AVERAGE VALUES OF THE FUNCTIONAL EQUATION FACTORS AT THE ZEROS OF DERIVATIVES OF DIRICHLET L-FUNCTIONS

by

Ertan Elma

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APPROVED BY:

Prof. Cem Yalçın Yıldırım . . . .
(Thesis Supervisor)

Prof. Nihat Sadık Değer . . . .

Prof. Kâzım İlhan İkeda . . . .

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ABSTRACT

AVERAGE VALUES OF THE FUNCTIONAL EQUATION FACTORS AT THE ZEROS OF DERIVATIVES OF DIRICHLET L-FUNCTIONS

In this work, average values of the functional equation factors of Riemann Zeta function and Dirichlet L-functions at the zeros of derivatives of Dirichlet L-functions are calculated.

ÖZET

FONKSİYONEL DENKLEM ÇARPANLARININ DIRICHLET L-FONKSİYONLARININ TÜREVLERİNİN SIFIRLARINDAKİ ORTALAMA DEĞERLERİ

Bu çalışmada, Dirichlet L-fonksiyonlarının ve Riemann Zeta fonksiyonunun fonksiyonel denklemlerindeki çarpanlar, Dirichlet L-fonksiyonlarının türevlerinin sıfırlarında hesaplanmıştır.

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LIST OF SYMBOLS

$s = \sigma + it$	A complex variable with $\sigma,\ t \in \mathbb{R}$
$e(\theta)$	$= e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$
$f^{(j)}(s)$	j^{th} derivative of $f(s)$, $f^{(0)} = f$
$L(s, \psi)$	A Dirichlet L-function
$N(T)$	The number of zeros $\rho_\zeta = \sigma_\zeta + i\gamma_\zeta$ of $\zeta(s)$ with $0 < \gamma_\zeta \leqslant T$
$N(T, \psi)$	The number of zeros $\rho_\psi = \sigma_\psi + i\gamma_\psi$ of $L(s, \psi)$ with $0 < \gamma_\psi \leqslant T$
$N_k(T)$	The number of zeros $\rho_{\zeta,k} = \sigma_{\zeta,k} + i\gamma_{\zeta,k}$ of $\zeta^{(k)}(s)$ with $0 < \gamma_{\zeta,k} \leqslant T$
$N_k(T, \psi)$	The number of zeros $\rho_{\psi,k} = \sigma_{\psi,k} + i\gamma_{\psi,k}$ of $L^{(k)}(s, \psi)$ with $0 < \gamma_{\psi,k} \leqslant T$
$\mu(n)$	The Möbius function
$\Gamma(s)$	The Gamma Function
$\zeta(s)$	The Riemann Zeta Function
$\Lambda(n)$	The von-Mangoldt Function
$\tau(\psi)$	The Gauss sum of ψ
$\varphi(n)$	The number of a, $1 \leqslant a \leqslant n$, for which $(a, n) = 1$; known as Euler's totient function
$\psi(n)$	A Dirichlet character
$\chi_\zeta(s)$	The factor in the unsymmetric form of the functional equation of $\zeta(s)$
$\chi_\psi(s)$	The factor in the unsymmetric form of the functional equation of $L(s, \psi)$
$f(x) = O(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is an absolute constant
$f(x) = O_k(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is a constant depending on k
$f(x) = O_q(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is a constant depending on q
$f(x) = O_{k,q}(g(x))$	$\lvert f(x)\rvert \leqslant Cg(x)$ where C is a constant depending on k and q
$f(x) \ll_k g(x)$	$f(x) = O_k(g(x))$
$f(x) \ll_q g(x)$	$f(x) = O_q(g(x))$
$f(x) \ll_{k,q} g(x)$	$f(x) = O_{k,q}(g(x))$

$f(x) \asymp g(x)$	$cf(x) \leqslant g(x) \leqslant Cf(x)$ for some positive absolute constants c and C
$f(x) \sim g(x)$	$\lim\limits_{x\to\infty} f(x)/g(x) = 1$

LIST OF ACRONYMS/ABBREVIATIONS

GRH	Generalized Riemann Hypothesis

1. INTRODUCTION

We summarize some well-known results about the Riemann zeta function and Dirichlet L-functions. These results can be found in [1–5].

1.1. The Riemann Zeta Function

The Riemann zeta function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\sigma > 1),$$

where $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. In 1860, Bernhard Riemann proved that the function $\zeta(s)$ can be analytically continued throughout the complex plane and the only pole of $\zeta(s)$ is at $s = 1$ with residue 1 and this pole is simple. Thus, the Laurent expansion of $\zeta(s)$ around $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \sum_{i=0}^{\infty} \gamma_i (s-1)^i,$$

and by differentiation, we have

$$\zeta^{(j)}(s) = \frac{(-1)^j j!}{(s-1)^{j+1}} + O_j(1), \quad (s \to 1). \tag{1.1}$$

Riemann also showed that the function $\zeta(s)$ satisfies the symmetric functional equation

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma\left(\frac{1}{2}(1-s)\right)\zeta(1-s),$$

which can be written as

$$\zeta(s) = \chi_\zeta(s)\zeta(1-s),$$

where

$$\chi_\zeta(s) := \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{\Gamma\left(\frac{1}{2}s\right)}$$
$$= 2^s \pi^{-1+s} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s).$$

It is well-known (see [10]) that

$$\chi_\zeta(s) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} \exp\left(-it \log \frac{|t|}{2\pi e} + \frac{i\pi}{4} \operatorname{sgn}(t)\right) \left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{1.2}$$

and that

$$\frac{\chi_\zeta'}{\chi_\zeta}(s) = -\log \frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right),$$

uniformly in $\alpha \le \sigma \le \beta$ and $|t| \ge 1$ for any fixed real numbers α and β. The famous Riemann Hypothesis asserts that all the zeros of the function $\zeta(s)$ in the critical strip, namely the strip $0 \le \sigma \le 1$, are on the line $\sigma = \frac{1}{2}$. Let $N_k(T)$ denote the number of zeros of the k^{th} derivative of $\zeta(s)$ inside the horizontal strip $0 < \gamma_{\zeta,k} < T$. It is known (see Chapter 10 in [1]) that

$$N_0(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T).$$

In addition, Berndt [6] showed that, for $k \ge 1$

$$N_k(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O_k(\log T).$$

1.2. Dirichlet L-Functions

A Dirichlet character ψ modulo $q \geqslant 1$ is a totally multiplicative function from $\mathbb{Z}$ to $\mathbb{C}$ with period q and is zero only at integers which are not relatively prime to q, and has absolute value 1 at integers relatively prime to q. By total multiplicativity we mean $\psi(mn) = \psi(m)\psi(n)$ and having period q means $\psi(n+q) = \psi(n)$ for any integers m, n. There exist $\varphi(q)$ distinct Dirichlet characters modulo q where φ is the Euler totient function. The principal Dirichlet character ψ_0 is defined by $\psi_0(n) = 1$ if $(n, q) = 1$ and zero otherwise.

A Dirichlet character modulo q may have a period smaller than q when we restrict it to integers relatively prime to q. In this case, we say ψ is imprimitive, otherwise we say ψ is primitive. If a Dirichlet character ψ modulo q is primitive, then we have $q \geqslant 3$ because for $q = 2$ the only Dirichlet character is the principal character. We would like to note that, if q is an odd prime number, then all Dirichlet characters modulo q are primitive.

Dirichlet characters modulo a given natural number q satisfy the orthogonality relations

$$\sum_{n \pmod{q}} \psi(n) = \begin{cases} \varphi(q) & \text{if } \psi = \psi_0, \\ 0 & \text{if } \psi \neq \psi_0, \end{cases}$$

and

$$\sum_{\psi \pmod{q}} \psi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv 1 \pmod{q}, \\ 0 & \text{if } n \not\equiv 1 \pmod{q}. \end{cases}$$

Moreover, for an integer a with $(a, q) = 1$, we have

$$\sum_{\psi \pmod{q}} \overline{\psi}(a)\psi(n) = \begin{cases} \varphi(q) & \text{if } n \equiv a \pmod{q}, \\ 0 & \text{if } n \not\equiv a \pmod{q}. \end{cases} \tag{1.3}$$

For a Dirichlet character ψ modulo q, we define the Gaussian sum $\tau(\psi)$ as

$$\tau(\psi) := \sum_{n=1}^{q} \psi(n) e\left(\frac{n}{q}\right),$$

where $e(\alpha) := e^{2\pi i\alpha}$. Note that

$$\overline{\tau(\psi)} = \psi(-1)\tau(\overline{\psi}).$$

Let ψ be a Dirichlet character modulo $q \geqslant 1$. Then a Dirichlet L-function is defined as

$$L(s,\psi) := \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s}, \quad (\sigma > 1).$$

The functional equation of a Dirichlet L-function associated with a primitive Dirichlet character ψ can be written as

$$L(s,\psi) = \chi_\psi(s)L(1-s,\overline{\psi}), \tag{1.4}$$

where

$$\chi_\psi(s) := \frac{\tau(\psi)}{i^{\mathfrak{a}}q^{\frac{1}{2}}}\pi^{-\frac{1}{2}+s}q^{\frac{1}{2}-s}\frac{\Gamma(\frac{1}{2}(\mathfrak{a}+1-s))}{\Gamma(\frac{1}{2}(\mathfrak{a}+s))},$$

and

$$\mathfrak{a} := \begin{cases} 0, & \text{if } \psi(-1) = 1 \\ 1, & \text{if } \psi(-1) = -1. \end{cases} \tag{1.5}$$

It is known (analogous to (1.2), see Lemma 2.1 in [5]) that for a primitive Dirichlet character ψ modulo $q \geq 3$,

$$\chi_\psi(s) = \frac{\tau(\psi)e^{\frac{i\pi}{4}}}{q^{\frac{1}{2}}}\left(\frac{2\pi}{qt}\right)^{\sigma-\frac{1}{2}} \exp\left(-it\log\frac{qt}{2\pi e}\right)\left(1+O\left(\frac{1}{t}\right)\right), \tag{1.6}$$

and

$$\chi_\psi(1-s) = \frac{\tau(\psi)e^{-\frac{i\pi}{4}}}{q^{\frac{1}{2}}}\left(\frac{2\pi}{qt}\right)^{\frac{1}{2}-\sigma} \exp\left(it\log\frac{qt}{2\pi e}\right)\left(1+O\left(\frac{1}{t}\right)\right), \tag{1.7}$$

in any fixed half plane $\alpha \leqslant \sigma \leqslant \beta$, $t \geq \delta > 0$ where δ is an arbitrarily small positive number. Moreover, (see Lemma 2.2 in [5]) for any fixed σ, and $|t| > 1$,

$$\frac{\chi'_\psi}{\chi_\psi}(s) = -\log \frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right), \tag{1.8}$$

From the functional equation, it can be inferred that for $\sigma < 0$, $L(s,\psi)$ has zeros at the poles of $\Gamma(\frac{1}{2}s + \mathfrak{a})$, that is when $\frac{1}{2}(s + \mathfrak{a})$ is a negative integer. These are be called the *trivial zeros* of $L(s,\psi)$. The trivial zeros of $L(s,\psi)$ are the negative even integers if $\psi(-1) = 1$, and are the negative odd integers if $\psi(-1) = -1$. The Generalized Riemann Hypothesis (GRH) asserts that all the zeros of the function $L(s,\psi)$ in the strip $0 \leq \sigma \leq 1$ are on the line $\sigma = \frac{1}{2}$. For a primitive Dirichlet character ψ modulo $q \geq 3$, it is known (see [2]) that the number of zeros of $L(s,\psi)$ in the rectangle $0 < \sigma < 1$, $|t| < T$ is equal to $\frac{T}{\pi}\log\frac{qT}{2\pi e} + O(\log qT)$. In [7], Yıldırım proved that for $k \geq 1$ and for any $\delta > 0$, $\exists M = M(k,\delta,\mathfrak{a})$ such that there is no zero of $L^{(k)}(s,\psi)$ in the region $|s| > q^M$, $\sigma < -\delta$, $|t| > \delta$. Moreover, there exists sufficiently large σ_k such that $L^{(k)}(s,\psi)$ has no zero in $\sigma > \sigma_k$. Now, for $k \geq 1$ let $N_k(T,\psi)$ denote the number of zeros of $L^{(k)}(s,\psi)$ in $-q^K < \sigma < \sigma_k$, $|t| < T$. It is shown in [7] that

$$N_k(T,\psi) = \frac{T}{\pi}\log\frac{qT}{2\pi em} + O\left(q^K \log T\right), \quad (T \to \infty), \tag{1.9}$$

where m denotes the smallest prime number that does not divide q.

1.3. Motivation of the Present Work and Main Results

In [8], Conrey and Ghosh studied the sum

$$\sum_{0<\gamma_{\zeta,k}<T} \chi_\zeta(\rho_{\zeta,k}), \tag{1.10}$$

and this sum is calculated in detail by Karabulut and Yıldırım in [9]. The result of the latter work is the following:

$$\sum_{0<\gamma_{\zeta,k}<T} \chi_\zeta(\rho_{\zeta,k}) = \mathcal{A}_k \frac{T}{2\pi} + O_k\left(\frac{T}{\log T}\right), \quad (T \to \infty),$$

where

$$\mathcal{A}_k := -\sum_{u=0}^{\infty}(-1)^u \sum_{v=1}^{k} \binom{k}{v} \sum_{\substack{i_1+\dots+i_k=u \\ i_1,\dots,i_k \geqslant 0}} \binom{u}{i_1,\dots i_k}$$
$$\times \prod_{w=1}^{k} (-1)^w w! \binom{k}{w}^{i_w} \frac{(-1)^v (v+1)!}{(i_1+2i_2+\dots+ki_k+v)!}.$$

In [9], it is also shown that

$$\mathcal{A}_k = -\left(k+1-\sum_{r=1}^{k} e^{-z_r}\right),$$

where z_r, $r = 1, ..., k$ are the zeros of $P_k(z) := \sum_{j=0}^{k} \frac{z^j}{j!}$, and $\mathcal{A}_0 = -1$. In this work, we calculate the following generalizations of the sum in (1.10):

$$\sum_{0<\gamma_{\psi_1,k}\leqslant T} \chi_{\psi_2}(\rho_{\psi_1,k}) \qquad \text{and} \qquad \sum_{0<\gamma_{\psi,k}\leqslant T} \chi_\zeta(\rho_{\psi,k}).$$

Using the method in [9], we obtained the following theorems.

Theorem 1.1. *Assume GRH. Let q be a fixed odd prime number and ψ_1, ψ_2 be any non-principal Dirichlet characters modulo q. Then, as $T \to \infty$, we have*

$$\sum_{0<\gamma_{\psi_1,k}\leqslant T} \chi_{\psi_2}(\rho_{\psi_1,k}) = \frac{\overline{\tau(\psi_2)}\tau(\psi_1)}{\varphi(q)} \mathcal{A}_k \frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right).$$

Theorem 1.2. *Assume GRH. Let q be a fixed odd prime number and ψ be any non-*

principal Dirichlet character modulo q. Then, as $T \to \infty$*, we have*

$$\sum_{0<\gamma_{\psi,k}\leqslant T} \chi_\zeta(\rho_{\psi,k}) \ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} \log T \log\log T.$$

We note that the assumption of GRH in both theorems can be removed at the cost of some more detailed analysis.

2. LEMMATA

Lemma 2.1. *For a positive integer m, and for $|t| \geqslant 1$, we have*

$$\frac{\chi_\psi^{(m)}}{\chi_\psi}(s) = \left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^m. \tag{2.1}$$

Proof. The case for $m = 1$ is mentioned in (1.8) and its proof can be found in Lemma 2.2 in [5]. Using the identity

$$\frac{\chi_\psi^{(m+1)}}{\chi_\psi}(s) = \frac{d}{ds}\left(\frac{\chi_\psi^{(m)}}{\chi_\psi}(s)\right) + \frac{\chi_\psi^{(m)}}{\chi_\psi}(s)\frac{\chi_\psi'}{\chi_\psi}(s),$$

and the induction hypothesis, we have

$$\begin{aligned}\frac{\chi_\psi^{(m+1)}}{\chi_\psi}(s) =& m\left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m-1} O\left(\frac{1}{|t|}\right)\\ &+\left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m}\left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)\\ =&\left(-\log\frac{q|t|}{2\pi} + O\left(\frac{1}{|t|}\right)\right)^{m+1}.\end{aligned}$$

Thus, the result follows by induction. □

Lemma 2.2. *(Vinogradov-Korobov, 1958) There exists a positive absolute constant δ_1 such that $\zeta(s) \neq 0$ throughout the region*

$$\sigma \geqslant 1 - \frac{\delta_1}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}}, \quad |t| \geqslant 3,$$

in which one has

$$\frac{\zeta'}{\zeta}(s) \ll (\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}.$$

Proof. See pp 119-120 of [11]. □

The following lemma is an exercise in [13] on page 444.

Lemma 2.3. *Suppose that $L(s,\psi) \neq 0$ for $\sigma > 1/2$, that is, assume GRH. Then*

$$\frac{L'}{L}(s,\psi) \ll ((\log q\tau)^{2-2\sigma}+1)\min\left(\frac{1}{|\sigma-1|}, \log\log q\tau\right),$$

uniformly for $\frac{1}{2}+\frac{1}{\log\log q\tau} \leq \sigma \leq \frac{3}{2}$ where $\tau := |t|+4$.

Proof. For $x \geq 2$ and $y \geq 2$, define

$$w(u) := \begin{cases} 1 & \text{if } 1 \leqslant u \leqslant x, \\ 1-\frac{\log\frac{u}{x}}{\log y} & \text{if } x \leqslant u \leqslant xy, \\ 0 & \text{if } u \geqslant xy. \end{cases}$$

Now, consider the sum,

$$\begin{aligned} \sum_{n\leqslant xy}\frac{w(n)\Lambda(n)\psi(n)}{n^s} &= \sum_{n\leqslant x}\frac{\Lambda(n)\psi(n)}{n^s} + \sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s} - \sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\frac{\log\frac{n}{x}}{\log y} \\ &= \sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s} + \frac{1}{\log y}\sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n}. \end{aligned}$$

Since the coefficients of $\frac{L'}{L}(s,\psi)$ are given by $\Lambda(n)\psi(n)$, we have, for $\sigma_0 > 1$,

$$\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s} = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} -\frac{L'}{L}(s+w,\psi)\frac{(xy)^w}{w}\,dw,$$

and

$$\frac{1}{\log y}\sum_{x<n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n} = \frac{1}{\log y}\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n} - \frac{1}{\log y}\sum_{n\leqslant x}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n}$$

$$
\begin{aligned}
=&\frac{1}{\log y}\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{xy}{n}-\sum_{n\leqslant xy}\frac{\Lambda(n)\psi(n)}{n^s}\\
&-\frac{1}{\log y}\sum_{n\leqslant x}\frac{\Lambda(n)\psi(n)}{n^s}\log\frac{x}{n}\\
=&\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}-\frac{L'}{L}(s+w,\psi)\frac{(xy)^w}{w^2}\,dw\\
&-\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}-\frac{L'}{L}(s+w,\psi)\frac{(xy)^w}{w}\,dw\\
&-\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}-\frac{L'}{L}(s+w,\psi)\frac{(x)^w}{w^2}\,dw.
\end{aligned}
$$

Combining these,

$$
\sum_{n\leqslant xy}\frac{w(n)\Lambda(n)\psi(n)}{n^s}=-\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{L'}{L}(s+w,\psi)\frac{(xy)^w-x^w}{w^2}\,dw.
$$

Now, using Cauchy's residue theorem,

$$
\begin{aligned}
-\frac{1}{2\pi i\log y}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{L'}{L}(s+w,\psi)\frac{(xy)^w-x^w}{w^2}\,dw=&-\frac{L'}{L}(s,\psi)-\sum_{\rho_\psi}\frac{(xy)^{\rho_\psi-s}-x^{\rho_\psi-s}}{(\rho_\psi-s)^2\log y}\\
&-(1-\mathfrak{a})\sum_{k=1}^{\infty}\frac{(xy)^{-2k-s}-x^{-2k-s}}{(2k+s)^2\log y}\\
&-\mathfrak{a}\sum_{k=1}^{\infty}\frac{(xy)^{-2k+1-s}-x^{-2k+1-s}}{(2k-1+s)^2\log y}.
\end{aligned}
$$

If $\sigma\geq 1/2$ then $|y^{\rho_\psi-s}-1|\leqslant 2$. Thus, for $\sigma>1/2$ we have

$$
\left|\sum_{\rho_\psi}\frac{(xy)^{\rho_\psi-s}-x^{\rho_\psi-s}}{(\rho_\psi-s)^2\log y}\right|\leqslant\frac{2x^{1/2-\sigma}}{\log y}\sum_{\rho_\psi}\frac{1}{|s-\rho_\psi|^2}.
$$

We know that (see Section 10.2 in [13])

$$
\frac{L'}{L}(s,\psi)=B(\psi)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\mathfrak{a}}{2}\right)-\frac{1}{2}\log\frac{q}{\pi}+\sum_{\rho_\psi}\left(\frac{1}{s-\rho_\psi}+\frac{1}{\rho_\psi}\right),
$$

and

$$\Re B(\psi) = -\sum_{\rho_\psi} \Re \frac{1}{\rho_\psi}.$$

Then,

$$\begin{aligned}
(\sigma - \frac{1}{2}) \sum_{\rho_\psi} \frac{1}{|s-\rho_\psi|^2} &= (\sigma - \frac{1}{2}) \sum_{\rho_\psi} \frac{1}{(\sigma - \frac{1}{2})^2 + (t - \gamma_\psi)^2} \\
&= \Re \left(\sum_{\rho_\psi} \frac{1}{s - \rho_\psi} \right) \\
&= \Re \left(\frac{L'}{L}(s,\psi) + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s+\mathfrak{a}}{2} \right) + \frac{1}{2} \log \frac{q}{\pi} \right) \\
&= \Re \frac{L'}{L}(s,\psi) + \frac{1}{2} \log \tau + \frac{1}{2} \log \frac{q}{\pi} + O(1).
\end{aligned}$$

Thus, for a complex number θ with $|\theta| \leqslant 1$, we have

$$\begin{aligned}
\frac{L'}{L}(s,\psi) = -\sum_{n \leqslant xy} \frac{w(n)\Lambda(n)\psi(n)}{n^s} + \frac{\theta 2 x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log y} \left| \Re \frac{L'}{L}(s,\psi) \right| \\
+ O\left(\frac{x^{\frac{1}{2}-\sigma} \log q\tau}{(\sigma - \frac{1}{2}) \log y} \right). \qquad (2.2)
\end{aligned}$$

Thus if

$$\frac{2x^{\frac{1}{2}-\sigma}}{(\sigma - \frac{1}{2}) \log y} \leqslant c < 1, \qquad (2.3)$$

then we have

$$\frac{L'}{L}(s,\psi) \ll \left| \sum_{n \leqslant xy} \frac{w(n)\Lambda(n)\psi(n)}{n^s} \right| + \frac{x^{\frac{1}{2}-\sigma} \log q\tau}{(\sigma - \frac{1}{2}) \log y}.$$

We take

$$y = \exp\left(\frac{1}{\sigma - \frac{1}{2}} \right), \quad x = \frac{(\log q\tau)^2}{y}.$$

Note that $\sigma \leq 3/2$ implies $y \geq 2$. Now we have ,

$$\frac{2x^{\frac{1}{2}-\sigma}}{(\sigma-\frac{1}{2})\log y} = 2e(\log q\tau)^{1-2\sigma}.$$

Thus for $\sigma \geqslant \frac{1}{2} + \frac{1}{\log\log q\tau}$ and $c = \frac{2}{e}$, inequality (2.3) holds. We observe that,

$$\sum_{n \leqslant xy} \frac{w(n)\Lambda(n)\psi(n)}{n^s} \ll \sum_{n \leqslant (\log q\tau)^2} \frac{\Lambda(n)}{n^{\frac{1}{2}}} \ll \log q\tau,$$

uniformly for $\sigma \geqslant \frac{1}{2}$. Thus we obtain

$$\frac{L'}{L}(s,\psi) \ll \log q\tau, \tag{2.4}$$

uniformly for $1/2 + 1/\log\log q\tau \leq \sigma \leq 3/2$. By using (2.4) in (2.2), we get

$$\left|\frac{L'}{L}(s,\psi)\right| \leq \sum_{n \leqslant (\log q\tau)^2} \frac{\Lambda(n)}{n^\sigma} + O((\log q\tau)^{2-2\sigma}), \tag{2.5}$$

uniformly for $1/2 + 1/\log\log q\tau \leq \sigma \leq 3/2$. Now we write

$$\sum_{n \leqslant (\log q\tau)^2} \frac{\Lambda(n)}{n^\sigma} = \sum_{0 \leq k \leq 2\log\log q\tau} \sum_{e^k \leq n < e^{k+1}} \frac{\Lambda(n)}{n^\sigma}. \tag{2.6}$$

By Chebyshev's estimate on the von Mangoldt function which states that $\sum_{n \leq z} \Lambda(n) \asymp z$, we get

$$\sum_{e^k \leq n < e^{k+1}} \frac{\Lambda(n)}{n^\sigma} \ll e^{k(1-\sigma)}, \tag{2.7}$$

and

$$\sum_{0 \leq k \leq 2\log\log q\tau} e^{k(1-\sigma)} \ll (\log q\tau)^{2-2\sigma},$$

if σ is away from 1 in the sense that $\frac{1}{|\sigma-1|} < \log\log q\tau$. If, on the other hand, σ is close to 1 such that $\frac{1}{|\sigma-1|} > \log\log q\tau$, then on the right hand side of (2.6) we sum at most $2\log\log q\tau$ terms which are $\ll 1$ by (2.7). Combining these results, we obtain

$$\frac{L'}{L}(s,\psi) \ll ((\log q\tau)^{2-2\sigma}+1)\min\left(\frac{1}{|\sigma-1|}, \log\log q\tau\right),$$

uniformly for $1/2 + 1/\log\log q\tau \leq \sigma \leq 3/2$. $\square$

The following lemma is Theorem 2 in [7].

Lemma 2.4. *Let $k \in \mathbb{N}$ and ψ be a primitive Dirichlet character modulo q, $q \geqslant 3$, and m be the smallest prime number that does not divide q. Then $|L^{(k)}(s,\psi)| > \frac{(\log m)^k}{2m^\sigma}$ for $\sigma > 1 + m\left(1+\sqrt{1+\frac{2k^2}{m\log m}}\right)$.*

Proof. Since m is the smallest prime number that does not divide q, m is in fact the smallest natural number $n > 1$ such that $\psi(n) \neq 0$. Thus we have

$$L^{(k)}(s,\psi) = (-1)^k \sum_{n=m}^{\infty} \frac{\psi(n)(\log n)^k}{n^s}, \quad (\sigma > 0),$$

and

$$\begin{aligned} |L^{(k)}(s,\psi)| &\geq \frac{(\log m)^k}{m^\sigma} - \sum_{n=m+1}^{\infty} \frac{(\log n)^k}{n^\sigma}, \quad (\sigma > 1), \\ &> \frac{(\log m)^k}{m^\sigma} - \int_m^\infty \frac{(\log x)^k}{x^\sigma}\,dx, \quad (\sigma > \frac{k}{\log m}), \\ &= \frac{(\log m)^k}{m^\sigma} - \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{((\sigma-1)\log m)^j}{j!}. \end{aligned}$$

Now we put $z := (\sigma-1)\log m$. If $\frac{(\log m)^k}{2m^\sigma} - \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}}\sum_{j=0}^{k}\frac{z^j}{j!} \geq 0$, then we obtain the desired result.

Note that

$$\frac{(\log m)^k}{2m^\sigma} - \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}}\sum_{j=0}^{k}\frac{z^j}{j!} = \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}}\left(\frac{(\log m)^k(\sigma-1)^{k+1}}{k!2m} - \sum_{j=0}^{k}\frac{z^j}{j!}\right)$$
$$= \frac{m^{1-\sigma}k!}{(\sigma-1)^{k+1}}\left(\frac{z^{k+1}}{k!2m\log m} - \sum_{j=0}^{k}\frac{z^j}{j!}\right),$$

and by [12] (Problem 16, Part III), there is only one $z \in \mathbb{R}^+$ such that $\frac{z^{k+1}}{k!.2m\log m} - \sum_{j=0}^{k}\frac{z^j}{j!} = 0$. We can assume $\sigma > 1 + \frac{k}{\log m}$, so that $z > k$. Then

$$\sum_{j=0}^{k}\frac{z^j}{j!} \le \frac{z^k}{k!} + \frac{kz^{k-1}}{(k-1)!} \le \frac{z^{k+1}}{k!2m\log m}, \tag{2.8}$$

the last inequality being true for $z^k 2m\log m + z^{k-1}k^2 2m\log m \le z^{k+1}$ which is equivalent to $0 \le z^2 - (2m\log m)z - 2k^2 m\log m$. Here the discriminant of the quadratic expression $z^2 - (2m\log m)z - 2k^2m\log m$ in z is $\Delta = 4m^2\log^2 m + 8k^2 m\log m$. Thus, for $z \ge m\log m + \sqrt{m^2\log^2 m + 2k^2 m\log m} = m\log m\left(1 + \sqrt{1 + \frac{2k^2}{m\log m}}\right)$ or in terms of σ, for $\sigma > 1 + m\left(1 + \sqrt{1 + \frac{2k^2}{m\log m}}\right)$ the result of the lemma holds. $\square$

The following lemma is Lemma 2 in [10] with r replaced by r/q.

Lemma 2.5. *Let $a > 1$, $q \in \mathbb{N}$ and A be large. We have, for $A \leqslant \frac{r}{q} \leqslant B \leqslant 2A$,*

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}} dt$$
$$= q^{-\frac{1}{2}}(2\pi)^{1-a} r^a e^{\frac{-ir}{q} + \frac{i\pi}{4}} + q^{a-\frac{1}{2}} E\left(\frac{r}{q}, A, B\right),$$

while for $\frac{r}{q} < A$ or $\frac{r}{q} > B$,

$$\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}} dt = q^{a-\frac{1}{2}} E\left(\frac{r}{q}, A, B\right).$$

Here

$$E\left(\frac{r}{q}, A, B\right) := O\left(A^{a-\frac{1}{2}}\right) + O\left(\frac{A^{a+\frac{1}{2}}}{|A-\frac{r}{q}| + A^{\frac{1}{2}}}\right) + O\left(\frac{B^{a+\frac{1}{2}}}{|B-\frac{r}{q}| + B^{\frac{1}{2}}}\right). \tag{2.9}$$

The following lemma is a slight modification of Lemma 2.1 in [9].

Lemma 2.6. *Let A be large and $m \in \mathbb{Z}$ with $|m| = o(\log A)$ and $a > 1$. We have, for $A \leqslant \frac{r}{q} \leqslant B \leqslant 2A$,*

$$\begin{aligned}&\int_A^B \exp\left(it \log \frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{qt}{2\pi}\right)^m dt \\ &\qquad = q^{-\frac{1}{2}}(2\pi)^{1-a} r^a e^{-ir/q + i\pi/4}\left(\log \frac{r}{2\pi}\right)^m + q^{a-\frac{1}{2}} E(r/q, A, B)(\log qA)^m,\end{aligned}$$

while for $\frac{r}{q} < A$ or $\frac{r}{q} > B$,

$$\int_A^B \exp\left(it \log \frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{qt}{2\pi}\right)^m dt = q^{a-\frac{1}{2}} K_0^{|m|} E(r/q, A, B)(\log qA)^m,$$

the constants implied in the O-terms do not depend on m, K_0 can be taken to be any fixed number > 1 when m is negative and to be 1 when m is non-negative, $E(r/q, A, B)$ is as given by (2.9).

Proof. The case when m is a positive integer is covered in Lemma 2.12 in [5]. Thus, we will consider the case m is negative. For $m = -1$, we have

$$\begin{aligned}\int_A^B \exp\left(it \log \frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log \frac{qt}{2\pi}\right)^{-1} dt &= \frac{1}{\log\left(\frac{r}{2\pi}\right)}\int_A^B \exp\left(it \log \frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}} dt \\ - \frac{1}{\log\left(\frac{r}{2\pi}\right)}\int_A^B \exp\left(it \log \frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}} \log \frac{qt}{r}\left(\log \frac{qt}{2\pi}\right)^{-1} dt,\end{aligned} \tag{2.10}$$

since $\left(\log \frac{qt}{2\pi}\right)^{-1} = \left(\log \frac{r}{2\pi}\right)^{-1}\left(1 - \frac{\log \frac{qt}{r}}{\log \frac{qt}{2\pi}}\right)$. The first integral on the right hand side of (2.10) can be calculated by Lemma 2.5. We apply integration by parts to the second integral on the right hand side of (2.10). Note that $\frac{d}{dt}\left(\exp\left(it\log\frac{qt}{re}\right)\right) = i\left(\exp\left(it\log\frac{qt}{re}\right)\right)\log\frac{qt}{r}$. Thus, we have

$$\begin{aligned}
&-\frac{1}{\log\left(\frac{r}{2\pi}\right)}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\log\frac{qt}{r}\left(\log\frac{qt}{2\pi}\right)^{-1}dt\\
&\qquad=-\frac{1}{\log\left(\frac{r}{2\pi}\right)}\left[-i\exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-1}\right]_A^B\\
&\qquad\quad+\frac{1}{\log\left(\frac{r}{2\pi}\right)}\int_A^B -i\exp\left(it\log\frac{qt}{re}\right)\frac{t^{a-\frac{3}{2}}}{\log\left(\frac{qt}{2\pi}\right)(2\pi)^{a-1/2}}\left(a-\frac{1}{2}-\frac{1}{\log\left(\frac{qt}{2\pi}\right)}\right)dt\\
&\qquad\ll\left|\frac{1}{\log\left(\frac{r}{2\pi}\right)}\right|A^{a-\frac{1}{2}}\left(\log qA\right)^{-1}.
\end{aligned}$$

This proves the statement for $m=-1$. In general, for a natural number m, let $I_m := \int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m}dt$, then we have

$$\begin{aligned}
&\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m}dt\\
&=\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m+1}\left(\log\frac{qt}{2\pi}\right)^{-1}dt\\
&=\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m+1}\left(\log\frac{r}{2\pi}\right)^{-1}\left(1-\frac{\log\frac{qt}{r}}{\log\frac{qt}{2\pi}}\right)dt\\
&=\left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m+1}\\
&\quad-\left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\log\frac{qt}{r}\left(\log\frac{qt}{2\pi}\right)^{-m}dt,\\
&=\left(\log\frac{r}{2\pi}\right)^{-1}I_{m-1}\\
&\quad-\left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\log\frac{qt}{r}\left(\log\frac{qt}{2\pi}\right)^{-m}dt. \qquad (2.11)
\end{aligned}$$

Using $\frac{d}{dt}\left(\exp\left(it\log\frac{qt}{re}\right)\right) = i\left(\exp\left(it\log\frac{qt}{re}\right)\right)\log\frac{qt}{r}$ and integration by parts, we see that the last integral in (2.11) is

$$
\begin{aligned}
=&i\left(\log\frac{r}{2\pi}\right)^{-1}\left[\exp\left(it\log\frac{qt}{re}\right)\left(\frac{qt}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^{-m}\right]_A^B \\
&-i\left(\log\frac{r}{2\pi}\right)^{-1}\int_A^B \exp\left(it\log\frac{qt}{re}\right)\frac{t^{a-3/2}}{(2\pi)^{a-1/2}}\left(\log\frac{qt}{2\pi}\right)^{-m}\left[a-\frac{1}{2}+m\left(\log\frac{qt}{2\pi}\right)^{-1}\right] \\
&\ll\left|\left(\log\frac{r}{2\pi}\right)^{-1}\right|A^{a-\frac{1}{2}}\left(\log qA\right)^{-m}. \qquad (2.12)
\end{aligned}
$$

Hence, using (2.11), (2.12) and induction, the result follows. □

Lemma 2.7. *(Lemma 2.13 in [5]) Let A be large and $A < B < 2A$, $E(r, A, B)$ be defined as in (2.9). Assume that $(b_n)_{n\geqslant 1}$ is a sequence of complex numbers such that $b_n \ll n^{\epsilon}$ for any $\epsilon > 0$. Then if $a > 1$,*

$$\sum_{n=1}^{\infty}\frac{b_n}{n^a}E(2\pi n/q, A, B) \ll_q A^{a-\frac{1}{2}}.$$

Lemma 2.8. *Let T be large, $a > 1$ be fixed, ψ be a primitive Dirichlet character modulo $q \geqslant 3$ and $(b_n)_{n\geqslant 1}$ be a sequence of complex numbers such that $b_n \ll n^{\epsilon}$ for any $\epsilon > 0$. For $m \in \mathbb{Z}$ with $|m| = o(\log T)$ as $T \to \infty$, we have*

$$
\begin{aligned}
\frac{1}{2\pi}\int_{T/2}^{T}\chi_\psi(1-(a+it))\left(\log\frac{qt}{2\pi}\right)^m\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\,dt =&\frac{\tau(\psi)}{q}\sum_{\frac{qT}{4\pi}<n\leqslant\frac{qT}{2\pi}} b_n\left(\log n\right)^m e^{-\frac{2\pi i n}{q}} \\
&+O_q(K_0^{|m|}T^{a-\frac{1}{2}}(\log T)^m).
\end{aligned}
$$

Proof. By (1.7),

$$\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\chi_\psi(1-(a+it))\left(\log\frac{qt}{2\pi}\right)^m dt$$

$$= \frac{1}{2\pi} \frac{\tau(\psi) e^{-\frac{i\pi}{4}}}{q^{\frac{1}{2}}} \int_{T/2}^{T} \left(\sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}} \right) \left(\frac{qt}{2\pi} \right)^{a-\frac{1}{2}} \exp\left(it \log \frac{qt}{2\pi e} \right) \left(\log \frac{qt}{2\pi} \right)^m dt$$

$$+ O_q \left(\int_{T/2}^{T} \left| \sum_{n=1}^{\infty} \frac{b_n}{n^a} \right| t^{a-\frac{3}{2}} (\log t)^m \, dt \right). \tag{2.13}$$

Since $b_n \ll n^\epsilon$ and $a > 1$, we have $\sum_{n=1}^{\infty} \frac{|b_n|}{n^a} \ll 1$. So the error term in (2.13) is $\ll T^{a-\frac{1}{2}} (\log T)^m$. The main term in (2.13) is

$$= \frac{1}{2\pi} \frac{\tau(\psi) e^{-\frac{i\pi}{4}}}{q^{\frac{1}{2}}} \left(\sum_{n=1}^{\infty} \frac{b_n}{n^a} \int_{T/2}^{T} \left(\frac{qt}{2\pi} \right)^{a-\frac{1}{2}} \left(\log \frac{qt}{2\pi} \right)^m \exp\left(it \log \frac{qt}{2\pi n e} \right) dt \right),$$

changing the order of summation and integration is justified by the absolute convergence of the series $\sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}}$. By taking $A = T/2$, $B = T$ and $r = 2\pi n$ in Lemma 2.6, we get

$$\begin{aligned}
&\frac{1}{2\pi} \frac{\tau(\psi) e^{-\frac{i\pi}{4}}}{q^{\frac{1}{2}}} \left(\sum_{n=1}^{\infty} \frac{b_n}{n^a} \int_{T/2}^{T} \left(\frac{qt}{2\pi} \right)^{a-\frac{1}{2}} \left(\log \frac{qt}{2\pi} \right)^m \exp\left(it \log \frac{qt}{2\pi n e} \right) dt \right) \\
&= \frac{1}{2\pi} \frac{\tau(\psi) e^{-\frac{i\pi}{4}}}{q^{\frac{1}{2}}} \left(\sum_{\frac{qT}{4\pi} < n \leqslant \frac{qT}{2\pi}} \frac{b_n}{n^a} q^{-\frac{1}{2}} (2\pi)^{1-a} (2\pi n)^a e^{-\frac{2\pi i n}{q} + \frac{\pi i}{4}} \left(\log \frac{2\pi n}{2\pi} \right)^m \right) \\
&\quad + \frac{1}{2\pi} \frac{\tau(\psi) e^{-\frac{i\pi}{4}}}{q^{\frac{1}{2}}} q^{a-\frac{1}{2}} K_0^{|m|} \left(\log \frac{qT}{2} \right)^m \sum_{n=1}^{\infty} \frac{b_n}{n^a} E\left(\frac{2\pi n}{q}, \frac{T}{2}, T \right) \\
&= \frac{\tau(\psi)}{q} \sum_{\frac{qT}{4\pi} < n \leqslant \frac{qT}{2\pi}} b_n (\log n)^m e^{-\frac{2\pi i n}{q}} + O_q\left(K_0^{|m|} T^{a-\frac{1}{2}} (\log T)^m \right),
\end{aligned}$$

where the last equation follows from Lemma 2.7. Hence, the desired result follows. □

Lemma 2.9. *Let T be large, $a > 1$, $q \geq 3$ be fixed, $(b_n)_{n \geqslant 1}$ be a sequence of complex numbers such that $b_n \ll n^\epsilon$ for any $\epsilon > 0$. For $m \in \mathbb{Z}$ with $|m| = o(\log T)$ as $T \to \infty$, we have*

$$\frac{1}{2\pi} \int_{T/2}^{T} \chi_\zeta(1 - (a + it)) \left(\log \frac{qt}{2\pi} \right)^m \sum_{n=1}^{\infty} \frac{b_n}{n^{a+it}} \, dt$$

$$= \sum_{\frac{T}{4\pi}<n\leqslant\frac{T}{2\pi}} b_n (\log qn)^m + O_q\left(K_0^{|m|} T^{a-\frac{1}{2}} (\log T)^m\right).$$

Proof. By (1.2), we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\chi_\zeta(1-(a+it))\left(\log\frac{qt}{2\pi}\right)^m dt\\
=&\frac{e^{-\frac{i\pi}{4}}}{2\pi}\int_{T/2}^{T}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}\right)\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\exp\left(it\log\frac{t}{2\pi e}\right)\left(\log\frac{qt}{2\pi}\right)^m dt\\
&+O_q\left(\int_{T/2}^{T}\left|\sum_{n=1}^{\infty}\frac{b_n}{n^a}\right| t^{a-\frac{3}{2}}(\log t)^m\,dt\right). \qquad (2.14)
\end{aligned}$$

Since $b_n \ll n^\epsilon$ and $a > 1$, we have $\sum_{n=1}^{\infty}\frac{|b_n|}{n^a} \ll 1$. So the error term in (2.14) is $\ll_q T^{a-\frac{1}{2}}(\log T)^m$. The main term in (2.14) is

$$= \frac{e^{-\frac{i\pi}{4}}}{2\pi}\left(\sum_{n=1}^{\infty}\frac{b_n}{n^a}\int_{T/2}^{T}\left(\frac{t}{2\pi}\right)^{a-\frac{1}{2}}\left(\log\frac{qt}{2\pi}\right)^m\exp\left(it\log\frac{t}{2\pi ne}\right)dt\right), \qquad (2.15)$$

changing the order of summation and integration is justified by absolute convergence of $\sum_{n=1}^{\infty}\frac{b_n}{n^{a+it}}$. By taking $A = T/2, B = T$ and $r = 2\pi nq$ in Lemma 2.6, the expression in (2.15) becomes

$$\begin{aligned}
&= \frac{e^{-\frac{i\pi}{4}}}{2\pi}\left(\sum_{\frac{T}{4\pi}<n\leqslant\frac{T}{2\pi}}\frac{b_n}{n^a}(2\pi)^{1-a}(2\pi n)^a e^{-2\pi in+\frac{\pi i}{4}}\left(\log\frac{q2\pi n}{2\pi}\right)^m\right)\\
&\quad+\frac{e^{-\frac{i\pi}{4}}}{2\pi}K_0^{|m|}\left(\log\frac{qT}{2}\right)^m\sum_{n=1}^{\infty}\frac{b_n}{n^a}E\left(2\pi n,\frac{T}{2},T\right)\\
&= \sum_{\frac{T}{4\pi}<n\leqslant\frac{T}{2\pi}} b_n(\log qn)^m + O_q\left(K_0^{|m|}T^{a-\frac{1}{2}}(\log T)^m\right),
\end{aligned}$$

the last equation follows from Lemma 2.7. ☐

Lemma 2.10. *For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, ψ a primitive Dirichlet character* *modulo* $q \geq 1$ *and* $\sigma > 1$, *define*

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v)}{n^s} := \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w}.$$

Then, for $\sigma > 1$

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v)\psi(n)}{n^s} = \frac{L^{(v+1)}}{L}(s, \psi) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s, \psi) \right)^{i_w} =: \sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v; \psi)}{n^s}. \tag{2.16}$$

Proof. For a natural number r, we have

$$\begin{aligned} \frac{\zeta^{(r)}}{\zeta}(s) = \zeta^{(r)}(s)\frac{1}{\zeta(s)} &= \left((-1)^r \sum_{n=1}^{\infty} \frac{(\log n)^r}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \right) \\ &= (-1)^r \sum_{n=1}^{\infty} \frac{\Lambda_r(n)}{n^s}, \quad (\sigma > 1), \end{aligned} \tag{2.17}$$

where

$$\Lambda_r(n) := (\mu * \log^r)(n) = \sum_{d|n} \mu(d)(\log(n/d))^r.$$

So,

$$\frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s) \right)^{i_w} = (-1)^{K+1} \sum_{n=1}^{\infty} \frac{\Lambda_{v+1} * \Lambda_1^{i_1 *} * ... * \Lambda_k^{i_k *}(n)}{n^s}, \tag{2.18}$$

where for a natural number t, $\Lambda_r(n)^{t*}$ denote the t times Dirichlet convolution of $\Lambda_r(n)$ with itself and $K := i_1 + 2i_2 + ... + ki_k + v$. Similarly, by using the total multiplicativity of $\psi(n)$, we have

$$\frac{L^{(r)}}{L}(s, \psi) = \left((-1)^r \sum_{n=1}^{\infty} \frac{(\log n)^r \psi(n)}{n^s} \right) \left(\sum_{n=1}^{\infty} \frac{\mu(n)\psi(n)}{n^s} \right)$$

$$= (-1)^r \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} \mu(d)\psi(d)\log(n/d)\psi(n/d)$$

$$= (-1)^r \sum_{n=1}^{\infty} \frac{\psi(n)}{n^s} \sum_{d|n} \mu(d)\log(n/d)$$

$$= (-1)^r \sum_{n=1}^{\infty} \frac{\Lambda_r(n)\psi(n)}{n^s}, \quad (\sigma > 1).$$

Thus, by using the total multiplicativity of $\psi(n)$ again,

$$\frac{L^{(v+1)}}{L}(s,\psi) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi)\right)^{i_w} = (-1)^{K+1} \sum_{n=1}^{\infty} \frac{\Lambda_{v+1} * \Lambda_1^{i_1*} * ... * \Lambda_k^{i_k*}(n)\psi(n)}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v)\psi(n)}{n^s}, \quad (\sigma > 1). \tag{2.19}$$

Hence, the desired result follows from (2.18) and (2.19). □

Lemma 2.11. *Using the notation given by (2.16), for* $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, ψ_1, ψ_2 *any primitive Dirichlet characters modulo* $q \geq 1$ *and* $\sigma > 1$, *we have*

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v; \psi_1)\psi_2(n)}{n^s} = \sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v; \psi_1\psi_2)}{n^s}.$$

Proof. The proof uses the total multiplicativity of Dirichlet characters and it is almost the same proof as in Lemma 2.10, hence it can be omitted. □

Lemma 2.12. *For* $k, i_1, i_2, ..., i_k, n \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$ *and* $\sigma > 1$, *we have*

$$\Lambda_{v+1} * \Lambda_1^{i_1*} * ... * \Lambda_k^{i_k*}(n) \ll (\log n)^{K+1},$$

where $K := i_1 + 2i_2 + ... + ki_k + v$.

Proof. As in (2.17),

$$\sum_{n=1}^{\infty} \Lambda_k(n) n^{-s} = (-1)^k \frac{\zeta^{(k)}}{\zeta}(s), \quad (\sigma > 1), \tag{2.20}$$

where $\Lambda_k(n) = \sum_{d|n} \mu(d)(\log \frac{n}{d})^k$. By the Möbius inversion formula, we have

$$(\log n)^k = \sum_{d|n} \Lambda_k(d). \tag{2.21}$$

Taking $k = 1$ in (2.20) shows that $\Lambda_1(n) = \Lambda(n)$ for any natural number n. By considering the coefficients of the identity

$$\left(\frac{\zeta^{(k)}}{\zeta}(s)\right)' = \frac{\zeta^{(k+1)}}{\zeta}(s) - \frac{\zeta^{(k)}}{\zeta}(s)\frac{\zeta'}{\zeta}(s), \tag{2.22}$$

we see

$$\Lambda_{k+1}(n) = \Lambda_k(n)\log n + \sum_{d|n} \Lambda_k(d)\Lambda_1(n/d). \tag{2.23}$$

By definition $\Lambda_1(n) \leq \log n$. Thus,

$$\begin{aligned}
\Lambda_{k+1}(n) &\leq \Lambda_k(n)\log n + \log n \sum_{d|n} \Lambda_k(d) \\
&= \Lambda_k(n)\log n + (\log n)^{k+1} \\
&\leq 2(\log n)^{k+1},
\end{aligned}$$

if we assume that $\Lambda_k(n) \leq (\log n)^k$. This induction argument shows that

$$\Lambda_k(n) \ll (\log n)^k, \tag{2.24}$$

for any natural number k. Now, we observe that for any natural numbers m, t

$$\begin{aligned}
\Lambda_m * \Lambda_t(n) &= \sum_{d|n} \Lambda_m(d)\Lambda_t(n/d) \\
&\ll (\log n)^t \sum_{d|n} \Lambda_m(d) \\
&= (\log n)^{m+t}.
\end{aligned}$$

Hence by an induction argument again, the desired result can easily be obtained. $\square$

Now we state Perron's formula which will be frequently used in the present work.

Lemma 2.13. *(Perron's Formula) Let* $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *converge absolutely for* $\sigma = \Re s > 1$ *and let* $|a_n| < C\Phi(n)$ *where* $C > 0$ *and for* $x \geq x_0$, $\Phi(x)$ *is monotonically increasing. Let further*

$$\sum_{n=1}^{\infty} |a_n| n^{-s} \ll (\sigma - 1)^{-\alpha},$$

as $\sigma \to 1^+$ *for some* $\alpha > 0$. *If* $w = u+iv$ *(u; v real) is arbitrary,* $b > 0$, $T > 0$, $u+b > 1$, *then*

$$\sum_{n \leq x} a_n n^{-w} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} A(s+w) \frac{x^s}{s} \, ds + O\left(\frac{x^b}{T(u+b-1)^{\alpha}}\right) + O\left(\frac{\Phi(x) x^{1-u} \log 2x}{T}\right) + O\left(\frac{\Phi(2x)}{x^u}\right),$$

and the estimate is uniform in x, T, b, *and* u *provided that* b *and* u *are bounded.*

Proof. See Appendix 3 in [3]. $\square$

Lemma 2.14. *Assume GRH. Let* q *be a fixed odd prime number and* ψ_1 *be a primitive Dirichlet character modulo* q, $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, $K := i_1 + 2i_2 + ... + ki_k + v$ *and* $\sigma > 1$. *If* ψ *is a Dirichlet character modulo* q *such that* $\psi \neq \psi_1$, *then we have*

$$\sum_{n \leq x} b_n(i_1, ..., i_k; v; \overline{\psi_1}) \psi(n) = O(x^{1 - \frac{1}{\log\log q(x+4)}} (\log x)(A(k) \log\log q(x+4))^{K+1}).$$

Moreover, if also $i_1 + ... + i_k \leqslant \frac{\log x}{\log\log x}$, *then we have*

$$\sum_{n \leq x} b_n(i_1, ..., i_k; v; \overline{\psi_1}) \psi_1(n) = S(i_1, ..., i_k; v) x (\log x)^K + E_b(i_1, ..., i_k; v),$$

where

$$S(i_1, ..., i_k; v) := \frac{(-1)^{K+1}(v+1)! \prod_{w=1}^{k}(w!)^{i_w}}{K!},$$

and

$$E_b(i_1, ..., i_k; v) := O_q\left((A(k)^K)\left((\log x)^{K+2} + \frac{x(\log x)^{K-1}}{(K-1)!} + \frac{x\,(\log x)^{(\frac{2}{3}+\epsilon)(K+3)}}{e^{\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon}}}\right)\right).$$

Proof. Let ψ be a Dirichlet character modulo q such that $\psi \neq \psi_1$. By Lemma 2.11, we have

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi(n)}{n^s} = \frac{L^{(v+1)}}{L}(s, \overline{\psi_1}\psi) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi)\right)^{i_w},$$

for $\sigma > 1$. By Lemma 2.12, we know that $|b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi(n)| \leq (\log n)^{K+1}$. Thus, we have

$$\sum_{n \leq x} b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi(n) = \frac{1}{2\pi i} \int_{1+\frac{1}{\log x}-ix}^{1+\frac{1}{\log x}+ix} \frac{L^{(v+1)}}{L}(s, \overline{\psi_1}\psi) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi)\right)^{i_w} \frac{x^s}{s}\, ds$$
$$+ O((A(k)\log x)^{K+2}), \tag{2.25}$$

by Lemma 2.13. Since q is assumed to be an odd prime number, all non-principal Dirichlet characters are primitive. Since $\psi \neq \psi_1$, $\overline{\psi_1}\psi$ is a non-principal and primitive Dirichlet character modulo q. Thus, the integrand in (2.25) has no poles inside the rectangle with vertices $1+\frac{1}{\log x}-ix$, $1+\frac{1}{\log x}+ix$, $1-\frac{1}{\log\log q(x+4)}+ix$ and $1-\frac{1}{\log\log q(x+4)}-ix$. Now, by (2.25) and the residue theorem, we have

$$\sum_{n \leq x} b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi(n) = I_1 + I_2 + I_3 + O((A(k)\log x)^{K+2}),$$

where

$$I_1 := -\frac{1}{2\pi i} \int\limits_{1+\frac{1}{\log x}+ix}^{1-\frac{1}{\log\log q(x+4)}+ix} \frac{L^{(v+1)}}{L}(s, \overline{\psi_1}\psi) \prod_{w=1}^{k} \frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi) \frac{x^s}{s}\, ds,$$

$$I_2 := -\frac{1}{2\pi i} \int\limits_{1-\frac{1}{\log\log q(x+4)}+ix}^{1-\frac{1}{\log\log q(x+4)}-ix} \frac{L^{(v+1)}}{L}(s, \overline{\psi_1}\psi) \prod_{w=1}^{k} \frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi) \frac{x^s}{s}\, ds,$$

$$I_3 := -\frac{1}{2\pi i} \int\limits_{1-\frac{1}{\log\log q(x+4)}-ix}^{1+\frac{1}{\log x}-ix} \frac{L^{(v+1)}}{L}(s, \overline{\psi_1}\psi) \prod_{w=1}^{k} \frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi) \frac{x^s}{s}\, ds.$$

Now, we will find upper bounds for I_1, I_2 and I_3 by using the result in Lemma 2.3. Along the line of integration in I_1, we have $s = \sigma + ix$ with $1 - \frac{1}{\log\log q(x+4)} \leq \sigma \leq 1 + \frac{1}{\log x}$. Note that for large x, $\frac{1}{2} + \frac{1}{\log\log q(x+4)} \leq 1 - \frac{1}{\log\log q(x+4)}$. Thus, we can use Lemma 2.3 and get

$$\begin{aligned}
\frac{L'}{L}(s, \overline{\psi_1}\psi) &\ll ((\log q\tau)^{2-2\sigma} + 1) \min\left(\frac{1}{|\sigma - 1|}, \log\log q\tau\right) \\
&\ll ((\log q(x+4))^{2-2\sigma} + 1) \log\log q(x+4) \\
&\ll ((\log q(x+4))^{2-2(1-1/\log\log q(x+4))} + 1) \log\log q(x+4) \\
&\ll \log\log q(x+4).
\end{aligned}$$

By Cauchy's theorem, for $r := 1/\log\log q(x+4)$,

$$\frac{d}{ds}\frac{L'}{L}(s, \overline{\psi_1}\psi) = \frac{1}{2\pi i} \int_{|w-s|=r} \frac{\frac{L'}{L}(w, \overline{\psi_1}\psi)}{(w-s)^2}\, dw \ll \frac{\log\log q(x+4)}{r^2} r = (\log\log q(x+4))^2,$$

and

$$\begin{aligned}
\frac{L''}{L}(s, \overline{\psi_1}\psi) &= \frac{d}{ds}\frac{L'}{L}(s, \overline{\psi_1}\psi) + \left(\frac{L'}{L}(s, \overline{\psi_1}\psi)\right)^2 \\
&\ll (\log\log q(x+4))^2 + (\log\log q(x+4))^2 \\
&\ll (\log\log q(x+4))^2.
\end{aligned}$$

Now, by using the identity $\frac{L^{(w+1)}}{L} = (\frac{L^{(w)}}{L})' + \frac{L^{(w)}}{L}\frac{L'}{L}$ and induction on w, we obtain

$$\frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi) \ll_w (\log\log q(x+4))^w, \tag{2.26}$$

along the line of integration in I_1. Since $|\frac{x^s}{s}| \ll 1$ along this line, we have

$$I_1 \ll (A(k)\log\log q(x+4))^{K+1}.$$

By symmetry, we also have

$$I_3 \ll (A(k)\log\log q(x+4))^{K+1}.$$

Now, we will find an upper bound for I_2. Along the upper part of the line of integration in I_2, we have $s = 1 - \frac{1}{\log\log q(x+4)} + it$ with $0 \le t \le x$. Note that for large x, we have $\frac{1}{2} + \frac{1}{3} < 1 - \frac{1}{\log\log q(x+4)}$ and clearly, the inequality $\frac{1}{2} + \frac{1}{\log\log q(t+4)} \le \frac{1}{2} + \frac{1}{3}$ is equivalent to $t \ge \frac{\exp(\exp 3)}{q} - 4$. Thus, for $t \ge \left|\frac{\exp(\exp 3)}{q} - 4\right| =: t_0$, we can use Lemma 2.3. We split the upper part of the integral in I_2 as $\int_0^{t_0} + \int_{t_0}^{x}$. Note that for $t \ge t_0$, we have

$$\begin{aligned}\frac{L'}{L}(s, \overline{\psi_1}\psi) &\ll ((\log q\tau)^{2-2(1-\frac{1}{\log\log q(x+4)})} + 1)\min\left(\frac{1}{|1 - \frac{1}{\log\log q(x+4)} - 1|}, \log\log q\tau\right)\\ &= ((\log q(x+4))^{\frac{2}{\log\log q(x+4)}} + 1)(\log\log q(x+4))\\ &\ll \log\log q(x+4),\end{aligned}$$

by Lemma 2.3. By the same argument as in (2.26), we have

$$\frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi) \ll_w (\log\log q(x+4))^w,$$

for $t \ge t_0$. Thus, the part $\int_{t_0}^{x}$ in I_2 is

$$\ll (A(k)\log\log q(x+4))^{K+1} x^{1-\frac{1}{\log\log q(x+4)}} \int\limits_{\frac{\exp(\exp 3)}{q}-4}^{x} \frac{dt}{\sqrt{(\frac{1}{\log\log q(x+4)})^2 + t^2}}$$

$$\ll (A(k)\log\log q(x+4))^{K+1} x^{1-\frac{1}{\log\log q(x+4)}} \left(\log\left(\sqrt{\left(\frac{1}{\log\log q(x+4)}\right)^2 + x^2} + x\right)\right)$$
$$\ll x^{1-\frac{1}{\log\log q(x+4)}} (\log x)(A(k)\log\log q(x+4))^{K+1}.$$

On the part $\int_0^{t_0}$ in I_2, we have $s = 1 - \frac{1}{\log\log q(x+4)} + it$, $0 \le t \le t_0$ and

$$\frac{L^{(v+1)}}{L}(s, \overline{\psi_1}\psi) \prod_{w=1}^{k} \frac{L^{(w)}}{L}(s, \overline{\psi_1}\psi) \frac{x^s}{s} \ll x^{1-\frac{1}{\log\log q(x+4)}}.$$

Thus, the part $\int_0^{t_0}$ in I_2 is $\ll x^{1-\frac{1}{\log\log q(x+4)}}$. Combining the results for $\int_0^{t_0}$ and $\int_{t_0}^{x}$ and using symmetry for the lower part of I_2, we obtain

$$I_2 \ll x^{1-\frac{1}{\log\log q(x+4)}} (\log x)(A(k)\log\log q(x+4))^{K+1}.$$

Combining the results for I_1, I_2, and I_3 finishes the proof of the first statement in the lemma.

To prove the second statement in the lemma, we consider the sum $\sum_{n\le x} b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi_1(n)$. Note that $\overline{\psi_1}\psi_1 = \psi_0$ where ψ_0 is the principal Dirichlet character modulo q and that

$$\begin{aligned} L(s,\psi_0) &= \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right) \\ &= \zeta(s) \left(1 - \frac{1}{q^s}\right) \\ &=: \zeta(s) A_q(s), \quad (\sigma > 1). \end{aligned} \tag{2.27}$$

By Lemma 2.11, we know that

$$\sum_{n=1}^{\infty} \frac{b_n(i_1, ..., i_k; v; \overline{\psi_1})\psi_1(n)}{n^s} = \frac{L^{(v+1)}}{L}(s, \psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s, \psi_0)\right)^{i_w}, \quad (\sigma > 1).$$

By Lemma 2.12 and 2.13, we have

$$\sum_{n\leqslant x} b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_1(n) = \frac{1}{2\pi i}\int\limits_{1+\frac{1}{\log x}-ix}^{1+\frac{1}{\log x}+ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds$$
$$+O((A(k)\log x)^{K+2}).$$

To evaluate the sum under consideration, we follow the steps used in the proof of Lemma 2.4 in [9]. Define $\delta_1(k) := \frac{\delta_1}{2^k}$ where δ_1 is the constant whose existence is given in Lemma 2.2 and let $\epsilon_0 > 0$ be a small fixed number. By the residue theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds \\ = \mathrm{Res}_{s=1}\left\{\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\},\end{aligned} \tag{2.28}$$

where C is the positively oriented boundary of the rectangle with vertices $1+\frac{1}{\log x}-ix$, $1+\frac{1}{\log x}+ix$, $1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix$ and $1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix$. Note that the only pole of the integrand in (2.28) inside C is at $s=1$ and the order of this pole is $K+1$. Thus,

$$\begin{aligned}\sum_{n\leqslant x} b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_1(n) = \mathrm{Res}_{s=1}\left\{\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\} \\ -\frac{1}{2\pi i}\left(I_1^0+I_2^0+I_3^0\right)+O((A(k)\log x)^{K+2}),\end{aligned} \tag{2.29}$$

where

$$I_1^0 := \int\limits_{1+\frac{1}{\log x}+ix}^{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds,$$

$$I_2^0 := \int\limits_{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+ix}^{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds,$$

$$I_3^0 := \int_{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}-ix}^{1+\frac{1}{\log x}-ix} \frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\,ds.$$

First we will find some upper bounds for I_1^0, I_2^0, I_3^0 and then evaluate the above residue. By (2.27) and Lemma 2.2, we have

$$\begin{aligned}\frac{L'}{L}(s,\psi_0) &= \frac{\zeta'}{\zeta}(s)+\frac{A_q'}{A_q}(s)\\ &\ll (\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}\\ &\ll (\log|t|)^{\frac{2}{3}+\epsilon_0},\end{aligned} \tag{2.30}$$

in the region

$$\sigma \geqslant 1-\frac{\delta_1}{(\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}}},\qquad |t|\geqslant 3. \tag{2.31}$$

By (2.30) and Cauchy's theorem for $r=\frac{1}{((\log|t|)^{\frac{2}{3}}(\log\log|t|)^{\frac{1}{3}})}$, we obtain

$$\begin{aligned}\frac{d}{ds}\frac{L'}{L}(s,\psi_0) &= \frac{1}{2\pi i}\int_{|w-s|=r}\frac{\frac{L'}{L}(s,\psi_0)}{(w-s)^2}\,dw\\ &\ll \frac{(\log|t|)^{\frac{2}{3}+\epsilon_0}}{r^2}r\\ &\ll \left((\log|t|)^{\frac{2}{3}+\epsilon_0}\right)^2,\end{aligned}$$

in the region given in (2.31). Now by using the identity $\frac{L^{(w+1)}}{L}=(\frac{L^{(w)}}{L})'+\frac{L^{(w)}}{L}\frac{L'}{L}$ and induction on w, we obtain

$$\frac{L^{(w)}}{L}(s,\psi_0)\ll_{w,q}(\log|t|)^{(\frac{2}{3}+\epsilon_0)w},\qquad \left(\sigma\geqslant 1-\frac{\delta_1}{2^w(\log|t|)^{\frac{2}{3}+\epsilon_0}},\ |t|\geqslant 3\right).$$

Hence, on I_1^0, I_3^0 and the part of I_2^0 of distance $\geqslant 3$ from the real line, we have

$$\frac{L^{(w)}}{L}(s,\psi_0)=O_{w,q}(\log x)^{(\frac{2}{3}+\epsilon_0)w}. \tag{2.32}$$

Since $|\frac{x^s}{s}| \ll 1$ on the the horizontal parts of the contour, we get

$$
\begin{aligned}
I_1^0 &\ll_q (A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+1)}, \\
I_3^0 &\ll_q (A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+1)}.
\end{aligned}
$$

For I_2^0, by symmetry it is enough to consider the part above the real axis which can be split up as $\int_0^3 + \int_3^x$. By (2.32), the second part is

$$
\begin{aligned}
&\ll x e^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}} (A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+1)} \int_3^x \frac{dt}{t} \\
&\ll x e^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}} (A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+3)}.
\end{aligned}
$$

Now we consider the first part, that is, the part $\int_0^3$. By (2.27), we have

$$
L^{(t)}(s,\psi_0) = \sum_{m=0}^{t} \binom{t}{m} \zeta^{(t-m)}(s) A_q^{(m)}(s),
$$

so that

$$
\frac{L^{(t)}}{L}(s,\psi_0) = \sum_{m=0}^{t} \binom{t}{m} \frac{\zeta^{(t-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s), \tag{2.33}
$$

and

$$
\begin{aligned}
&\frac{L^{(v+1)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w} \\
&\quad = \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \prod_{w=1}^{k} \left(\sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right)^{i_w}.
\end{aligned}
$$

Thus, by (1.1) as $s \to 1$.

$$
\begin{aligned}
\frac{L^{(v+1)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w} &\ll \left| \frac{\zeta^{(v+1)}}{\zeta}(s) \prod_{w=1}^{k} \left(\frac{\zeta^{(w)}}{\zeta}(s)\right)^{i_w} \right| \\
&\ll \left(\frac{A(k)}{|s-1|}\right)^{K+1}.
\end{aligned} \tag{2.34}
$$

Hence, by (2.34), the first part is

$$\ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\,(A(k))^{K+1}\int\limits_{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}^{1-\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}+3i}\frac{ds}{|s-1|^{K+1}}. \tag{2.35}$$

The integral in (2.35) is

$$\ll \int\limits_{0}^{\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}\frac{dt}{\left(\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}\right)^{K+1}}+\int\limits_{\frac{\delta_1(k)}{(\log x)^{\frac{2}{3}+\epsilon_0}}}^{3}\frac{dt}{t^{K+1}}$$
$$\ll \left(\frac{(\log x)^{\frac{2}{3}+\epsilon_0}}{\delta_1(k)}\right)^{K}.$$

Hence we have

$$I_2^0 \ll xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\,(A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+3)}\,.$$

Now by (2.29),

$$\begin{aligned}\sum_{n\leqslant x}b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_1(n) =& \operatorname{Res}_{s=1}\left\{\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\}\\ &+O\left((A(k)\log x)^{K+2}\right)\\ &+O\left(xe^{-\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon_0}}\,(A(k)\log x)^{(\frac{2}{3}+\epsilon_0)(K+3)}\right).\end{aligned} \tag{2.36}$$

The pole at $s=1$ is of order $K+1$, so the residue in (2.36) is

$$\begin{aligned}=&\frac{1}{K!}\frac{d^K}{ds^K}\left\{(s-1)^{K+1}\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right\}_{s=1}\\ =&\frac{1}{K!}\sum_{\substack{j_1+j_2+j_3=K\\ j_1,j_2,j_3\in\mathbb{N}}}\binom{K}{j_1,j_2,j_3}\frac{d^{j_1}}{ds^{j_1}}\end{aligned}$$

$$\times \left\{ (s-1)^{K+1} \frac{L^{(v+1)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0) \right)^{i_w} \frac{x^s}{s} \right\}_{s=1}$$
$$\times \frac{d^{j_2}}{ds^{j_2}} \{x^s\}_{s=1} \frac{d^{j_3}}{ds^{j_3}} \left\{ \frac{1}{s} \right\}_{s=1} . \tag{2.37}$$

Since $j_3 = K - j_1 - j_2$, the expresssion in (2.37) is

$$= \frac{1}{K!} \sum_{j_1 \leqslant K} \frac{K!}{j_1! j_2! (K-j_1-j_2)!} \frac{d^{j_1}}{ds^{j_1}} \left\{ \frac{L^{(v+1)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0) \right)^{i_w} \right\}_{s=1}$$
$$\times \sum_{j_2 \leqslant K-j_1} \frac{d^{j_2}}{ds^{j_2}} \{x^s\}_{s=1} \left\{ \frac{(-1)^{K-j_1-j_2}(K-j_1-j_2)!}{s^{K-j_1-j_2+1}} \right\}_{s=1}$$
$$= (-1)^K \sum_{j_1 \leqslant K} \frac{(-1)^{j_1}}{j_1!} \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^{K+1} \frac{L^{(v+1)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0) \right)^{i_w} \right\}_{s=1}$$
$$\times \sum_{j_2 \leqslant K-j_1} \frac{(-1)^{j_2}}{j_2!} \{x^s (\log x)^{j_2}\}_{s=1}$$
$$= x(-1)^K \sum_{j_1 \leqslant K} \frac{(-1)^{j_1}}{j_1!} \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^{K+1} \frac{L^{(v+1)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0) \right)^{i_w} \right\}_{s=1}$$
$$\times \sum_{j_2 \leqslant K-j_1} \frac{(-1)^{j_2}}{j_2!} (\log x)^{j_2}. \tag{2.38}$$

Now we consider three special cases for $i_t, 1 \leq t \leq k$ and $v \in \mathbb{N}$. If $i_1 = i_2 = ... = i_k = v = 0$, then the expression in (2.38) is

$$= x \left\{ (s-1) \frac{L'}{L}(s,\psi_0) \right\}_{s=1} = x \left\{ (s-1) \left(\frac{\zeta'}{\zeta}(s) + \frac{A_q'}{A_q}(s) \right) \right\}_{s=1} = -x.$$

If $i_1 = i_2 = ... = i_k = 0, v = 1$, then the expression in (2.38) is

$$= x(-1)^1 \sum_{j_1 \leqslant 1} \frac{(-1)^{j_1}}{j_1!} \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^2 \frac{L^{(2)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0) \right)^{i_w} \right\}_{s=1}$$
$$\times \sum_{j_2 \leqslant 1-j_1} \frac{(-1)^{j_2}}{j_2!} (\log x)^{j_2}$$
$$= (-x + x\log x) \left\{ (s-1)^2 \frac{\zeta'' + 2\zeta' A_q' + \zeta A_q''}{\zeta A_q}(s) \right\}_{s=1}$$

$$
\begin{aligned}
&+x\frac{d}{ds}\left\{(s-1)^2\frac{\zeta''+2\zeta' A_q'+\zeta A_q''}{\zeta A_q}(s)\right\}_{s=1}\\
&=2x\log x-2x-x\frac{2A_q'}{A_q}(1).
\end{aligned}
$$

If $i_1=1, i_2=i_3=...=i_k=v=0$, then the expression in (2.38) is

$$
\begin{aligned}
=&-x\sum_{j_1\leqslant 1}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^2\frac{L'}{L}(s,\psi_0)\frac{L'}{L}(s,\psi_0\right\}_{s=1}\sum_{j_2\leqslant 1-j_1}\frac{(-1)^{j_2}\log^{j_2}x}{j_2!}\\
=&-x\left\{(s-1)^2\left(\frac{L'}{L}(s,\psi_0)\right)^2\right\}_{s=1}(1-\log x)\\
&+x\frac{d}{ds}\left\{(s-1)^2\left(\frac{L'}{L}(s,\psi_0)\right)^2\right\}_{s=1}\\
=&-x\left\{(s-1)^2\left(\frac{\zeta'}{\zeta}+\frac{A_q'}{A_q}\right)^2(s)\right\}_{s=1}(1-\log x)\\
&+x\frac{d}{ds}\left\{(s-1)^2\left(\frac{\zeta'}{\zeta}+\frac{A_q'}{A_q}\right)^2(s)\right\}_{s=1}\\
=&x\log x-x-2x\frac{A_q'}{A_q}(1).
\end{aligned}
$$

In general, by (2.33), we have

$$
\begin{aligned}
&\mathrm{Res}_{s=1}\left(\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right)\\
&\quad=x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\right\}_{s=1}\\
&\quad\quad\times\sum_{j_2\leqslant K-j_1}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2}\\
&\quad=x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\\
&\quad\quad\times\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\quad\quad\times\left.\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}\sum_{j_2\leqslant K-j_1}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2}.
\end{aligned}
$$

Now we split the double sum over j_1, j_2 into four parts: The term with $j_1 = 0$, $j_2 = K$ (which gives the main term) and $j_1 = 0$, $j_2 = K - 1$ and $j_1 = 1$, $j_2 = K - 1$ and the remaining terms. Thus,

$$
\begin{aligned}
&\operatorname{Res}_{s=1}\left(\frac{L^{(v+1)}}{L}(s,\psi_0)\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\psi_0)\right)^{i_w}\frac{x^s}{s}\right)\\
&=x\frac{(\log x)^K}{K!}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\qquad\left.\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}\\
&\quad-x\frac{(\log x)^{K-1}}{(K-1)!}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\qquad\left.\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}\\
&\quad+x\frac{(\log x)^{K-1}}{(K-1)!}\frac{d}{ds}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\qquad\left.\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}\\
&\quad+x(-1)^K\sum_{j_1\leqslant K}\frac{(-1)^{j_1}}{j_1!}\frac{d^{j_1}}{ds^{j_1}}\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.\\
&\qquad\left.\times\prod_{w=1}^{k}\left(\sum_{n=0}^{w}\binom{w}{n}\frac{\zeta^{(w-n)}}{\zeta}(s)\frac{A_q^{(n)}}{A_q}(s)\right)^{i_w}\right\}_{s=1}\sum_{j_2\leqslant K-a(j_1)}\frac{(-1)^{j_2}}{j_2!}(\log x)^{j_2},
\end{aligned}
$$

where

$$
a(j_1)=\begin{cases}2, & \text{if } j_1=0,1\\ j_1, & \text{otherwise.}\end{cases}
$$

By (1.1) , we have

$$
\left\{(s-1)^{K+1}\sum_{m=0}^{v+1}\binom{v+1}{m}\frac{\zeta^{(v+1-m)}}{\zeta}(s)\frac{A_q^{(m)}}{A_q}(s)\right.
$$

$$\times \prod_{w=1}^{k} \left(\sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right)^{i_w} \Bigg\}_{s=1} = (-1)^{K+1}(v+1)! \prod_{w=1}^{k} (w!)^{i_w}.$$

Thus,

$$\begin{aligned}
&\mathrm{Res}_{s=1} \left(\frac{L^{(v+1)}}{L}(s,\psi_0) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\psi_0) \right)^{i_w} \frac{x^s}{s} \right) \\
&= \frac{(-1)^{K+1}(v+1)! \prod_{w=1}^{k}(w!)^{i_w}}{K!} x(\log x)^K \\
&\quad + \frac{(-1)^{K}(v+1)! \prod_{w=1}^{k}(w!)^{i_w}}{(K-1)!} x(\log x)^{K-1} \\
&\quad + x\frac{(\log x)^{K-1}}{(K-1)!} \frac{d}{ds} \Bigg\{ (s-1)^{K+1} \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \\
&\qquad \times \prod_{w=1}^{k} \left(\sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right)^{i_w} \Bigg\}_{s=1} \\
&\quad + x(-1)^K \sum_{j_1 \leqslant K} \frac{(-1)^{j_1}}{j_1!} \frac{d^{j_1}}{ds^{j_1}} \Bigg\{ (s-1)^{K+1} \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \\
&\qquad \times \prod_{w=1}^{k} \left(\sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right)^{i_w} \Bigg\}_{s=1} \\
&\qquad \times \sum_{j_2 \leqslant K - a(j_1)} \frac{(-1)^{j_2}}{j_2!} (\log x)^{j_2}. \qquad (2.39)
\end{aligned}$$

Instead of the sum over j_2 in (2.39), we use Taylor remainder theorem and write

$$\frac{1}{x} - \frac{(-1)^{K-a(j_1)+1}(\log x)^{K-a(j_1)+1}}{(K-a(j_1)+1)!} x^{-\theta},$$

for some $\theta \in [0,1]$. Then the fourth term in (2.39) can be bounded as

$$\begin{aligned}
\ll \sum_{j_1 \leqslant K} \frac{1}{j_1!} \Bigg| \frac{d^{j_1}}{ds^{j_1}} \Bigg\{ &(s-1)^{K+1} \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \\
&\times \prod_{w=1}^{k} \left(\sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right)^{i_w} \Bigg\}_{s=1} \Bigg|
\end{aligned}$$

$$+\, x(\log x)^{K+1} \sum_{j_1 \leqslant K} \frac{(\log x)^{-a(j_1)}}{j_1!(K-a(j_1)+1)!}$$
$$\times \left| \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^{K+1} \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \right.\right.$$
$$\left.\left. \times \prod_{w=1}^{k} \left(\sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right)^{i_w} \right\}_{s=1} \right| \tag{2.40}$$

By Cauchy's estimate on a disk of radius 1 centered at $s = 1$, we have

$$\left| \frac{d^{j_1}}{ds^{j_1}} \left\{ (s-1)^{K+1} \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \right.\right.$$
$$\left.\left. \times \prod_{w=1}^{k} \left(\sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right)^{i_w} \right\}_{s=1} \right|$$
$$\leqslant j_1! \max_{|s-1|=1} \left\{ \left| \sum_{m=0}^{v+1} \binom{v+1}{m} \frac{\zeta^{(v+1-m)}}{\zeta}(s) \frac{A_q^{(m)}}{A_q}(s) \right| \right.$$
$$\left. \times \prod_{w=1}^{k} \left| \sum_{n=0}^{w} \binom{w}{n} \frac{\zeta^{(w-n)}}{\zeta}(s) \frac{A_q^{(n)}}{A_q}(s) \right| \right\} \ll_q j_1!(A(k))^K. \tag{2.41}$$

By using (2.41), the fourth term in (2.39) can be majorized by

$$\ll (A(k))^K \left[\sum_{j_1 \leqslant K} 1 + x(\log x)^{K+1} \sum_{j_1 \leqslant K} \frac{(\log x)^{-a(j_1)}}{j_1!(K-a(j_1)+1)!} \right]$$
$$= (A(k))^K \left[K + x(\log x)^{K+1} \left(\frac{3(\log x)^{-2}}{(K-1)!} + \sum_{3 \leqslant j_1 \leqslant K} \frac{(\log x)^{j_1}}{(K-j_1+1)!} \right) \right].$$

The condition $i_1 + \ldots + i_k \leqslant \frac{\log x}{\log\log x}$ implies

$$K \leqslant \frac{k \log x}{\log\log x} + k, \tag{2.42}$$

so in the case that the last sum over j_1 is not void, we have

$$\sum_{3 \leqslant j_1 \leqslant K} \frac{(\log x)^{j_1}}{(K-j_1+1)!} = \frac{(\log x)^{-3}}{(K-2)!} \left[1 + \frac{K-2}{\log x} + \frac{(K-2)(K-3)}{(\log x)^2} + \ldots + \frac{(K-2)!}{1!(\log x)^{K-3}} \right]$$

$$
\begin{aligned}
&< \frac{(\log x)^{-3}}{(K-2)!}\left[1+\frac{2k}{\log\log x}+\left(\frac{2k}{\log\log x}\right)^2+...\right]\\
&< \frac{2(\log x)^{-3}}{(K-2)!}\\
&< \frac{3(\log x)^{-2}}{(K-1)!\log\log x},
\end{aligned}
$$

for large x. Thus, the expression in (2.40) is

$$
\ll \frac{(A(k))^K x(\log x)^{K-1}}{(K-1)!}. \tag{2.43}
$$

By (2.41), the upper bound in (2.43) dominates the second and third terms in the right-hand side of (2.39) with an appropriate $A(k)$. Combining (2.36), (2.39) and (2.43) finishes the proof. $\square$

Lemma 2.15. *Let ψ_1 be a primitive Dirichlet character modulo q, $q \geqslant 3$. For $k, i_1, i_2, ..., i_k \in \mathbb{N}$, $v \in \{0, 1, ..., k\}$, $K := i_1 + 2i_2 + ... + ki_k + v$ and $\sigma > 1$, define*

$$
\sum_{n=1}^{\infty}\frac{b_n(i_1,...,i_k;v;\overline{\psi_1})}{n^s} := \frac{L^{(v+1)}}{L}(s,\overline{\psi_1})\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{i_w}.
$$

If ψ is a Dirichlet character modulo q such that $\psi \neq \psi_1$, then we have

$$
\sum_{\frac{x}{2}<n\leq x}\frac{b_n(i_1,...,i_k;v;\overline{\psi_1})\psi(n)}{(\log n)^K} = O\left(\frac{x^{1-\frac{1}{\log\log q(x+4)}}(A(k)\log\log q(x+4))^{K+1}}{(\log x)^{K-1}}\right).
$$

Moreover, if also $i_1 + ... + i_k \leqslant \frac{\log x}{\log\log x}$, then we have

$$
\sum_{\frac{x}{2}<n\leq x}\frac{b_n(i_1,...,i_k;v;\overline{\psi_1})\psi_1(n)}{(\log n)^K} = S(i_1,...,i_k;v)\frac{x}{2} + O\left(\frac{E_b(i_1,...,i_k;v)}{(\log x)^K}\right),
$$

where

$$
S(i_1,...,i_k;v) := \frac{(-1)^{K+1}(v+1)!\prod_{w=1}^{k}(w!)^{i_w}}{K!},
$$

and

$$E_b(i_1, ..., i_k; v) := O_q\left((A(k)^K)\left((\log x)^{K+2} + \frac{x(\log x)^{K-1}}{(K-1)!} + \frac{x(\log x)^{(\frac{2}{3}+\epsilon)(K+3)}}{e^{\delta_1(k)(\log x)^{\frac{1}{3}-\epsilon}}}\right)\right).$$

Proof. We apply partial summation to the result of Lemma 2.14. $\square$

3. PROOF OF THEOREM 1.1

Let q be a fixed odd prime number, k be a natural number and ψ_1, ψ_2 be any two non-principal Dirichlet characters modulo q. Since q is assumed to be a prime number, ψ_1 and ψ_2 are primitive. We want to estimate the sum

$$S_k(A,B) := \sum_{A<\gamma_{\psi_1,k}\leqslant B} \chi_{\psi_2}(\rho_{\psi_1,k}),$$

for large A and B with $A < B \leqslant 2A$. By the residue theorem,

$$S_k(A,B) = \frac{1}{2\pi i}\int_C \chi_{\psi_2}(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1)\,ds,$$

for a suitable contour C. In [7], Yıldırım proved that for any $\delta > 0$, there exists $M = M(k,\delta,\mathfrak{a}_1)$ such that there is no zero of $L^{(k)}(s,\psi_1)$ in the region $|s| > q^M$, $\sigma < -\delta$, $|t| > \delta$ where $\mathfrak{a}_1$ is the number in (1.5) corresponding to ψ_1. By Lemma 2.4, we have zero-free half planes $\sigma > 1 + m\left(1+\sqrt{1+\frac{2k^2}{m\log m}}\right)$ for $L^{(k)}(s,\psi_1)$ where m is the smallest prime number that does not divide q.. Since q is assumed to be an odd prime number, m is equal to 2. Thus, we can take the contour C as the positively oriented boundary of the rectangle with vertices $\sigma'_{k,q}+iA$, $\sigma'_{k,q}+iB$, $-\delta+iB$, $-\delta+iA$ where $\sigma'_{k,q} > \sigma_{k,q} := 1 + 2\left(1+\sqrt{1+\frac{k^2}{\log 2}}\right)$ and δ is a fixed real number with $0 < \delta < \frac{1}{8}$. Throughout the proof, we neglect δ-independence in the constants implied by the O-symbols. Now, we have $S_k(A,B) = I_1 + I_2 + I_3 + I_4$, where

$$I_1 := \frac{1}{2\pi i}\int_{\sigma'_{k,q}+iA}^{\sigma'_{k,q}+iB} \chi_{\psi_2}(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1)\,ds,$$

$$I_2 := \frac{1}{2\pi i}\int_{\sigma'_{k,q}+iB}^{-\delta+iB} \chi_{\psi_2}(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi_1)\,ds,$$

$$I_3 := \frac{1}{2\pi i} \int_{-\delta+iB}^{-\delta+iA} \chi_{\psi_2}(s) \frac{L^{(k+1)}}{L^{(k)}}(s, \psi_1)\, ds,$$
$$I_4 := \frac{1}{2\pi i} \int_{-\delta+iA}^{\sigma'_{k,q}+iA} \chi_{\psi_2}(s) \frac{L^{(k+1)}}{L^{(k)}}(s, \psi_1)\, ds.$$

Note that we can take horizontal sides of the contour C to be a distance of $\gg \frac{1}{\log A}$ from any zero of $L^{(k)}(s, \psi_1)$ by adjusting A and B by an amount of ≤ 1. In view of (1.6), this means adding or deleting $O(\log A)$ number of terms each of size $\ll_q A^{\frac{1}{2}+\delta}$. Since $\frac{L^{(k+1)}}{L^{(k)}}(s, \psi_1) \ll (\log A)^2$ on the horizontal parts of C and $\frac{L^{(k+1)}}{L^{(k)}}(s, \psi_1) \ll 1$ on the right vertical line of C, again in view of (1.6), we see that the contribution from I_1, I_2 and I_4 is $\ll_{k,q} A^{\frac{1}{2}+\delta} \log^2 A$. Thus, we have

$$S_k(A, B) = I_3 + O_{k,q}(A^{\frac{1}{2}+\delta} \log^2 A). \tag{3.1}$$

By parametrization and conjugation, we have

$$\begin{aligned}
I_3 &= -\frac{1}{2\pi} \int_A^B \chi_{\psi_2}(-\delta + it) \frac{L^{(k+1)}}{L^{(k)}}(-\delta + it, \psi_1)\, dt \\
&= \overline{-\frac{1}{2\pi} \int_A^B \chi_{\overline{\psi_2}}(-\delta - it) \frac{L^{(k+1)}}{L^{(k)}}(-\delta - it, \overline{\psi_1})\, dt} \\
&= \overline{-\frac{1}{2\pi} \int_A^B \chi_{\overline{\psi_2}}(1 - (1 + \delta + it)) \frac{L^{(k+1)}}{L^{(k)}}(1 - (1 + \delta + it), \overline{\psi_1})\, dt} \\
&= \overline{-\frac{1}{2\pi i} \int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi_2}}(1 - s) \frac{L^{(k+1)}}{L^{(k)}}(1 - s, \overline{\psi_1})\, ds}.
\end{aligned} \tag{3.2}$$

Thus, by (3.1) and (3.2), we have

$$\overline{S_k(A, B)} = -\frac{1}{2\pi i} \int_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi_2}}(1 - s) \frac{L^{(k+1)}}{L^{(k)}}(1 - s, \overline{\psi_1})\, ds + O_{k,q}\left(A^{\frac{1}{2}+\delta} \log^2 A\right). \tag{3.3}$$

By (1.4) and Lemma 2.1, we have

$$
\begin{aligned}
L^{(k)}(s,\psi_1) =& \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \chi_{\psi_1}^{(i)} L^{(k-i)}(1-s,\overline{\psi_1}) \\
=& \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \chi_{\psi_1}(s) \left(-\ell + O\left(\frac{1}{|t|}\right)\right)^{i} L^{(k-i)}(1-s,\overline{\psi_1}) \\
=& \chi_{\psi_1}(s) \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \left(-\ell + O\left(\frac{1}{|t|}\right)\right)^{i} L^{(k-i)}(1-s,\overline{\psi_1}) \\
=& \chi_{\psi_1}(s) \left(1 + O\left(\frac{1}{|t|}\right)\right) \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (-\ell)^{i} L^{(k-i)}(1-s,\overline{\psi_1}) \\
=& \chi_{\psi_1}(s) \left(1 + O\left(\frac{1}{|t|}\right)\right) \left(-\ell + \frac{d}{ds}\right)^{k} L(1-s,\overline{\psi_1}),
\end{aligned}
\tag{3.4}
$$

where $\ell = \log \frac{q|t|}{2\pi}$. By differentiating both sides of (3.4) we see that

$$
\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi_1}) = -\left(\ell + \frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})\right)\left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{3.5}
$$

for $\sigma' \le \sigma \le \sigma''$ where σ' and σ'' are any fixed real numbers and

$$
\frac{G_k'}{G_k}(s,z,\psi_1) := \left(z + \frac{d}{dz}\right)^{k} L(s,\psi_1) = z^k L(s,\psi_1) + kz^k L'(s,\psi_1) + \ldots + L^{(k)}(s,\psi_1),
$$

and the differentiation in G_k' is with respect to s. We have

$$
\begin{aligned}
\frac{G_k'}{G_k}(s,z,\overline{\psi_1}) =& \frac{z^k L'(s,\overline{\psi_1}) + kz^{k-1} L''(s,\overline{\psi_1}) + \ldots + L^{(k+1)}(s,\overline{\psi_1})}{z^k L(s,\overline{\psi_1}) + kz^{k-1} L'(s,\overline{\psi_1}) + \ldots + L^{(k)}(s,\overline{\psi_1})} \\
=& \frac{z^k (L'(s,\overline{\psi_1}) + kz^{-1} L''(s,\overline{\psi_1}) + \ldots + z^{-k} L^{(k+1)}(s,\overline{\psi_1}))}{z^k (L(s,\overline{\psi_1}) + kz^{-1} L'(s,\overline{\psi_1}) + \ldots + z^{-k} L^{(k)}(s,\overline{\psi_1}))} \\
=& \frac{\sum_{v=0}^{k} \binom{k}{v} \frac{1}{z^v} \frac{L^{(v+1)}}{L}(s,\overline{\psi_1})}{1 + \sum_{w=1}^{k} \binom{k}{w} \frac{1}{z^w} \frac{L^{(w)}}{L}(s,\overline{\psi_1})}.
\end{aligned}
\tag{3.6}
$$

Since $\frac{L^{(w)}}{L}(s,\overline{\psi_1}) \ll_w 1$ when $\sigma \ge 1 + \delta$, we get

$$
\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s,\overline{\psi_1}) \ll_k \frac{1}{\log A}, \tag{3.7}
$$

and

$$\frac{G_k'}{G_k}(s,\ell,\overline{\psi_1}) \ll_k 1, \quad (\sigma \geq 1+\delta,\ A \leq t \leq B). \tag{3.8}$$

By writing (3.5) in (3.3), and then using (3.8) and Lemma 2.8 with $m = 1$ and $b_1 = 1$, $b_i = 0$, for $i > 1$, we obtain

$$\overline{S_k(A,B)} = \frac{1}{2\pi i} \int\limits_{1+\delta+iA}^{1+\delta+iB} \chi_{\overline{\psi_2}}(1-s) \frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})\, ds + O_{k,q}(A^{\frac{1}{2}+\delta} \log^2 A). \tag{3.9}$$

Now, we approximate $\frac{G_k'}{G_k}(s,\ell,\overline{\psi_1})$ by a Dirichlet series. By (3.7), in the region $\sigma \geq 1+\delta$, $A \leq t \leq B$ we can expand the denominator of $\frac{G_k'}{G_k}$ as a power series. Thus, we can write

$$\begin{aligned}
\left(1 + \sum_{w=1}^{k} \binom{k}{w} \frac{1}{z^w} \frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{-1} &= \sum_{u=0}^{\infty} (-1)^u \left(\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^u \\
&= \sum_{u \leq \frac{\log A}{\log\log A}} (-1)^u \left(\sum_{w=1}^{k} \binom{k}{w} \frac{1}{\ell^w} \frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^u \\
&\quad + O\left(\frac{1}{A}\right),
\end{aligned} \tag{3.10}$$

By using (3.6), (3.9) and (3.10), we obtain

$$\begin{aligned}
\overline{S_k(A,B)} = &\sum_{u \leq \frac{\log A}{\log\log A}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w} \\
&\times \frac{1}{2\pi} \int_A^B \frac{\chi_{\overline{\psi_2}}(-\delta - it)}{\ell^K} \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(1+\delta+it,\overline{\psi_1})\right)^{i_w} \frac{L^{(v+1)}}{L}(1+\delta+it,\overline{\psi_1})\, dt \\
&+ O_{k,q}(A^{\frac{1}{2}+\delta} \log^2 A),
\end{aligned}$$

where $K := i_1 + 2i_2 + \ldots + ki_k + v$.

By putting $A = T/2$, $B = T$ and using Lemma 2.8 with $m = -K$, we obtain

$$\overline{S_k(T/2,T)} = \sum_{u \le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \frac{\tau\left(\overline{\psi_2}\right)}{q} \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})e^{-\frac{2\pi i n}{q}}}{(\log n)^K} + O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T), \qquad (3.11)$$

where $\sum_{n=1}^{\infty} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})}{n^s} := \frac{L^{(v+1)}}{L}(s,\overline{\psi_1}) \prod_{w=1}^{k} \left(\frac{L^{(w)}}{L}(s,\overline{\psi_1})\right)^{i_w}$. In view of (2.19), we observe that for $n \in \mathbb{N}$ with $(n,q) > 1$, $b_n(i_1,\ldots,i_k;v;\overline{\psi_1}) = 0$. Also, note that every natural number n between $\frac{qT}{4\pi}$ and $\frac{qT}{2\pi}$ with $(n,q) = 1$ is congruent to a natural number $a \pmod q$ with $(a,q) = 1$ and if $n \equiv a \pmod q$, then $e(-\frac{n}{q}) = e(-\frac{a}{q})$. Thus, using the cancellation property of Dirichlet characters in (1.3), the sum over n in (3.11) can be written as

$$\sum_{\substack{a \pmod q \\ (a,q)=1}} \sum_{\substack{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi} \\ n \equiv a \pmod q}} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})}{(\log n)^K}$$
$$= \frac{1}{\varphi(q)} \sum_{\substack{a \pmod q \\ (a,q)=1}} e^{-\frac{2\pi i a}{q}} \sum_{\psi \pmod q} \overline{\psi}(a) \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi(n)}{(\log n)^K}.$$

Thus,

$$\overline{S_k(T/2,T)} = \sum_{u \le \frac{\log T}{\log\log T}} \sum_{v=0}^{k} (-1)^u \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \binom{k}{w}^{i_w}$$
$$\times \frac{\tau\left(\overline{\psi_2}\right)}{q\varphi(q)} \sum_{\substack{a \pmod q \\ (a,q)=1}} e^{-\frac{2\pi i a}{q}} \sum_{\psi \pmod q} \overline{\psi}(a) \sum_{\frac{qT}{4\pi} < n \le \frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi(n)}{(\log n)^K}$$
$$+ O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T). \qquad (3.12)$$

Now, we split the sum over Dirichlet characters ψ in (3.12) into two parts: The term with $\psi = \psi_1$ and the other terms.

Then,

$$\overline{S_k(T/2,T)} = \frac{\tau\left(\overline{\psi_2}\right)\overline{\tau(\psi_1)}}{q\varphi(q)} \sum_{u\le\frac{\log T}{\log\log T}} \sum_{v=0}^{k}(-1)^u\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k}\binom{k}{w}^{i_w}$$
$$\times \sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi_1(n)}{(\log n)^K}$$
$$+ E_{\psi\ne\psi_1} + O_{k,q}(T^{\frac{1}{2}+\delta}\log^2 T), \tag{3.13}$$

where

$$E_{\psi\ne\psi_1} := \sum_{u\le\frac{\log T}{\log\log T}} \sum_{v=0}^{k}(-1)^u\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k}\binom{k}{w}^{i_w}$$
$$\times \frac{\tau\left(\overline{\psi_2}\right)}{q\varphi(q)} \sum_{a \pmod q} e^{-\frac{2\pi i a}{q}} \sum_{\psi\ne\psi_1 \pmod q} \overline{\psi}(a) \sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}} \frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi(n)}{(\log n)^K}.$$

By Lemma 2.15 with $x=\frac{qT}{2\pi}$,

$$E_{\psi\ne\psi_1} \ll_{k,q} \sum_{u\le\frac{\log T}{\log\log T}} \sum_{v=0}^{k}\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}} \binom{u}{i_1,\ldots,i_k}$$
$$\times \frac{T^{1-\frac{1}{\log\log q(T+4)}}(A(k)\log\log q(T+4))^{K+1}}{(\log T)^{K-1}}$$
$$\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}}\log T\log\log q(T+4)$$
$$\times \sum_{u\le\frac{\log T}{\log\log T}} \sum_{v=0}^{k}\binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}} \binom{u}{i_1,\ldots,i_k} \left(\frac{A(k)\log\log q(T+4)}{\log T}\right)^K$$
$$\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}}\log T\log\log T$$
$$\times \sum_{u\le\frac{\log T}{\log\log T}} \left(\frac{A(k)\log\log q(T+4)}{\log T}\right)^u \sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}} \binom{u}{i_1,\ldots,i_k}$$
$$\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}}\log T\log\log T \sum_{u\le\frac{\log T}{\log\log T}} \left(\frac{A(k)\log\log q(T+4)}{\log T}\right)^u k^u$$

$$\begin{aligned}&\ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}}\log T\log\log T\\&\ll_{k,q}\frac{T}{\log T}.\end{aligned}\tag{3.14}$$

Thus, by (3.13) and (3.14), we have

$$\begin{aligned}\overline{S_k(T/2,T)}=&\frac{\tau\left(\overline{\psi_2}\right)\overline{\tau(\psi_1)}}{q\varphi(q)}\sum_{u\le\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}\\&\times\sum_{\frac{qT}{4\pi}<n\le\frac{qT}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi_1})\psi_1(n)}{(\log n)^K}\\&+O_{k,q}\left(\frac{T}{\log T}\right).\end{aligned}$$

By Lemma 2.15 with $x=\frac{qT}{2\pi}$, we get

$$\begin{aligned}\overline{S_k(T/2,T)}=&-\frac{\tau\left(\overline{\psi_2}\right)\overline{\tau(\psi_1)}}{\varphi(q)}\frac{T}{4\pi}\left(\sum_{u=0}^{\infty}-\sum_{u>\frac{\log T}{\log\log T}}\right)\\&\times\left[(-1)^u\sum_{v=0}^{k}\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\right.\\&\left.\times\prod_{w=1}^{k}\left\{(-1)^w w!\binom{k}{w}\right\}^{i_w}\frac{(-1)^v(v+1)!}{K!}\right]\\&+E_1+E_2+E_3+O_{k,q}\left(\frac{T}{\log T}\right),\quad\text{say.}\end{aligned}\tag{3.15}$$

By using Lemmas 2.14 and 2.15 and (2.42), we have

$$\begin{aligned}E_1&\ll_k\log^2 T\sum_{u\leqslant\frac{\log T}{\log\log T}}\sum_{v=0}^{k}\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}(A(k))^K\\&\ll_k(A(k))^{\frac{\log T}{\log\log T}}\log^2 T\sum_{u\leqslant\frac{\log T}{\log\log T}}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\\&=(A(k))^{\frac{\log T}{\log\log T}}\log^2 T\sum_{u\leqslant\frac{\log T}{\log\log T}}k^u\end{aligned}$$

$$
\begin{aligned}
&\ll_k (A(k))^{\frac{\log T}{\log\log T}} \log^2 T \\
&\ll_k T^{\epsilon},
\end{aligned}
$$

and

$$
\begin{aligned}
E_2 &\ll_k \frac{T}{\log T} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1,\ldots,i_k} \frac{(A(k))^K}{(K-1)!} \\
&\ll_k \frac{T}{\log T} \sum_{u \leqslant \frac{\log T}{\log\log T}} \frac{(A(k))^u}{(u-1)!} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1,\ldots,i_k} \\
&\ll_k \frac{T}{\log T} \left(1 + O\left(e^{A(k)}\right)\right) \\
&\ll_k \frac{T}{\log T},
\end{aligned}
$$

and

$$
\begin{aligned}
E_3 &\ll_k \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \sum_{u \leqslant \frac{\log T}{\log\log T}} \sum_{v=0}^{k} \binom{k}{v} \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1,\ldots,i_k} \left(\frac{A(k)}{(\log T)^{\frac{1}{3}-\epsilon}}\right)^K \\
&\ll_k \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \sum_{u \leqslant \frac{\log T}{\log\log T}} \left(\frac{A(k)}{(\log T)^{\frac{1}{3}-\epsilon}}\right)^u \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k \geq 0}} \binom{u}{i_1,\ldots,i_k} \\
&\ll_k \frac{T(\log T)^{2+3\epsilon}}{e^{\delta_1(k)(\log T)^{\frac{1}{3}-\epsilon}}} \frac{1}{1 - A(k)(\log T)^{-\frac{1}{3}+\epsilon}}.
\end{aligned}
$$

The sum over $u > \log T/\log\log T$ in (3.15) is

$$
\ll_k T \sum_{u > \frac{\log T}{\log\log T}} \frac{(A(k))^u}{u!} \ll T \frac{(A(k))^{\lceil \log T/\log\log T \rceil}}{\lceil \log T/\log\log T \rceil!} \ll T^{\frac{2\log\log\log T}{\log\log T}},
$$

by Stirling formula. Thus, we get

$$
\overline{S_k(T/2,T)} = \frac{\tau\left(\overline{\psi_2}\right)\overline{\tau(\psi_1)}}{\varphi(q)} \left[-\sum_{u=0}^{\infty} (-1)^u \sum_{v=0}^{k} \binom{k}{v} \right.
$$

$$\times \sum_{\substack{i_1+\ldots+i_k=u \\ i_1,\ldots,i_k\geq 0}} \binom{u}{i_1,\ldots,i_k} \prod_{w=1}^{k} \left\{(-1)^w w! \binom{k}{w}\right\}^{i_w} \frac{(-1)^v (v+1)!}{K!} \Bigg] \frac{T}{4\pi}$$

$$+ O_{k,q}\left(\frac{T}{\log T}\right)$$

$$= \frac{\tau\left(\overline{\psi_2}\right)\overline{\tau(\psi_1)}}{\varphi(q)} \frac{\mathcal{A}_k}{2} \frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right). \tag{3.16}$$

By (1.6) and (1.9), we have the trivial estimate

$$S_k(0, \sqrt{T}) \ll_{k,q} T^{\frac{3}{4}+\frac{\delta}{2}} \log T. \tag{3.17}$$

By using (3.16) with T replaced by $\frac{T}{2}$, $\frac{T}{4}$, ... down to almost $\sqrt{T}$ and adding them up, and also by using the trivial estimate (3.17), we obtain

$$\sum_{0<\gamma_{\psi_1,k}\leqslant T} \chi_{\psi_2}(\rho_{\psi_1,k}) = \frac{\overline{\tau\left(\overline{\psi_2}\right)\tau(\psi_1)}}{\varphi(q)} \mathcal{A}_k \frac{T}{2\pi} + O_{k,q}\left(\frac{T}{\log T}\right), \quad (T \to \infty),$$

upon GRH.

4. PROOF OF THEOREM 1.2

Let q be a fixed odd prime number and ψ be a non-principal primitive Dirichlet character modulo q. We want to estimate the sum

$$S_k(A,B) := \sum_{A<\gamma_{\psi,k}\leqslant B} \chi_\zeta(\rho_{\psi,k}),$$

for large A and $A < B \leqslant 2A$. By the residue theorem, we have

$$\begin{aligned} S_k(A,B) := \sum_{A<\gamma_{\psi,k}\leqslant B} \chi_\zeta(\rho_{\psi,k}) &= \frac{1}{2\pi i}\int_C \chi_\zeta(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi)\,ds \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where C can be taken as in the proof of Theorem 1.1, that is, C is the positively oriented boundary of the rectangle with vertices $\sigma'_{k,q}+iA$, $\sigma'_{k,q}+iB$, $-\delta+iB$, $-\delta+iA$ where $\sigma'_{k,q} > \sigma_{k,q} := 1+2\left(1+\sqrt{1+\frac{k^2}{\log 2}}\right)$, δ is a fixed real number with $0<\delta<\frac{1}{8}$ and

$$I_1 := \frac{1}{2\pi i}\int_{\sigma'_{k,q}+iA}^{\sigma'_{k,q}+iB} \chi_\zeta(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi)\,ds,$$

$$I_2 := \frac{1}{2\pi i}\int_{\sigma'_{k,q}+iB}^{-\delta+iB} \chi_\zeta(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi)\,ds,$$

$$I_3 := \frac{1}{2\pi i}\int_{-\delta+iB}^{-\delta+iA} \chi_\zeta(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi)\,ds,$$

$$I_3 := \frac{1}{2\pi i}\int_{-\delta+iA}^{\sigma'_{k,q}+iA} \chi_\zeta(s)\frac{L^{(k+1)}}{L^{(k)}}(s,\psi)\,ds.$$

Note that we can take horizontal sides of the contour C to be a distance of $\gg \frac{1}{\log A}$ from any zero of $L^{(k)}(s,\psi)$ by adjusting A and B by an amount of ≤ 1. In view of

(1.2), this means adding or deleting $O(\log A)$ number of terms each of size $\ll A^{\frac{1}{2}+\delta}$. Since $\frac{L^{(k+1)}}{L^{(k)}}(s,\psi) \ll (\log A)^2$ on the horizontal parts of C and $\frac{L^{(k+1)}}{L^{(k)}}(s,\psi) \ll 1$ on the right vertical line of C, again in view of (1.2), we see that the contribution from I_1, I_2 and I_4 is $\ll_{k,q} A^{\frac{1}{2}+\delta}\log^2 A$. Thus, we have $S_k(A,B) = I_3 + O_{k,q}(A^{\frac{1}{2}+\delta}\log^2 A)$. Now,

$$
\begin{aligned}
I_3 &= -\frac{1}{2\pi}\int_A^B \chi_\zeta(-\delta+it)\frac{L^{(k+1)}}{L^{(k)}}(-\delta+it,\psi)\,dt \\
&= \overline{-\frac{1}{2\pi}\int_A^B \chi_\zeta(-\delta-it)\frac{L^{(k+1)}}{L^{(k)}}(-\delta-it,\overline{\psi})\,dt} \\
&= \overline{-\frac{1}{2\pi}\int_A^B \chi_\zeta(1-(1+\delta+it))\frac{L^{(k+1)}}{L^{(k)}}(1-(1+\delta+it),\overline{\psi})\,dt} \\
&= \overline{-\frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \chi_\zeta(1-s)\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi})\,ds}.
\end{aligned}
$$

So,

$$
\overline{S_k(A,B)} = -\frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \chi_\zeta(1-s)\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi})\,ds + O_{k,q}\left(A^{\frac{1}{2}+\delta}\log^2 A\right). \tag{4.1}
$$

Now, we treat the factor $\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi})$ in (4.1) as in the proof of Theorem 1.1 and write

$$
\frac{L^{(k+1)}}{L^{(k)}}(1-s,\overline{\psi}) = -\left(\ell + \frac{G_k'}{G_k}(s,\ell,\overline{\psi})\right)\left(1+O\left(\frac{1}{|t|}\right)\right), \tag{4.2}
$$

where $\ell := \log\frac{q|t|}{2\pi}$, $\sigma' \le \sigma \le \sigma''$ for any fixed real numbers σ' and σ'' and

$$
\frac{G_k'}{G_k}(s,z,\psi) := \left(z+\frac{d}{dz}\right)^k L(s,\psi) = z^k L(s,\psi) + kz^k L'(s,\psi_1) + ... + L^{(k)}(s,\psi),
$$

and the differentiation in G_k' is with respect to s.

Then, we see that

$$\frac{G_k'}{G_k}(s,z,\overline{\psi}) = \frac{\sum_{v=0}^{k}\binom{k}{v}\frac{1}{z^v}\frac{L^{(v+1)}}{L}(s,\overline{\psi})}{1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{L^{(w)}}{L}(s,\overline{\psi})}. \tag{4.3}$$

Since $\frac{L^{(w)}}{L}(s,\overline{\psi}) \ll_w 1$ for $\sigma \geq 1+\delta$, we get

$$\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi}) \ll_k \frac{1}{\log A}, \tag{4.4}$$

and

$$\frac{G_k'}{G_k}(s,\ell,\overline{\psi}) \ll_k 1, \quad (\sigma \geq 1+\delta,\ A \leq t \leq B). \tag{4.5}$$

By writing (4.2) in (4.1) and then using Lemma 2.9 with $m = 1$ and $b_1 = 1$, $b_i = 0$, $i > 1$ and (4.5), we have

$$\overline{S_k(A,B)} = \frac{1}{2\pi i}\int_{1+\delta+iA}^{1+\delta+iB} \chi_\zeta(1-s)\frac{G_k'}{G_k}(s,\ell,\overline{\psi})\,ds + O_{k,q}(A^{\frac{1}{2}+\delta}\log^2 A). \tag{4.6}$$

By (4.4), for $\sigma \geq 1+\delta$, $A \leq t \leq B$, we can expand the denominator of $\frac{G_k'}{G_k}$ as a power series.

$$\begin{aligned}\left(1+\sum_{w=1}^{k}\binom{k}{w}\frac{1}{z^w}\frac{L^{(w)}}{L}(s,\overline{\psi})\right)^{-1} &= \sum_{u=0}^{\infty}(-1)^u\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi})\right)^u \\ &= \sum_{u\leq\frac{\log A}{\log\log A}}(-1)^u\left(\sum_{w=1}^{k}\binom{k}{w}\frac{1}{\ell^w}\frac{L^{(w)}}{L}(s,\overline{\psi})\right)^u \\ &\quad + O\left(\frac{1}{A}\right).\end{aligned} \tag{4.7}$$

By using (4.7) in (4.3), (4.6) reads as

$$\begin{aligned}\overline{S_k(A,B)} = &\sum_{u\le\frac{\log A}{\log\log A}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}\\ &\times\frac{1}{2\pi}\int_A^B\frac{\chi_\zeta(-\delta-it)}{\ell^K}\prod_{w=1}^{k}\left(\frac{L^{(w)}}{L}(1+\delta+it,\overline{\psi})\right)^{i_w}\frac{L^{(v+1)}}{L}(1+\delta+it,\overline{\psi})\,dt\\ &+O_{k,q}(A^{\frac12+\delta}\log^2 A),\end{aligned}$$

where $K := i_1 + 2i_2 + \ldots + ki_k + v$. By taking $A = \frac{T}{2}$, $B = T$ and using Lemma 2.9 with $m = -K$, we get,

$$\begin{aligned}\overline{S_k(T/2,T)} = &\sum_{u\le\frac{\log T}{\log\log T}}\sum_{v=0}^{k}(-1)^u\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}\\ &\times\sum_{\frac{T}{4\pi}<n\le\frac{T}{2\pi}}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi})}{(\log qn)^K}+O_{k,q}(T^{\frac12+\delta}\log^2 T),\end{aligned}\tag{4.8}$$

where $\sum_{n=1}^{\infty}\frac{b_n(i_1,\ldots,i_k;v;\overline{\psi})}{n^s} := \frac{L^{(v+1)}}{L}(s,\overline{\psi})\prod_{w=1}^{k}\frac{L^{(w)}}{L}(s,\overline{\psi})$ for $\sigma > 1$. Now, by (4.8) and Lemma 2.15 with $x = \frac{T}{2\pi}$, we obtain

$$\begin{aligned}\overline{S_k(T/2,T)} \ll_{k,q}\ &T^{1-\frac{1}{\log\log q(T+4)}}\sum_{u\le\frac{\log T}{\log\log T}}\sum_{v=0}^{k}\binom{k}{v}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\prod_{w=1}^{k}\binom{k}{w}^{i_w}\\ &\times\frac{(A(k)\log\log q(T+4))^{K+1}}{(\log T)^{K-1}}\\ \ll_{k,q}\ &T^{1-\frac{1}{\log\log q(T+4)}}\log T\log\log T\sum_{u\le\frac{\log T}{\log\log T}}\sum_{\substack{i_1+\ldots+i_k=u\\ i_1,\ldots,i_k\ge 0}}\binom{u}{i_1,\ldots,i_k}\\ &\times\left(\frac{(A(k)\log\log q(T+4))}{(\log T)}\right)^K\\ \ll_{k,q}\ &T^{1-\frac{1}{\log\log q(T+4)}}\log T\log\log T\sum_{u\le\frac{\log T}{\log\log T}}\left(\frac{(A(k)\log\log q(T+4))}{(\log T)}\right)^u\\ \ll_{k,q}\ &T^{1-\frac{1}{\log\log q(T+4)}}\log T\log\log T.\end{aligned}\tag{4.9}$$

By (1.2) and (1.9), we have the trivial estimate

$$S_k(0, \sqrt{T}) \ll_{k,q} T^{\frac{3}{4}+\frac{\delta}{2}} \log T. \tag{4.10}$$

By using (4.9) with T replaced by $\frac{T}{2}$, $\frac{T}{4}$, ... down to almost $\sqrt{T}$ and adding them up, and also by using the trivial estimate (4.10), we obtain

$$\sum_{0<\gamma_{\psi,k}\leqslant T} \chi_\zeta(\rho_{\psi,k}) \ll_{k,q} T^{1-\frac{1}{\log\log q(T+4)}} \log T \log\log T,$$

upon GRH.

5. CONCLUSION

Considering Theorem 1.1 and 1.2, the sum $\sum_{0<\gamma_{\psi_1,k}\leqslant T}\chi_{\psi_2}(\rho_{\psi_1,k})$ is calculated with a main term but for the sum $\sum_{0<\gamma_{\psi,k}\leqslant T}\chi_\zeta(\rho_{\psi,k})$, we do not obtain a main term. We conclude that, in general sense, for a fixed odd prime number q and non-principal Dirichlet characters ψ_1, ψ_2 that are of the same modulus q, the function χ_{ψ_2} is aware of the zeros $\rho_{\psi_1,k}$, however the function χ_ζ does not know much about these zeros.

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