6 | 5 | 2 | 5 |
| 12 | 0.3 | 0.3 | 8.94 | 14.97 | 2.22 | 6.76 | 14.63 | 0.71 | 6.02 | 14.60 | 0.40 | 6 | 5 | 2 | 5 |
| 12 | 0.3 | 0.6 | 10.59 | 21.60 | 4.19 | 9.29 | 21.28 | 3.14 | 8.60 | 20.95 | 2.79 | 6 | 5 | 2 | 5 |
| 12 | 0.3 | 1.0 | 10.18 | 19.69 | 3.10 | 8.69 | 19.30 | 3.01 | 8.12 | 18.90 | 2.58 | 6 | 5 | 2 | 5 |
| 12 | 0.6 | 0.3 | 6.76 | 15.17 | 0.28 | 5.69 | 14.26 | 0.00 | 5.36 | 13.50 | 1.00 | 6 | 5 | 2 | 5 |
| 12 | 0.6 | 0.6 | 11.39 | 19.82 | 3.57 | 10.39 | 18.87 | 3.27 | 10.09 | 17.92 | 2.66 | 6 | 5 | 2 | 5 |
| 12 | 0.6 | 1.0 | 10.41 | 15.77 | 3.99 | 9.38 | 14.73 | 2.13 | 9.11 | 14.60 | 2.05 | 6 | 5 | 2 | 5 |
| 12 | 0.9 | 0.3 | 8.25 | 15.41 | 1.74 | 6.99 | 15.09 | 1.12 | 6.01 | 14.82 | 1.04 | 6 | 5 | 2 | 5 |
| 12 | 0.9 | 0.6 | 9.81 | 16.03 | 2.10 | 9.30 | 15.01 | 1.33 | 8.68 | 14.69 | 0.36 | 6 | 5 | 2 | 5 |
| 12 | 0.9 | 1.0 | 9.18 | 16.34 | 4.29 | 8.30 | 15.85 | 0.23 | 7.65 | 15.57 | 0.65 | 6 | 5 | 2 | 5 |
| 15 | 0.3 | 0.3 | 4.75 | 9.53 | 1.55 | 4.02 | 8.64 | 1.54 | 3.52 | 8.25 | 1.45 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.3 | 0.6 | 7.31 | 10.48 | 2.70 | 6.92 | 10.13 | 1.89 | 6.57 | 9.90 | 1.61 | 6 | 1, 3, 4 | 2 | 3, 4 |
| 15 | 0.3 | 1.0 | 9.17 | 13.11 | 3.52 | 8.65 | 12.41 | 3.34 | 7.80 | 11.96 | 2.64 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.6 | 0.3 | 5.13 | 10.09 | 2.24 | 4.37 | 9.01 | 1.96 | 3.66 | 8.95 | 1.87 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.6 | 0.6 | 5.36 | 8.88 | 1.51 | 5.03 | 8.54 | 0.76 | 4.17 | 8.39 | 0.30 | 6 | 1, 3, 4 | 2 | 3, 4 |
| 15 | 0.6 | 1.0 | 7.63 | 13.53 | 2.31 | 6.99 | 12.69 | 2.15 | 6.90 | 12.37 | 1.30 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.9 | 0.3 | 6.56 | 9.11 | 0.61 | 5.61 | 8.35 | 0.96 | 5.55 | 7.49 | 0.91 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 15 | 0.9 | 0.6 | 8.66 | 13.70 | 2.17 | 7.73 | 13.25 | 1.79 | 6.84 | 13.17 | 1.20 | 6 | 1, 3, 4 | 2 | 3, 4 |
| 15 | 0.9 | 1.0 | 7.89 | 11.84 | 3.31 | 6.78 | 10.86 | 3.03 | 6.73 | 10.34 | 2.63 | 6 | 1, 3, 4, 5 | 2 | 3, 4, 5 |
| 20 | 0.3 | 0.3 | 9.58 | 15.01 | 5.30 | 7.96 | 12.43 | 0.48 | 7.81 | 11.76 | 0.28 | 6 | 5 | 2 | 5 |
| 20 | 0.3 | 0.6 | 14.78 | 19.60 | 7.63 | 13.85 | 19.92 | 6.78 | 13.26 | 19.53 | 5.78 | 6 | 1 | 3 | 4 |
| 20 | 0.3 | 1.0 | 11.21 | 19.12 | 5.70 | 10.54 | 14.91 | 5.15 | 9.61 | 13.92 | 4.81 | 3 | 1 | 3 | 4, 5 |
| 20 | 0.6 | 0.3 | 10.83 | 19.27 | 5.92 | 9.75 | 15.23 | 5.31 | 8.94 | 14.73 | 5.28 | 6 | 4, 5 | 2 | 4, 5 |
| 20 | 0.6 | 0.6 | 14.89 | 26.69 | 3.92 | 12.74 | 26.28 | 2.95 | 12.56 | 25.41 | 2.54 | 6 | 1 | 2 | 4, 5 |
| 20 | 0.6 | 1.0 | 17.61 | 26.64 | 6.26 | 15.96 | 23.04 | 5.45 | 15.15 | 22.86 | 5.19 | 6 | 1 | 2 | 3 |
| 20 | 0.9 | 0.3 | 13.08 | 22.10 | 4.04 | 12.14 | 20.92 | 3.45 | 11.65 | 20.55 | 2.54 | 6 | 1 | 2 | 5 |
| 20 | 0.9 | 0.6 | 13.97 | 21.32 | 7.81 | 13.03 | 19.77 | 7.02 | 12.03 | 19.47 | 6.16 | 6 | 3 | 2 | 3 |
| 20 | 0.9 | 1.0 | 18.92 | 34.98 | 8.01 | 18.02 | 27.38 | 5.76 | 17.35 | 27.09 | 5.64 | 4 | 1 | 4 | 5 |

Table 5.5: Performances of the seven implementations of Formulation 3C with respect to the gap (part 2).

| Number of nodes in the branch-and-cut tree | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1st cut set: Xpress cuts | | | 2nd cut set: Valid Ineq. (4.19) | | | 3rd cut set: Valid Ineq. (4.52) | | | 4th cut set: Valid Ineq. (4.55) | | |
| $\|V\|$ | Ed | Dd | Average | Max | Min | Average | Max | Min | Average | Max | Min | Average | Max | Min |
| 9 | 0.3 | 0.3 | 4.2 | 19 | 1 | 4.2 | 21 | 1 | 5.4 | 17 | 1 | 6.4 | 15 | 1 |
| 9 | 0.3 | 0.6 | 8.1 | 27 | 1 | 8.1 | 21 | 1 | 7.5 | 25 | 1 | 7.7 | 25 | 1 |
| 9 | 0.3 | 1.0 | 7.7 | 27 | 1 | 7.7 | 29 | 1 | 9.7 | 27 | 1 | 9.1 | 27 | 1 |
| 9 | 0.6 | 0.3 | 8.9 | 31 | 1 | 8.9 | 57 | 1 | 11.8 | 35 | 1 | 11.8 | 35 | 1 |
| 9 | 0.6 | 0.6 | 28.8 | 131 | 1 | 28.8 | 95 | 7 | 29.6 | 81 | 5 | 28.2 | 81 | 5 |
| 9 | 0.6 | 1.0 | 42.9 | 187 | 1 | 42.9 | 139 | 1 | 36.8 | 115 | 1 | 42.4 | 145 | 1 |
| 9 | 0.9 | 0.3 | 38.2 | 171 | 1 | 38.2 | 147 | 1 | 39.0 | 199 | 1 | 39.0 | 199 | 1 |
| 9 | 0.9 | 0.6 | 16.2 | 39 | 1 | 16.2 | 71 | 1 | 19.8 | 51 | 1 | 20.0 | 51 | 1 |
| 9 | 0.9 | 1.0 | 39.4 | 215 | 1 | 39.4 | 167 | 1 | 42.4 | 195 | 1 | 42.4 | 195 | 1 |
| 10 | 0.3 | 0.3 | 33.5 | 153 | 1 | 33.5 | 203 | 1 | 30.3 | 81 | 1 | 29.9 | 81 | 1 |
| 10 | 0.3 | 0.6 | 50.4 | 147 | 1 | 50.4 | 149 | 5 | 60.0 | 199 | 9 | 63.6 | 199 | 7 |
| 10 | 0.3 | 1.0 | 69.8 | 271 | 9 | 69.8 | 229 | 7 | 79.8 | 441 | 5 | 79.8 | 441 | 5 |
| 10 | 0.6 | 0.3 | 48.2 | 127 | 9 | 48.2 | 77 | 9 | 49.0 | 119 | 13 | 49.6 | 119 | 13 |
| 10 | 0.6 | 0.6 | 64.4 | 255 | 11 | 64.4 | 289 | 13 | 84.1 | 388 | 11 | 84.1 | 388 | 11 |
| 10 | 0.6 | 1.0 | 143.0 | 283 | 25 | 143.0 | 596 | 19 | 175.2 | 764 | 17 | 175.2 | 764 | 17 |
| 10 | 0.9 | 0.3 | 78.2 | 365 | 17 | 78.2 | 220 | 11 | 112.8 | 382 | 11 | 112.8 | 382 | 11 |
| 10 | 0.9 | 0.6 | 179.2 | 827 | 9 | 119.2 | 549 | 9 | 123.3 | 352 | 11 | 123.3 | 352 | 11 |
| 10 | 0.9 | 1.0 | 196.5 | 651 | 1 | 196.5 | 719 | 1 | 201.2 | 959 | 1 | 202.8 | 959 | 1 |
| 12 | 0.3 | 0.3 | 90.3 | 249 | 1 | 90.3 | 503 | 1 | 147.8 | 427 | 1 | 148.4 | 427 | 1 |
| 12 | 0.3 | 0.6 | 71.2 | 249 | 15 | 61.2 | 241 | 13 | 84.4 | 257 | 9 | 84.4 | 257 | 9 |
| 12 | 0.3 | 1.0 | 96.1 | 511 | 1 | 96.1 | 495 | 2 | 105.5 | 605 | 2 | 105.5 | 605 | 2 |
| 12 | 0.6 | 0.3 | 165.5 | 1232 | 1 | 165.5 | 697 | 1 | 189.4 | 1299 | 1 | 189.4 | 1299 | 1 |
| 12 | 0.6 | 0.6 | 315.6 | 971 | 17 | 315.6 | 961 | 15 | 356.2 | 1334 | 13 | 356.2 | 1334 | 13 |
| 12 | 0.6 | 1.0 | 242.9 | 681 | 11 | 242.9 | 911 | 11 | 316.7 | 1519 | 7 | 316.7 | 1519 | 7 |
| 12 | 0.9 | 0.3 | 137.0 | 553 | 13 | 137.0 | 763 | 19 | 158.0 | 529 | 10 | 158.9 | 529 | 19 |
| 12 | 0.9 | 0.6 | 141.0 | 291 | 9 | 141.0 | 349 | 3 | 136.0 | 321 | 3 | 136.0 | 321 | 3 |
| 12 | 0.9 | 1.0 | 188.3 | 1041 | 1 | 188.3 | 903 | 1 | 172.9 | 1210 | 1 | 172.9 | 1210 | 1 |
| 15 | 0.3 | 0.3 | 174.4 | 649 | 15 | 174.4 | 439 | 19 | 123.3 | 516 | 29 | 123.3 | 516 | 29 |
| 15 | 0.3 | 0.6 | 229.9 | 453 | 13 | 229.9 | 902 | 19 | 292.9 | 677 | 21 | 292.9 | 677 | 21 |
| 15 | 0.3 | 1.0 | 679.5 | 1735 | 31 | 379.5 | 1484 | 21 | 355.1 | 999 | 18 | 355.1 | 999 | 18 |
| 15 | 0.6 | 0.3 | 192.5 | 1054 | 19 | 192.5 | 858 | 33 | 459.6 | 1328 | 35 | 455.0 | 1328 | 35 |
| 15 | 0.6 | 0.6 | 216.7 | 775 | 27 | 216.7 | 485 | 19 | 222.3 | 803 | 25 | 222.3 | 803 | 25 |
| 15 | 0.6 | 1.0 | 575.0 | 2465 | 43 | 475.0 | 1939 | 25 | 519.7 | 2987 | 27 | 519.7 | 2987 | 27 |
| 15 | 0.9 | 0.3 | 583.6 | 1579 | 5 | 583.6 | 1724 | 9 | 620.1 | 1373 | 19 | 620.1 | 1373 | 19 |
| 15 | 0.9 | 0.6 | 1476.6 | 11155 | 17 | 1450.2 | 10569 | 15 | 1109.5 | 7917 | 9 | 1109.7 | 7917 | 9 |
| 15 | 0.9 | 1.0 | 441.3 | 1528 | 27 | 441.3 | 1149 | 23 | 524.8 | 1675 | 39 | 524.8 | 1675 | 39 |
| 20 | 0.3 | 0.3 | 2097.4 | 5901 | 3 | 1105.0 | 4633 | 1 | 1615.6 | 5043 | 7 | 1715.9 | 5839 | 7 |
| 20 | 0.3 | 0.6 | 2704.3 | 3714 | 395 | 1089.9 | 5480 | 361 | 2475.2 | 4718 | 439 | 2391.9 | 3282 | 439 |
| 20 | 0.3 | 1.0 | 946.2 | 1688 | 49 | 1497.3 | 2845 | 29 | 1638.1 | 3484 | 27 | 1011.5 | 1991 | 27 |
| 20 | 0.6 | 0.3 | 3186.0 | 7523 | 133 | 4422.0 | 5440 | 97 | 2812.8 | 11927 | 63 | 2188.1 | 5680 | 63 |
| 20 | 0.6 | 0.6 | 2084.1 | 5571 | 51 | 2264.3 | 2782 | 63 | 2023.3 | 3560 | 57 | 1889.6 | 4357 | 57 |
| 20 | 0.6 | 1.0 | 1127.2 | 2064 | 525 | 1879.6 | 1426 | 173 | 1029.5 | 1508 | 620 | 1135.4 | 1510 | 620 |
| 20 | 0.9 | 0.3 | 3082.0 | 6722 | 141 | 1999.1 | 5391 | 242 | 3120.3 | 8110 | 107 | 4093.9 | 8136 | 107 |
| 20 | 0.9 | 0.6 | 2103.4 | 3358 | 301 | 2548.5 | 2559 | 627 | 1808.7 | 2967 | 317 | 2289.5 | 3520 | 317 |
| 20 | 0.9 | 1.0 | 1094.8 | 1697 | 548 | 1094.8 | 1903 | 927 | 948.7 | 1305 | 686 | 1820.6 | 2341 | 1581 |

Table 5.6: Performances of the seven implementations of Formulation 3C with respect to the nodes explored in the tree search (part 1).

| | | | Number of nodes in the branch-and-cut tree | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 5th cut set: Valid Ineq. (4.66) | | | 6th cut set: Valid Ineq. (4.19), (4.52), (4.55), (4.66) | | | 7th cut set: Xpress Cuts + V.I.(4.19), (4.52), (4.55), (4.66) | | | Best | Worst |
| $\|V\|$ | Ed | Dd | Average | Max | Min | Average | Max | Min | Average | Max | Min | cut set | cut set |
| 9 | 0.3 | 0.3 | 8.5 | 21 | 1 | 5.6 | 13 | 1 | 6.2 | 19 | 1 | 1, 2 | 5 |
| 9 | 0.3 | 0.6 | 9.6 | 51 | 1 | 7.8 | 25 | 1 | 5.8 | 27 | 1 | 7 | 5 |
| 9 | 0.3 | 1.0 | 18.7 | 103 | 1 | 9.7 | 27 | 1 | 10.1 | 27 | 1 | 1, 2 | 5 |
| 9 | 0.6 | 0.3 | 40.6 | 173 | 5 | 11.8 | 35 | 1 | 15.2 | 31 | 1 | 1, 2 | 5 |
| 9 | 0.6 | 0.6 | 63.4 | 103 | 15 | 29.0 | 81 | 5 | 29.8 | 131 | 1 | 4 | 5 |
| 9 | 0.6 | 1.0 | 60.0 | 145 | 15 | 39.6 | 117 | 1 | 43.4 | 187 | 1 | 3 | 5 |
| 9 | 0.9 | 0.3 | 57.9 | 179 | 11 | 38.2 | 199 | 1 | 32.2 | 171 | 1 | 7 | 5 |
| 9 | 0.9 | 0.6 | 36.4 | 79 | 1 | 19.8 | 51 | 1 | 21.7 | 39 | 1 | 1, 2 | 5 |
| 9 | 0.9 | 1.0 | 103.8 | 235 | 7 | 42.4 | 195 | 1 | 38.4 | 215 | 1 | 7 | 5 |
| 10 | 0.3 | 0.3 | 52.1 | 205 | 8 | 32.5 | 81 | 1 | 46.2 | 153 | 1 | 4 | 5 |
| 10 | 0.3 | 0.6 | 71.0 | 221 | 3 | 56.4 | 199 | 5 | 42.2 | 147 | 1 | 7 | 5 |
| 10 | 0.3 | 1.0 | 67.8 | 343 | 1 | 80.0 | 441 | 7 | 62.6 | 271 | 9 | 7 | 6 |
| 10 | 0.6 | 0.3 | 43.2 | 119 | 13 | 48.2 | 119 | 13 | 37.8 | 127 | 9 | 7 | 4 |
| 10 | 0.6 | 0.6 | 72.7 | 269 | 8 | 87.3 | 388 | 11 | 79.4 | 255 | 11 | 1, 2 | 6 |
| 10 | 0.6 | 1.0 | 146.1 | 391 | 15 | 176.0 | 764 | 17 | 167.5 | 283 | 25 | 1, 2 | 6 |
| 10 | 0.9 | 0.3 | 97.6 | 382 | 13 | 110.8 | 382 | 11 | 112.8 | 365 | 17 | 1, 2 | 3, 4, 7 |
| 10 | 0.9 | 0.6 | 144.9 | 441 | 7 | 130.9 | 352 | 11 | 162.3 | 827 | 9 | 2 | 1 |
| 10 | 0.9 | 1.0 | 170.4 | 769 | 1 | 201.8 | 959 | 1 | 195.9 | 651 | 1 | 5 | 4 |
| 12 | 0.3 | 0.3 | 172.9 | 457 | 9 | 147.6 | 427 | 1 | 140.5 | 249 | 1 | 1, 2 | 5 |
| 12 | 0.3 | 0.6 | 71.6 | 209 | 9 | 85.2 | 257 | 9 | 68.0 | 249 | 15 | 2 | 6 |
| 12 | 0.3 | 1.0 | 133.1 | 605 | 2 | 105.5 | 605 | 2 | 126.7 | 511 | 1 | 1, 2 | 5 |
| 12 | 0.6 | 0.3 | 205.7 | 1299 | 1 | 189.6 | 1299 | 1 | 159.6 | 1232 | 1 | 7 | 5 |
| 12 | 0.6 | 0.6 | 348.0 | 1334 | 19 | 341.4 | 1334 | 13 | 327.2 | 971 | 17 | 1, 2 | 3, 4 |
| 12 | 0.6 | 1.0 | 325.5 | 1519 | 15 | 316.7 | 1519 | 7 | 387.0 | 681 | 11 | 1, 2 | 7 |
| 12 | 0.9 | 0.3 | 157.8 | 529 | 9 | 156.7 | 529 | 10 | 204.2 | 553 | 13 | 1, 2 | 7 |
| 12 | 0.9 | 0.6 | 158.5 | 417 | 7 | 136.4 | 321 | 7 | 166.1 | 291 | 9 | 3, 4 | 7 |
| 12 | 0.9 | 1.0 | 199.3 | 1210 | 11 | 172.9 | 1210 | 1 | 176.1 | 1041 | 1 | 3, 4, 6 | 5 |
| 15 | 0.3 | 0.3 | 123.3 | 516 | 29 | 123.3 | 516 | 29 | 126.4 | 649 | 15 | 3, 4, 5, 6 | 1, 2 |
| 15 | 0.3 | 0.6 | 315.2 | 677 | 21 | 292.9 | 677 | 21 | 379.1 | 453 | 13 | 1, 2 | 7 |
| 15 | 0.3 | 1.0 | 355.1 | 999 | 18 | 355.1 | 999 | 18 | 403.9 | 1735 | 31 | 3, 4, 5, 6 | 1 |
| 15 | 0.6 | 0.3 | 419.7 | 1328 | 35 | 459.6 | 1328 | 35 | 300.4 | 1054 | 19 | 1, 2 | 3, 6 |
| 15 | 0.6 | 0.6 | 223.4 | 803 | 21 | 222.3 | 803 | 25 | 222.3 | 775 | 27 | 1, 2 | 5 |
| 15 | 0.6 | 1.0 | 519.7 | 2987 | 27 | 519.7 | 2987 | 27 | 460.9 | 2465 | 43 | 7 | 1 |
| 15 | 0.9 | 0.3 | 620.1 | 1373 | 19 | 620.1 | 1373 | 19 | 667.3 | 1579 | 5 | 1, 2 | 7 |
| 15 | 0.9 | 0.6 | 1107.6 | 7917 | 9 | 1109.5 | 7917 | 9 | 1187.5 | 11891 | 17 | 5 | 1 |
| 15 | 0.9 | 1.0 | 519.4 | 1675 | 39 | 524.8 | 1675 | 39 | 471.3 | 1528 | 27 | 1, 2 | 3, 4, 6 |
| 20 | 0.3 | 0.3 | 1744.7 | 6045 | 89 | 1648.4 | 5164 | 7 | 1711.6 | 5901 | 3 | 2 | 1 |
| 20 | 0.3 | 0.6 | 2556.8 | 4741 | 439 | 2412.3 | 3532 | 439 | 2658.9 | 13353 | 395 | 2 | 1 |
| 20 | 0.3 | 1.0 | 1018.8 | 1768 | 27 | 1059.8 | 2447 | 27 | 1555.9 | 5642 | 49 | 1 | 3 |
| 20 | 0.6 | 0.3 | 2501.8 | 8817 | 63 | 2017.2 | 3971 | 63 | 1915.5 | 17588 | 133 | 7 | 2 |
| 20 | 0.6 | 0.6 | 1765.1 | 4412 | 57 | 1529.2 | 2701 | 57 | 1694.6 | 10280 | 51 | 6 | 2 |
| 20 | 0.6 | 1.0 | 1134.0 | 1567 | 620 | 1059.2 | 1567 | 620 | 944.0 | 5505 | 525 | 7 | 2 |
| 20 | 0.9 | 0.3 | 3020.0 | 6250 | 107 | 3383.4 | 6250 | 107 | 3177.3 | 24231 | 141 | 2 | 4 |
| 20 | 0.9 | 0.6 | 2353.6 | 4240 | 317 | 2511.4 | 4019 | 317 | 1564.5 | 9150 | 301 | 7 | 2 |
| 20 | 0.9 | 1.0 | 1026.5 | 1879 | 607 | 1938.2 | 2941 | 1554 | 1351.3 | 1697 | 548 | 3 | 6 |

Table 5.7: Performances of the seven implementations of Formulation 3C with respect to the nodes explored in the tree search (part 2).

# Conclusion

In this thesis, we introduced the Partitioning-Hub Location-Routing problem (PHLRP), which can be classified as a variant of the hub location problem. PHLRP consists of partitioning a network into sub-networks, locating at least one hub in each subnetwork and routing the traffic within the network such that all inter-subnetwork traffic is routed through the hubs and all intra-subnetwork traffic stays within the sub-networks all the way from the source to the destination. Obviously, besides the hub location component, PHLRP also involves a graph partitioning component and a routing component. PHLRP finds applications in the strategic planning or deployment of the Intermediate System-Intermediate System (ISIS) Internet Protocol networks and the Less-than-truck load freight distribution systems.

In Chapter 2, we introduced three IP formulations for solving PHLRP. The hub-location component and the graph partitioning components of PHLRP are modeled in the same way in all three formulations. More precisely, the hub-location component is represented by the $p$-median variables and constraints; and the graph partitioning component is represented by the size-constrained graph partitioning variables and constraints. The formulations differ from each other in the way the peculiar routing requirements of PHLRP are modeled. Formulations 1 and 2 are based on a three-layer replication of the original network under consideration. Formulation 3, which is more compact, employs a two-level replication of the underlying network. We conclude Chapter 2 by elaborating in Sections 2.5, 2.6 and 2.7 on some relaxations that can be applied to the formulations when we ignore the arc-capacity constraints and assumption (A3).

We devote Chapter 3 to analytical and empirical comparison of the three IP formulations. Our thorough analysis reveals that Formulation 1 is the tightest of the three formulations. In Section A.1 we also show analytically that the LP relaxations of Formulations 2 and 3 do not dominate each other.

On the other hand, our empirical comparison in a standard branch-and-cut framework that is provided by CPLEX shows that Formulation 2 performs by far worse than the other two formulations in solving PHLRP. Formulation 3 yields slightly better solution times on average than Formulation 1. That is, compactness of Formulation 3 outweighs tightness of Formulation 1.

Chapter 4 is in a way independent from the other chapters. In this chapter our focus is on the size-constrained graph partitioning polytope, which is defined as the convex hull of $\boldsymbol{w} \in \{0,1\}^{\binom{n}{2}}$ that satisfy (2.1a), (2.1b) and (2.1c). We analyze the polyhedral structure of this polytope and identify several classes of facet defining inequalities for it.

And, in Chapter 5 we wrap up our efforts for solving PHLRP. Namely, we present the results of our computational experiments in which we employ some inequalities that are highlighted in Chapter 4 in a branch-and-cut algorithm. We run our experiments on Formulation 3, which yields the best solution times in Section 3.2. We exploit the reductions that can be applied to Formulation 3 when the arc capacities are ignored and assumption (A3) is relaxed (as suggested in Section 2.7). In this reduced form of the formulation, the integrality restrictions on all flow variables and assignment variables (which represent assignment of local nodes to hubs) can be relaxed. Hence, the only remaining set of binary variables in the formulations are the partitioning variables and the hub-location variables. For this reason, the facets of the the size-constrained graph partitioning polytope constitute strong valid inequalities for the reduced forms of Formulations 1, 2 and 3.

Our experiments show that our approach brings significant improvements to the solution time of PHLRP when compared with the default branch-and-cut solver of XPress.

However, as it is also evident from our computational experience, our research in this thesis does not yield an efficient algorithm for solving large instances of PHLRP. Yet the outcomes of this thesis would provide a valuable basis for developing metaheuristics that can solve large, practical instances of the problem. Also, developing special algorithms for special graphs (e.g., trees) might provide effective solution methodologies for practical purposes.

## Future research directions

A curious look at the analytical and empirical comparison of the IP formulations that is presented in Chapter 3 reveals the need for further elaboration

on comparison of Formulations 2 and 3. We show in Section A.1 that LP relaxations of these two formulations do not dominate each other. However, empirical evidence shows that Formulation 2's LP relaxation gap is always much higher than that of Formulation 3. We believe that exploring this relationship between these two formulations might prove to be a very nice and elegant extension to the research presented in Chapter 3 and in Section A.1. We believe that a thorough polyhedral analysis of the polytope that is defined by (2.1a)-(2.1g), $w_e \in \{0,1\}^{\binom{n}{2}}$ for all $e \in E_n$ and $x_{u,u} \in \{0,1\}$ for all $u \in V$ would provide invaluable valid inequalities for solving PHLRP. In fact, analysis of this polytope would also be a very interesting research effort from a polyhedral analysis perspective. Because, it combines two polytopes defined in two different subspaces (i.e., the subspaces defined by $w_e$ and $x_{u,v}$) by means of one single set of constraints (i.e., (2.1g)). We believe that such an analysis would provide a valuable extension to the literature on the graph partitioning polytopes and the $p$-median polytope.

# Appendix

## A.1 Analytical comparison of Formulations 2 and 3

In this section, we recast Formulations 2 and 3 on the basis of two layers with no inter-area links. In other words, we rewrite them using the variables $w_{u,v}$, $x_{u,v}$, $\phi_{i,j}^{u,v}$ and $\gamma_{i,j}^{u,v}$ only, the former two and the latter two being defined exactly in the same manner as they are defined in Sections 2.1 and 2.4 respectively. Finally, we compare the formulations obtained in this manner with each other to see which one implies the other.

### A.1.1 Reformulating Formulation 2

The variables $w_{u,v}$, $x_{u,v}$ and $\gamma_{i,j}^{u,v}$ are already present in Formulation 2 (the latter as $g_{i,j}^{u,v}$). We introduce the variables $\phi_{i,j}^{u,v}$ into Formulation 2 by means of the following constraints:

$$\phi_{i,j}^{u,v} = f_{i,j}^{u,v} + h_{i,j}^{u,v} \qquad \forall (u,v) \in T, \forall (i,j) \in A. \tag{A.1}$$

Now, we replace the constraints (2.2c) of Formulation 2 with

$$\phi_{i,j}^{u,v} + \phi_{j,i}^{u,v} \leq w_{i,j} \qquad \forall (u,v) \in T, \forall (i,j) \in E, \tag{A.2}$$

and change the objective function as

$$\min \qquad \sum_{(u,v)\in T} \sum_{(i,j)\in A} c_{i,j}^{u,v} d_{u,v} (\phi_{i,j}^{u,v} + g_{i,j}^{u,v}). \tag{A.3}$$

Note that, these transformations induce no change in the LP relaxation objective value of Formulation 2. Now that the variables $f_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$ are no longer in the objective function, we can project them out from Formulation 2 and end

up with a reformulation that employs only the variables $w_{u,v}$, $x_{u,v}$, $\phi_{i,j}^{u,v}$ and $g_{i,j}^{u,v}$.

The following matrix representation (A.4) displays the relevant constraints to the projection of Formulation 2, that is, the constraints that involve $f_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$ variables (i.e., (2.6b), (2.6f), (2.6c) and (A.1)). In this representation, $\mathbf{0}_{l_1 \times l_2}$ represents an $l_1 \times l_2$ matrix of zeros, and $I_l$ represents an identity matrix of dimension $l$. The first, second and third blocks of constraints respectively represent (2.6b), (2.6f) and (2.6c); and the last two blocks of constraints stand for the equivalent inequality representation of (A.1).

$$\left[\begin{array}{c|c|c} -N & \mathbf{0}_{n\times|A|} & N \\ \hline \mathbf{0}_{n\times|A|} & \mathbf{0}_{n\times|A|} & N \\ \hline \mathbf{0}_{n\times|A|} & N & \mathbf{0}_{n\times|A|} \\ \hline -I_{|A|} & I_{|A|} & I_{|A|} \\ \hline I_{|A|} & -I_{|A|} & -I_{|A|} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\phi^{u,v}} \\ \hline \boldsymbol{f^{u,v}} \\ \hline \boldsymbol{h^{u,v}} \end{array}\right] \leq \left[\begin{array}{c} \boldsymbol{b}^1 \\ \hline \boldsymbol{b}^2 \\ \hline \boldsymbol{b}^3 \\ \hline \mathbf{0}_{|A|} \\ \hline \mathbf{0}_{|A|} \end{array}\right] \qquad \forall (u,v) \in T, \quad \text{(A.4)}$$

where

$$b_i^1 = \begin{cases} x_{u,i} - 1, & \text{if } i = u, \\ x_{u,i}, & \text{if } i \in V - \{u,v\}, \\ x_{u,i} + w_{u,v}, & \text{if } i = v; \end{cases}$$

$$b_i^2 = \begin{cases} x_{v,i}, & \text{if } i \in V - \{v\}, \\ x_{v,v} - 1 + w_{u,v}, & \text{if } i = v; \end{cases}$$

and

$$b_i^3 = \begin{cases} 1, & \text{if } i = u, \\ 0, & \text{if } i \in V - \{u,v\}, \\ -w_{u,v}, & \text{if } i = v. \end{cases}$$

We associate the dual vectors $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, $\boldsymbol{\beta} \in \mathbb{R}_+^n$, $\boldsymbol{\mu} \in \mathbb{R}_+^n$, $\boldsymbol{\lambda} \in \mathbb{R}_+^{|A|}$ and $\boldsymbol{\theta} \in \mathbb{R}_+^{|A|}$ with the first, second, third, fourth and the fifth blocks of constraints, respectively. Then, the projection cone corresponding to the $f_{i,j}^{u,v}$ and $h_{i,j}^{u,v}$ variables is determined by the inequalities

$$\lambda_{i,j} - \theta_{i,j} \geq (\alpha_j - \alpha_i) + (\beta_j - \beta_i) \qquad \forall (i,j) \in A, \qquad \text{(A.5a)}$$

$$\lambda_{i,j} - \theta_{i,j} \geq \mu_j - \mu_i \qquad \forall (i,j) \in A. \qquad \text{(A.5b)}$$

Namely, if we denote the projection cone by $W$,

$$W = \{(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta}) \in \mathbb{R}_+^{3n+2|A|} | \text{(A.5a) and (A.5b) are satisfied}\}.$$

The following proposition gives the extreme rays of $W$.

**Proposition A.1.1.** *Let* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta}) \in W$ *be a ray of* $W$*. Then* $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta})$ *is an extreme ray of* $W$ *if and only if it is equivalent to one of the following forms:*

1. $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\mu} = \mathbf{0}$*;*
   $\lambda_{i^*,j^*} = 1$ *for some* $(i^*, j^*) \in A$*,* $\lambda_{i,j} = 0$ *for* $(i,j) \in A - \{(i^*, j^*)\}$*;*
   $\boldsymbol{\theta} = 0$*.*

2. $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\mu} = \mathbf{0}$*;*
   $\lambda_{i^*,j^*} = \theta_{i^*,j^*} = 1$ *for some* $(i^*, j^*) \in A$*,* $\lambda_{i,j} = \theta_{i,j} = 0$ *for* $(i,j) \in A - \{(i^*, j^*)\}$*.*

3. $\alpha_i = 1$ *for* $i \in S_\alpha$*,* $\alpha_i = 0$ *otherwise,*
   $\beta_i = 1$ *for* $i \in S_\beta$*,* $\beta_i = 0$ *otherwise,*
   $\mu_i = 1$ *for* $i \in S_\mu$*,* $\mu_i = 0$ *otherwise,*
   $\lambda_{i,j} = 1$ *for* $(i,j) \in S_\lambda$*,* $\lambda_{i,j} = 0$ *otherwise,*
   $\theta_{i,j} = 1$ *for* $(i,j) \in S_\theta$*,* $\theta_{i,j} = 0$ *otherwise,*
   *where* $S_\alpha$*,* $S_\beta$*,* $S_\mu$*,* $S_\lambda$ *and* $S_\theta$ *comply with the following conditions C-1–C-8:*

   *C-1.* $S_\alpha, S_\beta, S_\mu \subset V$ *and at least one is non-empty.*

   *C-2.* $S_\alpha \cap S_\beta = \emptyset$*.*

   *In the sequel, let* $\tilde{V} = V - (S_\alpha \cup S_\beta \cup S_\mu)$*.*

   *C-3.* $\tilde{V} \neq \emptyset$*.*

   *C-4. if* $S_\alpha \neq \emptyset$ *and* $S_\mu \neq \emptyset$*, then* $S_\alpha \cap S_\mu \neq \emptyset$ *and* $A(S_\alpha \cap S_\mu, \tilde{V}) \neq \emptyset$*.*

   *C-5. if* $S_\beta \neq \emptyset$ *and* $S_\mu \neq \emptyset$*, then* $S_\beta \cap S_\mu \neq \emptyset$ *and* $A(S_\beta \cap S_\mu, \tilde{V}) \neq \emptyset$*.*

   *C-6.* $S_\lambda = A^-(S_\mu) \cup A^-(S_\alpha \cup S_\beta)$*.*

   *C-7.* $S_\theta \subseteq A((S_\alpha \cup S_\beta) \cap S_\mu, \tilde{V})$*.*

   *C-8. the subgraph induced by* $S_\alpha \cup S_\beta \cup S_\mu$ *is connected.*

*Proof.* *Sufficiency:* Pick a ray $\boldsymbol{r} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta}) \in W$ of type 1, i.e., $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\mu} = 0$, $\boldsymbol{\theta} = 0$, $\lambda_{i^*,j^*} = 1$ for some $(i^*, j^*) \in A$ and $\lambda_{i,j} = 0$ for $(i,j) \in A - \{(i^*, j^*)\}$. Suppose that $\boldsymbol{r}$ is not an extreme ray. Then there exist $\boldsymbol{r}^l = \left(\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\mu}^l, \boldsymbol{\lambda}^l, \boldsymbol{\theta}^l\right) \in W$ for $l = 1, 2$ that comply with the following two conditions:
**(E-1)** $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are not equivalent to $\boldsymbol{r}$ up to a constant; and,

**(E-2)** $\boldsymbol{r} = \frac{1}{2}\boldsymbol{r}^1 + \frac{1}{2}\boldsymbol{r}^2$.
Since $\boldsymbol{r}^1 \in W$ and $\boldsymbol{r}^2 \in W$, (E-2) implies that $\boldsymbol{\alpha}^l = \boldsymbol{\beta}^l = \boldsymbol{\mu}^l = 0$, $\boldsymbol{\theta}^l = 0$ and $\lambda^l_{i,j} = 0$ for $l = 1, 2$ and for all $(i, j) \in A - \{(i^*, j^*)\}$. But, then $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are equivalent to $\boldsymbol{r}$ up to a constant, which contradicts (E-1).

Now, suppose that $\boldsymbol{r}$ is not extreme and is of type 2, i.e., $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\mu} = 0$, $\lambda_{i^*,j^*} = \theta_{i^*,j^*} = 1$ for some $(i^*, j^*) \in A$ and $\lambda_{i,j} = \theta_{i,j} = 0$ for $(i, j) \in A - \{(i^*, j^*)\}$. Then there exist $\boldsymbol{r}^l = \left(\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\mu}^l, \boldsymbol{\lambda}^l, \boldsymbol{\theta}^l\right) \in W$ for $l = 1, 2$ that comply with (E-1) and (E-2). (E-2) implies that $\boldsymbol{\alpha}^l = \boldsymbol{\beta}^l = \boldsymbol{\mu}^l = 0$, $\lambda^l_{i,j} = \theta^l_{i,j} = 0$ for $l = 1, 2$ and for all $(i, j) \in A - \{(i^*, j^*)\}$. From (E-2), we can infer further that

$$\begin{aligned} \lambda^1_{i^*,j^*} &= 1 - \epsilon_\lambda, \qquad & \theta^1_{i^*,j^*} &= 1 - \epsilon_\theta, \\ \lambda^2_{i^*,j^*} &= 1 + \epsilon_\lambda, \qquad & \theta^2_{i^*,j^*} &= 1 + \epsilon_\theta \end{aligned}$$

for some $\epsilon_\lambda, \epsilon_\theta \in \mathbb{R}$. Since $\boldsymbol{r}^1 \in W$ and $\boldsymbol{r}^2 \in W$, (A.5a) and (A.5b) imply $\epsilon_\lambda = \epsilon_\theta$. But then $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ contradict (E-1).

Finally, suppose that $\boldsymbol{r}$ is not extreme and is of type 3, i.e., $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ are as given in the proposition statement for some $S_\alpha$, $S_\beta$, $S_\mu$, $S_\lambda$ and $S_\theta$. Then there exist $\boldsymbol{r}^l = \left(\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\mu}^l, \boldsymbol{\lambda}^l, \boldsymbol{\theta}^l\right) \in W$ for $l = 1, 2$ that comply with (E-1) and (E-2). (E-2) implies that

$$\begin{aligned} \alpha^l_i &= 0 \qquad && \text{for } l = 1, 2, \text{and for all } i \notin S_\alpha, \\ \beta^l_i &= 0 \qquad && \text{for } l = 1, 2, \text{and for all } i \notin S_\beta, \\ \mu^l_i &= 0 \qquad && \text{for } l = 1, 2, \text{and for all } i \notin S_\mu, \\ \lambda^l_{i,j} &= 0 \qquad && \text{for } l = 1, 2, \text{and for all } (i, j) \notin S_\lambda, \\ \theta^l_{i,j} &= 0 \qquad && \text{for } l = 1, 2, \text{and for all } (i, j) \notin S_\theta. \end{aligned}$$

(E-2) implies further that

$$\begin{aligned} \alpha^1_i &= 1 - \epsilon_{\alpha_i}, \qquad & \alpha^2_i &= 1 + \epsilon_{\alpha_i} \qquad & \forall i \in S_\alpha; \qquad & \text{(A.6a)} \\ \beta^1_i &= 1 - \epsilon_{\beta_i}, \qquad & \beta^2_i &= 1 + \epsilon_{\beta_i} \qquad & \forall i \in S_\beta; \qquad & \text{(A.6b)} \\ \mu^1_i &= 1 - \epsilon_{\mu_i}, \qquad & \mu^2_i &= 1 + \epsilon_{\mu_i} \qquad & \forall i \in S_\mu; \qquad & \text{(A.6c)} \\ \lambda^1_{i,j} &= 1 - \epsilon_{\lambda_{i,j}}, \qquad & \lambda^2_{i,j} &= 1 + \epsilon_{\lambda_{i,j}} \qquad & \forall (i, j) \in S_\lambda; \qquad & \text{(A.6d)} \\ \theta^1_{i,j} &= 1 - \epsilon_{\theta_{i,j}}, \qquad & \theta^2_{i,j} &= 1 + \epsilon_{\theta_{i,j}} \qquad & \forall (i, j) \in S_\theta; \qquad & \text{(A.6e)} \end{aligned}$$

where $\epsilon_{\alpha_i}, \epsilon_{\beta_j}, \epsilon_{\mu_k}, \epsilon_{\lambda_{l_1,l_2}}, \epsilon_{\theta_{l_3,l_4}} \in \mathbb{R}$ for all $i \in S_\alpha$, $j \in S_\beta$, $k \in S_\mu$, $(l_1, l_2) \in S_\lambda$ and $(l_3, l_4) \in S_\theta$.

In the rest of this proof, we mark the end of the proofs of the claims by "Q.E.D."; and, we mark the end of the proof of the proposition by the usual hollow square (i.e., □). Let $\tilde{S}_{\alpha\beta} = V - (S_\alpha \cup S_\beta)$, $\tilde{S}_\mu = V - S_\mu$.

*Claim 1.*

(i) $\epsilon_{\alpha_j} = \epsilon_{\lambda_{i,j}}$ for all $(i,j) \in A(\tilde{S}_{\alpha\beta}, S_\alpha)$;

(ii) $\epsilon_{\beta_j} = \epsilon_{\lambda_{i,j}}$ for all $(i,j) \in A(\tilde{S}_{\alpha\beta}, S_\beta)$;

(iii) $\epsilon_{\mu_j} = \epsilon_{\lambda_{i,j}}$ for all $(i,j) \in A^-(S_\mu)$.

*Proof:* We prove part (i) only; the proofs for the others are very similar. Consider an arbitrary node $j^* \in S_\alpha$ and an arbitrary arc $(i^*, j^*) \in A(\tilde{S}_{\alpha\beta}, \{j^*\})$. We have $\theta^l_{i^*,j^*} = 0$ for $l = 1, 2$, because $(i^*, j^*) \notin S_\theta$; and, $\alpha^l_{i^*} = \beta^l_{i^*} = \beta^l_{j^*} = 0$ for $l = 1, 2$, because $i^* \in \tilde{S}_{\alpha\beta}$ and $S_\alpha \cap S_\beta = \emptyset$. Plugging these values, (A.6a) and (A.6d) into the corresponding inequality of (A.5a) yields $\epsilon_{\alpha_{j^*}} = \epsilon_{\lambda_{i^*,j^*}}$. Q.E.D.

*Claim 2.*

(i) $\epsilon_{\alpha_i} = \epsilon_{\mu_i} = \epsilon_{\theta_{i,j}}$ for all $(i,j) \in S_\theta$ such that $i \in S_\alpha \cap S_\mu$.

(ii) $\epsilon_{\beta_i} = \epsilon_{\mu_i} = \epsilon_{\theta_{i,j}}$ for all $(i,j) \in S_\theta$ such that $i \in S_\beta \cap S_\mu$.

*Proof:* We prove only part (i); part (ii) is very similar. Consider an arbitrary arc $(i^*, j^*) \in S_\theta$. We know that $i^* \in S_\alpha \cap S_\mu$, $j^* \in \tilde{V}$ and $(i^*, j^*) \notin S_\lambda$, hence $\lambda^l_{i^*,j^*} = \beta^l_{i^*} = \beta^l_{j^*} = \alpha^l_{j^*} = \mu^l_{j^*} = 0$ for $l = 1, 2$. Plugging these values, (A.6a) and (A.6e) into the corresponding inequalities of (A.5a) and (A.5b) yields the result. Q.E.D.

*Claim 3.*

(i) $\epsilon_{\alpha_i} = \epsilon_{\alpha_j}$ for all $(i,j) \in A(S_\alpha \cap S_\mu, S_\alpha)$;

(ii) $\epsilon_{\alpha_i} = \epsilon_{\alpha_j}$ for all $(i,j) \in A(S_\alpha - S_\mu, S_\alpha - S_\mu)$;

(iii) $\epsilon_{\beta_i} = \epsilon_{\beta_j}$ for all $(i,j) \in A(S_\beta \cap S_\mu, S_\beta)$;

(iv) $\epsilon_{\beta_i} = \epsilon_{\beta_j}$ for all $(i,j) \in A(S_\beta - S_\mu, S_\beta - S_\mu)$;

(v) $\epsilon_{\mu_i} = \epsilon_{\mu_j}$ for all $(i,j) \in A(S_\alpha \cap S_\mu, S_\mu)$;

(vi) $\epsilon_{\mu_i} = \epsilon_{\mu_j}$ for all $(i,j) \in A(S_\beta \cap S_\mu, S_\mu)$;

(vii) $\epsilon_{\mu_i} = \epsilon_{\mu_j}$ for all $(i,j) \in A(S_\mu - (S_\alpha \cup S_\beta), S_\mu - (S_\alpha \cup S_\beta))$;

(viii) $\epsilon_{\alpha_i} = \epsilon_{\beta_j}$ for all $(i,j) \in A(S_\alpha, S_\beta - S_\mu)$;

(ix) $\epsilon_{\beta_i} = \epsilon_{\alpha_j}$ for all $(i,j) \in A(S_\beta, S_\alpha - S_\mu)$.

*Proof:* We prove only part (i); the others are very similar. Pick an arbitrary $(i^*, j^*) \in A(S_\alpha \cap S_\mu, S_\alpha)$. We know that $(i^*, j^*)$ is not in $S_\lambda$ and $S_\theta$, hence $\lambda^l_{i^*,j^*} = \theta^l_{i^*,j^*} = 0$ for $l = 1, 2$; and, $\beta^l_{i^*} = \beta^l_{j^*} = 0$ for $l = 1, 2$ since $S_\alpha \cap S_\beta = \emptyset$. Plugging these values and (A.6a) into the corresponding inequality of (A.5a) yields the result. Q.E.D.

*Claim 4.* $\epsilon_{\alpha_i} = \epsilon_{\beta_j} = \epsilon_{\mu_k} = \epsilon_{\lambda_{l_1,l_2}} = \epsilon_{\theta_{l_3,l_4}} = \xi \in \mathbb{R}$ for all $i \in S_\alpha$, $j \in S_\beta$, $k \in S_\mu$, $(l_1, l_2) \in S_\lambda$ and $(l_3, l_4) \in S_\theta$.
*Proof:* Suppose that $S_\alpha$, $S_\beta$ and $S_\mu$ are all non-empty. Due to the conditions C-3, C-4 and C-5 in the statement of the proposition, Claims 1 and 2 imply the following:

- For every $i^* \in S_\alpha \cap S_\mu$ and $(j^*, i^*) \in A(\tilde{V}, \{i^*\})$, $\epsilon_{\alpha_{i^*}} = \epsilon_{\mu_{i^*}} = \epsilon_{\lambda_{j^*,i^*}}$. If there exists $(i^*, k^*) \in S_\theta$, $\epsilon_{\alpha_{i^*}} = \epsilon_{\mu_{i^*}} = \epsilon_{\theta_{i^*,k^*}}$.

- For every $i^* \in S_\beta \cap S_\mu$ and $(j^*, i^*) \in A(\tilde{V}, \{i^*\})$, $\epsilon_{\beta_{i^*}} = \epsilon_{\mu_{i^*}} = \epsilon_{\lambda_{j^*,i^*}}$. If there exists $(i^*, k^*) \in S_\theta$, $\epsilon_{\beta_{i^*}} = \epsilon_{\mu_{i^*}} = \epsilon_{\theta_{i^*,k^*}}$.

Then, from condition C-8 and Claim 3, one can infer that $\epsilon_{\alpha_i} = \epsilon_{\beta_j} = \epsilon_{\mu_k} = \xi \in \mathbb{R}$ for all $i \in S_\alpha$, $j \in S_\beta$, $k \in S_\mu$. Finally, Claims 1 and 2 imply that $\epsilon_{\lambda_{l_1,l_2}} = \epsilon_{\theta_{l_3,l_4}} = \xi$ for all $(l_1, l_2) \in S_\lambda$ and $(l_3, l_4) \in S_\theta$. Q.E.D.

Claim 4 proves that $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are equivalent to $\boldsymbol{r}$ up to a constant, which contradicts (E-1).

*Necessity:* Pick a feasible non-zero ray $r = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \boldsymbol{\theta})$ in $W$. Suppose that $\boldsymbol{r}$ is extreme. Let $P_\alpha = \{i \in V : \alpha_i > 0\}$, $P_\beta = \{i \in V : \beta_i > 0\}$, $P_\mu = \{i \in V : \mu_i > 0\}$, $P_\lambda = \{(i,j) \in A : \lambda_{i,j} > 0\}$ and $P_\theta = \{(i,j) \in A : \theta_{i,j} > 0\}$.

*Claim 5:* If $P_\alpha = P_\beta = P_\mu = P_\theta = \emptyset$, then $|P_\lambda| = 1$.
*Proof:* Let $(i^*, j^*)$ be an arbitrary arc in $A$ and suppose that $(i^*, j^*) \in P_\gamma$. If

$|P_\lambda| > 1$, the two rays $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ defined in the following manner would be feasible and comply with (E-1)-(E-2):

$$\alpha_i^1 = 0 \ \ \forall i \in V, \qquad \alpha_i^2 = 0 \ \ \forall i \in V,$$
$$\beta_i^1 = 0 \ \ \forall i \in V, \qquad \beta_i^2 = 0 \ \ \forall i \in V,$$
$$\mu_i^1 = 0 \ \ \forall i \in V, \qquad \mu_i^2 = 0 \ \ \forall i \in V,$$
$$\lambda_{i,j}^1 = \begin{cases} 2\lambda_{i,j}, & \text{if } (i,j) = (i^*, j^*), \\ 0, & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} 2\lambda_{i,j}, & \text{if } (i,j) \in P_\lambda - \{(i^*, j^*)\}, \\ 0, & \text{otherwise,} \end{cases}$$
$$\theta_{i,j}^1 = 0 \ \ \forall (i,j) \in A, \qquad \theta_{i,j}^2 = 0 \ \ \forall (i,j) \in A.$$

Q.E.D.

Claim 5 proves that when $P_\alpha = P_\beta = P_\mu = P_\theta = \emptyset$, $r$ is an extreme ray of the first type.

*Claim 6:* If $P_\alpha = P_\beta = P_\mu = \emptyset$ and $P_\theta \neq \emptyset$, then $P_\lambda = P_\theta$.
*Proof:* The inequalities (A.5a) and (A.5b) dictate that $P_\lambda \supseteq P_\theta$. But if $P_\lambda - P_\theta \neq \emptyset$, the two rays $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ defined in the following manner would be feasible and comply with (E-1)-(E-2):

$$\alpha_i^1 = 0 \ \ \forall i \in V, \qquad \alpha_i^2 = 0 \ \ \forall i \in V,$$
$$\beta_i^1 = 0 \ \ \forall i \in V, \qquad \beta_i^2 = 0 \ \ \forall i \in V,$$
$$\mu_i^1 = 0 \ \ \forall i \in V, \qquad \mu_i^2 = 0 \ \ \forall i \in V,$$
$$\lambda_{i,j}^1 = \begin{cases} 2\lambda_{i,j}, & \text{if } (i,j) \in P_\theta, \\ 0, & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} 2\lambda_{i,j}, & \text{if } (i,j) \in P_\lambda - P_\theta, \\ 0, & \text{otherwise,} \end{cases}$$
$$\theta_{i,j}^1 = 2\theta_{i,j} \ \ \forall (i,j) \in P_\theta, \qquad \theta_i^2 = 0 \ \ \forall (i,j) \in A.$$

Q.E.D.

*Claim 7:* If $P_\alpha = P_\beta = P_\mu = \emptyset$ and $P_\theta \neq \emptyset$, then $|P_\lambda| = |P_\theta| = 1$.
*Proof:* Suppose that $|P_\lambda| = |P_\theta| \geq 2$. Let $(i^*, j^*)$ be an arc in $P_\lambda$. The two rays $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ defined in the following

manner would be feasible and comply with (E-1)-(E-2):

$$\alpha_i^1 = 0 \ \ \forall i \in V, \qquad \alpha_i^2 = 0 \ \ \forall i \in V,$$
$$\beta_i^1 = 0 \ \ \forall i \in V, \qquad \beta_i^2 = 0 \ \ \forall i \in V,$$
$$\mu_i^1 = 0 \ \ \forall i \in V, \qquad \mu_i^2 = 0 \ \ \forall i \in V,$$
$$\lambda_{i,j}^1 = \begin{cases} 2\lambda_{i,j}, & \text{if } (i,j) = (i^*,j^*), \\ 0, & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} 2\lambda_{i,j}, & \text{if } i \in P_\lambda - \{(i^*,j^*)\}, \\ 0, & \text{otherwise,} \end{cases}$$
$$\theta_{i,j}^1 = \begin{cases} 2\theta_{i,j}, & \text{if } (i,j) = (i^*,j^*), \\ 0, & \text{otherwise,} \end{cases} \qquad \theta_{i,j}^2 = \begin{cases} 2\theta_{i,j}, & \text{if } i \in P_\lambda - \{(i^*,j^*)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Q.E.D.

*Claim 8:* Suppose that $P_\alpha = P_\beta = P_\mu = \emptyset$ and $P_\theta \neq \emptyset$. Suppose further that $P_\lambda = P_\theta = \{(i^*,j^*)\}$. Then $\lambda_{i^*,j^*} = \theta_{i^*,j^*}$.
*Proof:* Suppose that $\lambda_{i^*,j^*} > \theta_{i^*,j^*}$. Then, two rays $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ defined in the following manner would be feasible and comply with (E-1)-(E-2):

$$\alpha_i^1 = 0 \ \ \forall i \in V, \qquad \alpha_i^2 = 0 \ \ \forall i \in V,$$
$$\beta_i^1 = 0 \ \ \forall i \in V, \qquad \beta_i^2 = 0 \ \ \forall i \in V,$$
$$\mu_i^1 = 0 \ \ \forall i \in V, \qquad \mu_i^2 = 0 \ \ \forall i \in V,$$
$$\lambda_{i,j}^1 = \begin{cases} 2(\lambda_{i,j} - \theta_{i,j}), & \text{if } (i,j) = (i^*,j^*), \\ 0, & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} 2\theta_{i,j}, & \text{if } (i,j) = (i^*,j^*), \\ 0, & \text{otherwise,} \end{cases}$$
$$\theta_{i,j}^1 = 0 \ \ \forall (i,j) \in A, \qquad \theta_{i,j}^2 = \begin{cases} 2\theta_{i,j}, & \text{if } (i,j) = (i^*,j^*), \\ 0, & \text{otherwise.} \end{cases}$$

Q.E.D.

Claims 6-8 prove that when $P_\alpha = P_\beta = P_\mu = \emptyset$ and $P_\theta \neq \emptyset$, $\boldsymbol{r}$ is an extreme ray of second type. In the sequel, we assume that at least one of $P_\alpha$, $P_\beta$ and $P_\mu$ is non-empty.

*Claim 9:* $P_\lambda \cap P_\theta = \emptyset$.
*Proof:* Consider an $(i^*,j^*) \in P_\lambda \cap P_\theta$ (i.e., $\lambda_{i^*,j^*} > 0$ and $\theta_{i^*,j^*} > 0$). We can perturb $\boldsymbol{r}$ by a suitably small value $\epsilon$ to obtain the the two rays $\boldsymbol{r}^1 =$

$(\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ where

$$\alpha_i^1 = \alpha_i \ \ \forall i \in V, \qquad \alpha_i^2 = \alpha_i \ \ \forall i \in V,$$
$$\beta_i^1 = \beta_i \ \ \forall i \in V, \qquad \beta_i^2 = \beta_i \ \ \forall i \in V,$$
$$\mu_i^1 = \mu_i \ \ \forall i \in V, \qquad \mu_i^2 = \mu_i \ \ \forall i \in V,$$
$$\lambda_{i,j}^1 = \begin{cases} \lambda_{i,j} + \epsilon, & \text{if } (i,j) = (i^*, j^*), \\ \lambda_{i,j}, & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} \lambda_{i,j} - \epsilon, & \text{if } (i,j) = (i^*, j^*), \\ \lambda_{i,j}, & \text{otherwise,} \end{cases}$$
$$\theta_{i,j}^1 = \begin{cases} \theta_{i,j} + \epsilon, & \text{if } (i,j) = (i^*, j^*), \\ \theta_{i,j}, & \text{otherwise,} \end{cases} \qquad \theta_{i,j}^2 = \begin{cases} \theta_{i,j} - \epsilon, & \text{if } (i,j) = (i^*, j^*), \\ \theta_{i,j}, & \text{otherwise.} \end{cases}$$

Obviously, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ satisfy (E-1)-(E-2). Q.E.D.

*Claim 10:*

(i) $\lambda_{i,j} = \max\{(\alpha_j - \alpha_i) + (\beta_j - \beta_i), \mu_j - \mu_i\} > 0$ for all $(i,j) \in P_\lambda$;

(ii) $\theta_{i,j} = -\max\{(\alpha_j - \alpha_i) + (\beta_j - \beta_i), \mu_j - \mu_i\} > 0$ for all $(i,j) \in P_\theta$.

*Proof:* We give the proof for only the first part, the proof for the second part is very similar.

Claim 9 implies that if $(i,j) \in P_\lambda$, then $\theta_{i,j} = 0$. Thus, by (A.5a) we have $\lambda_{i,j} \geq \max\{(\alpha_j - \alpha_i) + (\beta_j - \beta_i), \mu_j - \mu_i\}$ for all $(i,j) \in P_\lambda$. Suppose that there exists $(i^*, j^*) \in P_\lambda$ such that $\lambda_{i^*,j^*} > \max\{(\alpha_{j^*} - \alpha_{i^*}) + (\beta_{j^*} - \beta_{i^*}), \mu_{j^*} - \mu_{i^*}\}$. Then we can perturb $\boldsymbol{r}$ by a suitably small $\epsilon$ to obtain the rays $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ where

$$\alpha_i^1 = \alpha_i \ \ \forall i \in V, \qquad \alpha_i^2 = \alpha_i \ \ \forall i \in V,$$
$$\beta_i^1 = \beta_i \ \ \forall i \in V, \qquad \beta_i^2 = \beta_i \ \ \forall i \in V,$$
$$\mu_i^1 = \mu_i \ \ \forall i \in V, \qquad \mu_i^2 = \mu_i \ \ \forall i \in V,$$
$$\lambda_{i,j}^1 = \begin{cases} \lambda_{i,j} + \epsilon, & \text{if } (i,j) = (i^*, j^*), \\ \lambda_{i,j}, & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} \lambda_{i,j} - \epsilon, & \text{if } (i,j) = (i^*, j^*), \\ \lambda_{i,j}, & \text{otherwise,} \end{cases}$$
$$\theta_{i,j}^1 = \theta_{i,j} \ \ \forall (i,j) \in A, \qquad \theta_{i,j}^2 = \theta_{i,j} \ \ \forall (i,j) \in A.$$

Both $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are feasible and satisfy (E-1)-(E-2). Q.E.D.

*Claim 11:* $P_\alpha \cap P_\beta = \emptyset$.
*Proof:* Suppose $P_\alpha \cap P_\beta \neq \emptyset$ and suppose $i^* \in P_\alpha \cap P_\beta$. Then, one can find

$\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ where

$$\alpha_i^1 = \begin{cases} \alpha_i - \epsilon & \text{if } i = i^*; \\ \alpha_i & \text{otherwise.} \end{cases} \qquad \alpha_i^2 = \begin{cases} \alpha_i + \epsilon & \text{if } i = i^*; \\ \alpha_i & \text{otherwise.} \end{cases}$$

$$\beta_i^1 = \begin{cases} \beta_i + \epsilon & \text{if } i = i^*; \\ \beta_i & \text{otherwise.} \end{cases} \qquad \beta_i^2 = \begin{cases} \beta_i - \epsilon & \text{if } i = i^*; \\ \beta_i & \text{otherwise.} \end{cases}$$

$$\mu_i^1 = \mu_i \ \ \forall i \in V, \qquad \mu_i^2 = \mu_i \ \ \forall i \in V,$$

$$\lambda_{i,j}^1 = \lambda_{i,j} \ \ \forall (i,j) \in A, \qquad \lambda_{i,j}^2 = \lambda_{i,j} \ \ \forall (i,j) \in A,$$

$$\theta_{i,j}^1 = \theta_{i,j} \ \ \forall (i,j) \in A, \qquad \theta_{i,j}^2 = \theta_{i,j} \ \ \forall (i,j) \in A.$$

Obviously, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are both feasible and satisfy (E-1) and (E-2). Q.E.D.

<u>*Claim 12:*</u> If $\alpha_i = \beta_j = \mu_k = \zeta > 0$ for all $i \in P_\alpha$, $j \in P_\beta$, $k \in P_\mu$, then

(i) $P_\lambda = A^-(P_\mu) \cup A^-(P_\alpha \cup P_\beta)$ and $P_\theta \subseteq A\left((S_\alpha \cup S_\beta) \cap S_\mu, \tilde{V}\right)$.

(ii) $\lambda_{l_1,l_2} = \theta_{l_3,l_4} = \zeta > 0$ for all $(l_1,l_2) \in P_\lambda$ and $(l_3,l_4) \in P_\theta$.

*Proof:* Let $X_1 = A^-(P_\mu) \cup A^-(P_\alpha \cup P_\beta)$ and $X_2 = A\left((S_\alpha \cup S_\beta) \cap S_\mu, \tilde{V}\right)$. Claim 10 implies that for all $(i,j) \in X_1$, $\lambda_{i,j} > 0$. Hence $X_1 \subseteq P_\lambda$. But, $P_\lambda - X_1 = \emptyset$ because otherwise we would have an arc $(i^*,j^*) \in X_1$ for which $\lambda_{i^*,j^*} > 0$; and, existence of such an arc would contradict Claim 10 as one can verify that $\max\{(\alpha_j - \alpha_i) + (\beta_j - \beta_i), \mu_j - \mu_i\} \leq 0$ for all $(i,j) \in A - X_1$. Therefore $P_\lambda = X_1$.

Similarly, one can verify that for all $(i,j) \in A - X_2$, $\max\{(\alpha_j - \alpha_i) + (\beta_j - \beta_i), \mu_j - \mu_i\} \geq 0$. Hence, $P_\theta \subseteq X_2$. Note that, it is always feasible to assign $\theta_{i^*,j^*} = 0$ even when $\max\{(\alpha_{j^*} - \alpha_{i^*}) + (\beta_{j^*} - \beta_{i^*}), \mu_{j^*} - \mu_{i^*}\} < 0$ for some $(i^*,j^*) \in X_2 - P_\theta$.

Part (ii) is a direct consequence of Claim 10. Q.E.D.

<u>*Claim 13:*</u> $\alpha_i = \beta_j = \mu_k = \lambda_{l_1,l_2} = \theta_{l_3,l_4} = \zeta > 0$ for all $i \in P_\alpha$, $j \in P_\beta$, $k \in P_\mu$, $(l_1,l_2) \in P_\lambda$ and $(l_3,l_4) \in P_\theta$.

*Proof:* Consider $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ where

$$\alpha_i^1 = \begin{cases} \alpha_i + \epsilon & \text{if } i \in P_\alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad \alpha_i^2 = \begin{cases} \alpha_i - \epsilon & \text{if } i \in P_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_i^1 = \begin{cases} \beta_i + \epsilon & \text{if } i \in P_\beta, \\ 0 & \text{otherwise,} \end{cases} \qquad \beta_i^2 = \begin{cases} \beta_i - \epsilon & \text{if } i \in P_\beta, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_i^1 = \begin{cases} \mu_i + \epsilon & \text{if } i \in P_\mu, \\ 0 & \text{otherwise,} \end{cases} \qquad \mu_i^2 = \begin{cases} \mu_i - \epsilon & \text{if } i \in P_\mu, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_{i,j}^1 = \begin{cases} \lambda_{i,j} + \epsilon & \text{if } (i,j) \in P_\lambda, \\ 0 & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} \lambda_{i,j} - \epsilon & \text{if } (i,j) \in P_\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta_{i,j}^1 = \begin{cases} \theta_{i,j} + \epsilon & \text{if } (i,j) \in P_\theta, \\ 0 & \text{otherwise,} \end{cases} \qquad \theta_{i,j}^2 = \begin{cases} \theta_{i,j} - \epsilon & \text{if } (i,j) \in P_\theta, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $P_\lambda$ and $P_\theta$ are as given in part (ii) of Claim 12. Note that, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are both feasible and satisfy (E-2). If $\boldsymbol{r}$ has a non-zero entry which is not equal to $\zeta$, then $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ would satisfy (E-1) as well, which contradicts extremeness of $\boldsymbol{r}$. Q.E.D.

*Claim 14:* $P_\alpha$, $P_\beta$ and $P_\mu$ satisfy conditions C-3, C-4 and C-5.
*Proof:* We consider the most general case and suppose that $P_\alpha$, $P_\beta$ and $P_\mu$ are all non-empty. Suppose further that, either C-3 or C-4 does not hold and hence $A(P_\alpha \cap P_\mu, \tilde{V}) = \emptyset$. Let $\tilde{P}_{\alpha\beta} = V - (P_\alpha \cup P_\beta)$. Consider $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ where

$$\alpha_i^1 = 2\alpha_i \ \ \forall i \in V, \qquad \alpha_i^2 = 0 \ \ \forall i \in V,$$

$$\beta_i^1 = 0 \ \ \forall i \in V, \qquad \beta_i^2 = 2\beta_i \ \ \forall i \in V,$$

$$\mu_i^1 = 0 \ \ \forall i \in V, \qquad \mu_i^2 = 2\mu_i \ \ \forall i \in V,$$

$$\lambda_{i,j}^1 = \begin{cases} 2\lambda_{i,j} & \text{if } (i,j) \in A^-(\tilde{P}_{\alpha\beta}, P_\alpha), \\ 0 & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} 2\lambda_{i,j} & \text{if } (i,j) \in A^-(P_\mu) \cup A(\tilde{P}_{\alpha\beta}, P_\beta), \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta_{i,j}^1 = 0 \ \ \forall i \in V, \qquad \theta_{i,j}^2 = 2\theta_{i,j} \ \ \forall i \in V.$$

Both $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are feasible and satisfy (E-1) and (E-2). Note that, when $A(P_\alpha \cap P_\mu, \tilde{V}) \neq \emptyset$, it is not possible to write $\boldsymbol{r}$ as the sum of two such rays. We can argue in the same manner when $A(P_\beta \cap P_\mu, \tilde{V}) \neq \emptyset$. This completes the proof of Claim 14. Q.E.D.

*Claim 15:* $P_\alpha$, $P_\beta$ and $P_\mu$ satisfy condition C-8.
*Proof:* Let $G' = (V', E')$ be a connected component of the subgraph induced by $P_\alpha \cup P_\beta \cup P_\mu$. Let $P'_\alpha = V' \cap P_\alpha$, $P'_\beta = V' \cap P_\beta$, $P'_\mu = V' \cap P_\mu$, $P'_\lambda = A^-(P_\mu) \cup A^-(P_\alpha \cup P_\beta)$ and $P'_\theta = P_\theta \cap A\left((P'_\alpha \cup P'_\beta) \cap P'_\mu, \tilde{V}\right)$. Consider $\boldsymbol{r}^1 = (\boldsymbol{\alpha}^1, \boldsymbol{\beta}^1, \boldsymbol{\mu}^1, \boldsymbol{\lambda}^1, \boldsymbol{\theta}^1)$ and $\boldsymbol{r}^2 = (\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2, \boldsymbol{\mu}^2, \boldsymbol{\lambda}^2, \boldsymbol{\theta}^2)$ where

$$\alpha_i^1 = \begin{cases} 2\alpha_i & \text{if } i \in P'_\alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad \alpha_i^2 = \begin{cases} 2\alpha_i & \text{if } i \in P_\alpha - P'_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_i^1 = \begin{cases} 2\beta_i & \text{if } i \in P'_\beta, \\ 0 & \text{otherwise,} \end{cases} \qquad \beta_i^2 = \begin{cases} 2\beta_i & \text{if } i \in P_\beta - P'_\beta, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_i^1 = \begin{cases} 2\mu_i & \text{if } i \in P'_\mu, \\ 0 & \text{otherwise,} \end{cases} \qquad \mu_i^2 = \begin{cases} 2\mu_i & \text{if } i \in P_\mu - P'_\mu, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_{i,j}^1 = \begin{cases} 2\lambda_{i,j} & \text{if } (i,j) \in P'_\lambda, \\ 0 & \text{otherwise,} \end{cases} \qquad \lambda_{i,j}^2 = \begin{cases} 2\lambda_{i,j} & \text{if } (i,j) \in P_\lambda - P'_\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta_{i,j}^1 = \begin{cases} 2\theta_{i,j} & \text{if } (i,j) \in P'_\theta, \\ 0 & \text{otherwise,} \end{cases} \qquad \theta_{i,j}^2 = \begin{cases} 2\theta_{i,j} & \text{if } (i,j) \in P_\theta - P'_\theta, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are both feasible and satisfy (E-1) and (E-2). Q.E.D.

Claims 9-15 prove that when at least one of $P_\alpha$, $P_\beta$ and $P_\mu$ is non-empty, $\boldsymbol{r}$ is an extreme ray of type 3. □ □

The inequalities implied by the 1st and 2nd type extreme rays of $W$ are respectively as follows:

$$\begin{aligned} -\phi_{i,j}^{u,v} &\leq 0 \qquad \forall (i,j) \in A, \forall (u,v) \in T \\ \phi_{i,j}^{u,v} - \phi_{i,j}^{u,v} &\leq 0 \qquad \forall (i,j) \in A, \forall (u,v) \in T. \end{aligned}$$

These two sets of inequalities are obviously redundant. The inequalities implied by the 3th type extreme rays are

$$-(N\phi^{u,v})_{S_\alpha} + \phi(S_\theta) - \phi(S_\lambda) \leq \sum_{i \in S_\alpha} b_i^1 + \sum_{i \in S_\beta} b_i^2 + \sum_{i \in S_\mu} b_i^3 \qquad \forall (u,v) \in T. \quad \text{(A.7)}$$

To sum up, we can reformulate Formulation 2 over the subspace defined by $w_{u,v}$, $x_{u,v}$, $\phi_{i,j}^{u,v}$ and $g_{i,j}^{u,v}$ by replacing the objective function (2.6a) by (A.3), the constraints (2.2c) by (A.2), and the constraints (2.6b), (2.6f) and (2.6c) by (A.7). For the sake of simplicity, in the rest of this section we refer to this reformulation of Formulation 2 as Formulation 2P.

### A.1.2 Reformulating Formulation 3

Reformulating Formulation 3 over the subspace defined by $w_{u,v}$, $x_{u,v}$, $\phi_{i,j}^{u,v}$ and $\gamma_{i,j}^{u,v}$ is quite straightforward. We only need to project out the variables $\tau_i^{1,u,v}$ and $\tau_i^{2,u,v}$. Below, (A.8) is an equivalent matrix representation of constraints relevant to projection (i.e., (2.7e), (2.7f), (2.7g), (2.7h) and (2.7i), respectively). As with (A.4), $\mathbf{1}_{l_1\times l_2}$ and $\mathbf{0}_{l_1\times l_2}$ stand for $l_1 \times l_2$ matrices of zeros and ones, respectively; and, $I_l$ represents the identity matrix of dimension $l$. Note that, here the flow balance constraints (2.7e) and (2.7f) are equivalently rewritten in inequality form.

$$\left[\begin{array}{c|c|c|c} N & \mathbf{0}_{n\times|A|} & I_n & -I_n \\ \hline \mathbf{0}_{n\times|A|} & N & -I_n & I_n \\ \hline \mathbf{0}_{n\times|A|} & \mathbf{0}_{n\times|A|} & I_n & \mathbf{0}_{n\times n} \\ \hline \mathbf{0}_{n\times|A|} & \mathbf{0}_{n\times|A|} & \mathbf{0}_{n\times n} & I_n \\ \hline \mathbf{0}_{1\times|A|} & \mathbf{0}_{1\times|A|} & \mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} \\ \hline \mathbf{0}_{1\times|A|} & \mathbf{0}_{1\times|A|} & -\mathbf{1}_{1\times n} & \mathbf{0}_{1\times n} \end{array}\right] \left[\begin{array}{c} \boldsymbol{\phi}^{u,v} \\ \hline \boldsymbol{\gamma}^{u,v} \\ \hline \boldsymbol{\tau}^{1,u,v} \\ \hline \boldsymbol{\tau}^{2,u,v} \end{array}\right] \leq \left[\begin{array}{c} \boldsymbol{b}^4 \\ \hline \mathbf{0}_{n\times 1} \\ \hline \boldsymbol{b}^5 \\ \hline \boldsymbol{b}^6 \\ \hline 1-w_{u,v} \\ \hline w_{u,v}-1 \end{array}\right] \quad \forall (u,v) \in T, \tag{A.8}$$

where

$$\boldsymbol{b}_i^4 = \begin{cases} 1, & \text{if } i = u, \\ 0, & \text{if } i \in V - \{u,v\}, \\ -1, & \text{if } i = v; \end{cases}$$

$$\boldsymbol{b}_i^5 = x_{u,i} \qquad \forall i \in V;$$

and

$$\boldsymbol{b}_i^6 = x_{v,i} \qquad \forall i \in V.$$

We associate the dual variables $\alpha, \beta, \psi, \lambda \in \mathbb{R}^n$ with the first four blocks of rows, and the dual variables $\theta$ and $\zeta$ with the last two constraints. Let $W$ be the projection cone:

$$W = \{(\alpha, \beta, \psi, \lambda, \theta, \zeta) \in \mathbb{R}_+^{4n+2} | (A.9a) \text{ and } (A.9b) \text{ are satisfied}\},$$

where

$$\alpha_i - \beta_i + \psi_i + \theta - \zeta \geq 0 \qquad \forall i \in V, \tag{A.9a}$$

$$\beta_i - \alpha_i + \lambda_i \geq 0 \qquad \forall i \in V. \tag{A.9b}$$

**Proposition A.1.2.** *A ray $r = [\boldsymbol{\alpha}\ \boldsymbol{\beta}\ \boldsymbol{\psi}\ \boldsymbol{\lambda}\ \theta\ \zeta] \in \mathbb{R}^{4n+2}$ is an extreme ray of the projection cone $W$ if and only if it belongs to one of the following nine types:*

1. $\theta = 1$ *and 0 otherwise.*

2. $\psi_i = 1$ *for some* $i \in V$ *and 0 otherwise.*

3. $\lambda_i = 1$ *for some* $i \in V$ *and 0 otherwise.*

4. $\theta = \zeta = 1$ *and 0 otherwise.*

5. $\alpha_i = \lambda_i = 1$ *for some* $i \in V$ *and 0 otherwise.*

6. $\beta_i = \theta = 1$ *for* $i \in S$ *and 0 otherwise, for all* $S \subseteq V$*.*

7. $\beta_i = \psi_i = 1$ *for some* $i \in V$ *and 0 otherwise.*

8. $\zeta = 1$*,* $\psi_i = 1$ *for* $i \in S$*,* $\alpha_i = \lambda_i = 1$ *for* $i \in V - S$ *for all* $S \subseteq V$*.*

9. $\alpha_i = \beta_i = 1$ *for some* $i \in V$ *and 0 otherwise.*

*Proof.* Sufficiency: Pick a non-zero ray $\boldsymbol{r} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\psi}, \boldsymbol{\lambda}, \theta, \zeta) \in W$. If $\boldsymbol{r}$ is not an extreme ray of $W$, there exist $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l) \in W$ for $l = 1, 2$ that comply with the following two conditions:
**(E-1)** $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are non-zero and are not equivalent to $\boldsymbol{r}$ up to a constant; and,
**(E-2)** $\boldsymbol{r} = \frac{1}{2}\boldsymbol{r}^1 + \frac{1}{2}\boldsymbol{r}^2$.
In this part of the proof, we show that if $\boldsymbol{r}$ is of one of types 1-9 and not extreme, (E-1) and (E-2) contradict each other and can never hold simultaneously for $\boldsymbol{r}$, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$.

Suppose $\boldsymbol{r}$ is of type 1 and not extreme. Then, (E-2) implies for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ that $\boldsymbol{\alpha}^l = \boldsymbol{\beta}^l = \boldsymbol{\psi}^l = \boldsymbol{\lambda}^l = 0$ and $\zeta^l = 0$ for $l = 1, 2$. But, then $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are equivalent to $\boldsymbol{r}$ up to a constant, which contradicts (E-1). Arguing in the same manner, we can also show that $\boldsymbol{r}$ is extreme when it is of type 2 or 3.

Now, suppose $\boldsymbol{r}$ is a ray of type 4 and is not extreme. From (E-2) we can infer about $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ that $\boldsymbol{\alpha}^l = \boldsymbol{\beta}^l = \boldsymbol{\psi}^l = \boldsymbol{\lambda}^l = 0$, $\theta^l = 1 + (-1)^l \epsilon_\theta$, $\zeta^l = 1 + (-1)^l \epsilon_\zeta$ for $l = 1, 2$ and for some $\epsilon_\theta, \epsilon_\zeta \in \mathbb{R}$. Plugging these values into (A.9a) for $l = 1, 2$ yields $\epsilon_\theta = \epsilon_\zeta$. But then, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ would contradict (E-1). In the same manner, we can also show that $\boldsymbol{r}$ is extreme when it is of type 5, 7 and 9.

Next, suppose $\boldsymbol{r}$ is a ray of type 6 and is not extreme. (E-2) implies for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ that

$$\begin{aligned}
\boldsymbol{\alpha}^l &= 0 \qquad \text{for } l = 1, 2, \\
\beta_i^l &= \begin{cases} 1 + (-1)^l \epsilon_{\beta_i} & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} \qquad \text{for } l = 1, 2, \\
\boldsymbol{\psi}^l &= 0 \qquad \text{for } l = 1, 2, \\
\boldsymbol{\lambda}^l &= 0 \qquad \text{for } l = 1, 2, \\
\theta^l &= 1 + (-1)^l \epsilon_\theta \qquad \text{for } l = 1, 2, \\
\zeta^l &= 0 \qquad \text{for } l = 1, 2,
\end{aligned}$$

for some $S \subseteq V$ and $\epsilon_\theta, \epsilon_{\beta_i} \in \mathbb{R}$ for $i \in S$. Plugging these values into (A.9a) for $l = 1, 2$ yields $\epsilon_\theta = \epsilon_{\beta_i}$ for all $i \in S$, which contradicts (E-1).

Finally, suppose $\boldsymbol{r}$ is a ray of type 8 and is not extreme. Then, (E-2) implies for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ that

$$\begin{aligned}
\alpha_i^l &= \begin{cases} 1 + (-1)^l \epsilon_{\alpha_i} & \text{if } i \in V - S \\ 0 & \text{if } i \in S \end{cases} \qquad \text{for } l = 1, 2, \\
\boldsymbol{\beta}^l &= 0 \qquad \text{for } l = 1, 2, \\
\psi_i^l &= \begin{cases} 1 + (-1)^l \epsilon_{\psi_i} & \text{if } i \in S \\ 0 & \text{if } i \in V - S \end{cases} \qquad \text{for } l = 1, 2, \\
\lambda_i^l &= \begin{cases} 1 + (-1)^l \epsilon_{\lambda_i} & \text{if } i \in V - S \\ 0 & \text{if } i \in S \end{cases} \qquad \text{for } l = 1, 2, \\
\theta^l &= 0 \qquad \text{for } l = 1, 2, \\
\zeta^l &= 1 + (-1)^l \epsilon_\zeta \qquad \text{for } l = 1, 2,
\end{aligned}$$

for some $S \subseteq V$ and $\epsilon_{\alpha_i}, \epsilon_{\psi_j}, \epsilon_{\lambda_i}, \epsilon_\zeta \in \mathbb{R}$ for $i \in V - S$, $j \in S$. Plugging $\alpha_i^l$, $\psi_i^l$ and $\zeta^l$ values into (A.5a) for $l = 1, 2$ yields $\epsilon_{\alpha_i} = \epsilon_{\psi_j} = \epsilon_\zeta$ for $i \in V - S$, $j \in S$. And, plugging $\alpha_i^l$, $\lambda_i^l$ for $l = 1, 2$ into (A.5b) yields $\epsilon_{\alpha_i} = \epsilon_{\lambda_i} = \epsilon_\zeta$ for $i \in V - S$.

*Necessity:* Consider an extreme ray $\boldsymbol{r} = (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\psi}, \boldsymbol{\lambda}, \theta, \zeta)$ of $W$. Note that, if $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\psi} = \boldsymbol{\lambda} = 0$, and $\zeta = 0$, $\boldsymbol{r}$ is equal (up to a constant) to the extreme ray of type 1. In the rest of this proof, we argue similarly for other values of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\psi}$, $\boldsymbol{\lambda}$, $\zeta$, and show in a series of claims that $\boldsymbol{r}$ is necessarily equal (up to a constant) to a ray of one of types 2-9.

Let in the sequel $P_\alpha = \{i \in V : \alpha_i > 0\}$, $P_\beta = \{i \in V : \beta_i > 0\}$, $P_\psi = \{i \in V : \psi_i > 0\}$ and $P_\lambda = \{i \in V : \lambda_i > 0\}$. More precisely, in the

sequel,

| | | | |
|---|---|---|---|
| Claim 1 | show(s) $\boldsymbol{r}$ is of | type 4 | if $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\psi} = \boldsymbol{\lambda} = 0$ and $\zeta > 0$; |
| Claims 2, 3 | | type 2 or 3 | if $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi \neq V$; |
| Claims 4-6 | | type 8 | if $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi = V$; |
| Claims 7-9 | | type 6 or 7 | if $\boldsymbol{\alpha} = 0$, $\boldsymbol{\beta} \neq 0$; |
| Claims 10, 11 | | type 9 | if $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} \neq 0$; |
| Claims 12-16 | | type 8 | if $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$, $\boldsymbol{\psi} \neq 0$; |
| Claims 12, 17, 18 | | type 5 | if $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = \boldsymbol{\psi} = 0$. |

Like in the proof of Proposition A.1.1, we mark the end of the proofs of the claims by "Q.E.D."; and, we mark the end of the overall proof of the proposition by the usual hollow square (i.e., □). We now move on to the claims.

*Claim 1.* Suppose $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\psi} = \boldsymbol{\lambda} = 0$ and $\zeta > 0$. Then, $\theta = \zeta$.
*Proof:* (A.9a) implies $\theta \geq \zeta$. But having $\theta > \zeta$ would contradict $\boldsymbol{r}$ being extreme. To see this, consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$ where

$$\begin{aligned}
\boldsymbol{\alpha}^1 &= 0, & \boldsymbol{\alpha}^2 &= 0,\\
\boldsymbol{\beta}^1 &= 0, & \boldsymbol{\beta}^2 &= 0,\\
\boldsymbol{\psi}^1 &= 0, & \boldsymbol{\psi}^2 &= 0,\\
\boldsymbol{\lambda}^1 &= 0, & \boldsymbol{\lambda}^2 &= 0,\\
\theta^1 &= 2(\theta - \zeta), & \theta^2 &= 2\zeta,\\
\zeta^1 &= 0, & \zeta^2 &= 2\zeta.
\end{aligned}$$

When constructed in this manner, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ would both be feasible and comply with (E-1) and (E-2). Q.E.D.

Claim 1 demonstrates that $\boldsymbol{r}$ is equal (up to a constant) to the extreme ray of type 4 if $\boldsymbol{\alpha} = \boldsymbol{\beta} = \boldsymbol{\psi} = \boldsymbol{\lambda} = 0$, and $\zeta > 0$, .

*Claim 2.* Suppose $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$. Then, $\theta = \zeta = 0$ unless $P_\psi = V$.
*Proof:* If $P_\psi \neq V$, then the inequality (A.9a) corresponding to a $j \in V - P_\psi$

suggests $\theta \geq \zeta$. Then, $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$, where

$$\begin{aligned} \boldsymbol{\alpha}^1 &= 0, & \boldsymbol{\alpha}^2 &= 0, \\ \boldsymbol{\beta}^1 &= 0, & \boldsymbol{\beta}^2 &= 0, \\ \boldsymbol{\psi}^1 &= 2\boldsymbol{\psi}, & \boldsymbol{\psi}^2 &= 0, \\ \boldsymbol{\lambda}^1 &= 2\boldsymbol{\lambda}, & \boldsymbol{\lambda}^2 &= 0, \\ \theta^1 &= 0, & \theta^2 &= 2\theta, \\ \zeta^1 &= 0, & \zeta^2 &= 2\zeta, \end{aligned}$$

would both be feasible, satisfy (E-1) and (E-2) and hence contradict $\boldsymbol{r}$ being extreme. Q.E.D.

*Claim 3.* Suppose $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi \neq V$. Then $|P_\psi| + |P_\lambda| = 1$.
*Proof:* Since $P_\psi \neq V$, we know from Claim 2 that $\theta = \zeta = 0$. We prove the claim in two steps.

Step 1. Suppose both of $P_\psi$ and $P_\lambda$ are non-empty. (E-1) and (E-2) would hold for $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ where $l = 1, 2$, and

$$\begin{aligned} \boldsymbol{\alpha}^1 &= 0, & \boldsymbol{\alpha}^2 &= 0, \\ \boldsymbol{\beta}^1 &= 0, & \boldsymbol{\beta}^2 &= 0, \\ \boldsymbol{\psi}^1 &= 2\boldsymbol{\psi}, & \boldsymbol{\psi}^2 &= 0, \\ \boldsymbol{\lambda}^1 &= 0, & \boldsymbol{\lambda}^2 &= 2\boldsymbol{\lambda}, \\ \theta^1 &= 0, & \theta^2 &= 0, \\ \zeta^1 &= 0, & \zeta^2 &= 0. \end{aligned}$$

Step 2. Now, suppose without loss of generality that $P_\lambda$ is empty and $P_\psi$ is not. Let $i^* \in P_\psi$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$, where

$$\begin{aligned} &\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0, \\ &\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0, \\ &\psi_i^1 = \begin{cases} 2\psi_i & \text{if } i = i^*, \\ 0 & \text{otherwise,} \end{cases} && \psi_i^2 = \begin{cases} 2\psi_i & \text{if } i \in P_\psi - \{i^*\}, \\ 0 & \text{otherwise,} \end{cases} \\ &\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 0, \\ &\theta^1 = 0, && \theta^2 = 0, \\ &\zeta^1 = 0, && \zeta^2 = 0. \end{aligned}$$

If $|P_\psi| = 1$, $\boldsymbol{r}^2 = 0$. But otherwise, (E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$, and hence contradict $\boldsymbol{r}$ being extreme. Q.E.D.

From Claims 2 and 3, we can infer that if $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi \neq V$, $\boldsymbol{r}$ is equal (up to a constant) to an extreme ray of type 2 or 3.

*Claim 4.* Suppose $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi = V$. Then $\boldsymbol{\lambda} = 0$.
*Proof:* If $P_\lambda \neq \emptyset$, we can construct $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$, where

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\boldsymbol{\psi}^1 = 2\boldsymbol{\psi}, && \boldsymbol{\psi}^2 = 0,\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 2\boldsymbol{\lambda},\\
&\theta^1 = 2\theta, && \theta^2 = 0,\\
&\zeta^1 = 2\zeta, && \zeta^2 = 0.
\end{aligned}$$

Obviously, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ comply with (E-1) and (E-2). Q.E.D.

*Claim 5.* Suppose $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi = V$. Then $\theta = 0$.
*Proof:* Suppose $\theta > 0$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$, where

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\psi_i^1 = \psi_i - \epsilon \ \ \forall i \in V, && \psi_i^2 = \psi_i + \epsilon \ \ \forall i \in V,\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 0,\\
&\theta^1 = \theta + \epsilon, && \theta^2 = \theta - \epsilon,\\
&\zeta^1 = \zeta, && \zeta^2 = \zeta.
\end{aligned}$$

*Claim 6.* Suppose $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi = V$. Then $\psi_i = \zeta > 0$ for all $i \in V$.
*Proof:* First, note that $\zeta > 0$, because otherwise we could prove in a very similar manner to Step 2 of Claim 3 that $\boldsymbol{r}$ is not extreme. Now, for a sufficiently small $\epsilon > 0$, we can construct $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$, in the following

manner:

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\psi_i^1 = \psi_i + \epsilon \ \ \forall i \in V, && \psi_i^2 = \psi_i - \epsilon \ \ \forall i \in V,\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 0,\\
&\theta^1 = 0, && \theta^2 = 0,\\
&\zeta^1 = \zeta + \epsilon, && \zeta^2 = \zeta - \epsilon.
\end{aligned}$$

If there exists $j \in V$ such that $\psi_j \neq \zeta$, then $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ would satisfy (E-1) and (E-2), and hence contradict $\boldsymbol{r}$ being extreme. Otherwise, that is, if $\psi_i = \zeta$ for all $i \in V$, both $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ would equal (up to a constant) to $\boldsymbol{r}$. Q.E.D.

Claims 4, 5 and 6 demonstrate that when $\boldsymbol{\alpha} = \boldsymbol{\beta} = 0$ and $P_\psi = V$, $\boldsymbol{r}$ is an extreme ray of type 8.

*Claim 7.* Suppose $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\beta} \neq 0$. Then $\boldsymbol{\lambda} = 0$.
*Proof:* If $\boldsymbol{\lambda} = 0$, we can construct $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$, where

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = 2\boldsymbol{\beta}, && \boldsymbol{\beta}^2 = 0,\\
&\boldsymbol{\psi}^1 = 2\boldsymbol{\psi}, && \boldsymbol{\psi}^2 = 0,\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 2\boldsymbol{\lambda},\\
&\theta^1 = 2\theta, && \theta^2 = 0,\\
&\zeta^1 = 2\zeta, && \zeta^2 = 0.
\end{aligned}$$

Obviously, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ comply with (E-1) and (E-2). Q.E.D.

*Claim 8.* Suppose $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\beta} \neq 0$. Then $\zeta = 0$.
*Proof:* From Claim 7, we know $\boldsymbol{\lambda} = 0$. Suppose $\zeta > 0$. We investigate two cases.

1. $\theta > 0$. In this case, (E-1) and (E-2) would hold for $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$,

$l = 1, 2$, constructed in the following manner for a sufficiently small $\epsilon > 0$:

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = \boldsymbol{\beta}, && \boldsymbol{\beta}^2 = \boldsymbol{\beta},\\
&\boldsymbol{\psi}^1 = \boldsymbol{\psi}, && \boldsymbol{\psi}^2 = \boldsymbol{\psi},\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 0,\\
&\theta^1 = \theta + \epsilon, && \theta^2 = \theta - \epsilon,\\
&\zeta^1 = \zeta + \epsilon, && \zeta^2 = \zeta - \epsilon.
\end{aligned}$$

2. $\theta = 0$. In this case, (A.9a) dictate that $P_\psi = V$. Then, $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$ comply with (E-1) and (E-2), when constructed in the following manner for a sufficiently small $\epsilon$:

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = \boldsymbol{\beta}, && \boldsymbol{\beta}^2 = \boldsymbol{\beta},\\
&\psi^1 = \psi + \epsilon \ \ \forall i \in V, && \psi^2 = \psi - \epsilon \ \ \forall i \in V,\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 0,\\
&\theta^1 = 0, && \theta^2 = 0,\\
&\zeta^1 = \zeta + \epsilon, && \zeta^2 = \zeta - \epsilon.
\end{aligned}$$

Q.E.D.

*Claim 9.* Suppose that $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\beta} \neq 0$. Then, the following are true:

(i) If $\theta > 0$, then

  (a) $\boldsymbol{\psi} = 0$, and,
  (b) $\beta_i = \theta$ for all $i \in P_\beta$.

(ii) If $\theta = 0$, then

  (a) $P_\psi = P_\beta$,
  (b) $|P_\psi| = |P_\beta| = 1$, and,
  (c) $\beta_{i^*} = \psi_{i^*} > 0$ for $i^* \in P_\beta$.

*Proof:* From Claims 7 and 8, we know $\boldsymbol{\lambda} = 0$ and $\zeta = 0$. We prove the two cases separately.

(i) $\theta > 0$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$ where,

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\beta_i^1 = \begin{cases} \beta_i + \epsilon & \text{if } i \in P_\beta, \\ 0 & \text{if } i \notin P_\beta, \end{cases} && \beta_i^2 = \begin{cases} \beta_i - \epsilon & \text{if } i \in P_\beta, \\ 0 & \text{if } i \notin P_\beta, \end{cases}\\
&\boldsymbol{\psi}^1 = \boldsymbol{\psi}, && \boldsymbol{\psi}^2 = \boldsymbol{\psi},\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 0,\\
&\theta^1 = \theta + \epsilon, && \theta^2 = \theta - \epsilon,\\
&\zeta^1 = 0, && \zeta^2 = 0,
\end{aligned}$$

and $\epsilon$ is a sufficiently small positive real number. Note that, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ are both feasible and comply with (E-2). But, (E-1) holds for them only if (i)-(a) or (i)-(b) does not hold. Hence, if $\theta > 0$, $P_\psi$ is necessarily empty.

(ii) $\theta = 0$. In this case, (A.9a) dictate that $P_\psi \supseteq P_\beta$. Pick $i^* \in P_\beta$. Now, consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ , $l = 1, 2$, where

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 0, && \boldsymbol{\alpha}^2 = 0,\\
&\beta_i^1 = \begin{cases} \beta_i + \epsilon & \text{if } i = i^*, \\ \beta_i & \text{otherwise}, \end{cases} && \beta_i^2 = \begin{cases} \beta_i - \epsilon & \text{if } i = i^*, \\ \beta_i & \text{otherwise}, \end{cases}\\
&\psi_i^1 = \begin{cases} \psi_i + \epsilon & \text{if } i = i^*, \\ \psi_i & \text{otherwise}, \end{cases} && \psi_i^2 = \begin{cases} \psi_i - \epsilon & \text{if } i = i^*, \\ \psi_i & \text{otherwise}, \end{cases}\\
&\boldsymbol{\lambda}^1 = 0, && \boldsymbol{\lambda}^2 = 0,\\
&\theta^1 = 0, && \theta^2 = 0,\\
&\zeta^1 = 0, && \zeta^2 = 0,
\end{aligned}$$

and $\epsilon$ is a sufficiently small positive real number. Note that when any one of (ii)-(a), (ii)-(b) and (ii)-(c) does not hold, $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ comply with (E-1) and (E-2). Q.E.D.

From Claims 7,8 and 9, one can infer that when $\boldsymbol{\alpha} = 0$ and $\boldsymbol{\beta} \neq 0$, $\boldsymbol{r}$ is equal, up to a constant, to an extreme ray of type 6 or 7.

*Claim 10.* Suppose $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\beta} \neq 0$. Then $P_\alpha \cap P_\beta \neq \emptyset$.
*Proof:* Suppose $P_\alpha \cap P_\beta = \emptyset$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$

where,

$$\begin{aligned}
\boldsymbol{\alpha}^1 &= 2\boldsymbol{\alpha}, & \boldsymbol{\alpha}^2 &= 0,\\
\boldsymbol{\beta}^1 &= 0, & \boldsymbol{\beta}^2 &= 2\boldsymbol{\beta},\\
\boldsymbol{\psi}^1 &= 0, & \boldsymbol{\psi}^2 &= 2\boldsymbol{\psi},\\
\boldsymbol{\lambda}^1 &= 2\boldsymbol{\lambda}, & \boldsymbol{\lambda}^2 &= 0,\\
\theta^1 &= 0, & \theta^2 &= 2\theta,\\
\zeta^1 &= 0, & \zeta^2 &= 2\zeta.
\end{aligned}$$

(E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$. Q.E.D.

*Claim 11.* Suppose $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\beta} \neq 0$. Then, the following are true:

(i) $\boldsymbol{\psi} = 0$, $\boldsymbol{\lambda} = 0$,

(ii) $\theta = \zeta = 0$,

(iii) $P_\alpha = P_\beta$,

(iv) $|P_\alpha| = |P_\beta| = 1$,

(v) $\alpha_{i^*} = \beta_{i^*}$ for $i^* \in P_\alpha$.

*Proof:* From Claim 10 we know that $P_\alpha \cap P_\beta \neq \emptyset$. Pick arbitrarily $i^* \in P_\alpha \cap P_\beta$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$ where,

$$\begin{aligned}
\alpha_i^1 &= \begin{cases} \alpha_i + \epsilon & \text{if } i = i^*, \\ \alpha_i & \text{otherwise,} \end{cases} & \alpha_i^2 &= \begin{cases} \alpha_i - \epsilon & \text{if } i = i^*, \\ \alpha_i & \text{otherwise,} \end{cases}\\
\beta_i^1 &= \begin{cases} \beta_i + \epsilon & \text{if } i = i^*, \\ \beta_i & \text{otherwise,} \end{cases} & \beta_i^2 &= \begin{cases} \beta_i - \epsilon & \text{if } i = i^*, \\ \beta_i & \text{otherwise,} \end{cases}\\
\boldsymbol{\psi}^1 &= \boldsymbol{\psi}, & \boldsymbol{\psi}^2 &= \boldsymbol{\psi},\\
\boldsymbol{\lambda}^1 &= \boldsymbol{\lambda}, & \boldsymbol{\lambda}^2 &= \boldsymbol{\lambda},\\
\theta^1 &= \theta, & \theta^2 &= \theta,\\
\zeta^1 &= \zeta, & \zeta^2 &= \zeta,
\end{aligned}$$

and $\epsilon$ is a sufficiently small positive real number. (E-2) holds for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$. If $\boldsymbol{r}$ violates any of conditions (i)-(v), $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ satisfy (E-1) as well. Q.E.D.

Claims 10 and 11 imply that when $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\beta} \neq 0$, $\boldsymbol{r}$ is equal (up to a constant) to an extreme ray of type 9.

*Claim 12.* Suppose $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\beta} = 0$. Then, $P_\alpha = P_\lambda$.
*Proof:* Due to (A.5a), $P_\lambda \supseteq P_\alpha$. Let $i^* \in P_\lambda - P_\alpha$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$, where

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 2\boldsymbol{\alpha}, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = 2\boldsymbol{\beta}, && \boldsymbol{\beta}^2 = 0,\\
&\boldsymbol{\psi}^1 = 2\boldsymbol{\psi}, && \boldsymbol{\psi}^2 = 0,\\
&\lambda_i^1 = \begin{cases} 0 & \text{if } i = i^*,\\ 2\lambda_i & \text{otherwise,}\end{cases} && \lambda_i^2 = \begin{cases} 2\lambda_i & \text{if } i = i^*,\\ 0 & \text{otherwise,}\end{cases}\\
&\theta^1 = 2\theta, && \theta^2 = 0,\\
&\zeta^1 = 2\zeta, && \zeta^2 = 0.
\end{aligned}$$

(E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$. Q.E.D.

*Claim 13.* Suppose $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\psi} \neq 0$. Then, $\zeta > 0$.
*Proof:* Suppose $\zeta = 0$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$ for $l = 1, 2$ where,

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = 2\boldsymbol{\alpha}, && \boldsymbol{\alpha}^2 = 0,\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\boldsymbol{\psi}^1 = 0, && \boldsymbol{\psi}^2 = 2\boldsymbol{\psi},\\
&\boldsymbol{\lambda}^1 = 2\boldsymbol{\lambda}, && \boldsymbol{\lambda}^2 = 0,\\
&\theta^1 = 2\theta, && \theta^2 = 0,\\
&\zeta^1 = 0, && \zeta^2 = 0.
\end{aligned}$$

(E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$. Q.E.D.

*Claim 14.* Suppose $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\psi} \neq 0$. Then $\theta = 0$.
*Proof:* Suppose $\theta > 0$. We know from Claim 13 that $\zeta > 0$. Hence, $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$, constructed in the following manner for a suffi-

ciently small $\epsilon > 0$ would comply with (E-1) and (E-2):

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = \boldsymbol{\alpha}, && \boldsymbol{\alpha}^2 = \boldsymbol{\alpha},\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\boldsymbol{\psi}^1 = \boldsymbol{\psi}, && \boldsymbol{\psi}^2 = \boldsymbol{\psi},\\
&\boldsymbol{\lambda}^1 = \boldsymbol{\lambda}, && \boldsymbol{\lambda}^2 = \boldsymbol{\lambda},\\
&\theta^1 = \theta + \epsilon, && \theta^2 = \theta - \epsilon,\\
&\zeta^1 = \zeta + \epsilon, && \zeta^2 = \zeta - \epsilon.
\end{aligned}$$

Q.E.D.

<u>*Claim 15.*</u> Suppose $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\psi} \neq 0$. Then $P_\alpha \cap P_\psi = \emptyset$ and $P_\alpha \cup P_\psi = V$.
*Proof:* From Claims 12 and 14, we know that $P_\alpha = P_\lambda$ and $\theta = 0$. Hence, (A.9a) imply that $P_\alpha \cup P_\psi = V$. Suppose $P_\alpha \cap P_\psi \neq \emptyset$. Pick $i^* \in P_\alpha \cap P_\psi$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$, where

$$\begin{aligned}
&\alpha_i^1 = \begin{cases} \alpha_i + \epsilon & \text{if } i = i^*, \\ \alpha_i & \text{otherwise,} \end{cases} && \lambda_i^2 = \begin{cases} \alpha_i - \epsilon & \text{if } i = i^*, \\ \alpha_i & \text{otherwise,} \end{cases}\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\psi_i^1 = \begin{cases} \psi_i - \epsilon & \text{if } i = i^*, \\ \psi_i & \text{otherwise,} \end{cases} && \lambda_i^2 = \begin{cases} \psi_i + \epsilon & \text{if } i = i^*, \\ \psi_i & \text{otherwise,} \end{cases}\\
&\lambda_i^1 = \begin{cases} \lambda_i + \epsilon & \text{if } i = i^*, \\ \lambda_i & \text{otherwise,} \end{cases} && \lambda_i^2 = \begin{cases} \lambda_i - \epsilon & \text{if } i = i^*, \\ \lambda_i & \text{otherwise,} \end{cases}\\
&\theta^1 = 0, && \theta^2 = 0,\\
&\zeta^1 = \zeta, && \zeta^2 = \zeta,
\end{aligned}$$

and $\epsilon$ is a sufficiently small positive real number. Obviously, (E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$. Q.E.D.

<u>*Claim 16.*</u> Suppose $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\psi} \neq 0$. Then, $\alpha_i = \psi_j = \lambda_i = \zeta$ for all $i \in P_\alpha = P_\lambda$ and $j \in P_\psi$.
*Proof:* From Claims 12 and 14, we know that $P_\alpha = P_\lambda$ and $\theta = 0$. We know further from Claim 15 that $P_\alpha \cap P_\psi = \emptyset$ and $P_\alpha \cup P_\psi = V$. Consider

$\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$, where

$$\alpha_i^1 = \begin{cases} \alpha_i + \epsilon & \text{if } i \in P_\alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad \lambda_i^2 = \begin{cases} \alpha_i - \epsilon & \text{if } i \in P_\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

$$\boldsymbol{\beta}^1 = 0, \qquad \boldsymbol{\beta}^2 = 0,$$

$$\psi_i^1 = \begin{cases} \psi_i + \epsilon & \text{if } i \in P_\psi, \\ 0 & \text{otherwise,} \end{cases} \qquad \lambda_i^2 = \begin{cases} \psi_i - \epsilon & \text{if } i \in P_\psi, \\ 0 & \text{otherwise,} \end{cases}$$

$$\lambda_i^1 = \begin{cases} \lambda_i + \epsilon & \text{if } i \in P_\lambda, \\ 0 & \text{otherwise,} \end{cases} \qquad \lambda_i^2 = \begin{cases} \lambda_i - \epsilon & \text{if } i \in P_\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

$$\theta^1 = 0, \qquad \theta^2 = 0,$$

$$\zeta^1 = \zeta + \epsilon, \qquad \zeta^2 = \zeta - \epsilon,$$

and $\epsilon$ is a sufficiently small positive real number. (E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ only if the hypothesis of the claim does not hold. Q.E.D.

From Claims 12-16, one can infer that when $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\psi} \neq 0$, $\boldsymbol{r}$ is equal, up to a constant, to an extreme ray of type 8.

*Claim 17.* Suppose $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\psi} = 0$. Then, $\zeta = 0$.
*Proof:* Suppose $\zeta > 0$. From Claim 12, we know $P_\alpha = P_\lambda$. We consider two cases.

1. $P_\alpha = V$. Consider $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$, where

$$\alpha_i^1 = \alpha_i + \epsilon \ \ \forall i \in V, \qquad \alpha_i^2 = \alpha_i - \epsilon \ \ \forall i \in V,$$

$$\boldsymbol{\beta}^1 = 0, \qquad \boldsymbol{\beta}^2 = 0,$$

$$\boldsymbol{\psi}^1 = 0, \qquad \boldsymbol{\psi}^2 = 0,$$

$$\lambda_i^1 = \lambda_i + \epsilon \ \ \forall i \in V, \qquad \lambda_i^2 = \lambda_i - \epsilon \ \ \forall i \in V,$$

$$\theta^1 = \theta, \qquad \theta^2 = \theta,$$

$$\zeta^1 = \zeta + \epsilon, \qquad \zeta^2 = \zeta - \epsilon,$$

   and $\epsilon$ is a sufficiently small positive real number. (E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$.

2. $P_\alpha \subset V$. Inequality (A.9a) corresponding to a $i \in V - P_\alpha$ implies that $\theta \geq \zeta$. Then, $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$ constructed in the

following manner for a sufficiently small $\epsilon > 0$ would comply with (E-1) and (E-2):

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = \boldsymbol{\alpha}, && \boldsymbol{\alpha}^2 = \boldsymbol{\alpha},\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\boldsymbol{\psi}^1 = 0, && \boldsymbol{\psi}^2 = 0,\\
&\boldsymbol{\lambda}^1 = \boldsymbol{\lambda}, && \boldsymbol{\lambda}^2 = \boldsymbol{\lambda},\\
&\theta^1 = \theta + \epsilon, && \theta^2 = \theta - \epsilon,\\
&\zeta^1 = \zeta + \epsilon, && \zeta^2 = \zeta - \epsilon.
\end{aligned}$$

Q.E.D.

*Claim 18.* Suppose $\boldsymbol{\alpha} \neq 0$, $\boldsymbol{\beta} = 0$ and $\boldsymbol{\psi} = 0$. Then,

(i) $\theta = 0$,

(ii) $|P_\alpha| = |P_\lambda| = 1$, and,

(iii) $\alpha_i = \lambda_i$ for $i \in P_\alpha$.

*Proof:* From Claims 12 and 17 we know that $P_\alpha = P_\lambda$ and $\zeta = 0$. Pick $i^* \in P_\alpha$. We can construct $\boldsymbol{r}^l = (\boldsymbol{\alpha}^l, \boldsymbol{\beta}^l, \boldsymbol{\psi}^l, \boldsymbol{\lambda}^l, \theta^l, \zeta^l)$, $l = 1, 2$, such that

$$\begin{aligned}
&\alpha_i^1 = \begin{cases} \alpha_i + \epsilon & \text{if } i = i^*, \\ \alpha_i & \text{otherwise,} \end{cases} && \lambda_i^2 = \begin{cases} \alpha_i - \epsilon & \text{if } i = i^*, \\ \alpha_i & \text{otherwise,} \end{cases}\\
&\boldsymbol{\beta}^1 = 0, && \boldsymbol{\beta}^2 = 0,\\
&\boldsymbol{\psi}^1 = 0, && \boldsymbol{\psi}^2 = 0,\\
&\lambda_i^1 = \begin{cases} \lambda_i + \epsilon & \text{if } i = i^*, \\ \lambda_i & \text{otherwise,} \end{cases} && \lambda_i^2 = \begin{cases} \lambda_i - \epsilon & \text{if } i = i^*, \\ \lambda_i & \text{otherwise,} \end{cases}\\
&\theta^1 = \theta, && \theta^2 = \theta,\\
&\zeta^1 = 0, && \zeta^2 = 0.
\end{aligned}$$

Obviously, (E-1) and (E-2) hold for $\boldsymbol{r}^1$ and $\boldsymbol{r}^2$ only if any of conditions (i)-(iii) fail to hold. Q.E.D.

Claims 12, 17 and 18 demonstrate that when $\boldsymbol{\alpha} \neq 0$ and $\boldsymbol{\beta} = \boldsymbol{\psi} = 0$, $\boldsymbol{r}$ is equivalent up to a constant to the extreme ray of type 5. □ □

The inequalities implied by these extreme rays are respectively as follows:

$$0 \leq 1 - w_{u,v}, \tag{A.10a}$$
$$0 \leq x_{u,i} \qquad \forall i \in V, \tag{A.10b}$$
$$0 \leq x_{v,i} \qquad \forall i \in V, \tag{A.10c}$$
$$0 \leq 0, \tag{A.10d}$$
$$(N\phi^{u,v})_i \leq b_i^4 + x_{v,i} \qquad \forall i \in V, \forall (u,v) \in T, \tag{A.10e}$$
$$(N\gamma^{u,v})_S \leq 1 - w_{u,v} \qquad \forall S \subseteq V, \forall (u,v) \in T, \tag{A.10f}$$
$$(N\gamma^{u,v})_i \leq x_{u,i} \qquad \forall i \in V, \forall (u,v) \in T, \tag{A.10g}$$
$$(N\phi^{u,v})_S \leq w_{u,v} - 1 + \sum_{l \in V-S} x_{u,l} + \sum_{l \in S} x_{v,l} \quad \forall S \subseteq V, \forall (u,v) \in T, \tag{A.10h}$$
$$(N\phi^{u,v})_i + (N\gamma^{u,v})_i \leq b_i^4 \qquad \forall i \in V, \forall (u,v) \in T. \tag{A.10i}$$

To sum up, projection of Formulation 3 over the subspace of $w_{u,v}$, $x_{u,v}$, $\phi_{i,j}^{u,v}$ and $\gamma_{i,j}^{u,v}$ variables replaces (2.7e)-(2.7i) by (A.10e)-(A.10i) (as (A.10a)-(A.10d) are obviously redundant). For the sake of simplicity, in the rest of this chapter we refer to this reformulation of Formulation 3 as Formulation 3P.

### A.1.3 Comparison

Formulation 2P consists of (2.1a)-(2.1g), (2.2b), (2.6d), (2.6e), (A.2) and (A.7). All of these inequalities except (2.6d), (2.6e) and (A.7) are already present in Formulation 3P. Below, we present several lemmas which state that (2.6d) and (2.6e) are dominated by Formulation 3P. We start with a technical lemma.

**Lemma A.1.1.** *Inequalities (A.10i) imply*

$$(N\phi^{u,v})_i + (N\gamma^{u,v})_i = b_i^4. \tag{A.11}$$

*Proof.* Inequalities (A.10i) imply

$$(N\phi^{u,v})_{V-i} + (N\gamma^{u,v})_{V-i} \leq \sum_{l \in V-i} b_l^4 \qquad \forall i \in V.$$

The result follows immediately, because $(N\phi^{u,v})_{V-i} = -(N\phi^{u,v})_i$, $(N\gamma^{u,v})_{V-i} = -(N\gamma^{u,v})_i$ and $\sum_{l \in V-i} b_i^4 = -b_l^4$ for all $i \in V$. □

Now follows the lemma on domination of (2.6d) and (2.6e).

**Lemma A.1.2.** *(2.6d) and (2.6e) are dominated by Formulation 3P.*

*Proof.* We present the proof for only (2.6e). For (2.6d), it is very similar. Note that, due to (2.1d), we can rewrite (A.10h) as

$$(N\phi^{u,v})_S \leq w_{u,v} - \sum_{l\in S} x_{u,l} + \sum_{l\in S} x_{v,l} \qquad \forall S \subseteq V, \forall (u,v) \in T. \tag{A.12}$$

Inequalities (A.11) and (A.12) imply

$$(N\gamma^{u,v})_i \geq b_i^4 + x_{u,i} - x_{v,i} - w_{u,v} \qquad \forall i \in V, \forall (u,v) \in T,$$

which dominates (2.6e) for every $(u,v) \in T$ and $i \in V - \{v\}$. And, inequalities (A.10e) and (A.11) imply

$$(N\gamma^{u,v})_i \geq -x_{v,i} \qquad \forall i \in V, \forall (u,v) \in T,$$

which, due to (2.1g), dominates (2.6e) for all $(u,v) \in T$ and $i = v$. □

Now, in Lemma A.1.3 we state some *sufficient* conditions on $S_\alpha$, $S_\beta$ and $S_\mu$ under which the inequalities (A.7) are dominated by Formulation 3P.

**Lemma A.1.3.** *(i) When $S_\alpha \neq \emptyset$ and $S_\beta = S_\mu = \emptyset$, (A.7) is dominated by Formulation 3P if $\{u,v\} \nsubseteq S_\alpha$.*

*(ii) When $S_\beta \neq \emptyset$ and $S_\alpha = S_\mu = \emptyset$, (A.7) is dominated by Formulation 3P if $\{u,v\} \nsubseteq S_\beta$.*

*(iii) When $S_\mu \neq \emptyset$ and $S_\alpha = S_\beta = \emptyset$, (A.7) is dominated by Formulation 3P.*

*Proof.* (i) In this case, (A.7) turn into

$$-\phi(A^+(S_\alpha)) \leq \sum_{i\in S_\alpha} b_i^1.$$

When $u \notin S_\alpha$, $\sum_{i\in S} b_i^1 \geq 0$, hence (A.7) become redundant. When $u \in S_\alpha$ and $v \notin S_\alpha$, inequalities (2.7e) imply that

$$\phi(A^+(S_\alpha)) + \sum_{i\in S_\alpha} \tau_i^{1,u,v} \geq \sum_{i\in S_\alpha} b_i^4 \geq 1,$$

which, by (2.7g), implies

$$\phi(A^+(S_\alpha)) \geq \sum_{i\in V-S_\alpha} \tau_i^{1,u,v}. \tag{A.13}$$

Summing up the inequalities (2.7h) for $i \in S_\alpha$ and multiplying by -1, we obtain

$$-\sum_{i \in S_\alpha} \tau_i^{1,u,v} \geq -\sum_{i \in S_\alpha} x_{u,i}.$$

Now, adding this inequality to (2.7g), we get

$$\sum_{i \in V - S_\alpha} \tau_i^{1,u,v} \geq 1 - w_{u,v} - \sum_{i \in S_\alpha} x_{u,i}.$$

This inequality and (A.13) yield the result.

(ii) In this case, the inequalities (A.7) turn into the following:

$$-\phi(A^-(S_\beta)) \leq \sum_{i \in S_\beta} b_i^2. \tag{A.14}$$

When $v \notin S_\beta$, $\sum_{i \in S_\beta} b_i^2 \geq 0$, hence (A.7) become redundant. When $v \in S_\beta$ and $u \notin S_\beta$, the inequality

$$(N\phi^{u,v})_{S_\beta} \leq \sum_{i \in S_\beta} (b_i^4 + x_{v,i})$$

obtained by summing up (A.10e) for $i \in S_\beta$ dominates (A.14).

(iii) In this case, (A.7) turn into

$$-\phi(A^-(S_\mu)) \leq \sum_{i \in S_\mu} b_i^3.$$

Note that, $\sum_{i \in S_\mu} b_i^3$ can be negative only when $u \notin S_\mu$ and $v \in S_\mu$. Now, letting $S = V - S_\mu$ in (A.10f) yields

$$-(N\gamma^{u,v})_{S_\mu} \leq 1 - w_{u,v}. \tag{A.15}$$

Summing up inequalities (A.10i) for $i \in S_\mu$ gives

$$(N\phi^{u,v})_{S_\mu} + (N\gamma^{u,v})_{S_\mu} \leq \sum_{i \in S_\mu} b_i^3 = -1,$$

since we assume $u \notin S_\mu$ and $v \in S_\mu$. This inequality and (A.15) gives the result.

□

Note that the conditions stated in Lemma A.1.3 are sufficient conditions. They do not necessarily imply that Formulation 2P is implied by Formulation 3P. In fact, below we present an example solution for an example instance, which illustrates that a feasible point for Formulation 3 may violate inequalities (A.7). From Lemma A.1.3 and this example we can conclude that neither of Formulation 2P and Formulation 3P dominates the other.

Consider the graph in Figure 3.1. Let $n = 6$, $F_L = 2$, $F_U = 3$ and $N = 2$. For the sake of simplicity assume that $T = \{(5, 6)\}$. Now, consider the following solution $\tilde{\boldsymbol{z}} = (\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\tau}}^{\mathbf{1}}, \tilde{\boldsymbol{\tau}}^{\mathbf{2}})$:

$$\tilde{\boldsymbol{x}} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 0.4 & 0.2 & 0.2 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0.2 & 0.2 & 0 & 0 \\ 0.2 & 0.2 & 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.4 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \end{array}\right] \end{array};$$

$$\tilde{w}_{h,l} = \begin{cases} 0.2, & \text{if } \{h, l\} = \{5, 6\}, \\ 0.4, & \text{otherwise (i.e., } \forall \{h, l\} \in E - \{5, 6\}); \end{cases}$$

$$\tilde{\phi}_{i,j}^{5,6} = \begin{cases} 0.4, & \text{if } (i, j) = (5, 2), \\ 0.4, & \text{if } (i, j) = (2, 6), \\ 0.4, & \text{if } (i, j) = (5, 1), \\ 0.4, & \text{if } (i, j) = (1, 6), \\ 0.2, & \text{if } (i, j) = (5, 4), \\ 0.2, & \text{if } (i, j) = (3, 6), \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{\gamma}_{i,j}^{5,6} = \begin{cases} 0.2, & \text{if } (i, j) = (4, 3), \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{\tau}_i^{1,5,6} = \begin{cases} 0.25, & \text{if } i = 1, \\ 0.25, & \text{if } i = 2, \\ 0.05, & \text{if } i = 3, \\ 0.25, & \text{if } i = 4, \\ 0, & \text{otherwise;} \end{cases}$$

$$\tilde{\tau}_i^{2,5,6} = \begin{cases} 0.25, & \text{if } i = 1, \\ 0.25, & \text{if } i = 2, \\ 0.25, & \text{if } i = 3, \\ 0.05, & \text{if } i = 4, \\ 0, & \text{otherwise.} \end{cases}$$

This solution is feasible for Formulation 3. But, it violates inequality (A.7) for $S_\alpha = \{1, 2, 5, 6\}, S_\beta = S_\mu = \emptyset$, because $\tilde{\phi}\left(A^+(S_\alpha)\right) = 0.2$ and $\sum_{i \in S_\alpha} \tilde{x}_{5,i} = 0.5$ and $w_{5,6} = 0.2$.

Lemmas A.1.2, A.1.3 and this example solution demonstrates that Formulation 2P is not implied by Formulation 3P. Furthermore, the example solution presented in Section 3.1.1 shows that Formulation 3P is not implied by Formulation 2P, as it is feasible for Formulation 2 but not for (A.10i) in Formulation 3P (obviously, to see this we need to set $\phi_{i,j}^{u,v}$ to $f_{i,j}^{u,v} + h_{i,j}^{u,v}$ and $\gamma_{i,j}^{u,v}$ to $g_{i,j}^{u,v}$).

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