SOURCE IDENTIFICATION PROBLEM FOR A TELEGRAPH EQUATION

by

Fatma ÇEKİÇ

A thesis submitted to

the Graduate School of Sciences and Engineering

of

Fatih University

in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

June 2015
Istanbul, Turkey

APPROVAL PAGE

This is to certify that I have read this thesis written by Fatma ÇEKİÇ and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science in Mathematics.

Prof. Allaberen ASHYRALYEV
Thesis Supervisor

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Feyzi BAŞAR
Head of Department

Examining Committee Members

Prof. Allaberen ASHYRALYEV

Prof. Feyzi BAŞAR

Asst. Prof. Özgür YILDIRIM

It is approved that this thesis has been written in compliance with the formatting rules laid down by the Graduate School of Sciences and Engineering.

Prof. Nurullah ARSLAN
Director

June 2015

SOURCE IDENTIFICATION PROBLEM FOR A TELEGRAPH EQUATION

Fatma ÇEKİÇ

M.S. Thesis – Mathematics
June 2015

Thesis Supervisor: Prof. Allaberen ASHYRALYEV

ABSTRACT

In this thesis, the source identification problem for a telegraph equation with unknown parameter p

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) = p + f(t)\ (0 \leq t \leq T), \\ \\ u(0) = \varphi, u'(0) = \psi, u(T) = \xi \end{cases}$$

in a Hilbert space H with the self-adjoint positive definite operator A is considered. The stability estimates for the solution of this problem is established. A first and second order of accuracy difference schemes for the approximate solution of this problem are presented. Stability estimates for the solution of these difference schemes are established. In applications, three source identification problems for telegraph equations are investigated. The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

Keywords: Source Identification Problem, Telegraph Equation, Difference Schemes, Stability, Hilbert Space.

BİR TELEGRAF DENKLEMİ İÇİN KAYNAK İDENTİFİKASYON PROBLEMİ

Fatma ÇEKİÇ

Yüksek Lisans Tezi – Matematik
Haziran 2015

Tez Danışmanı: Prof. Dr. Allaberen ASHYRALYEV

ÖZ

Bu tezde, bir H Hilbert uzayında özeşlenik pozitif tanımlı A operatörlü, bilinmeyen p parametreli telegraf denklemi için kaynak identifikasyon problemi

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) = p + f(t)\ (0 \le t \le T), \\ u(0) = \varphi, u'(0) = \psi, u(T) = \xi \end{cases}$$

ele alınmıştır. Bu bilinmeyen parametreli identifikasyon probleminin çözümü için kararlılık kestirimleri kurulmuştur. Bu problemin yaklaşık çözümü için birinci ve ikinci dereceden yakınsaklı fark şemaları sunulmuştur. Bu fark şemalarının çözümü için kararlılık kestirimleri kurulmuştur. Uygulamalarda, bilinmeyen parametreli telegraf denklemleri için üç tane identifikasyon problemi incelenmiştir. Bu fark şemalarının çözümü için bulunan teorik sonuçlar, sayısal örneklerle desteklenmiştir.

Anahtar Kelimeler: Kaynak İdentifikasyon Problemi, Telegraf Denklemi, Fark Şemaları, Kararlılık, Hilbert Uzayı.

To my family

ACKNOWLEDGEMENT

I would like to express my sincere gratitude to my supervisor Prof. Dr. Allaberen ASHYRALYEV for his guidance, helps and immense knowledge.

I am really thankful to my colleagues and friends who have willingly helped me out with their abilities.

Most importantly, I want to thank my dear family, especially my father and brother for working their best to give me the education, the opportunities and freedoms to pursue my dreams and interests.

Last but not least, I also would like to devote my special thanks to my love Serhat for his moral support and precious love, who was always standing by me in my hard times during this work.

TABLE OF CONTENTS

LIST OF TABLES

TABLE

CHAPTER 1

INTRODUCTION

The differential equations with parameters play a very important role in many branches of science and engineering. Some examples were given in temperature overspecification by Dehghan (Dehghan, 2001), robotics, chemistry (chromatography) by Kimura and Suzuki (Kimura and Suzuki, 1993), physics (optical tomography) by Gryazin, Klibanov, and Lucas (Gryazin et al., 1999).

The source identification problem for partial differential equations have been studied extensively by many researchers (see, (Ashyralyev and Ashyralyyeva, 2015); (Eidelman, 1984); (Hasanov, 2010); (Orlovsky and Piskarev, 2013a); (Orlovsky and Piskarev, 2009); (Orlovsky and Piskarev, 2013b); (Ashyralyev and Ashyralyyev, 2014); (Ashyralyyev, 2014b); (Ashyralyyev, 2014a); (Ashyralyev, 2010); (Serov and Pivrinta, 2006); (V.B. Shakhmurov, 2013); (Daoudi-Merzagui and Tabet, 2013); (Safari et al., 2013); (Tinaztepe et al., 2014); (Akyildiz et al., 2013); (Ashyralyev and Agirseven, 2014); (Ozbilge and Demir, 2013b); (Ozbilge and Demir, 2013a); (Ashyralyev et al., 2012); (Erdogan and Uygun, 2012); (Ashyralyev and Sobolevskii, 2004); (Ashyralyyev and Dedeturk, 2013); (Erdogan and Ashyralyev, 2014); (Ashyralyev and Erdogan, 2014); (Erdogan and Sazaklioglu, 2014); (Erdogan, 2012); (Ashyralyev and Urun, 2013b); (Ashyralyev and Urun, 2014); (Ashyralyev and Urun, 2013a); (Wu and Wu, 2014); (Blasio and Lorenzi, 2007) and the references therein). However, such problems were not well-investigated in general.

Our goal in this work is to investigate difference schemes for approximately solving telegraph equations with parameter. It is known that various boundary value problems for telegraph equations with parameter can be reduced to the boundary value problem for the differential equation with parameter p

$$\begin{cases} \frac{d^2u(t)}{dt^2} + \alpha\frac{du(t)}{dt} + Au(t) = p + f(t)\ (0 \le t \le T), \\ \\ u(0) = \varphi, u_t(0) = \psi, u(T) = \xi \end{cases} \tag{1.1}$$

in a Hilbert space H with self-adjoint positive definite operator A and $A \ge \delta I$. Here $\delta > 0, \alpha > 0$ and

$$\delta > \frac{\alpha^2}{4}.$$

The pair $\{u(t), p\}$ is called a solution of problem (1.1) if the following conditions are satisfied:

i) $u(t)$ is twice continuously differentiable function on $[0,T]$. The derivatives at the endpoints of the segment are understood as the appropriate unilateral derivatives.

ii) The element $u(t)$ belongs to $D(A)$ for all $t \in [0,T]$, and the function $Au(t)$ is continuous on $[0,T]$.

iii) $u(t)$ satisfies the equation and boundary conditions (1.1), $p \in H$.

It is clear that for finding a solution $u(t)$ of problem (1.1) it is useful to apply the substitution

$$u(t) = v(t) + A^{-1}p, \tag{1.2}$$

where $v(t)$ is the solution of the following nonlocal boundary value problem for the differential equation

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = f(t)\ (0 \le t \le T), \\ \\ v(T) = v(0) + \xi - \varphi, v_t(0) = \psi \end{cases} \tag{1.3}$$

and p is the unknown element defined by formulas

$$p = A\left(\xi - v(T)\right) \text{ or } p = A\left(\varphi - v(0)\right). \tag{1.4}$$

So, we consider the algorithm which includes three stages for solving problem (1.1):

In the first stage, we consider problem (1.3) and we will obtain $v(t)$.

In the second stage, we will put $t = 0$ or $t = T$ and find $v(0)$ or $v(T)$. Then using (1.4), we will obtain p.

In the third stage, we will use formula (1.2) for obtaining the solution $u(t)$ of problem (1.1). Moreover, we have one more possibility. Actually, we can obtain $u(t)$ by putting p into the problem (1.1) and solving it.

Let us see how to apply the analytical methods, namely, Fourier series and Fourier transform methods for obtaining the solution of source identification problem for telegraph equations on some examples.

Example 1.1. *Obtain the Fourier series solution of the initial-boundary value problem for telegraph equation with unknown source function $p(x)$*

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + f(t,x),\ 0 < t < 1,\ 0 < x < \pi, \\ \\ f(t,x) = (2t^2 + 4t + 3)\sin x,\ 0 < t < 1,\ 0 < x < \pi, \\ \\ u(0,x) = \sin x,\ u(1,x) = 2\sin x,\ u_t(0,x) = 0,\ 0 \le x \le \pi, \\ \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1. \end{cases} \tag{1.5}$$

Solution. For the solution of problem (1.5), it can be used the Fourier series method (method of separation of variables). To do this, let $v(t,x)$ be the solution of the following problem

$$\begin{cases} \frac{\partial^2 v(t,x)}{\partial t^2} + 2\frac{\partial v(t,x)}{\partial t} - \frac{\partial^2 v(t,x)}{\partial x^2} + v(t,x) = (2t^2 + 4t + 3)\sin x,\ 0 < t < 1, 0 < x < \pi, \\ \\ v(0,x) - v(1,x) = -\sin x,\ v_t(0,x) = 0,\ 0 \le x \le \pi, \\ \\ v(t,0) = v(t,\pi) = 0,\ 0 \le t \le 1. \end{cases}$$

We seek formula for the solution of this problem by the Fourier series method

$$v(t,x)=\sum_{k=1}^{\infty}A_k(t)\sin(kx).$$

We have that

$$\sum_{k=1}^{\infty}A_k^{''}(t)sin(kx)+2\sum_{k=1}^{\infty}A_k^{'}(t)\sin(kx)+\sum_{k=1}^{\infty}A_k(t)k^2\sin(kx)$$

$$+\sum_{k=1}^{\infty}A_k(t)\sin(kx)=(2t^2+4t+3)\sin x$$

and

$$\sum_{k=1}^{\infty}A_k(0)\sin(kx)-\sum_{k=1}^{\infty}A_k(1)\sin(kx)=-\sin x,$$

$$\sum_{k=1}^{\infty}A_k^{'}(0)\sin(kx)=0.$$

Equating the coefficients of $\sin(kx)$ for $k=1,2,\dots,$ we get

$$\begin{cases} A_1^{''}(t)+2A_1^{'}(t)+2A_1(t)=2t^2+4t+3,\ 0<t<1, \\ \\ A_1(0)-A_1(1)=-1,\ A_1^{'}(0)=0 \end{cases}$$

for $k=1$ and

$$\begin{cases} A_k^{''}(t)+2A_k^{'}(t)+(k^2+1)A_k(t)=0,\ 0<t<1, \\ \\ A_k(0)=A_k(1),\ A_k^{'}(0)=0 \end{cases}$$

for $k\neq 1$.

Let $k\neq 1$. Then,

$$A_k^{''}(t)+2A_k^{'}(t)+(k^2+1)A_k(t)=0$$

and the characteristic equation for this differential equation is

$$r^2+2r+(k^2+1)=0.$$

The roots of this equation are

$$r_{1,2}=-1\pm ki.$$

Therefore, the general solution of this differential equation is

$$A_k(t) = c_1 e^{-t}\cos(kt) + c_2 e^{-t}\sin(kt).$$

Using the conditions $A_k(0) = A_k(1)$ and $A_k^{'}(0) = 0$, we get $c_1 = c_2 = 0$. Therefore,

$$A_k(t) = 0 \text{ for all } k \neq 1.$$

Now, let $k = 1$. Then,

$$A_1^{''}(t) + 2A_1^{'}(t) + 2A_1(t) = 2t^2 + 4t + 3.$$

We know that the general solution will be of the form

$$A_1(t) = A_1^c(t) + A_1^p(t)$$

where the complementary solution $A_1^c(t)$ is the solution of the homogeneous differential equation

$$(A_1^c(t))^{''} + 2(A_1^c(t))^{'} + 2A_1^c(t) = 0.$$

Similarly, we can obtain

$$A_1^c(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t.$$

Now, for the particular solution let

$$A_1^p(t) = b_1 t^2 + b_2 t + b_3.$$

So, differentiating and putting it into the differential equation, we get

$$2b_1 + 2(2b_1 t + b_2) + 2(b_1 t^2 + b_2 t + b_3) = 2t^2 + 4t + 3$$

or

$$2b_1 t^2 + (4b_1 + b_2)t + 2(b_1 + b_2 + b_3) = 2t^2 + 4t + 3.$$

Equating the coefficients of t^k for $k = 0, 1, 2$, we get

$$b_1 = 1,\ b_2 = 0 \text{ and } b_3 = \frac{1}{2}.$$

It follows that

$$A_1^p(t) = t^2 + \frac{1}{2}.$$

Therefore,

$$A_1(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t + t^2 + \frac{1}{2}.$$

Using the conditions $A_1(0) - A_1(1) = -1$ and $A_1^{'}(0) = 0$, we can easily obtain $c_1 = c_2 = 0$. From that it follows

$$A_1(t) = t^2 + \frac{1}{2}.$$

Therefore,

$$v(t,x) = A_1(t) \sin x = \left(t^2 + \frac{1}{2}\right) \sin x.$$

Now, for obtaining $p(x)$, we will use formula (1.4). In this example, we have the differential operator A defined by the formula

$$Au(x) = -u^{''}(x) + u(x)$$

with domain

$$D(A) = \left\{u(x), u^{'}(x), u^{''}(x) \in L_2\left[0,\pi\right],\ u(0) = u(\pi) = 0\right\}.$$

Then, using formula (1.4), we get

$$\begin{aligned} p(x) &= A\varphi - Av(0) \\ &= \sin x + \sin x - \frac{1}{2}\sin x - \frac{1}{2}\sin x \\ &= \sin x. \end{aligned}$$

So,

$$p(x) = \sin x.$$

Putting it into (1.5), we get the following problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = (2t^2 + 4t + 4)\sin x,\ 0 < t < 1,\ 0 < x < \pi, \\ u(0,x) = \sin x,\ u(1,x) = 2\sin x,\ u_t(0,x) = 0,\ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi) = 0,\ 0 \le t \le 1. \end{cases}$$

We seek the solution of this problem by the Fourier series method

$$u(t,x) = \sum_{k=1}^{\infty} B_k(t) \sin(kx).$$

We have that

$$\sum_{k=1}^{\infty} B_k''(t)\sin(kx) + 2\sum_{k=1}^{\infty} B_k'(t)\sin(kx) + \sum_{k=1}^{\infty} B_k(t)k^2\sin(kx)$$

$$+\sum_{k=1}^{\infty} B_k(t)\sin(kx) = (2t^2+4t+4)\sin x$$

and

$$\sum_{k=1}^{\infty} B_k(0)\sin(kx) = \sin x,$$

$$\sum_{k=1}^{\infty} B_k'(0)\sin(kx) = 0.$$

Equating the coefficients of $\sin(kx)$ for $k = 1, 2, \ldots,$ we get

$$\begin{cases} B_1''(t) + 2B_1'(t) + 2B_1(t) = 2t^2+4t+4, \ 0 < t < 1, \\ \\ B_1(0) = 1, \ B_1'(0) = 0 \end{cases}$$

for $k = 1$ and

$$\begin{cases} B_k''(t) + 2B_k'(t) + (k^2+1)B_k(t) = 0, \ 0 < t < 1, \\ \\ B_k(0) = 0, \ B_k'(0) = 0 \end{cases}$$

for $k \neq 1$.

Let $k \neq 1$. Then,

$$B_k''(t) + 2B_k'(t) + (k^2+1)B_k(t) = 0.$$

By the same manner, we can obtain

$$B_k(t) = b_1 e^{-t}\cos(kt) + b_2 e^{-t}\sin(kt).$$

Using the conditions $B_k(0) = 0$ and $B_k'(0) = 0$, we get $b_1 = b_2 = 0$. Therefore,

$$B_k(t) = 0 \text{ for all } k \neq 1.$$

Now, let $k = 1$. Then,

$$B_1''(t) + 2B_1'(t) + 2B_1(t) = 2t^2+4t+4.$$

We know that the general solution will be of the form

$$B_1(t) = B_1^c(t) + B_1^p(t).$$

Similarly, we can obtain the complementary solution

$$B_1^c(t) = b_1 e^{-t} \cos t + b_2 e^{-t} \sin t.$$

Now, for the particular solution let

$$B_1^p(t) = d_1 t^2 + d_2 t + d_3.$$

So, differentiating and putting it into the differential equation, we get

$$2d_1 + 2(2d_1 t + d_2) + 2(d_1 t^2 + d_2 t + d_3) = 2t^2 + 4t + 4$$

or

$$2d_1 t^2 + (4d_1 + d_2)t + 2(d_1 + d_2 + d_3) = 2t^2 + 4t + 4.$$

Equating the coefficients of t^k for $k = 0, 1, 2$, we get

$$d_1 = d_3 = 1 \text{ and } d_2 = 0.$$

It follows that

$$B_1^p(t) = t^2 + 1.$$

Therefore,

$$B_1(t) = b_1 e^{-t} \cos t + b_2 e^{-t} \sin t + t^2 + 1.$$

Using the conditions $B_1(0) = 1$ and $B_1^{'}(0) = 0$, we can easily obtain $b_1 = b_2 = 0$. Therefore,

$$B_1(t) = t^2 + 1.$$

and

$$u(t, x) = B_1(t) \sin x = \left(t^2 + 1\right) \sin x$$

is the solution of initial-boundary value problem for telegraph equation (1.5).

Note that using similar procedure one can obtain the solution of the following identification problem for a multidimensional telegraph equation

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \alpha \frac{\partial u(t,x)}{\partial t} - \sum\limits_{r=1}^{n} a_r \frac{\partial^2 u(t,x)}{\partial x_r^2} + \delta u(t,x) = p(x) + f(t,x), \\ x = (x_1, \dots x_n) \in \overline{\Omega}, 0 < t < T, \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), u(T,x) = \xi(x), x \in \overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial n} = 0, x \in S, \end{cases}$$

where $a_r \geq a \geq 0, \delta > 0$ and $f(t,x)(t \in [0,T], x \in \overline{\Omega}), \varphi(x), \psi(x), \xi(x), (x \in \overline{\Omega})$ are given smooth functions. Here Ω is the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1, 1 \leq k \leq n)$ with the boundary

$$S, \overline{\Omega} = \Omega \cup S.$$

Here $\frac{\partial}{\partial n}$ indicates differentiation in the direction of the exterior normal to S.

Example 1.2. *Obtain the Fourier transform solution of the identification problem for a telegraph equation*

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + f(t,x),\ 0 < t < 1,\ -\infty < x < \infty, \\ f(t,x) = \left((2-4x^2)e^{-t} - 1\right) e^{-x^2},\ 0 < t < 1,\ -\infty < x < \infty, \\ u(0,x) = e^{-x^2},\ u(1,x) = e^{-1}e^{-x^2},\ u_t(0,x) = -e^{-x^2},\ \ -\infty < x < \infty. \end{cases} \tag{1.6}$$

Solution. For the solution of problem (1.6), it can be used the Fourier transform method. To do this, let $v(t,x)$ be the solution of the following problem

$$\begin{cases} \frac{\partial^2 v(t,x)}{\partial t^2} + 2\frac{\partial v(t,x)}{\partial t} - \frac{\partial^2 v(t,x)}{\partial x^2} + v(t,x) = f(t,x), 0 < t < 1, -\infty < x < \infty, \\ f(t,x) = \left((2-4x^2)e^{-t} - 1\right) e^{-x^2}, 0 < t < 1,\ -\infty < x < \infty, \\ v(0,x) - v(1,x) = (1-e^{-1})e^{-x^2},\ v_t(0,x) = -e^{-x^2},\ \ -\infty < x < \infty. \end{cases} \tag{1.7}$$

We denote that

$$v(t,s) = \mathbf{F}\left\{v(t,x)\right\}.$$

Taking Fourier transform of both sides of the differential equation (1.7) and given conditions, we obtain

$$\begin{cases} v_{tt}(t,s) + 2v_t(t,s) + (s^2+1)v(t,s) = (s^2e^{-t}-1)\,\mathbf{F}\left\{e^{-x^2}\right\}, \\ \\ 0<t<1,\ -\infty<x<\infty, \\ \\ v(0,s)-v(1,s) = (1-e^{-1})\mathbf{F}\left\{e^{-x^2}\right\},\ v_t(0,s) = \mathbf{F}\left\{-e^{-x^2}\right\},\ \ -\infty<x<\infty. \end{cases}$$

Then, to solve the problem we need to separate $v(t,s)$ into two parts

$$v(t,s) = v^c(t,s) + v^p(t,s),$$

where $v^c(t,s)$ is the solution of homogeneous equation

$$v_{tt}(t,s) + 2v_t(t,s) + (s^2+1)v(t,s) = 0$$

and $v^p(t,s) = A(s)e^{-t} + B(s)$ is the solution of nonhomogeneous equation

$$v_{tt}(t,s) + 2v_t(t,s) + (s^2+1)v(t,s) = \left(s^2e^{-t}-1\right)\mathbf{F}\left\{e^{-x^2}\right\}.$$

It is easy to see that

$$v^c(t,s) = c_1e^{-t}\cos st + c_2e^{-t}\sin st$$

and

$$A(s) = \mathbf{F}\left\{e^{-x^2}\right\} \text{ and } B(s) = -\frac{1}{s^2+1}\mathbf{F}\left\{e^{-x^2}\right\}.$$

From that, it follows

$$v^p(t,s) = \mathbf{F}\left\{e^{-x^2}\right\}\left(e^{-t} - \frac{1}{s^2+1}\right).$$

Using the conditions $v(0,s)-v(1,s) = (1-e^{-1})\mathbf{F}\left\{e^{-x^2}\right\}$ and $v_t(0,s) = \mathbf{F}\left\{-e^{-x^2}\right\}$, we can easily obtain $c_1 = c_2 = 0$. So,

$$v(t,s) = \mathbf{F}\left\{e^{-x^2}\right\}\left(e^{-t} - \frac{1}{s^2+1}\right).$$

Now, for obtaining $p(x)$, we will use formula (1.2). It implies that

$$p(x) = Au(t) - Av(t)$$

or

$$p(x) = -u''(t) + u(t) - (-v''(t) + v(t)).$$

Let us denote $\mathbf{F}\{p(x)\} = p(s)$. Taking Fourier transform of both sides and putting $t = 0$, we obtain

$$p(s) = (s^2+1)\left[u(0,s) - v(0,s)\right].$$

Since we have that

$$u(0,s) = \mathbf{F}\left\{e^{-x^2}\right\}$$

and

$$v(0,s) = \mathbf{F}\left\{e^{-x^2}\right\}\left(1 - \frac{1}{s^2+1}\right),$$

we obtain

$$\begin{aligned} p(s) &= (s^2+1)\left[\mathbf{F}\left\{e^{-x^2}\right\} - \mathbf{F}\left\{e^{-x^2}\right\}\left(1 - \frac{1}{s^2+1}\right)\right] \\ &= \mathbf{F}\left\{e^{-x^2}\right\}. \end{aligned}$$

Now, taking inverse Fourier transform, we get

$$p(x) = \mathbf{F}^{-1}\left\{\mathbf{F}\left\{e^{-x^2}\right\}\right\} = e^{-x^2}.$$

For obtaining $u(t,x)$, we will use formula

$$p(s) = (s^2+1)\left[u(t,s) - v(t,s)\right],$$

from that it follows

$$\begin{aligned} u(t,s) &= \frac{p(s)}{s^2+1} + v(t,s) \\ &= \frac{\mathbf{F}\left\{e^{-x^2}\right\}}{s^2+1} + \left[\mathbf{F}\left\{e^{-x^2}\right\}\left(e^{-t} - \frac{1}{s^2+1}\right)\right] \\ &= \mathbf{F}\left\{e^{-x^2}\right\}e^{-t}. \end{aligned}$$

Finally, taking inverse Fourier transform, we get

$$u(t,x) = \mathbf{F}^{-1}\left\{\mathbf{F}\left\{e^{-x^2}\right\}e^{-t}\right\} = e^{-x^2}e^{-t}$$

is the solution of the identification problem for telegraph equation (1.6).

Note that using the same manner one obtains the solution of the following identification problem for the $2m-th$ order multidimensional telegraph equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}+\alpha\frac{\partial u}{\partial t}-\sum\limits_{|r|=2m} a_r\frac{\partial^{|r|}u}{\partial x_1^{r_1}\ldots\partial x_n^{r_n}}+\delta u=p(x)+f(t,x), \\ 0\leq t\leq T, x,r\in\mathbb{R}^n, |r|=r_1+\ldots+r_n, \\ u(0,x)=\varphi(x), u_t(0,x)=\psi(x), u(T,x)=\xi(x), x\in\mathbb{R}^n, \end{cases}$$

where $a_r\geq a\geq 0,\delta>0$ and $f(t,x)(t\in[0,T],x\in\mathbb{R}^n),\varphi(x),\psi(x),\xi(x),(x\in\mathbb{R}^n)$ are given smooth functions.

However,the analytical methods described above, namely Fourier series method and Fourier transform method can be used only when the differential equation has constant coefficients.

Let us briefly describe the contents of the various sections of the thesis. It consists of five chapters.

First chapter is the introduction.

In **second chapter,** the main theorem on stability of problem (1.1) is established. In applications, theorems on the stability inequalities for the solution of three source identification problems for the telegraph equations are established.

In **third chapter**, the first and second order of accuracy difference schemes for the approximate solution of problem (1.1) are presented. Stability estimates for the solution of these difference schemes are established. In applications, stability estimates for the solution of difference schemes for three source identification problems are established.

In **fourth chapter,** the methods are illustrated by numerical examples.

Finally, **fifth chapter** is conclusion.

CHAPTER 2

SOURCE IDENTIFICATION PROBLEM FOR THE TELEGRAPH EQUATION

2.1 THE MAIN THEOREM

Let H be a Hilbert space, A be a positive definite self-adjoint operator with $A \geq \delta I$, where $\delta > 0$. Let $\alpha > 0$ and

$$\delta > \frac{\alpha^2}{4}. \tag{2.1}$$

Throughout this paper, $\{c(t), t \geq 0\}$ is a strongly continuous cosine operator-function defined by the formula

$$c\left(t\right) = \frac{e^{itB^{1/2}} + e^{-itB^{1/2}}}{2}.$$

Then, from the definition of the sine operator-function $s\left(t\right)$

$$s(t)u = \int\limits_0^t c(s)u \; ds$$

it follows that

$$s\left(t\right) = B^{-1/2}\frac{e^{itB^{1/2}} - e^{-itB^{1/2}}}{2i}.$$

Here $B = A - \frac{\alpha^2}{4}I$. For the theory of cosine operator-function, we refer to (Fattorini, 1985) and (Piskarev and Shaw, 1997). Now, let us give some lemmas that will be needed below.

Lemma 2.1. *The estimates hold:*

$$\left\|c(t)\right\|_{H\to H} \leq 1, \left\|B^{1/2}s(t)\right\|_{H\to H} \leq 1, \left\|B^{-1/2}\right\|_{H\to H} \leq \frac{1}{\sqrt{\delta - \frac{\alpha^2}{4}}} \tag{2.2}$$

Lemma 2.2. *Assume that*

$$1 > \left(1 + \frac{\frac{\alpha}{2}}{\sqrt{\delta - \frac{\alpha^2}{4}}}\right) e^{-\frac{\alpha}{2}T}. \tag{2.3}$$

Then, the operator

$$I - \left(c(T) + \frac{\alpha}{2}s(T)\right) e^{-\frac{\alpha}{2}T}$$

has inverse

$$P = \left\{I - \left(c(T) + \frac{\alpha}{2}s(T)\right) e^{-\frac{\alpha}{2}T}\right\}^{-1}$$

and the following estimate

$$||P||_{H\to H} \leq M \tag{2.4}$$

holds, where $M = M(\delta, \alpha) > 0$.

Proof. The proof of estimate (2.4) is based on the estimate

$$\left\|c(T) + \frac{\alpha}{2}s(T)\right\|_{H\to H} \leq 1 + \frac{\frac{\alpha}{2}}{\sqrt{\delta - \frac{\alpha^2}{4}}}. \tag{2.5}$$

Using the triangly inequality and the estimate (2.2), we obtain

$$\left\|c(T) + \frac{\alpha}{2}s(T)\right\|_{H\to H} \leq \|c(T)\|_{H\to H} + \frac{\alpha}{2}\|s(T)\|_{H\to H} \leq 1 + \frac{\frac{\alpha}{2}}{\sqrt{\delta - \frac{\alpha^2}{4}}}.$$

Lemma 2.2 is proved.

In this chapter, we will investigate boundary value problem (1.1) for the differential equation with parameter p. The main theorem on stability of problem (1.1) is established. In applications, theorems on the stability inequalities for the solution of three source identification problems for the telegraph equations are established.

Firstly, the solvability of problem (1.1) in the space $C(H)$ of the continuous H-valued functions $\varphi(t)$ defined on $[0, T]$, equipped with the norm

$$\|u\|_{C(H)} = \max_{0\leq t\leq T} ||\varphi(t)||_H$$

is investigated. We will prove the following main theorem on continuously dependents of the solution on the given data.

Theorem 2.1. *Suppose that* $\varphi,\xi \in D(A)$ *and* $\psi \in D(A^{\frac{1}{2}})$. *Let conditions (2.1) and (2.3) are satisfied and* $f(t)$ *be continuously differentiable function on* $[0,T]$. *Then, for the solution* $(u(t),p)$ *of problem (1.1) in* $C(H)\times H$ *the following stability inequalities*

$$\|u\|_{C(H)} + \| A^{-1}p \|_H \leq M(\delta,\alpha)\left[\|\varphi\|_H + \|\xi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \|f\|_{C(H)}\right],$$

$$\left\|\frac{d^2u}{dt^2}\right\|_{C(H)} + \|Au\|_{C(H)} + \|p\|_H$$

$$\leq M(\delta,\alpha)\left[\|A\varphi\|_H + \|\varphi\|_H + \left\|A^{\frac{1}{2}}\psi\right\|_H + \|A\xi\|_H + \|\xi\|_H + \max_{0\leq t\leq T}\|f'(t)\|_H + \|f(0)\|_H\right]$$

hold, where $M(\delta,\alpha)$ *is independent of* $f(t)$, $t\in[0,T]$ *and* φ,ψ,ξ.

Proof of Theorem 2.1 is based on formulas (1.2), (1.4) and the following theorem on well-posedness of nonlocal boundary value problem (1.3).

Theorem 2.2. *Suppose that the assumptions of Theorem 2.1 hold. Then, for the solution* $v(t)$ *of problem (1.3) in* $C(H)$ *the stability estimates*

$$\|v\|_{C(H)} \leq M(\delta,\alpha)\left[\|\varphi\|_H + \|\psi\|_H + \|\xi\|_H + \|f\|_{C(H)}\right], \tag{2.6}$$

$$\left\|\frac{d^2v}{dt^2}\right\|_{C(H)} + \|Av\|_{C(H)} \tag{2.7}$$

$$\leq M(\delta,\alpha)\left[\|A\varphi\|_H + \|\varphi\|_H + \left\|A^{\frac{1}{2}}\psi\right\|_H + \|A\xi\|_H + \|\xi\|_H + \max_{0\leq t\leq T}\|f'(t)\|_H + \|f(0)\|_H\right]$$

hold, where $M(\delta,\alpha)$ *does not depend on* $f(t)$, $t\in[0,T]$ *and* φ,ψ,ξ.

Proof. First, we obtain the formula for solution of problem (1.3) under the assumption (2.1). We have the following formula

$$v(t) = e^{-\frac{\alpha}{2}t}c(t)\,v(0) + \frac{\alpha}{2}e^{-\frac{\alpha}{2}t}s(t)v(0) + e^{-\frac{\alpha}{2}t}s(t)\psi \tag{2.8}$$

$$+\int_0^t e^{-\frac{\alpha}{2}(t-z)}s(t-z)f(z)dz$$

for the mild solution of initial value problem

$$\begin{cases} \frac{d^2v(t)}{dt^2} + \alpha\frac{dv(t)}{dt} + Av(t) = f(t) \ (0 \le t \le T), \\ \\ v(0) \text{ is given}, v'(0) = \psi. \end{cases}$$

Applying condition $v(T) = v(0) + \xi - \varphi$, and formula (2.8), we get

$$v(0) = \left(c(T) + \frac{\alpha}{2}s(T)\right)e^{-\frac{\alpha}{2}T}v(0) + e^{-\frac{\alpha}{2}T}s(T)\psi \tag{2.9}$$
$$+\lambda\int_0^T e^{-\frac{\alpha}{2}(T-z)}s(T-z)f(z)dz + \varphi - \xi.$$

By Lemma 2.2, under the assumption (2.3), there exists of inverse

$$P = \left\{I - \left(c(T) + \frac{\alpha}{2}s(T)\right)e^{-\frac{\alpha}{2}T}\right\}^{-1}.$$

Therefore, using (2.9), we obtain

$$v(0) = P\left\{e^{-\frac{\alpha}{2}T}s(T)\psi + \lambda\int_0^T e^{-\frac{\alpha}{2}(T-z)}s(T-z)f(z)dz + \varphi - \xi\right\}. \tag{2.10}$$

Consequently, the solution of problem (1.3) satisfy formulas (2.8) and (2.10).

Second, we obtain estimate (2.6). Using formulas (2.10), (2.8), and estimates (2.2), we obtain

$$\|v(0)\|_H \le M_1(\delta,\alpha)\left[\|\varphi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \|\xi\|_H + \max_{0\le t\le T}\|f(t)\|_H\right],$$

$$\max_{0\le t\le T}\|v(t)\|_H \le M_2(\delta,\alpha)\left[\|v(0)\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \max_{0\le t\le T}\|f(t)\|_H\right].$$

Estimate (2.6) follows from these estimates.

Third, we obtain estimate (2.7). Applying A to formula (2.10) and estimates (2.2), we get

$$\|Av(0)\|_H \le \|P\|_{H\to H}\left\{\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}e^{-\frac{\alpha}{2}T}\left\|B^{\frac{1}{2}}s(T)\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H + \|A\varphi\|_H + \|A\xi\|_H\right.$$

$$+\left\|AB^{-1}\right\|_{H\to H}\left[\|f(T)\|_H+e^{-\frac{\alpha}{2}T}\|c(T)\|_{H\to H}\|f(0)\|_H\right]$$

$$\times\left\|AB^{-1}\right\|_{H\to H}\int\limits_0^T e^{-\frac{\alpha}{2}(T-z)}\parallel c(T-z)\parallel_{H\to H}\left[\frac{\alpha}{2}\parallel f(z)\parallel_H+\parallel f'(z)\parallel_H\right]dz$$

$$\leq M_1(\delta,\alpha)\left\{\|A\varphi\|_H+\|A\xi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int\limits_0^T\|f'(t)\|_H\,dt\right\}. \quad (2.11)$$

Applying A to formula (2.8) and using an integration by parts, we can write the formula

$$Av(t)=e^{-\frac{\alpha}{2}t}c(t)Av(0)+\frac{\alpha}{2}e^{-\frac{\alpha}{2}t}A^{\frac{1}{2}}s(t)A^{\frac{1}{2}}v(0)+A^{\frac{1}{2}}e^{-\frac{\alpha}{2}t}s(t)A^{\frac{1}{2}}\psi$$

$$+e^{-\frac{\alpha}{2}t}AB^{-1}\left[e^{\frac{\alpha}{2}t}f(t)-c(t)f(0)-\int\limits_0^t e^{\frac{\alpha}{2}z}c(t-z)\left[\frac{\alpha}{2}f(z)+f'(z)\right]dz\right].$$

Using the last formula and estimates (2.2), we obtain

$$\|Au(t)\|_H\leq\|c(t)\|_{H\to H}\,e^{-\frac{\alpha}{2}t}\|Av(0)\|_H$$

$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|\frac{\alpha}{2e^{\frac{\alpha}{2}t}}\right|\parallel A^{\frac{1}{2}}v(0)\parallel_H$$

$$+\parallel B^{\frac{1}{2}}s(t)\parallel_{H\to H}\left\|A^{1/2}B^{-\frac{1}{2}}\right\|_{H\to H}\left|e^{-\frac{\alpha}{2}t}\right|\parallel A^{\frac{1}{2}}\psi\parallel_H$$

$$+\left\|AB^{-1}\right\|_{H\to H}\left[\|f(t)\|_H+e^{-\frac{\alpha}{2}t}\|c(t)\|_{H\to H}\|f(0)\|_H\right]$$

$$+\left\|AB^{-1}\right\|_{H\to H}\int\limits_0^t e^{-\frac{\alpha}{2}(t-z)}\parallel c(t-z)\parallel_{H\to H}\left[\frac{\alpha}{2}\parallel f(z)\parallel_H+\parallel f'(z)\parallel_H\right]dz$$

$$\leq M_3(\delta,\alpha)\left[\|Av(0)\|_H+\parallel A^{\frac{1}{2}}\psi\parallel_H+\|f(0)\|_H+\max_{0\leq t\leq T}\parallel f'(t)\parallel_H\right]$$

for any $t\in[0,T]$. Then, we get

$$\max_{0\leq t\leq T}\parallel Av(t)\parallel_H \quad (2.12)$$

$$\leq M_3(\delta,\alpha)\left[\parallel Av(0)\parallel_H+\parallel A^{\frac{1}{2}}\psi\parallel_H+\|f(0)\|_H+\max_{0\leq t\leq T}\parallel f'(t)\parallel_H\right].$$

Estimate

$$\max_{0\leq t\leq T}\|Au(t)\|_H$$

$$\leq M_4(\delta,\alpha)\left\{\|A\varphi\|_H+\|A\xi\|_H+\left\|A^{1/2}\psi\right\|_H+\|f(0)\|_H+\int\limits_0^T\|f'(t)\|_H\,dt\right\}$$

follows from estimates (2.7), (2.11) and (2.12). Finally, estimate for $\max\limits_{0\leq t\leq T}\left\|\frac{d^2u}{dt^2}\right\|_H$ follows from the last estimate and the triangle inequality. Theorem 2.2 is proved.

Now, we will consider three applications of Theorem 2.1.

First, we consider the nonlocal boundary value problem for telegraph equations

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-(a(x)u_x)_x+\delta u(t,x)=p(x)+f(t,x), \\ \\ 0<t<T, 0<x<l, \\ \\ u(0,x)=\varphi(x), u_t(0,x)=\psi(x), u(T,x)=\xi(x), 0\leq x\leq l, \\ \\ u(t,0)=u(t,l), u_x(0,x)=u_x(t,l), 0\leq t\leq T. \end{cases} \tag{2.13}$$

Problem (2.13) has a unique smooth solution $(u(t,x),p(x))$ for the smooth $a(x)\geq a>0$, $x\in(0,l)$, $\delta>0$, $a(l)=a(0)$, $\varphi(x),\psi(x),\xi(x),(x\in[0,l]$ and $f(t,x)$ $(t\in(0,T),x\in(0,l))$ functions. This allows us to reduce boundary value problem (2.13) to abstract boundary value problem (1.1) in a Hilbert space $H=L_2[0,1]$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x)=-(a(x)u_x)_x+\delta u \tag{2.14}$$

with domain

$$D(A^x)=\{u(x):u(x),u_x(x),(a(x)u_x)_x\in L_2[0,1],u(1)=u(0),u_x(1)=u_x(0)\}.$$

Theorem 2.3. *Let conditions (2.1) and (2.3) are satisfied. Then, for the solution* $\{u(t,x),p(x))\}$ *of problem (2.13), we have the following stability inequalities*

$$\| u \|_{C(L_2[0,1])}+\| (A^x)^{-1} p \|_{L_2[0,1]} \tag{2.15}$$

$$\leq M(\delta,\alpha)\left[\| \varphi \|_{L_2[0,1]}+\| \psi \|_{L_2[0,1]}+\| \xi \|_{L_2[0,1]}+\max_{0\leq t\leq T}\|f(t)\|_{L_2[0,1]}\right],$$

$$\max_{0\leq t\leq T}\|u''\|_{L_2[0,1]}+\|u\|_{C(W_2^2[0,1])}+\|p\|_{L_2[0,1]} \tag{2.16}$$

$$\leq M(\delta,\alpha)\left[\| \varphi \|_{W_2^2[0,1]}+\| \psi \|_{W_2^1[0,1]}+\max_{0\leq t\leq T}\|f'(t)\|_{L_2[0,1]}\right.$$

$$+\|\xi\|_{W_2^2[0,1]}+\|f(0)\|_{L_2[0,1]}\Big],$$

where $M(\delta,\lambda)$ *is independent of* $\varphi(x)$, $\psi(x)$, $\xi(x)$,*and* $f(t,x)$.*Here, the Sobolev space* $W_2^2[0,1]$ *is defined as the set of all functions* f *defined on* $[0,1]$ *such that* f *and second order derivative function* $f\,''$ *is both locally integrable in* $L_2\,[0,1]$, *equipped with the norm*

$$\parallel f\parallel_{W_2^2[0,1]}=\left(\int\limits_0^1 |f(x)|^2\,dx\right)^{\frac{1}{2}}+\left(\int\limits_0^1 |f_{xx}(x)|^2\,dx\right)^{\frac{1}{2}},$$

and the Sobolev space $W_2^1[0,1]$ *is defined as the set of all functions* f *defined on* $[0,1]$ *such that* f *and first order derivative function* $f\,'$ *is both locally integrable in* $L_2\,[0,1]$, *equipped with the norm*

$$\parallel f\parallel_{W_2^1[0,1]}=\left(\int\limits_0^1 |f(x)|^2\,dx\right)^{\frac{1}{2}}+\left(\int\limits_0^1 |f_{x}(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Proof. Problem (2.13) can be written in abstract form

$$\begin{cases} \frac{d^2u(t)}{dt^2}+\alpha\frac{du(t)}{dt}+Au(t)=f(t)\ (0\le t\le T), \\ \\ u(0)=\varphi, u'(0)=\psi, u(T)=\xi \end{cases} \tag{2.17}$$

in a Hilbert space $L_2[0,l]$ of all square integrable functions defined on $[0,l]$ with self-adjoint positive definite operator $A=A^x$ defined by formula (2.14). Here, $f(t)=f(t,x)$ and $u(t)=u(t,x)$ are known and unknown abstract functions defined on $[0,l]$ with the values in $H=L_2[0,l]$. Therefore, estimates (2.15) and (2.16) follow from estimates of Theorem 2.1.

Second, let $\Omega\subset R^n$ be a bounded open domain with smooth boundary S, $\overline{\Omega}=\Omega\cup S$. In $[0,T]\times\Omega$, we consider the nonlocal boundary value problem for the telegraph equation

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})x_r=p(x)+f(t,x), \\ \\ x=(x_1,...,x_n)\in\Omega, 0<t<T, \\ \\ u(0,x)=\varphi(x), \frac{\partial u(0,x)}{\partial t}=\psi(x), u(T,x)=\xi(x), x\in\overline{\Omega}, \\ \\ u(t,x)=0, x\in S, 0\leq t\leq T, \end{cases} \tag{2.18}$$

where $\alpha_r(x)$, $(x\in\Omega)$, $\varphi(x)$, $\psi(x)$, $\xi(x)$, $\left(x\in\overline{\Omega}\right)$ and $f(t,x), (t\in(0,T))\,, x\in\Omega$ are given smooth functions and $\alpha_r(x)>0, \delta\geq 0$. We introduce the Hilbert spaces $L_2(\overline{\Omega})$ of the all square integrable functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{L_2(\overline{\Omega})}=\left\{\int\cdots\int\limits_{x\in\overline{\Omega}}|f(x)|^2dx_1\cdots dx_n\right\}^{1/2}.$$

Problem (2.18) has a unique smooth solution $(u(t,x),p(x))$ for the smooth functions $\varphi(x)$, $\psi(x)$, $a_r(x)$ and $f(t,x)$. This allows us to reduce the problem (2.18) to the abstract boundary value problem (1.1) in the Hilbert space $H=L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x)=-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r} \tag{2.19}$$

with domain

$$D(A^x)=\left\{u(x):u(x),u_{x_r}(x),(a_r(x)u_{x_r})_{x_r}\in L_2(\overline{\Omega}),1\leq r\leq n,u(x)=0,x\in S\right\}.$$

Theorem 2.4. *Let conditions (2.1) and (2.3) are satisfied. Then, for the solution* $\{u(t,x),p(x))\}$ *of problem (2.18) the stability inequalities*

$$\| u \|_{C(L_2(\overline{\Omega}))}+\| (A^x)^{-1}p \|_{L_2(\overline{\Omega})}$$

$$\leq M(\delta,\alpha)\left[\| \xi \|_{L_2(\overline{\Omega})}+\| \varphi \|_{L_2(\overline{\Omega})}+\| \psi \|_{L_2(\overline{\Omega})}+\max_{0\leq t\leq T}\|f(t)\|_{L_2(\overline{\Omega})}\right],$$

$$\max_{0\leq t\leq T}\|u''\|_{L_2(\overline{\Omega})}+\|u\|_{C(W_2^2(\overline{\Omega}))}+\|p\|_{L_2(\overline{\Omega})}$$

$$\leq M(\delta,\alpha)\Bigg[\| \varphi \|_{W_2^2(\overline{\Omega})}+\| \psi \|_{W_2^1(\overline{\Omega})}+\max_{0\leq t\leq T}\|f'(t)\|_{L_2(\overline{\Omega})}$$

$$+\|\xi\|_{W_2^2(\overline{\Omega})}+\|f(0)\|_{L_2(\overline{\Omega})}\Big]$$

hold, where $M(\delta,\alpha)$ *does not depend on* $\varphi(x)$, $\psi(x)$, $\xi(x)$ *and* $f(t,x)$.*Here and in future, the Sobolev space* $W_2^2(\overline{\Omega})$ *is defined as the set of all functions* f *defined on* $\bar{\Omega}$ *such that* f *and all second order partial derivative functions* f_{x_r,x_r}, $r=1,...n$ *is both locally integrable in* $L_2(\overline{\Omega})$,*equipped with the norm*

$$\| f \|_{W_2^2(\overline{\Omega})}=\| f \|_{L_2(\overline{\Omega})}+\left(\int\cdots\int_{x\in\overline{\Omega}}\sum_{r=1}^{n}|f_{x_r,x_r}|^2\,dx_1\cdots dx_n\right)^{1/2},$$

and the Sobolev space $W_2^1(\overline{\Omega})$ *is defined as the set of all functions* f *defined on* $\bar{\Omega}$ *such that* f *and all first order partial derivative functions* f_{x_r}, $r=1,...n$ *is both locally integrable in* $L_2(\overline{\Omega})$,*equipped with the norm*

$$\| f \|_{W_2^1(\overline{\Omega})}=\| f \|_{L_2(\overline{\Omega})}+\left(\int\cdots\int_{x\in\overline{\Omega}}\sum_{r=1}^{n}|f_{x_r}|^2\,dx_1\cdots dx_n\right)^{1/2}.$$

The proof of Theorem 2.4 is based on Theorem 2.1 and the symmetry properties of the operator A^x defined by formula (2.19) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 2.5. *For the solutions of the elliptic differential problem (Sobolevskii, 1975)*

$$\begin{cases} A^x u(x)=\omega(x), x\in\Omega, \\ u(x)=0, x\in S, \end{cases}$$

the following coercivity inequality holds

$$\sum_{r=1}^{n}\|u_{x_rx_r}\|_{L_2(\overline{\Omega})}\leq M_1||\omega||_{L_2(\overline{\Omega})}.$$

Here M_1 *does not depend on* $\omega(x)$.

Third, in $[0,T]\times\Omega$, the boundary value problem for the multidimensional telegraph equation

$$\begin{cases} u_{tt}(t,x)+\alpha u_t(t,x)-\sum\limits_{r=1}^{n}(a_r(x)u_{x_r})x_r+\delta u=p(x)+f(t,x), \\ x=(x_1,...,x_n)\in\Omega, 0<t<T, \\ u(0,x)=\varphi(x), \frac{\partial u(0,x)}{\partial t}=\psi(x), u(T,x)=\xi(x), x\in\overline{\Omega}, \\ \frac{\partial u(t,x)}{\partial\vec{n}}=0, x\in S, 0\le t\le T \end{cases} \tag{2.20}$$

with the Neumann condition is considered. Here, $\vec{n}$ is the normal vector to S, $a_r(x)\ge a>0$, $(x\in\Omega)$, $\varphi(x)$, $\psi(x),\xi(x)$ $(x\in\overline{\Omega})$, and $f(t,x)$ $(t\in(0,T),\ x\in\Omega)$ are given smooth functions and $\delta>0$. Problem (2.20) has a unique smooth solution $(u(t,x),\ p(x))$ for the smooth functions $\varphi(x)$, $\psi(x)$, $\xi(x), a_r(x)$ and $f(t,x)$. This allows us to reduce the problem (2.20) to the abstract boundary value problem (1.1) in the Hilbert space $H=L_2(\overline{\Omega})$ with a self-adjoint positive definite operator A^x defined by formula

$$A^x u(x)=-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}+\delta u \tag{2.21}$$

with domain

$$D(A^x)=\left\{u(x):u(x),u_{x_r}(x),(a_r(x)u_{x_r})_{x_r}\in L_2(\overline{\Omega}),1\le r\le n,\frac{\partial u\left(x\right)}{\partial\vec{n}}=0,x\in S\right\}.$$

Theorem 2.6. *Let conditions (2.1) and (2.3) are satisfied. Then, for the solution* $\{u(t,x),p(x))\}$ *of problem (2.20), the following stability inequalities*

$$\parallel u\parallel_{C(L_2(\overline{\Omega}))}+\parallel(A^x)^{-1}p\parallel_{L_2(\overline{\Omega})}$$

$$\le M(\delta,\alpha)\left[\parallel\xi\parallel_{L_2(\overline{\Omega})}+\parallel\varphi\parallel_{L_2(\overline{\Omega})}+\parallel\psi\parallel_{L_2(\overline{\Omega})}+\max_{0\le t\le T}\|f(t)\|_{L_2(\overline{\Omega})}\right],$$

$$\max_{0\le t\le T}\|u''\|_{L_2(\overline{\Omega})}+\|u\|_{C(W_2^2(\overline{\Omega}))}+\|p\|_{L_2(\overline{\Omega})}$$

$$\le M(\delta,\alpha)\left[\parallel\varphi\parallel_{W_2^2(\overline{\Omega})}+\parallel\psi\parallel_{W_2^1(\overline{\Omega})}+\max_{0\le t\le T}\|f'(t)\|_{L_2(\overline{\Omega})}\right.$$

$$\left.+\parallel\xi\parallel_{W_2^2(\overline{\Omega})}+\|f(0)\|_{L_2(\overline{\Omega})}\right],$$

hold, where $M(\delta,\alpha)$ *does not depend on* $\varphi(x)$, $\psi(x),\xi(x)$ *and* $f(t,x)$.

The proof of Theorem 2.6 is based on Theorem 2.1 and the symmetry properties of the operator A^x defined by formula (2.20) and the following theorem on the coercivity inequality for the solution of the elliptic differential problem in $L_2(\overline{\Omega})$.

Theorem 2.7. *For the solutions of the elliptic differential problem*

$$\begin{cases} A^x u(x) = \omega(x), x \in \Omega, \\ \frac{\partial u(x)}{\partial \vec{n}} = 0, x \in S, \end{cases}$$

the following coercivity inequality holds (Sobolevskii, 1975)

$$\sum_{r=1}^{n} \|u_{x_r x_r}\|_{L_2(\overline{\Omega})} \leq M_1(\delta) \|\omega\|_{L_2(\overline{\Omega})}.$$

Here $M_1(\delta)$ *is independent of* $\omega(x)$.

In the next chapter, the first and second order of accuracy difference schemes for the approximate solution of problem (1.1) are studied. Stability estimates for the solution of these difference schemes are established. In applications, difference schemes for the approximate solution of three boundary value problems (2.13), (2.18) and (2.20) are presented. Stability estimates for the solution of these difference schemes are established.

CHAPTER 3

STABLE TWO-STEP DIFFERENCE SCHEMES FOR TELEGRAPH EQUATIONS WITH AN UNKNOWN PARAMETER

We consider the stable two-step first order of accuracy difference scheme for approximately solving boundary value problem (1.1)

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_k}{\tau}+Au_{k+1}=p+f_k, \\ \\ f_k=f(t_{k+1}), 1\leq k\leq N-1, N\tau=T, \\ \\ u_0=\varphi, \frac{u_1-u_0}{\tau}+\tau Bu_1=\frac{1}{1+\frac{\alpha\tau}{2}}\psi, u_N=\xi. \end{cases} \tag{3.1}$$

We are interested to study the stability of solutions of the difference scheme (3.1) under the assumption (2.3). We have not been able to obtain the discrete analogue of estimates of Theorem 2.1 under the assumption (2.3) for the solution of the difference scheme (3.1). Nevertheless, we can established the discrete analogue of estimates of Theorem 2.1 under the more strong assumption than (2.3).

It is clear that

$$u_k=v_k+A^{-1}p, \tag{3.2}$$

$$p=A\left(\xi-v_N\right), \tag{3.3}$$

where v_k is the solution of the following difference scheme for approximately solving boundary value problem (1.3)

$$\begin{cases} \frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}+\alpha\frac{v_{k+1}-v_k}{\tau}+Av_{k+1}=f_k, \\ \\ f_k=f(t_{k+1}), 1\le k\le N-1, N\tau=T, \\ \\ v_N=v_0+\xi-\varphi, \frac{v_1-v_0}{\tau}+\tau B\left(v_1+\varphi-v_0\right)=\frac{1}{1+\frac{\alpha\tau}{2}}\psi. \end{cases} \tag{3.4}$$

Now, let us give some lemmas that will be needed below.

Lemma 3.1. *The estimates hold:*

$$\|R\|_{H\to H}\le\frac{1}{1+\frac{\alpha\tau}{2}}, \left\|\widetilde{R}\right\|_{H\to H}\le\frac{1}{1+\frac{\alpha\tau}{2}}, \left\|\tau B^{\frac{1}{2}}R\right\|_{H\to H}\le 1, \left\|\tau B^{\frac{1}{2}}\widetilde{R}\right\|_{H\to H}\le 1. \tag{3.5}$$

Here

$$R=\left((1+\frac{\alpha\tau}{2})I-i\tau B^{\frac{1}{2}}\right)^{-1}, \widetilde{R}=\left((1+\frac{\alpha\tau}{2})I+i\tau B^{\frac{1}{2}}\right)^{-1}.$$

Proof. Applying the spectral representation of self-adjoint positive definite operator, we get

$$\|R\|_{H\to H}\le\sup_{\delta\le\mu<\infty}\frac{1}{\left|1+\frac{\alpha\tau}{2}-i\mu\right|}=\sup_{\delta\le\mu<\infty}\frac{1}{\sqrt{\left(1+\frac{\alpha\tau}{2}\right)^2+\mu^2}}\le\frac{1}{1+\frac{\alpha\tau}{2}}.$$

By the similar manner, it is easy to see that the estimates in (3.5) hold.

Lemma 3.2. *The operator*

$$\left[I-\frac{1}{2}[R^{N-1}+\widetilde{R}^{N-1}]\right]$$

has inverse

$$P_\tau=\left\{I-\frac{1}{2}[R^{N-1}+\widetilde{R}^{N-1}]\right\}^{-1}$$

and the following estimate

$$||P_\tau||_{H\to H}\le M \tag{3.6}$$

holds, where $M=M(\delta,\alpha)>0$.

Proof. The proof of estimate (3.6) is based on the estimate

$$\left\| \frac{1}{2}[R^{N-1} + \widetilde{R}^{N-1}] \right\|_{H \to H} \le \frac{1}{\left(1 + \frac{\alpha\tau}{2}\right)^{N-1}}.$$

Using the triangly inequality and the estimate (3.5), we obtain

$$\left\| \frac{1}{2}[R^{N-1} + \widetilde{R}^{N-1}] \right\|_{H \to H} \le \frac{1}{2} \left\| R^{N-1} \right\|_{H \to H} + \frac{1}{2} \left\| \widetilde{R}^{N-1} \right\|_{H \to H} \le \frac{1}{\left(1 + \frac{\alpha\tau}{2}\right)^{N-1}}.$$

Lemma 3.2 is proved.

In this chapter, the solvability of problem (3.1) in the space $C_\tau(H)$ of the H-valued mesh functions $\varphi^\tau = \{\varphi_k\}_0^N$ defined on grid space

$$[0,T]_\tau = \{t_k = k\tau, 0 \le k \le N, N\tau = T\},$$

equipped with the norm

$$\|u\|_{C_\tau(H)} = \max_{0 \le k \le N} ||\varphi_k||_H$$

is investigated. We will prove the following main theorem on continuously dependents of the solution on the given data.

Theorem 3.1. *Suppose that $\varphi, \xi \in D(A), \psi \in D(A^{\frac{1}{2}})$ and (2.1) is satisfied. Let*

$$1 > \frac{1}{\left(\left(1 + \frac{\alpha\tau}{2}\right)^2 + \tau^2\left(\delta - \frac{\alpha^2}{4}\right)\right)^{\frac{N}{2}}}. \tag{3.7}$$

Then, for the solution $\left\{\{u_k\}_0^N, p\right\}$ of problem (3.1) in $C_\tau(H) \times H$ the following stability inequalities

$$\max_{0 \le k \le N} \|u_k\|_H + \| A^{-1}p \|_H$$
$$\le M(\delta,\alpha)\left[\|\varphi\|_H + \|\xi\|_H + \left\|A^{-\frac{1}{2}}\psi\right\|_H + \max_{1 \le k \le N-1} \left\|A^{-\frac{1}{2}} f_k\right\|_H\right],$$
$$\max_{1 \le k \le N-1} \left\| \frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} \right\|_H + \max_{0 \le k \le N} \|Au_k\|_H + \|p\|_H$$
$$\le M(\delta,\alpha)\left[\|A\varphi\|_H + \|\varphi\|_H + \left\|A^{\frac{1}{2}}\psi\right\|_H + \|A\xi\|_H + \|\xi\|_H \right.$$
$$\left. + \sum_{s=2}^{N-1} \|f_s - f_{s-1}\|_H + \|f_1\|_H\right]$$

hold, where $M(\delta,\alpha)$ is independent of f_k and φ, ψ, ξ.

Proof of Theorem 3.1 is based on formulas (3.2) and (3.3) and the following theorem on well-posedness of nonlocal boundary value problem (1.3).

Theorem 3.2. *Suppose that the assumptions of Theorem 3.1 hold. Then, for the solution* $\{v_k\}_0^N$ *of problem (3.4) in* $C_\tau(H)$ *the stability estimates*

$$\max_{0\le k\le N}\|v_k\|_H$$

$$\le M(\delta,\alpha)\left[\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\|\xi\|_H+\max_{1\le k\le N-1}\left\|A^{-\frac{1}{2}}f_k\right\|_H\right], \tag{3.8}$$

$$\max_{1\le k\le N-1}\left\|\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}\right\|_H+\max_{0\le k\le N}\|Av_k\|_H \tag{3.9}$$

$$\le M(\delta,\alpha)\left[\|A\varphi\|_H+\|\varphi\|_H+\left\|A^{\frac{1}{2}}\psi\right\|_H+\|A\xi\|_H+\|\xi\|_H\right.$$

$$\left.+\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H\right]$$

hold, where $M(\delta,\alpha)$ *does not depend on* f_k *and* φ,ψ,ξ.

Proof. First, we will obtain the formula for the solution of problem (3.4). We can rewrite (3.4) into the following difference problem

$$\begin{cases} v_{k-1}-(2+\alpha\tau)v_k+\left((1+\alpha\tau)I+\tau^2\left(B+\frac{\alpha^2}{4}I\right)\right)v_{k+1} \\ =\tau^2 f_k, 1\le k\le N-1, \\ v_N=v_0+\xi-\varphi, \frac{v_1-v_0}{\tau}+\tau B(v_1+\varphi-v_0)=\frac{1}{1+\frac{\alpha\tau}{2}}\psi. \end{cases} \tag{3.10}$$

It is clear that there exist a unique solution of this initial value problem

$$\begin{cases} v_{k-1}-(2+\alpha\tau)v_k+\left((1+\alpha\tau)I+\tau^2\left(B+\frac{\alpha^2}{4}I\right)\right)v_{k+1} \\ =\tau^2 f_k, 1\le k\le N-1, \\ v_0, v_1 \text{ are given} \end{cases}$$

the following formula holds (see (Ashyralyev and Sobolevskii, 2004))

$$v_k=R\widetilde{R}(\widetilde{R}-R)^{-1}[R^{k-1}-\widetilde{R}^{k-1}]v_0$$

$$+(\widetilde{R}-R)^{-1}(\widetilde{R}^k-R^k)\left[v_0-\tau^2BR\widetilde{R}\varphi+R\widetilde{R}\frac{\tau}{1+\frac{\alpha\tau}{2}}\psi\right]$$

$$+\sum_{s=1}^{k-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau^2f_s, 2\leq k\leq N. \tag{3.11}$$

Applying formula (3.11) and conditiion

$$v_N=v_0+\xi-\varphi,$$

we can obtain the formula for the solution of (3.10). Actually, we have that

$$R\widetilde{R}(\widetilde{R}-R)^{-1}[R^{N-1}-\widetilde{R}^{N-1}]v_0$$

$$+(\widetilde{R}-R)^{-1}(\widetilde{R}^N-R^N)\left[v_0-\tau^2BR\widetilde{R}\varphi+R\widetilde{R}\frac{\tau}{1+\frac{\alpha\tau}{2}}\psi\right]$$

$$+\sum_{s=1}^{N-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{N-s}-R^{N-s}\right]\tau^2f_s=v_0+\xi-\varphi.$$

From that it follows that

$$\left\{I-\frac{1}{2}\left[R^{N-1}+\widetilde{R}^{N-1}\right]\right\}v_0$$

$$=(2iB^{\frac{1}{2}})^{-1}(\widetilde{R}^N-R^N)\left[-\tau B\varphi+\frac{1}{1+\frac{\alpha\tau}{2}}\psi\right] \tag{3.12}$$

$$+\sum_{s=1}^{N-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{N-s}-R^{N-s}\right]\tau^2f_s-\xi+\varphi.$$

By Lemma 3.2, under assumption (2.3), there exists of inverse

$$P_\tau=\left\{I-\frac{1}{2}\left[R^{N-1}+\widetilde{R}^{N-1}\right]\right\}^{-1}.$$

Therefore, using (3.12), we obtain

$$v_0=P_\tau\left\{(2iB^{\frac{1}{2}})^{-1}(\widetilde{R}^N-R^N)\left[-\tau B\varphi+\frac{1}{1+\frac{\alpha\tau}{2}}\psi\right]\right. \tag{3.13}$$

$$\left.+\sum_{s=1}^{N-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{N-s}-R^{N-s}\right]\tau^2f_s-\xi+\varphi\right\}.$$

Consequently, the solution of problem (3.4) satisfy formulas (3.13), and

$$v_1=v_0-\tau^2BR\widetilde{R}\varphi+R\widetilde{R}\frac{\tau}{1+\frac{\alpha\tau}{2}}\psi \tag{3.14}$$

and also (3.11). Second, we obtain estimate (3.8). Using the triangle inequality, formulas (3.11), (3.13), (3.14), and estimates (3.5), we obtain

$$\|v_0\|_H \le \|P_\tau\|_{H\to H}\left\{\frac{1}{2}\left[\left\|\tau B^{\frac{1}{2}}\widetilde{R}^N\right\|_{H\to H}+\left\|\tau B^{\frac{1}{2}}R^N\right\|_{H\to H}\right]\|\varphi\|_H\right.$$

$$+\frac{1}{1+\frac{\alpha\tau}{2}}\frac{1}{2}\left[\left\|\widetilde{R}^N\right\|_{H\to H}+\|R^N\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H$$

$$+\sum_{s=1}^{N-1}\frac{1}{2}\left[\left\|\widetilde{R}^{N-s}\right\|_{H\to H}+\|R^{N-s}\|_{H\to H}\right]\tau\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H$$

$$+\|\xi\|_H+\|\varphi\|_H\}$$

$$\le M(\delta,\alpha)\left[\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\|\xi\|_H+\max_{1\le k\le N-1}\left\|A^{-\frac{1}{2}}f_k\right\|_H\right],$$

$$\|v_1\|_H\le\|v_0\|_H+\left\|\tau^2BR\widetilde{R}\right\|_{H\to H}\|\varphi\|_H$$

$$+\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\frac{1}{1+\frac{\alpha\tau}{2}}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H$$

$$\le M_1(\delta,\alpha)\left[\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\|\xi\|_H+\max_{1\le k\le N-1}\left\|A^{-\frac{1}{2}}f_k\right\|_H\right],$$

$$\|v_k\|_H\le\frac{1}{2}\left[\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\|R^{k-1}\|_{H\to H}\right]\|v_0\|_H$$

$$+\frac{1}{2}\left[\left\|\widetilde{R}^{k}\right\|_{H\to H}+\|R^{k}\|_{H\to H}\right]\left[\tau\left\|B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\|\varphi\|_H\right.$$

$$\left.+\frac{1}{1+\frac{\alpha\tau}{2}}\left\|R\widetilde{R}\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}\psi\right\|_H\right]$$

$$+\sum_{s=1}^{k-1}\frac{1}{2}\left[\left\|\widetilde{R}^{k-s}\right\|_{H\to H}+\|R^{k-s}\|_{H\to H}\right]\tau\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{-\frac{1}{2}}f_s\right\|_H$$

$$\le M_3(\delta,\alpha)\left[\|\varphi\|_H+\left\|A^{-\frac{1}{2}}\psi\right\|_H+\|\xi\|_H+\max_{1\le k\le N-1}\left\|A^{-\frac{1}{2}}f_k\right\|_H\right],2\le k\le N.$$

Estimate (3.8) is proved. Third, we obtain establish estimate (3.9). Using the Abel's formula, we can write

$$\sum_{s=1}^{k-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau^2f_s$$

$$=\tau^2R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left\{(I-\widetilde{R})^{-1}\left[\widetilde{R}f_{k-1}-\widetilde{R}^kf_1+\sum_{s=1}^{k-2}\widetilde{R}^{k-s}\left(f_s-f_{s+1}\right)\right]\right.$$

$$\left.-(I-R)^{-1}\left[Rf_{k-1}-R^kf_1+\sum_{s=1}^{k-2}R^{k-s}\left(f_s-f_{s+1}\right)\right]\right\}.$$

Since

$$I-\widetilde{R}=\tau\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\widetilde{R}, I-R=\tau\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)R, \widetilde{R}-R=\left(-2i\tau B^{\frac{1}{2}}\right)\widetilde{R}R,$$

$$(I-\widetilde{R})(I-R)=\widetilde{R}R\tau^{2}\left(\frac{\alpha^{2}}{4}I+B\right)=A\tau^{2}\widetilde{R}R,$$

$$\widetilde{R}(I-\widetilde{R})^{-1}-R(I-R)^{-1}=\left(\widetilde{R}-R\right)(I-\widetilde{R})^{-1}(I-R)^{-1}=\left(\widetilde{R}-R\right)\left(A\tau^{2}\widetilde{R}R\right)^{-1},$$

$$\widetilde{R}^{k}(I-\widetilde{R})^{-1}-R^{k}(I-R)^{-1}=\left(\widetilde{R}^{k}(I-R)-R^{k}(I-\widetilde{R})\right)(I-\widetilde{R})^{-1}(I-R)^{-1}$$

$$=\left\{\tau\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k}-\tau\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k}\right\}\left(A\tau^{2}\widetilde{R}R\right)^{-1},$$

we have that

$$\sum_{s=1}^{k-1}R\widetilde{R}\left(\widetilde{R}-R\right)^{-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right]\tau^{2}f_{s}$$

$$=A^{-1}\left\{f_{k-1}-\left(\left(-2i\tau B^{\frac{1}{2}}\right)\widetilde{R}R\right)^{-1}\left(\widetilde{R}-R\right)^{-1}\left\{\left[\widetilde{R}^{k}-R^{k}\right]-\widetilde{R}R\left[\widetilde{R}^{k-1}-R^{k-1}\right]\right\}f_{1}\right.$$

$$\left.+\left(\left(-2i\tau B^{\frac{1}{2}}\right)\widetilde{R}R\right)^{-1}\sum_{s=1}^{k-2}\left\{\tau\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)R\widetilde{R}^{k-s}-\tau\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)\widetilde{R}R^{k-s}\right\}(f_{s}-f_{s+1})\right\}$$

$$=A^{-1}\left\{f_{k-1}-\left(-2iB^{\frac{1}{2}}\right)^{-1}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-1}\right\}f_{1}\right.$$

$$\left.+\left(-2iB^{\frac{1}{2}}\right)^{-1}\sum_{s=1}^{k-2}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-s-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-s-1}\right\}(f_{s}-f_{s+1})\right\}.$$

Using this formula and applying A to the formulas (3.11), (3.13), we can write

$$v_{0}=P_{\tau}\left\{A(2iB^{\frac{1}{2}})^{-1}(\widetilde{R}^{N}-R^{N})\left[-\tau B\varphi+\frac{1}{1+\frac{\alpha\tau}{2}}\psi\right]-A\xi+A\varphi\right. \tag{3.15}$$

$$+f_{N-1}-\left(-2iB^{\frac{1}{2}}\right)^{-1}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{N-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{N-1}\right\}f_{1}$$

$$\left.+\left(-2iB^{\frac{1}{2}}\right)^{-1}\sum_{s=1}^{N-2}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{N-s-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{N-s-1}\right\}(f_{s}-f_{s+1})\right\},$$

$$Av_{k}=AR\widetilde{R}(\widetilde{R}-R)^{-1}[R^{k-1}-\widetilde{R}^{k-1}]v_{0} \tag{3.16}$$

$$+A(\widetilde{R}-R)^{-1}(\widetilde{R}^{k}-R^{k})\left[v_{0}-\tau^{2}BR\widetilde{R}\varphi+R\widetilde{R}\frac{\tau}{1+\frac{\alpha\tau}{2}}\psi\right]$$

$$+f_{k-1}-\left(-2iB^{\frac{1}{2}}\right)^{-1}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-1}\right\}f_{1}+\left(-2iB^{\frac{1}{2}}\right)^{-1}$$

$$\times\sum_{s=1}^{k-2}\left\{\left(\frac{\alpha}{2}-iB^{\frac{1}{2}}\right)\widetilde{R}^{k-s-1}-\left(\frac{\alpha}{2}+iB^{\frac{1}{2}}\right)R^{k-s-1}\right\}(f_{s}-f_{s+1}), 2\le k\le N.$$

Using the triangle inequality, formulas (3.15), (3.16) and estimates (3.5), we obtain

$$\|Av_0\|_H \le \|P_\tau\|_{H\to H}\left\{\frac{1}{2}\left[\left\|\tau B^{\frac{1}{2}}\widetilde{R}^N\right\|_{H\to H}+\left\|\tau B^{\frac{1}{2}}R^N\right\|_{H\to H}\right]\|A\varphi\|_H\right.$$

$$+\frac{1}{1+\frac{\alpha\tau}{2}}\frac{1}{2}\left[\left\|\widetilde{R}^N\right\|_{H\to H}+\left\|R^N\right\|_{H\to H}\right]\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H$$

$$+\|f_{N-1}\|_H+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\left\{\left\|\widetilde{R}^{N-1}\right\|_{H\to H}+\left\|R^{N-1}\right\|_{H\to H}\right\}\|f_1\|_H$$

$$\left.+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\sum_{s=1}^{N-2}\left\{\left\|\widetilde{R}^{N-s-1}\right\|_{H\to H}+\left\|R^{N-s-1}\right\|_{H\to H}\right\}\|f_s-f_{s+1}\|_H\right\}$$

$$+\|A\xi\|_H+\|A\varphi\|_H\}$$

$$\le M_4(\delta,\alpha)\left[\|A\varphi\|_H+\|\varphi\|_H+\left\|A^{\frac{1}{2}}\psi\right\|_H+\|A\xi\|_H+\|\xi\|_H\right.$$

$$\left.+\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H\right],$$

$$\|Av_1\|_H \le \|Av_0\|_H+\left\|\tau^2 BR\widetilde{R}\right\|_{H\to H}\|A\varphi\|_H$$

$$+\left\|\tau B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\frac{1}{1+\frac{\alpha\tau}{2}}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H$$

$$\le M_5(\delta,\alpha)\left[\|A\varphi\|_H+\|\varphi\|_H+\left\|A^{\frac{1}{2}}\psi\right\|_H+\|A\xi\|_H+\|\xi\|_H\right.$$

$$\left.+\sum_{s=2}^{N-1}\|f_s-f_{s-1}\|_H+\|f_1\|_H\right],$$

$$\|Av_k\|_H \le \frac{1}{2}\left[\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|R^{k-1}\right\|_{H\to H}\right]\|Av_0\|_H$$

$$+\frac{1}{2}\left[\left\|\widetilde{R}^{k}\right\|_{H\to H}+\left\|R^{k}\right\|_{H\to H}\right]\left[\tau\left\|B^{\frac{1}{2}}R\widetilde{R}\right\|_{H\to H}\|A\varphi\|_H\right.$$

$$\left.+\frac{1}{1+\frac{\alpha\tau}{2}}\left\|R\widetilde{R}\right\|_{H\to H}\left\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\right\|_{H\to H}\left\|A^{\frac{1}{2}}\psi\right\|_H\right]$$

$$+\|f_{k-1}\|_H+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\left\{\left\|\widetilde{R}^{k-1}\right\|_{H\to H}+\left\|R^{k-1}\right\|_{H\to H}\right\}\|f_1\|_H$$

$$\left.+\frac{1}{2}\left[\frac{\alpha}{2}\left\|B^{-\frac{1}{2}}\right\|_{H\to H}+1\right]\sum_{s=1}^{k-2}\left\{\left\|\widetilde{R}^{k-s-1}\right\|_{H\to H}+\left\|R^{k-s-1}\right\|_{H\to H}\right\}\|f_s-f_{s+1}\|_H\right\}$$

$$\le M_5(\delta,\alpha)\left[\|A\varphi\|_H+\|\varphi\|_H+\left\|A^{\frac{1}{2}}\psi\right\|_H+\|A\xi\|_H+\|\xi\|_H\right.$$

$$+\sum_{s=2}^{N-1}\left\|f_s-f_{s-1}\right\|_H+\left\|f_1\right\|_H\Bigg],2\leq k\leq N.$$

Theorem 3.2 is proved.

Now, we consider the second order of accuracy difference schemes for approximately solving boundary value problem (1.1)

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{A}{2}(u_{k+1}+u_{k-1})\\ \\ =p+f_k,f_k=f(t_k),1\leq k\leq N-1,N\tau=T,\\ \\ u_0=\varphi,u_N=\xi,\\ \\ \frac{u_1-u_0}{\tau}+\frac{\tau}{4}Bu_1+\frac{1}{1+\frac{\alpha}{4}\tau}\left(\frac{1}{4}B-\frac{\alpha\tau B}{16}+\frac{\alpha^2}{8}I\right)\tau u_0\\ \\ =\frac{1-\frac{\alpha}{4}\tau}{1+\frac{\alpha}{4}\tau}\left(\psi+\frac{\tau}{2}f_0\right),f_0=f(0),\end{cases}\tag{3.17}$$

$$\begin{cases}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+\alpha\frac{u_{k+1}-u_{k-1}}{2\tau}+\frac{A}{2}u_k+\frac{A}{4}(u_{k+1}+u_{k-1})\\ \\ =p+f_k,f_k=f(t_k),1\leq k\leq N-1,N\tau=T,\\ \\ u_0=\varphi,u_N=\xi,\\ \\ \frac{u_1-u_0}{\tau}+\frac{\tau}{4}Bu_1+\frac{1}{1+\frac{\alpha}{4}\tau}\left(\frac{1}{4}B-\frac{\alpha\tau B}{16}+\frac{\alpha^2}{8}I\right)\tau u_0\\ \\ =\frac{1-\frac{\alpha}{4}\tau}{1+\frac{\alpha}{4}\tau}\left(\psi+\frac{\tau}{2}f_0\right),f_0=f(0).\end{cases}\tag{3.18}$$

Note that applying operator approach we can establish stability estimates for the solution of difference schemes (3.17) and (3.18).

Now, we consider applications of Theorem 3.1. First, we consider the nonlocal boundary value problem (2.13). The discretization of problem (2.13) is carried out in two steps. In the first step, we consider the discretization in x. To the differential operator A^x defined by the formula (2.14), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x)=\{-(a(x)\varphi_{\bar{x}})_{x,n}+\delta\varphi_n\}_1^{M-1} \tag{3.19}$$

acting in the space of grid functions $\varphi^h(x)=\{\varphi_n\}_0^M$ satisfying the conditions $\varphi_0=\varphi_M$, $\varphi_1-\varphi_0=\varphi_M-\varphi_{M-1}$. It is well-known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we reach the boundary value problem

$$\begin{cases} u_{tt}^h(t,x)+\alpha u_t^h(t,x)+A_h^x u^h(t,x)=p^h(x)+f^h(t,x), \\ \\ 0<t<T, x\in[0,l]_h, \\ \\ u^h(0,x)=\varphi^h(x), u_t^h(0,x)=\psi^h(x), x\in[0,l]_h, \\ \\ u^h(T,x)=\xi^h(x), x\in[0,l]_h. \end{cases} \tag{3.20}$$

In the second step, we replace (3.20) with difference scheme (3.1)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\alpha\frac{u_{k+1}^h(x)-u_k^h(x)}{\tau}+A_h^x u_{k+1}^h(x)=p^h(x)+f_k^h(x), \\ \\ f_k^h(x)=f^h(t_{k+1},x),\ t_k=k\tau,\ 1\le k\le N-1,\ x\in[0,l]_h, N\tau=T, \\ \\ u_0^h(x)=\varphi^h(x), u_N^h(x)=\xi^h(x) \\ \\ \frac{u_1^h(x)-u_0^h(x)}{\tau}+\left(A_h^x-\frac{\alpha^2}{4}I_h\right)\tau u_1^h(x)=\frac{1}{1+\frac{\alpha}{2}\tau}\psi^h(x), x\in[0,l]_h. \end{cases} \tag{3.21}$$

Theorem 3.3. *Suppose that (2.1) and (3.7) are satisfied. Then, for the solution* $\left\{\left\{u_k^h\left(x\right)\right\}_0^N,p^h(x)\right\}$ *of problem* (3.21) *the following stability estimates*

$$\max_{1\le k\le N}\|u_k^h\|_{L_{2h}}+\|\left(A_h^x\right)^{-1}p^h\|_{L_{2h}}$$

$$\le M_1(\delta,\alpha)\left\{\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\psi^h\|_{L_{2h}}+\|\xi^h\|_{L_{2h}}+\|\varphi^h\|_{L_{2h}}\right\},$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^2}\le M_2(\delta,\alpha)\left\{\max_{2\le k\le N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.$$

$$\left.+\|f_1^h\|_{L_{2h}}+\|\psi^h\|_{W_{2h}^1}+\|\varphi^h\|_{W_{2h}^2}+\|\xi^h\|_{W_{2h}^2}\right\}$$

hold, where $M_1(\delta,\alpha)$ *and* $M_2(\delta,\alpha)$ *do not depend on* $\varphi^h(x)$, $\psi^h(x)$, $\xi^h(x)$ *and* $f_k^h(x)$, $1\le k\le N-1$.

Proof. Difference scheme (3.21) can be written in abstract form

$$\begin{cases} \frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}+\alpha\frac{u_{k+1}^h-u_k^h}{\tau}+A_h u_{k+1}^h=p^h+f_k^h, \\ \\ 1\le k\le N-1, N\tau=T, \\ \\ u_0^h=\varphi^h, u_N^h=\xi^h, \frac{u_1^h-u_0^h}{\tau}+\left(A_h-\frac{\alpha^2}{4}I_h\right)u_1^h=\frac{1}{1+\frac{\alpha}{2}\tau}\psi^h \end{cases} \tag{3.22}$$

in a Hilbert space L_{2h} with self-adjoint positive definite operator $A_h=A_h^x$ by formula (3.19).

Here, $f_k^h=f_k^h(x)$ and $u_k^h=u_k^h(x)$ are known and unknown abstract mesh functions defined on $[0,l]_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem3.3 follow from estimates of Theorem3.1. Thus, Theorem3.3 is proved.

Second, we consider boundary value problem (2.18). The discretization of problem (2.18) is carried out in two steps. In the first step, we define the grid space

$$\overline{\Omega}_h=\{x=x_r=(h_1j_1,\cdots,h_nj_n), j=(j_1,\cdots,j_n), 0\le j_r\le N_r,$$
$$N_rh_r=1, r=1,\cdots,n\}, \Omega_h=\overline{\Omega}_h\cap\Omega, S_h=\overline{\Omega}_h\cap S$$

and introduce the Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ of the grid functions $\varphi^h(x)=\{\varphi(h_1j_1,\cdots,h_nj_n)\}$ defined on $\overline{\Omega}_h$ equipped with the norm

$$\|\varphi^h\|_{L_{2h}}=\left(\sum_{x\in\overline{\Omega}_h}\left|\varphi^h(x)\right|^2h_1\cdots h_n\right)^{\frac{1}{2}}.$$

To the differential operator A^x defined by the formula (2.19), we assign the difference operator A_h^x by the formula

$$A_h^xu^h=-\sum_{r=1}^n\left(\alpha_r(x)u_{x_r}^h\right)_{x_r,j_r}, \tag{3.23}$$

where A_h^x is known as self-adjoint positive definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $u^h(x)=0$ for all $x\in S_h$. With the help of the difference operator A_h^x, we arrive at the following boundary value problem

$$\begin{cases} u_{tt}^h(t,x)+\alpha u_t^h(t,x)+A_h^x u^h(t,x)=p^h\left(x\right)+f^h(t,x), \\ \\ 0<t<T, x\in\Omega_h, \\ \\ u^h(0,x)=\varphi^h(x), u^h(T,x)=\xi^h(x), u_t^h(0,x)=\psi^h(x), x\in\Omega_h. \end{cases} \tag{3.24}$$

In the second step, we replace (3.24) with the difference scheme (3.1)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\alpha\frac{u_{k+1}^h(x)-u_k^h(x)}{\tau}+A_h^x u_{k+1}^h(x)=p^h\left(x\right)+f_k^h(x), \\ \\ f_k^h(x)=f^h(t_{k+1},x),\ t_k=k\tau,\ 1\le k\le N-1,\ x\in\Omega_h, N\tau=T, \\ \\ u_0^h(x)=\varphi^h(x), u_N^h(x)=\xi^h(x), \\ \\ \frac{u_1^h(x)-u_0^h(x)}{\tau}+\left(A_h^x-\frac{\alpha^2}{4}I_h\right)\tau u_1^h(x)=\frac{1}{1+\frac{\alpha}{2}\tau}\psi^h(x), x\in\Omega_h \end{cases} \tag{3.25}$$

for an infinite system of ordinary differential equations.

Theorem 3.4. *Suppose that (2.1) and (3.7) are satisfied. Then, for the solution* $\left\{\left\{u_k^h\left(x\right)\right\}_0^N, p^h(x)\right\}$ *of problem* (3.25) *the following stability estimates*

$$\max_{1\le k\le N}\|u_k^h\|_{L_{2h}}+\|\left(A_h^x\right)^{-1}p^h\|_{L_{2h}}$$

$$\le M_1(\delta,\alpha)\left\{\max_{1\le k\le N-1}\|f_k^h\|_{L_{2h}}+\|\ \psi^h\ \|_{L_{2h}}+\|\ \xi^h\ \|_{L_{2h}}+\|\ \varphi^h\ \|_{L_{2h}}\right\},$$

$$\max_{1\le k\le N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{1\le k\le N}\|u_k^h\|_{W_{2h}^2}\le M_2(\delta,\alpha)\left\{\max_{2\le k\le N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.$$

$$\left.+\|\ f_1^h\ \|_{L_{2h}}+\|\ \psi^h\ \|_{W_{2h}^1}+\|\ \xi^h\ \|_{W_{2h}^2}+\|\ \varphi^h\ \|_{W_{2h}^2}\right\}$$

hold, where $M_1(\delta,\alpha)$ *and* $M_2(\delta,\alpha)$ *do not depend on* $\varphi^h(x)$, $\psi^h(x)$, $\xi^h(x)$ *and* $f_k^h(x)$, $1\le k\le N-1$.

Proof. Difference scheme (3.25) can be written in abstract form (3.22) in a Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h=A_h^x$ by formula (3.23).

Here, $f_k^h = f_k^h(x)$ and $u_k^h = u_k^h(x)$ are known and unknown abstract mesh functions defined on $\overline{\Omega}_h$ with the values in $H = L_{2h}$. Therefore, estimates of Theorem3.4 follow from estimates of Theorem3.1 and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3.5. *For the solutions of the elliptic difference problem (Sobolevskii, 1975)*

$$\begin{cases} A_h^x u^h(x) = \omega^h(x), \ x \in \Omega_h, \\ \\ u^h(x) = 0, \ x \in S_h, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n} \left\| {u^h}_{x_r x_{\overline{r}}} \right\|_{L_{2h}} \le M_3 ||\omega^h||_{L_{2h}},$$

where M_3 does not depend on h and ω^h.

Third, we consider the boundary value problem (2.20). The discretization of problem (2.20) is carried out in two steps.

To the differential operator A^x defined by the formula (2.21), we assign the difference operator A_h^x by the formula

$$A_h^x u^h = -\sum_{r=1}^{n} \left(\alpha_r(x) u_{x_r}^h \right)_{x_r, j_r} + \delta u^h, \tag{3.26}$$

where A_h^x is known as self-adjoint positive definite operator in L_{2h}, acting in the space of grid functions $u^h(x)$ satisfying the conditions $D^h u^h(x) = 0$ for all $x \in S_h$, where $D^h u^h(x)$ is the second order of approximation of $\frac{\partial u(x)}{\partial \vec{n}}$.With the help of the difference operator A_h^x, we arrive at the following boundary value problem

$$\begin{cases} u_{tt}^h(t,x) + \alpha u_t^h(t,x) + A_h^x u^h(t,x) = p^h(x) + f^h(t,x), \\ \\ 0 < t < T, x \in \Omega_h, \\ \\ u^h(0,x) = \varphi^h(x), u^h(T,x) = \xi^h(x), u_t^h(0,x) = \psi^h(x), x \in \Omega_h. \end{cases} \tag{3.27}$$

In the second step, we replace (3.27) with the difference scheme (3.1)

$$\begin{cases} \frac{u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)}{\tau^2}+\alpha\frac{u_{k+1}^h(x)-u_k^h(x)}{\tau}+A_h^x u_{k+1}^h(x)=p^h\left(x\right)+f_k^h(x), \\ \\ f_k^h(x)=f^h(t_{k+1},x),\ t_k=k\tau,\ 1\leq k\leq N-1,\ x\in\Omega_h, N\tau=T, \\ \\ u_0^h(x)=\varphi^h(x), u_N^h(x)=\xi^h(x), \\ \\ \frac{u_1^h(x)-u_0^h(x)}{\tau}+\left(A_h^x-\frac{\alpha^2}{4}I_h\right)\tau u_1^h(x)=\frac{1}{1+\frac{\alpha}{2}\tau}\psi^h(x), x\in\Omega_h \end{cases} \tag{3.28}$$

for an infinite system of ordinary differential equations.

Theorem 3.6. *Suppose that (2.1) and (3.7) are satisfied. Then, for the solution* $\left\{\left\{u_k^h\left(x\right)\right\}_0^N, p^h(x)\right\}$ *of problem* (3.28) *the following stability estimates*

$$\max_{1\leq k\leq N}\|u_k^h\|_{L_{2h}}+\|\left(A_h^x\right)^{-1}p^h\|_{L_{2h}}$$

$$\leq M_1(\delta,\alpha)\left\{\max_{1\leq k\leq N-1}\|f_k^h\|_{L_{2h}}+\|\ \psi^h\ \|_{L_{2h}}+\|\ \xi^h\ \|_{L_{2h}}+\|\ \varphi^h\ \|_{L_{2h}}\right\},$$

$$\max_{1\leq k\leq N-1}\left\|\frac{u_{k+1}^h-2u_k^h+u_{k-1}^h}{\tau^2}\right\|_{L_{2h}}+\max_{1\leq k\leq N}\|u_k^h\|_{W_{2h}^2}\leq M_2(\delta,\alpha)\left\{\max_{2\leq k\leq N-1}\left\|\frac{1}{\tau}\left(f_k^h-f_{k-1}^h\right)\right\|_{L_{2h}}\right.$$

$$\left.+\|\ f_1^h\ \|_{L_{2h}}+\|\ \psi^h\ \|_{W_{2h}^1}+\|\ \xi^h\ \|_{W_{2h}^2}+\|\ \varphi^h\ \|_{W_{2h}^2}\right\}$$

hold, where $M_1(\delta,\alpha)$ *and* $M_2(\delta,\alpha)$ *do not depend on* $\varphi^h(x),\psi^h(x)$, $\xi^h(x)$ *and* $f_k^h(x)$, $1\leq k\leq N-1$.

Proof. Difference scheme (3.28) can be written in abstract form (3.22) in a Hilbert space $L_{2h}=L_2(\overline{\Omega}_h)$ with self-adjoint positive definite operator $A_h=A_h^x$ by formula (3.26).

Here, $f_k^h=f_k^h(x)$ and $u_k^h=u_k^h(x)$ are known and unknown abstract mesh functions defined on $\overline{\Omega}_h$ with the values in $H=L_{2h}$. Therefore, estimates of Theorem2.7 follow from estimates of Theorem2.1 and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3.7. *For the solutions of the elliptic difference problem (Sobolevskii, 1975)*

$$\begin{cases} A_h^x u^h(x) = \omega^h(x), \ x \in \Omega_h, \\ \\ D^h u^h(x) = 0, \ x \in S_h, \end{cases}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{n} \left\| {u^h}_{x_r \overline{x_r}} \right\|_{L_{2h}} \leq M_3 ||\omega^h||_{L_{2h}},$$

where M_3 *does not depend on* h *and* ω^h*.*

Note that the difference schemes of the second order of accuracy with respect to one variable for approximate solutions of boundary value problems (2.13), (2.18) and (2.20) generated by difference schemes (3.17) and (3.18) can be constructed. This approach permit us to establish the stability estimates for the solution of these difference schemes.

In applications, one test example is considered. The theoretical statements for the solution of these difference schemes are supported by the result of the numerical experiment.

CHAPTER 4

NUMERICAL RESULTS

For the numerical result, we consider the initial-boundary value problem

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + 2\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) = p(x) + (e^{-t} - 1)\sin x, \\ \\ 0 < x < \pi, 0 < t < 1, \\ \\ u(0,x) = \sin x, u(1,x) = e^{-1}\sin x, u_t(0,x) = -\sin x, 0 \leq x \leq \pi, \\ \\ u(t,0) = u(t,\pi) = 0, 0 \leq t \leq 1 \end{cases} \tag{4.1}$$

for the telegraph equation. The exact solution of the given problem is

$$u(t,x) = e^{-t}\sin x$$

and the unknown function

$$p(x) = \sin x.$$

Firstly, for simplicity we denote that

$$f(t,x) = (e^{-t} - 1)\sin x.$$

For the approximate solution of the problem (4.1), we consider the set $[0,1]_\tau \times [0,\pi]_h$ of a family of grid points depending on the small parameters τ and h

$$[0,1]_\tau \times [0,\pi]_h = \left\{ (t_k, x_n) : \begin{array}{c} t_k = k\tau, 1 \leq k \leq N-1, N\tau = 1, \\ x_n = nh, 1 \leq n \leq M-1, Mh = \pi \end{array} \right\}.$$

For the solution of (4.1), the difference scheme of the first order of accuracy in t and second order of accuracy in x

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^k}{\tau}-\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+u_n^{k+1}=\theta_n^{k+1}, \\ \theta_n^{k+1}=f(t_{k+1},x_n)+p(x_n), x_n=nh, t_{k+1}=(k+1)\tau, \\ 1\leq k\leq N-1, 1\leq n\leq M-1, \\ u_n^0=\sin(x_n), u_n^N=e^{-1}\sin(x_n), \frac{u_n^1-u_n^0}{\tau}=-\sin(x_n), 0\leq n\leq M, \\ u_0^k=u_M^k=0, 0\leq k\leq N \end{cases} \tag{4.2}$$

and two types of second order of accuracy in t and x

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\frac{1}{2}\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\frac{1}{2}\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\frac{1}{2}\left(u_n^{k+1}+u_n^{k-1}\right)=\theta_n^k, \\ \theta_n^k=f(t_k,x_n)+p(x_n), x_n=nh, t_k=k\tau, \\ 1\leq k\leq N-1, 1\leq n\leq M-1, \\ u_n^0=\sin(x_n), x_n=nh, \\ \frac{u_n^1-u_n^0}{\tau}=-\sin(x_n)+\frac{\tau}{2}\frac{u_n^2-2u_n^1+u_n^0}{\tau^2}, u_n^N=e^{-1}\sin(x_n), 0\leq n\leq M, \\ u_0^k=u_M^k=0,\ 0\leq k\leq N, \end{cases} \tag{4.3}$$

$$\begin{cases} \frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}+2\frac{u_n^{k+1}-u_n^{k-1}}{2\tau}-\frac{1}{2}\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}-\frac{1}{4}\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2} \\ -\frac{1}{4}\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+\frac{1}{2}u_n^k+\frac{1}{4}\left(u_n^{k+1}+u_n^{k-1}\right)=\theta_n^k, \\ \theta_n^k=f(t_k,x_n)+p(x_n), x_n=nh, t_k=k\tau, \\ 1\le k\le N-1, 1\le n\le M-1, \\ u_n^0=\sin x_n, x_n=nh, \\ \frac{u_n^1-u_n^0}{\tau}=-\sin(x_n)+\frac{\tau}{2}\frac{u_n^2-2u_n^1+u_n^0}{\tau^2}, u_n^N=e^{-1}\sin(x_n), 0\le n\le M, \\ u_0^k=u_M^k=0, 0\le k\le N \end{cases} \tag{4.4}$$

are constructed.

For obtaining values of $p(x_n)$ at grid points, from the following equation

$$\begin{gathered} p(x_n)=-e^{-1}\frac{\sin(x_{n+1})-2\sin(x_n)+\sin(x_{n-1})}{h^2}+e^{-1}\sin(x_n)+ \\ +\frac{v_{n+1}^N-2v_n^N+v_{n-1}^N}{h^2}-v_n^N,\ x_n=nh,\ 1\le n\le M-1 \end{gathered} \tag{4.5}$$

where v_s^k, $s=n\pm 1$ is the solution of the first and second order of accuracy difference schemes

$$\begin{cases} \frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2}+2\frac{v_n^{k+1}-v_n^k}{\tau}-\frac{v_{n+1}^{k+1}-2v_n^{k+1}+v_{n-1}^{k+1}}{h^2}+v_n^{k+1}=f(t_{k+1},x_n), \\ f(t_{k+1},x_n)=(e^{-t_{k+1}}-1)\sin x_n, x_n=nh, t_{k+1}=(k+1)\tau, \\ 1\le k\le N-1, 1\le n\le M-1, \\ v_n^N-v_n^0=(e^{-1}-1)\sin(x_n), x_n=nh, \\ \frac{v_n^1-v_n^0}{\tau}=-\sin(x_n),\ 0\le n\le M, \\ v_0^k=v_M^k=0,\ 0\le k\le N, \end{cases} \tag{4.6}$$

$$\begin{cases} \frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2}+2\frac{v_n^{k+1}-v_n^{k-1}}{2\tau}-\frac{1}{2}\frac{v_{n+1}^{k+1}-2v_n^{k+1}+v_{n-1}^{k+1}}{h^2} \\ -\frac{1}{2}\frac{v_{n+1}^{k-1}-2v_n^{k-1}+v_{n-1}^{k-1}}{h^2}+\frac{1}{2}\left(v_n^{k+1}+v_n^{k-1}\right)=f(t_k,x_n), \\ f(t_k,x_n)=(e^{-t_k}-1)\sin x_n, x_n=nh, t_k=k\tau, \\ 1\le k\le N-1,\ 1\le n\le M-1, \\ v_n^N-v_n^0=(e^{-1}-1)\sin(x_n), x_n=nh, \\ \frac{v_n^1-v_n^0}{\tau}=-\sin(x_n)+\frac{\tau}{2}\frac{v_n^2-2v_n^1+v_n^0}{\tau^2},\ 0\le n\le M, \\ v_0^k=v_M^k=0,\ 0\le k\le N \end{cases} \tag{4.7}$$

and

$$\begin{cases} \frac{v_n^{k+1}-2v_n^k+v_n^{k-1}}{\tau^2}+2\frac{v_n^{k+1}-v_n^{k-1}}{2\tau}-\frac{1}{2}\frac{v_{n+1}^{k}-2v_n^{k}+v_{n-1}^{k}}{h^2}-\frac{1}{4}\frac{v_{n+1}^{k+1}-2v_n^{k+1}+v_{n-1}^{k+1}}{h^2} \\ -\frac{1}{4}\frac{v_{n+1}^{k-1}-2v_n^{k-1}+v_{n-1}^{k-1}}{h^2}+\frac{1}{2}v_n^k+\frac{1}{4}\left(v_n^{k+1}+v_n^{k-1}\right)=f(t_k,x_n), \\ f(t_k,x_n)=(e^{-t_k}-1)\sin(x_n), x_n=nh, t_k=k\tau, \\ 1\le k\le N-1, 1\le n\le M-1, \\ v_n^N-v_n^0=(e^{-1}-1)\sin(x_n), x_n=nh, \\ \frac{v_n^1-v_n^0}{\tau}=-\sin(x_n)+\frac{\tau}{2}\frac{v_n^2-2v_n^1+v_n^0}{\tau^2}, 0\le n\le M, \\ v_0^k=v_M^k=0, 0\le k\le N. \end{cases} \tag{4.8}$$

generated by difference schemes (4.2),(4.3) and (4.4), respectively.

4.1 THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

Applying the first order of accuracy difference scheme (4.6), we obtain $(N+1)\times(N+1)$ system of linear equations and we can rewrite this system as the following form

$$\begin{cases} \left(-\frac{1}{h^2}\right)v_{n+1}^{k+1}+\left(\frac{1}{\tau^2}+\frac{2}{\tau}+\frac{2}{h^2}+1\right)v_n^{k+1}+\left(-\frac{2}{\tau^2}-\frac{2}{\tau}\right)v_n^k \\ +\left(\frac{1}{\tau^2}\right)v_n^{k-1}+\left(-\frac{1}{h^2}\right)v_{n-1}^{k+1}=f(t_{k+1},x_n), \\ f(t_{k+1},x_n)=(e^{-t_{k+1}}-1)\sin x_n, x_n=nh, t_{k+1}=(k+1)\tau, \\ 1\le k\le N-1, 1\le n\le M-1, \\ v_n^N-v_n^0=(e^{-1}-1)\sin(x_n), x_n=nh, \\ \frac{v_n^1-v_n^0}{\tau}=-\sin(x_n),\ 0\le n\le M, \\ v_0^k=v_M^k=0,\ 0\le k\le N. \end{cases} \tag{4.9}$$

We denote

$$a=-\frac{1}{h^2},\ b=\frac{1}{\tau^2}+\frac{2}{\tau}+\frac{2}{h^2}+1,\ c=-\frac{2}{\tau^2}-\frac{2}{\tau},\ d=\frac{1}{\tau^2},$$

$$\varphi_n^k=\begin{cases} (e^{-1}-1)\sin(x_n)\ ,\ k=0, \\ f(t_{k+1},x_n)\qquad ,1\le k\le N-1, \\ -\sin(x_n)\qquad ,\ k=N, \end{cases}$$

$$\varphi_n=\begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ d & c & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d & c & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d & c & b & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & b \\ -\frac{1}{\tau} & \frac{1}{\tau} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

and $C = A$,

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$v_s = \begin{bmatrix} v_s^0 \\ v_s^1 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n-1, n, n+1.$$

Then, (4.6) can be written as

$$\begin{cases} Av_{n+1} + Bv_n + Cv_{n-1} = D\varphi_n, \ 1 \le n \le M-1, \\ \\ v_0 = \tilde{0}, \ v_M = \tilde{0}. \end{cases}$$

So, we have the second order difference equation with respect to n with matrix coefficients. By using the Gauss elimination method, we can reach to the solution of v_n^k, $0 \le k \le N$, $0 \le n \le M$.

For the solution of the matrix equations, we seek the solution as of the form

$$\begin{cases} v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \ n = M-1, \ldots, 2, 1, \\ \\ v_0 = \tilde{0}, \ v_M = \tilde{0} \end{cases}$$

where α_j and β_j, $j = 1, \ldots, M$ are calculated as

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1}(A),$$

$$\beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n),$$

with α_1 is $(N+1) \times (N+1)$ and β_1 is $(N+1) \times 1$ zero matrix.

Then, using equation (4.5), values of $p(x_n)$ at grid points are obtained. Replacing $p(x_n)$ in (4.2), $(N+1) \times (N+1)$ system of linear equations and it can be written in the matrix form

$$\begin{cases} A_2 u_{n+1} + B_2 u_n + C_2 u_{n-1} = D\gamma_n, \ 1 \le n \le M-1, \\ \\ u_0 = \tilde{0}, \ u_M = \tilde{0} \end{cases}$$

where $C_2 = C$, $A_2 = A$,

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d & c & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d & c & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d & c & b & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & b \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n-1, n, n+1,$$

$$\gamma_n^k = \begin{cases} \sin(x_n) & , \ k = 0, \\ f(t_{k+1}, x_n) + p(x_n) & , 1 \le k \le N-1, \\ e^{-1}\sin(x_n) & , \ k = N, \end{cases} ,$$

$$\gamma_n = \begin{bmatrix} \gamma_n^0 \\ \gamma_n^1 \\ \vdots \\ \gamma_n^N \end{bmatrix}_{(N+1)\times 1} .$$

Again, applying the modified Gauss elimination method we can reach to the solution of u_n^k, $0 \le k \le N$, $0 \le n \le M$.

4.2 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME (1ST TYPE)

Applying the second order of accuracy difference scheme (4.7), we obtain $(N+1)\times(N+1)$ system of linear equations and we can rewrite this system as the following form

$$\begin{cases} \left(-\frac{1}{2h^2}\right)v_{n+1}^{k+1}+\left(-\frac{1}{2h^2}\right)v_{n+1}^{k-1}+\left(\frac{1}{\tau^2}+\frac{1}{\tau}+\frac{1}{h^2}+\frac{1}{2}\right)v_n^{k+1}+\left(-\frac{2}{\tau^2}\right)v_n^k \\ +\left(\frac{1}{\tau^2}-\frac{1}{\tau}+\frac{1}{h^2}+\frac{1}{2}\right)v_n^{k-1}+\left(-\frac{1}{2h^2}\right)v_{n-1}^{k+1}+\left(-\frac{1}{2h^2}\right)v_{n-1}^{k-1}=f(t_k,x_n), \\ f(t_k,x_n)=(e^{-t_k}-1)\sin x_n, x_n=nh, t_k=k\tau, \\ 1\le k\le N-1,\ 1\le n\le M-1, \\ v_n^N-v_n^0=(e^{-1}-1)\sin(x_n), x_n=nh, \\ \frac{v_n^1-v_n^0}{\tau}=-\sin(x_n)+\frac{\tau}{2}\frac{v_n^2-2v_n^1+v_n^0}{\tau^2},\ 0\le n\le M, \\ v_0^k=v_M^k=0,\ 0\le k\le N. \end{cases} \tag{4.10}$$

We denote

$$a=-\frac{1}{2h^2},\ b=\frac{1}{\tau^2}+\frac{1}{\tau}+\frac{1}{h^2}+\frac{1}{2},\ c=-\frac{2}{\tau^2},\ d=\frac{1}{\tau^2}-\frac{1}{\tau}+\frac{1}{h^2}+\frac{1}{2},$$

$$\varphi_n^k=\begin{cases}(e^{-1}-1)\sin(x_n)\ ,\ k=0,\\ f(t_k,x_n)\qquad ,1\le k\le N-1,\\ -\sin(x_n)\qquad ,\ k=N,\end{cases}$$

$$\varphi_n=\begin{bmatrix}\varphi_n^0\\ \varphi_n^1\\ \vdots\\ \varphi_n^N\end{bmatrix}_{(N+1)\times 1},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a & 0 & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & 0 & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & a & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a & 0 & a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ d & c & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d & c & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d & c & b & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & b \\ -\frac{3}{2\tau} & \frac{2}{\tau} & -\frac{1}{2\tau} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)},$$

and $C = A$,

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$v_s = \begin{bmatrix} v_s^0 \\ v_s^1 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n-1, n, n+1.$$

Then, (4.7) can be written as

$$\begin{cases} Av_{n+1} + Bv_n + Cv_{n-1} = D\varphi_n, \; 1 \leq n \leq M-1, \\ \\ v_0 = \tilde{0}, \; v_M = \tilde{0}. \end{cases}$$

So, we have the second order difference equation with respect to n with matrix coefficients. By using the Gauss elimination method, we can reach to the solution of v_n^k, $0 \leq k \leq N$, $0 \leq n \leq M$.

For the solution of the matrix equations, we seek the solution as of the form

$$\begin{cases} v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \; n = M-1, \ldots, 2, 1, \\ \\ v_0 = \tilde{0}, \; v_M = \tilde{0} \end{cases}$$

where α_j and β_j, $j = 1, \ldots, M$ are calculated as

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1}(A),$$

$$\beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n),$$

with α_1 is $(N+1) \times (N+1)$ and β_1 is $(N+1) \times 1$ zero matrix.

Then, using equation (4.5), values of $p(x_n)$ at grid points are obtained. Replacing $p(x_n)$ in (4.3), $(N+1) \times (N+1)$ system of linear equations and it can be written in the matrix form

$$\begin{cases} A_2 u_{n+1} + B_2 u_n + C_2 u_{n-1} = D\gamma_n, \; 1 \leq n \leq M-1, \\ \\ u_0 = \tilde{0}, \; u_M = \tilde{0} \end{cases}$$

where $C_2 = C$, $A_2 = A$,

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d & c & b & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & d & c & b & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & d & c & b & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & b \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n-1, n, n+1,$$

$$\gamma_n^k = \begin{cases} \sin(x_n) & , \ k = 0, \\ f(t_k, x_n) + p(x_n) & , 1 \leq k \leq N-1, \\ e^{-1}\sin(x_n) & , \ k = N, \end{cases} ,$$

$$\gamma_n = \begin{bmatrix} \gamma_n^0 \\ \gamma_n^1 \\ \vdots \\ \gamma_n^N \end{bmatrix}_{(N+1)\times 1} .$$

Again, applying the modified Gauss elimination method we can reach to the solution of u_n^k, $0 \leq k \leq N$, $0 \leq n \leq M$.

4.3 THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME (2ND TYPE)

Applying the second order of accuracy difference scheme (4.8), we obtain $(N+1)\times(N+1)$ system of linear equations and we can rewrite this system as the following form

$$\begin{cases} \left(-\frac{1}{4h^2}\right)v_{n+1}^{k+1}+\left(-\frac{1}{2h^2}\right)v_{n+1}^{k}+\left(-\frac{1}{4h^2}\right)v_{n+1}^{k-1}+\left(\frac{1}{\tau^2}+\frac{1}{\tau}+\frac{1}{2h^2}+\frac{1}{4}\right)v_{n}^{k+1} \\ +\left(-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2}\right)v_{n}^{k}+\left(\frac{1}{\tau^2}-\frac{1}{\tau}+\frac{1}{2h^2}+\frac{1}{4}\right)v_{n}^{k-1} \\ +\left(-\frac{1}{4h^2}\right)v_{n-1}^{k+1}+\left(-\frac{1}{2h^2}\right)v_{n-1}^{k}+\left(-\frac{1}{4h^2}\right)v_{n-1}^{k-1}=f(t_k,x_n), \\ f(t_k,x_n)=(e^{-t_k}-1)\sin x_n, x_n=nh, t_k=k\tau, \\ 1\le k\le N-1,\ 1\le n\le M-1, \\ v_n^N-v_n^0=(e^{-1}-1)\sin(x_n), x_n=nh, \\ \frac{v_n^1-v_n^0}{\tau}=-\sin(x_n)+\frac{\tau}{2}\frac{v_n^2-2v_n^1+v_n^0}{\tau^2},\ 0\le n\le M, \\ v_0^k=v_M^k=0,\ 0\le k\le N \end{cases} \tag{4.11}$$

We denote

$$a=-\frac{1}{4h^2},\ b=-\frac{1}{2h^2},\ c=\frac{1}{\tau^2}+\frac{1}{\tau}+\frac{1}{2h^2}+\frac{1}{4},\ d=-\frac{2}{\tau^2}+\frac{1}{h^2}+\frac{1}{2},\ e=\frac{1}{\tau^2}-\frac{1}{\tau}+\frac{1}{2h^2}+\frac{1}{4},$$

$$\varphi_n^k=\begin{cases}(e^{-1}-1)\sin(x_n)\ , & k=0, \\ f(t_k,x_n) & ,1\le k\le N-1, \\ -\sin(x_n) & ,\ k=N,\end{cases}$$

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \vdots \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1} ,$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a & b & a & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & b & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & a & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a & b & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a & b & a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ e & d & c & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e & d & c & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & e & d & c & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & e & d & c \\ -\frac{3}{2\tau} & \frac{2}{\tau} & -\frac{1}{2\tau} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

and $C = A$,

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)} ,$$

$$v_s = \begin{bmatrix} v_s^0 \\ v_s^1 \\ \vdots \\ v_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n-1, n, n+1.$$

Then, (4.8) can be written as

$$\begin{cases} Av_{n+1} + Bv_n + Cv_{n-1} = D\varphi_n, \; 1 \le n \le M-1, \\ \\ v_0 = \tilde{0}, \; v_M = \tilde{0}. \end{cases}$$

So, we have the second order difference equation with respect to n with matrix coefficients. By using the Gauss elimination method, we can reach to the solution of v_n^k, $0 \le k \le N$, $0 \le n \le M$.

For the solution of the matrix equations, we seek the solution as of the form

$$\begin{cases} v_n = \alpha_{n+1} v_{n+1} + \beta_{n+1}, \; n = M-1, \ldots, 2, 1, \\ \\ v_0 = \tilde{0}, \; v_M = \tilde{0} \end{cases}$$

where α_j and β_j, $j = 1, \ldots, M$ are calculated as

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1}(A),$$

$$\beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n),$$

with α_1 is $(N+1) \times (N+1)$ and β_1 is $(N+1) \times 1$ zero matrix.

Then, using equation (4.5), values of $p(x_n)$ at grid points are obtained. Replacing $p(x_n)$ in (4.4), $(N+1) \times (N+1)$ system of linear equations and it can be written in the matrix form

$$\begin{cases} A_2 u_{n+1} + B_2 u_n + C_2 u_{n-1} = D\gamma_n, \; 1 \le n \le M-1, \\ \\ u_0 = \tilde{0}, \; u_M = \tilde{0} \end{cases}$$

where $C_2 = C$, $A_2 = A$,

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ e & d & c & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e & d & c & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & e & d & c & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & d & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & e & d & c \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}_{(N+1)\times(N+1)},$$

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1} \qquad \text{for } s = n-1, n, n+1,$$

$$\gamma_n^k = \begin{cases} \sin(x_n) & , \ k = 0, \\ f(t_k, x_n) + p(x_n) & , \ 1 \le k \le N-1, \\ e^{-1}\sin(x_n) & , \ k = N, \end{cases} ,$$

$$\gamma_n = \begin{bmatrix} \gamma_n^0 \\ \gamma_n^1 \\ \vdots \\ \gamma_n^N \end{bmatrix}_{(N+1)\times 1} .$$

Again, applying the modified Gauss elimination method we can reach to the solution of u_n^k, $0 \le k \le N$, $0 \le n \le M$.

The MATLAB implementation used for these computations is given in Appendix. Computed results are given in Error Analysis.

4.4 ERROR ANALYSIS

The results of the numerical analysis are introduced. The numerical solutions are recorded for different values of N and M and u_n^k represents the numerical solutions of these difference schemes at (t_k, x_n). Table 4.1 is constructed for $N = M = 20, 40$ and 80, respectively.

The errors are computed by the following formula

$$E = \max_{\substack{1 \leq k \leq N \\ 1 \leq n \leq M}} \left| u(t_k, x_n) - u_n^k \right| .$$

Table 4.1 Error analysis for the exact solution $u(t, x)$.

$\tau = \frac{1}{N}, h = \frac{pi}{M}$	$N = M = 20$	$N = M = 40$	$N = M = 80$
The difference scheme (4.2)	0.0050	0.0024	0.0012
The difference scheme (4.3)	2.5733×10^{-4}	6.4987×10^{-5}	1.6338×10^{-5}
The difference scheme (4.4)	2.4384×10^{-4}	6.1511×10^{-5}	1.5457×10^{-5}

The results of computer calculations show that the second order of accuracy difference schemes are more accurate than first order of accuracy difference scheme. Moreover, by using the second type of second order of accuracy difference scheme the solution accuracy increases faster than the first type of second order of accuracy difference scheme.

Also for finding the unknown function $p(x)$, an extra condition is needed. To support the numerical result, Table 4.2 is constructed for the error of $p(x)$ at the nodes in maximum norm.

Table 4.2 Error analysis for $p(x)$.

$\tau = \frac{1}{N}, h = \frac{pi}{M}$	$N = M = 20$	$N = M = 40$	$N = M = 80$
The difference scheme (4.2)	0.0356	0.0167	0.0081
The difference scheme (4.3)	0.0016	4.1575×10^{-4}	1.0448×10^{-4}
The difference scheme (4.4)	7.8862×10^{-4}	2.0283×10^{-4}	5.0741×10^{-5}

CHAPTER 5

CONCLUSION

This work is devoted to study the stability of identification problem for a telegraph equation with an unknown parameter. The following original results are obtained:

- The abstract theorem on the stability estimate for the solution of the source identification problem for a telegraph equation is proved.
- Stability estimates for the solution of three source identification problems for the telegraph equation are obtained.
- The first and second order of accuracy difference schemes for the approximate solution of the source identification problem for the telegraph equation are presented.
- Theorems on the stability estimates for the solution of difference schemes for the approximate solution of identification problem for telegraph equation are proved.
- Stability estimates for the solution of difference schemes for three source identification problems for telegraph equation are obtained.
- The Matlab implementation of these difference schemes are presented.
- The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples.

REFERENCES

Akyildiz, F., S.Tatar, and Ulusoy, S., "Existence and uniqueness for a nonlinear inverse reaction-diffusion problem with a nonlinear source in higher dimensions", *Mathematical Methods in the Applied Sciences*, Vol. 36, pp. 2397–2402, 2013.

Ashyralyev, A., "On a problem of determining the parameter of a parabolic equation", *Ukranian Mathematical Journal*, Vol. 62, pp. 1200–1210, 2010.

Ashyralyev, A. and Agirseven, D., "On source identification problem for a delay parabolic equation", *Nonlinear Analysis: Modelling and Control*, Vol. 19, pp. 335–349, 2014.

Ashyralyev, A. and Ashyralyyev, C., "On the problem of determining the parameter of an elliptic equation in a Banach space", *Nonlinear Analysis-Modelling and Control*, Vol. 19, pp. 350–366, 2014.

Ashyralyev, A. and Ashyralyyeva, M.A., "On source identification problem for a hyperbolic-parabolic equation", *Contemporary Analysis and Applied Mathematics*, Vol. 3, No. 1, , 2015.

Ashyralyev, A. and Erdogan, A., "Well-posedness of the right-hand side identification problem for a parabolic equation", *Ukranian Mathematical Journal*, Vol. 66, pp. 165–177, 2014, DOI: 10.1007/s11253-014-0920-0.

Ashyralyev, A., Erdogan, A., and Demirdag, O., "On the determination of the right-hand side in a parabolic equation", *Applied Numerical Mathematics*, Vol. 62, pp. 1672–1683, 2012.

Ashyralyev, A. and Sobolevskii, P., *New Difference Schemes for Partial Differential Equations*, Birkhuser Verlag, Basel, Boston, Berlin, 2004.

Ashyralyev, A. and Urun, M., "Determination of a control parameter for the difference Schrodinger equation", *Abstract and Applied Analysis*, Vol. 2013, , 2013a, Article Number: 548201(2013), DOI: 10.1155/2013/548201.

Ashyralyev, A. and Urun, M., "Determination of a control parameter for the Schrdinger equation", *Contemporary Analysis and Applied Mathematics*, Vol. 1, pp. 156–166, 2013b.

Ashyralyev, A. and Urun, M., "A second order of accuracy difference scheme for Schrodinger equations with an unknown parameter", *Filomat*, Vol. 28, pp. 981–993, 2014, DOI: 10.2298/FIL1405981A.

Ashyralyyev, C., “High order approximation of the inverse elliptic problem with Dirichlet-Neumann conditions”, *Filomat*, Vol. 28, pp. 947–962, 2014a.

Ashyralyyev, C., “High order of accuracy difference schemes for the inverse elliptic problem with Dirichlet condition”, *Boundary Value Problems*, Vol. 2014, , 2014b, Article Number: 5, DOI: 10.1186/1687-2770-2014-5.

Ashyralyyev, C. and Dedeturk, M., “A finite difference method for the inverse elliptic problem with the Dirichlet condition”, *Contemporary Analysis and Applied Mathematics*, Vol. 1, pp. 132–155, 2013.

Blasio, G.D. and Lorenzi, A., “Identification problems for parabolic delay differential equations with measurement on the boundary”, *Journal of Inverse and Ill-Posed Problems*, Vol. 15, pp. 709–734, 2007.

Daoudi-Merzagui, N. and Tabet, Y., “Existence of multiple positive solutions for a nonlocal boundary value problem with sign changing nonlinearities”, *Filomat*, Vol. 27, pp. 485–497, 2013.

Dehghan, M., “Determination of a control parameter in the two-dimensional diffusion equation”, *Applied Numerical Mathematics*, Vol. 37, pp. 489–502, 2001.

Eidelman, Y., *Boundary Value Problems for Differential Equations with Parameters*, Ph.D. Thesis, Voronezh State University, 1984, in Russian.

Erdogan, A., “A note on the right-hand side identification problem arising in biofluid mechanics”, *Abstract and Applied Analysis*, Vol. 2012, , 2012, Article Number: 548508(2012), DOI: 10.1155/2012/5485082012.

Erdogan, A. and Ashyralyev, A., “On the second order implicit difference schemes for a right hand side identification problem”, *Applied Mathematics and Computation*, Vol. 226, pp. 212–228, 2014.

Erdogan, A. and Sazaklioglu, A., “A note on the numerical solution of an identification problem for observing two-phase flow in capillaries”, *Mathematical Methods in the Applied Sciences*, Vol. 37, pp. 2393–2405, 2014.

Erdogan, A. and Uygun, H., “A note on the inverse problem for a fractional parabolic equation”, *Abstract and Applied Analysis*, Vol. 2012, , 2012, Article Number: 276080 (2012), DOI: 10.1155/2012/276080.

Fattorini, H.O., *Second Order Linear Differential Equations in Banach Spaces*, Notas de Matematica, North-Holland, 1985.

Gryazin, Y., Klibanov, M., and Lucas, T., “Imaging the diffusion coefficient in a parabolic inverse problem in optical tomography”, *Inverse Problems*, Vol. 25, pp. 373–397, 1999.

Hasanov, A., “Identification of unknown diffusion and convection coefficients in ion transport problems from flux data: an analytical approach”, *Journal of Mathematical Chemistry*, Vol. 48, pp. 413–423, 2010.

Kimura, T. and Suzuki, T., “A parabolic inverse problem arising in a mathematical model for chromatography”, *SIAM Journal on Applied Mathematics*, Vol. 53, pp. 1747–1761, 1993.

Orlovsky, D. and Piskarev, S., “On approximation of inverse problems for abstract elliptic problems”, *Journal of Inverse and Ill-Posed Problems*, Vol. 17, pp. 765–782, 2009.

Orlovsky, D. and Piskarev, S., “The approximation of Bitzadze-Samarsky type inverse problem for elliptic equations with Neumann conditions”, *Contemporary Analysis and Applied Mathematics*, Vol. 1, pp. 118–131, 2013a.

Orlovsky, D. and Piskarev, S., “Approximation of the Bitsadze-Samarskii inverse problem for an elliptic equation with the dirichlet conditions”, *Differential Equations*, Vol. 49, pp. 895–907, 2013b.

Ozbilge, E. and Demir, A., “Identification of the unknown coefficient in a quasi-linear parabolic equation by a semigroup approach”, *Journal of Inequalities and Applications*, Vol. 2013, , 2013a, Article number 212, DOI: 10.1186/1029-242X-2013-212.

Ozbilge, E. and Demir, A., “Semigroup approach for identification of the unknown diffusion coefficient in a linear parabolic equation with mixed output data”, *Boundary Value Problems*, Vol. 2013, , 2013b, Article Number 43, DOI: 10.1186/1687-2770-2013-43.

Piskarev, S. and Shaw, Y., “On certain operator families related to cosine operator function”, *Taiwanese Journal of Mathematics*, Vol. 1, pp. 3585–3592, 1997.

Safari, A.R., Mekhtiyev, M., and Sharifov, Y., “Maximum principle in the optimal control problems for systems with integral boundary conditions and its extension”, *Abstract and Applied Analysis*, Vol. 2013, , 2013, Article Number 946910, DOI: 10.1155/2013/946910.

Serov, V. and Pivrinta, L., “Inverse scattering problem for two-dimensional Schrdinger operator”, *Journal of Inverse and Ill-Posed Problems*, Vol. 14, pp. 295–305, 2006.

Sobolevskii, P.E., *Difference Methods for the Approximate Solution of Differential Equations*, Voronezh State University Press, Voronezh, Russia, 1975, in Russian.

Tinaztepe, R., S.Tatar, and Ulusoy, S., “Identification of the density dependent coefficient in an inverse reaction-diffusion problem from a single boundary data”, *Electronic Journal of Differential Equations*, Vol. 2014, , 2014, Article Number 21.

V.B. Shakhmurov, A.S., “Abstract parabolic problems with parameter and application”, *Applied Mathematics and Computation*, Vol. 219, pp. 9561–9571, 2013.

Wu, B. and Wu, S., “Existence and uniqueness of an inverse source problem for a fractional integrodifferential equation”, *Computers & Mathematics with Applications*, Vol. 68, pp. 1123–1136, 2014.

APPENDIX A

THE MATLAB IMPLEMENTATIONS

A.1 MATLAB IMPLEMENTATION FOR THE FIRST ORDER OF ACCURACY DIFFERENCE SCHEME

```
function firstorder(N,M)
if nargin¡1; N=20; M=20; end
tau=1/N; h=pi/M;
a=(-1)/(hˆ2);
b=1/(tauˆ2)+2/(tau)+2/(hˆ2)+1;
c=((-2)/(tauˆ2))-(2/(tau));
d=1/(tauˆ2);
’Finding V’;
A=zeros(N+1,N+1);
for i=2:N; A(i,i+1)=a;end; A;
C=A;
B=zeros(N+1,N+1);
for i=2:N;
B(i,i)=c;
B(i,i-1)=d;
B(i,i+1)=b; end;
B(1,1)=-1; B(1,N+1)=1;
B(N+1,1)=-1/tau; B(N+1,2)=1/tau; B;
for i=1:N+1; D(i,i)=1; end; D;
```

```
fii=zeros(N+1,M+1);
for j=2:M;
for k=2:N;
t=k*tau; x=(j-1)*h;
fii(k,j:j)=(exp(-t)-1)*sin(x); end;
fii(1,j:j)=(exp(-1)-1)*sin(x);
fii(N+1,j:j)=-sin(x); end;
’Gauss Elimination’;
D=eye(N+1);
alpha2=zeros(N+1,N+1); betha2=zeros(N+1,1);
for j=3:M+1;
alphaj=inv(B+C*alphaj-1)*(-A);
bethaj=inv(B+C*alphaj-1)*(D*(fii(:,j-1))-C*bethaj-1); end;
for k=1:N+1; V(k,1)=0; V(k,M+1)=0; end;
for n=M:-1:2;
V(:,n)=alphan+1*V(:,n+1)+bethan+1; end;
’Finding p’;
p(1)=0; p(M+1)=0;
for n=2:M;
p(n)=2*exp(-1)*sin((n-1)*h)...
+(V(N+1,n+1)-2*V(N+1,n)+V(N+1,n-1))/(h^2)-V(N+1,n);
end;
’Exact solution of p’;
for j=1:M+1; esp(j)=sin((j-1)*h); end;
maxerrorp=max(max(abs(esp-p)))
’Finding U’;
clear alpha;
clear betha;
clear gamma;
clear B;
B=zeros(N+1,N+1);
for i=2:N;
```

```
B(i,i)=c;
B(i,i-1)=d;
B(i,i+1)=b; end;
B(1,1)=1; B(N+1,N+1)=1;
gamma=zeros(N+1,M+1);
for j=2:M;
for k=2:N;
t=k*tau; x=(j-1)*h;
gamma(k,j:j)=(exp(-t)-1)*sin(x)+p(j); end;
gamma(1,j)=sin(x);
gamma(N+1,j)=exp(-1)*sin(x); end;
’Modified Gauss 2’;
D=eye(N+1);
alpha2=zeros(N+1,N+1); betha2=zeros(N+1,1);
for j=3:M+1;
alphaj=inv(B+C*alphaj-1)*(-A);
bethaj=inv(B+C*alphaj-1)*(D*(gamma(:,j-1))-C*bethaj-1); end;
for k=1:N+1; U(k,1)=0; U(k,M+1)=0; end;
for n=M:-1:2; U(:,n)=alphan+1*U(:,n+1)+bethan+1; end;
’Exact Solution of This Problem’;
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau; x=(j-1)*h;
es(k,j)=exp(-t)*sin(x); end; end;
figure; surf(es); rotate3d;
title(’EXACT SOLUTION’);
axis tight;
figure; surf(U); rotate3d;
title(’THE DIFFERENCE SCHEMES SOLUTION’);
axis tight;
’Error Analysis’;
maxerror=max(max(abs(es-U)))
```

A.2 MATLAB IMPLEMENTATION FOR THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME (1ST TYPE)

```
function secondorder(N,M)
if nargin¡1; N=20; M=20; end
tau=1/N; h=pi/M;
a=(-1)/(2*(hˆ2));
b=1/(tauˆ2)+1/(tau)+1/(hˆ2)+1/2;
c=(-2)/(tauˆ2);
d=1/(tauˆ2)-1/(tau)+1/(hˆ2)+1/2;
’Finding V’;
A=zeros(N+1,N+1);
for i=2:N; A(i,i+1)=a; A(i,i-1)=a; end; A;
C=A;
B=zeros(N+1,N+1);
for i=2:N;
B(i,i)=c;
B(i,i-1)=d;
B(i,i+1)=b; end;
B(1,1)=-1; B(1,N+1)=1;
B(N+1,1)=-3/(2*(tau)); B(N+1,2)=2/tau; B(N+1,3)=-1/(2*(tau)); B;
for i=1:N+1; D(i,i)=1; end; D;
fii=zeros(N+1,M+1);
for j=2:M;
for k=2:N;
t=(k-1)*tau; x=(j-1)*h;
fii(k,j:j)=(exp(-t)-1)*sin(x); end;
fii(1,j:j)=(exp(-1)-1)*sin(x);
fii(N+1,j:j)=-sin(x); end;
’Gauss Elimination’;
D=eye(N+1);
alpha2=zeros(N+1,N+1); betha2=zeros(N+1,1);
```

```
for j=3:M+1;
alphaj=inv(B+C*alphaj-1)*(-A);
bethaj=inv(B+C*alphaj-1)*(D*(fii(:,j-1))-C*bethaj-1); end;
for k=1:N+1; V(k,1)=0; V(k,M+1)=0; end;
for n=M:-1:2;
V(:,n)=alphan+1*V(:,n+1)+bethan+1; end;
’Finding p’;
p(1)=0; p(M+1)=0;
for n=2:M;
p(n)=2*exp(-1)*sin((n-1)*h)...
+(V(N+1,n+1)-2*V(N+1,n)+V(N+1,n-1))/(h^2)-V(N+1,n);
end;
’Exact solution of p’;
for j=1:M+1; esp(j)=sin((j-1)*h); end;
maxerrorp=max(max(abs(esp-p)))
’Finding U’;
clear alpha;
clear betha;
clear gamma;
clear B;
B=zeros(N+1,N+1);
for i=2:N;
B(i,i)=c;
B(i,i-1)=d;
B(i,i+1)=b; end;
B(1,1)=1; B(N+1,N+1)=1;
gamma=zeros(N+1,M+1);
for j=2:M;
for k=2:N;
t=(k-1)*tau; x=(j-1)*h;
gamma(k,j:j)=(exp(-t)-1)*sin(x)+p(j); end;
gamma(1,j)=sin(x);
```

```
gamma(N+1,j)=exp(-1)*sin(x); end;
'Modified Gauss 2';
D=eye(N+1);
alpha2=zeros(N+1,N+1); betha2=zeros(N+1,1);
for j=3:M+1;
alphaj=inv(B+C*alphaj-1)*(-A);
bethaj=inv(B+C*alphaj-1)*(D*(gamma(:,j-1))-C*bethaj-1); end;
for k=1:N+1; U(k,1)=0; U(k,M+1)=0; end;
for n=M:-1:2; U(:,n)=alphan+1*U(:,n+1)+bethan+1; end;
'Exact Solution of This Problem';
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau; x=(j-1)*h;
es(k,j)=exp(-t)*sin(x); end; end;
figure; surf(es); rotate3d;
title('EXACT SOLUTION');
axis tight;
figure; surf(U); rotate3d;
title('THE DIFFERENCE SCHEMES SOLUTION');
axis tight;
'Error Analysis';
maxerror=max(max(abs(es-U)))
```

A.3 MATLAB IMPLEMENTATION FOR THE SECOND ORDER OF ACCURACY DIFFERENCE SCHEME (2ND TYPE)

```
function secondorder2(N,M)
if nargin¡1; N=20; M=20; end
tau=1/N; h=pi/M;
a=(-1)/(4*(hˆ2));
```

```
b=(-1)/(2*(h^2));
c=1/(tau^2)+1/(tau)+1/(2*(h^2))+1/4;
d=-2/(tau^2)+1/(h^2)+1/2;
e=1/(tau^2)-1/(tau)+1/(2*(h^2))+1/4;
’Finding V’;
A=zeros(N+1,N+1);
for i=2:N;
A(i,i)=b;
A(i,i+1)=a;
A(i,i-1)=a; end; A;
C=A;
B=zeros(N+1,N+1);
for i=2:N;
B(i,i)=d;
B(i,i-1)=e;
B(i,i+1)=c; end;
B(1,1)=-1; B(1,N+1)=1;
B(N+1,1)=-3/(2*(tau)); B(N+1,2)=2/tau; B(N+1,3)=-1/(2*(tau)); B;
for i=1:N+1; D(i,i)=1; end; D;
fii=zeros(N+1,M+1);
for j=2:M;
for k=2:N;
t=(k-1)*tau; x=(j-1)*h;
fii(k,j:j)=(exp(-t)-1)*sin(x); end;
fii(1,j:j)=(exp(-1)-1)*sin(x);
fii(N+1,j:j)=-sin(x); end;
’Gauss Elimination’;
D=eye(N+1);
alpha2=zeros(N+1,N+1); betha2=zeros(N+1,1);
for j=3:M+1;
alphaj=inv(B+C*alphaj-1)*(-A);
bethaj=inv(B+C*alphaj-1)*(D*(fii(:,j-1))-C*bethaj-1); end;
```

```
for k=1:N+1; V(k,1)=0; V(k,M+1)=0; end;
for n=M:-1:2;
V(:,n)=alphan+1*V(:,n+1)+bethan+1; end;
’Finding p’;
p(1)=0; p(M+1)=0;
for n=2:M;
p(n)=2*exp(-1)*sin((n-1)*h)...
+(V(N+1,n+1)-2*V(N+1,n)+V(N+1,n-1))/(h^2)-V(N+1,n); end;
’Exact solution of p’;
for j=1:M+1; esp(j)=sin((j-1)*h); end;
maxerrorp=max(max(abs(esp-p)))
’Finding U’;
clear alpha;
clear betha;
clear gamma;
clear B;
B=zeros(N+1,N+1); for i=2:N;
B(i,i)=d;
B(i,i-1)=e;
B(i,i+1)=c; end;
B(1,1)=1; B(N+1,N+1)=1;
gamma=zeros(N+1,M+1);
for j=2:M;
for k=2:N;
t=(k-1)*tau; x=(j-1)*h;
gamma(k,j:j)=(exp(-t)-1)*sin(x)+p(j); end;
gamma(1,j)=sin(x);
gamma(N+1,j)=exp(-1)*sin(x); end;
’Modified Gauss 2’;
D=eye(N+1);
alpha2=zeros(N+1,N+1); betha2=zeros(N+1,1);
for j=3:M+1;
```

```
alphaj=inv(B+C*alphaj-1)*(-A);
bethaj=inv(B+C*alphaj-1)*(D*(gamma(:,j-1))-C*bethaj-1); end;
for k=1:N+1; U(k,1)=0; U(k,M+1)=0; end;
for n=M:-1:2; U(:,n)=alphan+1*U(:,n+1)+bethan+1; end;
’Exact Solution of This Problem’;
for j=1:M+1;
for k=1:N+1;
t=(k-1)*tau; x=(j-1)*h;
es(k,j)=exp(-t)*sin(x);end; end;
figure; surf(es); rotate3d;
title(’EXACT SOLUTION’);
axis tight;
figure; surf(U); rotate3d;
title(’THE DIFFERENCE SCHEMES SOLUTION’); axis tight;
’Error Analysis’;
maxerror=max(max(abs(es-U)))
```