Supersymmetry, black holes and holography in three dimensions

Alkaç, Gökhan

Supersymmetry, Black Holes and Holography in Three Dimensions

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by

Gökhan Alkaç

born on 24 March 1987
in Çorum, Turkey

Supervisor
Prof. E.A. Bergshoeff

Co-supervisor
Prof. D. Roest

Assessment committee
Prof. M.A.M van de Weijgaert
Prof. K. Stelle
Prof. S. Deger

To those who are brave enough to fight for truth and justice

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Contents

List of Abbreviations

ΛCDM	Λ-Cold Dark Matter
AdS	Anti-de Sitter
BD	Boulware-Deser
BIK	Born-Infeld type extension of K-gravity
BINMG	Born-Infeld type extension of NMG
BTZ	Banados-Teitelboim-Zanelli
CS	Chern-Simons
CFT	Conformal Field Theory
CNMG	NMG with a cosmological constant
EH	Einstein-Hilbert
FP	Fierz-Pauli
GR	General Relativity
GMG	Generalized Massive Gravity
NMG	New Massive Gravity
MMG	Minimal Massive Gravity
SM	Standard Model
TMG	Topologically Massive Gravity
vDVZ	van Dam-Veltman-Zakharov

Chapter 1

Introduction

1.1 Introduction

Among the four fundamental interactions that are known to exist in nature, it is probably gravitation that seems the most puzzling to the theoretical physicist. Our attempts at a theoretical understanding of gravitation began with Newton's inverse square law, which led to a successful description of the orbital motion of planets in our solar system. With the confirmation of the existence of a new planet predicted by Newton's theory, which we now call Neptune, it became the standard way to describe gravitational phenomena.

This remained the case until Einstein proposed a new theory of gravitation, the general theory of relativity (GR), where gravitation is seen as a manifestation of the curvature of spacetime generated by matter. In addition to explaining the well-known discrepancy in Mercury's orbit, Einstein's theory changed our view of space, time, and matter radically with its remarkable predictions such as the existence of black holes. However, when the theory was treated within the framework of quantum field theory, which was successfully employed for mathematically consistent formulations of the other three fundamental interactions, it was realized that it was non-renormalizable and therefore would lose its predictive power at high energies.

Presently, we have two different models that we use to describe our universe at the smallest and the largest scales. On the one hand, there is the Standard Model of particle physics (SM) which, under certain assumptions, explains the behavior of sub-atomic particles to an astonishingly high level of accuracy. On the other, there is the Standard Model of Big Bang Cosmology (ΛCDM), which utilizes GR at the cosmological scales with plenty of impressive phenomenological successes: To name just a few, we can list the explanation of the galactic rotation curves, the cosmic microwave background, and the accelerating expansion of our universe. Nevertheless, it is commonly believed that some fundamental principles are still missing in this picture for various reasons.

The problem of non-renormalizability of GR prevents us from embedding gravitation into the SM and having a complete description including all the fundamental interactions at the smallest possible scales. Furthermore, the ΛCDM predicts the existence of dark matter and dark energy, which seems to be problematic in several aspects. No direct experimental evidence for dark matter has been found yet, and this form of non-baryonic matter is not included in the SM. While the value of the cosmological constant introduced to account for the dark energy effects is predicted to be extremely small in the ΛCDM, the SM prediction in the form of Casimir energy turns out to be many orders of magnitude higher.

Many different approaches have been followed with the hope of shedding light on the issues we mentioned above. String theory, which has the virtue of unifying all the fundamental interactions, and loop quantum gravity are the two main attempts at the quantization of gravity. There also have been proposals to modify the SM and GR. It is the modifications of GR that will play a central role in the majority of discussions in this thesis. Therefore, we will present a review of the main developments, successes and problems of this approach in order to motivate further investigations.

A simple way to modify Einstein's gravity is to add higher curvature terms to the Einstein-Hilbert (EH) action. The resulting theory still possesses the same gauge symmetries since it is defined through a Lorentz invariant action. As an important result in terms of the issue of renormalizability of gravity, it was shown by Stelle that a renormalizable gravity theory can be obtained by adding higher curvature terms to the EH action. However, the resulting theory is non-unitary, giving rise to negative norm states in the scattering matrix [1,2]. As can be seen from this classical example, even though adding higher curvature terms can improve the renormalizability properties, there are serious restrictions on the theories due to unitarity considerations. One technical note that will be useful in our analysis later is that the tree-level unitarity is assured if the theory has no ghosts (particles with negative kinetic energy) and no tachyons (particles with $m^2 < 0$ in flat space) in its particle spectrum. This is the natural first step for constraining conceivable modified theories of gravity.

Considering the fluctuations of the metric around the flat space and taking the weak field limit in GR yield the unique self-consistent theory of a massless spin-2 particle, which we call a graviton, with the linearized diffeomorphisms as its gauge symmetry. One can also start from this unique field theory and consider all the possible higher order interaction terms respecting the gauge symmetries of the theory. It is possible to show that this procedure yields GR as the non-linear completion of the field theory of massless spin-2, revealing a fundamental connection between the gauge invariance and the equivalence principle [3–5]. From this field theory perspective, it seems natural then to consider the case of massive spin-2 and its implications at the non-linear level. This is the main idea of the Massive Gravity research program that has attracted considerable attention recently [6, 7]. The introduction of a massive graviton yields a Yukawa-type gravitational potential with a length scale determined by the graviton mass, which might potentially help solve the phenomenological problems related to gravity on galactic and cosmological scales.

The classical theory of massive spin-2 particles was first constructed by Fierz and Pauli [8]. A 4D massive spin-2 particle carries five local degrees of freedom which can be decomposed into helicity 0 (scalar), helicity ± 1 (vector) and helicity ± 2 (tensor) modes. It was shown by van Dam, Veltman [9], and independently by Zakharov [10] that it is not possible to recover the linearized GR, in the massless limit of the Fierz-Pauli (FP) theory because of the additional scalar mode. In the limit, one obtains a massless graviton from the tensor mode and the vector mode decouples. However, the scalar mode remains coupled to the trace of the energy-momentum tensor, resulting in discrepancies from the massless spin-2 theory. For example, the effective potential energy between two point sources in the massless limit of the FP theory is

$$U = -\frac{4}{3}\frac{Gm_1 m_2}{r}, \tag{1.1.1}$$

failing to recover the correct Newtonian potential because of the extra attraction produced by the scalar mode. This is the famous van Dam-Veltman-Zakharov (vDVZ) discontinuity.

A resolution of the vDVZ discontinuity was offered by Vainshtein, who showed that higher order interactions in the massive spin-2 theory leads to an effective screening of the scalar mode that confines the scalar fluctuations within a region defined by the Vainshtein radius [11]. In the case of the FP theory, the Vainshtein radius diverges and the effect of the scalar mode is seen at arbitrarily large distances, which is nothing but an artifact of the linearized theory that disappears, when the non-linear effects are taken into account. Along with these developments, however, Boulware and Deser considered the most general tensor theory of gravity as a non-linear completion of the FP theory and proved that six local degrees of freedom propagate in such theories where one extra scalar degree of freedom arising from the nonlinear effects is always a ghost, known as the Boulware-Deser (BD) ghost [12]. Until recently, the conclusion was that it is not possible to realize the idea of massive gravity at the nonlinear level without this scalar ghost mode, and therefore, that it was not possible to construct any phenomenological model in a mathematically consistent way.

The first advance regarding the existence of the scalar ghost was achieved through the work of de Rham, Gabadadze and Tolley, who analyzed the problem in the so-called decoupling limit which makes the dynamics of the scalar mode and its coupling to the tensor mode more manifest. After introducing a generalization of the FP theory where the ghost mode can be removed up to the fifth order in nonlinearities [13], they managed to obtain a nonlinear theory of massive gravity which is free of the BD ghost to all orders [14]. The generalization of this result beyond the decoupling limit was achieved in a series of papers by Hassan and Rosen: They first provided a systematic construction of nonlinear massive gravity actions where the interaction term is formed out of elementary symmetric polynomials of the square root matrix $\sqrt{g^{-1}f}$, $f_{\mu\nu}$ being the reference metric that is used to put the interactions into the required form [15]. Later, they produced a proof of the removal of the scalar ghost in nonlinear massive gravities to all orders and beyond the decoupling limit, by making use of this construction [16]. Although the reference metric was taken to be a Minkowski metric in this analysis, it was shown that the result holds for a generic choice of reference metric [17]. Along with the invention of a bimetric massive gravity model with a dynamical reference metric [18], the idea of massive gravity has again become an active field of research with potential applications especially in cosmology (see reviews [6,7] for recent developments).

In this thesis, we will follow a different approach and study various gravitational theories in three spacetime dimensions (3D) with different applications in mind. The main idea will be that although the phenomelogical motivations mentioned previously are lost, one can use 3D gravity theories as a theoretical laboratory to improve our understanding of certain physical ideas in a relatively simpler setup. In the next section, we will review the main developments that motivate our study of gravitational theories in 3D.

1.2 Why Three Dimensions?

The first property of Einstein's gravity in 3D that will be of interest to us is its local triviality, which can be seen from the decomposition of the Riemann tensor in terms of the Ricci tensor and the metric as

$$R_{\mu\nu\rho\sigma} = 2(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) - R\, g_{\mu[\rho}g_{\sigma]\nu}. \tag{1.2.1}$$

This decomposition is possible in 3D since the Riemann tensor and the Ricci tensor have $\frac{1}{12}d(d-1)$ and $d(d+1)$ independent components on a d-dimensional spacetime and the two match for $d = 3$. One immediate consequence is that for any Einstein space, $R_{\mu\nu} = \lambda g_{\mu\nu}$ with constant λ, the curvature of the spacetime is directly determined by the local matter distribution and the constant λ. Hence, there are no dynamical degrees of freedom that can be propagated through gravitational waves. The theory can also be formulated as a topological gauge theory in the Chern-Simons form [19, 20].

Despite being locally trivial, 3D Einstein gravity with a negative cosmological constant has been shown to admit a black hole solution, the Banados-Teitelboim-Zanelli (BTZ) black hole [21]. The solution is locally indistinguishable from Anti-de Sitter (AdS) spacetime but possesses all the properties of a black hole. It has a well-defined event horizon and it is characterized by only its mass and angular momentum, which are manifested as global charges and make it globally different than AdS spacetime.

Another very interesting property of 3D Einstein gravity with a negative cosmological constant was revealed by Brown and Hanneaux. They showed that the global charges corresponding to the asymptotic symmetries of the theory yield a Poisson bracket algebra with a central extension. Under certain boundary conditions, the resulting algebra turns out to be two copies of the Virasoro algebra, which is the algebra satisfied by the generators of the local conformal transformations on the Hilbert space of 2D Conformal Field Theories (CFT) [22]. Hence, it demonstrates an intriguing relationship between the dynamics of the classical graviton in 3D and the representations in the Hilbert space of a 2D CFT, which is quantum mechanical. The holographic nature of this result is very notable and it is now considered an early sign of the AdS/CFT correspondence [23, 24], which is based on the ideas introduced by 't Hooft and Susskind [25, 26].

The holographic principle in its modern formulation suggests an exact equivalence of a quantum gravity theory on the d-dimensional AdS spacetime and a gauge theory defined on the (d-1)-dimensional conformal boundary of the AdS spacetime. It is a weak/strong-type duality, which makes it possible to study both theories in different regimes. Although the idea was made concrete as the duality between type-IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}_5$ and the $\mathcal{N} = 4$ super Yang-Mills theory in the large-N_c limit, it is conjectured to hold for general gravity theories on AdS spacetime. By studying a weakly coupled theory, one can extract information about a strongly coupled theory that is conjectured to be dual to it. This approach opens up a new door to a better understanding of strongly coupled gauge theories, which has found many applications in very different areas.

Since the algebra of local conformal transformations in 2D are infinitedimensional, 2D CFTs are much easier to analyze compared to higher dimensional cases. They also find an application in string theory where the excitations of the string are described by a 2D CFT from the world-sheet perspective. Therefore, they are the most studied and best-known examples of field theories with conformal symmetry, which makes the study of holography in 3D gravity theories particularly worthwhile. One example that should be specifically mentioned is Strominger's computation of the Bekenstein-Hawking entropy of the BTZ black hole using the asymptotic growth of the number states of a 2D CFT [27]. This result reflects the fact that the holographic principle can yield surprisingly useful results even in the cases where the details of the dual boundary field theory are not known. Without any direct reference to string theory or supersymmetry, the assumption of a gravity theory on AdS_3 with a 2D CFT dual provides a microscopic explanation for the entropy of the BTZ black hole.

One other advantage of working in 3D is the possibility of constructing theories with a massive spin-2 mode which are unitary. The first example of such theories was obtained by adding a gravitational Chern-Simons term to the Einstein-Hilbert action. The resulting theory, called Topologically Massive Gravity (TMG), has third order field equations describing a single helicity +2 or -2 mode, depending on the sign of the Chern-Simons coupling [28, 29]. This is a natural result of the fact that the action is supplemented by a parity odd term. In order to ensure the positivity of the energy of this mode, the Einstein-Hilbert term must be used with the opposite sign, as compared to the usual Einstein gravity, which is commonly called the "wrong sign" in the literature.

A theory of 3D massive gravity without parity violating terms was constructed by Bergshoeff, Hohm and Townsend [30, 31]. This theory, called New Massive Gravity (NMG), is obtained by the addition of a certain combination of curvature squared terms ($K = R_{\mu\nu}^2 - \frac{3}{8}R^2$) to the Einstein-Hilbert term, leading to fourth order field equations. The theory has two helicity ± 2 massive mode since it preserves parity. In addition to the massive mode, the theory also has a massless mode as a result of its field equations being fourth order in derivatives. Like its 4D analogues, the theory suffers from an Ostrogadski-type instability, that is, kinetic terms for the massive and the massless modes carry opposite signs in the action, and this implies that one of them is a ghost. Since the massless mode does not propagate any degrees of freedom in 3D, one can use the wrong sign for the Einstein-Hilbert term and ensure the unitarity of the massive mode. This is a special feature that cannot be achieved in higher dimensions.

Although both TMG and NMG can be made unitary on flat spacetime by choosing the wrong sign for the Einstein-Hilbert term, there still appear some problems when the theories are defined on AdS spacetime. The BTZ black hole becomes a solution of these theories and the wrong sign of the massless mode make the mass of the BTZ black hole, a global charge still dependent on the massless mode, negative. Additionally, making the bulk theory unitary yields a negative central charge for the boundary CFT, resulting in negative norm states corresponding to the boundary graviton modes. The resolution of the unitarity problem in the bulk is in direct conflict with the unitarity of the boundary CFT.

In order to solve the bulk-boundary unitarity conflict in higher derivative extensions of Einstein's gravity in 3D, an extension of TMG, Minimal Massive Gravity (MMG), was proposed [32]. The theory was obtained through a Chern-Simons-like formulation where the Lagrangian density is formed by a dreibein, a spin connection and additional auxiliary fields. The field equations can be written by using the metric alone but it is not possible to obtain the field equations from a Lorentz invariant action formed only by the metric. It is a very special type of theory where the field equations are covariantly conserved only on-shell. It was shown that there is a region at the parameter space of this theory where both the bulk and the boundary theories are unitary.

The theory defined by the pure quadratic part of NMG exhibits interesting properties. It was first shown in [33] to possess a single, massless propagating mode. This result was also confirmed in [31], where the authors showed that the linearized action of the theory reduces to the Maxwell action for a vector field, which is on-shell equivalent to a massless scalar in 3D. Since the action of the theory is formed by the scalar ($K = R_{\mu\nu}^2 - \frac{3}{8}R^2$), it is referred to as K-gravity. The study of exact solutions of K-gravity led to the first black hole solutions in 3D gravity which are asymptotically locally flat, and which can also be deformed to have a non-spherically symmetric horizons, leading to the so-called "black flower" solutions [34].

A Born-Infeld type extension of NMG (BINMG) was also invented in [35]. It is defined through an action that consists of the determinant of the cosmological Einstein tensor. When expanded in the small curvature, it results in the NMG at the quadratic order. Remarkably, it also yields deformations of NMG at the cubic and the fourth order which are obtained by demanding the existence of a holographic c-theorem [36, 37].

For an extensive discussion of Einstein's gravity in three dimensions, with many details of classical and quantum aspects, we refer the interested readers to the book [38].

1.3 Outline of This Thesis

In Chapter 2, some background material that is useful to understand the later chapters is provided. This includes a review of the unitarity analysis of different 3D modified gravity models, a brief introduction to the conformal construction of supergravity theories, the main aspects of classical supersymmetric solutions of gravity theories, and the algebraic classification of 3D spacetimes.

In Chapter 3, we investigate the $\mathcal{N} = 2$ supersymmetric extension of 3D gravity theories with actions up to quadratic curvature invariants, which includes Einstein's gravity, TMG, NMG, and K-gravity as special cases. The $\mathcal{N} = 2$ supersymmetry allows for two different off-shell formulations of supergravity, $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ formulations, which yield different physical properties for theories with higher derivative terms. After presenting the construction of all the supersymmetric invariants, we show that the unitarity on the AdS background survives only in the $\mathcal{N} = (1,1)$ case and the supersymmetric vacuum does not provide any improvement regarding the non-unitarity of the boundary CFT.

Chapter 4 is devoted to supersymmetric solutions of the $\mathcal{N} = (1,1)$ NMG. We show that, contrary to the $\mathcal{N} = 1$ case, which supports only solutions with a null Killing vector, the $\mathcal{N} = (1,1)$ supersymmetry allows for solutions with a timelike Killing vector, including different deformed AdS backgrounds and a Lifshitz spacetime. Finally, the supersymmetry properties of some black hole solutions of NMG are discussed.

In Chapter 5, we study the asymptotically locally flat black hole solutions of K-gravity from a different perspective. By using the decomposition of the field equations into two natural tensors, we classify all the possible conformally flat solutions of the theory and provide new types of solutions. We also show that the known solutions are still solutions of the Born-Infeld type and the Chern-Simons type extensions of the theory. Furthermore, the modification of the conserved charges and the thermodynamical quantities in these extensions will be discussed.

In Chapter 6, we provide a summary and give some possible directions for future research.

Chapter 2

Background Material

In this chapter, our purpose is to set the stage for the following chapters. In Section 2.1, we introduce the most studied modified gravity theories in 3D and we review the unitarity analysis around flat spacetime background. Section 2.2 is an introduction to the method we employ to construct supergravity theories in Chapter 3. In Section 2.3, some basic concepts regarding the classical solutions of supergravity theories is presented, which is a preparation for Chapter 4. Section 2.4 introduces the algebraic classification of 3D spacetimes which we will make use of in Chapter 5.

2.1 3D Gravity Theories

2.1.1 Unitarity around Flat Spacetime

We will introduce various 3D gravity theories by reviewing the canonical analysis on flat spacetime since it gives an opportunity to summarize some of their basic properties. This analysis was performed in [31,39] and we will mainly follow the discussion and the notation of [39]. Let us consider the following most general Lorentz invariant action with at most four derivatives of the metric,

$$I = I_{EH} + I_{CS} + I_{\alpha+\beta}, \tag{2.1.1}$$

with

$$\begin{aligned} I_{EH} &= \frac{1}{\kappa}\int d^3x\,\sqrt{-g}\,R, \\ I_{CS} &= -\frac{1}{2\mu}\int d^3x\,\sqrt{-g}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\lambda\sigma}\left(\partial_\mu\Gamma^{\sigma}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\beta}{}_{\nu\rho}\right) \\ I_{\alpha+\beta} &= \int d^3x\,\sqrt{-g}\,\left(\alpha R^2 + \beta R^2_{\mu\nu}\right). \end{aligned} \tag{2.1.2}$$

The first term is the usual Einstein-Hilbert (EH) term giving rise to the Einstein equations. The second term is the gravitational Chern-Simons (CS) term in 3D and the second term is a combination of quadratic curvature invariants with arbitrary coefficients. We will investigate the fluctuations around the Minkowski spacetime by expanding the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.1.3}$$

where $\eta_{\mu\nu}$ is the metric of Minkowski spacetime. The fluctuation $h_{\mu\nu}$ is an arbitrary symmetric tensor with 6 independent components in 3D. The action for $h_{\mu\nu}$ can be easily obtained by linearizing the corresponding field equations and then integrating them as

$$\begin{aligned} I_{EH} &= -\frac{1}{2\kappa}\int d^3x\, h_{\mu\nu}\,\mathcal{G}_L^{\mu\nu}, \\ I_{CS} &= -\frac{1}{2\mu}\int d^3x\,\epsilon_{\mu\alpha\beta}\,\mathcal{G}_L^{\alpha\nu}\partial^\mu h^\beta{}_\nu \\ I_{\alpha+\beta} &= -\frac{1}{2}\int d^3x\, h_{\mu\nu}\left[(2\alpha+\beta)\left(\eta^{\mu\nu}\Box-\partial^\mu\partial^\nu\right)R_L+\beta\Box\mathcal{G}_L^{\mu\nu}\right]. \end{aligned} \tag{2.1.4}$$

The linearized Einstein and Ricci tensors, and curvature scalar are given as

$$\mathcal{G}_L^{\mu\nu} = R_L^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}R_L, \qquad R_L=\partial_\alpha\partial_\beta h^{\alpha\beta}-\Box h,$$
$$R_L^{\mu\nu}=\frac{1}{2}\left(\partial_\sigma\partial^\mu h^{\nu\sigma}+\partial_\sigma\partial^\nu h^{\mu\sigma}-\Box h^{\mu\nu}-\partial^\mu\partial^\nu h\right), \qquad h=\eta^{\mu\nu}h_{\mu\nu}, \tag{2.1.5}$$

where $\Box=\partial_\mu\partial^\mu=-\partial_0^2+\nabla^2$ and the indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. As will be seen soon, it is useful to decompose $h_{\mu\nu}$ in terms of the following six arbitrary functions of $(t,\vec{x})$,

$$\begin{aligned} &h_{ij}\equiv\left(\delta_{ij}+\hat{\partial}_i\hat{\partial}_j\right)\phi-\hat{\partial}_i\hat{\partial}_j\chi+\left(\epsilon_{ik}\hat{\partial}_k\hat{\partial}_j+\epsilon_{jk}\hat{\partial}_k\hat{\partial}_i\right)\xi, \\ &h_{0i}\equiv-\epsilon_{ij}\partial_j\eta+\partial_i N_L, \qquad h_{00}\equiv N, \end{aligned} \tag{2.1.6}$$

where $\hat{\partial}_i\equiv\partial_i/\sqrt{-\nabla^2}$. This will allow us to identify the auxiliary fields and the dynamical fields easily. Since the linearized action is invariant under the gauge transformations $\delta_\zeta h_{\mu\nu}=\partial_\mu\zeta_\nu+\partial_\nu\zeta_\mu$, which are the linearized diffeomorphisms, we should work with the gauge invariant quantities. As explained in [39], the field ϕ is already gauge invariant and we have two more gauge invariant quantities that can be formed out of the arbitrary functions introduced in (2.1.6) as

$$q\equiv\nabla^2N-2\nabla^2\dot{N}_L+\ddot{\chi}, \qquad \varphi\equiv\dot{\xi}-\nabla^2\eta. \tag{2.1.7}$$

Now, with the help of the set of gauge invariant quantities (ϕ,q,φ), it is easy to identify the dynamical degrees of freedom and study the unitarity of the theories defined in (2.1.2). In terms of these gauge invariant combinations, the linearized actions (2.1.1) become

$$
\begin{aligned}
I_{EH} &= \frac{1}{2\kappa}\int d^3x\,\left(\phi q+\varphi^2\right),\\
I_{CS} &= \frac{1}{2\mu}\int d^3x\,\varphi\left(q+\Box\phi\right),\\
I_{\alpha+\beta} &= \frac{1}{2}\int d^3x\,\left[\left(2\alpha+\beta\right)\left(q-\Box\phi\right)^2+\beta\left(q\Box\phi+\frac{1}{2}\,\varphi\Box\varphi\right)\right].
\end{aligned}
\tag{2.1.8}
$$

Elimination of the free fields from the first two actions show that the EH term and the CS term do not propagate any degrees of freedom when they are alone. The most natural extension is to consider the combination of the two, which yields the Topologically Massive Gravity (TMG) which was introduced in [28, 29]. The resulting action, after the elimination of free fields, is

$$
I_{TMG}=-\frac{1}{2\kappa}\int d^3x\,\phi\Box\phi-\frac{\mu^2}{\kappa^2}\phi, \tag{2.1.9}
$$

which shows that the action describes the propagation of a single massive mode of helicity $+2$ or -2. For the unitarity, we should ensure the absence of ghosts and tachyons (see Section 1.1 for definitions) and it yields

$$
\kappa<0,\qquad m_g^2=-\frac{|\mu|}{\kappa}>0, \tag{2.1.10}
$$

which is the origin of the "wrong sign" of the Einstein-Hilbert term mentioned in the Introduction. Note that depending on the sign of the parity violating Chern-Simons term, the theory describes only one of the possible helicity states

Another extension of Einstein's theory with dynamical degrees of freedom can be obtained by combining the Einstein-Hilbert term and the four-derivative term in (2.1.1). For $2\alpha+\beta\neq 0$, it is possible to eliminate the field q and end up with the action

$$
\begin{aligned}
I &= I_\phi+I_\varphi,\\
I_\phi &= \frac{1}{2}\int d^3x\,\left[\frac{\beta\left(8\alpha+3\beta\right)}{4\left(2\alpha+\beta\right)}\left(\Box\phi\right)^2+\frac{\left(4\alpha+\beta\right)}{2\kappa\left(2\alpha+\beta\right)}\phi\Box\phi-\frac{1}{4\kappa^2\left(2\alpha+\beta\right)}\phi^2\right],\\
I_\varphi &= \frac{\beta}{2}\int d^3x\,\left(\varphi\Box\varphi+\frac{1}{\kappa\beta}\varphi^2\right),
\end{aligned}
\tag{2.1.11}
$$

from which one can immediately see that for the combination $8\alpha+3\beta=0$ the four-derivative term disappears and the theory, as expected from a parity invariant theory, describes the two helicity degrees of freedom of a massive spin-2 field in 3D. The resulting theory, called New Massive Gravity [30,31], has the following linearized action

$$\begin{aligned} I_{NMG} &= I_\phi + I_\varphi, \\ I_\phi &= -\frac{1}{2\kappa}\int d^3x \left(\phi\Box\phi + \frac{1}{\kappa\beta}\phi^2\right), \\ I_\varphi &= \frac{\beta}{2}\int d^3x \left(\varphi\Box\varphi + \frac{1}{\kappa\beta}\varphi^2\right). \end{aligned} \tag{2.1.12}$$

The conditions for the unitarity are

$$\kappa < 0, \qquad \beta > 0, \qquad m_\phi^2 = m_\varphi^2 = \frac{1}{\kappa\beta} > 0. \tag{2.1.13}$$

As we see, the wrong-sign for the EH term is required again.

When the most general combination of the terms given in (2.1.2) are considered, one sees that the combination $8\alpha + 3\beta = 0$ is again forced to remove the higher derivative term in the linearized action and one obtains the Generalized Massive Gravity (GMG) with the following linearized action

$$\begin{aligned} I_{GMG} &= \frac{\beta}{2}\int d^3x \left[\varphi\Box\varphi - \left(m_g^2 + \frac{1}{\mu^2\beta^2}\right)\varphi^2\right] + \frac{2m_g^2}{\beta\mu}\varphi\phi \\ &\quad + m_g^2\left(\phi\Box\phi - m_g^2\phi^2\right), \end{aligned} \tag{2.1.14}$$

with $m_g^2 = \frac{1}{\kappa\beta}$ and it can be diagonalized to obtain

$$I_{GMG} = \frac{\beta}{2}\int d^3x \left(\Psi_+\Box\Psi_+ - m_+^2\Psi_+^2 + \Psi_-\Box\Psi_- - m_-^2\Psi_-^2\right), \tag{2.1.15}$$

where

$$m_\pm^2 = m_g^2 + \frac{1}{2\mu^2\beta^2} \pm \frac{1}{\mu\beta}\sqrt{m_g^2 + \frac{1}{4\mu^2\beta^2}}. \tag{2.1.16}$$

Unitarity is ensured if

$$\kappa < 0, \qquad \beta > 0, \qquad m_\pm^2 > 0. \tag{2.1.17}$$

Note that since the theory is now parity violating, the $+2$ and the -2 helicity modes have different masses.

The theory defined by only the quadratic curvature invariants given in (2.1.2) with the choice $8\alpha + 3\beta = 0$ has also interesting properties [33], some of which will be studied in Chapter 4. Since its action is defined by the curvature invariant $K = R_{\mu\nu}^2 - \frac{3}{8}R^2$, we will refer to it as K-gravity. Its linearized action is

$$I_K = \frac{\beta}{2}\int d^3x\, \varphi\Box\varphi, \tag{2.1.18}$$

which describes the propagation of a massless excitation around flat spacetime and it is unitary provided that $\beta > 0$.

2.1.2 Bulk-Boundary Unitarity Problem: NMG as an Example

We will present the unitarity properties of NMG on an AdS spacetime and discuss the problems related to its CFT dual. We start with the following action for NMG with a cosmological term (CNMG)

$$I_{CNMG} = \frac{1}{\kappa}\int d^3x\,\sqrt{-g}\left[\sigma R + \frac{1}{m^2}K - 2\lambda m^2\right], \tag{2.1.19}$$

where

$$K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2, \tag{2.1.20}$$

which corresponds to the choices

$$\beta = \frac{1}{\kappa m^2}, \qquad\qquad 8\alpha + 3\beta = 0 \tag{2.1.21}$$

in the action (2.1.2) we considered before. The parameter $\sigma = \pm 1$ keeps track of the sign of Newton's constant and he cosmological term is introduced with the dimensionless parameter λ. The analysis of the unitarity is more involved on an AdS spacetime and it leads to richer possibilities as first shown in [31], whose main results will be summarized now. The fluctuations around the AdS spacetime is studied by expanding the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{2.1.22}$$

where $\bar{g}_{\mu\nu}$ is the metric of AdS_3. As a maximally symmetric spacetime, it satisfies

$$\bar{R}_{\mu\nu\rho\sigma} = \Lambda(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\rho}\bar{g}_{\nu\sigma}), \tag{2.1.23}$$

where the proportionality constant is determined by the AdS radius as $\Lambda = -\frac{1}{\ell^2}$. It is a solution of the CNMG defined by (2.1.19) if

$$\Lambda = -2m^2 \left[\sigma \pm \sqrt{1+\lambda}\right]. \tag{2.1.24}$$

The theory describes the propagation of a massive graviton with the mass

$$m_g^2 = -\sigma m^2 + \frac{\Lambda}{2}, \tag{2.1.25}$$

which reduces to the flat space result 2.1.13 in the limit $\ell \to \infty$ as expected. The unitarity is achieved for the following choice of parameters:

- $m^2 > 0 \qquad \sigma = -1 \qquad \Lambda > -2m^2 \qquad 0 < \lambda < 3$
- $m^2 < 0 \qquad \sigma = +1 \qquad \Lambda > 2m^2 \qquad \lambda > 3.$

The "right sign" for the EH term is allowed this time as long as the constraints on the AdS radius and the cosmological parameter are satisfied.

As first shown in the case of EH theory [22], the gravitational theories with an AdS$_3$ vacuum admit an asymptotic symmetry group at the boundary which consists of two copies of the Virasoro algebra, the symmetry algebra for 2D CFT. The norm of the states corresponding to the boundary gravitons are guaranteed to be positive if the central charge of the algebra is positive, which is required for a well-defined boundary field theory. The central charge is also related to the entropy of the BTZ black hole through the Cardy formula

$$S = \frac{A_{BTZ}}{4G_3}\Omega, \tag{2.1.26}$$

where the 3D Newton's constant is related to our parameter by $\kappa = 16\pi G_3$ and A_{BTZ} is the standard horizon area of the BTZ black hole. The dimensionless parameter Ω is proportional to the central charge c as

$$\Omega = \frac{2G_3}{3\ell}c, \tag{2.1.27}$$

which is equal to 1 for Einstein's theory with a cosmological constant. Therefore, the positivity of the central charge is also required for the positivity of the entropy of BTZ black hole.

For any parity preserving higher-derivative gravity theory, the central charge c can be determined from the formula

$$c = \frac{\ell}{2G_3} g_{\mu\nu} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}}, \qquad (2.1.28)$$

where $\mathcal{L}$ is the Lagrangian density without the prefactor $\frac{1}{\kappa}$. It gives the following value of the central charge for the CNMG theory defined by the Lagrangian (2.1.19)

$$c = \frac{3\ell}{2G_3} \left(\sigma - \frac{\Lambda}{2m^2} \right) = \frac{3\ell}{2G_3} \left(\sigma + \frac{1}{2m^2 \ell^2} \right), \qquad (2.1.29)$$

which should be positive for the unitarity of the boundary CFT and the positivity of the entropy of the BTZ black hole.

For the first choice of the parameters $(m^2, \sigma = -1)$, the central charge is positive for $\Lambda < -2m^2$. However, it violates the condition for the unitarity of the bulk theory. For the second choice $(m^2, \sigma = -1)$, the positivity of the central charge together with the expression (2.1.24) yields

$$2 > \sqrt{1 + \lambda}, \qquad (2.1.30)$$

which is true for

$$0 < \lambda < 3, \qquad (2.1.31)$$

conflicting with the condition of the unitarity of the bulk theory. Therefore, there is no choice of parameters for which both the bulk and the boundary theory are unitary. Although we discussed the case of NMG here, this conflict of bulk-boundary unitarity is a generic feature of modified gravity theories we discussed in the previous subsection, which include TMG and GMG.

2.2 Conformal Construction of Supergravity Theories

In Chapter 3, we will present the supersymmetrization of higher curvature theories defined by the action (2.1.2). Since the field equations are much more complicated compared to Einstein's theory, we will employ an off-shell formulation where the algebra closes on the fields without imposing field equations. A particularly effective method for obtaining off-shell supergravity theories is the superconformal method, where one starts from a conformally extended version of the superalgebra that is desired to be imposed on the theory. In this section. we will review this technique by following the textbook [40], where a detailed account with many applications can be found.

2.2.1 Conformal Symmetry

We define a conformal transformation as a coordinate transformation which changes the line element of Minkowski space $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ only by a scale factor. For an infinitesimal coordinate transformation $x^{'\mu} = x^\mu - \xi^\mu$ in D-dimensions, this implies the conformal Killing equation for the Minkowski metric

$$\mathcal{L}_\xi\, \eta_{\mu\nu} = f(x)\eta_{\mu\nu}, \tag{2.2.1}$$

where the Lie derivative of the Minkowski metric along the direction of ξ is proportional to metric itself and the scale factor $f(x)$ can be obtained from the trace of the equation, yielding

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = \frac{2}{D}\,(\partial\cdot\xi)\eta_{\mu\nu}, \tag{2.2.2}$$

In 2D, this equation has infinitely many solutions giving rise to the Virasoro algebra of string theory. For $D > 2$, the most general solution is given by

$$\xi^\mu(x) = a^\mu + \lambda^{\mu\nu}x_\nu + \lambda_D x^\mu + \left(x^2\lambda_K^\mu - 2x^\mu x\cdot\lambda_K\right), \tag{2.2.3}$$

Note that if we require the invariance of the metric and set the scale factor to zero, we only get the first two terms, which are the D translations a^μ and the $\frac{D(D-1)}{2}$ Lorentz transformations $\lambda^{\mu\nu}$ of the Poincaré group. In addition to these, we now have one scale transformation (dilatation) λ_D and the D special conformal transformations λ_K^μ. A general infinitesimal conformal transformation can be written in terms of the transformation parameters and generators as

$$\delta(\epsilon) = \epsilon^A T_A = a^\mu P_\mu + \frac{1}{2}\lambda^{\mu\nu}M_{\mu\nu} + \lambda_D D + \lambda_K^\mu K_\mu, \tag{2.2.4}$$

where the effect of each generator can be explicitly read off from the coordinate transformation (2.2.3). By computing the Lie bracket of the transformations, one obtains the conformal algebra with the following non-vanishing commutators

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= 4\eta_{[\mu[\rho}M_{\sigma]\nu]}, & [P_\mu, M_{\nu\rho}] &= 2\eta_{\mu[\nu}P_{\rho]},\\
[K_\mu, M_{\nu\rho}] &= 2\eta_{\mu[\nu}K_{\rho]}, & [P_\mu, K_\nu] &= 2\left(\eta_{\mu\nu}D + M_{\mu\nu}\right),\\
[D, P_\mu] &= P_\mu, & [D, K_\mu] &= -K_\mu.
\end{aligned} \tag{2.2.5}$$

This is isometric to the $SO(D,2)$ algebra with the definition

$$M^{\hat{\mu}\hat{\nu}} = \begin{pmatrix} M^{\mu\nu} & \frac{1}{2}(P^\mu - K^\mu) & \frac{1}{2}(P^\mu + K^\mu) \\ -\frac{1}{2}(P^\nu - K^\nu) & 0 & -D \\ -\frac{1}{2}(P^\nu + K^\nu) & D & 0 \end{pmatrix}, \tag{2.2.6}$$

where the incides are raised by the $SO(D,2)$ metric

$$\hat{\eta} = \mathrm{diag}\,(-1, 1, \ldots, 1, -1) \tag{2.2.7}$$

2.2.2 Gauge Fields and Constraints

Our aim now is to realize the conformal symmetry as a set of transformations acting on a set of fields which live on a dynamical spacetime. In order to describe the dynamics of the spacetime, we introduce the vielbein through the relation

$$g_{\mu\nu}(x) = e_\mu^a(x)\eta_{ab}e_\nu^a(x), \tag{2.2.8}$$

where η_{ab} is the metric of D-dimensional Minkowski spacetime. Any x-dependent matrix $\Lambda^a{}_b(x)$ which preserves the metric defines a local Lorentz transformation.

In theories of gravity, we use local indices for transformation parameters and generators and interpret the vielbein e_μ^a as the gauge field of local coordinate transformations with the parameter $\xi^a = e_\mu^a \xi^\mu$. By gauging also the other symmetries contained in the conformal group defined by (2.2.5), one obtains the following transformation rules for the gauge fields.

$$\begin{aligned} \delta e_\mu^a &= -\lambda^{ab} e_{\mu b} - \lambda_D e_\mu^a, \\ \delta \omega_\mu{}^{ab} &= \partial_\mu \lambda^{ab} + 2\omega_{\mu c}{}^{[a}\lambda^{b]c} - 4\lambda_K^{[a} e_\mu^{b]}, \\ \delta b_\mu &= \partial_\mu \lambda_D + 2\lambda_K^a e_{\mu a}, \\ \delta f_\mu^a &= \partial_\mu \lambda_K^a - \lambda_K^a b_\mu + \omega_\mu{}^{ab}\lambda_{Kb} - \lambda^{ab} f_{\mu b} + \lambda_D f_\mu^a. \end{aligned} \tag{2.2.9}$$

Now, the generators (P_a, M_{ab}, D, K_a) carry local indices (if possible) and they correspond to local transformation parameters $\left(\xi^a, \lambda^{ab}, \lambda_D, \lambda_K^a\right)$ with corresponding gauge fields $\left(e_\mu^a, \omega_\mu{}^{ab}, b_\mu, f_\mu^a\right)$ respectively. The Lorentz transformations easily follow from the index structure. For the dilatations, it is common to refer to the scale factor of a field as its Weyl weight. For example, the gauge fields we consider here have the Weyl weights $(-1, 0, 0, 1)$.

We want to end up with a gravity theory where the only independent field is vielbein. Therefore, other gauge fields should be determined by imposing constraints or gauge fixing. The gauge field of local Lorentz transformations in Einstein's theory, the spin connection ${\omega_\mu}^{ab}$, is a composite field determined by the vielbein e_μ^a. In order to achieve this, we will impose the following constraint on the curvature of translations as

$$R_{\mu\nu}(P^a) = 2\left(\partial_{[\mu} + b_{[\mu}\right) e^a_{\nu]} + 2{\omega_{[\mu}}^{ab} e_{\nu]b} = 0, \tag{2.2.10}$$

whose solution is

$$\begin{aligned} {\omega_\mu}^{ab}(e,b) &= {\omega_\mu}^{ab}(e) + 2e_\mu^{[a} e^{b]\nu} b_\nu, \\ {\omega_\mu}^{ab}(e) &= 2e^{\nu[a}\partial_{[\mu} {e_{\nu]}}^{b]} - e^{\nu[a} e^{b]\sigma} e_{\mu c}\, \partial_\nu e^c_\sigma. \end{aligned} \tag{2.2.11}$$

As we will discuss soon, the gauge field of dilatations b_μ will be set to zero by gauge fixing and our constraint (2.2.10) becomes the Cartan's first structure equation without torsion,

$$de^a + {\omega^a}_b \wedge e^b = T^a = 0, \tag{2.2.12}$$

which yields the spin connection of the Poincaré group ${\omega_\mu}^{ab}(e)$ given in (2.2.11).

For the gauge field of special conformal translations f_μ^a, we impose the following constraint on the curvature of local Lorentz transformations as

$$e_b^\nu R_{\mu\nu}(M^{ab}) = 0, \tag{2.2.13}$$

where

$$\begin{aligned} R_{\mu\nu}(M^{ab}) &= {R_{\mu\nu}}^{ab} + 8 f_{[\mu}^{[a} e_{\nu]}^{b]}, \\ {R_{\mu\nu}}^{ab} &= 2\partial_{[\mu} {\omega_{\nu]}}^{ab}(e,b) + 2{\omega_{[\mu}}^{ac}(e,b) {{\omega_{\nu]}}^b}_c(e,b). \end{aligned} \tag{2.2.14}$$

Note that the ${R_{\mu\nu}}^{ab}$ will become the curvature tensor of local Lorentz transformations in the Poincaré group after the gauge fixing ($b_\mu = 0$), which is given by Cartan's second structure equation as

$$R^{ab} = d\omega^{ab} + {\omega^a}_c \wedge \omega^{bc}. \tag{2.2.15}$$

The solution for the gauge field f_μ^a is given by

$$2(D-2) f_\mu^a = -{R_\mu}^a + \frac{1}{2(D-1)} e_\mu^a R, \tag{2.2.16}$$

with

$$R_{\mu\nu} = R_{\rho\mu}(M^{ab})e_a^\rho e_{\nu b}, \qquad\qquad R = R_\mu{}^\mu. \tag{2.2.17}$$

Let us now consider the transformation of the gauge field b_μ under the special conformal transformation given by

$$\delta_K(b_\mu) = 2\lambda_{K\mu}, \tag{2.2.18}$$

which is a shift defined at each spacetime point. This shift can be chosen such that the value of the field vanishes everywhere. Therefore, we can gauge-fix the special conformal transformations by imposing

$$b_\mu = 0. \tag{2.2.19}$$

The transformation law of a gauge-fixed field gives a relation between the gauge-fixed symmetry parameter and the parameters of the remaining symmetries, which is called a decomposition law. Here, in the case of the field b_μ, it takes the form

$$\lambda_{K\mu} = -\frac{1}{2}\partial_\mu\lambda_D. \tag{2.2.20}$$

2.2.3 Einstein's Gravity from Local Conformal Group

We will now obtain Einstein's theory by coupling the gauge multiplet of the conformal symmetry to a scalar field ϕ which transforms under the dilatations as

$$\delta\phi = \omega\lambda_D\phi, \tag{2.2.21}$$

where the Weyl weight ω is kept arbitrary for the time being.

For the construction of the action, we will need covariant quantities whose transformation does not yield the derivative of the symmetry parameters. When the gauge group includes the local translations, the covariant derivative of a field is defined with a local index as follows

$$\begin{aligned} \mathcal{D}_a\phi &= \epsilon_a^\mu\,\mathcal{D}_\mu\phi, \\ \mathcal{D}_\mu\phi &= \left(\partial_\mu - \delta(B_\mu)\right)\phi, \\ &= \left(\partial_\mu - B_\mu^A T_A\right)\phi, \end{aligned} \tag{2.2.22}$$

where the translations are omitted in the sum over gauge group indices.

According to our prescription (2.2.22), we can now compute the covariant derivative of the field ϕ as

$$\mathcal{D}_a\phi = e_a^\mu(\partial_\mu - \omega b_\mu)\phi, \tag{2.2.23}$$

whose transformation

$$\delta\mathcal{D}_a\phi = (\omega+1)\lambda_D\mathcal{D}_a\phi - \lambda_a^b\mathcal{D}_b\phi - 2\omega\lambda_{Ka}\phi \tag{2.2.24}$$

does not depend on the derivative of the transformation parameters which are compensated by the gauge fields. This makes it very easy to compute such quantities in practice. One can focus only on the terms with the gauge field e_μ^a since all the other gauge fields are guaranteed to not appear in the final result.

From the transformation (2.2.24), we can find

$$\mathcal{D}_\mu\mathcal{D}_a\phi = \left(\partial_\mu - (\omega+1)b_\mu\right)\mathcal{D}_a\phi + \omega_{\mu a}{}^b\mathcal{D}_b\phi + 2\omega f_{\mu a}\phi, \tag{2.2.25}$$

from which we can construct the following scalar quantity

$$\square^C\phi = \eta^{ab}\mathcal{D}_a\mathcal{D}_b\phi = e^{\mu a}\mathcal{D}_\mu\mathcal{D}_a\phi. \tag{2.2.26}$$

We will use this quantity to write down a kinetic term for the scalar field ϕ. Its variation yields

$$\delta\square^C\phi = (\omega+2)\lambda_D\square^C\phi + (2D - 4\omega + 4)\lambda_K^a\mathcal{D}_a\phi. \tag{2.2.27}$$

In order to ensure its invariance under the special conformal transformations, we choose

$$\omega = \frac{1}{2}D - 1. \tag{2.2.28}$$

Now a Weyl invariant quantity can be formed by using the determinant of the vielbein e, which is required to have a Lorentz invariant volume element. The quantity $e\,\phi\square^C\phi$ has the Weyl weight $-D + 2\omega + 2 = 0$ and therefore suitable for the construction of the invariant Lagrangian

$$I = -\frac{1}{2}\int d^Dx\, e\,\phi\square^C\phi. \tag{2.2.29}$$

Using the expression for the composite field f_μ^a (2.2.16) and gauge fixing the special conformal transformations ($b_\mu = 0$) yields

$$I = \int d^Dx\, e\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{D-2}{8(D-1)}R\phi^2\right) \tag{2.2.30}$$

Finally, the gauge fixing of local dilatations

$$\phi = \sqrt{\frac{8(D-1)}{(D-2)\kappa}} \tag{2.2.31}$$

gives the EH action in D dimensions

$$I = \frac{1}{\kappa} \int d^D x \, e \, R. \tag{2.2.32}$$

In summary, we realized the local conformal group with two independent fields e_μ^a and b_μ and two dependent fields $\omega_\mu{}^{ab}$ and f_μ^a which are determined the constraints (2.2.10-2.2.13). Then, a conformally invariant action was constructed by introducing a scalar field ϕ. After gauge fixing the special conformal transformations and dilatations, we obtained an action which is invariant under the local Poincaré group, which is the EH action.

One might worry that the scalar kinetic term in (2.2.30) appears with the wrong sign and it is unphysical. Indeed, this is a general feature of conformal methods where the additional fields are conformally coupled to gravity. In order to see why this is not a problem, one can start from the EH action (2.2.32) and perform the field redefinition

$$g_{\mu\nu} \to \phi^{\frac{4}{D-2}} g_{\mu\nu}. \tag{2.2.33}$$

The result is the Weyl invariant action (up to a multiplicative constant) given in (2.2.30). Therefore, it still describes only the dynamics of the spin-2 field and the physically important sign is that of the Ricci scalar, which is already correctly chosen.

2.2.4 Supergravity Theories from Superconformal Groups

Having obtained Einstein's gravity from the local conformal group, we can outline our main strategy for the construction of supergravity theories as a natural supersymmetric generalization. We will start from a superconformal algebra which includes the conformal algebra $SO(D,2)$ as a subalgebra. The generators of a superconformal algebra can be considered as matrix elements defined on a vector space and they take the following generic form

$$\begin{pmatrix} \text{conformal algebra} & Q, S \\ Q, S & \text{R-symmetry} \end{pmatrix}, \tag{2.2.34}$$

Q and S represent the fermionic generators of the algebra. R-symmetry generators are the bosonic generators with the characteristic property that they commute with the rest of the bosonic generators but not with the fermionic generators. We will construct actions invariant under the local conformal group by coupling the superconformal gauge multiplet (Weyl multiplet) to a matter multiplet. After gauge-fixing the redundant symmetries, we will find actions invariant under the generators of the form

$$\begin{pmatrix} \text{Poincaré algebra} & Q \\ Q & \text{R-symmetry} \end{pmatrix} , \tag{2.2.35}$$

which will be the supersymmetric invariants that we will use to construct the desired supergravity theories.

2.2.5 $\mathcal{N} = 1$ Poincaré Supergravity with Cosmological Constant in Three Dimensions

We now present the construction of Poincaré supergravity with cosmological constant in 3D by using the superconformal method [41–43], which is a simple but very useful example to understand our strategy to construct massive supergravity theories in Chapter 3. The $\mathcal{N} = 1$ Weyl multiplet in three dimensions consists of the fields

$$(e_\mu{}^a, \psi_\mu, b_\mu, \omega_\mu{}^{ab}, f_\mu{}^a, \phi_\mu) , \tag{2.2.36}$$

where $e_\mu{}^a$ is the dreibein, ψ_μ is the gauge field of Q-supersymmetry (gravitino) represented by a vector-spinor, b_μ is the dilatation gauge field, $\omega_\mu{}^{ab}$ is the spin connection, $f_\mu{}^a$ is the gauge field of special conformal transformations and ϕ_μ is the gauge field of S-supersymmetry represented by a vector-spinor.[1] The corresponding transformation parameters are

$$(\xi^a, \epsilon, \Lambda, \Lambda_D, \Lambda^{ab}, \Lambda_K^a, \eta) . \tag{2.2.37}$$

Similar to the bosonic case, we want to end up with the independent fields $(e_\mu{}^a, \psi_\mu, b_\mu)$, where the gravitino is included as the gauge field of the supersymmetry. Therefore, we express the gauge fields $\omega_\mu{}^{ab}, \phi_\mu, f_\mu{}^a$ in terms of the independent fields by imposing the constraints

$$\widehat{R}^a_{\mu\nu}(P) = 0 , \qquad e^\nu{}_b \widehat{R}_{\mu\nu}{}^{ab}(M) = 0 , \qquad \gamma^\nu \widehat{R}_{\mu\nu}(Q) = 0 , \tag{2.2.38}$$

[1] Two-component Majarona spinors are used.

where the supercovariant curvatures associated with translations, Lorentz rotations and Q-supersymmetry are defined as

$$
\begin{aligned}
\widehat{R}_{\mu\nu}{}^{a}(P) &= 2(\partial_{[\mu} + b_{[\mu)}\, e_{\nu]}{}^{a} + 2\,\omega_{[\mu}{}^{ab} e_{\nu]b} - \frac{1}{2}(\bar{\psi}_{[\mu}\gamma^{a}\psi_{\nu]} + h.c.)\ , \\
\widehat{R}_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^{b} + 8 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]} - \frac{1}{2}\bar{\psi}_{\mu}\gamma^{ab}\phi_{\nu} - \frac{1}{2}\bar{\phi}_{\mu}\gamma^{ab}\psi_{\nu} \\
\widehat{R}_{\mu v}(Q) &= 2\partial_{[\mu}\psi_{\nu]} + \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\,\psi_{\nu]} + b_{[\mu}\psi_{\nu]} - 2\,\gamma_{[\mu}\phi_{\nu]}\ . \qquad (2.2.39)
\end{aligned}
$$

The solutions in terms of the independent fields are given by

$$
\begin{aligned}
\omega_{\mu}{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\,\partial_{\nu}e_{\sigma}{}^{c} + 2e_{\mu}{}^{[a}b^{b]} + \frac{1}{2}\bar{\psi}_{\mu}\gamma^{[a}\psi^{b]} \\
&\quad + \frac{1}{2}\bar{\psi}^{[a}\gamma^{b]}\psi_{\mu} + \frac{1}{2}\bar{\psi}^{[a}\gamma_{\mu}\psi^{b]}\ , \\
\phi_{\mu} &= -\gamma^{a}\widehat{R}'_{\mu a}(Q) + \frac{1}{4}\gamma_{\mu}\gamma^{ab}\widehat{R}'_{ab}(Q)\ , \\
f_{\mu}^{a} &= -\frac{1}{2}\widehat{R}'_{\mu}{}^{a}(M) + \frac{1}{8}e_{\mu}{}^{a}\widehat{R}'(M)\ , \qquad (2.2.40)
\end{aligned}
$$

where we introduced the prime notation in the curvatures which mean that the field which is solved through the constraint equation is excluded from the associated curvature expression. The transformation rules for the independent fields are given by

$$
\begin{aligned}
\delta e_{\mu}{}^{a} &= -\Lambda^{a}{}_{b}\, e_{\mu}{}^{b} - \Lambda_{D} e_{\mu}{}^{a} + \phi 12\bar{\epsilon}\,\gamma^{a}\psi_{\mu}\ , \\
\delta\psi_{\mu} &= -\frac{1}{4}\Lambda^{ab}\gamma_{ab}\psi_{\mu} - \frac{1}{2}\Lambda_{D}\psi_{\mu} + \mathcal{D}_{\mu}\epsilon - \gamma_{\mu}\eta\ , \\
\delta b_{\mu} &= \partial_{\mu}\Lambda_{D} + 2\Lambda_{K\mu} + \frac{1}{2}\bar{\epsilon}\,\phi_{\mu} - \frac{1}{2}\bar{\eta}\,\psi_{\mu}\ , \qquad (2.2.41)
\end{aligned}
$$

which defines the 3D $\mathcal{N} = 1$ Weyl multiplet with $2 + 2$ (bosonic + fermionic) off-shell degrees of freedom.

We also need a scalar multiplet which will be coupled to the Weyl multiplet and give rise to a supersymmetric invariant after gauge fixing. It consists of a real physical scalar A, a Majarona fermion χ and an auxiliary scalar F with

$2+2$ degrees of freedom. The fields transform as

$$\begin{aligned}
\delta A &= \frac{1}{2}\bar{\epsilon}\chi + \omega\Lambda_D A\,, \\
\delta\chi &= \not{D}A\epsilon - \frac{1}{2}F\epsilon + 2\omega A\eta + \left(\omega + \frac{1}{2}\right)\Lambda_D\chi\,, \\
\delta F &= -\bar{\epsilon}\not{D}\chi + 2\left(\omega - \frac{1}{2}\right)\bar{\eta}\chi + (\omega+1)\,\Lambda_D F\,,
\end{aligned} \tag{2.2.42}$$

where the supercovariant derivatives are given by

$$\begin{aligned}
\mathcal{D}_\mu A &= (\partial_\mu - \omega b_\mu)\,A - \frac{1}{2}\bar{\psi}_\mu\chi \\
\mathcal{D}_\mu\chi &= \left(\partial_\mu - (\omega + \frac{1}{2})b_\mu\right)\chi - \not{D}A\psi_\mu + \frac{1}{2}F\psi_\mu - 2\omega A\phi_\mu
\end{aligned} \tag{2.2.43}$$

Properties of the Weyl and the scalar multiplet are summurized in Table 2.1 The action for a scalar multiplet with components (A,χ,F) which is invariant under the superconformal group is given by

$$S = \int d^D x\, e\left(F - \bar{\psi}_\mu\gamma^\mu\chi - Z\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right), \tag{2.2.44}$$

For the invariance under the dilatations, the highest weight component F should have the Weyl weight 3.

multiplet	field	type	off-shell	D
Weyl	${e_\mu}^a$	dreibein	2	1
	ψ_μ	gravitino	2	$\frac{1}{2}$
Scalar	A	scalar	1	ω
	χ	spinor	2	$\omega+\frac{1}{2}$
	F	scalar	1	$\omega+1$

Table 2.1
This table summarizes some properties of the basic multiplets of the $3D$ superconformal tensor calculus. The fourth and fifth columns indicate the number of off-shell degrees of freedom represented by the fields and the dilatation weights respectively.

In order to construct Poincaré supergravity, we will form such a scalar multiplet out of two scalar multiplets by employing the so called multiplication rule. One can start with two scalar multiplets (A_i, χ_i, F_i), $i = 1, 2$ and obtain a multiplet whose lowest component having Weyl weight $\omega = \omega_1 + \omega_2$ as follows

$$\begin{aligned} A &= A_1 A_2, \\ \chi &= A_1 \chi_2 + A_2 \chi_1, \\ F &= A_1 F_2 + A_2 F_1 + \bar{\chi}_1 \chi_2. \end{aligned} \tag{2.2.45}$$

Starting from a scalar multiplet (ϕ, λ, S) with Weyl weight of $\frac{1}{2}$, it is easy to show that the fields $\big(S, -2\not{D}\lambda, 4\Box^C \phi\big)$ form a multiplet with Weyl weight $\frac{3}{2}$. It is called the kinetic multiplet corresponding to the multiplet (ϕ, λ, S) since it contains the kinetic terms of the fields in the first multiplet. Their multiplication gives a multiplet with the desired properties so the result can be used in our action formula (2.2.44), which yields

$$\begin{aligned} S &= \int d^D x\, e \Big(4\phi \Box^C \phi + S^2 - 2\bar{\lambda}\not{D}\lambda + 2\phi \tilde{\bar{\psi}}_\mu \gamma^\mu \not{D}\lambda \\ & \qquad -S\bar{\psi}_\mu \gamma^\mu \lambda - \phi S \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu \Big). \end{aligned} \tag{2.2.46}$$

In order to obtain Poincaré supergravity theory, we gauge fix the extra transformations by imposing

$$\phi = 1, \qquad \lambda = 0, \qquad b_\mu = 0. \tag{2.2.47}$$

The first one fixes dilatation, the second fixes the S-supersymmetry and the last one fixes the special conformal transformations. As a result, we obtain the following decomposition rules

$$\begin{aligned} \Lambda_D &= 0, \\ \Lambda_{K\mu} &= \frac{1}{2}\bar{\eta}\psi_\mu - \frac{1}{2}\bar{\epsilon}\phi_\mu, \\ \eta &= \frac{1}{2} S \epsilon. \end{aligned} \tag{2.2.48}$$

Using the gauge fixing conditions (2.2.47) in the action (2.2.46) gives the final result for the action of Poincare supergravity

$$S_{EH} = \int d^D x\, e \left(R - 2S^2 - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho \right) \tag{2.2.49}$$

up to a multiplicative constant.

With the help of the decomposition rules (2.2.48), one can show that this action is invariant under the following supersymmetry transformations

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta\psi_\mu &= D_\mu\epsilon - \frac{1}{2}S\gamma_\mu\epsilon, \\
\delta S &= \frac{1}{4}\bar{\epsilon}\gamma^{\mu\nu}\psi_{\mu\nu} - \frac{1}{4}S\bar{\epsilon}\gamma^\mu\psi_\mu\,,
\end{aligned} \tag{2.2.50}$$

where

$$D_\mu\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\right)\epsilon, \qquad \psi_{\mu\nu} = 2D_{[\mu}\psi_{\nu]} \tag{2.2.51}$$

The Lagrangian for a supersymmetric cosmological constant can be found by multiplying four copies of the scalar multiplet (ϕ, λ, S). The resulting multiplet has the following components $(\phi^4, 4\phi^3\lambda, 4\phi^3 S + 6\phi^2\tilde{\lambda}\lambda)$. Using this multiplet in the action formula (2.2.44), imposing the gauge fixing conditions (2.2.47), and multiplying the action by $\frac{1}{2}$ gives

$$S_C = \int d^D x\, e\left(S - \frac{1}{2}\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right) \tag{2.2.52}$$

We can now construct the action of off-shell Poincaré supergravity with cosmological constant by combining the two supersymmetric invariants given in (2.2.49-2.2.52)

$$\begin{aligned}
S_{EH+C} &= \frac{1}{\kappa}\int d^D x\, e\left(R - 2S^2 - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho\right) \\
&\qquad + 4M\left(S - \frac{1}{2}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right)
\end{aligned} \tag{2.2.53}$$

where we introduced the dimensionfull parameter M to have correct units. The field equation for the auxiliary fields S gives

$$S = M \tag{2.2.54}$$

and therefore it can be eliminated from the theory, which gives rises to the action for on-shell AdS supergravity

$$S_{EH+C} = \frac{1}{\kappa}\int d^D x\, e\left(R + 2M^2 - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - 2M\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right) \tag{2.2.55}$$

which is invariant under the on-shell transformation rules

$$\begin{aligned} \delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\ \delta\psi_\mu &= D_\mu\epsilon - \frac{1}{2}M\gamma_\mu\epsilon\,, \end{aligned} \tag{2.2.56}$$

where the AdS radius L is defined by

$$M^2 = \frac{1}{L^2}. \tag{2.2.57}$$

In the case of higher derivative gravity theories, the field equations are more complicated and therefore it is practically very difficult to obtain the on-shell closure of the superalgebra. With the use of auxiliary fields, the closure can be achieved off-shell with one important difference. In general, the auxiliary fields in the supermultiplet cannot be eliminated from the theory and they become physical fields which propagate dynamical degrees of freedom. We will see examples of this in Chapter 3.

2.3 Classical Solutions of Supergravity Theories

This section is devoted to general concepts regarding the supersymmetric solutions of supergravity theories which will be the subject of Chapter 4. Most of the discussion is based on the explanations given in the textbook [40]. We can start by considering a purely bosonic theory to understand the main logic. In the case of Einstein's theory, for example, one has an action which is invariant under general coordinate transformations with arbitrary infinitesimal parameters $\xi^\mu(x)$. The simplest classical solution is the Minkowski spacetime with metric $g_{\mu\nu} = \eta_{\mu\nu}$. The solution is invariant only for the Killing vectors of Minkowski spacetime which determine the isometry group of the spacetime. For $(d+1)$-dimensional Minkowski spacetime, it is the Poincaré group ISO$(1,d)$ generated by $\frac{(d+1)(d+2)}{2}$ linearly independent Killing vectors. $\frac{d(d-1)}{2}$ of them describe SO$(1,d)$ rotations and $(d+1)$ of them are translations. One can study the fluctuations around this classical background and the dynamics of these fluctuations is covariant under the isometry group. The linearized Einstein's theory on a Minkowski background is probably the best known example in this regard.

In the case of supergravity theories, the action is invariant under local supersymmetry transformations parametrized by arbitrary spinor functions $\epsilon(x)$.

In general, one is interested in classical solutions that can be considered as backgrounds so that fluctuations around it can be studied quantum mechanically. Similar to the bosonic case, the possible solutions are not invariant under all the symmetries of the action but they preserve a subset of local supersymmetries that also form a superalgebra. For a solution that preserves some of the supersymmetries, the supersymmetry variations of fields should vanish. They take the following generic form

$$\begin{aligned} \delta B(x) &= \bar{\epsilon}(x) f_1(B(x)) F(x) + \mathcal{O}(F^3) = 0, \\ \delta F(x) &= f_2(B(x))\epsilon(x) + \mathcal{O}(F^2) = 0, \end{aligned} \tag{2.3.1}$$

where $B(x)$ and $F(x)$ represent the bosonic and fermionic fields in the theory respectively. The functions $f_1(B(x))$ and $f_2(B(x))$ contain γ-matrices and spacetime derivatives. The first equation is automatically satisfied since the fermions vanish in the classical limit. Therefore, it is only the second equation which provides a nontrivial constraint through the term linear in fermions as

$$\delta F(x)|_{\text{linear}} = f_2(B(x))\epsilon(x) = 0, \tag{2.3.2}$$

which is a first-order differential equation for the transformation parameter $\epsilon(x)$. For a theory with N supercharges, one can find n linearly independent solution to this equation ($n \leqslant N$), which are called Killing spinors. It is said that the solution preserves a fraction $\frac{n}{N}$ of supersymmetries. Requiring to preserve some of the supersymmetries restricts the configurations of the bosonic fields such that it greatly simplifies the solution of the field equations.

Investigation of the Killing spinor equation which arises from the transformation of the gravitino gives a nontrivial condition for the existence of the Killing spinor. The gravitino field generically transform under supersymmetry as

$$\delta\psi_\mu = \hat{D}_\mu \epsilon = D_\mu \epsilon + \cdots = 0\,, \tag{2.3.3}$$

which is the standard Lorentz covariant derivative of the transformation parameter $\epsilon(x)$ modified by extra terms depending on the properties of the supermultiplet under consideration. Since it should vanish for a Killing spinor, the following commutator should also vanish

$$[\hat{D}_\mu, \hat{D}_\nu]\epsilon = \left(\frac{1}{4} R_{\mu\nu}{}^{ab}\gamma_{ab} + \cdots\right)\epsilon \tag{2.3.4}$$

which includes a term with the standard Lorentz curvature and possible modifications. This condition is called the integrability condition for Killing spinors since it is a necessary condition for the Killing spinor equation (2.3.3) to be satisfied. For the case of a Minkowski spacetime, it becomes trivial since there is no extra term added to the Lorentz curvature term and the curvature tensor vanishes. However, in general, it provides a strong constraint on the curved spacetimes which can support supersymmetry.

In the analysis of Killing spinors, the bosonic field configurations are constrained by the fermionic transformation rules, giving rise to certain differential and algebraic relations between the components of the bosonic fields in the theory. Since the fermionic fields are set to zero, one has equations that are linear in $\epsilon(x)$ and no fermionic bilinears or higher order fermionic terms appear in the Killing spinor equations. Therefore, there is nothing which can distinguish whether the Killing spinors are commuting or non-commuting throughout the analysis. One can just assume commuting spinors and work with them to find the Killing spinors, which determines the number of preserved supersymmetries, and to find the relations satisfied by the bosonic fields.

One can make use of this nice feature to relate the Killing spinors to the Killing vectors of the spacetime which supports the supersymmetric solutions. Since the anti-commutator of supersymmetries closes on spacetime translations, such a relation is hardly surprising and it is indeed the translational Killing vectors of the spacetime which can be obtained from the Killing spinors. Let us consider the case of 3D $\mathcal{N} = 1$ supergravity that we introduced in the previous section as an example. The Killing spinor equation that follows from the supersymmetry transformation of the gravitino, which is given in (2.2.50), as

$$\delta\psi_\mu = D_\mu\epsilon - \frac{1}{2}S\gamma_\mu\epsilon = 0\,. \tag{2.3.5}$$

Assuming commuting spinors, it is easy to show that the vector K^μ which is defined by

$$K^\mu = \bar{\epsilon}\gamma^\mu\epsilon\,, \tag{2.3.6}$$

is a Killing vector which satisfies

$$\nabla_{(\mu}K_{\nu)} = 0. \tag{2.3.7}$$

Note that the vector K^μ vanishes identically for non-commuting spinors. The norm of this Killing vector can be computed with the help of Fierz identities as

$$K^\mu K_\mu = -(\bar{\epsilon}\epsilon)^2 = 0\,. \tag{2.3.8}$$

This implies that the spacetimes which supports $\mathcal{N} = 1$ supersymmetry in 3D can only have null translational Killing vectors, which is a severe restriction on the possible supersymmetric solutions. In Chapter 4, we will see that increasing the number of supercharges will lead to the possibility of time-like translational Killing vectors, yielding a richer set of possibilities.

A similar structure to the Killing spinor analysis also appears in the approach of Bogomol'nyi, Prasad and Sommerfield (BPS) to the magnetic monopole solutions where the components of the bosonic fields are constrained by first order differential equations. It is for this reason that the supersymmetric solutions are also sometimes referred to as BPS solutions in the literature.

In the case of extended supersymmetry, the subalgebra formed by the residual supersymmetries allows a central extension in the closure of the anticommutator of the supersymmetry generators. Choosing the physical states of the theory as the eigenstates of the central charges, it is possible to show that there is a bound on the mass of the physical states given as

$$M \geq |\mathcal{Z}| \,, \tag{2.3.9}$$

which is called the BPS bound. For supersymmetric black hole solutions, the central charge can be expressed in terms of global charges in the theory such as electric/magnetic charges and the solutions saturate the BPS bound, i.e. $M = |\mathcal{Z}|$, which correspond to the extremal black hole solutions. This will play an important role in our discussion of supersymmetric black holes in Chapter 4.

2.4 Algebraic Classification of 3D Spacetimes

In this section, we discuss the algebraic classification of 3D spacetimes which we will make use of in Chapter 5. We mainly follow the discussion given in [44] (see [45] for a very detailed treatment of exact solutions and their classifications.) Although several different classification schemes are possible, the algebraic classification is very useful to understand how solutions appear in the context of higher derivative gravity theories and to look for new type of solutions. The main idea is to consider certain curvature tensors, which we call $T^a{}_b$, as linear mappings between vectors. The eigenvalue equation for the linear map $T^a{}_b$

$$T^a{}_b V^b = \lambda V^a, \tag{2.4.1}$$

where λ's are the eigenvalues with the corresponding eigenvectors V^a, provides a way to classify spacetimes which is based on the eigenvalues and their (algebraic and geometric) multiplicities.

2.4.1 Comparison of 3D and 4D

In 4D, the Riemann tensor has 20 independent components which can be considered as the 10 independent components of the Weyl tensor $C_{\mu\nu\rho\sigma}$, the 9 independent components of the traceless Ricci tensor $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ and 1 component of the Ricci scalar R, which yields two distinct independent algebraic classifications of curvature: the Petrov and the Segre classifications.

In the Petrov classification, the 10 real components of the Weyl tensor is written as a traceless and symmetric 3×3 complex matrix $C^a{}_b$, which is a linear map between complex 3-vectors. Since the Weyl tensor vanishes in 3D, there is no direct way to apply this classification scheme in 3D. However, the fact that the Weyl tensor vanishes for conformally flat spacetimes provides an analogy. This role of Weyl tensor in 4D is played by the Cotton tensor $C^a{}_b$ in 3D. The Petrov classification in 3D is based on the algebraic classification of the Cotton tensor $C^a{}_b$, considered as a linear map between real 3-vectors [46–50].

The Segre classification is an algebraic classification based on the traceless Ricci tensor $\tilde{R}^a{}_b = R^a{}_b - \frac{1}{D}\delta^a_b R$, $(D = 3, 4)$ which defines a linear map between real D-vectors. In 3D, it has been studied in [47, 48, 50, 51].

2.4.2 Petrov-Segre Classification in TMG

For the case of TMG that we introduced in Section 2.1, a special feature arises. In order to see that, we start by writing its action as

$$I = \frac{1}{\kappa}\int d^3x\,\sqrt{-g}\left[R - 2\Lambda + \frac{1}{2\mu}\epsilon^{\lambda\mu\nu}\Gamma^\rho{}_{\lambda\sigma}\left(\partial_\mu\Gamma^\sigma{}_{\rho\nu} + \frac{2}{3}\Gamma^\sigma{}_{\mu\tau}\Gamma^\tau{}_{\nu\rho}\right)\right], \quad (2.4.2)$$

where Λ is the cosmological constant and μ is the mass parameter. The fields equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \quad (2.4.3)$$

where $C_{\mu\nu}$ is the Cotton tensor (symmetric and traceless) defined by

$$C_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta}\nabla_\alpha S_{\beta\nu}, \quad (2.4.4)$$

with the Schouten tensor $S_{\mu\nu}$ given by

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R. \tag{2.4.5}$$

Taking the trace of field equations (2.4.3) gives $R = 6\Lambda$, which allows one to write the field equations in the following form

$$\widetilde{R}_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0, \tag{2.4.6}$$

where $\widetilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{3} g_{\mu\nu} R$ is the traceless Ricci tensor as we defined before. Therefore, for the solutions of TMG, the traceless Ricci tensor and the Cotton tensor differ only by a proportionality constant and the Petrov and the Segre classifications coincide. We will refer to it as the Petrov-Segre classification. Since the Cotton tensor includes the derivative of the Ricci tensor, the Segre classification, which is based on the traceless Ricci tensor, is more fundamental. Therefore, we will first use the Segre classification and then specify the corresponding Petrov type of the spacetime under consideration by using the equivalence of the two.

For the Petrov-Segre classification of 3D spacetimes, which is summarized in Table 2.2, we need to introduce a null frame (ℓ^a, m^a, n^a) where ℓ, n are null vectors and m is a unit spacelike vector. They have vanishing inner products except

$$\ell^a n_a = -1, \qquad m^a m_a = 1, \tag{2.4.7}$$

which makes it possible to write down the flat metric η_{ab} as

$$\eta_{ab} = -\ell_a n_b - n_a \ell_b + m_a m_b. \tag{2.4.8}$$

We also need the unit timelike vector t and the unit spacelike vector z defined as

$$t^a = \frac{\ell^a + n^a}{\sqrt{2}}, \qquad z^a = \frac{\ell^a - n^a}{\sqrt{2}}. \tag{2.4.9}$$

The remaining ingredients are two real functions $\alpha(x)$, $\beta(x)$ and two real parameters $\lambda = \pm 1$, $\tau = \pm 1$.

As mentioned before, the algebraic type of a spacetime is determined by the eigenvalues and their algebraic/geometric multiplicities. One way to encode this information is to find the Jordan normal form of the matrix $\widetilde{R}^a{}_b$. By a

Petrov	Segre	Canonical $\widetilde{R}_{ab}$	Scalar Invariants
O	$[(11,1)]$	0	$I = J = 0$
N	$[(12)]$	$\lambda \ell_a \ell_b$	$I = J = 0$
D$_t$	$[(11),1]$	$\alpha(\eta_{ab} + 3t_a t_b)$	$I^3 = 6J^2 \neq 0$
D$_s$	$[1(1,1)]$	$\alpha(\eta_{ab} - 3m_a m_b)$	$I^3 = 6J^2 \neq 0$
III	$[3]$	$2\tau \ell_{(a} m_{b)}$	$I = J = 0$
II	$[12]$	$\alpha(\eta_{ab} - 3m_a m_b) + \lambda \ell_a \ell_b$	$I^3 = 6J^2 \neq 0$
I$_{\mathbb{R}}$	$[11,1]$	$\alpha(\eta_{ab} - 3m_a m_b) - \beta(\ell_a \ell_b + n_a n_b)$	$I^3 > 6J^2$
I$_{\mathbb{C}}$	$[1z\bar{z}]$	$\alpha(\eta_{ab} - 3m_a m_b) - \beta(\ell_a \ell_b - n_a n_b)$	$I^3 < 6J^2$

TABLE 2.2
Petrov-Segre Classification for TMG and the corresponding canonical form the the traceless Ricci tensor $\tilde{R}_{ab}$

similarity transformation, which preserves the eigenvalue structure, any square matrix can be put into the Jordan normal form:

$$\mathsf{M} = \begin{pmatrix} \mathsf{J}_1 & & \\ & \ddots & \\ & & \mathsf{J}_\mathsf{p} \end{pmatrix},$$

which is block diagonal. Each Jordan block J_i is λ_i times an identity matrix and 1's on the superdiagonal as

$$\mathsf{J}_\mathsf{i} = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}.$$

In the Segre classification, the numbers $1, 2, 3$ give the size of Jordan blocks. The round brackets are used to indicate that the Jordan blocks in the brackets correspond to the same eigenvalue. A comma is to used to distinguish between the spacelike and the timelike eigenvectors, where the sizes of the Jordan blocks corresponding to the spacelike eigenvectors are written before the comma. This also reflects itself in the Petrov classification such that there are two distinct type D spacetimes: the one-dimensional eigenspace is spacelike for type D_s spacetimes and timelike for type D_t spacetimes. Since the matrix $\widetilde{R}^a{}_b$ is not symmetric in general, it can have complex eigenvalues, which leads to a split in the Petrov classification again: type $\mathrm{I}_{\mathbb{R}}$ spacetimes have three real distinct eigenvalues and type $\mathrm{I}_{\mathbb{C}}$ spacetimes have one real and two complex conjugate eigenvalues. They are denoted as $[11, 1]$ and $[1z\bar{z}]$ in the Segre notation respectively. Instead of computing the eigenvectors, one can also make use of the fact that each Jordan normal form has a distinct minimal polynomial (see [44] for details).

However, we will use another method to identify the Petrov-Segre type of a spacetime, which is based on the fact that determining the eigenvalues and their algebraic multiplicities is equivalent to computing the following scalar invariants

$$I = \widetilde{R}^a{}_b \widetilde{R}^b{}_a = \mathrm{tr}(\widetilde{R}^2), \qquad J = \widetilde{R}^a{}_b \widetilde{R}^b{}_c \widetilde{R}^c{}_a = \mathrm{tr}(\widetilde{R}^3). \tag{2.4.10}$$

As can be seen from the Table 2.2, certain types of spacetimes have the same value for these scalar invariants. In such cases, one can check the canonical form of the traceless Ricci tensor to complete the classification.

Note that the Petrov and the Segre classifications are equivalent for TMG because its field equations relates the Cotton tensor to the traceless Ricci tensor with a proportionality constant. Therefore, the equivalence is not true for a generic higher curvature 3D gravity theory. For some cases, the solutions of TMG are also solutions of some other gravity theory (see [52–55] for examples in NMG and its Born-Infeld type extension) and the terminology of the Petrov-Segre classification remains exact. In chapter 5, we will study the solutions of K-gravity which are not in general solutions of TMG but we will stick to this terminology although the equivalence does not hold any more. What is meant should be clear from Table 2.2.

Chapter 3

Massive $\mathcal{N} = 2$ Supergravity Theories in Three Dimensions

3.1 Introduction

As we briefly explained in Chapter 1, the fact that Einstein's gravity has no local dynamics in 3D leads to a search for modified theories of gravity with dynamical degrees of freedom which are expected to reveal richer physical properties. Although there is a wide class of theories which have the property of tree-level unitarity, they also suffer from the so called bulk-boundary unitarity problem, which we reviewed for the case of NMG in Section 2.1. It is possible that the supersymmetric vacua in any of these theories might behave better than the purely bosonic versions and even provide a solution to this problem. It is also known that, although claimed to be renormalizable first [56], NMG was shown to be nonrenormalizable [57]. For the the supersymmetric extension of the theory, it is natural to expect an improvement in this regard. Off-shell supergravity invariants up to and including four derivatives are known for $\mathcal{N} = (1,0)$ supergravity in 3D [58]. Some of their properties, such as their supersymmetric vacua and spectrum about AdS_3 vacuum, have been studied [58–60]. However, no improvement in the bulk-boundary issue has been achieved.

Our aim in this chapter is to construct higher derivative supergravity invariants with $\mathcal{N} = 2$ supersymmetry and to look for their ghost free combinations. We will focus our attention to supergravity theories which admit anti-de Sitter space as a vacuum solution. The underlying supersymmetry algebra is $OSp(p,q)$, whose bosonic part is given by $O(2,2) \oplus SO(p) \times SO(q)$ [20,61,62]. This leads to two distinct off-shell $\mathcal{N} = 2$ supergravities namely, $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravities. The conformal $\mathcal{N} = 2$ supergravity, and the two-derivative invariants were considered in [63–66]. Off-shell matter-coupled suprgravity theories were investigated in the superspace framework in [67–70]. The on-shell construction and the matter couplings of the three dimensional $\mathcal{N} = 2$ supergravity were studied in [71–73]. Here we shall provide all the four derivative off-shell invariants of the $3D$, $\mathcal{N} = 2$ supergravity. The method we shall employ is the superconformal tensor calculus, which was introduced in Section 2.2. The $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supergravities arise from the coupling of the Weyl multiplet to a compensating scalar or vector multiplet, respectively, followed by fixing some of the conformal symmetries. In the case of $\mathcal{N} = (2,0)$ supersymmetry, we shall employ a map between the Yang-Mills and the supergravity multiplet to construct the supersymmetric completion of the Ricci-squared term.

Taking into account the new invariants we construct here, we end up with a seven parameter action with $\mathcal{N} = (1,1)$ supersymmetry and a six parameter action with $\mathcal{N} = (2,0)$ supersymmetry. We find that the former admits a four parameter subfamily which admits an AdS vacuum solution around which the spectrum of small fluctuations is ghost-free. In the latter case, however, we find that such a scenario is not possible. This turns out to be due to the fact that a particular type of dimension four invariant that exists for the $\mathcal{N} = (1,1)$ model does not seem to exist for the $\mathcal{N} = (2,0)$ model. The existence of the supersymmetric cosmological extension of the $(2,0)$ supergravity, which does not exist in the parent $\mathcal{N} = 1$ new minimal supergravity in $4D$, is not sufficient for the existence of a ghost-free supersymmetric AdS_3 vacuum.

This chapter is organized as follows. In Section 3.2, we give a brief introduction to the superconformal formalism, and introduce the Weyl multiplet, the scalar and the vector multiplet of the $D = 3$, $\mathcal{N} = 2$ theory in the context of conformal supergravity. We then provide a multiplication rule for scalar multiplets, and construct composite scalar and vector multiplets. Subsequently, we proceed to construct various superconformal actions for these matter multiplets. In Section 3.3, we consider the scalar multiplet actions constructed in section 3.2 and gauge fix the superconformal symmetries to obtain $\mathcal{N} = (1,1)$ cosmological Poincaré supergravity theory as well as the supersymmetric completion of the $R^2_{\mu\nu}, R^2$ and the off-diagonal $(RS^n + h.c)$, where S is the auxiliary scalar of the $\mathcal{N} = (1,1)$ Poincaré multiplet. We then present the $\mathcal{N} = (1,1)$ generalized massive supergravity, and analyze the bosonic spectrum around a maximally supersymmetric AdS_3 vacuum. In Section 3.4, we repeat the same analysis for the vector multiplet, and construct the $\mathcal{N} = (2,0)$ cosmological Poincaré supergravity and the supersymmetric R^2 and off-diagonal RD invariants, where D is the auxiliary scalar of the $\mathcal{N} = (2,0)$ Poincaré multiplet. In this section, we observe that certain fields in the $\mathcal{N} = (2,0)$ Poincaré multiplet transform in the same manner as the fields of a Yang-Mills multiplet with $G = SO(2,1)$. Using this correspondence, we construct the supersymmetric $R^2_{\mu\nu}$ invariant. Subsequently, we discuss the ghost-free $\mathcal{N} = (2,0)$ generalized massive gravity theory, and analyze the spectrum around a maximally supersymmetric Minkowski background. In Section 3.5, we give conclusion and discussions. Finally, the details of the complex spinor conventions and Fierz identities are given in an appendix. The material presented in this chapter is based on [74].

3.2 3D $\mathcal{N} = 2$ Superconformal Tensor Calculus

In this section we shall describe the Weyl multiplet based on the superconformal algebra $OSp(4|2)$ in three dimensions. We will then present the off-shell scalar and vector multiples which will be used in the subsequent sections as compensators. The rules for combining these multiplets to obtain new (composite) multiplets and action formula will follow. The action formula will be used in the following sections together with the composite multiplet formula to obtain several off-shell supergravity invariants. Finally, we shall also record for completeness the Chern-Simons invariant which does not require any compensating multiplet coupling since it is superconformal invariant by itself [63].

3.2.1 The Weyl and Compensating Multiplets

Weyl multiplet. The $\mathcal{N} = 2$ Weyl multiplet in three dimensions is based on the conformal superalgebra $OSp(4|2)$ and consists of the fields

$$(e_\mu{}^a, \psi_\mu, V_\mu, b_\mu, \omega_\mu{}^{ab}, f_\mu{}^a, \phi_\mu) \ , \tag{3.2.1}$$

where $e_\mu{}^a$ is the Dreibein, ψ_μ is the gravitino represented by a Dirac vector-spinor, V_μ is the $U(1)$ R-symmetry gauge field, b_μ is the dilatation gauge field, $\omega_\mu{}^{ab}$ is the spin connection, $f_\mu{}^a$ is the conformal boost gauge field and ϕ_μ is the special supersymmetry gauge field represented by a Dirac vector-spinor. The corresponding gauge parameters are

$$(\xi^a, \epsilon, \Lambda, \Lambda_D, \Lambda^{ab}, \Lambda_K^a, \eta) \ . \tag{3.2.2}$$

The gauge fields $\omega_\mu{}^{ab}, \phi_\mu, f_\mu{}^a$ can be expressed in terms of the remaining fields by imposing the constraints [63]

$$\widehat{R}^a_{\mu\nu}(P) = 0 \ , \qquad \widehat{R}_{\mu\nu}{}^{ab}(M) = 0 \ , \qquad \widehat{R}_{\mu\nu}(Q) = 0 \ , \tag{3.2.3}$$

where the supercovariant curvatures associated with translations, Lorentz rotations and supersymmetry are defined as

$$\begin{aligned}
\widehat{R}_{\mu\nu}{}^a(P) &= 2(\partial_{[\mu} + b_{[\mu})\, e_{\nu]}{}^a + 2\,\omega_{[\mu}{}^{ab} e_{\nu]b} - \frac{1}{2}(\bar{\psi}_{[\mu}\gamma^a\psi_{\nu]} + h.c.) \ , \\
\widehat{R}_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^b + 8 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]} - \frac{1}{2}\bar{\psi}_\mu\gamma^{ab}\phi_\nu - \frac{1}{2}\bar{\phi}_\mu\gamma^{ab}\psi_\nu + h.c \\
\widehat{R}_{\mu v}(Q) &= 2\partial_{[\mu}\psi_{\nu]} + \frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\,\psi_{\nu]} + b_{[\mu}\psi_{\nu]} - 2\,\gamma_{[\mu}\phi_{\nu]} - 2\mathrm{i}V_{[\mu}\psi_{\nu]} \ .
\end{aligned} \tag{3.2.4}$$

These constraints together with the Bianchi identity for $\widehat{R}_{\mu\nu}(P)$ imply that the curvature associated with dilatation also vanishes, *viz.* $\widehat{R}_{\mu\nu}(D) = 0$. Solving the constraints (3.2.3) gives

$$\begin{aligned}
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\,\partial_\nu e_\sigma{}^c + 2e_\mu{}^{[a}b^{b]} + \frac{1}{2}\bar{\psi}_\mu\gamma^{[a}\psi^{b]} \\
&\quad +\frac{1}{2}\bar{\psi}^{[a}\gamma^{b]}\psi_\mu + \frac{1}{2}\bar{\psi}^{[a}\gamma_\mu\psi^{b]}\ , \\
\phi_\mu &= -\gamma^a\widehat{R}'_{\mu a}(Q) + \frac{1}{4}\gamma_\mu\gamma^{ab}\widehat{R}'_{ab}(Q)\ , \\
f_\mu^a &= -\frac{1}{2}\widehat{R}'_\mu{}^a(M) + \frac{1}{8}e_\mu{}^a\widehat{R}'(M)\ ,
\end{aligned} \tag{3.2.5}$$

where the prime in the curvatures used in (3.2.5) means that the term including the field we are solving for is excluded. The transformation rules for the independent fields are given by

$$\begin{aligned}
\delta e_\mu{}^a &= -\Lambda^a{}_b\, e_\mu{}^b - \Lambda_D e_\mu{}^a + \frac{1}{2}\bar{\epsilon}\,\gamma^a\psi_\mu + h.c.\ , \\
\delta\psi_\mu &= -\frac{1}{4}\Lambda^{ab}\gamma_{ab}\psi_\mu - \frac{1}{2}\Lambda_D\psi_\mu + \mathcal{D}_\mu\epsilon - \gamma_\mu\eta + \mathrm{i}\Lambda\psi_\mu\ , \\
\delta b_\mu &= \partial_\mu\Lambda_D + 2\Lambda_{K\mu} + \frac{1}{2}\bar{\epsilon}\,\phi_\mu - \frac{1}{2}\bar{\eta}\,\psi_\mu + h.c\ , \\
\delta V_\mu &= \partial_\mu\Lambda + \frac{1}{2}\mathrm{i}\bar{\epsilon}\,\phi_\mu + \frac{1}{2}\mathrm{i}\bar{\eta}\,\psi_\mu + h.c.\ ,
\end{aligned} \tag{3.2.6}$$

where

$$\mathcal{D}_\mu\epsilon = \Big(\partial_\mu + \frac{1}{2}b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \mathrm{i}V_\mu\Big)\epsilon\ . \tag{3.2.7}$$

Finally, we give the transformation rule for ϕ_μ for later convenience

$$\delta\phi_\mu \;=\; \cdots + \mathrm{i}\gamma^\nu\widehat{F}_{\mu\nu}\epsilon - \frac{1}{4}\mathrm{i}\gamma_\mu\gamma\cdot\widehat{F}_{\mu\nu}\epsilon\ , \tag{3.2.8}$$

where we have displayed the supercovariant terms and the ellipses refer to the remaining terms implied by the $OSp(4|2)$ algebra, and $\widehat{F}_{\mu\nu}$ is given by

$$\widehat{F}_{\mu\nu} \;=\; 2\partial_{[\mu}V_{\nu]} - \mathrm{i}\bar{\psi}_{[\mu}\phi_{\nu]} - \mathrm{i}\bar{\phi}_{[\mu}\psi_{\nu]}\ . \tag{3.2.9}$$

Scalar multiplet. The off-shell $\mathcal{N}=2$ scalar multiplet with 4+4 degrees of freedom consists of a physical complex scalar A, a Dirac fermion χ and an auxiliary complex scalar $\mathcal{F}$ with the following transformation rules [1]

$$\begin{aligned}
\delta A &= \frac{1}{2}\bar{\epsilon}\chi + w\Lambda_D A - \mathrm{i}w\Lambda A \ , \\
\delta\chi &= \not{\mathcal{D}}A\,\epsilon - \frac{1}{2}\mathcal{F}\,(B\epsilon)^* + 2wA\,\eta + (w+\frac{1}{2})\Lambda_D\chi + \mathrm{i}(-w+1)\Lambda\chi \ , \\
\delta\mathcal{F} &= -\tilde{\epsilon}\not{\mathcal{D}}\chi + 2(w-\frac{1}{2})\tilde{\eta}\,\chi + (w+1)\Lambda_D\mathcal{F} + \mathrm{i}(-w+2)\Lambda\,\mathcal{F} \ , \qquad (3.2.10)
\end{aligned}$$

where the supercovariant derivatives are given by

$$\begin{aligned}
\mathcal{D}_\mu A &= (\partial_\mu - w\,b_\mu + \mathrm{i}wV_\mu)A - \frac{1}{2}\bar{\psi}_\mu\chi \ , \\
\mathcal{D}_\mu\chi &= (\partial_\mu - (w+\tfrac{1}{2})b_\mu + \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab} + \mathrm{i}(w-1)V_\mu)\chi - \not{\mathcal{D}}A\psi_\mu \\
&\quad + \frac{1}{2}\mathcal{F}(B\psi_\mu)^* - 2wA\phi_\mu \ . \qquad (3.2.11)
\end{aligned}$$

Note that the lowest component has Weyl weight w and $U(1)_R$ weight $-w$. Another multiplet with its lowest component having Weyl weight w and $U(1)_R$ weight w can be obtained by charge conjugation

$$\begin{aligned}
\delta A^* &= \frac{1}{2}\tilde{\epsilon}\,(B\chi)^* + w\Lambda_D A^* + iw\Lambda A^* \ , \\
\delta\,(B\chi)^* &= \not{\mathcal{D}}A^*\,(B\epsilon)^* - \frac{1}{2}F^*\epsilon + 2wA^*\,(B\eta)^* + (w+\frac{1}{2})\Lambda_D\,(B\chi)^* \\
&\quad + \mathrm{i}\,(w-1)\,\Lambda\,(B\chi)^* \ , \\
\delta\mathcal{F}^* &= -\bar{\epsilon}\not{\mathcal{D}}\,(B\chi)^* + 2(w-\frac{1}{2})\bar{\eta}(B\chi)^* + (w+1)\Lambda_D\mathcal{F}^* \\
&\quad + \mathrm{i}(w-2)\Lambda\,\mathcal{F}^* \ , \qquad (3.2.12)
\end{aligned}$$

where the supercovariant derivatives are

$$\begin{aligned}
\mathcal{D}_\mu A^* &= (\partial_\mu - wb_\mu - \mathrm{i}wV_\mu)A^* - \frac{1}{2}\tilde{\psi}_\mu(B\chi)^* \ , \\
\mathcal{D}_\mu(B\chi)^* &= \left(\partial_\mu - (w+\frac{1}{2})b_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \mathrm{i}(w-1)V_\mu\right)(B\chi)^* \\
&\quad - \not{\mathcal{D}}A^*(B\psi_\mu)^* + \frac{1}{2}\mathcal{F}^*\psi_\mu - 2wA^*(B\phi_\mu)^* \, , \\
\mathcal{D}_\mu P^* &= (\partial_\mu - \frac{1}{2}b_\mu - (w-2)\mathrm{i}\,V_\mu)P^* + \bar{\psi}_\mu\not{\mathcal{D}}(B\chi)^* \\
&\quad - 2(w-\frac{1}{2})\bar{\phi}_\mu(B\chi)^* \, . \qquad (3.2.13)
\end{aligned}$$

[1] See Appendix 3.A for the definition of $\widetilde{\eta}$ and the constant matrix B.

Multiplet	Field	Type	Off-shell	w	q
Weyl	$e_\mu{}^a$	dreibein	2	-1	0
	ψ_μ	gravitino	4	$-\frac{1}{2}$	1
	V_μ	$U(1)_R$ gauge field	2	0	0
Scalar	A	complex scalar	2	w_A	$-w_A$
	χ	Dirac spinor	4	$w_A + \frac{1}{2}$	$-w_A + 1$
	$\mathcal{F}$	complex auxiliary	2	$w_A + 1$	$-w_A + 2$
Vector	ρ	real scalar	1	1	0
	C_μ	gauge field	2	0	0
	λ	Dirac spinor	4	$\frac{3}{2}$	1
	D	real auxiliary	1	2	0

TABLE 3.1
Properties of the $3D, \mathcal{N} = 2$ Weyl and compensating multiplets where (w, q) label the dilatation weight and the $U(1)_R$ charge, respectively.

Vector multiplet. The off-shell vector $\mathcal{N} = 2$ vector multiplet with $4 + 4$ degrees of freedom consists of a gauge field C_μ, a scalar ρ, a spinor λ and an auxiliary scalar D. Their transformation rules are given by

$$\begin{aligned}
\delta C_\mu &= \frac{1}{2}\,\bar\epsilon\,\gamma_\mu\lambda - \frac{1}{4}\mathrm{i}\,\rho\,\bar\epsilon\,\psi_\mu + h.c.\ , \\
\delta\rho &= (\mathrm{i}\bar\epsilon\,\lambda + h.c.) + \Lambda_D\rho\ , \\
\delta\lambda &= -\frac{1}{4}\gamma^{\mu\nu}\widehat{G}_{\mu\nu}\,\epsilon + \frac{1}{2}\mathrm{i}D\epsilon - \frac{1}{4}\mathrm{i}\not{D}\rho\,\epsilon - \frac{1}{2}\mathrm{i}\rho\,\eta + \mathrm{i}\Lambda\,\lambda + \frac{3}{2}\Lambda_D\lambda\ , \\
\delta D &= (-\frac{1}{2}\mathrm{i}\bar\epsilon\not{D}\lambda + \frac{1}{2}\mathrm{i}\bar\eta\lambda + h.c.) + 2\Lambda_D D\ ,
\end{aligned} \tag{3.2.14}$$

Components	w	q
(ξ, φ, M)	$\frac{5}{2}$	$-\frac{5}{2}$
(Z, Ω, F)	2	-2
(ϕ, ζ, S)	$\frac{1}{2}$	$-\frac{1}{2}$
(σ, ψ, N)	0	0
(Φ, Ψ, P)	$-\frac{1}{2}$	$\frac{1}{2}$

Table 3.2
Compensating scalar multiplets with (w, q) denoting the Weyl weight and $U(1)_R$ charge of the lowest component scalar field.

where

$$
\begin{aligned}
\mathcal{D}_\mu \rho &= (\partial_\mu - b_\mu)\,\rho + \left(-\mathrm{i}\bar{\psi}_\mu \lambda + h.c.\right) \ , \\
\mathcal{D}_\mu \lambda &= (\partial_\mu - \frac{3}{2} b_\mu + \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab} - \mathrm{i}V_\mu)\lambda + \frac{1}{4}\gamma^{\rho\sigma}\widehat{G}_{\rho\sigma}\,\psi_\mu \\
&\quad -\frac{1}{2}\mathrm{i}D\psi_\mu + \frac{1}{4}\mathrm{i}\not{\mathcal{D}}\rho\,\psi_\mu + \frac{1}{2}\mathrm{i}\rho\,\phi_\mu \ , \\
\widehat{G}_{\mu\nu} &= 2\partial_{[\mu}C_{\nu]} + (-\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda + \frac{1}{2}\mathrm{i}\rho\,\bar{\psi}_\mu\psi_\nu + h.c.) \ . \qquad (3.2.15)
\end{aligned}
$$

As we shall discuss in subsection 3.4.3, the nonabelian versions of (3.2.14) and (3.2.15) can be obtained by taking the fields of the vector multiplet in the adjoint representation of a Lie group G, and imposing the closure of the algebra accordingly.

3.2.2 Combination of Local Supermultiplets

To provide a supersymmetric completion of Poincaré supergravity and of the higher dimensional invariants, we need to produce multiplets with different weights. We give now general rules to do so and we introduce all composite multiplets that will be needed to construct invariant actions.

Scalar Multiplets. We will construct composite scalar multiplets using the multiplication rules for scalar multiplets. One can start with two scalar multiplets (A_i, χ_i, F_i), $i = 1, 2$ and obtain a multiplet whose lowest component which has Weyl weight $w = w_1 + w_2$ and $U(1)_R$ weight $q = q_1 + q_2$ as follows

$$\begin{aligned} A &= A_1 A_2 \ , \\ \chi &= A_1 \chi_2 + A_2 \chi_1 , \\ F &= A_1 F_2 + A_2 F_1 + \tilde{\chi}_1 \chi_2 \ . \end{aligned} \tag{3.2.16}$$

It is also possible to use the inverse of the multiplication rule (3.2.16) to obtain a multiplet with Weyl weight $w = w_1 - w_2$ and $U(1)_R$ weight $q = q_1 - q_2$

$$\begin{aligned} A &= A_1 A_2^{-1} \ , \\ \chi &= A_2^{-1} \chi_1 - A_1 A_2^{-2} \chi_2 \ , \\ F &= A_2^{-1} F_1 - A_1 A_2^{-2} F_2 - A_2^{-2} \tilde{\chi}_2 \chi_1 + A_1 A_2^{-3} \tilde{\chi}_2 \chi_2 \ . \end{aligned} \tag{3.2.17}$$

Given the scalar multiplet (see table 3.1),

$$\Sigma = (\phi, \zeta, S) \ , \tag{3.2.18}$$

the associated inverse multiplet has the components

$$\Sigma^{-1} \equiv (\Phi, \Psi, P) = \Big(\phi^{-1} \ , \ \ -\phi^{-2} \zeta, \ \ -\phi^{-2} S + \phi^{-3} \tilde{\zeta} \zeta \Big) \ . \tag{3.2.19}$$

as can be seen by considering the multiplication of the unit multiplet $(A_1, \chi_1, F_1) = (1, 0, 0)$, which has weights $(\omega, c) = (0, 0)$, with the multiplet $(A_2, \chi_2, F_2) = \Sigma$, by means of the formula (3.2.17).

Next, we note that a scalar multiplet (ϕ, ζ, S) with weights $(w, q) = (\frac{1}{2}, -\frac{1}{2})$ has the corresponding kinetic multiplet with weights $(w, q) = (\frac{3}{2}, -\frac{3}{2})$ given by

$$\mathcal{K} = \big(S^*, -2 \not{\mathcal{D}} (B\zeta)^*, 4 \Box^C \phi^* \big) \ , \tag{3.2.20}$$

where

$$\begin{aligned} \Box^C \phi^* \ &= \ \left(\partial^a - \frac{3}{2} b^a - \frac{i}{2} V^a \right) \mathcal{D}_a \phi^* + {\omega_a}^{ab} \mathcal{D}_b \phi^* + f_a^a \phi^* \\ &+ \frac{1}{2} \tilde{\phi}_a \gamma^a (B\zeta)^* - \frac{1}{2} \tilde{\psi}^a \mathcal{D}_a (B\zeta)^* \ . \end{aligned} \tag{3.2.21}$$

Using the above multiplets as building blocks and using the product formula (3.2.16) we can construct a number of multiplets which will be useful in building actions. To begin with, we consider the four-fold product of Σ obtaining

$$\Sigma^4 : \qquad (Z, \Omega, F) = \Big(\phi^4, \ \ 4\phi^3 \zeta, \ \ 4\phi^3 S + 6\phi^2 \tilde{\zeta} \zeta \Big) \ . \tag{3.2.22}$$

Note that Z has weights $(w,q)=(2,-2)$ and will be useful to construct a cosmological constant invariant. Another multiplet with the same weights $(2,-2)$ is

$$\begin{aligned}\Sigma\times\mathcal{K}\quad : \quad Z' &= \phi S^* \ ,\\ \Omega' &= \zeta S^* - 2\phi\not{\mathcal{D}}(B\zeta)\ ,\\ F' &= 4\phi\Box^C\phi^* + |S|^2 - 2\tilde{\zeta}\not{\mathcal{D}}(B\zeta)^*\ .\end{aligned} \tag{3.2.23}$$

We will use this multiplet to construct the Einstein-Hilbert action. A composite neutral multiplet with $(w,q)=(0,0)$ can be obtained as follows

$$\begin{aligned}&\mathcal{K}\times\Sigma^{-3}\quad :\\ \sigma &= \phi^{-1}S^* \ ,\\ \psi &= -2\phi^{-3}\not{\mathcal{D}}\,(B\zeta)^* - 3\phi^{-4}S^*\zeta\ ,\\ N &= 4\phi^{-3}\Box^C\phi - 3\phi^{-4}\,|S|^2 + 6\phi^{-4}\tilde{\zeta}\not{\mathcal{D}}\,(B\zeta)^*\\ &\quad +6\phi^{-5}S^*\tilde{\lambda}\zeta\ ,\end{aligned} \tag{3.2.24}$$

which can be used to produce new scalar multiplets without changing the weights of the original multiplets. Examples are:

$$\begin{aligned}&(\sigma,\psi,N)^n\times(Z,\Omega,F)\quad :\\ Z^{(n)} &= \sigma^n Z\ ,\\ \Omega^{(n)} &= n\sigma^{n-1}Z\psi + \sigma^n\Omega\ ,\\ F^{(n)} &= \sigma^n F + n\sigma^{n-1}ZN + n(n-1)\sigma^{n-2}Z\tilde{\psi}\psi\\ &\quad +n\sigma^{n-1}\tilde{\psi}\Omega\ .\end{aligned} \tag{3.2.25}$$

$$(\sigma,\psi,N)\times\Sigma: \qquad (\phi',\zeta',S') = \Big(\sigma\phi\ ,\quad \sigma\zeta+\phi\psi\ ,\quad \sigma S+\phi N+\tilde{\zeta}\psi\Big)\ . \tag{3.2.26}$$

Finally, we construct the multiplet (ξ,φ,M), with weights $(\frac{5}{2},-\frac{5}{2})$, in terms of the components of the multiplet (Φ,Ψ,P) as follows

$$\begin{aligned}\xi &= \Box^c P^*\ ,\\ \varphi &= -2\Box^c\not{\mathcal{D}}(B\Psi)^* - 2\mathrm{i}\gamma^\nu\mathcal{D}^\mu\widehat{F}_{\mu\nu}(B\lambda)^* + 2\mathrm{i}\gamma^\nu\widehat{F}_{\mu\nu}\mathcal{D}^\nu(B\lambda)^*\\ &\quad +\mathrm{i}\gamma^{\mu\nu}\not{\mathcal{D}}\widehat{F}_{\mu\nu}(B\lambda)^* + \tfrac{5}{2}\mathrm{i}\gamma^{\mu\nu}\widehat{F}_{\mu\nu}\not{\mathcal{D}}(B\lambda)^*\ ,\\ M &= 4\Box^c\Box^c\Phi^* - 8\mathrm{i}\mathcal{D}^a\widehat{F}_{ab}\mathcal{D}^b\Phi^* - 2\widehat{F}_{ab}\widehat{F}^{ab}\Phi^* + \text{fermions}\ ,\end{aligned} \tag{3.2.27}$$

where we have omitted the complicated fermionic expressions in the composite formula for M as we shall be interested in the bosonic part of an action formula for which this multiplet will be used. With this multiplet we will produce a Ricci tensor squared invariant.

Vector Multiplets. For the construction of the n vector multiplet action, we first introduce a real function $C_{IJ}(\rho)$, which is a function of the vector multiplet scalars ρ^I, and the n vector multiplets are labeled by $I, J, \ldots = 1, 2, \ldots, n$. The lowest component of a vector multiplet can then be composed as

$$\rho_I = C_{IJ} D^J + C_{IJK}\, \bar{\lambda}^J \lambda^K \ . \tag{3.2.28}$$

The label I is fixed, and it differs from the indices that are being summed over. We also define

$$C_{IJK} = \frac{\partial C_{IJ}}{\partial \rho^K}\ , \quad C_{IJKL} = \frac{\partial^2 C_{IJ}}{\partial \rho^K\, \partial \rho^L}\ , \quad C_{IJKLM} = \frac{\partial^3 C_{IJ}}{\partial \rho^K\, \partial \rho^L\, \partial \rho^M}\ . \tag{3.2.29}$$

In order to ensure that ρ_I is the scalar of a superconformal vector multiplet, we impose that the conformal weight of C_{IJ} is $\omega(C_{IJ}) = -1$, and following constraints are satisfied

$$C_{IJK} = C_{I(JK)}\ , \quad C_{IJK}\, \rho^K = -C_{IJ}\ . \tag{3.2.30}$$

Furthermore, additional constraints are needed to ensure that λ_I, D_I and $\widehat{G}_{\mu\nu I}$ are also the elements of a superconformal vector multiplet

$$C_{IJKL}\, \rho^L = -2\, C_{IJK}\ , \quad C_{IJKLM}\, \rho^M = -3\, C_{IJKL}\ . \tag{3.2.31}$$

Applying a sequence of Q- and S-transformations, we find the components of the composite vector multiplet as

$$\begin{aligned}
\rho_I &= C_{IJ}\, D^J + C_{IJK}\, \bar{\lambda}^J \lambda^K\ ,\\
\lambda_I &= \frac{1}{2} C_{IJK}\, D^J \lambda^K - \frac{1}{2} C_{IJ}\, \not{\mathcal{D}} \lambda^J - \frac{1}{4} \mathrm{i}\, C_{IJK}\, \gamma^{\mu\nu} \widehat{G}^J_{\mu\nu} \lambda^K\\
&\quad - \frac{1}{4} C_{IJK}\, \not{\mathcal{D}} \rho^J \lambda^K + C_{IJKL}\, \lambda^L \bar{\lambda}^J \lambda^K\\
D_I &= \frac{1}{2} C_{IJK}\, D^J D^K + \tfrac{1}{4} C_{IJ}\, \Box^C \rho^J - \frac{1}{4} C_{IJK}\, \widehat{G}^J_{\mu\nu}\, \widehat{G}^{\mu\nu K}\\
&\quad + \frac{1}{8} C_{IJK}\, \mathcal{D}_\mu \rho^J \mathcal{D}^\mu \rho^K - \frac{1}{2} C_{IJK}\, \bar{\lambda}^J \not{\mathcal{D}} \lambda^K + \frac{1}{2} C_{IJK}\, \overline{\mathcal{D}_\mu \lambda^J} \gamma^\mu \lambda^K\ ,\\
&\quad - \frac{1}{2} \mathrm{i}\, C_{IJKL}\, \bar{\lambda}^L \gamma^{\mu\nu} \widehat{G}^J_{\mu\nu} \lambda^K + C_{IJKL}\, D^J \bar{\lambda}^K \lambda^L + C_{IJKLM}\, \bar{\lambda}^J \lambda^K \bar{\lambda}^L \lambda^M\ ,\\
\widehat{G}_{\mu\nu I} &= \tfrac{1}{2} \mathcal{D}_\sigma \Big(\epsilon_{\lambda\mu\nu}\, C_{IJ}\, \widehat{G}^{\sigma\lambda J} \Big) + 2\mathrm{i} \mathcal{D}_{[\mu} \Big(C_{IJK}\, \bar{\lambda}^J \gamma_{\nu]} \lambda^K \Big)\\
&\quad - \frac{1}{4} C_{IJ}\, \rho^J \widehat{F}_{\mu\nu}\ ,
\end{aligned} \tag{3.2.32}$$

where the superconformal d'Alambertian for ρ^I is given by

$$\begin{aligned}\Box^C\rho^I &= \left(\partial^a - 2b^a + \omega_b{}^{ba}\right)\mathcal{D}_a\rho^I + 2f_a^a\rho^I \\ &\quad + \left(-\mathrm{i}\bar{\psi}^a\mathcal{D}_a\lambda^I + i\bar{\phi}_a\gamma^a\lambda^I + h.c.\right) . \end{aligned} \tag{3.2.33}$$

Note that $\widehat{G}_{\mu\nu I}$ satisfies the Bianchi identity due to the constraints (3.2.30).

The composition formula (3.2.32) can be truncated to a map between two vector multiplets by choosing $C_{21} = \rho^{-1}$, in which case one obtains, for the bosonic fields,

$$\begin{aligned}\rho' &= \rho^{-1}D - \rho^{-2}\bar{\lambda}\lambda , \\ D' &= -\frac{1}{2}\rho^{-2}D^2 + \frac{1}{4}\rho^{-1}\Box^C\rho + \frac{1}{4}\rho^{-2}\widehat{G}_{\mu\nu}\widehat{G}^{\mu\nu} - \frac{1}{8}\rho^{-2}\mathcal{D}_\mu\rho\,\mathcal{D}^\mu\rho \\ \widehat{G}'_{\mu\nu} &= \frac{1}{2}\mathcal{D}_\sigma\left(\epsilon_{\lambda\mu\nu}\rho^{-1}\widehat{G}^{\sigma\lambda}\right) - \frac{1}{4}\widehat{F}_{\mu\nu} , \end{aligned} \tag{3.2.34}$$

where 1 labels the multiplet $(\rho, C_\mu, \lambda, D)$, and 2 labels the multiplet $(\rho', C'_\mu, \lambda', D')$. Another composite multiplet is obtained by choosing $C_{31} = -\rho^{-2}\rho'$ and $C_{32} = \rho^{-1}$ in the composition formula (3.2.32). The bosonic components of the composite multiplet $(\rho'', \lambda'', C''_\mu, D'')$, labeled by 3, can then be written as

$$\begin{aligned}\rho'' &= -\rho^{-2}\rho' D + \rho^{-1}D' , \\ D'' &= \rho^{-3}\rho' D^2 - \rho^{-2}DD' - \frac{1}{4}\rho^{-2}\rho'\Box^C\rho + \frac{1}{4}\rho^{-1}\Box^C\rho' \\ &\quad -\frac{1}{2}\rho^{-3}\rho'\widehat{G}_{\mu\nu}\widehat{G}^{\mu\nu} + \frac{1}{2}\rho^{-2}\widehat{G}'_{\mu\nu}\widehat{G}^{\mu\nu} + \frac{1}{4}\rho^{-3}\rho'\mathcal{D}_\mu\rho\,\mathcal{D}^\mu\rho \\ &\quad -\frac{1}{4}\rho^{-2}\mathcal{D}_\mu\rho'\mathcal{D}^\mu\rho , \\ \hat{G}''_{\mu\nu} &= \frac{1}{2}\epsilon_{\lambda\mu\nu}\mathcal{D}_\sigma\left(-\rho^{-2}\rho'\widehat{G}^{\sigma\lambda} + \rho^{-1}\widehat{G}'^{\sigma\lambda}\right) . \end{aligned} \tag{3.2.35}$$

3.2.3 Action Formulae

In this section, we collect the action formulae for scalar and vector multiplets that we shall use in the subsequent sections when constructing supergravity models. The construction of the supergravity invariants require coupling the Weyl multiplet to at least one compensating multiplet. We, therefore, consider two classes of supersymmetric invariants in accordance with the compensating multiplet under consideration. Finally, we also present the superconformal completion of the Chern-Simons action [63], which does not require a compensating multiplet.

Scalar Multiplet Actions. We start with the action for a scalar multiplet (Z, Ω, F)

$$e^{-1}\mathcal{L}_F = Re\left(F - \tilde{\psi}_\mu\gamma^\mu\Omega - Z\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\right) , \qquad (3.2.36)$$

which is invariant under dilatations and $U(1)_R$ transformations since the highest component field F has weight $(w, q) = (3, 0)$.

The composite multiplet given in (3.2.25) can be used in the action formula (3.2.36) to produce

$$\begin{aligned} e^{-1}\mathcal{L}_{F(n)} &= \mathrm{Re}\Big(\sigma^n F + n\,\sigma^{n-1} ZN + n(n-1)\sigma^{n-2} Z\tilde{\psi}\,\psi + n\,\sigma^{n-1}\tilde{\psi}\Omega \\ &\quad -n\sigma^{n-1} Z\,\tilde{\psi}_\mu\gamma^\mu\psi - \sigma^n\tilde{\psi}_\mu\gamma^\mu\,\Omega - \sigma^n Z\,\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\Big) , \qquad (3.2.37) \end{aligned}$$

which we shall use below to obtain an action providing a supersymmetric completion of RS^n.

Next, we use the components of the composite scalar multiplet multiplet (3.2.23) in the action formula (3.2.36) which yields the following action that will be used to construct the supersymmetric completions of the Einstein-Hilbert term as well as the R^2 term

$$\begin{aligned} e^{-1}\mathcal{L}_{\mathrm{K}} &= 4\phi\,\Box^C\phi^* + |S|^2 - 2\,\tilde{\zeta}\,\not{\mathcal{D}}\,(B\zeta)^* + 2\,\phi\,\tilde{\psi}_\mu\gamma^\mu\not{\mathcal{D}}\,(B\zeta)^* \\ &\quad -S^*\tilde{\psi}_\mu\gamma^\mu\zeta - \phi\,S^*\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\ . \qquad (3.2.38) \end{aligned}$$

We next consider the scalar multiplets (ξ, φ, M) and (Φ, Ψ, P). Using the multiplication rule (3.2.16), the action describing the coupling of these multiplets is given by

$$\begin{aligned} e^{-1}\mathcal{L}_{\xi\Phi} &= \mathrm{Re}\Big(\xi P + \Phi M + \tilde{\Psi}\varphi - \Phi\,\tilde{\psi}_\mu\gamma^\mu\varphi - \xi\tilde{\psi}_\mu\gamma^\mu\Psi \\ &\quad -\Phi\,\xi\,\tilde{\psi}_\mu\gamma^{\mu\nu}\psi_\nu\Big)\ . \qquad (3.2.39) \end{aligned}$$

Using the composite expressions (3.2.27), the bosonic part of the action that gives rise to the $R^2_{\mu\nu}$ invariant is given by

$$\begin{aligned} e^{-1}\mathcal{L}_\Phi &= \mathrm{Re}\Big(4\Phi\,\Box^C\Box^C\Phi^* + P\Box^C P^* - 8\mathrm{i}\Phi\,\mathcal{D}^a\widehat{F}_{ab}\mathcal{D}^b\Phi^* \\ &\quad -2\widehat{F}_{ab}\,\widehat{F}^{ab}\Phi\Phi^*\Big)\ . \qquad (3.2.40) \end{aligned}$$

Vector Multiplet Actions. Supersymmetric Lagrangians for vector multiplet can be constructed starting from an action formula which describes the coupling of two vector multiplets as follows

$$
\begin{aligned}
e^{-1}\mathcal{L}_{DD'} \quad = \quad & \rho D' + \rho' D + 2\left(\bar{\lambda}\lambda' + h.c.\right) - 2\epsilon^{\mu\nu\rho}C_\mu \partial_\nu C'_\rho \\
& -\frac{1}{2}\mathrm{i}\left(\rho\,\bar{\psi}_\mu\gamma^\mu\lambda' + \rho'\bar{\psi}_\mu\gamma^\mu\lambda + h.c.\right) \\
& -\frac{1}{8}\left(\rho\rho'\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + h.c.\right)\ .
\end{aligned} \tag{3.2.41}
$$

As a special case, one can set the primed and the un-primed multiplet equal to each other, obtaining [69]

$$
\begin{aligned}
e^{-1}\mathcal{L}_D \quad = \quad & 2\rho D - \epsilon^{\mu\nu\rho}C_\mu\, G_{\nu\rho} + 4\bar{\lambda}\,\lambda - \mathrm{i}\left(\rho\,\bar{\psi}_\mu\gamma^\mu\lambda + h.c.\right) \\
& -\frac{1}{4}(\rho^2\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + h.c)\ .
\end{aligned} \tag{3.2.42}
$$

Using the composite multiplets (3.2.34) in this action formula, we also obtain the conformal vector multiplet action

$$
\begin{aligned}
e^{-1}\mathcal{L}_V \quad = \quad & \frac{1}{4}\Box^C\rho + \frac{1}{2}\rho^{-1}D^2 - \frac{1}{8}\rho^{-1}\partial_\mu\rho\,\partial^\mu\rho - \frac{1}{4}\rho^{-1}G_{\mu\nu}\,G^{\mu\nu} \\
& +\frac{1}{2}\epsilon^{\mu\nu\rho}\,C_\mu\,\partial_\nu V_\rho\ ,
\end{aligned} \tag{3.2.43}
$$

up to fermionic terms.

Considering the coupling of a primed and double-primed multiplets in accordance with the action formula (3.2.41), and employing the composite expressions (3.2.35) result in an action that will be used in the construction of a supersymmetric completion of the R^2 term,

$$
\begin{aligned}
e^{-1}\mathcal{L}_{VV'} \quad = \quad & \rho^{-3}(\rho')^2 D^2 - 2\rho^{-2}\rho' D D' + \rho^{-1}(D')^2 + \tfrac{1}{4}\rho^{-1}\rho'\Box^c\rho' \\
& -\frac{1}{4}\rho^{-2}\rho'^2\Box^c\rho + \frac{1}{4}\rho^{-3}\rho'^2\mathcal{D}_\mu\rho\,\mathcal{D}^\mu\rho - \frac{1}{4}\rho'\rho^{-2}\mathcal{D}_\mu\rho'\mathcal{D}^\mu\rho \\
& -\frac{1}{2}\rho^{-3}(\rho')^2\widehat{G}_{\mu\nu}\,\widehat{G}^{\mu\nu} + \rho'\rho^{-2}\widehat{G}'_{\mu\nu}\,\widehat{G}^{\mu\nu} \\
& -\frac{1}{2}\rho^{-1}\widehat{G}'_{\mu\nu}\,\widehat{G}'^{\mu\nu}\,,
\end{aligned} \tag{3.2.44}
$$

where we have given only the terms that contributes to the bosonic part of the action. More generally, we obtain the most general 2−derivative vector multiplet coupling, by using the action formula (3.2.41), as

$$
\begin{aligned}
e^{-1}\mathcal{L}_{V_I} \;=\; & \frac{1}{4}C_{IJ}\,\rho^I\Box^c\rho^J+\frac{1}{8}C_{IJK}\,\rho^I\mathcal{D}_\mu\rho^J\mathcal{D}^\mu\rho^K-\frac{1}{2}C_{IJ}\,\widehat{G}^I_{\mu\nu}\widehat{G}^{\mu\nu J}\\
& -\frac{1}{4}C_{IJK}\,\rho^I\widehat{G}^J_{\mu\nu}\widehat{G}^{\mu\nu K}+C_{IJ}\,D^ID^J+\frac{1}{2}C_{IJK}\,\rho^ID^JD^K\\
& +\frac{1}{4}C_{IJ}\,\rho^J\epsilon^{\mu\nu\rho}C^I_\mu F_{\nu\rho}\,. \qquad (3.2.45)
\end{aligned}
$$

Note that the index I is fixed to represent a certain multiplet by construction due to (3.2.32), and summing over I indices correspond to summing different off-shell invariants.

Finally, there also exists an action that constitutes the superconformal completion of the Lorentz Chern-Simons term. It is given by [63]

$$
\begin{aligned}
\mathcal{L}_{\mathrm{CS}} \;=\; & -\frac{1}{4}\,\varepsilon^{\mu\nu\rho}\left[R_{\mu\nu}{}^{ab}(\omega)\,\omega_{\rho ab}+\frac{2}{3}\omega_\mu{}^{ab}\,\omega_{\nu b}{}^c\,\omega_{\rho ca}\right]+\varepsilon^{\mu\nu\rho}F_{\mu\nu}V_\rho\\
& -\bar{R}^\mu\,\gamma_\nu\gamma_\mu R^\nu \qquad (3.2.46)
\end{aligned}
$$

where the Hodge dual of the gravitino curvature is defined by

$$
R^\mu=\varepsilon^{\mu\nu\rho}(D_\nu(\omega)-\mathrm{i}V_\nu)\psi_\rho\;. \qquad (3.2.47)
$$

The supersymmetric Chern-Simons action is invariant under the Weyl multiplet transformation rules (3.2.6). Therefore, it can be used for both $\mathcal{N}=(1,1)$ and $\mathcal{N}=(2,0)$ supergravities.

3.3 $\mathcal{N}=(1,1)$ Supergravity Models

The off-shell $\mathcal{N}=(1,1)$ Poincaré supergravity and the supersymmetric completion of the cosmological term are already given in the literature, and they are also referred to as Type I minimal supergravity or three dimensional old minimal supergravity [63–65, 67]. Here we shall derive them from the superconformal tensor calculus point of view, which will also serve to establish our notation and conventions. Using the superconformal tensor calculus we shall construct three new invariants, namely the supersymmetric completion of the R^2 and $R^2_{\mu\nu}$ terms and of $(RS^2+\text{h.c.})$, where S is the complex auxiliary field. The last invariant is key to the construction of a ghost-free massive supergravity theory with $\mathcal{N}=(1,1)$ supersymmetry.

3.3.1 $\mathcal{N}=(1,1)$ Cosmological Poincaré Supergravity

The off-shell Poincaré supergravity action is readily obtained from the action formula (3.2.38) by fixing the dilatation, conformal boost and special supersymmetry transformation, by imposing

$$\phi = 1 \ , \qquad \zeta = 0 \ , \qquad b_\mu = 0 \ . \tag{3.3.1}$$

The first one fixes the dilatation and $U(1)_R$ transformation, the second fixes the S-supersymmetry and the last one fixes the special conformal transformations. Maintaining these gauge conditions imply that

$$\begin{aligned} \Lambda_D &= \mathrm{i}\Lambda = 0 \ , \\ \Lambda_{K\mu} &= \frac{1}{4}\bar{\eta}\,\psi_\mu - \frac{1}{4}\bar{\epsilon}\,\phi_\mu + h.c. \ , \\ \eta &= -\frac{1}{2}\mathrm{i}\gamma^\nu V_\nu \epsilon + \frac{1}{2}S\,(B\epsilon)^* \ . \end{aligned} \tag{3.3.2}$$

These decomposition laws imply the super supersymmetry transformation rules

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu + h.c., \\ \delta\psi_\mu &= D_\mu(\omega)\,\epsilon - \frac{1}{2}\mathrm{i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon - \frac{1}{2}S\gamma_\mu\,(B\epsilon)^* \, , \\ \delta V_\mu &= \frac{1}{8}\mathrm{i}\bar{\epsilon}\,\gamma^{\nu\rho}\gamma_\mu\,(\psi_{\nu\rho} - \mathrm{i}V_\sigma\gamma^\sigma\gamma_\nu\,\psi_\rho - S\gamma_\nu\,(B\psi_\rho)^*) + h.c., \\ \delta S &= -\frac{1}{4}\tilde{\epsilon}\,\gamma^{\mu\nu}\,(\psi_{\mu\nu} - \mathrm{i}V_\sigma\,\gamma^\sigma\gamma_\mu\psi_\nu - S\gamma_\mu\,(B\psi_\nu)^*)\ , \end{aligned} \tag{3.3.3}$$

where

$$D_\mu(\omega)\epsilon = (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab})\epsilon \ , \qquad \psi_{\mu\nu} = 2D_{[\mu}(\omega)\psi_{\nu]} \ . \tag{3.3.4}$$

Using the gauge fixing conditions (3.3.1) in the action (3.2.38) gives the action of Poincaré supergravity

$$e^{-1}\mathcal{L}_{EH} = R + 2V^2 - 2\,|S|^2 - \left(\bar{\psi}_\mu\,\gamma^{\mu\nu\rho}D_\nu(\omega)\psi_\rho + h.c.\right) \ . \tag{3.3.5}$$

where $V^2 := V_\mu V^\mu$. Next, we construct the supersymmetric cosmological term by using the multiplet (Z,Ω,F) given in Eq. (3.2.22) in the action formula (3.2.36), imposing the gauge fixing conditions (3.3.1), and multiplying the action by $1/2$. We obtain

$$e^{-1}\mathcal{L}_C = S - \frac{1}{4}\tilde{\psi}_\mu\,\gamma^{\mu\nu}\,\psi_\nu + h.c. \tag{3.3.6}$$

3.3.2 $\mathcal{N} = (1,1)$ Higher Dimensional Invariants

We begin with the construction of the R^2 invariant. To this end, we employ the composite scalar multiplet Σ' from (3.2.26) in the action formula (3.2.38). In the resulting action we use the composite neutral multiplet from (3.2.24). Subsequently we fix the extra gauge symmetries as in (3.3.1). These are straightforward manipulations which give the full R^2 invariant whose bosonic part is given by

$$\begin{aligned} e^{-1}\mathcal{L}_{R^2} &= R^2 + 16\,|S|^4 + 4(V^2)^2 + 6R\,|S|^2 + 4RV^2 + 12\,|S|^2\,V^2 \\ &\quad -16\partial_\mu S\,\partial^\mu S^* - 8\mathrm{i}V^\mu S^* \overleftrightarrow{\partial_\mu} S + 16\,(\nabla_\mu V^\mu)^2 \ , \end{aligned} \tag{3.3.7}$$

where $S^* \overleftrightarrow{\partial_\mu} S = S^* \partial_\mu S - S \partial_\mu S^*$.

To construct the supersymmetric $R^2_{\mu\nu}$ invariant, we employ the action formula (3.2.40). Substituting for the components of the multiplet (Φ, Ψ, P) given in (3.2.19), and imposing gauge-fixing conditions (3.3.1), lead to the following supersymmetric completion of the Ricci tensor squared

$$\begin{aligned} e^{-1}\mathcal{L}_{R^2_{\mu\nu}+R^2} &= R_{\mu\nu}R^{\mu\nu} - \frac{23}{64}R^2 - \tfrac{1}{32}R\,|S|^2 - R_{\mu\nu}V^\mu V^\nu + \frac{5}{16}RV^2 + \frac{1}{16}(V^2)^2 \\ &\quad -\frac{25}{16}V^2\,|S|^2 - \frac{1}{4}\partial_\mu S\partial^\mu S^* - \frac{5}{8}\mathrm{i}V^\mu S^* \overleftrightarrow{\partial_\mu} S + \frac{1}{4}\,(\nabla_\mu V^\mu)^2 \\ &\quad -F_{\mu\nu}F^{\mu\nu} \ , \end{aligned} \tag{3.3.8}$$

where we have exhibited the bosonic part of the Lagrangian. The R^2 dependent part can be removed by adding $\frac{23}{64}\mathcal{L}_{R^2}$ to this Lagrangian, obtaining

$$\begin{aligned} e^{-1}\mathcal{L}_{R^2_{\mu\nu}} &= R_{\mu\nu}R^{\mu\nu} - R_{\mu\nu}V^\mu V^\nu + \frac{7}{4}RV^2 + \frac{17}{8}R\,|S|^2 + \frac{23}{4}|S|^4 - F_{\mu\nu}F^{\mu\nu} \\ &\quad +6\,(\nabla_\mu V^\mu)^2 + \frac{3}{2}(V^2)^2 + \tfrac{11}{4}V^2\,|S|^2 - 6\partial_\mu S\partial^\mu S^* \\ &\quad -\frac{7}{2}\mathrm{i}V^\mu S^* \overleftrightarrow{\partial_\mu} S \ . \end{aligned} \tag{3.3.9}$$

Next, we construct the supersymmetric completion of the RS^n term. To this end, we employ the action formula (3.2.37), in which we substitute for the components of the multiplets (σ, ψ, N) and (Z', Ω', F') which are given in (3.2.24) and (3.2.23), respectively. Imposing the gauge fixing conditions

(3.3.1) in the resulting Lagrangian, and dividing by an overall constant factor of $-(n+1)$, we obtain

$$e^{-1}\mathcal{L}^{(n)} = \frac{1}{2}\left[R + \frac{2(3n-1)}{n+1}|S|^2 + 2V^2 - 4i\nabla_\mu V^\mu\right] S^n + h.c.\ , \tag{3.3.10}$$

where we have given the bosonic part of the result. Note that the $n = 0$ case agrees with the Poincaré supergravity action (3.3.5) which we obtained by an alternative procedure.

3.3.3 $\mathcal{N} = (1, 1)$ Generalized Massive Supergravity

We now consider a combination of the invariants up to dimension four, namely,

$$\begin{aligned} I = \frac{1}{\kappa^2}\int d^3x \Big[\frac{1}{2}M\mathcal{L}_C + \sigma\mathcal{L}_{EH} + \frac{1}{\mu}\mathcal{L}_{CS} + \frac{1}{\nu}\mathcal{L}_{RS} \\ + \frac{1}{m^2}\mathcal{L}_{R^2_{\mu\nu}} + c_1\mathcal{L}_{R^2} + c_2\mathcal{L}_{RS^2}\Big]\ , \end{aligned} \tag{3.3.11}$$

where $(\sigma, M, \mu, \nu, m^2, c_1, c_2)$ are arbitrary real constants. Defining

$$S = A + \mathrm{i}B\ , \tag{3.3.12}$$

where A and B are real scalar fields, the $\mathcal{N} = (1, 0)$ supersymmetric truncation is achieved by setting $V_\mu = 0$ and $B = 0$. In that case generalized massive gravity (GMG) model is defined by setting

$$\nu = \infty\ , \qquad c_1 = -\frac{3}{8m^2}\ , \qquad c_2 = \frac{1}{8m^2}\ . \tag{3.3.13}$$

With these choices of the coupling constants the model expanded around a supersymmetric AdS_3 vacuum propagates only helicity ± 2 and $\pm 3/2$ states with AdS energies that respect perturbative unitarity. We shall define the $\mathcal{N} = (1, 1)$ supersymmetric version of the GMG model by choosing the coupling constants as in (3.3.13), since the quadratic action obtained by expanding around the supersymmetric AdS_3 vacuum contains the $\mathcal{N} = (1, 0)$ sector as an independent subsector. In this case the complete Lagrangian becomes

$$
\begin{aligned}
e^{-1}\mathcal{L}_{GMG} &= \sigma(R+2V^2-2|S|^2)+MA \\
&\quad -\frac{1}{4\mu}\left[\epsilon^{\mu\nu\rho}\left(R_{\mu\nu}{}^{ab}\,\omega_{\rho ab}+\frac{2}{3}\omega_{\mu}{}^{ab}\,\omega_{\nu b}{}^{c}\,\omega_{\rho ca}\right)-8\epsilon^{\mu\nu\rho}V_\mu\partial_\nu V_\rho\right] \\
&\quad +\frac{1}{m^2}\Big[R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2-R_{\mu\nu}V^\mu V^\nu-F_{\mu\nu}F^{\mu\nu}+\frac{1}{4}R(V^2-B^2) \\
&\qquad +\frac{1}{6}|S|^2(A^2-4B^2)-\tfrac{1}{2}V^2(3A^2+4B^2) \\
&\qquad -2V^\mu B\partial_\mu A\Big]\,.
\end{aligned} \tag{3.3.14}
$$

Remarkably, all terms proportional to $|\partial S|^2, RA^2, (\nabla_\mu V^\mu)^2$ and $(V_\mu V^\mu)^2$ have cancelled. The cancellation of the $|\partial S|^2$ and RA^2 require c_1 and c_2 to have the values given in (3.3.13), and it is crucial for having a ghost-free propagation of massive modes, as we shall see below.

Notwithstanding that the fields A and B do not propagate, their elimination yields highly nonlinear interactions, including those which take the form of an infinite power series in the Ricci curvature scalar R. In that sense, the notion of a supersymmetric GMG model is extended here, compared to the case of the $\mathcal{N}=(1,0)$ supersymmetric version where the single auxiliary field, a real scalar, can be eliminated from the action by means of its algebraic equation of motion, yielding the standard bosonic GMG action. Nonetheless, in both cases the action contains the combination $(R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2)$, and if we take this feature to be the defining one for an extended definition of super GMG models, it is clear that such an extension is not unique. In the models under consideration, there is no need for eliminating the auxiliary fields, even when they are non-propagating, unless their field equations are algebraic ones.

Turning to the model with parameters chosen as in (3.3.13), we shall focus on a maximally supersymmetric AdS vacuum and determine the spectrum of fluctuations around it. In view of the results of [75], the following background is maximally supersymmetric

$$
\bar{R}_{\mu\nu}=-\frac{2}{\ell^2}\bar{g}_{\mu\nu}\,,\qquad \bar{A}=-\frac{1}{\ell}\,,\qquad \bar{V}_\mu=0\,,\qquad \bar{B}=0\,, \tag{3.3.15}
$$

where $\bar{g}_{\mu\nu}$ is the AdS_3 metric, and ℓ is the AdS_3 radius which must obey the equation

$$
4\sigma+\ell M+\frac{2}{3\ell^2m^2}=0\,. \tag{3.3.16}
$$

Let us define the fluctuation fields around this vacuum as

$$\begin{aligned} g_{\mu\nu} &= \bar{g}_{\mu\nu}\left(1+\frac{1}{3}h\right)+H_{\mu\nu}\ , \qquad \bar{g}^{\mu\nu}H_{\mu\nu}=0\ , \\ A &= \bar{A}+a\ , \qquad B=\bar{B}+b\ , \qquad V_\mu=\bar{V}_\mu+v_\mu\ , \end{aligned} \tag{3.3.17}$$

and choose the gauge condition

$$\bar{\nabla}^\mu H_{\mu\nu}=0\ . \tag{3.3.18}$$

The linearized field equations then take the form

$$\begin{aligned} &[\mathcal{D}(1)\,\mathcal{D}(-1)\,\mathcal{D}(\eta_+)\,\mathcal{D}(\eta_-)\,H]_{\mu\nu}=-\frac{1}{3\ell^2}\left(\bar{\nabla}_\mu\bar{\nabla}_\nu-\frac{1}{3}\bar{g}_{\mu\nu}\bar{\Box}\right)h\ , \\ &\frac{\Omega}{m^2}\left(\ell^2\bar{\Box}-3\right)h=0\ , \qquad \frac{\Omega}{m^2}\,a=0\ , \qquad \frac{\Omega}{m^2}\,b=0\ , \\ &\frac{\Omega}{m^2}\,[\mathcal{D}(\eta_+)\,\mathcal{D}(\eta_-)v]_\mu=0\ , \end{aligned} \tag{3.3.19}$$

where

$$\eta_\pm=\Omega^{-1}\left(-\frac{\ell m^2}{2\mu}\pm\sqrt{\frac{\ell^2m^4}{4\mu^2}-\Omega}\right)\ , \qquad \Omega\equiv\sigma\,\ell^2m^2-\tfrac{1}{2}\ , \tag{3.3.20}$$

and $\mathcal{D}(\eta)$ is a first-order linear differential operator, parametrized by a dimensionless constant η, that acts on a rank-$s\geq 1$ totally symmetric, traceless and divergence-free tensor as follows

$$\left[\mathcal{D}(\eta)\,\varphi^{(s)}\right]_{\mu_1\cdots\mu_s}=[\mathcal{D}(\eta)]_{\mu_1}{}^{\rho}\,\varphi^{(s)}_{\rho\mu_2\cdots\mu_s}\ , \qquad [\mathcal{D}\,(\eta)]_\mu{}^{\nu}=\ell^{-1}\,\delta_\mu^\nu+\frac{\eta}{\sqrt{|\bar{g}|}}\,\varepsilon_\mu{}^{\tau\nu}\bar{\nabla}_\tau\ . \tag{3.3.21}$$

The equations for $H_{\mu\nu}$ and h agree precisely with those arising in the $\mathcal{N}=(1,0)$ GMG model [58, 59] whose spectrum was studied in detail in [60], extending earlier results of [76] for the bosonic model. For "non-critical" values of the couplings summarized by the condition

$$m^{-2}\Omega(\eta_+-\eta_-)(|\eta_+|-1)(|\eta_-|-1)\neq 0. \tag{3.3.22}$$

These equations describe the UIRs of $SO(2,2)$ with lowest weight (E_0,s). $\ell^{-1}E_0$ is the lowest energy, and s is the helicity, where their values given by

$$(E_0,s):\quad (2,2)\ ,\quad (2,-2)\ ,\qquad \left(1+\frac{1}{|\eta_+|},\frac{2\eta_+}{|\eta_+|}\right)\ ,\quad \left(1+\frac{1}{|\eta_-|},\frac{2\eta_-}{|\eta_-|}\right)\ . \tag{3.3.23}$$

The new degrees of freedom arising here are furnished by the field v_μ. From Eq. (3.3.19) it follows that the propagating modes have the representation content

$$(E_0, s): \quad \left(1 + \frac{1}{|\eta_+|}, \frac{\eta_+}{|\eta_+|}\right) \ , \quad \left(1 + \frac{1}{|\eta_-|}, \frac{\eta_-}{|\eta_-|}\right) \ . \tag{3.3.24}$$

Together with the spin-2 modes displayed in (3.3.23), these form the bosonic content of a massive spin-2 supermultiplet of $\mathcal{N} = (2,0)$ supersymmetry in three dimensions. The structure of this multiplet is similar to the one studied in detail in [77]. The critical versions the $\mathcal{N} = (2,0)$ GMG model arise if the following condition is satisfied

$$m^{-2}\Omega(\eta_+ - \eta_-)(|\eta_+| - 1)(|\eta_-| - 1) = 0 \ . \tag{3.3.25}$$

We shall not examine these points here but we note that the spin-2 sector at critical point has been analyzed in considerable detail in [58]. As for the spin-1 sector, it follows a pattern similar to the one discussed in great detail in [77], in the context of a parent supergravity theory whose off-shell degrees of freedom coincide with those of $\mathcal{N} = (2,0)$ supergravity in three dimensions upon a circle reduction.

3.4 $\mathcal{N} = (2,0)$ Supergravity Models

This section is devoted to the construction of $\mathcal{N} = (2,0)$ supergravity invariants. The Poincaré supergravity and its cosmological extension has already been given in [67, 68], and it is also referred to as Type II minimal supergravity. In this section, we first introduce our gauge fixing choices, and construct the Poincaré supergravity and the supersymmetric cosmological constant based on the conformal vector multiplet actions discussed in section 3.2. We then proceed to the four-derivative invariants and construct the supersymmetric R^2 invariant by the same method. Finally, establishing an analogy between the non-abelian vector multiplet and the Poincaré multiplet, we construct the $R^2_{\mu\nu}$ invariant, and discuss the ghost-free maximally supersymmetric vacuum of the four-derivative extended theory.

3.4.1 $\mathcal{N} = (2,0)$ Cosmological Poincaré Supergravity

The off-shell Poincaré supergravity is obtained from the action formula (3.2.43) and by gauge fixing the superconformal transformations by imposing the following gauge conditions

$$\rho = 1, \qquad \lambda = 0, \qquad b_\mu = 0 \ , \tag{3.4.1}$$

where the first choice fixes dilatations, the second one fixes the S-supersymmetry and the third fixes the special conformal symmetry. These gauge choices are maintained provided that

$$\begin{aligned} \Lambda_D &= 0 \ , \\ \Lambda_{K\mu} &= -\frac{1}{4}\bar{\epsilon}\,\phi_\mu + \frac{1}{4}\bar{\eta}\,\psi_\mu + h.c. \ , \\ \eta &= \frac{1}{2}\mathrm{i}\gamma\cdot\widehat{G}\,\epsilon + D\epsilon \ . \end{aligned} \tag{3.4.2}$$

We thus end up with the new minimal Poincaré multiplet consisting of a dreibein e_μ^a, a gravitino ψ_μ, a $U(1)_R$ symmetry gauge field V_μ, a vector gauge field C_μ and an auxiliary scalar D. The resulting local supersymmetry transformation rules are given as

$$\begin{aligned} \delta e_\mu{}^a &= \frac{1}{2}\,\bar{\epsilon}\,\gamma^a\,\psi_\mu + h.c., \\ \delta\psi_\mu &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab} - \mathrm{i}V_\mu\right)\epsilon - \frac{1}{2}\mathrm{i}\,\gamma_\mu\gamma\cdot\widehat{G}\epsilon - \gamma_\mu\,D\epsilon, \\ \delta C_\mu &= -\frac{1}{4}\mathrm{i}\,\bar{\epsilon}\,\psi_\mu + h.c., \\ \delta V_\mu &= -\frac{1}{2}\mathrm{i}\bar{\epsilon}\,\gamma^\nu\widehat{\psi}_{\mu\nu} + \frac{1}{8}\mathrm{i}\bar{\epsilon}\,\gamma_\mu\gamma\cdot\widehat{\psi} - \frac{1}{2}\bar{\epsilon}\,\gamma\cdot\widehat{G}\,\psi_\mu + \mathrm{i}D\bar{\epsilon}\,\psi_\mu + h.c., \\ \delta D &= -\frac{1}{16}\bar{\epsilon}\,\gamma\cdot\widehat{\psi} + h.c., \end{aligned} \tag{3.4.3}$$

where the $U(1)_R$ covariant gravitino field strength is given by

$$\widehat{\psi}_{\mu\nu} = 2\left(\partial_{[\mu} + \frac{1}{4}\omega_{[\mu|}{}^{ab}\,\gamma_{ab} - \mathrm{i}V_{[\mu}\right)\psi_{\nu]} - \mathrm{i}\gamma_{[\mu}\gamma\cdot\widehat{G}\psi_{\nu]} - 2D\,\gamma_{[\mu}\psi_{\nu]} \ . \tag{3.4.4}$$

Substituting the gauge fixing conditions (3.4.1) into the Lagrangian (3.2.43), and rescaling with a factor of -16, we obtain the following Poincaré supergravity Lagrangian

$$e^{-1}\mathcal{L}_{EH} = R - 2G^2 - 8D^2 - 8\epsilon^{\mu\nu\rho}\,C_\mu\,\partial_\nu V_\rho \ , \tag{3.4.5}$$

where we have defined

$$G_\mu := \epsilon_{\mu\nu\rho} G^{\nu\rho} \ , \qquad G^2 := G_\mu G^\mu \, . \tag{3.4.6}$$

Consequently, G_μ is a covariantly conserved tensor $\nabla^\mu G_\mu = 0$. A supersymmetric cosmological constant can be added to the Poincaré supergravity Lagrangian (3.4.5), which can be obtained from the action formula (3.2.42), and imposing the gauge fixing choices (3.4.1). We then obtain

$$e^{-1}\mathcal{L}_C = 2D - \epsilon^{\mu\nu\rho}\, C_\mu\, G_{\nu\rho} - \left(\frac{1}{8}\bar{\psi}_\mu\, \gamma^{\mu\nu}\, \psi_\nu + h.c.\right) \, . \tag{3.4.7}$$

3.4.2 $\mathcal{N} = (2,0)$ RD and R^2 Invariants

For the construction of the RD invariant, we consider the vector multiplet action (3.2.42) for the primed vector multiplet $(\rho', C'_\mu, \lambda', D')$. Using the composite expressions given in (3.2.34) and fixing the redundant superconformal symmetries by using the gauge fixing choices (3.4.1), gives the supersymmetric completion of the RD action

$$e^{-1}\mathcal{L}_{RD} = RD + 8D^3 - 2G^{\mu\nu}\left(F_{\mu\nu} + \nabla_\mu G_\nu + 2DG_{\mu\nu}\right) + \frac{1}{2}\epsilon^{\mu\nu\rho}\, V_\mu\, F_{\nu\rho} \ , \tag{3.4.8}$$

where we have rescaled the Lagrangian with an overall factor of -8. Note that although the RD invariant and the Lorentz-Chern-Simons invariant (3.2.46) have the same conformal $\epsilon^{\mu\nu\rho}V_\mu F_{\nu\rho}$ term, the RD invariant is not conformally invariant as can be understood from the existence of the Ricci scalar. Such non-conformal invariants are studied in detail in the context of Chern-Simons contact terms in three dimensions [78–80].

Next, we construct the supersymmetric completion of R^2. Using the composition formula (3.2.34) and employing the gauge fixing choices (3.4.1) in the action formula (3.2.44), we obtain

$$e^{-1}\mathcal{L}_{R^2} = (R + 24D^2 + 2G^2)^2 - 8\left(F_{\mu\nu} + 2\nabla_{[\mu}G_{\nu]} + 4DG_{\mu\nu}\right)^2 + 64D\Box D \, . \tag{3.4.9}$$

3.4.3 $\mathcal{N} = (2,0)$ $R^2_{\mu\nu}$ Invariant

The supersymmetric completion of the Ricci tensor-squared term is most conveniently obtained by establishing a map between the Yang-Mills and supergravity multiplets. To do so, we begin by gauge fixing the nonabelian version of the transformation rules (3.2.14) in accordance with (3.4.1), obtaining

$$
\begin{aligned}
\delta C_\mu^I &= \frac{1}{2}\bar\epsilon\,\gamma_\mu\lambda^I - \frac{1}{4}\mathrm{i}\rho^I\bar\epsilon\,\psi_\mu + h.c.\ ,\\
\delta\rho^I &= \mathrm{i}\bar\epsilon\lambda^I + h.c.\ ,\\
\delta\lambda^I &= -\frac{1}{4}\gamma^{\mu\nu}\widehat{G}^I_{\mu\nu}\epsilon + \frac{1}{2}\mathrm{i}D^I\epsilon - \frac{1}{4}\mathrm{i}\widehat{\not{D}}\rho^I\epsilon - \frac{1}{2}\mathrm{i}\rho^I D\epsilon + \frac{1}{4}\rho^I\gamma\cdot\widehat{G}\epsilon\ ,\\
\delta D^I &= \Big(-\frac{1}{2}\mathrm{i}\bar\epsilon\,\widehat{\not{D}}\lambda^I + \frac{1}{2}\mathrm{i}D\bar\epsilon\lambda^I - \frac{1}{4}\bar\epsilon\gamma\cdot\widehat{G}\lambda^I\\
&\quad +\frac{1}{4}g\,\bar\epsilon\,f_{JK}{}^I\,\rho^J\lambda^K\Big) + h.c.\,. \qquad (3.4.10)
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{D}_\mu\rho^I &= \partial_\mu\rho^I + \big(-\mathrm{i}\bar\psi_\mu\,\lambda^I + h.c.\big) + g\,f_{JK}{}^I\,C_\mu^J\,\rho^K\ ,\\
\widehat{D}_\mu\lambda^I &= (\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\,\gamma_{ab} - \mathrm{i}V_\mu)\lambda^I + \frac{1}{4}\gamma^{\rho\sigma}\widehat{G}^I_{\rho\sigma}\,\psi_\mu - \frac{1}{2}\mathrm{i}D^I\psi_\mu + \frac{1}{4}\mathrm{i}\widehat{\not{D}}\rho^I\psi_\mu\\
&\quad +\frac{1}{2}\mathrm{i}\rho^I D\,\psi_\mu - \frac{1}{4}\rho^I\gamma\cdot\widehat{G}\psi_\mu + g\,f_{JK}{}^I C_\mu^J\,\lambda^K\ ,\\
\widehat{G}^I_{\mu\nu} &= 2\partial_{[\mu}C^I_{\nu]} - \left(\bar\psi_{[\mu}\gamma_{\nu]}\lambda^I - \frac{i}{2}\rho^I\bar\psi_\mu\,\psi_\nu + h.c.\right)\\
&\quad +g\,f_{JK}{}^I\,C_\mu^J\,C_\nu^K\,. \qquad (3.4.11)
\end{aligned}
$$

We will next show that the following set of fields

$$
(\Omega_\mu{}^{-ab}\,,\widehat{G}^{ab}\,,\widehat{\psi}^{ab}\,,\widehat{F}^{ab}(V_+,\omega,\Omega^-)) \qquad (3.4.12)
$$

transform as a Yang-Mills multiplet $(C_\mu^I\,,\rho^I\,,\lambda^I\,,D^I)$, where the ab index pair plays the role of Yang-Mills index. The definitions of the torsionful spin connection $\Omega_\mu{}^{-ab}$, the gravitino field strength $\widehat{\psi}_{ab}$, and the modified $U(1)_R$ gauge field are given by

$$
\begin{aligned}
\Omega_\mu{}^{ab\,\pm} &= \omega_\mu{}^{ab} \pm 2\varepsilon_\mu{}^{ab}D\ , &(3.4.13)\\
\widehat{\psi}_{ab} &= 2\nabla_{[a}(\omega,\Omega^+,V)\psi_{b]} - \mathrm{i}\gamma_{[a}\gamma\cdot\widehat{G}\psi_{b]}\ , &(3.4.14)\\
V_{a+} &= V_a + \frac{1}{2}\epsilon_a{}^{bc}\widehat{G}_{bc}\ , &(3.4.15)
\end{aligned}
$$

where in the definition of $\widehat{\psi}^{ab}$, the connection ω rotates the Lorentz vector index while the connection Ω^+ rotates the Lorentz spinor index.

First, we calculate the transformation rules for ${\omega_\mu}^{ab}$, D and $\widehat{G}_{ab}$

$$
\begin{aligned}
\delta {\omega_\mu}^{ab} &= -\frac{1}{4}\bar{\epsilon}\,\gamma_\mu \widehat{\psi}_{ab} + \frac{1}{2}\bar{\epsilon}\,\gamma^{[a}\widehat{\psi}^{b]}{}_\mu + D\bar{\epsilon}\,\gamma_{ab}\,\psi_\mu \\
&\quad -\mathrm{i}\bar{\epsilon}\,\psi_\mu \widehat{G}^{ab} + h.c.\,, \qquad (3.4.16)\\
\delta D &= -\frac{1}{16}\bar{\epsilon}\,\gamma\cdot\widehat{\psi} + h.c.\ , \qquad (3.4.17)\\
\delta \widehat{G}_{ab} &= -\frac{1}{4}\mathrm{i}\bar{\epsilon}\,\widehat{\psi}_{ab} + h.c.\ . \qquad (3.4.18)
\end{aligned}
$$

From the first two equations, we observe that

$$
\delta {\Omega_\mu}^{-ab} = -\frac{1}{2}\bar{\epsilon}\,\gamma_\mu \widehat{\psi}^{ab} - \mathrm{i}\bar{\epsilon}\,\psi_\mu \widehat{G}^{ab} + h.c.\ . \qquad (3.4.19)
$$

Next, we compute the transformation rule for the gravitino curvature

$$
\begin{aligned}
\delta\widehat{\psi}_{ab} &= \frac{1}{4}\gamma^{cd}\,\widehat{R}_{abcd}(\Omega^+)\epsilon - \mathrm{i}\widehat{F}_{ab}(V)\epsilon - 2\mathrm{i}\nabla_{[a}(\omega)\widehat{G}_{b]c}\gamma^c\epsilon \\
&\quad -\mathrm{i}\nabla_{[a}(\omega)\,\widehat{G}^{cd}\varepsilon_{b]cd} + 2\mathrm{i}D\,\widehat{G}_{ab}\epsilon - \widehat{G}_{ab}\gamma\cdot\widehat{G}\epsilon \\
&\quad +\mathrm{i}\widehat{G}\gamma_{ab}\,\gamma\cdot\widehat{G}\epsilon\,, \qquad (3.4.20)
\end{aligned}
$$

where $\widehat{R}_{abcd}(\Omega^+)$ represents a torsionful supercovariant Riemann tensor. Using the definition of V_+ given in (3.4.15), the Bianchi identity $\nabla_{[a}\widehat{G}_{bc]} = 0$ and $\widehat{R}_{abcd}(\Omega^+) = \widehat{R}_{cdab}(\Omega^-)$, we rewrite the transformation rule for the gravitino curvature as

$$
\begin{aligned}
\delta\widehat{\psi}_{ab} &= \frac{1}{4}\gamma^{cd}\,\widehat{R}_{cdab}(\Omega^-)\epsilon - \mathrm{i}\widehat{F}_{ab}(V_+)\epsilon + \mathrm{i}\not{\nabla}(\Omega^-)\widehat{G}_{ab}\,\epsilon \\
&\quad -\widehat{G}_{ab}\,\gamma\cdot\widehat{G}\epsilon\ , \qquad (3.4.21)
\end{aligned}
$$

where in $\nabla_\mu(\Omega^-)\widehat{G}_{ab}$, the connection Ω^- rotates both a and b indices. Finally, defining $\widehat{F}_{ab}(V_+,\omega,\Omega^-)$ where ω rotates the Lorentz vector index b, whereas the connection Ω^- rotates the index c in the covariant derivative acting on $\widehat{G}_{bc}$, we have

$$
\begin{aligned}
\delta\widehat{\psi}_{ab} &= \frac{1}{4}\gamma^{cd}\,\widehat{R}_{cdab}(\Omega^-)\epsilon - \mathrm{i}\widehat{F}_{ab}(V_+,\omega,\Omega^-)\epsilon + \mathrm{i}\not{\nabla}(\Omega^-)\widehat{G}_{ab}\,\epsilon \\
&\quad -\widehat{G}_{ab}\,\gamma\cdot\widehat{G}\,\epsilon + 2\mathrm{i}DG_{ab}\,\epsilon\,. \qquad (3.4.22)
\end{aligned}
$$

Finally, we consider the transformation rule for $\widehat{F}_{ab}(V_+,\omega,\Omega^-)$

$$
\begin{aligned}
\delta\widehat{F}_{ab}(V_+,\omega,\Omega^-) &= \frac{1}{4}\mathrm{i}\bar\epsilon\,\nabla\!\!\!\!/(\omega,\Omega^-)\widehat{\psi}_{ab} - \frac{1}{4}\mathrm{i}D\bar\epsilon\,\widehat{\psi}_{ab} + \frac{1}{8}\bar\epsilon\,\gamma\cdot\widehat{G}\,\widehat{\psi}_{ab} \\
&\quad -\mathrm{i}\bar\epsilon\,\widehat{G}_{c[a}\widehat{\psi}_{b]}{}^{c} + h.c.\,, \qquad (3.4.23)
\end{aligned}
$$

where in $\nabla_c(\omega,\Omega^-)\widehat{\psi}_{ab}$ the connection ω acts on the spinor index, whereas Ω^- acts on both a and b indices.

Comparing the transformation rules (3.4.18), (3.4.19), (3.4.22) and (3.4.23) with those of the nonabelian vector multiplet, we find the following correspondence

$$
\Omega_\mu{}^{-ab} \leftrightarrow C_\mu^I\,,\quad 4\widehat{G}^{ab} \leftrightarrow \rho^I\,,\quad -\widehat{\psi}^{ab} \leftrightarrow \lambda^I\,,\quad 2\widehat{F}^{ab}(V_+,\omega,\Omega^-) \leftrightarrow D^I \qquad (3.4.24)
$$

With these results, we now turn to the supersymmetric completion of the Ricci squared term. To this end, we first construct the following Lagrangian

$$
\begin{aligned}
e^{-1}\mathcal{L}_{YM} &= \frac{1}{4}\Big(G^I_{\mu\nu} - \rho^I G_{\mu\nu}\Big)\Big(G^{\mu\nu I} - \rho^I G^{\mu\nu}\Big) \\
&\quad -\frac{1}{2}(D^I - \rho^I D)^2 + \frac{1}{8}D_\mu\rho^I D^\mu\rho^I\,, \qquad (3.4.25)
\end{aligned}
$$

describing the bosonic sector of a Yang-Mills multiplet coupling to supergravity. This is obtained by generalizing the superconformal invariant action (3.2.44) and then fixing gauges according to (3.4.1). It is now straightforward to use the map (3.4.24) which gives the bosonic part of the supersymmetric completion of the Riemann squared action

$$
\begin{aligned}
e^{-1}\mathcal{L}_{Riem^2} &= \frac{1}{4}\Big(R_{\mu\nu ab}(\Omega^-) - 4G_{ab}\,G_{\mu\nu}\Big)\Big(R^{\mu\nu ab}(\Omega^-) - 4G^{ab}\,G^{\mu\nu}\Big) \\
&\quad -2\Big(F_{ab}(V_+,\omega,\Omega^-) - 2DG_{ab}\Big)\Big(F^{ab}(V_+,\omega,\Omega^-) - 2DG^{ab}\Big) \\
&\quad +2\nabla_\mu(\Omega^-)G_{ab}\,\nabla^\mu(\Omega^-)G^{ab}\,. \qquad (3.4.26)
\end{aligned}
$$

Finally, expanding the torsion terms and using the definition of the three-dimensional Riemann tensor

$$
R_{\mu\nu ab} = \epsilon_{\mu\nu\rho}\,\epsilon_{abc}\Big(R^{\rho c} - \frac{1}{2}e^{\rho c}R\Big)\,, \qquad (3.4.27)
$$

we obtain the supersymmetric completion of the Ricci squared action

$$
\begin{aligned}
e^{-1}\mathcal{L}_{R^2_{\mu\nu}} &= R_{\mu\nu}\,R^{\mu\nu} - \frac{1}{4}R^2 + 4RD^2 + RG^2 - 2R_{\mu\nu}\,G^\mu\,G^\nu + 48D^4 + 8D\Box D \\
&\quad +8D^2G^2 + (G^2)^2 - 2(F_{\mu\nu} + \nabla_{[\mu}G_{\nu]})^2 \\
&\quad -(\nabla_\mu G_\nu + 4DG_{\mu\nu})^2\,, \qquad (3.4.28)
\end{aligned}
$$

where we recall that $G_\mu := \epsilon_{\mu\nu\rho} G^{\nu\rho}$. If desired, a term proportional to $\mathcal{L}_{R^2}$ from (3.4.9) can be added to this result to obtain the invariant in which the only curvature squared term is that of the Ricci tensor.

We conclude this subsection with some comments on the existence of an off-shell RD^2 invariant. Considering the vector multiplet action (3.2.45) and the composite formulae (3.2.34) and (3.2.35), we find the following choices for C_{IJ} to obtain a supersymmetric completion for the RD^2 term:

1. The supersymmetric completion of the RD^2 term can be obtained by supersymmetrizing the $\Box^c(\rho^{-3}D^2)$ term. In order to do so, we can consider two vector multiplets: $(\rho, C_\mu, \lambda, D)$ labeled by 1, and $(\rho'', C''_\mu, \lambda'', D'')$ labeled by 3, and set $C_{13} = \rho^{-1}$. Making this choice, we find that all the terms in the Lagrangian (3.2.45) cancel each other out, thus, not giving rise to an RD^2 invariant.

2. Alternatively, one can consider the supersymmetric completion of $\rho^{-2}D^2\Box^c\rho$ which gives rise to an RD^2 term after gauge fixing. Such a model can be obtained by considering two vector multiplets: $(\rho, C_\mu, \lambda, D)$ labeled by 1, and $(\rho', C'_\mu, \lambda', D')$ labeled by 2, and set $C_{22} = \rho^{-1}$. Making this choice, however, we find that the resulting action is the R^2 action given in (3.4.9).

3. Another alternative is the supersymmetric completion of $\rho^{-2}D\Box^c(\rho^{-1}D)$. This construction also corresponds to the choice $C_{22} = \rho^{-1}$, and coincides with the R^2 action given in (3.4.9)

In view of these arguments, it is not clear to us how the supersymmetric completion of RD^2 as an off-shell invariant independent of the R^2 and $R^2_{\mu\nu}$ invariants can be obtained within the tensor calculus framework presented in section 3.2.

3.4.4 $\mathcal{N} = (2,0)$ Generalized Massive Supergravity

We now consider a combination of the invariants up to dimension four, namely,

$$I = \frac{1}{\kappa^2}\int d^3x \Big[M\mathcal{L}_C + \sigma\mathcal{L}_{EH} + \frac{1}{\mu}\mathcal{L}_{CS} + \frac{1}{\nu}\mathcal{L}_{RD} + \frac{1}{m^2}\mathcal{L}_{R^2_{\mu\nu}} + c\,\mathcal{L}_{R^2}\Big]\,, \tag{3.4.29}$$

where $(\sigma, M, \mu, \nu, m^2, c)$ are arbitrary real constants. This action is invariant under the off-shell supersymmetry transformation rules given in (3.4.3). If we consider the defining feature of a super GMG model to be that it contains the term $R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2$, such an extension is clearly not not unique, as discussed earlier. Focusing on maximally supersymmetric backgrounds and ghost free fluctuations around it, we begin by noting that the metric for such backgrounds is AdS or Minkowski. In the former case, D must be non-vanishing, and this is problematic for ghost-freedom due the presence of the RD^2 term in the action. Such a term is akin to the RA^2 term in the $\mathcal{N} = (1,1)$ model which we are able to eliminate. In the case of a Minkowski background, the presence of the RD^2 term is harmless. Thus, to obtain a maximally supersymmetric Minkowski background, we are led to consider the model with the following choice of parameters

$$M = 0 \ , \qquad \nu = \infty \ , \qquad c = -\frac{1}{8m^2} \ . \tag{3.4.30}$$

In this case, the total Lagrangian becomes

$$\begin{aligned} e^{-1}\mathcal{L}_{GMG} \ = \ & \sigma\left(R - 2G_\mu G^\mu - 8D^2 - 4G^\mu V_\mu\right) \\ & -\frac{1}{4\mu}\epsilon^{\mu\nu\rho}\left[R_{\mu\nu}{}^{ab}\,\omega_{\rho ab} + \tfrac{2}{3}\omega_\mu{}^{ab}\,\omega_{\nu b}{}^{c}\,\omega_{\rho ca} - 8V_\mu\partial_\nu V_\rho\right] \\ & +\frac{1}{m^2}\Big[R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2 - 2RD^2 - R_{\mu\nu}\,G^\mu G^\nu + \frac{1}{2}RG^2 \\ & \qquad -24D^4 + \frac{1}{2}(G^2)^2 - 4D^2G^2 - F_{\mu\nu}F^{\mu\nu} \\ & \qquad +8DG^{\mu\nu}\left(F_{\mu\nu} + \nabla_\mu G_\nu\right)\Big]. \end{aligned} \tag{3.4.31}$$

For the maximally supersymmetric background, the fields (D, V_μ, C_μ) are vanishing. Therefore, the analysis of the linearized fluctuations for spin-2 modes around this background is the same as that of standard GMG model, amounting to the purely gravitational part of the action above. Thus, we know that the system describes two massive helicity ± 2 modes with masses [81]

$$m_\pm^2 = -\sigma m^2 + \frac{m^4}{2\mu^2}\left[1 \pm \sqrt{1 - \frac{4\sigma\mu^2}{m^2}}\right] \ . \tag{3.4.32}$$

Ghosts are absent for $m^2 > 0$ and $\sigma \leq 0$ [30, 59, 81]. Next, we note that the linearized fluctuation of the field D vanishes. Denoting the linearized vector fluctuations of (V_μ, C_μ) by the same symbols and choosing the Lorentz gauges

$\partial_\mu V^\mu = 0$ and $\partial_\mu C^\mu = 0$, one finds that their linearized field equations are given as

$$\frac{1}{m^2}\Box V^\mu - \frac{1}{\mu}\epsilon^{\mu\nu\rho}\,\partial_\nu V_\rho + \sigma G^\mu = 0\ , \qquad F_{\mu\nu} + 2\partial_{[\mu}G_{\nu]} = 0\ . \tag{3.4.33}$$

A simple manipulation of these equations gives

$$\left[(\Box + \sigma m^2)\delta^\rho_\mu\,\delta^\sigma_\nu - \frac{\sigma m^2}{\mu}\epsilon_{[\mu}{}^{\rho\sigma}\partial_{\nu]}\right]\begin{pmatrix} F_{\rho\sigma} \\ G_{\rho\sigma}\end{pmatrix} = 0\ . \tag{3.4.34}$$

Diagonalizing the mass matrix one finds that the masses for V^μ and C^μ are given by the formula (3.4.32). Thus, we have found the bosonic sector of two massive spin-2 multiplets of $\mathcal{N} = (2,0)$ supersymmetry.

3.5 Conclusions

In this chapter, we have completed the construction of all off-shell Poincaré supergravity invariants up to mass dimension four and with $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supersymmetry. We have utilized superconformal tensor calculus except for the supersymmetric completion of the Ricci tensor squared invariant with $\mathcal{N} = (2,0)$ supersymmetry, where we have employed a map between the Yang-Mills multiplet and the Poincaré multiplet. The resulting Lagrangians with $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supersymmetry contain seven and six free parameters respectively, each of which corresponds to separate off-shell invariants. We have determined the relation between the parameters so that the spectrum of fluctuations about a maximally symmetric vacuum solution is ghost-free. For ghost-free fluctuations about anAdS_3 vacuum, a certain type of off-diagonal invariants with mass dimension four, namely RS^2 for $\mathcal{N} = (1,1)$ supersymmetry and RD^2 for $\mathcal{N} = (2,0)$ supersymmetry, without curvature squared terms in their supersymmetric completion, play a crucial role. We have constructed the former, but surprisingly we have found that the latter does not seem to exist. Consequently, the $\mathcal{N} = (2,0)$ model does not seem to have a supersymmetric AdS vacuum with ghost-free spectrum, even though it does admit a supersymmetric Minkowski vacuum that gives a ghost-free massive spin-2 multiplet.

Since $\mathcal{N} = (1,1)$ generalized massive gravity admits a maximally supersymmetric AdS_3 vacuum, one should expect a holographically dual superconformal field theory. Here, we do not attempt to calculate the central charge as in the original argument of Brown-Henneaux [22], but consider the bosonic truncation of the $\mathcal{N} = (1,1)$ generalized massive gravity (3.3.14) as in [58, 59]. For parity-preserving theories with higher derivative extensions, the left and right central charges are given by [82, 83]

$$c_L = c_R = \frac{\ell}{2G_3} g_{\mu\nu} \frac{\partial(e^{-1}\mathcal{L})}{\partial R_{\mu\nu}} \,. \tag{3.5.1}$$

Parity-violating terms result in a difference in the left and right central charges. Given the Lagrangian (3.3.11) with parameter choices (3.3.13), the only parity violating contribution comes from the Lorentz Chern-Simons term and it is given by $\pm\frac{3}{2G_3\mu}$.Therefore, the central charges read

$$c_{L,R} = \frac{3\ell}{2G_3}\Big(\sigma + \frac{1}{2m^2\ell^2} \pm \frac{1}{\mu\ell}\Big) \,. \tag{3.5.2}$$

Note that this result precisely matches with the three dimensional $\mathcal{N} = (1,0)$ model [59] since the vacuum expectation values for the R-symmetry gauge field V_μ and the imaginary part of the auxiliary scalar S vanish, as can be seen in Eq. (3.3.15). Therefore, increasing the number of supercharges from $\mathcal{N} = 1$ to $\mathcal{N} = 2$ do not seem to provide a solution to the bulk-boundary problem.

Appendix

3.A Complex Spinor Conventions

We use the metric signature is $(-,+,+)$. The gamma matrices satisfy the Clifford algebra, i.e. $\left\{\gamma^a,\gamma^b\right\}=2\eta^{ab}$, and the identities

$$(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0, \qquad (\gamma^\mu)^T=-C\gamma^\mu C^{-1}, \qquad (\gamma^\mu)^*=B\gamma^\mu B^{-1}\ , \tag{3.A.1}$$

where C is the charge conjugation matrix and B is a unitary matrix with properties

$$CC^\dagger=1\ , \qquad CC^*=-1\ , \qquad C^T=-C\ . \tag{3.A.2}$$

$$C=iB\gamma^0\ , \qquad BB^\dagger=1\ , \qquad BB^*=1, \qquad B^T=B\ . \tag{3.A.3}$$

For Dirac spinors, there are two different definitions of the conjugate which are given by [65]

$$\bar\epsilon=\mathrm{i}\epsilon^\dagger\gamma^0, \qquad \tilde\epsilon=\overline{(B\epsilon)^*}\ . \tag{3.A.4}$$

For Majorana spinors, we impose the reality condition $\epsilon^*=B\epsilon$. The Majorana conjugation $\bar\epsilon=\epsilon^TC$ is equivalent to Dirac conjugation $\bar\epsilon=\mathrm{i}\epsilon^\dagger\gamma^0$.

In order to obtain the flipping rules for bilinears formed by Dirac spinors, it is useful to decompose a Dirac spinor into two Majorana spinors as $\epsilon_D=\epsilon_{M1}+\mathrm{i}\epsilon_{M2}$. As a result, we have

$$\begin{aligned}(B\epsilon_D)^* &= \epsilon_{M1}-\mathrm{i}\epsilon_{M2},\\ \bar\epsilon_D &= \bar\epsilon_{M1}-\mathrm{i}\bar\epsilon_{M2},\\ \tilde\epsilon_D &= \bar\epsilon_{M1}+\mathrm{i}\bar\epsilon_{M2},\end{aligned} \tag{3.A.5}$$

from which one obtains

$$\bar\epsilon_1\Gamma\,(B\epsilon_2)^*=\alpha\,\bar\epsilon_2\Gamma\,(B\epsilon_1)^*\ , \qquad \tilde\epsilon_1\Gamma\epsilon_2=\alpha\,\tilde\epsilon_2\Gamma\epsilon_1\ , \tag{3.A.6}$$

where Γ is any element of the Clifford algebra and α is the corresponding numerical factor in the Majorana flipping relations. Using the decomposition, one also gets

$$\tilde{\epsilon}_1 \Gamma \left(B\epsilon_2\right)^* = \alpha\, \bar{\epsilon}_2 \Gamma \epsilon_1 \ . \tag{3.A.7}$$

Note that this time we get a different type of bilinear, which becomes an important issue in the closure of the algebra on the scalar multiplet. Namely, a QQ commutation leads to a translation parameter

$$\xi_3^\mu = \frac{1}{2}\tilde{\epsilon}_2 \gamma^\mu \left(B\epsilon_1\right)^* - \frac{1}{2}\tilde{\epsilon}_1 \gamma^\mu \left(B\epsilon_2\right)^* \ , \tag{3.A.8}$$

which can be shown to be identical to the usual translation parameter

$$\xi_3^\mu = \frac{1}{2}\bar{\epsilon}_2 \gamma^\mu \epsilon_1 - \frac{1}{2}\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \ , \tag{3.A.9}$$

by using Eq. (3.A.7).

The charge conjugation of a spinor is defined by $\lambda^C = B^{-1}\lambda^* = (B\lambda)^*$ and the complex conjugation of bilinears is given by

$$\left(\bar{\chi}\Gamma\lambda\right)^* \equiv \left(\bar{\chi}\Gamma\lambda\right)^C = \overline{\chi^C}\Gamma^C\lambda^C = \tilde{\chi}\Gamma^C \left(B\lambda\right)^* \ , \tag{3.A.10}$$

$$\left(\tilde{\chi}\Gamma\lambda\right)^* \equiv \left(\tilde{\chi}\Gamma\lambda\right)^C = \widetilde{\chi^C}\Gamma^C\lambda^C = \bar{\chi}\Gamma^C \left(B\lambda\right)^* \ , \tag{3.A.11}$$

where the charge conjugation of matrices determined by $\left(\Gamma_1\Gamma_2\right)^C = \Gamma_1^C\Gamma_2^C$ and $\gamma_\mu^C = \gamma_\mu$.

3.B Fierz Identities

Elements of the Clifford algebra in $3D$ are $\left\{\Gamma^A = \mathbb{1}, \gamma^\mu\right\}$ with the orthogonality relation $Tr\left(\Gamma^A\Gamma_B\right) = 2\,\delta^A_B$. Therefore, any 2-dimensional matrix can be expanded in the basis $\left\{\Gamma^A\right\}$ as $M = \frac{1}{2}\sum_A \mathrm{Tr}\left(M\Gamma_A\right)\Gamma^A$. As a result, the Fierz identity in $3D$ is given by

$$\bar{\chi}_1\, \chi_2\, \epsilon = -\frac{1}{2}\left(\bar{\chi}_1\, \epsilon\, \chi_2 + \bar{\chi}_1\, \gamma^a \epsilon\, \gamma_a \chi_2\right) \ , \tag{3.B.1}$$

from which one obtains

$$\begin{aligned} \bar{\chi}_1\, \gamma^a \chi_2\, \gamma_a \epsilon &= -\bar{\chi}_1\, \chi_2\, \epsilon - 2\bar{\chi}_1 \epsilon\, \chi_2 \ , && (3.B.2) \\ \bar{\chi}_1\, \gamma^{ab} \chi_2\, \gamma_{ab} \epsilon &= 2\bar{\chi}_1\, \chi_2\, \epsilon + 4\bar{\chi}_1\, \epsilon\, \chi_2 \ . && (3.B.3) \end{aligned}$$

Whenever flipping relations are applicable, one can also obtain additional identities by antisymmetrizing (3.B.2)–(3.B.3) with respect to $1 \longleftrightarrow 2$

$$\begin{aligned}\tilde{\chi}_1\, \gamma^a \chi_2\, \gamma_a \epsilon &= -\tilde{\chi}_1\, \epsilon\, \chi_2 + \tilde{\chi}_2\, \epsilon\, \chi_1 \ , & (3.B.4)\\ \tilde{\chi}_1\, \gamma^{ab} \chi_2\, \gamma_{ab} \epsilon &= 2\tilde{\chi}_1\, \epsilon\, \chi_2 - 2\tilde{\chi}_2\, \epsilon\, \chi_1 \ . & (3.B.5)\end{aligned}$$

These identities are also true for bilinears of the type $\bar{\epsilon}_1 \Gamma\, (B\epsilon_2)^*$. Using (3.B.4) in (3.B.1) we also obtain

$$\tilde{\chi}_1\, \chi_2\, \epsilon = -\tilde{\chi}_1\, \epsilon\, \chi_2 - \tilde{\chi}_2\, \epsilon\, \chi_1 \ . \tag{3.B.6}$$

Chapter 4

Supersymmetric Backgrounds and Black Holes in $\mathcal{N} = (1, 1)$ Cosmological New Massive Supergravity

4.1 Introduction

In this chapter, using an off-shell Killing spinor analysis, we perform a systematic investigation of the supersymmetric background and black hole solutions of the $\mathcal{N} = (1,1)$ Cosmological New Massive Gravity model that we constructed in the previous chapter. By applying the ideas which were discussed in Section 2.3, we classify the supersymmetric solutions with respect to the norm of the translational Killing vector of the spacetime that supports at least some of the supersymmetries. The solutions with a null Killing vector are the same solutions that one finds in the $\mathcal{N} = 1$ model but we also find new solutions with a time-like Killing vector that are absent in the $\mathcal{N} = 1$ case. An example of such a solution is a Lifshitz spacetime. We also consider the supersymmetry properties of the so-called rotating hairy BTZ black holes and logarithmic black holes in an AdS_3 background. Furthermore, we study the possibility of the existence of supersymmetric Lifshitz black hole solution under certain assumptions. The material presented in this chapter is based on [84].

Supergravity in 3D has a long history. Especially in its realization with minimal, $\mathcal{N} = 1$, supersymmetry the theory has been established a long time ago both in on-shell and off-shell formulations as well as in the superconformal framework [42,43,85–87]. Despite of all the achievements in the supersymmetric constructions of the theory, the three-dimensional Poincaré (super)gravity by itself is of not much physical interest as the field equations of the theory imply that the spacetime curvature is zero, hence no physical degrees of freedom propagate.

TMG, which is obtained by supplementing the Poincaré theory with a parity-breaking Lorentz-Chern-Simons term, leads to a non-trivial dynamics of the gravitational field describing a helicity $+2$ or -2 state [28]. The $\mathcal{N} = 1$ supersymmetric completion of TMG was constructed in [88,89], and the supersymmetric background and black hole solutions of this supersymmetric theory were studied in [90]. Similar to TMG, the parity even NMG theory also provides dynamics to the three-dimensional gravity theory corresponding in this case to two states of helicity $+2$ and -2 [30]. As we discussed in Section 2.3, the supersymmetric background configurations of both $\mathcal{N} = 1$ TMG and $\mathcal{N} = 1$ NMG are severely restricted due to the spinor structure of the $\mathcal{N} = 1$ supersymmetry, which allows only planar-wave type solutions with a null Killing vector as well as maximally supersymmetric AdS_3 and Minkowski backgrounds [58,59].

In the previous chapter, we have formulated all four-derivative extension of the three-dimensional $\mathcal{N} = (1,1)$ off-shell cosmological Poincaré supergravity theory. For a discussion of this construction (and more) from a superspace point of view, see [91]. Extending the $\mathcal{N}=1$ theory with more supersymmetry cannot affect the dynamics of the Poincaré supergravity theory. It merely extends the size of the $\mathcal{N}=1$ Poincaré multiplet, consisting of a Dreibein $e_\mu{}^a$, a gravitino ψ_μ and a scalar A with an additional gravitino, an auxiliary vector V_μ and a pseudo-scalar B (see Section 4.2 for details). As we will show in this chapter, the merit of the $\mathcal{N}=(1,1)$ theory is that the spinors of the theory are Dirac instead of Majorana spinors, which allows a larger variety of supersymmetric background solutions than in the $\mathcal{N}=1$ case [67, 75, 92].

The main aim of this chapter is to study the supersymmetric backgrounds as well as black hole solutions of the $\mathcal{N}=(1,1)$ cosmological NMG, or shortly $\mathcal{N}=(1,1)$ CNMG, theory using the off-shell Killing spinor analysis. The power of the off-shell analysis is reflected by the fact that, once the conditions on the possible field configurations are obtained by using the off-shell supersymmetry transformation rules, one can use them to study the solutions of any model which respects the same set of transformation rules. This might include higher derivative corrections and/or matter couplings.

We begin our study in Section 4.2 with a brief review of $\mathcal{N}=(1,1)$ CNMG and its off-shell transformation rules. As a typical property of off-shell supergravity theories, the auxiliary fields of the theory start to propagate [1] when the Poincaré supergravity is extended with higher-order curvature terms. Assuming that the supersymmetric theory admits at least one Killing spinor, we present the Killing spinor equation and its integrability condition. We then review the implications of the existence of an off-shell Killing spinor as presented in [75]. The existence of such a spinor imposes numerous algebraic as well as differential identities on the metric, the vector V_μ and the scalars A and B of the theory. These identities are the backbone of our analysis that we present in the remainder of the chapter. The Killing spinor equation in particular implies that the background solutions can be put into two categories depending on whether the Killing vector that is formed out of the Killing spinors is *null* or *timelike*.

In Section 4.3, we investigate the solutions that admit a null Killing vector. In this case, the analysis for finding supersymmetric solutions simply reduces to the one corresponding to the $\mathcal{N}=1$ theory [59] as the vector V_μ and the pseudo-scalar B are set to zero due to the algebraic and differential constraints,

[1] There are exceptional cases where the auxiliary fields do not propagate such as in $\mathcal{N}=1$ CNMG [59].

which are consequences of the existence of a Killing spinor. The solution are, therefore, of the pp-wave type, like in the $\mathcal{N} = 1$ case.

In the timelike case, which we present in Section 4.4, the supersymmetric solutions are categorized according to the values of the components V_a of the vector in a flat basis. We find that the $\mathcal{N} = (1, 1)$ CNMG theory allows all solutions of $\mathcal{N} = (1, 1)$ TMG [75] with shifted parameters. Furthermore, we find additional $AdS_2 \times \mathbb{R}$ and Lifshitz solutions. These extra solutions are possible because of the fact that in $\mathcal{N} = (1, 1)$ TMG, the vector equation gives rise to a second order algebraic equation for the components V_a, whereas in the case of $\mathcal{N} = (1, 1)$ CNMG, the resulting equation is cubic, allowing more solutions.

In Section 4.5, we investigate the supersymmetric black holes with AdS_3 and Lifshitz backgrounds. We first show that the rotating hairy BTZ black hole of [93], which is a generalization of the well-known BTZ black hole [21] obtained by introducing a *gravitational hair* parameter, and the *logarithmic black hole* of [94] are solutions of the $\mathcal{N} = (1, 1)$ CNMG theory for the extremal cases (see Section 2.3 for a brief explanation of the extremal solutions in the context of supersymmetric theories). This is also true for the rotating BTZ black hole which can be obtained by setting the hair parameter b to zero. Given the Lifshitz solution, we then analyze whether we can find an extremal Lifshitz black hole. As the theory is ungauged, one can hope to saturate the BPS bound with the *massive vector hair* V_μ. As opposed to the pseudo-supersymmetry analysis of the Einstein-Weyl theory in four dimensional $\mathcal{N} = 1$ supergravity [95], this is not the case in $\mathcal{N} = (1, 1)$ CNMG. Furthermore, we will show that a simple rotating black hole ansatz does not satisfy the Killing spinor equation and the field equations of the theory simultaneously.

Finally, in Section 4.6, we present our conclusions and discuss further directions.

4.2 $\mathcal{N} = (1, 1)$ Cosmological New Massive Supergravity

The field content of the $\mathcal{N} = (1, 1)$ supergravity theory consists of the dreibein $e_\mu{}^a$, the gravitino ψ_μ, a complex scalar S, and a vector V_μ. The model we shall study is a particular combination of supersymmetric invariants up to dimension four that leads to a model that, when expanded around a supersymmetric AdS_3 vacuum, propagates only helicity ± 2 and $\pm 3/2$ states with energies defined on an AdS spacetime that respect perturbative unitarity. This model is called cosmological New Massive Gravity (CNMG). Here we focus on the bosonic part of the supersymmetric CNMG Lagrangian which is given by

$$\begin{aligned}
e^{-1}\mathcal{L}_{\rm CNMG} &= \sigma(R+2V^2-2|S|^2)+4MA \\
&+\frac{1}{m^2}\Big[R_{\mu\nu}R^{\mu\nu}-\frac{3}{8}R^2-R_{\mu\nu}V^\mu V^\nu-F_{\mu\nu}F^{\mu\nu}+\frac{1}{4}R(V^2-B^2) \\
&\qquad+\frac{1}{6}|S|^2(A^2-4B^2)-\frac{1}{2}V^2(3A^2+4B^2) \\
&\qquad-2V^\mu B\partial_\mu A\Big]\,, \qquad (4.2.1)
\end{aligned}$$

where (σ, M, m^2) are arbitrary real constants and we have defined $S = A + {\rm i}B$. The action corresponding to this Lagrangian is invariant under the following off-shell supersymmetry transformation rules [2]

$$\begin{aligned}
\delta e_\mu{}^a &= \frac{1}{2}\bar\epsilon\gamma^a\psi_\mu + h.c.\,, \\
\delta\psi_\mu &= D_\mu(\widehat\omega)\,\epsilon-\frac{1}{2}{\rm i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon-\frac{1}{2}S\gamma_\mu\epsilon^\star\,, \\
\delta V_\mu &= \frac{1}{8}{\rm i}\bar\epsilon\,\gamma^{\nu\rho}\gamma_\mu\left(\psi_{\nu\rho}-{\rm i}V_\sigma\gamma^\sigma\gamma_\nu\,\psi_\rho-S\gamma_\nu\psi^\star_\rho\right)+h.c.\,, \\
\delta S &= -\frac{1}{4}\tilde\epsilon\,\gamma^{\mu\nu}\left(\psi_{\mu\nu}-{\rm i}V_\sigma\,\gamma^\sigma\gamma_\mu\psi_\nu-S\gamma_\mu\psi^\star_\nu\right)\,, \qquad (4.2.2)
\end{aligned}$$

where $\tilde\epsilon = \overline{\epsilon^\star}$, $\widehat\omega$ is the super-covariant spin-connection and

$$D_\mu(\widehat\omega)\epsilon = (\partial_\mu+\frac{1}{4}\widehat\omega_\mu{}^{ab}\,\gamma_{ab})\epsilon\,, \qquad \psi_{\mu\nu} = 2D_{[\mu}(\widehat\omega)\psi_{\nu]}\,. \qquad (4.2.3)$$

The transformation rules (4.2.2) are off-shell as the algebra closes on these fields without imposing the field equations corresponding to the Lagrangian (4.2.1).

In order to determine the supersymmetric backgrounds allowed by a model with the transformation rules (4.2.2), one considers the Killing spinor equation

$$\mathcal{D}_\mu\epsilon = \partial_\mu\epsilon+\frac{1}{4}\widehat\omega_\mu{}^{ab}\,\gamma_{ab}\epsilon-\frac{1}{2}{\rm i}V_\nu\,\gamma^\nu\gamma_\mu\,\epsilon-\frac{1}{2}S\gamma_\mu\epsilon^\star = 0\,. \qquad (4.2.4)$$

[2]In this paper, we follow the conventions of [75], with the only difference being that the S we are using here is replaced by $S \to -Z$.

Any Killing spinor ϵ satisfying this equation must also satisfy the integrability condition

$$
\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu]\epsilon &= \frac{1}{4}\Big(R_{\mu\nu}{}^{\rho\sigma} + 2\delta^\rho_\mu\delta^\sigma_\nu(A^2+B^2) + 2\delta^\rho_\mu\delta^\sigma_\nu V^2 - 4\mathrm{i}\delta^\sigma_{[\nu}\nabla_{\mu]}V^\rho \\
&\quad -4\delta^\sigma_{[\nu}V_{\mu]}V^\rho\Big)\gamma_{\rho\sigma}\epsilon - \delta^\sigma_{[\nu}\Big(\partial_{\mu]}A + BV_{\mu]}\Big)\gamma_\sigma\epsilon^* \\
&\quad -\mathrm{i}\delta^\sigma_{[\nu}\Big(\partial_{\mu]}B - AV_{\mu]}\Big)\gamma_\sigma\epsilon^* - \frac{1}{2}\mathrm{i}F_{\mu\nu}\epsilon \\
&\quad +\mathrm{i}\epsilon_{\mu\nu\rho}V^\rho(A+\mathrm{i}B)\epsilon^* = 0\,.
\end{aligned}
\tag{4.2.5}
$$

Considering the field equations for A, B, V_μ and $g_{\mu\nu}$,

$$
\begin{aligned}
0 &= 4M - 4\sigma A + \frac{1}{m^2}\left[\frac{2}{3}A^3 - B^2A - 3V^2A + 2\left(\nabla\cdot V\right)B + 2V^\mu\partial_\mu B\right], \\
0 &= 4\sigma B + \frac{1}{m^2}\left[\frac{1}{2}RB + A^2B + \frac{8}{3}B^3 + 4V^2B + 2V^\mu\partial_\mu A\right], \\
0 &= 4\sigma V_\mu - \frac{1}{m^2}\left[2R_{\mu\nu}V^\nu + 4\nabla_\nu F_\mu{}^\nu + V_\mu\left(3A^2 + 4B^2 - \frac{R}{2}\right) + 2B\partial_\mu A\right], \\
0 &= \sigma\Big(R_{\mu\nu} + 2V_\mu V_\nu - \frac{1}{2}g_{\mu\nu}[R + 2V^2 - 2(A^2+B^2)]\Big) - 2g_{\mu\nu}MA \\
&\quad + \frac{1}{m^2}\Bigg[\Box R_{\mu\nu} - \frac{1}{4}\nabla_\mu\nabla_\nu R + \frac{9}{4}RR_{\mu\nu} - 4R^\rho_\mu R_{\nu\rho} - 2F_\mu{}^\rho F_{\nu\rho} \\
&\quad + \frac{1}{4}RV_\mu V_\nu - 2R^\rho_{(\mu}V_{\nu)}V_\rho - \frac{1}{2}\Box(V_\mu V_\nu) + \nabla_\rho\nabla_{(\mu}(V_{\nu)}V^\rho) \\
&\quad + \frac{1}{4}R_{\mu\nu}(V^2 - B^2) - \frac{1}{4}\nabla_\mu\nabla_\nu(V^2 - B^2) - \frac{1}{2}V_\mu V_\nu(3A^2 + 4B^2) \\
&\quad - 2BV_{(\mu}\partial_{\nu)}A - \frac{1}{2}g_{\mu\nu}\Big(\frac{13}{8}R^2 + \frac{1}{2}\Box R - 3R^2_{\rho\sigma} - R_{\rho\sigma}V^\rho V^\sigma \\
&\quad + \nabla_\rho\nabla_\sigma(V^\rho V^\sigma) - F^2_{\rho\sigma} + \frac{1}{4}R(V^2 - B^2) - \frac{1}{2}\Box(V^2 - B^2) \\
&\quad + \frac{1}{6}(A^2+B^2)(A^2-4B^2) - \frac{1}{2}V^2(3A^2+4B^2) - 2BV^\rho\partial_\rho A\Big)\Bigg],
\end{aligned}
\tag{4.2.6}
$$

it can be seen that for cosmological Poincaré supergravity, i.e. $m \to \infty$, A, B and V_μ can be eliminated algebraically. In this case, the integrability condition (4.2.5) reduces to

$$
\Big(R_{\mu\nu}{}^{\rho\sigma} + 2\delta^\rho_\mu\delta^\sigma_\nu M^2\Big)\gamma_{\rho\sigma}\epsilon = 0\,,
\tag{4.2.7}
$$

which implies that the maximally supersymmetric background is either Minkowski with $M = 0$, or AdS_3 with radius $1/M^2$. More solutions, with less supersymmetry, can be obtained by imposing projection conditions on ϵ. Note that even with the higher derivative contributions, the maximally supersymmetric solution is still given by the same background solution with a shifted value of the cosmological constant. The reason for this is that the expectation value of A receives a contribution from the higher derivative corrections whereas B and V_μ do not and, therefore, can still be set to zero.

In the case of cosmological Poincaré supergravity, the auxiliary fields can be eliminated from the theory, resulting in an on-shell supergravity theory with the field content $(e_\mu{}^a, \psi_\mu)$. However, with the higher derivative contributions added, the massive vector and the real scalars become dynamical and hence cannot be solved algebraically. These 'auxiliary' fields play a crucial role in determining the supersymmetric backgrounds allowed by the CNMG Lagrangian (4.2.1).

Now that we have clarified the maximally supersymmetric backgrounds, let us proceed to the case where we have at least one unbroken supersymmetry. In order to do so, we will briefly review the implications of an off-shell Killing spinor following the discussion of [75]. From the symmetries of the gamma matrices, one finds the following identities for a commuting Killing spinor ϵ

$$\bar{\epsilon}\epsilon^\star = \tilde{\epsilon}\epsilon = 0\,. \tag{4.2.8}$$

Thus, non-vanishing spinor bilinears can be defined as follows

$$\bar{\epsilon}\epsilon = -\tilde{\epsilon}\epsilon^\star \equiv \mathrm{i}f, \quad \bar{\epsilon}\gamma_\mu\epsilon = \tilde{\epsilon}\gamma_\mu\epsilon^\star \equiv K_\mu\,, \quad \bar{\epsilon}\gamma_\mu\epsilon^\star \equiv L_\mu = S_\mu + \mathrm{i}T_\mu, \tag{4.2.9}$$

where f is a real function and $K_\mu\,(L_\mu)$ is a real (complex) vector. Using the Fierz identities for commuting spinors, one can show that

$$K_\mu K^\mu = -f^2\,, \qquad K_\mu\gamma^\mu\epsilon = \mathrm{i}f\epsilon\,. \tag{4.2.10}$$

The first equation implies that the vector is either null or timelike. Using the Killing spinor equation (4.2.4) one finds that

$$\nabla_{(\mu}K_{\nu)} = 0\,, \tag{4.2.11}$$

showing that K_μ is a Killing vector. Finally, we may derive the following differential identities following from the Killing spinor equation (4.2.4)

$$\partial_{[\mu}K_{\nu]} = \epsilon_{\mu\nu\rho}\Big(-fV^\rho - \frac{1}{2}\left(SL^\rho + S^\star(L^\star)^\rho\right)\Big)\,, \tag{4.2.12}$$

$$\partial_\mu f = -\epsilon_{\mu\nu\rho}V^\nu K^\rho - \frac{1}{2}\mathrm{i}\left(SL_\mu - S^\star L^\star_\mu\right)\,. \tag{4.2.13}$$

We refer to [75] for readers interested in the derivation of these Killing spinor identities and of other implications of the existence of a Killing spinor.

4.3 The Null Killing Vector

We first consider the case that the function f introduced in Eq. (4.2.9) is zero everywhere, i.e. $f = 0$. This implies that K_μ is a null Killing vector. The case that $f \neq 0$ will be discussed in the next section. In our conventions, a Majorana spinor field has all real components. This being said, the first spinor bilinear equation in (4.2.9) leads to a Dirac spinor ϵ that is proportional to a real spinor ϵ_0 up to a phase factor characterized by an angle θ [75],

$$\epsilon = e^{-\mathrm{i}\frac{\theta}{2}}\,\epsilon_0\,. \tag{4.3.1}$$

The above equation implies that $L_\mu = e^{\mathrm{i}\theta} K_\mu$. Taking this into account, the differential equation (4.2.12) reads

$$\partial_{[\mu}K_{\nu]} = -\mathrm{Re}(\mathrm{S}e^{\mathrm{i}\theta})\,\epsilon_{\mu\nu\rho}\,\mathrm{K}^\rho\ \ . \tag{4.3.2}$$

Contracting this equation with K^μ we find that

$$K^\mu\,\nabla_\mu\,K_\nu = 0\,. \tag{4.3.3}$$

The same equation also implies that $K \wedge dK = 0$, i.e. K is hypersurface orthogonal. Thus, there exist functions u and P of the three-dimensional spacetime coordinates such that

$$K_\mu\,dx^\mu = P du\,. \tag{4.3.4}$$

Eq. (4.3.3) implies that K is tangent to affinely parameterized geodesics in the surface of constant P. One can, then, choose coordinates (u, v, x) such that v is an affine parameter along these geodesics, i.e.

$$K^\mu\,\partial_\mu = \frac{\partial}{\partial v}\,. \tag{4.3.5}$$

By virtue of our choice for K_μ the metric components further simplify to

$$g_{uv} = P(u,x), \qquad g_{vv} = g_{xv} = 0\,, \tag{4.3.6}$$

where $P = P(u,x)$ since we demand the null direction to be along the v direction. All these choices yield a metric of the following generic form

$$ds^2 = h_{ij}(x,u)\, dx^i\, dx^j + 2P(x,u)\, du\, dv\,, \tag{4.3.7}$$

where $x^i = (x,u)$. Without loss of generality, this metric can be cast in the following form by a coordinate transformation [59, 90]

$$ds^2 = dx^2 + 2P(x,u)\, du\, dv + Q(x,u)\, du^2\,, \tag{4.3.8}$$

with $\sqrt{|g|} = P$. With these results in hand, the auxiliary fields of the theory should satisfy the following constraints [75]

$$\begin{aligned} V_\mu &= -\frac{1}{2}\partial_\mu\, \theta(x,u)\,, \\ Se^{\mathrm{i}\theta} + S^\star e^{-\mathrm{i}\theta} &= \partial_x \log P(x,u)\,. \end{aligned} \tag{4.3.9}$$

In the next subsection we will investigate the solutions of CNMG under the assumption that $f = 0$.

4.3.1 The General Solution

To find the general solution with $f = 0$ we set S to be a constant, to be precise we set $A = -\frac{1}{l}$ and $B = 0$. Using (4.2.13) we obtain

$$\epsilon_{\mu\nu\rho}\, V^\nu K^\rho = -\frac{1}{l} K_\mu \sin\theta(u,x)\,. \tag{4.3.10}$$

The u component of this equation reads

$$\frac{1}{l} K_u \sin\theta(u,x) = P(u,x) V_x\,, \tag{4.3.11}$$

where we have used that $\varepsilon_{xuv} = 1$. Provided that the function $P(u,x)$ is nowhere vanishing, it is straightforward to integrate the first (vector) equation in (4.3.9) and obtain

$$\theta(u,x) = \arctan\left(\frac{2\, c(u)\, e^{-2x/l}}{1 - c^2(u)\, e^{-4x/l}}\right), \tag{4.3.12}$$

for arbitrary $c(u)$. From the second (scalar) equation in (4.3.9) we deduce that

$$-\frac{2}{l}\cos\theta(u,x) = \partial_x \log P(u,x)\,, \tag{4.3.13}$$

which, upon using Eq. (4.3.12), yields

$$P(x,u) = P(u)[\,e^{2x/l} + e^{-2x/l}c^2(u)]\,, \tag{4.3.14}$$

where $P(u)$ is an arbitrary function of u. We may set $P(u)$ to unity without loss of generality [59]. Using Eqs. (4.3.13) and (4.3.14) in the vector field equation (4.3.12), we deduce that $c(u) = 0$ and $\theta(u,x) = n\pi$. In order to fix n we use the trace of the gravity equation and find that $\theta(u,x) = \pi$.

We thus find that the metric (4.3.8) takes the following final form

$$ds^2 = dx^2 + 2\,e^{2x/l}\,du\,dv + Q(x,u)\,du^2\,. \tag{4.3.15}$$

This is the general form of a pp-wave metric. Taking the limit $l \to \infty$ gives rise to the pp-wave in a Minkowski background. Setting $l = 1$ and substituting $A = -1$, $B = 0$, $V_x = V_u = V_v = 0$ into the metric field equation, we find that $Q(x,u)$ satisfies the following differential equation

$$(2 + 4\sigma m^2)\,Q' - (9 + 2\sigma m^2)\,Q'' + 8Q''' - 2Q'''' = 0\,, \tag{4.3.16}$$

where the prime denotes a derivative with respect to x. The most general solution of this differential equation is given by

$$\begin{aligned} Q(x,u) \;&=\; e^{(1-\sqrt{\frac{1}{2}-\sigma m^2})x}\,C_1(u) + e^{(1+\sqrt{\frac{1}{2}-\sigma m^2})x}\,C_2(u) \\ &\quad + e^{2x}\,C_3(u) + C_4(u)\,, \end{aligned} \tag{4.3.17}$$

where the functions $C_i(u)\,, i = 1, \cdots, 4$, are arbitrary functions of u. We note that this expression for $Q(x,u)$ matches with that of [58,96]. It differs, however, with the expression given in [59]. This is due to the fact that the off-diagonal coupling of gravity to the scalar A was included in the supersymmetric New Massive Gravity model studied in [59], whereas such a term is absent in our case, see Eq. (4.2.1).

The solution for $Q(x,u)$ given in (4.3.17) has a redundancy [90]. To make this redundancy manifest we consider the following coordinate transformation

$$x = \widetilde{x} - \tfrac{1}{2}\log a'\,, \qquad u = a(\widetilde{u})\,, \qquad v = \widetilde{v} - \tfrac{1}{4}e^{-2\widetilde{x}}\,\frac{a''}{a'} + b(\widetilde{u})\,, \tag{4.3.18}$$

where $a(\widetilde{u})$ and $b(\widetilde{u})$ are arbitrary functions of $\widetilde{u}$ and the prime denotes a derivative with respect to $\widetilde{u}$. By choosing the function $a(\widetilde{u})$ and $b(\widetilde{u})$ such that the differential equations

$$\Big(\frac{a''}{a'}\Big)' - \frac{1}{2}\Big(\frac{a''}{a'}\Big)^2 - 2(a')^2\,\widetilde{C}_4(\widetilde{u}) = 0\,, \qquad b' + \frac{1}{2}a'\,\widetilde{C}_3(\widetilde{u}) = 0\,, \tag{4.3.19}$$

are satisfied the functions $\widetilde{C}_3$ and $\widetilde{C}_4$ can be set to zero. This implies that, without loss of generality, we may set $C_3 = C_4 = 0$. In addition to this, we get

$$\begin{aligned}
\widetilde{C}_1(\widetilde{u}) &= C_1(a(\widetilde{u}))\,[a'(\widetilde{u})]^{\frac{1}{2}(3+\sqrt{\frac{1}{2}-\sigma m^2})}\,, \\
\widetilde{C}_2(\widetilde{u}) &= C_2(a(\widetilde{u}))\,[a'(\widetilde{u})]^{\frac{1}{2}(3-\sqrt{\frac{1}{2}-\sigma m^2})}\,.
\end{aligned} \tag{4.3.20}$$

There are two special values of parameters which must be handled separately. These are the cases $\sigma m^2 = \pm\frac{1}{2}$. The reason for this is that for the $\sigma m^2 = \frac{1}{2}$ case the function C_1 degenerates with C_2 whereas for the $\sigma m^2 = -\frac{1}{2}$ case the function C_1 degenerates with C_4 while the function C_2 degenerates with C_3. Therefore, we solve the field equation (4.3.16) for these special cases, and display the solutions $Q(x,u)$ for these special values of the parameters explicitly:

$$\begin{aligned}
\sigma m^2 = \frac{1}{2} \quad &: \qquad Q(x,u) = e^x\,D_1(u) + x\,e^x\,D_2(u) + e^{2x}\,D_3(u) + D_4(u)\,, \\
\sigma m^2 = -\frac{1}{2} \quad &: \qquad Q(x,u) = x\,e^{2x}\,D_1(u) + x\,D_2(u) + e^{2x}\,D_3(u) \\
&\qquad\qquad +D_4(u)\,.
\end{aligned} \tag{4.3.21}$$

Here $D_i(u)\,,\ i = 1,\ldots,4$, are arbitrary functions of u. Setting $D_3 = D_4 = 0$, we are led to the following cases:

$$\begin{aligned}
\sigma m^2 \neq \pm\frac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\,e^{2x}\,du\,dv \\
&\qquad +\Big(e^{(1-\sqrt{\frac{1}{2}-\sigma m^2})x}\,D_1(u) + e^{(1+\sqrt{\frac{1}{2}-\sigma m^2})x}\,D_2(u)\Big)\,du^2\,, \\
\sigma m^2 = \frac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\,e^{2x}\,du\,dv + \Big(e^x\,D_1(u) + x\,e^x\,D_2(u)\Big)\,du^2\,, \\
\sigma m^2 = -\frac{1}{2} \quad &: \quad ds^2 = dx^2 + 2\,e^{2x}\,du\,dv \\
&\qquad +\Big(x\,e^{2x}\,D_1(u) + x\,D_2(u)\Big)\,du^2\,.
\end{aligned} \tag{4.3.22}$$

These pp-wave solutions coincide with the solutions of $\mathcal{N} = 1$ CNMG [58]. Having found the most general solutions for the null case, we will continue in the next subsection with determining the amount of supersymmetry that these solutions preserve by working out the Killing spinor equation (4.2.4).

4.3.2 Killing Spinor Analysis

In order to construct the Killing spinors for the pp-wave metric (4.3.15) we introduce the following orthonormal frame [90]

$$e^0 = e^{\frac{2x}{l}-\beta}\, dv, \qquad e^1 = e^{\beta} du + e^{\frac{2x}{l}-\beta}\, dv, \qquad e^2 = dx\,, \tag{4.3.23}$$

where $Q(u,x) = e^{2\beta(u,x)}$. It follows that the components of the spin-connection are given by

$$\begin{aligned}
\omega_{01} &= -\dot{\beta}\, du - \Big(\beta' - \frac{1}{l}\Big) dx\,, \\
\omega_{02} &= -\Big(\beta' - \frac{1}{l}\Big) e^{\beta}\, du - \frac{1}{l}\, e^{\frac{2x}{l}-\beta}\, dv\,, \\
\omega_{12} &= \beta'\, e^{\beta}\, du + \frac{1}{l}\, e^{\frac{2x}{l}-\beta}\, dv\,,
\end{aligned} \tag{4.3.24}$$

where

$$\dot{\beta} \equiv \frac{\partial\beta}{\partial u}\,, \qquad \beta' \equiv \frac{\partial\beta}{\partial x}\,. \tag{4.3.25}$$

For the null case, the Killing spinor equation (4.2.4) then reads

$$0 = d\epsilon + \frac{1}{4}\,\omega_{ab}\,\gamma^{ab}\epsilon + \frac{1}{2l}\,\gamma_a\, e^a\, \epsilon^\star\,. \tag{4.3.26}$$

We make the following choice of the γ matrices

$$\gamma_0 = \mathrm{i}\sigma_2\,, \qquad \gamma_1 = \sigma_1\,, \qquad \gamma_2 = \sigma_3\,, \tag{4.3.27}$$

where σ_i's are the standard Pauli matrices. With this choice the Killing spinor equation reads

$$\begin{aligned}
0 = d\epsilon \;&+\; \frac{1}{2}\Big(\dot{\beta}\,\sigma_3\,\epsilon - e^{\beta}\beta'(\sigma_1 + \mathrm{i}\sigma_2)\,\epsilon + \frac{1}{l} e^{\beta}\sigma_1\,(\epsilon + \epsilon^\star)\Big) du \\
&-\; \frac{1}{2l}\, e^{\frac{2x}{l}-\beta}\,(\sigma_1 + \mathrm{i}\sigma_2)\,(\epsilon - \epsilon^\star)\, dv \\
&+\; \frac{1}{2}\,\Big(\beta'\sigma_3\,\epsilon - \frac{1}{l}\,\sigma_3\,(\epsilon - \epsilon^\star)\Big)\, dx\,.
\end{aligned} \tag{4.3.28}$$

Decomposing a Dirac spinor into two Majorana spinors as $\epsilon = \xi + \mathrm{i}\zeta$, i.e.

$$\epsilon = \begin{pmatrix} \xi_1 + \mathrm{i}\zeta_1 \\ \xi_2 + \mathrm{i}\zeta_2 \end{pmatrix}\,, \tag{4.3.29}$$

we find the following equations for the components

$$
\begin{aligned}
0 &= d\xi_1 + \frac{1}{2}\dot{\beta}\,\xi_1\,du - e^{\beta}(\beta' - \frac{1}{l})\,\xi_2\,du + \frac{1}{2}\,\xi_1\,\beta'\,dx\,,\\
0 &= d\xi_2 + \frac{1}{l}\,e^{\beta}\,\xi_1\,du - \frac{1}{2}\,\dot{\beta}\,\xi_2\,du - \frac{1}{2}\,\beta'\,\xi_2\,dx\,,\\
0 &= d\zeta_1 + \frac{1}{2}\,\dot{\beta}\,\zeta_1\,du - e^{\beta}\,\beta'\,\zeta_2\,du - \frac{2}{l}\,e^{\frac{2x}{l}-\beta}\,\zeta_2\,dv + \frac{1}{2}\,(\beta' - \frac{2}{l})\,\zeta_1\,dx\,,\\
0 &= d\zeta_2 - \frac{1}{2}\,\dot{\beta}\,\zeta_2\,du - \frac{1}{2}\,(\beta' - \frac{2}{l})\,\zeta_2\,dx\,.
\end{aligned}
\tag{4.3.30}
$$

The first two equations are uniquely solved by $\xi_1 = \xi_2 = 0$. For the last two equations, the solution for a generic function $\beta(u, x)$ is given by

$$\zeta_1 = e^{-\frac{1}{2}\beta + \frac{x}{l}}\,, \qquad \zeta_2 = 0\,. \tag{4.3.31}$$

There is an additional solution for the special case that $\beta = x$. It is given by

$$\zeta_1 = (u + 2v)e^{\frac{1}{2}x}\,, \qquad \zeta_2 = e^{-\frac{1}{2}x}\,. \tag{4.3.32}$$

This solution corresponds to the first case given in Eq. (4.3.22) with $D_1(u) = 0$ and $D_2(u) = 1$. There is, however, a problem with this solution. One must choose $\sigma m^2 = -\frac{1}{2}$ for this solution and this conflicts with the condition imposed on this pp-wave solution when we classified the different solutions in the previous subsection, which is given in Eq. (4.3.22). Therefore, we conclude that the pp-wave Killing spinor equation is uniquely solved by

$$\xi_1 = \xi_2 = \zeta_2 = 0\,, \qquad \zeta_1 = e^{-\frac{1}{2}\beta + \frac{x}{l}}\,. \tag{4.3.33}$$

This implies that the pp-wave solutions all preserve 1/4 of the supersymmetries. Note that in the Minkowski limit $l \to \infty$, the equations for ξ and ζ degenerate. Thus, the number of Killing spinors are the same for both AdS and Minkowski pp-wave solutions.

We conclude this section by noting that when $D_1 = D_2 = 0$, the metric reduces to

$$ds^2 = dx^2 + 2e^{2x/l}\,du\,dv = dx^2 + e^{2x/l}\,(-dt^2 + d\phi^2)\,, \tag{4.3.34}$$

which is the AdS_3 metric in a Poincaré patch. In this case, we have

$$e^0 = e^{x/l}\,dt\,, \qquad e^1 = e^{x/l}\,d\phi\,, \qquad e^2 = dx\,, \tag{4.3.35}$$

which implies that

$$\omega_{02} = -\frac{1}{l}\, e^{x/l}\, dt\,, \qquad \omega_{12} = \frac{1}{l}\, e^{x/l}\, d\phi\,. \tag{4.3.36}$$

The Killing spinor equation then turns into

$$\begin{aligned} 0 &= d\epsilon - \frac{1}{2l}\, e^{x/l}\Big(\sigma_1\epsilon - \mathrm{i}\sigma_2\epsilon^\star\Big)dt - \frac{1}{2l}\, e^{x/l}\Big(\mathrm{i}\sigma_2\epsilon - \sigma_1\epsilon^\star\Big)d\phi \\ &\quad + \frac{1}{2l}\,\sigma_3\,\epsilon^\star dx\,. \end{aligned} \tag{4.3.37}$$

Decomposing the Dirac spinor into two Majorana spinors as $\epsilon = \xi + \mathrm{i}\zeta$, see Eq. (4.3.29), the Killing spinor equation gives rise to the following equations

$$\begin{aligned} 0 &= d\xi_1 + \frac{1}{2l}\,\xi_1 dx\,, \\ 0 &= d\xi_2 - \frac{1}{l}\, e^{x/l}\,\xi_1 dt + \frac{1}{l}\, e^{x/l}\,\xi_1 d\phi - \tfrac{1}{2l}\,\xi_2 dx\,, \\ 0 &= d\zeta_1 - \frac{1}{l}\, e^{x/l}\,\zeta_2\, dt - \frac{1}{l}\, e^{x/l}\,\zeta_2\, d\phi - \frac{1}{2l}\,\zeta_1 dx\,, \\ 0 &= d\zeta_2 + \frac{1}{2l}\,\zeta_2\, dx\,. \end{aligned} \tag{4.3.38}$$

Making use of the fact that the ξ and ζ equations are decoupled from each other, we find the following four independent solutions:

1. $\xi_1 = 0, \quad \xi_2 = e^{\frac{x}{2l}}, \quad \zeta_1 = \zeta_2 = 0,$

2. $\xi_1 = e^{-\frac{x}{2l}}, \quad \xi_2 = \frac{1}{l}\, e^{\frac{x}{2l}}\,(t - \phi), \quad \zeta_1 = \zeta_2 = 0,$

3. $\xi_1 = \xi_2 = 0, \quad \zeta_1 = e^{\frac{x}{2l}}, \quad \zeta_2 = 0,$

4. $\xi_1 = \xi_2 = 0, \quad \zeta_1 = \frac{1}{l}\, e^{\frac{x}{2l}}\,(t + \phi), \quad \zeta_2 = e^{-\frac{x}{2l}},$

Therefore, the AdS_3 solution has a supersymmetry enhancement with four Killing spinors.

4.4 The Timelike Killing Vector

In this section, we shall consider the case that $f \neq 0$ and hence that K is a timelike Killing vector field. Introducing a coordinate t such that $K^\mu \partial_\mu = \partial_t$, the metric can be written as [75]

$$ds^2 = -e^{2\varphi(x,y)}\,(dt + B_\alpha(x,y)\,dx^\alpha)^2 + e^{2\lambda(x,y)}(dx^2 + dy^2)\ , \tag{4.4.1}$$

where $\lambda(x,y)$ and $\varphi(x,y)$ are arbitrary functions and B_α $(\alpha = x, y)$ is a vector with two components. The Dreibein corresponding to this metric is naturally written as

$$e^t{}_0 = f^{-1}\,, \qquad e^t{}_i = -f^2 W_i\,, \qquad e^\alpha{}_0 = 0\,, \qquad e^\alpha{}_i = e^{-\lambda}\delta_i^\alpha\,, \tag{4.4.2}$$

where we have defined $f \equiv e^\varphi$ and $W_\alpha = e^{2\varphi-\lambda}B_\alpha$. We write $\mu = (t, \alpha)$ for the curved indices and $a = (0, i)$ for the flat ones, respectively. We also require that all functions occurring in the metric (4.4.2) are independent of the coordinate t. Taking all these things into account, the components of the spin connection ω_{abc} in the flat basis read as follows,

$$\begin{aligned}
\omega_{00i} &= -e^{-\lambda}\,f^{-1}\partial_i f\ ,\\
\omega_{0ij} &= -\omega_{ij0} \;=\; f\,e^{-2\lambda}\,\partial_{[i}\left(W_{j]}e^\sigma f^{-2}\right)\ ,\\
\omega_{ijk} &= 2e^{-\lambda}\,\delta_{i[j}\partial_{k]}\lambda\ .
\end{aligned} \tag{4.4.3}$$

Following [75], it can be shown that the existence of a timelike Killing spinor leads to the following relations between the auxiliary fields V_μ, S and the metric functions

$$V_0 = \frac{1}{2}\,\epsilon^{ij}\,\omega_{ij0}\ , \tag{4.4.4}$$

$$V_1 - iV_2 = \mathrm{i}e^{-\lambda}\,\partial_z\,(\varphi - \lambda + ic)\ , \tag{4.4.5}$$

$$S = \mathrm{i}e^{-\lambda-ic}\,\partial_z\,(\varphi + \lambda - ic)\ , \tag{4.4.6}$$

$$\epsilon^{ij}\partial_i B_j = -2V_0\,e^{2\lambda-\varphi}\ , \tag{4.4.7}$$

where $c(x,y)$ is an arbitrary time-independent real function and $z = x + iy$ denotes a complex coordinate.

At this stage we have paved the way for constructing supersymmetric background solutions by exploiting the Killing spinor identities. Making an ansatz for the vector field V_μ we can now solve eqs. (4.4.4)–(4.4.7) and determine the metric functions λ and φ. Following the same logic in [75], we now look for solutions for with the following field configuration as

$$S = \Lambda\,, \qquad V_a = \text{const}\,, \qquad V_2 = 0\,, \qquad c = 0\,. \tag{4.4.8}$$

With these choices, the non-vanishing components of the spin connection given in Eq. (4.4.3) in a flat basis read as follows

$$\begin{aligned} \omega_{002} &= -(\Lambda + V_1)\,, \qquad \omega_{112} \;=\; \Lambda - V_1\,, \\ \omega_{120} &= \omega_{201} \;=\; -\,\omega_{012} \;=\; V_0\,. \end{aligned} \tag{4.4.9}$$

Note that, by setting $V_2 = c = 0$, we can solve for λ and φ using eqs. (4.4.5) and (4.4.6) and their integrability conditions. Furthermore, B_y can be set to zero by a gauge choice. As a result, we obtain the following differential equations for the functions φ, λ and B_x

$$e^{-\lambda}\partial_y\varphi = V_1 + \Lambda, \tag{4.4.10}$$

$$e^{-\lambda}\partial_y\lambda = \Lambda - V_1, \tag{4.4.11}$$

$$\partial_y B_x = 2V_0\, e^{2\lambda-\varphi}, \tag{4.4.12}$$

with $\partial_x\varphi = \partial_x\lambda = 0$.

It is worth emphasizing that so far we have not used the equations of motion, we have only considered the constraints that follow from supersymmetry. The solutions of Eqs. (4.4.10)–(4.4.12) will bifurcate depending on the value of the vector component V_1. In the next subsection we will classify the supersymmetric solutions of the CNMG Lagrangian (4.2.1) with respect to the value of this vector field component by imposing the field equations.

4.4.1 Classification of Supersymmetric Background Solutions

In this subsection, we first integrate the differential equations (4.4.10)–(4.4.12) depending on the different values of the vector field components V_a, which yields the metric functions λ and φ. Next, we impose the field equations and determine the couplings. The results for the different cases are given in three subsubsections. For the convenience of the reader, we have summarized all supersymmetric background solutions allowed by the theory (4.2.1) in Table 4.1.

	V^2	V_0	V_1	Equation	Solve STMG?
Round AdS_3	0	0	0	4.4.15	✓
$AdS_2 \times \mathbb{R}$	> 0	0	Λ	4.4.18	✗
Null-Warped AdS_3	0	$\pm\Lambda$	Λ	4.4.21	✓
Spacelike Squashed AdS_3	> 0	$< \Lambda$	Λ	4.4.25	✓
Timelike Streched AdS_3	< 0	$> \Lambda$	Λ	4.4.27	✓
AdS_3 pp-wave	0	V_0	εV_0	4.4.34	✓
Lifshitz	> 0	0	$\neq 0$ and $\neq \Lambda$	4.4.39	✗

Table 4.1
Classification of supersymmetric background solutions of the $\mathcal{N} = (1,1)$ CNMG. The solutions are classified with respect to the values of the components of the auxiliary vector V_a, and compared with the solutions of the $\mathcal{N} = (1,1)$ TMG (STMG).

The case $V_1 = 0$

We start with the simplest case, i.e. $V_1 = 0$. The supersymmetry constraint equations (4.4.10)–(4.4.12) yield

$$\lambda = -\log(-\Lambda y), \qquad \varphi = \log(-\frac{1}{\Lambda y}), \qquad B_x = -\frac{2V_0}{\Lambda}\log(-\Lambda y). \quad (4.4.13)$$

The vector equation (4.2.6) then implies $V_0 = 0$ for $\Lambda \neq 0$. Finally, from the scalar equation we fix M to be

$$M = -\frac{\Lambda^3}{6m^2} + \Lambda\sigma. \quad (4.4.14)$$

Thus, the metric becomes

$$ds^2 = \frac{l^2}{y^2}\left(-dt^2 + dx^2 + dy^2\right), \quad (4.4.15)$$

which describes the **round $\mathbf{AdS_3}$ spacetime** with $l = -\frac{1}{\Lambda}$, see Table 4.1.

The case $V_1 = \Lambda \neq 0$

For $V_1 = \Lambda$, we obtain

$$\lambda = 0, \qquad \varphi = 2\Lambda y, \qquad B_x = -\frac{V_0}{\Lambda}e^{-2\Lambda y}. \quad (4.4.16)$$

The vector and the scalar field equation lead to the following subclasses **A, B** and **C** which we describe below.

A. $V_0 = 0,\quad \Lambda = -2\sqrt{\frac{m^2\sigma}{7}},\quad M = \frac{7\Lambda^3}{12m^2} + \Lambda\sigma$

With this choice of parameters the metric reads

$$ds^2 = -e^{4\Lambda y}dt^2 + dx^2 + dy^2\,. \tag{4.4.17}$$

After a simple coordinate transformation $y = \frac{\log r}{2\Lambda}$, $x = \frac{x'}{2\Lambda}$ the metric is brought into the following form

$$ds^2 = \frac{l^2}{4}(-r^2dt^2 + \frac{dr^2}{r^2} + dx^2)\,, \tag{4.4.18}$$

which is $\mathbf{AdS_2} \times \mathbb{R}$. This background also appeared in the bosonic version of NMG, although given in different coordinates [31, 97].

B. $V_0 = \pm\Lambda,\quad \Lambda = -\sqrt{\frac{-2m^2\sigma}{7}},\quad M = -\frac{\Lambda^3}{6m^2} + \Lambda\sigma$

This choice of parameters leads to the metric

$$ds^2 = -e^{4\Lambda y}dt^2 \pm 2e^{2\Lambda y}dtdx + dy^2\,. \tag{4.4.19}$$

Performing a coordinate transformation

$$y = l\log u, \qquad t = lx^-, \qquad x = \pm\frac{lx^+}{2}, \tag{4.4.20}$$

the metric (4.4.19) can be put into the more familiar form [98]

$$ds^2 = l^2\left[\frac{du^2 + dx^+dx^-}{u^2} - \left(\frac{dx^-}{u^2}\right)^2\right], \tag{4.4.21}$$

which is a **null warped** $\mathbf{AdS_3}$.

C. $V_0 = \pm\sqrt{\frac{7\Lambda^2 - 4m^2\sigma}{21}},\quad M = -\frac{\Lambda^3}{3m^2} + \frac{8\Lambda\sigma}{7}$

Using these values for the parameters and fixing the value of V_0 we deduce from the vector equation that

$$ds^2 = \frac{V^2}{\Lambda^2}\Big(dx + \frac{V_0\Lambda}{V^2}\,e^{2\Lambda y}\,dt\Big)^2 - \frac{\Lambda^2}{V^2}\,e^{4\Lambda y}\,dt^2 + dy^2\,. \tag{4.4.22}$$

After making a coordinate transformation $\frac{V_0\Lambda}{V^2}e^{2\Lambda y} = \frac{1}{z}$, the metric reads

$$ds^2 = \frac{V^2}{\Lambda^2}\Big(dx + \frac{dt}{z}\Big)^2 - \frac{1}{z^2}\frac{V^2}{\Lambda^2}\frac{dt^2}{\nu^2} + \frac{dy^2}{4\Lambda^2 z^2}\,, \tag{4.4.23}$$

where $\nu^2 = 1 - \frac{V^2}{\Lambda^2} < 1$.

This is not yet the end of the story for this subclass: provided that $V^2 > 0$, which implies $7\Lambda^2 + 2m^2\sigma > 0$, we have $1 > \nu^2 > 0$. By making a coordinate transformation

$$x = \frac{x'\nu}{2V}\,, \qquad t = \frac{t'\nu}{2V}\,, \tag{4.4.24}$$

the metric (4.4.22) can be cast into the following form

$$ds^2 \;=\; \frac{l^2}{4}\Big[\frac{-dt^2 + dz^2}{z^2} + \nu^2\Big(dx + \frac{dt}{z}\Big)^2\Big]\,, \tag{4.4.25}$$

which is the metric of **spacelike squashed AdS$_3$** with squashing parameter ν^2.

For $V^2 < 0$, i.e. $7\Lambda^2 + 2m^2\sigma < 0$, we perform a coordinate transformation

$$x = \frac{x'}{2}\sqrt{\frac{-\nu^2}{V^2}}\,, \qquad t = \frac{t'}{2}\sqrt{\frac{-\nu^2}{V^2}}\,, \tag{4.4.26}$$

after which the metric (4.4.22) can be written in the following form

$$ds^2 = \frac{l^2}{4}\Big[\frac{dt^2 + dz^2}{z^2} - \nu^2\Big(dx + \frac{dt}{z}\Big)^2\Big]\,, \tag{4.4.27}$$

where $\nu^2 > 1$. The metric (4.4.27) is one of the incarnations of a **timelike stretched AdS$_3$** background.

The case $V_1 \neq \Lambda$ and $V_1 \neq 0$

This class of solutions has $V_1 \neq \Lambda$ and $V_1 \neq 0$. The calculation of the metric functions follows the computations performed in the previous subsubsections with the additional definitions

$$\sigma = -\log(z), \qquad \varphi = \log(z^\alpha), \qquad B_x = -\frac{V_0}{V_1}z^{-(1+\alpha)}, \tag{4.4.28}$$

where

$$z \equiv (V_1 - \Lambda)y, \qquad \alpha \equiv \frac{V_1 + \Lambda}{V_1 - \Lambda}\,. \tag{4.4.29}$$

Using the components of the vector equation, we find

$$V_0(V_0^2 - V_1^2)(V_1 - \Lambda) = 0\,. \tag{4.4.30}$$

From Eq. (4.4.30) it is straightforward to see that this subclass has two different branches, i.e. $V_0 = 0$ and $V_1 = \varepsilon V_0$ with $\varepsilon^2 = 1$. We will discuss these two branches as separate cases **A** and **B** below.

A. $V_1 = \varepsilon V_0\,, \varepsilon = \pm 1\,, \quad V_0 = -\varepsilon\Lambda \pm \sqrt{\dfrac{\Lambda^2 - 2m^2\sigma}{2}}$

With this choice of parameters the vector equation gives rise to

$$2V_0^2 + 4\varepsilon V_0 + \Lambda^2 + 2m^2\sigma = 0\,. \tag{4.4.31}$$

The parameter M can be solved by using the field equation for A as follows,

$$M = \frac{-\Lambda^3}{6m^2} + \Lambda\sigma\,. \tag{4.4.32}$$

Plugging in the metric functions, we obtain the following expression for the metric

$$ds^2 = -z^{2\alpha}(-dt + 2\varepsilon z^{-1-\alpha}dx)dt + \frac{1}{(V_1 - \Lambda)^2}\frac{dz^2}{z^2}\,.$$

Performing the coordinate transformation [75]

$$z = u^{\frac{(\Lambda - V_1)}{\Lambda}}\,, \qquad t = lx^-\,, \qquad x = \frac{\varepsilon l x^+}{2}\,, \tag{4.4.33}$$

this metric can be written as follows

$$ds^2 = l^2\left[\frac{du^2 + dx^+dx^-}{u^2} - u^{2(\frac{\Lambda - V_1}{\Lambda})}\left(\frac{dx^-}{u^2}\right)^2\right]\,. \tag{4.4.34}$$

This is the metric of a **AdS$_3$ pp-wave**. Note that the limit $V_1 \to \Lambda$ is well defined and gives rise to the *minus* null warped AdS_3 metric of Eq. (4.4.21), as expected.

B. $V_0 = 0,\quad V_1 = \dfrac{\alpha+1}{\alpha-1},\quad M = \frac{\Lambda(9V_1^2-2\Lambda^2)}{12m^2} + \Lambda\sigma$

The final spacetime we consider appears for $V_0 = 0$. Rather than solving the vector equation for V_1 as we did in the previous cases, we set $V_1 = \dfrac{\alpha+1}{\alpha-1}$ using Eq. (4.4.29). The field equations further imply that

$$(1-14\alpha-7\alpha^2)\Lambda^2 + 4m^2(-1+\alpha)^2\sigma = 0, \tag{4.4.35}$$

whose solution is given by

$$\Lambda = -\sqrt{\frac{4m^2\sigma(\alpha-1)^2}{(1-14\alpha-7\alpha^2)}}\,. \tag{4.4.36}$$

Here, we would like to restrict our attention to $\alpha < 0$, as α will be minus the Lifshitz exponent, thus giving rise to spacetimes with a positive Lifshitz exponent

(1) $\alpha < \dfrac{1}{7}(-7-2\sqrt{14})$ then $m^2\sigma > 0$,

(2) $\dfrac{1}{7}(-7-2\sqrt{14}) < \alpha < 0$ then $m^2\sigma < 0$.

Provided that the vector field components are chosen as discussed, we obtain the **Lifshitz** metric

$$ds^2 = l_L^2\Big[-y^{2\alpha}dt^2 + \frac{1}{y^2}(dx^2+dy^2)\Big]\,, \tag{4.4.37}$$

where l_L is the Lifshitz radius which is defined by

$$l_L^2 = \frac{1}{(V_1-\Lambda)^2}\,. \tag{4.4.38}$$

We have redefined t as $t \to (V_1-\Lambda)^{2\alpha+2}t$. Note that in the limit $V_1 \to 0$ one obtains the round AdS_3 metric given in Eq. (4.4.15). Taking $y = \frac{1}{r}$ gives the metric in the standard form

$$ds^2 = l_L^2\Big(-r^{-2\alpha}dt^2 + r^2dx^2 + \frac{1}{r^2}dr^2\Big)\,, \tag{4.4.39}$$

where l_L^2 and V_1 are given in terms of α and Λ as[3]

$$l_L^2 = \Big(\frac{\alpha - 1}{2\Lambda}\Big)^2 . \tag{4.4.40}$$

As shown in [75], all the supersymmetric backgrounds that we have found in this section except the AdS_3 metric preserve 1/4 of the supersymmetries.

4.5 Supersymmetric Black Holes

In this section, we discuss the supersymmetry aspects of black hole solutions of CNMG in a AdS_3 or Lifshitz background. The existence of a Killing spinor is highly restricted due to the global requirement that the angular coordinate ϕ should be periodic. As shown in [75], the spacelike squashed AdS_3 solution can be interpreted as an extremal black hole upon making a coordinate transformation. Therefore, in this section we will discuss three specific cases of black hole solutions. We start our discussion in subsection 4.5.1 with a generalization of the BTZ black hole, and show that the periodicity condition implies the extremality of the black hole. In the next subsection we investigate the 'logarithmic' black hole given in [94], and show that, the logarithmic black hole is also supersymmetric. Finally, in a third subsection we investigate the possible black holes in a Lifshitz background.

4.5.1 The Rotating Hairy BTZ Black Hole and its Killing Spinors

The CNMG Lagrangian (4.2.1) admits the following rotating black hole solution [93]

$$ds^2 = -N^2 F^2 dt^2 + \frac{dr^2}{F^2} + r^2 \left(d\phi + N^\phi dt\right)^2 \ , \tag{4.5.1}$$

where N, N^ϕ and F are functions of the radial coordinate r, given by

$$\begin{aligned}
N^2 &= \left[1 + \frac{b}{4H}\left(1 - \Xi^{\frac{1}{2}}\right)\right]^2 \ , \\
N^\phi &= -\frac{\mathcal{J}}{2\mathcal{M}r^2}\left(\mathcal{M} - bH\right) \ , \\
F^2 &= \frac{H^2}{r^2}\left[H^2 + \frac{b}{2}\left(1 + \Xi^{\frac{1}{2}}\right)H + \frac{b^2}{16}\left(1 - \Xi^{\frac{1}{2}}\right)^2 - \mathcal{M}\ \Xi^{\frac{1}{2}}\right] \ ,
\end{aligned} \tag{4.5.2}$$

[3]Note that the standard Lifshitz exponent z in the literature is given by $z = -\alpha$.

and

$$H = \left[r^2 - \frac{1}{2}\mathcal{M}\left(1-\Xi^{\frac{1}{2}}\right) - \frac{b^2}{16}\left(1-\Xi^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} . \tag{4.5.3}$$

Here, we have set the AdS_3 radius $l = 1$, $\Xi := 1 - \mathcal{J}^2/\mathcal{M}^2$, and the rotation parameter $\mathcal{J}/\mathcal{M}$ is bounded in terms of the AdS radius according to

$$-1 \leq \mathcal{J}/\mathcal{M} \leq 1 . \tag{4.5.4}$$

The parameter b is the gravitational hair, and for $b = 0$ one recovers the BTZ black hole [21]. Since we impose the global requirement that ϕ should be periodic, i.e. $0 \leq \phi \leq 2\pi$, the vacuum of the BTZ black hole with gravitational hair, defined by $\mathcal{M} = \mathcal{J} = b = 0$, admits only two Killing spinors. In order to see that, we consider the Killing spinor (4.3.38). Since the equations for ξ_1 and ζ_2 enforce exponential solutions for ξ_1 and ζ_2, we cannot find a solution for ξ_2 and ζ_1 that is periodic in ϕ. Therefore, finding a periodic solution requires setting $\xi_1 = \zeta_2 = 0$. This implies that only two of the solutions of equations (4.3.38) are valid.

Introducing the following orthonormal frame for the metric

$$e^0 = NF dt , \qquad e^1 = r d\phi + rN^\phi dt , \qquad e^2 = F^{-1} dr , \tag{4.5.5}$$

the spin-connection components are given by

$$\begin{aligned}
\omega_{01} &= \frac{1}{2}\frac{rN^{\phi\prime}}{FN} dr , \qquad \omega_{02} = \left(-FNF' + \frac{r^2 N^\phi N^{\phi\prime}}{2N} - F^2 N'\right) dt + \frac{r^2 N^{\phi\prime}}{2N} d\phi , \\
\omega_{12} &= \frac{1}{2} F(2N^\phi + rN^{\phi\prime}) dt + F d\phi ,
\end{aligned} \tag{4.5.6}$$

and hence the Killing spinor equation reads

$$\begin{aligned}
0 = d\epsilon &+ \frac{1}{2}\Big(-\frac{rN^{\phi\prime}}{2FN}\sigma_3\epsilon + \frac{1}{F}\sigma_3\epsilon^\star\Big) dr + \frac{1}{2}\Big(\frac{r^2N^{\phi\prime}}{2N}\sigma_1\epsilon - \mathrm{i}F\sigma_2\epsilon + r\sigma_1\epsilon^\star\Big) d\phi \\
&+ \frac{1}{2}\Bigg[\Big(-FNF' + \frac{r^2N^\phi N^{\phi\prime}}{2N} - F^2N'\Big)\sigma_1\epsilon - \mathrm{i}\Big(FN^\phi + \frac{1}{2}rFN'\Big)\sigma_2\epsilon \\
&\qquad + \mathrm{i}NF\sigma_2\epsilon^\star + rN^\phi\sigma_1\epsilon^\star\Bigg] dt .
\end{aligned} \tag{4.5.7}$$

Decomposing the Dirac spinor into two Majorana spinors like in Eq. (4.3.29), we obtain the following equations

$$
\begin{aligned}
0 &= d\xi_1 + \frac{1}{4N}\Big(N^\phi\left[2N(r-F)+r^2N^{\phi\prime}\right] \\
&\quad - FN(-2N+2rF'+rN^{\phi\prime}+2FN')\Big)\xi_2\,dt \\
&\quad + \frac{1}{4N}\Big(2N(r-F)+r^2N^{\phi\prime}\Big)\xi_2\,d\phi + \frac{1}{4FN}\Big(2N-rN^{\phi\prime}\Big)\xi_1\,dr\,, \\
0 &= d\xi_2 + \frac{1}{4N}\Big(N^\phi\left[2N(r+F)+r^2N^{\phi\prime}\right] \\
&\quad + FN(-2N-2rF'+rN^{\phi\prime}-2FN')\Big)\xi_1\,dt \\
&\quad + \frac{1}{4N}\Big(2N(r+F)+r^2N^{\phi\prime}\Big)\xi_1\,d\phi + \frac{1}{4FN}\Big(-2N+rN^{\phi\prime}\Big)\xi_2\,dr\,, \\
0 &= d\zeta_1 + \frac{1}{4N}\Big(N^\phi\left[-2N(r+F)+r^2N^{\phi\prime}\right] \\
&\quad - FN(2N+2rF'+rN^{\phi\prime}+2FN')\Big)\zeta_2\,dt \\
&\quad + \frac{1}{4N}\Big(-2N(r+F)+r^2N^{\phi\prime}\Big)\zeta_2\,d\phi - \frac{1}{4FN}\Big(2N+rN^{\phi\prime}\Big)\zeta_1\,dr\,, \\
0 &= d\zeta_2 + \frac{1}{4N}\Big(N^\phi\left[-2N(r-F)+r^2N^{\phi\prime}\right] \\
&\quad + FN(2N-2rF'+rN^{\phi\prime}-2FN')\Big)\zeta_1\,dt \\
&\quad + \frac{1}{4N}\Big(-2N(r-F)+r^2N^{\phi\prime}\Big)\zeta_1\,d\phi \\
&\quad + \frac{1}{4FN}\Big(2N+rN^{\phi\prime}\Big)\zeta_2\,dr\,.
\end{aligned} \tag{4.5.8}
$$

From these equations it follows that for the generic case not all the $d\phi$ components can be set to zero, which is the requirement for finding a periodic Killing spinor. Therefore, we turn our attention to the extremal solutions with $\mathcal{M} = |\mathcal{J}|$. For this case we find the following Killing spinors that are periodic in ϕ

(1) $\underline{\mathcal{M} = -\mathcal{J}}$

$$
\xi_1 = \zeta_1 = \zeta_2 = 0\,, \qquad \xi_2 = \frac{b+\sqrt{-b^2+8J+16r^2}}{\sqrt{r}}\,, \tag{4.5.9}
$$

(2) $\underline{\mathcal{M}=\mathcal{J}}$

$$\xi_1 = \xi_2 = \zeta_2 = 0\,, \qquad \zeta_1 = \frac{b+\sqrt{-b^2-8J+16r^2}}{\sqrt{r}}\,. \tag{4.5.10}$$

Note that for zero hair, i.e. $b \to 0$, one re-obtains the Killing spinors for a BTZ black hole.

4.5.2 The 'Logarithmic' Black Hole

The supersymmetric CNMG Lagrangian (4.2.1) also admits the following so-called 'logarithmic' black hole solution [94]

$$ds^2 = -\frac{4\rho^2}{l^2 f^2(\rho)}\,dt^2 + f^2(\rho)\Big(d\phi - \varepsilon\,\frac{q\,l\ln[\frac{\rho}{\rho_0}]}{f^2(\rho)}\,dt\Big)^2 + \frac{l^2}{4\rho^2}\,d\rho^2\,, \tag{4.5.11}$$

where $q \leq 0$ and $0 < \phi < 2\pi$. The function $f^2(\rho)$ is defined by

$$f^2(\rho) = 2\rho + q\,l^2\ln[\frac{\rho}{\rho_0}]\,, \tag{4.5.12}$$

and the parameter $\varepsilon = \pm 1$ determines the direction of the rotation since

$$M = 2q\,, \qquad J = 2\,\varepsilon\,lq\,. \tag{4.5.13}$$

Setting $q = 0$ and making the coordinate transformation $\rho = r^2/2$ we obtain an AdS_3 background with ϕ being periodic. This implies that the background of the 'logarithmic' black hole preserves only half of the supersymmetries like in the case of the rotating hairy BTZ black hole in the previous subsection.

We now determine the explicit expressions for the Killing spinors. Introducing the following orthonormal frame for the metric

$$e^0 = \frac{2\rho}{l f(\rho)}\,dt\,, \qquad e^1 = f(\rho)\,d\phi - \frac{lq\,\varepsilon\ln[\frac{\rho}{\rho_0}]}{f(\rho)}\,dt\,, \qquad e^2 = \frac{l}{2\rho}d\rho\,, \tag{4.5.14}$$

we find the following expressions for the spin-connection components

$$\begin{aligned}
\omega_{01} &= -\frac{l^2 q\,\varepsilon}{4\rho^2 f(\rho)}\Big[f(\rho) - 2\rho\,f'(\rho)\ln[\frac{\rho}{\rho_0}]\Big]\,d\rho\,, \qquad \omega_{12} = -\frac{q\,\varepsilon}{f(\rho)}\,dt + \frac{2\rho\,f'(\rho)}{l}\,d\phi\,,\\
\omega_{02} &= -\frac{1}{2l^2\,\rho\,f^2(\rho)}\bigg(f(\rho)\Big[8\rho^2 - l^4q^2\ln[\frac{\rho}{\rho_0}]\Big] + 2\rho f'(\rho)\Big[-4\rho^2 + l^4q^2\ln[\frac{\rho}{\rho_0}]\Big]\bigg)dt\\
&\quad -lq\,\epsilon\,\Big(\frac{f(\rho)}{2\rho} + \ln[\frac{\rho}{\rho_0}]\,f'(\rho)\Big)\,d\phi\,.
\end{aligned} \tag{4.5.15}$$

Using these expressions in the Killing spinor equation (4.2.4), we find that the Killing spinors of the logarithmic black hole are given by

i. <u>$\varepsilon = 1$</u>

$$\xi_1 = \xi_2 = \zeta_2 = 0\,, \qquad \zeta_1 = \sqrt{\frac{\rho}{\rho_0}}\Big(\frac{1}{2r + l^2 q \ln[\frac{\rho}{\rho_0}]}\Big)^{1/4}\,, \tag{4.5.16}$$

ii. <u>$\varepsilon = -1$</u>

$$\xi_1 = \zeta_1 = \zeta_2 = 0\,, \qquad \xi_2 = \sqrt{\frac{\rho}{\rho_0}}\Big(\frac{1}{2r + l^2 q \ln[\frac{\rho}{\rho_0}]}\Big)^{1/4}\,. \tag{4.5.17}$$

This result may be somewhat surprising considering the expectation that the only existing supersymmetric black hole in an AdS_3 background is an extremal BTZ black hole [58]. However, unlinke the rotating BTZ black hole, the "logarithmic" black hole does not have a non-extremal limit $J \neq M$. Thus, one cannot recover a static, non-supersymmetric black hole from the $J \to 0$ limit of the "logarithmic" black hole. Therefore, this particular case evades the argument presented in [58].[4]

4.5.3 Searching For a Supersymmetric Lifshitz Black Hole

In this section, we briefly present our attempts to find a supersymmetric Lifshitz black hole. Following [95], we first try to saturate the BPS bound using the vector field V_μ, since it can, in principle, contribute as a massive vector hair. In order to do so, we consider the following metric ansatz

$$ds^2 = l_L^2\Big(- a dt^2 + r^2 dx^2 + \frac{1}{f} dr^2\Big)\,, \tag{4.5.18}$$

where the functions a and f depend on the coordinate r only. With this Ansatz for the metric, one can show that the Killing spinor equation imposes the following constraint on these functions

$$\frac{a'\sqrt{f}}{a} + \frac{2\sqrt{f}}{r} + 2(\alpha - 1) = 0\,. \tag{4.5.19}$$

[4]We thank Paul Townsend for a clarifying discussion on this exceptional case.

Having obtained this constraint, we next turn to the vector equation (4.2.6). Using the metric ansatz (4.5.18), the V_0 and V_2 components of the vector equation are automatically satisfied, while the V_1 component reads

$$\begin{aligned} 0 &= (1+\alpha)\Big[r^2 f a' + 2a^2\Big(-8f + r[2r(-1+5\alpha+\alpha^2)+5f']\Big) \\ &\qquad -ra\Big(ra'f' + 2f(-5a' + ra'')\Big)\Big]. \end{aligned} \tag{4.5.20}$$

Imposing the Killing spinor constraint (4.5.19) to simplify the vector equation, we obtain

$$-7r\sqrt{f}(\alpha-1) - 11f + r\Big(r(7\alpha-2)+3f'\Big) = 0\,. \tag{4.5.21}$$

As we wish to find a solution for f which has a double root at $r=r_0$, which is a necessary condition for an extremal black hole, we need to be able to eliminate the f terms in the vector equation. Using the fact that the Killing spinor constraint (4.5.19) can be cast into the following form

$$\sqrt{f}(1-\alpha) = -\frac{1}{2}\Big(\frac{a'}{a}+\frac{2}{r}\Big)f\,, \tag{4.5.22}$$

the vector equation can be written as

$$\frac{7}{2}r\Big(\frac{a'}{a}-\frac{8}{7r}\Big)f + r\Big(r(7\alpha-2)+3f'\Big) = 0\,, \tag{4.5.23}$$

which has the following solution

$$a = r^{8/7}\,, \qquad f = r^2 - r_0^2\,. \tag{4.5.24}$$

However, using this equation in the Killing spinor constraint (4.5.19), we find that $r_0 = 0$. A further check with the metric equation also imposes $r_0 = 0$. Therefore, although the Killing spinor equation allows the existence of a supersymmetric black hole, we find that the vector and metric equations are incompatible with that possibility.

Alternatively, one may try to start with a rotating Lifshitz black hole using the following metric ansatz

$$ds^2 = l_L^2\left[-r^{-2\alpha}F(r)dt^2 + \Big(rdx + r^{-\alpha}G(r)dt\Big)^2 + \frac{1}{r^2F(r)}dr^2\right]\,, \tag{4.5.25}$$

where $F(r)$ and $G(r)$ are arbitrary functions that depend on the coordinate r only. In this case the Killing spinor equation constrains the function $F(r)$ to be of the form

$$F(r) = 1 + ar^{-2+2\alpha}\,, \tag{4.5.26}$$

where a is a constant. Furthermore, the vector equation constraints the function $G(r)$ by the following differential equation

$$21r^4G'^2 - 42r^3(\alpha+1)GG' + 21r^2(1+\alpha)^2G^2 + 4ar^{2\alpha}(6\alpha - 11) = 0\,. \tag{4.5.27}$$

Using the solutions of this differential equation, along with the expression (4.5.26) in the gravity equation, we find that it takes us back to the Lifshitz background, not allowing a rotating black hole solution.

The result of this subsection is somewhat expected, considering the fact that for the only rotating Lifshitz solution known to us [99], the couplings are determined by using a stationary Lifshitz spacetime which has a rotation term. This is not allowed by the given matter configuration of the $\mathcal{N} = (1,1)$ CNMG theory.

Finally, we would like to comment that as our attempts to find a supersymmetric Lifshitz black hole has failed with the parity-even theory under our consideration (4.2.1), one may consider to modify the CNMG by adding a parity violating Lorentz-Chern-Simons term, which gives rise to the so-called $\mathcal{N} = (1,1)$ Generalized Massive Gravity (GMG). In that case, we found that the vector equation is modified in such a way that the Lifshitz background is no longer a solution with the field configuration given in (4.4.8).

4.6 Conclusions

Using the off-shell Killing spinor analysis, we have investigated the supersymmetric backgrounds of the $\mathcal{N} = (1,1)$ CNMG model given by the Lagrangian (4.2.1). The background solutions are classified according to the norm of the Killing vector constructed out of Killing spinors. There are two cases only. First of all, when the Killing vector is null, the $\mathcal{N} = (1,1)$ analysis reduces to that of the $\mathcal{N} = 1$ CNMG model, since the null choice forces the auxiliary massive vector V_μ and the auxiliary pseudo-scalar B to vanish. Therefore, the solution is of the pp-wave type which preserves 1/4 of the supersymmetries. In the AdS_3 limit, there is a supersymmetry enhancement, and the AdS_3 solution is maximally supersymmetric.

As a second case, we investigated the case that the Killing vector is taken to be timelike. In particular, we did consider a special class of solutions in which the pseudo-scalar B vanishes. In that case all the supersymmetric solutions can be classified in terms of the components V_a of the massive vector in the flat basis. A subclass of these solutions, with different parameters, are also solutions of the supersymmetric TMG model, see Table 4.1. In addition to these solutions, we found that the $\mathcal{N} = (1,1)$ CNMG model possesses Lifshitz and $AdS_2 \times \mathbb{R}$ solutions. All these background solutions preserve 1/4 of the supersymmetries.

Next, we investigated three cases of black hole solutions in an AdS_3 or Lifshitz background. In the case of AdS_3, we studied the rotating hairy BTZ black hole and the logarithmic black hole. We found that in general the rotating hairy BTZ black hole is not supersymmetric due to the fact that the periodicity condition on the ϕ coordinate and the periodic Killing spinors only arise when the black hole is extremal. In the case of the logarithmic black hole, we found that only the extremal black hole solution exists, which is supersymmetric by its own nature. Finally, we analyzed the conditions for the existence of a supersymmetric Lifshitz black hole, and showed that it does not exist given the field configuration of the $\mathcal{N} = (1,1)$ CNMG model.

Chapter 5

Asymptotically Locally Flat Black Holes in $2 + 1$ Dimensions

5.1 Introduction

Black holes in 3D with AdS and deformed AdS asymptotics have been known for a long time, the BTZ black hole [21] being the most famous example. However, it was recently realized that K-gravity, which we introduced Section 2.1, allows a wide class of interesting black hole solutions which are asymptotically locally flat. In this chapter, we study different types of such solutions including black flower solutions (asymptotically flat black holes with deformed horizons), static black holes, rotating black holes and the dynamical black flowers (black holes with radiative gravitons) K-gravity. We show how they appear in K-gravity and we show that they are also solutions to the infinite order extended version of the NMG, that is the Born-Infeld extension of NMG with the EH term deleted. The same metrics also solve the topologically extended versions of these theories, with modified conserved charges and thermodynamical quantities, such as the Wald entropy. Besides these solutions we find new conformally flat radiating type solutions to these extended gravity models. We show that these metrics do not arise in Einstein's gravity coupled to physical perfect fluids. The material in this chapter is based on [100].

Our main tool will be the recent observation [101, 102] that the field equations of K-gravity split into two natural parts as $K_{\mu\nu} = J_{\mu\nu} + H_{\mu\nu}$, whose explicit forms will be given below. The H-tensor is the 3 dimensional version of the Bach tensor that vanishes for conformally flat (and conformally Einstein) geometries. On the other hand the J-tensor does not have any derivatives of the curvature and appeares as an interesting, bulk and boundary unitary deformation of topologically massive gravity (TMG) [29] to the minimal massive gravity (MMG) [32]. All the types of black flower solution will appear as a conformally flat solution ($H_{\mu\nu} = 0$), that also has vanishing J-tensor. This realization and dissection of the K-gravity's equations are crucial to upgrade the solution to other extended theories of gravity, such as the Born-Infeld extension of NMG, without the Einstein term (which we shall call BIK-gravity) and their Chern-Simons modified versions. We will be able to show that there are in fact 3 types of such solutions and no more: Type-D solutions to which the black flower, static and rotating metrics belong, Type-II solutions of which the dynamical black flower solution is an example and Type-N solutions.

5.2 Black Holes in Purely quadratic theory: K-gravity

Black hole solutions were already found in [34, 103, 104] partly inspired from earlier solutions [105–107] but as stated above, we shall make the appearance of the solution more transparent to be able to upgrade it to other extended theories, classify the solutions and show that there are no other possible classes under the given conditions. The action of K-gravity is given by

$$I = \frac{1}{16\pi G}\int d^3x\,\sqrt{-g}\,K, \qquad K \equiv R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2, \tag{5.2.1}$$

where the 3D Newton's constant G has dimensions of mass which makes the theory power-counting super-renormalizable. The theory has a massless ghost-free graviton about its unique maximally symmetric, flat vacuum. The source-free field equations ($K_{\mu\nu} = 0$) read

$$\begin{aligned} 0 &= 2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{2}g_{\mu\nu}\Box R + 4R_{\mu\rho\nu\sigma}R^{\rho\sigma} - g_{\mu\nu}R_{\rho\sigma}R^{\rho\sigma} \\ &\quad -\frac{3}{2}RR_{\mu\nu} + \frac{3}{8}g_{\mu\nu}R^2. \end{aligned} \tag{5.2.2}$$

They look somewhat cumbersome but as observed in [101], these field equations split into two natural pieces

$$K_{\mu\nu} = J_{\mu\nu} + H_{\mu\nu} = 0, \tag{5.2.3}$$

where the J and H-tensors are defined as

$$J^{\mu\nu} \equiv \frac{1}{2}\eta^{\mu\rho\sigma}\eta^{\nu\tau\alpha}S_{\rho\tau}S_{\sigma\alpha}, \qquad H_{\mu\nu} \equiv \frac{1}{2}\eta_\mu{}^{\alpha\beta}\nabla_\alpha C_{\beta\nu} + \frac{1}{2}\eta_\nu{}^{\alpha\beta}\nabla_\alpha C_{\beta\mu}, \tag{5.2.4}$$

with $\eta^{\nu\tau\eta}$ being the completely antisymmetric tensor,

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R \tag{5.2.5}$$

the Schouten tensor and

$$C_{\mu\nu} = \eta_\mu{}^{\alpha\beta}\nabla_\alpha S_{\beta\nu} \tag{5.2.6}$$

the Cotton tensor. Observe that $H_{\mu\nu}$ is traceless and so the trace of the field equations yield

$$K = g^{\mu\nu}K_{\mu\nu} = g^{\mu\nu}J_{\mu\nu}. \tag{5.2.7}$$

As for the covariant divergence of these tensors one has:

$$\nabla_\mu H^{\mu\nu} = -\nabla_\mu J^{\mu\nu} = \eta^{\nu\alpha\beta} S_{\alpha\sigma} C_\beta{}^\sigma. \tag{5.2.8}$$

As we shall see, this compact form of writing the equations is not merely about aesthetics, but it will help us understand the solution better since we know which parts of the equations vanish on their own.

It is clear from the above discussion that *all* the solutions of K-gravity satisfy the on-shell vanishing of K- (and so the action)

$$R_{\mu\nu} R^{\mu\nu} = \frac{3}{8} R^2, \tag{5.2.9}$$

which can be recast as

$$I_1 \equiv \tilde{R}^\nu_\mu \tilde{R}^\mu_\nu = \frac{1}{24} R^2, \tag{5.2.10}$$

where $\tilde{R}_{\mu\nu}$ is the traceless Ricci tensor and I_1 is a curvature invariant which we shall use to perform the Segre classification (see Section 2.4 for details) of the solutions along with the Ricci scalar R and a third curvature invariant

$$I_2 \equiv \tilde{R}^\nu_\mu \tilde{R}^\rho_\nu \tilde{R}^\mu_\rho. \tag{5.2.11}$$

Now consider *all* the solutions of the theory with $H_{\mu\nu} = 0$, which implies $J_{\mu\nu} = 0$ as well yielding

$$J_{\mu\nu} = -\widetilde{R}_{\mu\rho} \widetilde{R}^\rho_\nu + \frac{1}{3} g_{\mu\nu} I_1 + \frac{1}{12} R \widetilde{R}_{\mu\nu} = 0, \tag{5.2.12}$$

where we wrote the explicit form of this tensor and used the fact that its trace must vanish. At this stage the solutions bifurcate into two main classes: $R \neq 0$ and $R = 0$. Let us consider these cases separately as they will lead to spaces of different types.

5.2.1 The $R \neq 0$ case

For this case the traceless Ricci tensor can be expressed using Eq. (5.2.12) as

$$\widetilde{R}_{\mu\nu} = \frac{12}{R} \Big(\widetilde{R}_{\mu\rho} \widetilde{R}^\rho_\nu - \frac{1}{3} g_{\mu\nu} I_1 \Big). \tag{5.2.13}$$

Multiplying this with $\widetilde{R}^{\mu\nu}$, one finds that the two curvature invariants are related by

$$I_1 = \frac{12}{R} I_2, \qquad I_2 = \frac{R^3}{288}, \qquad I_1^3 = 6 I_2^2. \tag{5.2.14}$$

Hence according to the Segre classification, all such solutions are either type-D (metrics for localized objects) or type-II (metrics for localized objects with gravitational radiation). Of course, this analysis does not yet tell us that there are solutions to the theory. We must search for them explicitly. The best way to start with is a circularly symmetric ansatz:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \tag{5.2.15}$$

Computing $H_{\mu\nu} = 0$, one finds that the metric function satisfies

$$f(r) = br - \mu + cr^2, \tag{5.2.16}$$

which yields the most general spherically symmetric and conformally flat metric. Inserting this solution to the remaining piece of the equations, $J_{\mu\nu} = 0$ equation, one finds that $c = 0$ is required. Hence the following, conformally flat, static and asymptotically locally flat metric

$$ds^2 = -(br - \mu)dt^2 + \frac{dr^2}{br - \mu} + r^2 d\phi^2, \tag{5.2.17}$$

is a solution of K-gravity with two constants b and μ. It is a static, asymptotically flat black hole as discussed in [34, 103, 105] with a circular Schwarzschild horizon at

$$r_s = \frac{\mu}{b}, \tag{5.2.18}$$

as long as $\mu > 0$ and $b > 0$. One can easily see that its traceless Ricci tensor is of Type-D_s given by

$$\tilde{R}_{\mu\nu} = -\frac{R}{12}\Big(g_{\mu\nu} - 3\xi_\mu \xi_\nu\Big), \tag{5.2.19}$$

where the scalar curvature and the vector ξ_μ are given as

$$R = -\frac{2b}{r}, \qquad \xi_\mu = -r\delta^\phi_\mu, \tag{5.2.20}$$

with $\xi_\mu \xi^\mu = 1$. The fact that ξ_μ is a space-like vector (hence the subscript s in D_s) is important as we shall note below. Note also that the scalar curvature R and the invariants I_1, I_2 blow up at the singularity $r = 0$, hidden safely inside the Schwarzschild horizon r_s. Going to the Euclidean version, near the event horizon one finds that the Hawking temperature of the black hole is

$$T = \frac{b}{4\pi}. \tag{5.2.21}$$

Hence the dimensional parameter b is related to the mass of the black hole while the dimensionless parameter μ is a gravitational hair [34].

Other solutions can be generated from the above static black hole. After the transformation

$$u = t - r^*, \qquad dr^* = \frac{dr}{br - \mu}, \tag{5.2.22}$$

described in [34], the static metric becomes

$$ds^2 = -(br - \mu)du^2 - 2dudr + r^2 d\phi^2. \tag{5.2.23}$$

With this form of the metric, one can introduce a deformation $[h(u, \phi)]$ along the spacelike Killing vector ∂_ϕ as follows

$$ds^2 = -(br - \mu)du^2 - 2dudr + (r - h(u, \phi))^2 d\phi^2. \tag{5.2.24}$$

Inserting this ansatz into $H_{\mu\nu} = 0$, one finds that the following linear equation must be satisfied[1]

$$\partial_u(\partial_u + \frac{b}{2})h = 0. \tag{5.2.25}$$

The solution of this equation is given by

$$h(u, \phi) = A(\phi) + B(\phi)e^{-\frac{b}{2}u}, \tag{5.2.26}$$

where $A(\phi)$ and $B(\phi)$ are arbitrary periodic functions of the angular coordinate ϕ. As a simple example of a black flower solution, see Figure 5.1. For this solution one has $J_{\mu\nu} = 0$ automatically and the equations of K-gravity are solved. Let us rewrite the metric in terms of the original coordinates to get a better picture of its nature. This leads to the following expression

$$ds^2 = -(br - \mu)dt^2 + \frac{dr^2}{br - \mu} + \Big(r - h(t, r, \phi)\Big)^2 d\phi^2, \tag{5.2.27}$$

where

$$h(t, r, \phi) = A(\phi) + B(\phi)e^{-\frac{b}{2}t}(br - \mu)^{1/2}. \tag{5.2.28}$$

This solution is again Type-D_s or Type-II with the spacelike vector given by

$$\xi_\mu = \Big(h(t, r, \phi) - r\Big)\delta^\phi_\mu. \tag{5.2.29}$$

[1] Note that one can actually take a more general metric such as $ds^2 = -g(r)du^2 - 2dudr + (r - h(u, \phi))^2 d\phi^2$ and find solutions to the $H_{\mu\nu} = 0$ equation, but, it turns out that unless $g(r) = br - \mu$, one does not have $J_{\mu\nu} = 0$.

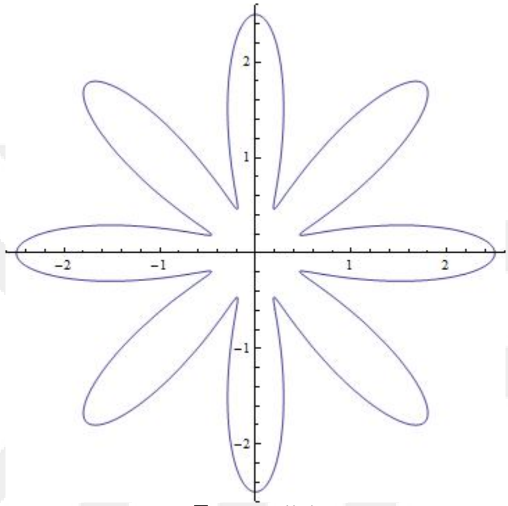

FIGURE 5.1
A picture of the black flower solution for the specific choice of dimensionless function $A(\phi) = \cos(8\phi) + 1.5$.

When $B(\phi)$=0, the time dependence drops out and one has a static, circularly nonsymmetric black hole (black flower) which is Type-D_s. But when $B(\phi) \neq 0$, then the spacetime is dynamical and the black flower is formed only at $t \to \infty$ for $b > 0$. One can interpret it either as a black hole radiating gravitons or as in-falling self-gravitating gravitons forming a black flower in the far future.

The full metric is given by

$$g_{\mu\nu} = g^S_{\mu\nu} + \xi_\mu \xi_\nu, \tag{5.2.30}$$

where $g^S_{\mu\nu}$ refers to the static circularly symmetric solution. This form suggests that one can write it in a double Kerr-Schild way as

$$g_{\mu\nu} = \eta_{\mu\nu} + f(r)\lambda_\mu \lambda_\nu + \xi_\mu \xi_\nu, \tag{5.2.31}$$

with $\eta_{\mu\nu}$ the flat metric in the u, r, ϕ coordinates and

$$f(r) = br - \mu, \qquad \lambda_\mu = \delta^u_\mu. \tag{5.2.32}$$

While formally, this is correct, since the ξ-vector is not null, it is not possible to employ this form to linearize field equations as done in [108].

For completeness we note that in the u, r, ϕ coordinates one has

$$\xi_\mu = \Big(h(u,\phi) - r\Big)\delta^\phi_\mu, \tag{5.2.33}$$

which has vanishing divergence $\nabla_\mu \xi^\mu = 0$, but ξ^μ is not a Killing vector. Let us note that for both static asymptotically flat black hole and the black flower, as well as the dynamical black flower, the Einstein tensor reads

$$G_{\mu\nu} = -\frac{R}{4}\Big(g_{\mu\nu} - \xi_\mu \xi_\nu\Big). \tag{5.2.34}$$

Defining the right-hand side as the energy-momentum tensor $T_{\mu\nu}$ one has

$$T_{\mu\nu} = -\frac{R}{4}\Big(g_{\mu\nu} - \xi_\mu \xi_\nu\Big), \tag{5.2.35}$$

and one might wonder if these solutions appear in Einstein's gravity coupled to perfect fluid [109, 110] with the pressure (P) and mass density (ρ) given by

$$P = -\frac{R}{4}, \qquad \rho = \frac{R}{2}. \tag{5.2.36}$$

While *formally* this is a strong energy condition-violating fluid, since the fluid velocity ξ_μ is spacelike these solutions do not exist in Einstein's theory coupled to perfect fluids. Before we move on to the other case, let us note that for the black flower solution, the scalar curvature is given by

$$R = -\frac{2b}{r - h(u,\phi)}. \tag{5.2.37}$$

In Figure 5.1, we have depicted an example of a simple black flower [$B(\phi) = 0$.] One must be careful though in the interpretation: from (5.2.37), it is clear that the scalar curvature is singular for the values of the radial coordinate $r = A(\phi)$ and hence one must restrict to $r > A(\phi)$. On the other hand, it is also clear from (5.2.27) that the event horizon is at

$$r_s = \frac{b}{\mu}. \tag{5.2.38}$$

If the singularity is pushed to the origin of the coordinates, then the horizon takes a noncircular shape. Another way to see is to look at the induced metric at the horizon

$$g_{\phi\phi} = (r_s - A(\phi))^2 \tag{5.2.39}$$

as suggested in [34].

5.2.2 The $R = 0$ case

For this case one has

$$\frac{1}{3}g_{\mu\nu}I_1 = \widetilde{R}_{\mu\rho}\widetilde{R}^{\rho}_{\nu}. \tag{5.2.40}$$

Multiplying this with $\widetilde{R}^{\mu\nu}$, one arrives at

$$I_2 = 0. \tag{5.2.41}$$

Furthermore, from Eq. (5.2.10), one finds that

$$I_1 = 0. \tag{5.2.42}$$

Hence, according to the Segre classification, all such solutions are Type-N_s with vanishing curvature invariants

$$R_{\mu\nu} = \xi_\mu \xi_\nu, \tag{5.2.43}$$

where ξ_μ is a null vector. A new example of this type of solutions is given by

$$ds^2 = F^2(t, x, y)(-dt^2 + dx^2 + dy^2), \tag{5.2.44}$$

where

$$F^2(t, x, y) = (at + bx + cy)^q, \qquad b = \pm\sqrt{a^2 + c^2}, \tag{5.2.45}$$

where q is an arbitrary real number.

5.3 The Born-Infeld extension of K-Gravity

Let us consider the Born-Infeld extension of K-gravity given by the action [35, 37, 111]

$$I = -\frac{\gamma^2}{4\pi G}\int d^3x \left[\sqrt{-\det\left(g_{\mu\nu} + \frac{\sigma}{\gamma}G_{\mu\nu}\right)} - (1 - \frac{\sigma}{4\gamma}R)\sqrt{-g}\right], \tag{5.3.1}$$

where γ is the BI parameter with $[\gamma] = mass^2$. This parameter is needed to be able to define a BI extension of gravity. It controls the scale where the higher derivative terms significantly change the structure of the theory. At the lowest order in the curvature expansion, this BI action reproduces that of K-gravity and γ plays no role. But in general there are two scales: the 3D Planck scale $M_p = 16\pi G$ and $\sqrt{\gamma}$. Of course one expects the latter to be several orders of magnitude smaller than the former. The theory has a unique maximally symmetric vacuum, that is the flat spacetime, and a massless unitary graviton about the vacuum. To find field equations of (5.3.1), we define a new tensor as

$$M_{\mu\nu} = g_{\mu\nu} + \frac{\sigma}{\gamma} G_{\mu\nu}. \tag{5.3.2}$$

Using this definition, (5.3.1) can be written as

$$I = -\frac{\gamma^2}{4\pi G} \int d^3x \Big[\sqrt{-M} - (1 - \frac{\sigma}{4\gamma} R) \sqrt{-g} \Big]. \tag{5.3.3}$$

Let us focus on the variation of the first part of the action;

$$\begin{aligned} I_M &= \int d^3x \sqrt{-M} \\ \delta I_M &= \frac{1}{2} \int d^3x \sqrt{-M} \sum_{\mu,\nu} (M^{-1})_{\mu\nu}\, \delta M_{\mu\nu} \\ &= \frac{1}{2} \int d^3x \sqrt{-g} \left(\sqrt{\frac{M}{g}} \sum_{\mu,\nu} (M^{-1})_{\mu\nu}\, \delta M_{\mu\nu} \right) \\ &= \frac{1}{2} \int d^3x \sqrt{-g}\, B^{\mu\nu} \delta M_{\mu\nu}, \end{aligned} \tag{5.3.4}$$

where the B tensor is defined by

$$B^{\mu\nu} = \sqrt{\frac{M}{g}} (M^{-1})_{\mu\nu}, \tag{5.3.5}$$

and $(M^{-1})_{\mu\nu}$ are the elements of the inverse of the matrix $M_{\mu\nu}$. Using this form of the variation is particularly useful for the finding field equations, which are given by

$$\begin{aligned} &- B_{\mu\nu} + g_{\mu\nu} + \frac{\sigma}{\gamma} \Big(- \nabla_\alpha \nabla_{(\mu} B_{\nu)}{}^{\alpha} + \frac{1}{2} (\nabla_\alpha \nabla_\beta B^{\alpha\beta} - \Box B) g_{\mu\nu} \\ &+ \frac{1}{2} \Box B_{\mu\nu} + \frac{1}{2} B_{\mu\nu} R + \frac{1}{2} \nabla_\mu \nabla_\nu B - \frac{1}{2} B R_{\mu\nu} + \frac{1}{2} G_{\mu\nu} \Big) = 0, \end{aligned} \tag{5.3.6}$$

where $B = g_{\mu\nu} B^{\mu\nu}$. We refer to [112] for a similar form of these equations. The metrics (5.2.17) and (5.2.27) solve the BIK equations without a change of the metric functions. Let us rewrite the Born-Infeld extension of K-gravity as follows

$$I_{BIK} = -\frac{\gamma^2}{4\pi G}\int d^3x\sqrt{-g}\Big(F(R,S_1,S_2)+\frac{\sigma}{4\gamma}R\Big), \tag{5.3.7}$$

with

$$\begin{aligned} F(R,S_1,S_2) &= \sqrt{1-\frac{\sigma}{2\gamma}(R+\frac{\sigma}{\gamma}S_1-\frac{1}{12\gamma^2}S_2)}-1, \\ S_1 = R^2_{\alpha\beta}-\frac{R^2}{2}, &\qquad S_2 = 8R^{\alpha\beta}R_{\alpha\sigma}R^{\sigma}{}_{\beta}-6RR^2_{\alpha\beta}+R^3. \end{aligned} \tag{5.3.8}$$

We shall need these expressions in the computation of the entropy and the mass of the black holes.

5.3.1 Some useful formulas for the calculation of entropy and mass

What we present here is common knowledge, hence we do not go into details but just quote the final results. We will calculate the geometrical entropy using the Wald formula where the black hole entropy is computed to be the Noether charge on the horizon [113–115]. There exists a simplified form of this formula given by [82, 116, 117]

$$S = -2\pi\int d\phi\sqrt{g_{\phi\phi}}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu}}\epsilon_{\mu}{}^{\alpha}\epsilon_{\nu\alpha}. \tag{5.3.9}$$

Here, the binormal $\epsilon_{\nu\beta}$ is defined through the time-like killing vector $\chi^{\mu} = (-1,0,-\Omega)$, with $\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}}$ being the angular velocity of the event horizon. One has the relation

$$\epsilon_{\mu\nu} = \frac{1}{\kappa}\nabla_{\mu}\chi_{\nu}, \tag{5.3.10}$$

where κ is the surface gravity defined as

$$\kappa = \sqrt{-\frac{1}{2}\nabla^{\mu}\chi^{\nu}\nabla_{\mu}\chi_{\nu}}. \tag{5.3.11}$$

In the presence of the Chern-Simons term which we will define below in (5.3.18), the black hole entropy is given in [118, 119]

$$S = \frac{2\pi}{\mu_{CS}}\int dx^{\sigma}\Gamma^{\nu}_{\mu\sigma}\epsilon^{\mu}{}_{\nu}\,. \tag{5.3.12}$$

Using the first law of thermodynamics, we also compute the mass of the static and rotating asymptotically locally flat black holes. It is an interesting property of these solutions that the angular velocity of the horizon Ω vanishes and the first law becomes $TdS = dM$. As a result, there is no way to compute the angular momentum with the help of the first law. For K-gravity, it was computed in [34]; however the angular momentum for the BIK theory remains to be computed by a direct method.

To compute the entropy and the mass of black holes in BIK gravity, we need the following expression

$$\frac{\partial F}{\partial R_{\mu\nu}} = \frac{-\sigma}{4\gamma(F+1)}\Big([1-\frac{\sigma}{\gamma}R+\frac{1}{2\gamma^2}(R^2_{\rho\sigma}-\frac{1}{2}R^2)]g^{\mu\nu} + \frac{2}{\gamma}(\sigma+\frac{1}{2\gamma}R)R^{\mu\nu}-\frac{2}{\gamma^2}R_{\mu\rho}R^{\rho}{}_{\nu}\Big). \quad (5.3.13)$$

For the static black hole (5.2.17) and the black flower solution (5.2.27), one finds through the above methods that

$$S = \frac{\pi b}{4G}, \qquad M = \frac{b^2}{32G}, \quad (5.3.14)$$

which are exactly the same values found for the K-gravity [34]. K- gravity has a rotating solution [93, 105] given as

$$ds^2 = -NFdu^2 - 2N^{1/2}dudr + (r^2+r_0^2)(d\phi + N^{\phi}du)^2, \quad (5.3.15)$$

with

$$F = br - \mu, \qquad N = \frac{(8r+a^2b)^2}{64(r^2+r_0^2)},$$
$$N^{\phi} = -\frac{a}{2}\Big(\frac{br-\mu}{r^2+r_0^2}\Big), \qquad r_0^2 = \frac{a^2}{4}\Big(\mu+\frac{a^2b^2}{16}\Big). \quad (5.3.16)$$

In [104], this solution is obtained by taking the flat-space limit of NMG and a contracted conformal field theory with the BMS_3 symmetries is proposed as a dual description of K-gravity. This is a type-D_s metric with the following properties

$$\tilde{R}_{\mu\nu} = -\frac{R}{12}\Big(g_{\mu\nu} - 3\xi_{\mu}\xi_{\nu}\Big),$$
$$\xi_{\mu} = \left\{0, -\frac{1}{2}a\sqrt{\frac{1}{r_0^2+r^2}}, -\frac{a^2b}{8} - r\right\}, \qquad \xi_{\mu}\xi^{\mu} = 1. \quad (5.3.17)$$

With this form of the Ricci tensor, one can show that the rotating metric, remarkably, also solves the BIK field equations which are a lot more complicated field equations. The calculation of the entropy and the mass also yields the same result as the static and the black flower cases (5.3.14).

5.3.2 Topologically extended K-gravity and BIK gravity

The Lagrangian density of topologically extended K-gravity is given by [29]

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}\Bigg\{\frac{1}{16\pi G}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2) \\ &\qquad + \frac{1}{2\mu_{CS}}\,\eta^{\mu\nu\alpha}\Gamma^{\beta}{}_{\mu\sigma}\Big(\partial_\nu\Gamma^{\sigma}{}_{\alpha\beta} + \frac{2}{3}\Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\alpha\beta}\Big)\Bigg\}, \end{aligned} \tag{5.3.18}$$

Let us calculate the mass and entropy of the static and rotating black holes.

1. The theory has an asymptotically flat black hole solution. Using the static and asymptotically locally flat metric (5.2.17), one obtains

$$S = \frac{\pi b}{4G}, \qquad M = \frac{b^2}{32G}. \tag{5.3.19}$$

2. The theory has a rotating black hole solution. Using the rotating asymptotically locally flat metric (5.3.15), one obtains

$$S = \pi b\Big(\frac{1}{4G} + \frac{2\pi a}{\mu_{CS}}\Big), \qquad M = b^2\Big(\frac{1}{32G} + \frac{3\pi a}{4\mu_{CS}}\Big). \tag{5.3.20}$$

The Lagrangian density of topologically extended BIK gravity is given by

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}\Bigg\{-\frac{\gamma^2}{4\pi G}\Big(F(R,K,S) + \frac{\sigma}{4\gamma}R\Big) \\ &\qquad + \frac{1}{2\mu_{CS}}\,\eta^{\mu\nu\alpha}\Gamma^{\beta}{}_{\mu\sigma}\Big(\partial_\nu\Gamma^{\sigma}{}_{\alpha\beta} + \frac{2}{3}\Gamma^{\sigma}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\alpha\beta}\Big)\Bigg\}, \end{aligned} \tag{5.3.21}$$

has the same entropy and mass expressions for these solutions. Hence the Chern-Simons term does its job and changes the thermodynamics and conserved charges of the rotating solution. Note that , for a particular choice of the Chern-Simons parameter μ_{CS}, the mass or the entropy of the black hole vanishes.

5.4 Conclusion

In this chapter, we have studied the recently found asymptotically flat black holes (static ones, rotating ones, the black flower solution) and the dynamical space-times of the purely quadratic part of the three dimensional new massive gravity. After identifying their types, according to the traceless Ricci tensor, we were able to upgrade these solutions to the Born-Infeld extension of the same theory which in principle has an infinite powers of curvature. As expected, the static black hole, rotating black hole and the black flower are type-D, albeit with a spacelike vector, hence they do not appear as solutions to Einstein's theory coupled to perfect "physical fluids". The dynamical black flower solution is of type-II, representing the appearance of a static black hole in the far future. It either forms due to the infalling gravitons or outgoing gravitons from a black hole. We also gave some new type-N solutions.

Chapter 6

Summary and Outlook

In this thesis, we have investigated a wide variety of 3D gravity theories from different perspectives, which include their supersymmetric completion in addition to background and black hole solutions. Our starting point was the fact that 3D Einstein gravity does not propagate any local degrees of freedom. In order to introduce local dynamics, theories defined by Lagrangians with higher curvature terms were considered.

In 3D, it is possible to avoid Ostragadski type instabilities, which generally appear in higher derivative gravity theories, and obtain theories that still preserve unitarity. In Chapter 2, considering Lagrangians with at most four derivatives of the metric, we reviewed those theories that are unitary around flat spacetime. These theories are also unitary around an AdS spacetime for a certain range in parameter space. However, they are expected to be holographically dual to a boundary CFT, and the boundary CFT turns out to be non-unitary for the range of parameters that make bulk gravity theory unitary. We introduced this bulk-boundary unitarity conflict by taking NMG as an example. The main tools and concepts used in the subsequent chapters are also explained in this chapter. The superconformal method for constructing supergravity theories, which is particularly useful in the higher derivative theories, is explained in detail. Aspects of supersymmetric solutions, which are important for the purpose of this thesis, are recapitulated. In the end, the algebraic classification of 3D spacetimes is discussed with a particular emphasis on higher derivative theories.

In Chapter 3, the $\mathcal{N} = 2$ supersymmetric extension of 3D higher derivative gravity theories up to four derivatives of the metric is constructed mainly by using the superconformal method. Studying the two possibilities for the off-shell closure of the algebra, $\mathcal{N} = (1,1)$ and $\mathcal{N} = (2,0)$ supersymmetries, we find that only the first one allows for a ghost free AdS vacuum. Assuming the formula derived for the case of pure gravity theories still holds, we show that the bulk-boundary unitarity problem is not solved by supersymmetry. It is still useful to check the validity of the formula for the central charge for matter-coupled gravity theories, which still leaves some hope for a solution to the problem. It might be the case that the non-minimal couplings between graviton and matter fields lead to a different expression for the central charge. The general class of off-shell supergravities constructed in this chapter leads to a number of interesting possibilities for future work. One can perform a systematic study of supersymmetric solutions, where a special case is the subject of Chapter 4. Although we constructed vector multiplet actions by using an arbitrary function of vector multiplet scalars, as given in equation (3.2.45), we did not consider such constructions for the scalar multiplet. It would be interesting to consider the coupling of an arbitrary number of scalar multiplets and vector multiplets, since that would enable us to construct a much larger class of supergravity Lagrangians [120]. The composite expression we derived for both scalar and vector multiplets can also be used to construct matter-coupled higher derivative supergravity models. Three dimensional matter coupled theories such as these have attracted considerable attention in the context of rigid supersymmetric theories on three-manifolds [80, 121]. One important possibility arises from the fact that the R-symmetry remains as a local symmetry in the $\mathcal{N} = (2,0)$ theory. One can use this setup to obtain an Einstein-Maxwell theory where the R-symmetry is dynamically gauged, which should admit new types of supersymmetric solutions.

In Chapter 4, supersymmetric solutions of the $\mathcal{N} = (1,1)$ NMG with cosmological constant were studied by employing the off-shell Killing spinor analysis, which was previously performed to study the solutions for the case of TMG. The analysis is still valid in our case since the bosonic field configurations are constrained only by supersymmetry without any reference to field equations. Contrary to the $\mathcal{N} = 1$ extension of the theory, where only spacetimes with a null Killing vectors are allowed, it is possible to obtain supersymmetric solutions with a time-like Killing vector. Various possible background solutions are extensively studied here and a comparison with the case of TMG is made. Embedding of some black hole solutions of NMG into a supersymmetric setup is also investigated. There are numerous directions one can consider for future study. An intriguing problem is to find a supersymmetric Lifshitz black hole. Although our trials with the current model have failed, it is natural to consider different approaches. For instance, one could saturate the BPS bound with a $U(1)$ charge. This can be achieved by coupling the $\mathcal{N} = (1,1)$ CNMG model to an off-shell vector multiplet and repeat the analysis performed in this chapter. It is also worth mentioning that the same procedure can be applied to the $\mathcal{N} = (2,0)$ CNMG model. This model has a different field content consisting of two auxiliary vectors and a real scalar as well as the graviton and the gravitino. Given that the $\mathcal{N} = (2,0)$ theory with matter couplings has new supersymmetric solutions [62], we expect that the $\mathcal{N} = (2,0)$ CNMG model will exhibit different supersymmetric solutions. Therefore, it would be interesting to see what the consequences of the different field content are for the supersymmetric solutions of the model.

In Chapter 5, we studied the solutions of K-gravity based on the observation that the field equations can be split into two parts one of which is the 3D Bach tensor that vanishes for conformally Einstein spaces and one of which is purely algebraic in the powers of the curvature. With this approach, the conformally Einstein solutions of the theory can be classified easily. We show that solutions of only three algebraic types are possible: Type-D, Type-II and Type-N. Since these solutions are conformally flat, we extend them trivially to be the solutions to the Chern-Simons modified version of the theory. It is also shown that solutions survive in the Born-Infeld type modification. For these different possibilities, physical properties of the black hole solutions, conserved charges and entropy, are presented. In future work, an exhaustive reach for all solutions with nontrivial 3D Bach tensor (namely, $H_{\mu\nu} \neq 0$) could be done in the K-gravity and the BIK gravity along the lines NMG and other extended 3D theories [52–54, 110, 122].

All the work that has been presented in this thesis is based on one central theme: Imagining a lower dimensional universe makes Einstein's theory trivial from many perspectives, but it also creates a whole new playground for ideas to modify it, which are not possible otherwise. Modifications that we have considered yield, among many other things, different theoretical possibilities for achieving gravitational dynamics, a richer structure of black hole physics, along with the opportunity to study the holographic principle in a more general setup. One might argue that this should be enough to pursue such a research program. However, it seems that all these attempts always bring some new interesting features at a price. They can also be seen as just different ways to understand how unique Einstein's theory is. It is, of course, up to the reader to decide.

List of Publications

- G. Alkac, L. Basanisi, E. Kilicarslan and B. Tekin, "Unitarity Problems in $3D$ Gravity Theories," Phys. Rev. D **96**, no. 2, 024010 (2017) [arXiv:1703.03630 [hep-th]].

- G. Alkac, S. Chakrabortty and P. Chaturvedi, "Holographic P-wave Superconductors in 1+1 Dimensions," to appear in Phys. Rev. D, [arXiv:1610.08757 [hep-th]].

- G. Alkac, E. Kilicarslan and B. Tekin, "Asymptotically flat black holes in 2+1 dimensions," Phys. Rev. D **93**, no. 8, 084003 (2016) [arXiv:1601.06696 [hep-th]].

- G. Alkac, L. Basanisi, E. A. Bergshoeff, D. O. Devecioglu and M. Ozkan, "Supersymmetric backgrounds and black holes in $\mathcal{N} = (1,\ 1)$ cosmological new massive supergravity," JHEP **1510**, 141 (2015) [arXiv:1507.06928 [hep-th]].

- G. Alkac, L. Basanisi, E. A. Bergshoeff, M. Ozkan and E. Sezgin, "Massive $\mathcal{N} = 2$ supergravity in three dimensions," JHEP **1502**, 125 (2015) [arXiv:1412.3118 [hep-th]].

- G. Alkac and D. O. Devecioglu, "Covariant Symplectic Structure and Conserved Charges of New Massive Gravity," Phys. Rev. D **85**, 064048 (2012) [arXiv:1202.1905 [gr-qc]].

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Samenvatting

Van de vier fundamentele interacties waarvan we het bestaan in de natuur af weten, is het waarschijnlijk zwaartekracht die het meest verwarrend is voor theoretisch natuurkundigen. Onze eerste poging tot een theoretisch begrip van zwaartekracht begon met Newton's inverse kwadraat-wet, deze leidde tot een succesvolle beschrijving van de banen van de planeten in ons zonnestelsel. Met de bevestiging van het bestaan van een nieuwe planeet die voorspelt werd door Newton's theorie, die we nu Neptunus noemen, werd Newton's theorie de standaard manier om zwaartekrachtfenomenen te beschrijven.

Dit bleef het geval tot dat Einstein een nieuwe zwaartekrachttheorie voorstelde, de algemene relativiteitstheorie (ART), hier wordt zwaartekracht gezien als een manifestatie van de door materie gegenereerde buiging van ruimtetijd. In toevoeging tot het verschaffen van een verklaring voor de afwijking van de baan van Mercurius, veranderde Einstein's theorie op radicale wijze de manier waarop we ruimte, tijd en materie beschouwen door middel van opmerkzame voorspellingen zoals het bestaan van zwarte gaten. Echter wanneer deze theorie werd behandeld binnen het raamwerk van kwantum veldentheorie, wat succesvol werd toegepast om een wiskundig consistente beschrijving te geven voor de drie andere fundamentele interacties, realiseerde men dat deze niet renormaliseerbaar is en daardoor zijn voorspellingskracht zou verliezen bij hoge energie.

Op het huidige moment bezitten wij twee verschillende modellen om ons universum te beschrijven op respectievelijk de kleinste en grootste schaal. Aan de ene kant hebben wij het standaard model van de deeltjesfysica (SM) die, onder bepaalde omstandigheden, het gedrag verklaart van sub-atomische deeltjes tot een verbazingwekkend hoge nauwkeurigheid. Aan de andere kant beschikken we over het standaard model van de oerknal kosmologie (ΛCDM), die gebruik maakt van ART op kosmologische schaal en een scala aan fenomenologische successen met zich meebrengt. Om een paar te noemen: de verklaring van sterrenstelsel rotatiekrommen, de kosmische achtergrondstraling en de versnelde uitbreiding van ons universum. Desalniettemin, wordt gebruikelijk aangenomen dat voor verscheidene redenen een aantal fundamentele principes nog steeds aan dit beeld ontbreken.

Het probleem van niet-renomaliseerbaarheid verhindert ons om ART in het standaard model te verwerken en daarmee een volledige beschrijving te verwezenlijken van alle fundamentele interacties op de kleinst mogelijke schaal. Daarnaast, het ΛCDM model voorspelt het bestaan van donkere materie en donkere energie, waarvan beide problematische aspecten bezitten. Op het moment is er nog geen experimenteel bewijs voor donkere materie, en deze vorm van niet-baryonische materie is niet in het standaard model verwezenlijkt. Tegelijkertijd is de waarde van de kosmologische constante, geïntroduceerd om rekening te houden met donkere energie effecten, erg klein binnen het ΛCDM model. De standaard model-voorspelling daarentegen stelt dat de Casimir energie vele ordes van grote hoger zou moeten liggen.

Een verscheidenheid aan aanpakken zijn gevolgd met de hoop om licht te werpen op deze eerdergenoemde problemen. Snaartheorie, wiens voordeel is dat het alle fundamentele interacties unificeert, en loop-kwantumzwaartekracht zijn de voornaamste kandidaten voor de kwantisatie van zwaartekracht. Daarnaast bestaan er voorstellen om zowel het standaard model alswel ART aan te passen. Het is de aanpassing van ART die een centrale rol zal spelen in het grootste deel van de discussie in deze these.

Een simpele manier om Einstein's zwaartekracht aan te passen is om hogere krommingstermen toe te voegen aan de Einstein-Hilbert actie. De resulterende theorie bezit nog steeds dezelfde ijksymmetriën aangezien het gedefinieerd is door middel van een Lorentz invariante actie.

Een belangrijk resultaat gerelateerd aan het renormalisatie probleem van zwaartekracht is, zoals aangetoond door Stelle, dat een renormaliseerbare zwaartekracht theorie verkregen kan worden door hogere krommingstermen toe te voegen aan de Einstein-Hilbert actie. Echter, de resulterende theorie in niet unitair en schend daardoor behoud van waarschijnlijkheid. Zoals dit klassieke voorbeeld aantoont, ondanks dat hogere krommingstermen de renormaliseerbaarheid eigenschappen van een theorie kunnen verbeteren, bestaan er zware restricties waaraan een theorie moet voldoen om unitariteit in acht te nemen.

Door de fluctuaties van de metriek om vlakke ruimte te beschouwen en het zwakke veld limiet van ART te nemen verkrijgen we de unieke zelf-consistente theorie van een massaloos spin-2 deeltje, wat we het graviton noemen. Als startpunt had men ook deze unieke veldentheorie kunnen nemen en alle mogelijke hogere krommingstermen beschouwen die de ijksymmetriën van de theorie handhaven. Het is mogelijk om aan te tonen dat deze procedure ART oplevert als de niet-lineaire vollediging van de veldentheorie van het massaloze spin-2 veld, dit onthult een fundamentele connectie tussen de ijkinvariantie en het equivalentie principe. Vanuit dit veldentheorie perspectief lijkt het natuurlijk om het geval van een massief spin-2 veld en zijn implicaties te beschouwen op niet-lineair niveau. Dit is het achterliggende idee van het 'Massive Gravity' onderzoeksprogramma wat veel aandacht heeft getrokken de laatste tijd. De introductie van een massief graviton leidt tot een Yukawa-type zwaartekracht-potentiaal, een op lange afstand exponentieel afstervende potentiaal met een lengte schaal bepaald door de graviton massa, deze kan mogelijk helpen met het oplossen van fenomenologische problemen geassocieerd met zwaartekracht op galactische en kosmologische schaal.

In deze these bestuderen we verscheidene zwaartekracht theoriën in 3 ruimtetijd dimensies met verschillende toepassingen in gedachten. Het hoofdidee is dat ondanks dat de eerder genoemde fenomenologische motivaties verloren raken, men 3D zwaartekracht theoriën kan beschouwen als een theoretisch laboratorium om ons begrip van bepaalde fysische ideeën te verbeteren in een relatieve simpelere opstelling.

Een belangrijke eigenschap van Einstein's zwaartekracht in 3D is zijn lokale trivialiteit. Voor elke 3D Einstein ruimte is kromming van ruimtetijd direct vastgesteld door de lokale materiedistributie. Hierdoor bestaan er geen dynamische vrijheidsgraden die door zwaartekrachtgolven kunnen propageren.

Ondanks zijn lokale trivialiteit is aangetoond dat 3D Einstein zwaartekracht met negatieve kosmologische constante zwarte gat-oplossingen toestaat, het Banados-Teitelboim-Zanelli (BTZ) zwarte gat. De oplossing is lokaal niet onderscheidbaar van Anti-de Sitter (AdS) ruimtetijd maar beschikt over alle eigenschappen van een zwart gat. Het beschikt over een goed gedefinieerde horizon en wordt uitsluitend gekarakteriseerd door zijn massa en hoekmoment, dit zijn globale ladingen die de ruimte onderscheid van AdS ruimtetijd.

Een andere interessante eigenschap van 3D Einstein zwaartekracht met een negatieve kosmologische constante werd onthuld door Brown en Henneaux. Zij hebben aangetoond dat de globale ladingen die overeen komen met de asymptotische symmetriën van de theorie een Poisson algebra opleveren met een centrale extensie. Onder bepaalde randvoorwaarden blijkt de resulterende algebra overeen te komen met twee kopieën van de Virasoro algebra, dit is de algebra waar de generatoren van de lokale hoekgetrouwe transformaties op de Hilbertruimte van 2D hoekgetrouwe veldentheorie (CFT). hierdoor wordt een intrigerende relatie tussen de dynamica van het klassieke graviton in 3D en the representaties van de Hilbert ruimte van een 2D CFT, die inherent kwantum mechanisch is. De holografische natuur van dit resultaat is opmerkelijk, en nu wordt het beschouwd als een voorteken van de AdS/CFT correspondentie, die is gebaseerd op ideeën geïntroduceerd door 't Hooft en Susskind.

Het holografisch principe in de moderne formulatie suggereert een exacte equivalentie van een kwantum zwaartekracht theorie op de d-dimensionale AdS ruimtetijd en een ijktheorie gedefinieerd op de (d-1)-dimensionale hoekgetrouwe rand van de AdS ruimtetijd. Het is een zwak/sterk-type dualiteit, wat het mogelijk maakt om beide theorieën te bestuderen in verschillende regimes. Ondanks dat het idee concreet gemaakt werd als de dualiteit tussen type-IIB snaren theorie op $AdS_5 \times S_5$ en de $\mathcal{N} = 4$ super Yang-Mills theorie in de grote N_c limiet, wordt vermoed dat de dualiteit standhoud voor algemene zwaartekracht theorieën op AdS ruimtetijd. Door het bestuderen van een zwak gekoppelde theorie kan men informatie extraheren over de vermoedelijk duale sterk gekoppelde theorie. Deze aanpak opent de deur naar het beter begrijpen van sterk gekoppelde ijktheorieën, waar veel toepassingen voor zijn in verschillende gebieden.

Omdat de algebra van lokale hoekgetrouwe transformaties in 2D oneindig dimensionaal is, zijn 2D CFTs veel gemakkelijker om te analyseren dan hoger dimensionale gevallen. Ze worden ook toegepast in snaren theorie waar de excitaties van de snaar worden beschreven door een 2D CFT vanuit het wereldlaken perspectief. Daarom zijn ze de meest bestudeerde en best bekende voorbeeld van velden theorieën met hoekgetrouwe symmetrieën, waardoor het bestuderen van holografie in 3D zwaartekracht theorieën in het bijzonder de moeite waard maakt. Een voorbeeld dat in het bijzonder genoemd zou moeten worden is Stromingers berekening van de Bekenstein-Hawking entropie van het BTZ zwarte gat door middel van staten in de 2D CFT. Dit resultaat reflecteert het feit dat het holografische principe tot verrassende resultaten kan leiden, zelfs in het geval waar de details van de duale velden theorie op de rand niet bekend zijn. Zonder enige directe referentie aan snaren theorie of supersymmetrie, de aanname van een zwaartekracht theorie op AdS met een duale 2D CFT levert een microscopische verklaring voor de entropie van een BTZ zwart gat.

Een ander voordeel van het werken in 3D is de mogeljkheid om theorieën te construeren met een massief spin-2 deeltje wat unitair is. Het eerste voorbeeld van zo een theorie werd verkregen door een zwaartekracht Chern-Simons term toe te voegen aan de Einstein-Hilbert actie. De resulterende theorie, genoemd Topologische Massieve Zwaartekracht (TMZ), heeft derde orde veld vergelijkingen die de enkele +2 of -2 heliciteit staten beschrijven, afhankelijk van het teken van de Chern-Simons koppeling.

Een andere unitaire theorie van 3D massieve zwaartekracht die niet pariteit schendende termen heeft was geconstrueerd door Bergshoeff, Hohm en Townsend. Deze theorie, Nieuwe Massieve Zwaartekracht (NMZ) genoemd, wordt verkregen door een zekere combinatie van gekwadrateerde kromming termen toe te voegen ($K = R_{\mu\nu}^2 - \frac{3}{8}R^2$) aan de Einstein-Hilbert term, wat leidt tot vierde orde veld vergelijkingen. De theorie heeft twee heliciteit ± 2 massieve staten omdat het pariteit behoudt.

Ondanks dat TMZ en NMZ allebei unitair gemaakt kunnen worden op vlakke ruimtetijd, zijn er nog enkele problemen als dit gedaan wordt op AdS ruimtetijd. Het BTZ zwarte gat wordt een oplossing van deze theorieën, en de parameters die voor unitariteit zorgen, maken de massa van het zwarte gat negatief. Ook zorgt het unitair maken van de bulk theorie ervoor dat de centrale lading van de rand CFT negatief wordt, wat de unitariteit van de rand theorie verpest. Unitariteit van de bulk is in direct conflict met unitariteit op de rand.

De theorie die bepaald wordt puur door het kwadratische gedeelte van NMZ laat interessante eigenschappen zien. Als eerst werd aangetoond dat het een enkele, massaloze propagerende staat heeft. Omdat de actie van de theorie is gevormd door een scalar ($K = R_{\mu\nu}^2 - \frac{3}{8}R^2$), wordt eraan gerefereerd als K-zwaartekracht. De studie naar exacte oplossingen van K-zwaartekracht leidde tot de eerste zwarte gat oplossingen in 3D zwaartekracht die asymptotisch lokaal vlak zijn, en die ook gedeformeerd kunnen worden tot het hebben van een niet sferisch symmetrische horizon, de zogenoemde "zwarte bloem" oplossingen.

In dit proefschrift werden deze 3D gemodificeerde zwaartekracht theorieën vanuit verschillende perspectieven bestudeerd, inclusief hun supersymmetrische theorie in toevoeging tot achtergrond en zwarte gat oplossingen.

In hoofdstuk twee wordt achtergrond materiaal behandeld dat nuttig is om de latere hoofdstukken te begrijpen. Inclusief een herhaling van de unitariteit analyse van de verschillende 3D gemodificeerde zwaartekracht modellen, een korte introductie tot hoekgetrouwe constructies van superzwaartekracht theorieën, het voornaamste aspect an klassieke suppersymmetrische oplossingen van zwaartekracht theorieën en de algebraïsche classificatie van 3D ruimtetijden.

In hoofdstuk drie onderzoeken we de $\mathcal{N} = 2$ supersymmetrische extensies van 3D zwaartekracht theorieën met acties die tot op kwadratische orde kromming invariant zijn, inclusief Einstein zwaartekracht, en TMZ, NMZ en K-zwaartekracht als bijzondere gevallen. De $\mathcal{N} = 2$ supersymmetrie laat twee verschillende formulaties toe van superzwaartekracht, $\mathcal{N} = (1,1)$ en $\mathcal{N} = (2,0)$ toe, die niet op de massaschil zitten en die verschillende fysische eigenschappen hebben voor theorieën met hogere afgeleide termen. Na het presenteren van de constructie van alle supersymmetrische invarianten laten we zien dat unitariteit op de AdS achtergrond alleen overleeft in het $\mathcal{N} = (1,1)$ geval en het supersymmetrische vacuum geeft geen verbetering ten opzichte van het niet unitair zijn van de rand CFT.

Hoofdstuk vier is gewijd aan supersymmetrische oplossingen van de $\mathcal{N} = (1,1)$ NMZ. We laten zien dat, in tegenstelling tot het $\mathcal{N} = 1$ geval, wat alleen oplossingen heeft met een nul Killing vector, $\mathcal{N} = (1,1)$ supersymmetrie oplossingen toelaat met een Killing vector in de tijdrichting, inclusief verschillende gedeformeerde AdS achtergronden en een Lifshitz ruimtetijd. Uiteindelijk worden de supersymmetrische eigenschappen van enkele zwarte gat oplossingen van NMZ bediscussieerd.

In hoofdstuk vijf bestuderen we de asymptotisch vlakke zwarte gat oplossingen van K-zwaartekracht vanuit een ander perspectief. Door het gebruik van een decompositie van de veld vergelijkingen in twee natuurlijke tensoren, classificeren we alle mogelijke oplossingen van de theorie en geven we nieuwe types van oplossingen. We laten ook zien dat de bekende oplossingen nog steeds oplossingen zijn van het Born-Infeld type en de Chern-Simons type extensies van de theorie. Voorts worden de modificaties van de behouden lading en de thermodynamische eigenschappen bediscussieerd.

In hoofdstuk zes geven we een samenvatting en geven we mogelijke richtingen aan voor toekomstig onderzoek.