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# Multicenter black holes in string theory

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*I am a part of all that I have met;  
Yet all experience is an arch wherethrough  
Gleams that untravelled world, whose margin fades  
For ever and for ever when I move,  
My spirit yearning in desire  
To follow knowledge like a sinking star,  
Beyond the utmost bound of human thought.*

*Come, my friends,  
'Tis not too late to seek a newer world.  
Push off, and sitting well in order smite  
The sounding furrows; for my purpose holds  
To sail beyond the sunset, and the baths  
Of all the western stars, until I die.  
It may be that the gulfs will wash us down:  
It may be we shall touch the Happy Isles,  
And see the great Achilles, whom we knew*

*Tho' much is taken, much abides; and though  
We are not now that strength which in old days  
Moved earth and heaven; that which we are, we are;  
One equal temper of heroic hearts,  
Made weak by time and fate, but strong in will  
To strive, to seek, to find, and not to yield.*

from *Ulysses* by Alfred Tennyson

Voor de Belgische belastingbetaler,  
In de hoop dat u ooit uw strandlaken kan uitspreiden  
op de onbekende kusten die wij verkennen.  
Het was een eer om voor u te mogen werken.



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# Chapter 1

## Introduction

Dear reader, welcome to this thesis. In this work the author has tried to present some of the research he has done in the previous four years of his PhD studies. Before we get ourselves into some of the necessary technical background of the next chapters, it might be appropriate to first discuss some general motivations for this work. In this chapter we review some relevant features of gravity, black holes and string theory, that will lead us to some natural questions and problems. The author hopes his research has made a contribution, even if modest, towards the answer of these epic riddles. At the end of this chapter we will be in a position to give an overview of the subjects that are covered in more detail in this thesis.

The discussion in this chapter will be non technical and is aimed at all people with some physics background. Even those readers with more of a physics interest than background are encouraged to give the chapter a try. The author would also like to point them to the Dutch summary in appendix C, which has a slightly lower density of physics jargon.

### 1.1 Gravity: *primus inter pares*

At present our most accurate descriptions of nature rely on the existence of four fundamental forces. Of these forces gravity is surely the most familiar. Maybe less appreciated is that at the same time it is probably also the most exotic of the four.

#### **The fantastic four**

The gravitational force is the most common force of nature as it continuously affects our everyday life. It makes us have to *get up* in the morning and it allows us to *fall* asleep in the evening. Furthermore, as was first realized by Newton,

this force that keeps us with our feet on the ground also dictates the motion of the planets around the Sun. As Newton explained, gravity is an attractive force between masses. It is proportional to the product of these masses and decreases quadratically with the separation between them. At the beginning of the last century Einstein refined Newton's theory and rewrote the laws of gravity as a relativistic field theory, known as general relativity. We don't need to remind the reader that general relativity as well as Newtonian gravity explained and predicted an amazing plethora of physical phenomena, tested experimentally to very high accuracy.

Of the remaining three fundamental forces electro-magnetism is known to most people these days, even if only as the force that makes their hairdryer blow. The electro-magnetic interaction was well understood by physicists a bit more than a century after gravity and summarized by Maxwell in the famous laws named after him. The existence of two additional forces, the strong and weak nuclear force, was only discovered in the second half of the last century. These nuclear forces only have effect on very short distance scales, of the order of an atomic nucleus i.e. a millionth of a billionth of a centimeter. These last three forces can be described together as a single relativistic field theory called a Yang-Mills theory, with gauge group  $SU(3) \times SU(2) \times U(1)$ . The fundamental description of these three interactions is essentially quantummechanical as the strong and weak force are only observed in the microscopic regime.

Naively one could just add the Einstein-Hilbert action of general relativity to that of Yang-Mills theory, to come to a description of all forces. Sadly such a description is inconsistent microscopically, as general relativity breaks down when quantummechanics becomes important.

### The specialty of gravity

We will come back to the difficulties in describing gravity quantummechanically in a moment. Let us first discuss why gravity is a bit special among the fundamental interactions, even before quantization.

A first striking feature of gravity is that it is so weak. To illustrate this point, think about a petanque player who picks up his or her balls with a magnet on a string. This simple thought experiment reveals that the magnetic force exerted by the tiny magnet on the metal ball is stronger than the gravitational force exerted on the ball *by the whole earth!!* To make this more precise one could try to attach a number to this relative difference in strength. Consider e.g. the ratio of the gravitational attraction to the electric force between two electrons. This ratio is an astonishingly small number of order  $10^{-40}$ .

Gravity still plays a major role in our lives as its weakness is compensated by its sheer abundance. More precisely, another property unique to gravity is that it is universal. We mean that everything, i.e. all forms of matter, the other three forcefields and even the gravitational field itself, interacts gravitationally. This happens because the source of the gravitational field is energy and, as far as our description of nature goes, everything of physical relevance carries energy.

This is in sharp contrast to the other forces where we know that they couple to a certain number of fields while they don't interact with some others.

Another feature that strengthens the influence of gravity is that it is (up to cosmological scales<sup>1</sup>) always attractive. Contrary to gravity, for electro-magnetism and even more so for the strong force, sources naturally arrange themselves in neutral combinations. For electro-magnetism this is the well known attraction between opposite charges. For the strong force this goes under the name of confinement, a phenomenon observed experimentally but less understood theoretically. Furthermore the fields propagating the weak nuclear force are massive, making the weak force exponentially smaller with the distance. The combination of these facts has a major consequence. In spite of its weakness with respect to the other forces at large enough distances gravity is the only force of relevance as on this scale all objects are neutral to the other forces. Combined with the fact that everything couples to gravity and furthermore that all these contributions are 'additive' implies that at astronomical distances gravity dominates the interactions between objects.

Another important curiosity about gravity is the equivalence between the gravitational mass and the inertial mass. This implies that, for gravity, dynamics and kinematics are intimately tied together. Einstein was the first to realize the full consequences of this fact. It leads to the identification of the gravitational field as a dynamical space-time metric determining the kinematical properties of all interactions. So although in principle one can see gravitation as just another field theory, and actually as a gauge theory on the tangent bundle of space-time, the fact that gravity is intertwined with the causal structure of the theory makes it special and already hints that it might be rather unique when studied quantummechanically.

### Quid quantum gravity?

The most pragmatic approach to quantizing gravity is to first restrict to small perturbations around a given gravitational background. These perturbations, called gravitons, are the gravitational analog of the gauge bosons propagating the other three forces. As a side note: another difference making gravity special among the forces is that the graviton carries spin 2 where the other gauge bosons have spin 1. Using these gravitons one can now set up a perturbative quantization in textbook fashion. Propagators can be defined, Feynman rules and diagrams worked out and scattering amplitudes calculated. That is to say, one can calculate the diagrams but one needs to introduce a cutoff to keep them finite. Unlike the other interactions, which are renormalizable, gravity cannot be made cutoff independent in this perturbative scheme unless one introduces an

<sup>1</sup> The most recent measurements of the energy content of our universe indicate that 70% is made up of *dark energy*. The equation of state of this dark energy is such that although it has positive energy it has negative pressure, making the universe expand faster and faster. This expansion acts as gravitational repulsion but is only of relevance on cosmological distances.

infinite number of arbitrary constants, the price being the loss of all predictability.

This non-renormalizability of the theory is essentially due to the fact that the gravitational coupling constant  $G_4$  has the dimension of inverse length square. It is not too difficult to see by dimensional analysis that the divergence of higher loop diagrams necessarily becomes worse and worse. The modern interpretation of a non-renormalizable field theory is as an effective description that is valid at most up to energies where the amplitudes become of order one. In case of gravity this energy is given by the so called Planck energy<sup>2</sup>  $G_4^{-1/2} \approx 2 \times 10^9 \text{ J} \approx 3 \times 10^{28} \text{ eV}$ . To give an idea of the incredible size of this energy: it is about a hundred million billion times the rest energy of the top quark, the most massive elementary particle known today. So in the most conservative scheme the scale where quantum gravity effects start to play a role is incredibly small. To really hammer the message home; the Planck length is  $l_p \approx 2 \times 10^{-35} \text{ m}$ . To get a feeling of how small this is consider first that the size of a proton compares to the width of a hair as that width compares to the distance Brussels-Edinburgh. The Planck length compares to the size of a proton as the width of a hair to a lightyear!<sup>3</sup> These huge/tiny numbers are just another expression of the fact that gravity is so weak compared to the other forces.

Clearly the non-renormalizability presents us with a puzzle: we know all matter around us is composed of particles that are inherently quantummechanical, so how do these quantummechanical particles source a gravitational field of which we only have a valid classical description? Faced however with the fact that this question seems only relevant at scales out of direct experimental reach for the foreseeable future<sup>4</sup>, one might wonder why anyone should care about quantum gravity at the present time. The author sees four immediate motivations: black holes, cosmology, unification and just plain curiosity.

The last motivation is evident and hopefully shared by the reader as much as by the author. The argument of unification is that the quantization of gravity could also have consequences for other aspects in fundamental physics, like the other forces and elementary particles which could be detectable. String theory and especially supersymmetry illustrate this and we will shortly return to this point when discussing the theory of superstrings later in section 1.3. Cosmology is another area where quantum gravity may play an important and maybe even observable role. We will not touch upon that active area of research in this thesis. The main idea is however clear: given that our universe started out with a Big Bang and that early enough in its evolution space-time was incredibly small and

<sup>2</sup> We work in units where  $c = \hbar = 1$ .

<sup>3</sup> A lightyear itself compares to the distance Brussels-Edinburgh as that distance Brussels-Edinburgh to the width of a hair.

<sup>4</sup> Note that there are some theoretical scenarios, see e.g. [1, 2], where the effective Planck mass is actually of the order of a TeV. This would make quantum gravity effects detectable at LHC. Although this would be more than great if realized in nature, there are at the moment no real compelling reasons for these scenarios to be realized. As such we will most of the time reason in the 'worst case scenario' of a tiny Planck length.

energetic, quantum gravity must have been more than relevant. The remaining motivation, that of black holes, will be the subject of this thesis. As we will discuss in the next section black holes have some features that are inherently linked to quantum gravity. In this way they lead to some theoretical predictions that any valid theory of quantum gravity should satisfy. In this sense one might think of black holes as a theoretical 'experimental testing ground' for quantum gravity.

## 1.2 Black holes: gravity at its strongest

Despite its extreme weakness gravity can still be so strong that it can hold all signals trapped. This phenomenon appears when enough matter is accumulated in a certain region of space, and is called a black hole.

### Rebel with a causal horizon

The gravitational field is a metric on space-time that determines the trajectories of matter moving around. The Einstein equations dynamically associate a given metric to a certain energy distribution. The net result is then indeed the gravitational attraction of this energy, already approximately described by Newton. The fact that all other interactions in nature also couple to gravity implies they have to follow the routes set out by the gravitational field. In this sense the gravitational field determines the *causal structure* of space-time. The idea of causality was one of the main ingredients in special relativity. The speed of light is taken as an absolute upper bound on the propagation of any signal. This implies that all signals have to follow so called time-like or null curves in space-time. In the Minkowski space of special relativity this translates to the fact that an event at a certain point in space-time can only influence processes that take place in its future lightcone. If we take into account gravity, which deforms Minkowski space to a more general curved Lorentzian manifold, this 'future lightcone' can become some rather involved subset of space-time. When enough energy has accumulated in a small enough spatial region, the causal structure can be so strongly deformed that no signals can escape that region. Such an object appears completely black to an outside observer, as not even light can escape it, and was thus given the name *black hole*.

Over the years many different types of black holes have been theoretically described; e.g. charged black holes, black strings, black saturns and of course the multicentered black holes of this thesis' title. In this introduction we will focus on the simplest of them all, the original Schwarzschild black hole. Schwarzschild was the first to find a solution to the equations that Einstein wrote down for the gravitational field. The interpretation of the Schwarzschild solution is that of the gravitational field of a spherical distribution of mass, outside that sphere. The Schwarzschild metric is for example the relevant metric in our solar system, as it describes the gravitational field of the Sun. Indeed the orbits of the planets are

to a very high accuracy trajectories of shortest distance in such a Schwarzschild metric. The metric can be understood as a gravitational throat with the Sun at its center. There appears a drastically new feature in the Schwarzschild metric when the spherical mass distribution crosses some critical bound. This happens when the total mass  $m$  is concentrated in a sphere of radius smaller than some critical value called the Schwarzschild radius, which depends on the mass:  $r_S = 2m G_4$ . In this case there is a sphere surrounding the central mass where the acceleration needed to remain at a fixed radial distance, i.e. to not fall into the throat, becomes infinite. This sphere is called the horizon of the black hole.

The presence of such a horizon has two important consequences. First as inside this horizon one is forced to fall ever deeper into the throat the spherical mass sourcing the gravitational field has to collapse to a point. The metric extended inside the horizon is known as the full Schwarzschild solution<sup>5</sup> or Schwarzschild black hole and describes the gravitational field of a point mass. The second important consequence is that the horizon is a kind of one-way membrane, i.e. it can be freely crossed in one direction but once past it no signal can return outside. As we will further discuss below, there are many more unexpected theoretical consequences of the presence of a horizon.

Let us first get some feeling of the Schwarzschild radius by giving some numbers. The Schwarzschild radius of the Sun is roughly 3 km, which should be compared to its actual radius of 700 000 km. For the Earth the Schwarzschild radius is a bit less than a centimeter. Note that although these two examples suggest that to form a black hole one has to compress a lot of matter to a very high density this is not the case. If one takes the mass of the black hole large enough the density of matter just before collapse in a black hole can be arbitrarily small. It is not difficult to calculate that the critical mass density inside a sphere of Schwarzschild radius is given by:  $\rho_S = \left(\frac{32\pi m^2 G_4^3}{3}\right)^{-1}$ . This is large for masses of the order of the Sun because  $G_4$  is so small, but if the total mass is large enough a black hole could just as well form in a region with the density of air. The amount of air (at atmospheric pressure) needed is that of a billion solar masses.

### The experimental status of black holes

Before we continue our theoretical discussion of black holes it might be interesting to pause a moment at their experimental status. In the last decade enormous progress has been made in the search for black holes. Experts nowadays state that the existence of black holes has been proven beyond doubt [3, 4]. For readers that share the authors unfamiliarity with black hole experiments it might be good to discuss what is exactly meant by that claim.

<sup>5</sup> The solution describing the gravitational field outside a spherical mass larger than the Schwarzschild radius, as e.g. for our Sun, is often referred to as the exterior Schwarzschild solution.

The best evidence comes from Sagittarius A\*, a black hole of a few million solar masses in the center of our Galaxy [4]. By measuring orbital velocities of surrounding stars the existence of a large mass is observed and this mass resides in a region that can maximally be a few thousand times the Schwarzschild radius. Furthermore Sagittarius A\* sends out radio signals which at millimeter wavelengths point at a massive source with a size of 5 to 20 times that of the event horizon. There are no known astrophysical explanations for such a concentration of mass other than a black hole [4, 5].

Black holes for which there exist evidence come in two types. First there are the supermassive black holes, these are black holes that sit in the center of galaxies and weigh of the order of a million to a billion solar masses. The example of Sagittarius A\* in the center of our Galaxy falls in this class. The existence of supermassive black holes has been proven beyond doubt by dynamical mass measurements and furthermore they are the best known explanation for active galactic nuclei (AGN) [5]. Recently the experimental community has shifted from narrowing down the error bars on individual supermassive black holes into survey studies trying to understand general properties of these black holes and the galaxies around them. A first example is the correlation between the mass of these black holes and the mass of the host galaxy, an observation which hints at a relation between massive black holes and galaxy formation.

A second type of black holes that has been observed are the so called stellar mass black holes. The existence of these black holes can be considered *confirmed* since the beginning of the 1990's [6] and since then roughly 20 examples have been found. As their name suggests stellar mass black holes have masses of the order 4 to 15 times the mass of the Sun. They appear as a dark component of a binary system, i.e. there is a star orbiting them. Although only twenty cases have been discovered rough estimates predict the existence of a few thousand to a few billion such stellar black holes in our galaxy alone [6].

The existence of black holes in the mass regime in between stellar and supermassive remains an open question for the moment. There are some X-ray sources that might hint at black holes of a few thousand solar masses but this is for the moment only speculation [3].

Quite exciting is the prospect of experimental black hole studies beginning to really start exploring the properties of black holes. One of the most interesting, from a theoretical point of view, is probably the hope of measuring the spin of black holes [3]. As there is a bound on the rotation of such Kerr black holes this might be a place to start comparing with theoretical predictions. It has even been suggested [7] that this might lead to some evidence in favor of string theory. Clearly both the experimental and theoretical status of those ideas are premature but nonetheless they are tantalizing.

### Black holes are hot

On the theoretical side the most surprising discovery about black holes is that they behave thermodynamically. For example, when merging different black

holes the resulting horizon area will always be greater than the sum of the original horizons. This is similar to the second law of thermodynamics where entropy always has to increase. Of course just this rough 'similarity' is not very convincing and wouldn't be so striking if one couldn't also formulate the exact analogies of the first and zeroth law of thermodynamics for black holes. We will review this statement more precisely in the next chapter. Let it suffice here to say that the area of a black hole horizon has exactly the same properties as entropy.

Bekenstein was the first to suggest that this has to be more than just an analogy, but rather that to every black hole one should indeed associate an entropy [8, 9]. His most basic argument is simply that otherwise we would have to give up the second law of thermodynamics. This is due to the fact that e.g. a Schwarzschild black hole is uniquely determined by its mass. E.g. both an ox and a car of the same weight would be indistinguishable when collapsed to a black hole. So unless we associate to the black hole an entropy larger than any other object with the same amount of mass the second law of thermodynamics is invalidated by collapse into a black hole. It turns out that an entropy proportional to the horizon area does exactly that. As an example consider the entropy associated to some radiation confined inside its Schwarzschild sphere, but treated by standard thermodynamics, i.e. neglecting gravity. Rewriting some standard formulas in thermodynamics reveals that the entropy goes like ( $k_B = 1$ )

$$S_r \sim \left(m\sqrt{G_4}\right)^{\frac{3}{2}}. \quad (1.1)$$

General relativity tells us however the radiation is no longer radiation but should have collapsed to a black hole as it was inside its Schwarzschild radius. The entropy associated to this black hole is then

$$S_{bh} \sim \frac{A}{G_4} \sim m^2 G_4. \quad (1.2)$$

So we see that the entropy of the black hole is larger<sup>6</sup> than the entropy of the radiation before collapse, hence saving the second law of thermodynamics.

Still, all these arguments might not be completely convincing, especially as the numerical coefficient of proportionality between the area and the entropy remained unknown for a while. This changed by the famous calculation of Hawking [10]. Hawking studied the effect of the black hole on the quantization of matter in its neighborhood. Note that this is a semiclassical calculation in the sense that he assumed the gravitational background to be fixed and classical while he treated the test particles in this background quantummechanically. In this

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<sup>6</sup> Do note that the entropy of the radiation would be smaller if the total mass was of the order of the Planck mass. Such a black hole would however have a radius of a Planck length and, as we already argued, we know we can't trust general relativity anymore in this regime. Stated in reverse we could take this observation as additional evidence for a modification of general relativity at small enough distances.

setup there is thus no need for a theory of quantum gravity. The result of his calculation is that there is a continuous flux of outgoing radiation produced by the black hole. So quantummechanically there is a possibility to escape from a black hole! Physically the radiation can be understood as originating in pair production near the horizon. When by a vacuum fluctuation near the horizon a particle and its anti partner are produced, one of them can fall into the horizon while the other escapes to infinity, resulting effectively in radiation sent out by the black hole. Another equivalent point of view is to think of the radiation as particles that tunnel through the horizon. If one does the calculation it turns out that the radiation is purely thermal, i.e. the black hole radiates in a perfect black body spectrum<sup>7</sup>. It should not come as a big surprise that the entropy and temperature associated to this black radiator agree exactly with the ones derived for the black hole from the analogy between the laws of black hole mechanics and that of thermodynamics. The famous formulae obtained by Hawking are in case of a Schwarzschild black hole:

$$S = \frac{4\pi m^2 G_4}{\hbar}, \quad T = \frac{\hbar}{8\pi m G_4}. \quad (1.3)$$

Here we reinstated the factors of  $\hbar$  to show that these effects are inherently quantummechanical.

Again, let us appreciate these formulas a bit more by providing some numbers to compare. For a black hole with the mass of the Sun the Hawking temperature is  $T \approx 6 \times 10^{-8}$  K, in other words, extremely cold. This holds of course even more so for the supermassive black holes which are still a few million times colder. We can also ask the opposite question; what would be the mass of a black hole glowing so we can see it with our eyes. The temperature of e.g. the Sun's surface, 5778 Kelvin, corresponds to a black hole of mass  $m \approx 2 \times 10^{19}$  kg, this is a bit less than a thousandth of the mass of the moon. Such a black hole would have a Schwarzschild radius of a few nanometer. The fact that this roughly corresponds to the wavelength of the peak of the black body spectrum is no coincidence. Using Wiens law one can calculate that the ratio of the peak wavelength emitted by a black hole and its Schwarzschild radius is a number of order 1, more precisely

$$\frac{\lambda_{max}}{r_S} = \frac{8\pi^2}{x} \approx 16, \quad (1.4)$$

where  $x = 5(1 - e^x)$ . This cute little result is quite satisfactory in that it seems in accord with the intuition that particles with a de Broglie wavelength of at least the Schwarzschild radius are the ones that mainly escape.

<sup>7</sup> Note that although at its horizon a black hole emits radiation as a perfect black body, the spectrum as measured at infinity is slightly different. This is due to the gravitational potential outside the horizon. Due to this potential the radiation loses energy when traveling to infinity and this shifts the frequency of the emitted radiation. This energy loss is furthermore frequency dependent, so the spectrum that is measured at spatial infinity has received a frequency dependent shift. This effect can be accounted for by introducing a greybody factor.

Because their temperature is inversely proportional to their size the smaller the black hole the more it radiates. One can calculate that the life time of a black hole is proportional to the third power of its mass:

$$t \sim m^3 G_4^2. \quad (1.5)$$

So for a solar mass black hole the evaporation time is  $t \approx 2 \times 10^{67}$  yr, incredibly much longer than the age of the universe. The mass of a black hole with the lifetime of a second is roughly 200 ton.

The fact that black holes have entropy and temperature leads to some natural questions and puzzles. First there is the so called information paradox. According to Hawkings semiclassical analysis, two black holes, one formed out of ox and the other out of car, evaporate in a purely thermal way. So after evaporation both the ox and the car have transformed into indistinguishable radiation. This is of course in contradiction with conservation of information, or more mathematically, the unitary evolution we know from quantummechanics. So either quantummechanics has to be modified or a full theory of quantum gravity has to modify Hawkings result. Although we wont discuss the information paradox further let us mention that quantum gravity as it follows from string theory favors the latter explanation [11].

The entropy associated to black holes leads to a second puzzle. As Boltzmann taught us, entropy in thermodynamics is directly related to the number of degenerate microstates corresponding to a given macrostate. So it is only natural to wonder what these microstates of a given black hole are. It seems impossible that these microstates can have a description in general relativity as there exist theorems that e.g. the Schwarzschild solution is the only static solution with spherical symmetry. The fact that the entropy has a quantum origin ( $S = \frac{A}{4\hbar G_4}$ ) hints that the microstates should probably be found in a proper description of quantum gravity. As we will discuss later string theory succeeds in this challenge, which can be interpreted as a nontrivial test on any proposal for a theory of quantum gravity.

### 1.3 String theory: the theory of a lot

Although there are some that hope that string theory might provide the theory of everything, this is optimistic and certainly premature. It is fair to say though that by today string theory really is the theory of a lot. It started out as a proposal to describe the strong force, but was discarded as that. It was then realized that quantized strings describe quantized spin two fields and thus provide a theory for quantum gravity. But apart from gravity, strings also describe more standard spin one gaugefields, scalars and fermions. So really quantized strings contain all the different kinds of matter observed today. Therefore it might be possible that string theory is a fundamental description of nature underlying all of the physics we observe today. More recently it became clear that string theory and gauge theory are related on an additional level, there exists a dual

description of strings on a  $d$ -dimensional space-time as a gauge theory (without gravity) on the  $(d - 1)$ -dimensional boundary.

### A theory of strings

As its name suggests string theory [12, 13, 14, 15, 16] started out as a theory of quantized relativistic strings. Classically these strings are two dimensional Lorentzian surfaces that are embedded in a fixed background space-time. Their dynamics is given by the so called Nambu-Goto action which just implies that these strings are volume minimizing. There is a single dimensionful parameter in the theory, the tension of the string

$$T = \frac{1}{\alpha'} = \frac{1}{l_s^2}, \quad (1.6)$$

which measures 'how badly' the string wants to minimize its worldvolume. Upon quantization of this action one finds a spectrum of particles organized in an infinite tower of masses that go in steps of  $\frac{1}{l_s}$ . Clearly  $l_s$ , which can be thought of as the typical length of the strings, has to be very small, as we know to a very high accuracy that the elementary particles are pointlike. A conservative estimate would be to take the string length around the Planck length. Given the small size of the strings, already the first level of the tower, with masses of order  $\frac{1}{l_s}$ , is incredibly massive and can thus be effectively neglected in a first approximation. In the purely bosonic setup we sketched above there is actually an excitation with a negative mass squared, which signals an instability of the theory. This tachyon is removed from the spectrum when adding fermions to the worldvolume of the string and demanding supersymmetry, a symmetry between the bosonic and fermionic sector of the theory. At the massless level the excitations arrange themselves naturally in scalar, fermionic and gauge field degrees of freedom of the background space-time. A big surprise is that in the case of closed strings (i.e. strings with periodic boundary conditions) the spectrum also contains the degrees of freedom of a massless spin two field on the background. So we see that the quantization of closed strings unavoidably leads to the graviton excitations we discussed above.

Interactions are introduced in the theory by allowing different strings to split or join. In this way one gets a picture very reminiscent of Feynman diagrams. There are a few essential differences however. First of all there is only a single 'type' of string and secondly the coupling constant,  $g_s$ , that measures the loop order, is actually a dynamical field of the theory. It is related to a massless scalar string excitation called the dilaton:  $g_s = e^\phi$ . This last point is important as it implies string theory is free of adjustable dimensionless parameters as the dilaton has to satisfy the dynamics of the theory. Another essential difference with the scattering formalism in ordinary quantum field theory is that the loop expansion of string theory is finite<sup>8</sup> order by order in  $g_s$ . This can intuitively be related

<sup>8</sup> We refer here to string perturbation theory around Minkowski space. String theory on more general backgrounds is less understood.

to the absence of pointlike vertices. This underlying perturbative scattering of strings now translates to the scattering of the massless string excitations that are fields in space-time. At low energies these scattering amplitudes give rise to interactions that can effectively be described by what is known as a supergravity. It is essential that this supergravity contains Einsteins general relativity.

The upshot of this whole story is that from string theory we recover the equations of motion of general relativity, but now from a theory that describes the gravitational excitations consistently in a quantum mechanical way, as the string theory amplitudes are finite while those of the naive quantumfield theory description of gravity are nonrenormalizable.

### A theory of branes

More recently it was realized that strings are really only one of the ingredients of the theory. It turns out that a more careful description of open strings shows the existence of higher (and lower) dimensional generalizations of strings. Just as a string is a one dimensional generalization of a point particle one can think of membranes and even higher dimensional generalizations. All these different objects are conveniently denoted by the common name  $p$ -brane.

A specific kind of  $p$ -brane are the D-branes, where the D stands for Dirichlet. At  $g_s = 0$ , i.e. no string loop interactions, these D-branes are rigid hypersurfaces in space-time on which the end points of the open strings are defined to stick. When  $g_s$  becomes non zero however the D-branes themselves become dynamic through the strings that end on them. They behave as very heavy objects with a mass of order  $\frac{1}{g_s}$ . At large  $g_s$ , where our perturbative theory of strings breaks down, the D-branes become light and should be the right perturbative degrees of freedom. So we see that strings are only an appropriate description of the 'full' theory in a specific perturbative regime. In another regime the theory, that was historically baptized string theory, could more naturally be called brane theory.

### A theory of M

The above example is only a single instance of a feature named duality. When two theories that seemingly describe very different physics, e.g. the theory of strings and the theory of branes, turn out to describe the same physics but in different regimes one calls them dual.

When superstring theory was first written down it was found that there are five different ways to do so, all theories which are defined on a ten dimensional space-time. These five different theories go under the names of type I, type IIA, type IIB, heterotic  $SO(32)$  and heterotic  $E_8 \times E_8$ . In the last decade of the previous century it was discovered that there is a whole web of dualities between these five different theories and an additional theory living in 11 space-time dimensions, denoted M-theory. Although we are far from a mathematical proof, all these dualities seem to suggest that there is a single underlying theory which in various regimes is described by the different ten dimensional

string theories or the eleven dimensional M-theory. So again broadening our definition: nowadays 'string theory' is the name referring to a puzzle of which many, but far from all, pieces are known. The hopes are that all these pieces will one day nicely fit together to form a single fundamental theory.

### A theory of our world?

Despite many theoretical successes and 'giant leaps' it is still unclear if string theory really is the theory describing our world. That after almost forty years<sup>9</sup> since the first calculations on string theory still not a single experimental prediction has been made, shows that clearly the connection to the real world is the Achilles heel of the theory.

At first thought, the above claims of 'unique theory', 'single dimensionful parameter', 'no freely adjustable dimensionless parameters' seem to be in contradiction with a theory that doesn't make unique predictions. At second thought this is however not so strange and maybe even what one should expect. Consider for example the 'unique' equation  $z^{10} = 1$ , this equation has no freely adjustable parameters, however it still has 10 equivalent solutions. String theory is faced with a similar problem. Its equations give rise to an incredible amount of solutions, often called the landscape of string theory vacua [18], and furthermore it is hard to identify if one (or some) have the exact properties of our universe. The good point is that most solutions have the qualitative features of our universe: gravity, gaugefields etc. The bad point is that the simplest solutions can be immediately ruled out quantitatively.

The main reason why it is so difficult to start searching for a solution corresponding to our world is that string theory predicts the number of space-time dimensions to be 10. This can only be in agreement with our observations if 6 of these dimensions are so small we didn't notice them yet in experiment. But although we have a lot of experimental constraints on our four dimensional world there is a great freedom of choice in the internal dimensions that correspond to rough features of our universe as we observe it.

Faced with this multitude of possible solutions there are a few roads along which one could hope to make progress. Many aspects of string theory are also theoretically poorly understood, especially the non perturbative formulation of the theory. So there might still be some undiscovered physics hidden in the theory that dynamically selects a vacuum. In other words: we might be missing some subtleties of the theory that imply that of the many vacua known today, only a single one (or a few) are proper vacua of the theory, see e.g. [19].

A second line of thought could be that maybe we need to add a new physical principle to string theory that would then select a unique vacuum. This might sound interesting and promising but for the moment it is unclear what such a principle could be.

The third road, which gains more and more support, is that all these different

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<sup>9</sup> E.g. Veneziano published his famous amplitude [17] in 1968.

vacua really are all valid solutions to the theory. In this case we might be forced to reconsider the quantities we want to calculate and the type of predictions we want to make. One could for example try to make statistical statements instead of studying solution by solution [20].

Not many have the hope of seeing direct evidence for strings in the near future, although there are some scenarios in which direct evidence of strings, such as extra dimensions, could be found at LHC, the new generation particle collider that is about to be turned on. Most would however agree that there are no natural reasons to expect these scenarios to be realized. There are some discoveries however, mainly that of supersymmetry, that would not be direct evidence but nonetheless be a strong encouragement for string theory. Furthermore there are some hopes of evidence for the theory from cosmology.

Apart from strings as a fundamental description of nature it could still have its uses in the real world as a dual description of strong dynamics. Actually AdS/CFT is for the moment theory that describes best the heavy ion collisions at RHIC, see [21] for a review.

### A theory of black holes

An important piece of evidence that string theory is the correct theory of quantum gravity comes from the theoretical study of black holes. After years of research into the possible microstates underlying black hole entropy, Strominger and Vafa finally succeeded in such a calculation in [22]. They considered a five dimensional black hole that in string theory can be described at small  $g_s$  as composed out of D-branes. Counting all possible D-branes states with the same conserved charges as the black hole, they found a number that in the limit of large charge asymptotes exactly to the exponential of the entropy.

Seemingly essential in their calculation is that the black hole is supersymmetric, or at least extremal. This implies that it has temperature zero and thus doesn't radiate. The system is thus stable and furthermore the entropy is purely configurational. There has been some work where people generalized the calculation to near extremal black holes and where also a microscopic interpretation of the Hawking radiation could be given see [23, 24] for some reviews. But for arbitrary black holes, let alone the Schwarzschild black hole, no derivations have been found to date.

Another feature of the Strominger-Vafa approach to black hole entropy is that the states are counted in a regime where gravity can be neglected. Due to properties of supersymmetry the number of states<sup>10</sup> is guaranteed to remain the same in the regime of stronger gravity. The downside of this approach is however that it doesn't lead to a picture or understanding of these microstates in the regime where we have a black hole. More recently another approach, dubbed the *fuzzball proposal* [25, 26] has been developed. It turns out that at least in supergravity there do exist solutions that carry the same charges and

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<sup>10</sup> More precisely the index.

symmetries as the black hole but don't have a horizon. There have been some partial successes and the hope is that quantizing the set of such solutions leads to the entropy of the corresponding black hole.

## 1.4 Overview of the thesis

The subject of this thesis will be a variety of aspects of multicentered black holes. In case one extends general relativity in a supersymmetric fashion to so called  $\mathcal{N} = 2$  supergravity, it is possible to have bound states of black holes. These multicentered black holes consist of single black holes that are separated from each other at a fixed distance. They were first described in [27, 28] and have since lead to a number of both answers and questions. First of all they exhibit very rich physics and are thus certainly of interest on their own. But they also play an important role in the larger picture.

In the microscopic regime at small  $g_s$  the BPS states of the  $\mathcal{N} = 2$  theory are interpreted as D-brane states, in the macroscopic supergravity regime things are a little more subtle though. For a given charge, some BPS states are realized as a single black hole and some as multicentered black holes, there are even states that only exist as multicentered black holes in the supergravity regime. This leads immediately to the question how the microscopic counting of black hole states, that we were so euphoric about above, can give the correct result. How do we know that the states we count microscopically correspond macroscopically to a single black hole and not to some complicated multicentered solution? In most cases this problem is evaded though, as the number of microstates corresponding to multicentered black holes is exponentially subleading compared to the microstates corresponding to the single black hole of that charge. As the entropy is derived taking only the leading contributions into account the 'multicentered microstates' are of no relevance. But there exists specific regions in charge space however where most microscopic BPS states correspond to a multicentered configuration and the roles are actually reversed, a fact dubbed the entropy enigma in [29, 30]. The entropy enigma seems add odds with many black hole lore and was demystified more recently in the authors work [31] in collaboration with de Boer, Denef, El-Showk and Messamah. As we will discuss in much more depth in chapter 4 the entropy enigma corresponds to a well known thermodynamic instability of black holes in AdS space, but in disguise. Apart from the entropy enigma [31] also discusses some other puzzles that appear when discussing the multicenter solutions in the context of AdS/CFT.

Multicentered black holes also seem to play a role in the understanding of black hole microstates in the supergravity regime. This subject has not fully been explored and is still very much in progress. There has been quite some work on the five dimensional version of the multicentered black holes and their possible role in the fuzzball program, see e.g. [32] for a starting reference. Also in a more recent proposal for black hole microstates initiated by [33] multicentered black holes seem to occupy an essential role. In a collaboration with Denef,

Gaiotto, Strominger and Yin, the author showed in [34] that the quantization of a certain multicentered configuration may lead to an explanation of the entropy of a single center black hole with the same total charge. This is discussed in depth in chapter 5. More generally, so called scaling multicentered black holes, bound black holes that are continuously connected to a single center black holes, seem to play an essential role in the understanding of black hole microstates but many questions remain unanswered for the moment.

Let us also mention the work of [29] to point at the relevance of multicentered black holes, here those black hole bound states were essential in derivation of the OSV conjecture [35].

### Overview of the chapters to come

In chapter 2 we spend some time introducing the necessary tools to discuss multicentered black holes. This chapter is mainly a review of well established facts. Before introducing supersymmetric black holes in supergravity we first discuss shortly more standard black holes. Then we present the  $\mathcal{N} = 2$  supergravity theories that are the natural habitat of the multicentered black holes. First we show how this  $\mathcal{N} = 2$  theory is related to string theory and how black hole solutions are related to the D-branes of that string theory. The second part of that chapter is devoted to a detailed study of the supersymmetric black holes in  $\mathcal{N} = 2$  supergravity. We present the solutions and discuss the attractor mechanism which will be of relevance when we discuss the multicentered generalizations. In section 2.5 we discuss two examples of interest, the D4D2D0 black hole and the purely fluxed D6 black hole. We end chapter 2 by quickly reviewing the microscopic derivation of the D4D0 black hole entropy as in [36].

The third chapter of the thesis discusses the general properties of multicentered black holes. First we gain some insight in the dynamics that make multicentered black holes possible. This is done by studying a test particle in a single center black hole background. As we will see in section 3.1 there are bound states of this particle at a fixed distance to the black hole. Considering that same system but now taking also the full effects of the 'test' particle into account then leads to the multicentered black holes. Using the BPS equations we derive the supergravity solutions that describe the multicentered black holes. After that we are in a position to discuss the physical properties of these black hole bound states. We will review their degrees of freedom, that are captured by a so called solution space, and introduce the attractor flow conjecture, probably the best way to classify the multicentered solutions. In section 3.4 we present some original work presented in [31]: two examples that illustrate important properties, only appreciated fully recently. In the first example we illustrate the difference between marginal and threshold stability and in the second example we show the existence of bound states of centers without D6-charge, which will have important consequences in chapter 4. We end chapter 3 by discussing in some more detail the microscopic picture of the bound black holes.

Chapter 4 contains the authors work reported in [31]. Here we first define a

decoupling limit for the multicentered black holes which places them in asymptotic  $\text{AdS}_3 \times S^2$  space. We calculate their boundary charges as a first step towards an interpretation in the dual CFT. In section 4.3 we present some examples. A first interesting one is the NS vacuum of the theory, global  $\text{AdS}_3$ , which turns out to be the decoupling limit of the bound state of a  $\text{D6}\overline{\text{D6}}$  pair. In the second example we review the entropy enigma in AdS space. This leads us to the next section where we explain that the entropy enigma is nothing but the well known instability of AdS black holes towards localization in the compact dimensions [37]. In the last section of this chapter we then further discuss the many implications the existence of multicentered solutions in AdS space has on the CFT description. There appear some puzzles and we make a first attempt at resolving some of those by a more careful analysis of the MSW string [36].

In the last chapter we first review the alternative derivation of the D4D0 entropy by [33] and the different ingredients needed for it. As we will discuss the microstates in this setup take the form of D2-branes wrapping the D4 black hole horizon. This review combines many different results and is presented in the authors conventions making it consistent with the rest of the thesis. In the second part of chapter 5 we present the authors work [34] where a refined version of [33] is studied by replacing the D4 'core' by a  $\text{D6}\overline{\text{D6}}$  pair. As we will discuss there, this scaling multicenter solution has some advantages over the original single center D4 black hole for the D2-analysis.

Finally there are three appendices. The first shortly points out our conventions and specifically the relations between these conventions in IIA respectively M-theory. The second appendix contains some technical details on the calculation of boundary charges in the presence of Chern-Simons terms, needed in chapter 4. Appendix C gives a summary of the thesis in Dutch.

**Note** Apart from the work presented in this thesis the author has also contributed to research in string theory not related to the multicentered black holes. In collaboration with Lucca Martucci, Jan Rosseel and Antoine Van Proeyen he investigated fermionic actions for D-branes in [38]. Together with Eric Bergshoeff, Jelle Hartong, André Ploegh and Jan Rosseel a relation between pseudo-supersymmetry and  $\ast$ -theories of supergravity was developed [39].



# Chapter 2

## Supersymmetric black holes

In this first of the more technical chapters we will set the stage for the rest of the thesis. During its history, string theory has grown from the basic idea of quantizing the oscillations of a string into a large and beautiful independent field of research. In its expansion it has borrowed but also produced insights in gravity, field theory and all kinds of advanced mathematics.

In this thesis we will restrict our attention to the rather tiny subtopic of multicentered black holes in supergravity/string theory but still there are many aspects of all those different ingredients of string theory that will play a role. As such the idea to give a completely self contained introduction to the necessary background, let alone in a single chapter, is unrealistic. Still we will present a rather large portion of essential tools and concepts all the time trying to balance completeness versus relevance. The cost might be a certain lack of precision now and then, for further information the reader is guided to the original references and the many excellent review papers and books. It is the authors hope that this introductory chapter will give a good idea to the novice of the different ingredients involved and serves as a reminder introducing our conventions to the expert.

Section 2.1 will very briefly review some basic aspects of black holes and their thermodynamics. The goal of this section is to point out some properties of the simplest black holes so they can later be compared/contrasted to the more complicated supersymmetric black holes.

In the second section we provide some of the necessary supergravity and special geometry background needed to discuss supersymmetric black holes.

The core of this introductory chapter is section 2.3. Here the relevant supersymmetric black holes will be introduced and described. This is reviewed in considerable detail, since it furnishes the basis of understanding the multicentered black holes of the next chapter, as well conceptually as technically.

In the fourth section we make a short digression on a different but equivalent

viewpoint of the four dimensional black holes. Namely, we will lift them to five dimensions, a technique that will be useful and in certain cases even necessary in the next chapters.

The best insight is often most easily gained by studying some well chosen examples. So in section 2.5 we discuss two examples illustrating some important features of the theory discussed before.

Finally we come to one of the nicest results on four dimensional supersymmetric black holes: a microscopic statistical derivation of their entropy. Section 2.6 will review this derivation following the approach of Maldacena-Strominger-Witten (MSW). This also gives the necessary background when in chapter 4 we will discuss more subtle aspects of their model. In this spirit we comment shortly on the more modern interpretation of MSW as an example of AdS<sub>3</sub>/CFT<sub>2</sub>.

## 2.1 Introduction to Black Holes

Black holes have been one of the most intriguing physical phenomena since their earliest description in general relativity. Although first discarded by some as mere theoretical aberrations their real physical existence is by now supported by various experimental data. In this first section we will quickly review some basic notions of the simplest black holes in four dimensions and will remind the reader of their thermodynamic behaviour.

### The Schwarzschild black hole

The unique spherically symmetric solution to the Einstein equations in vacuum was first written down by Schwarzschild and corresponds to the following metric:

$$ds^2 = - \left( 1 - \frac{2mG_4}{r} \right) dt^2 + \left( 1 - \frac{2mG_4}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (2.1)$$

At large enough  $r$  the above metric approaches that of flat Minkowski space arbitrarily close and is therefore called *asymptotically flat*. Near spatial infinity the Newtonian approximation thus becomes accurate, allowing the identification of  $G_4$  with Newton's constant and  $m$  with the gravitational mass. This shows the metric above can be interpreted as the gravitational field sourced by a point particle of mass  $m$  at the 'origin'  $r = 0$ .

Not unlike the electric field of a point source also in this gravitational analog the potential blows up near the source at  $r = 0$ . The divergence of the metric (2.1) at  $r = 0$  corresponds to a real physical singularity, as can be verified by calculating the following curvature invariant

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G_4^2 m^2}{r^6}. \quad (2.2)$$

In the form above the Schwarzschild metric (2.1) reveals another special value of the radial coordinate: at  $r = r_S = 2mG_4$  the  $g_{rr}$  term blows up. Although

this is a mere coordinate singularity that can be removed by using appropriate coordinates,  $r_S$  has a deep physical significance. On the sphere defined by  $t = \text{cst}$ ,  $r = r_S$  the timelike Killing vector  $\partial_t$  becomes null. Such a surface is called an *event horizon* and it encloses a region of space-time from where no timelike curves lead all the way to future infinity. In more physical words, the event horizon is a physical barrier that can only be crossed in one direction, once inside there is no way outside. As no light rays can escape a region enclosed by an event horizon it will appear completely black for an outside observer, hence space-times with such a horizon are generically called *black holes*.

### The Kerr-Newman black hole

The Schwarzschild solution (2.1) is the simplest example of black hole, as it is static, spherically symmetric and purely gravitational. A more general family of black holes is given by the so called Kerr-Newman metric [40]

$$ds^2 = -\frac{\Delta}{\Lambda} (dt - a \sin^2 \theta d\phi)^2 + \frac{\Lambda}{\Delta} dr^2 + \Lambda d\theta^2 + \frac{\sin^2 \theta}{\Lambda} ((r^2 + a^2)d\phi - adt)^2, \quad (2.3)$$

where we have defined the functions

$$\Delta = r^2 - 2mG_4 r + a^2 + G_4 q^2, \quad \Lambda = r^2 + a^2 \cos^2 \theta. \quad (2.4)$$

One can easily verify that in case we choose  $a = q = 0$  the metric (2.3) reduces to the Schwarzschild solution (2.1).

When  $a \neq 0$  the solution is no longer static;  $\partial_t$  remains a Killing vector however and thus the solution is still stationary. This stationarity can be interpreted as the space-time rotating with a constant angular velocity. This can be made concrete by an asymptotic expansion in  $r$  where the angular momentum  $J = am$  of the solution is identified. So in case  $q = 0$  the metric (2.3) describes the gravitational field of a spinning massive point particle.

If one takes  $q \neq 0$  the metric above is no longer a solution of the vacuum Einstein equations. Consider however an electro-magnetic field of the form

$$\begin{aligned} \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu &= \frac{q}{\Lambda^2} (r^2 - a^2 \cos^2 \theta) dr \wedge (dt - a \sin^2 \theta d\phi) \\ &\quad + \frac{q\sqrt{G_4}}{\Lambda^2} ar \sin 2\theta d\theta \wedge ((r^2 + a^2)d\phi - adt). \end{aligned} \quad (2.5)$$

One can check that this Maxwell field combined with the Kerr-Newman metric (2.3) solves the equations of motion of gravity minimally coupled to electro-magnetism, which can be obtained from the Einstein-Maxwell Lagrangian

$$S = \frac{1}{16\pi G_4} \int \sqrt{-g} \mathcal{R} - \frac{1}{16\pi} \int F \wedge \star F. \quad (2.6)$$

The electric charge is readily identified with  $q$  by calculating

$$q = \frac{1}{4\pi} \int \star F. \quad (2.7)$$

Thus we conclude that the most generic Kerr-Newman solution (2.3, 2.5) describes a charged spinning black hole.

The metric (2.3) has a curvature singularity at  $\Lambda = 0$ . In case  $G_4 m^2 < q^2 + \frac{a^2}{G_4}$  this singularity is in causal contact with future infinity. Such a *naked singularity* is deemed unphysical by a principle called *cosmic censorship* and solutions corresponding to these values for the parameters are discarded.

The singularity of the metric (2.3) is shielded by two horizons in the conjugate part of parameter space, where

$$G_4 m^2 \geq q^2 + \frac{a^2}{G_4}. \quad (2.8)$$

For parameters satisfying this bound the singularity is causally disconnected from future infinity and the solution makes physical sense. The two horizons correspond to the radii

$$r_{\pm} = \left( m \pm \sqrt{m^2 - \frac{q^2}{G_4} - \frac{a^2}{G_4^2}} \right) G_4. \quad (2.9)$$

The special case where both horizons coincide and which corresponds to the lower bound for the mass is called *extremal*:

$$m^2 = \frac{q^2}{G_4} + \frac{a^2}{G_4^2}. \quad (2.10)$$

As we will discuss later in this chapter, in supersymmetric generalizations of the Einstein-Maxwell theory (2.6) black holes always satisfy a bound analogous to (2.9). Furthermore, those black holes that preserve some supersymmetry of the theory saturate this bound and are thus closely related to the extremal black holes of Einstein-Maxwell theory.

### Generic black holes

The Kerr-Newman solution (2.3, 2.5) is the unique stationary axially symmetric solution to the action (2.6) [40]. A generic black hole somewhere in our universe will however not be stationary at all. Rather it will be a very dynamical object changing its shape quite drastically in time. First of all it will not have existed for eternity but it will have formed out of the collapse of a certain amount of matter inside its Schwarzschild radius. It will then keep on changing as it interacts with its environment and accumulates other matter floating around. The metric describing such a dynamic black hole will of course be much more complicated than the Kerr-Newman solution as it won't exhibit any of the symmetries the metric (2.3) has. Clearly an analytic description of these 'realistic' black holes is too much to hope for.

Imagine however that at a certain point such a generic dynamic black hole has absorbed all matter close to it and can approximately be described as floating

in an empty surrounding. If one lets this black hole evolve long enough it will radiate away all irregularities of its horizon through gravitational waves until it settles down into the stationary spherically symmetric Kerr-Newman form (2.3) [40].

It is quite remarkable that in 4 dimensions when only considering gravity and Maxwell's electro-magnetism as long range forces, the final equilibrium state of any black hole is completely determined by just three parameters: mass  $m$ , angular momentum  $J$  and charge  $q$ . This fact that finally all black holes look completely similar although they could be formed in totally different processes is often stated with the catchy phrase '*black holes have no hair*'.

### Black hole thermodynamics

With the knowledge that all black holes evolve to the stationary Kerr-Newman form one can start a series of thought experiments. Imagine starting from a Kerr-Newman black hole characterized by  $(m, J, q)$ . One can now toss some test particles, carrying some charge and angular momentum, into the horizon. After this perturbation the original Kerr-Newman black hole will evolve to a new one given by  $(m + \delta m, J + \delta J, q + \delta q)$ . From studying this process in detail, see e.g. [41], it follows that the perturbations have to satisfy the following equation

$$dm = \Omega dJ + \Phi dq + T dS, \quad (2.11)$$

where  $\Omega = \frac{a}{(r_+^2 + a^2)}$  is the angular velocity of the black hole and  $\Phi = \frac{qr_+}{(r_+^2 + a^2)}$  can be interpreted as its electric potential.

The quantity  $S$  turns out to be proportional to the area  $A$  of the external horizon and  $T$  proportional to the gravitational potential at that horizon, often called the *surface gravity*  $\kappa$ . Furthermore the changes of the black hole always obey

$$\delta S \geq 0. \quad (2.12)$$

Another fact one can prove [40, 41] is that for a generic black hole

$$\kappa \text{ is constant along the horizon.} \quad (2.13)$$

It is now a striking observation that the equations (2.11), (2.12) and (2.13) are exact analogies of respectively the first, second and zeroth law of thermodynamics. Under this parallel  $T$  and  $S$  correspond to respectively the temperature and entropy of the thermodynamic system. Bekenstein was the first who argued that this correspondence is not a mere coincidence but that  $S$  and  $T$  really *are* the temperature and entropy of the black hole [8, 9]. Not much later Hawking consolidated this fact by deriving the precise numerical proportionality factors between  $S$  and  $A$  and between  $T$  and  $\kappa$  in [10]. His celebrated formulas for what are now called the Bekenstein-Hawking entropy and the Hawking temperature are:

$$S = \frac{A}{4G_4\hbar}, \quad T = \frac{\kappa\hbar}{2\pi}. \quad (2.14)$$

Here we explicitly wrote the factors of  $\hbar = 1$  to emphasize the role of quantum mechanics in the derivation. Although all our previous discussions in this section were completely within the framework of classical general relativity, Hawking's calculation relies on semi-classical arguments. He considered the quantum properties of particles on a classical black hole background. Doing so he realized that black holes emit the same thermal radiation as a black body of temperature  $T$  would do. Intuitively one can understand this effect best by imagining particle-anti-particle pair creation near the horizon. It can happen that one of the two particles created in such a vacuum fluctuation will disappear behind the horizon while its partner can escape to infinity. Doing a detailed calculation shows that this effect gives rise to a steady outgoing flux of perfect blackbody radiation.

### The search for black hole microstates

The result of Hawking puts the thermodynamic properties of black holes on rather firm footing. It furthermore naturally leads to an interesting question, that of a statistical mechanical interpretation of this thermodynamic behaviour. As Boltzmann taught us entropy is naturally interpreted as a degeneracy of  $N$  states such that  $S \sim \log N$ . So what are these different states in case of the black hole? A first guess, different classical black hole solutions all carrying the same 'macroscopic charges'  $(m, q, a)$ , is discarded by the no hair theorem. As Hawking's calculation indicates, the microstates have probably to be found in quantum gravity since the temperature and entropy are inherently quantum mechanical. For supersymmetric black holes the entropy was calculated by counting microstates in string theory [22]. This has later been generalized to some extremal and nearly extremal black holes as well, see [23] for a review. In this thesis we will review the counting for a set of 4 dimensional supersymmetric black holes in section 2.6, but first we will introduce those supersymmetric black holes and the supergravities they are solutions to.

## 2.2 Embedding $\mathcal{N} = 2$ supergravity in string theory

As was discussed in the introduction a fruitful way of progress in black hole physics has been the study of supersymmetric black holes in the context of string theory. Supersymmetry is a symmetry between bosons and fermions that brings a lot of analytic control, see e.g. [42] for a detailed introduction. A gauged version of this symmetry leads to generalizations of general relativity called supergravity, that apart from gravity also contains fermions, gaugefields and often scalar fields. Many supergravities can be understood as low energy descriptions of string theory. Sadly a detailed review of string theory falls outside the scope of this thesis, for the interested reader there exist by now many excellent textbooks [12, 13, 14, 15, 16].

There are many known black objects in various corners of the string landscape and among these many are known to be connected by dualities. Over the years a lot of these settings and objects have been explored in detail and this has led to new insights in these systems themselves but also in many properties shared by black objects in general. The setting we choose to continue this exploration is that of 4 dimensional  $\mathcal{N} = 2$  supergravity, which we will discuss in some detail in this section, and in this thesis we will furthermore restrict our attention to black holes that preserve half the supersymmetry of this theory.

It has been known for a while now that  $\mathcal{N} = 2$  supergravities can be naturally embedded in string theory, see [43] for a review and further reference. It turns out that such an embedding is actually not unique as these theories can be connected to both Type II A and Type IIB supergravity in ten dimensions. As we will review in this section a great class of  $\mathcal{N} = 2$  theories in 4-dimensions can be seen as the low energy description of ten dimensional Type IIA supergravity when one assumes six of those dimensions to be a small compact manifold that has the property that it is Calabi-Yau.

It turns out that the 4 dimensional theory describing Type IIA compactified on a Calabi-Yau  $X$  is the same as this for IIB compactified on the so called *mirror* Calabi-Yau  $\tilde{X}$ . These two ways to describe  $\mathcal{N} = 2$  supergravity in 4 dimensions is yet another example of a duality, and it goes under the name of *mirror symmetry*, see [44] for a pedagogical review. The freedom of choosing between different pictures of a single problem, such that certain aspects that are very complicated in one picture are much more natural in the other, has led to many new insights, both in physics and mathematics [44].

Given these two possible ways of embedding  $\mathcal{N} = 2$  supergravity in ten dimensions we are free to choose the one that suits our purposes best. In this thesis we will choose to work in the IIA picture as it is the most natural in much of the following chapters where we will make the link with M-theory, see section 2.4. The special geometry of the vector multiplet, a main focus of this chapter, finds its most natural description in the IIB frame. For consistency and to not have to dwell too much on the intricacies of mirror symmetry we will however also in this chapter work in the IIA frame. The cost is only a minor loss of intuition that doesn't outweigh the benefit of having all the chapters in a single frame.

**Note** This section might be rather abstract as it discusses the general theory and framework we will need later on. The reader that has some difficulties digesting these sections might want to jump straight to section 2.3 and then later return to the present section to look up certain notations and concepts.

### 2.2.1 Type IIA supergravity on a Calabi-Yau manifold

We will first quickly review Type IIA supergravity and the concept of compactification separately. Then we will argue for the need of the internal manifold to be Calabi-Yau and remind the reader of the main properties of these manifolds.

Finally we will connect these concepts and give some more details on the compactification of Type IIA on a Calabi-Yau manifold. In the next subsection we will discuss the special geometry behind this compactification.

### Type IIA supergravity

Type IIA supergravity is one of the two possible supergravity theories with  $\mathcal{N} = 2$  supersymmetries in ten dimensions and it is the low energy limit of Type IIA string theory [15]. Type IIA differs from type IIB in that the two supersymmetries have opposite chirality where in type IIB they have the same chirality. Note that in ten dimensions a real irreducible spinor has 16 components and so the type II theories have 32 conserved supercharges.

Related to the difference in chirality of the fermionic sector, Type IIA also describes different bosonic fields than Type IIB. The bosonic fields they share form the so called Neveu-Schwarz (NS) sector. This NS sector consists of a spin 2 graviton  $g_{\mu\nu}$ , a spin 1 abelian two form gaugefield  $B_2$  and a scalar  $\phi$  called the dilaton. This dilaton  $\phi$  can be interpreted as an effective space-time dependent string coupling via  $g_s^{\text{eff}}(x) = e^{\phi(x)} g_s$ . There is also a Ramond-Ramond (RR) sector in the bosonic part of the theory, in the case of IIA it consists of spin 1 odd form gauge potentials  $C_1$  and  $C_3$ . Since the theory is supersymmetric there should be an equal amount of fermionic degrees of freedom, those are given by two opposite chirality spin 3/2 gravitinos  $\psi_\mu^\pm$  and similarly two spin 1/2 dilatinos  $\lambda^\pm$ . Up to second order in derivatives the action is completely constrained by supersymmetry and the part without fermions is given by

$$S_{\text{IIA}} = \frac{2\pi}{g_s^2 l_s^8} \int \sqrt{-g} \left( \mathcal{R} - \frac{1}{2} (\nabla\phi)^2 \right) - \frac{\pi}{g_s^2 l_s^4} \int e^{-\phi} H \wedge \star H \quad (2.15)$$

$$- \frac{\pi}{l_s^6} \int e^{\frac{3}{2}\phi} F_2 \wedge \star F_2 - \frac{\pi}{l_s^2} \int F_4 \wedge \star F_4 + \pi \int C_3 \wedge dC_3 \wedge H.$$

Here the field strengths are defined in terms of the potentials introduced above as

$$F_2 = dC_1, \quad H = dB_2, \quad F_4 = dC_3 - H \wedge C_1. \quad (2.16)$$

Note that we presented the action (2.15) in the *Einstein-frame* as this is the form most suited for compactification since especially in four dimensions we are used to the Einstein-Hilbert form of the gravitational action. From the string theory perspective the theory appears more natural in the so called *string-frame*, the two ways of writing the action are related to a (local) conformal rescaling of the various fields, for more details we refer to standard textbooks as [14, 15].

### Dimensional reduction

The idea that a theory in a certain number of space-time dimensions can sometimes be effectively described by a lower dimensional theory was first illustrated

by Kaluza and Klein [45, 46]. They showed that at purely gravitational theory in five dimensions is equivalent to a theory of gravity coupled to a Maxwell and scalar field in four dimensions, if the fifth dimension is a circle and one considers energies smaller than the inverse radius of that circle.

The concept that compact dimensions can be integrated out at low enough energies can be made more familiar by a simple example. Assume space-time has 10 dimensions but that it has the following direct product structure:

$$\mathcal{M}_{1,9} = \mathcal{M}_{1,3} \times \mathcal{M}_6 \quad (2.17)$$

where we assume  $\mathcal{M}_6$  to be a compact manifold. Consider a free massless scalar field  $\phi(x, y)$  where  $x$  are coordinates on  $\mathcal{M}_4$  and  $y$  on  $\mathcal{M}_6$ . For a manifold of the form (2.17) the ten dimensional d'Alembertian is of the form:

$$\square_{10} = \square_4 + \Delta_6. \quad (2.18)$$

Given a set of eigenstates of the Laplacian  $\Delta_6$ , i.e.  $\Delta_6 \tilde{\phi}_n(y) = \lambda_n \tilde{\phi}_n(y)$  ( $\lambda_0 = 0$ ), we can make the following decomposition:

$$\phi(x, y) = \sum_n \phi_n(x) \tilde{\phi}_n(y). \quad (2.19)$$

In this way the ten dimensional equations of motion reduce to equations of motion for the 4 dimensional part:

$$\square_{10} \phi(x, y) = 0 \quad \Leftrightarrow \quad \square_4 \phi_n(x) + \lambda_n \phi_n(x) = 0. \quad (2.20)$$

So we see that we can describe the ten dimensional scalar field equivalently by an infinite set of massive four dimensional scalar fields where the masses squared are given by the eigenvalues  $\lambda_n$ . These masses will be of order  $\frac{1}{L}$ , where  $L$  is the size of the compactification manifold, and so if we restrict ourselves to energies much smaller than this *compactification scale*  $L^{-1}$ , we can effectively neglect the scalars  $\phi_n$  with  $\lambda_n > 0$ . From now on we will only consider the massless spectrum of a compactification and decompose the fields as

$$\phi(x, y) = \phi(x) \phi_0(y), \quad (2.21)$$

where  $\phi_0$  is a harmonic field on the internal manifold.

This discussion is straightforwardly generalized to  $n$ -form gaugefields, where the internal component will be a harmonic  $n$ -form, and to fermions. As on compact Riemannian manifolds harmonic forms are completely classified by the cohomology, the massless field content of the low energy theory is completely determined by the cohomology of the internal manifold. This will be very useful in determining the theory in lower dimensions as we won't need too much geometric details of the internal manifold to carry out the compactification.

### Calabi-Yau manifolds

By compactifying 6 of the ten dimensions of Type IIA supergravity we will thus find an effective four dimensional theory. It is clear that many different internal manifolds are possible, all leading to different four dimensional theories. We are specifically interested in obtaining  $\mathcal{N} = 2$  supergravity. As  $\mathcal{N} = 2$  supersymmetry in four dimensions has 8 conserved supercharges while Type IIA has 32 conserved supercharges, we see that to match the two pictures the compactification should break exactly  $1/4$  of the total supersymmetry of IIA. This condition of breaking 24 supercharges while preserving 8 others is very restrictive on the internal manifold. As reviewed in [15] this is only possible if the manifold is equipped with a metric that has a unique covariantly constant spinor. This condition of a unique covariantly constant spinor is equivalent to the maybe more familiar requirement that the holonomy group should be  $SU(3)$  (in our case of a six dimensional internal manifold). Note that in case the holonomy group is a proper subgroup of  $SU(3)$  the amount of supersymmetry preserved in four dimensions is larger. We will not consider those cases here although in principle they are a special case. So the requirement of matching the supersymmetries of the theories already fixes the internal manifold to be of a very restricted type, that of  $SU(3)$  holonomy.

As is explained in e.g. [13], any 6 dimensional metric with a holonomy group that is a subgroup of  $SU(3)$  is Ricci-flat and Kähler and vice versa. So in the search for suitable compactification manifolds we should look for Kähler manifolds equipped with a Ricci-flat Kähler metric. There are however almost no explicit examples of such metrics. Luckily there is a famous existence theorem in mathematics, conjectured by Calabi and proven by Yau, that any Kähler manifold with a vanishing first Chern class has a unique Ricci flat metric for each Kähler class and complex structure [47]. So although we can't explicitly construct the Ricci flat metrics we know they exist on the many known manifolds that satisfy the topological constraints of being Kähler and having zero first Chern class and that are called *Calabi-Yau manifolds*. More specifically we will work with Calabi-Yau manifolds of complex dimension three and we will often refer to these shortly as Calabi-Yau three-folds or just  $CY_3$ , furthermore we will always restrict ourselves to cases of  $SU(3)$  holonomy. In this thesis Calabi-Yau manifold will thus mean a Ricci-flat Kähler manifold with holonomy group  $SU(3)$  (and not a subgroup!).

Apart from their defining properties  $CY_3$  manifolds have some other interesting features [47]. First of all they admit a constant holomorphic three-form, which is equivalent to the canonical bundle being trivial. The fact that will be of most use to us, is that their cohomology is very simple. All the Betti-numbers of a  $CY_3$  are determined by two numbers  $h^{2,1}$  and  $h^{1,1}$  and are summarized in the Hodge diamond, see table 2.1.

A classic example of a  $CY_3$  is given by the *quintic hypersurface* in  $\mathbb{CP}^4$ ,

|          |   |           |           |           |   |
|----------|---|-----------|-----------|-----------|---|
| $H^0(X)$ |   |           | 1         |           |   |
| $H^1(X)$ |   | 0         |           | 0         |   |
| $H^2(X)$ |   | 0         | $h^{1,1}$ |           | 0 |
| $H^3(X)$ | 1 | $h^{2,1}$ |           | $h^{2,1}$ | 1 |
| $H^4(X)$ |   | 0         | $h^{1,1}$ |           | 0 |
| $H^5(X)$ |   | 0         |           | 0         |   |
| $H^6(X)$ |   |           | 1         |           |   |

**Table 2.1** The Hodge diamond of Calabi-Yau threefold  $X$  is particularly simple. In this table  $h^{a,b} = \dim H^{a,b}(X)$  and horizontally we have made a split in the different components under the Dolbeault decomposition  $H^n(X) = \bigoplus_{a+b=n} H^{a,b}(X)$ . Note that the only non trivial odd cycles are three cycles as  $\dim H^1(X) = \dim H^2(X) = 0$  and that all two/four forms are of type  $(1,1)/(2,2)$  as  $h^{2,0} = h^{0,2} = 0$ .

defined by the algebraic equation

$$\sum_{i=1}^5 z_i^5 = 0, \quad (2.22)$$

where the  $z_i$  are projective coordinates on  $\mathbb{CP}^4$ . The Hodge numbers of the quintic are  $h^{1,1} = 1$  and  $h^{2,1} = 101$ , see e.g. [15].

### Dimensional reduction of Type IIA on a Calabi-Yau manifold

We now have all ingredients to perform the dimensional reduction of IIA on a Calabi-Yau manifold, leading to a 4 dimensional  $\mathcal{N} = 2$  supergravity, we will roughly follow [48]. As discussed above, if we restrict our attention to the massless fields in 4 dimensions we can expand the ten dimensional fields as four dimensional fields times an internal harmonic part. In case we compactify on a Calabi-Yau threefold  $X$ , we expand the gaugefields of IIA as

$$\begin{aligned} C_1 &= A^0(x), \quad B_2 = B_2(x) + B^A(x)D_A, \quad A = 1, \dots, h^{1,1}(X), \\ C_3 &= A^A(x) \wedge D_A + \xi^K(x)\alpha_K - \xi_K(x)\beta^K, \quad K = 1, \dots, h^{2,1}(X). \end{aligned} \quad (2.23)$$

Here the  $D_A$  form a harmonic basis of  $H^2(X) = H^{(1,1)}(X)$  and the  $(\alpha_K, \beta^K)$  form a real symplectic harmonic basis of  $H^3(X)$ , i.e. they satisfy

$$\int_X \alpha_K \wedge \beta^L = \delta_K^L. \quad (2.24)$$

Note that in this decomposition we made use of the simple form of the Hodge diamond of a Calabi-Yau threefold, Table 2.1.

For the metric we make a direct product ansatz:

$$ds_{10}^2 = g_{ij}(x)dx^i dx^j + g_{a\bar{b}}(x, y)dy^a d\bar{y}^{\bar{b}}. \quad (2.25)$$

In this decomposition  $g_{a\bar{b}}$  is a Ricci flat Kähler metric on our Calabi-Yau manifold  $X$ . By the Calabi-Yau theorem this metric is unique given a complex and Kähler structure. Consider however the variations

$$g_{a\bar{b}} \rightarrow g_{a\bar{b}} + \delta g_{a\bar{b}} + \delta g_{\bar{a}b}. \quad (2.26)$$

In case these variations keep the metric Ricci-flat they correspond to massless modes in four dimensions. By the uniqueness of the Ricci-flat metric on  $X$  such deformations have to be in one to one correspondence with changes of the complex and Kähler structures on  $X$ . As discussed in e.g. [49], the deformations (2.26) that leave the metric Ricci flat correspond to harmonic  $(1,1)$  and  $(2,1)$  forms through the map

$$\begin{aligned} \omega^{1,1} &= i\delta g_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}} = J^A D_A, \\ \omega^{2,1} &= \Omega_{\mu\nu}{}^{\bar{\lambda}} \delta g_{\bar{\lambda}\bar{\kappa}} dz^\mu dz^\nu d\bar{z}^{\bar{\kappa}} = v^K \alpha_K, \end{aligned} \quad (2.27)$$

where  $\Omega$  is the constant holomorphic three-form that exists on every  $\text{CY}_3$  as mentioned above. Note that we furthermore decomposed the harmonic forms in the bases introduced in (2.23). The coefficients  $v^K$  and  $J^A$  behave as massless scalar fields in four dimensions. The  $h^{2,1}$  complex scalars  $v^K$  parameterize changes of the complex structure on  $X$  and are thus called *complex structure moduli*, the real  $J^A$  parameterize changes of the Kähler form and are referred to as *Kähler moduli*. There are  $h^{1,1}$  of the latter as can be read from (2.27).

So the decompositions (2.23), (2.25) and (2.27) give us all the bosonic massless modes describing Type IIA compactified on a  $\text{CY}_3$ . As this compactification preserves 8 of the 32 supercharges as mentioned before the 4 dimensional fields can be neatly organized into  $\mathcal{N} = 2$  multiplets, giving us a nice compact way to summarize all the different fields present. First note that the Kähler moduli that are real scalars appear naturally in combination with the B-field moduli as a single complex scalar:

$$t^A = B^A + iJ^A. \quad (2.28)$$

The different four dimensional massless modes arranged in the different multiplets are shown in Table 2.2.

The four dimensional action can now be obtained by plugging the definitions (2.23), (2.25) and (2.27) into the Type IIA action (2.15). After a lengthy computation one finds an action of the form<sup>1</sup> [48]

$$\begin{aligned} S_{4d} &= \frac{2\pi}{l_4^2} \int \sqrt{-g} \mathcal{R} - \frac{64\pi^2}{l_4^2} \int (G_{AB} dt^A \wedge \star d\bar{t}^B + h_{XY} dq^X \wedge \star dq^Y) \\ &\quad + \frac{1}{16\pi} \int \left( \text{Im} \mathcal{N}_{\bar{A}\bar{B}}(t) F^{\bar{A}} \wedge \star F^{\bar{B}} + \text{Re} \mathcal{N}_{\bar{A}\bar{B}}(t) F^{\bar{A}} \wedge F^{\bar{B}} \right). \end{aligned} \quad (2.29)$$

<sup>1</sup> Note that to exactly derive the  $\mathcal{N} = 2$  action as presented here one also has to perform a Weyl rescaling of the 4d metric and fix the correct conventions, see e.g. [50] for subtleties some more details.

|                   |           |                             |
|-------------------|-----------|-----------------------------|
| gravity multiplet | 1         | $(g_{\mu\nu}, A^0)$         |
| vector multiplets | $h^{1,1}$ | $(A^A, t^A)$                |
| hyper multiplets  | $h^{2,1}$ | $(v^K, \xi^K, \xi_K)$       |
| tensor multiplet  | 1         | $(B_2, \phi, \xi^0, \xi_0)$ |

**Table 2.2** *The different massless bosonic fields of 4 dimensional  $\mathcal{N} = 2$  supergravity arranged into their multiplet structure. Note that the tensor field of the tensor multiplet can be dualized into a scalar. The tensor multiplet then becomes an additional hypermultiplet called the universal hypermultiplet.*

In this action we conveniently grouped the hyperscalars and the dualized tensor multiplet as  $q^X$  and defined the field strengths  $F^{\tilde{A}} = dA^{\tilde{A}}$ , with  $\tilde{A} = (0, A)$ . Indeed, as implied by our demand of preserved supersymmetries, this action (2.29) is the bosonic part of a  $\mathcal{N} = 2$  supergravity in 4 dimensions [51]. The precise expressions for the various coefficients in (2.29) and the geometry behind this action are rather involved and deserve a separate discussion. This will be the topic of the next subsection.

### 2.2.2 Special Geometry

In this subsection we will review some more details of the structure behind the action (2.29). The kinetic terms of the scalars are given by a non-linear sigma-model. I.e. the scalars  $t^A$  and  $q^X$  are maps from space-time to a manifold called target space. It is important to note that in (2.29) the target space is a direct product of a hyperscalar target space and a separate target space for the vectormultiplet scalars. As the couplings  $N_{\tilde{A}\tilde{B}}$  only depend on the vectormultiplet scalars, and the hyperscalars thus only appear through their kinetic terms, there exists a consistent truncation of the theory where all the hyperscalars are set to arbitrary constants.

The black hole solutions we will consider in this thesis are such solutions with constant hyperscalars and as such we will not dwell on the target space geometry of the hypermultiplets and focus on the geometry of the vectormultiplet sector.

There is a rich structure called *special Kähler geometry* underlying the vectormultiplet action. It would lead us a bit too far to introduce this concept from first principles and in full generality. We will take a more practical approach and we will only consider some formulae and concepts that we will need later on. Furthermore we will restrict to the special case of Type IIA Calabi-Yau compactifications in the large volume limit. For the more general theory and further reference there exist many excellent papers and reviews in the literature, a starting few could be [52, 53, 49].

### The even cohomology of a CY<sub>3</sub>

As will become clear along the way many objects of relevance will be most naturally thought of as taking values in the even cohomology of the internal Calabi-Yau manifold  $X$ :  $H^{2*}(X)$ . Before we continue let us remark that we will be a bit sloppy in distinguishing between elements in homology vs cohomology, relying on their relation through Poincaré duality<sup>2</sup>. Furthermore the even homology  $H_{2*}(X)$  is naturally defined over the integers in  $H_{2*}(X, \mathbb{Z})$ , Poincaré duality maps this naturally to a discrete version of cohomology,  $H^{2*}(X, \mathbb{Z})$ .

The even cohomology  $H^{2*}(X)$  has dimension  $2 + 2h^{1,1}$  as can be read from Table 2.1. We can expand an arbitrary element  $E \in H^{2*}(X)$  as

$$E = E^0 + E^A D_A + E_A D^A + E_0 dV, \quad (2.31)$$

here  $dV$  is the normalized volume form on  $X$ , the  $D_A$  denote a basis of  $H^2(X, \mathbb{Z})$  and the  $D^A$  a dual basis of  $H^4(X, \mathbb{Z})$ , i.e.

$$\int_X D_A \wedge D^B = \delta_A^B. \quad (2.32)$$

Sometimes, but not very often after this section, when we want to work in a notation that reveals the underlying symplectic structure more, we use the basis

$$E = \tilde{E}^{\tilde{A}} D_{\tilde{A}} + \tilde{E}_{\tilde{A}} D^{\tilde{A}}. \quad (2.33)$$

Here  $\tilde{A} = (0, A)$  and we take  $D_0 = 1$  and  $D^0 = -dV$ . Note that this induces a subtle minus sign<sup>3</sup> in the identification  $(\tilde{E}^0, \tilde{E}^A, \tilde{E}_A, \tilde{E}_0) = (E^0, E^A, E_A, -E_0)$ .

It will be very useful to define the *intersection numbers* of  $X$ :

$$D_{ABC} = \int_X D_A \wedge D_B \wedge D_C. \quad (2.34)$$

Furthermore we define an antisymmetric topological intersection product  $\langle \cdot, \cdot \rangle$  on  $H^{2*}(X)$  by

$$\langle E_1, E_2 \rangle = \int_X E_1 \wedge E_2^*. \quad (2.35)$$

<sup>2</sup> To every  $n$ -cycle  $\sigma \in H_n(X)$  we can associate a Poincaré dual  $6-n$  form  $\tilde{\sigma} \in H^{6-n}(X)$  that satisfies for any  $n$ -form  $\omega$ :

$$\int_{\sigma} \omega = \int_X \omega \wedge \tilde{\sigma}. \quad (2.30)$$

This map  $\sigma \mapsto \tilde{\sigma}$  is one to one and as such we will no longer notationally distinguish between  $\sigma$  and its dual  $\tilde{\sigma}$ , leaving it to the context to determine whether the element in cohomology or homology is intended.

<sup>3</sup> The awkward minus sign comes about because we want the symplectic covariant  $\langle D_{\tilde{A}}, D^{\tilde{B}} \rangle = \delta_{\tilde{A}}^{\tilde{B}}$  to hold. See also the last line of (2.36) which takes a standard symplectic form.

Here  $E^*$  is nothing but  $E$  with the sign of the 2- and 6-form components changed:  $E^* = E^0 - E^A D_A + E_A D^A - E_0 dV$ . We can write out the intersection product in components as

$$\begin{aligned} \langle E_1, E_2 \rangle &= -E_1^0 E_2^0 + E_1^A E_A^2 - E_A^1 E_2^A + E_0^1 E_2^0, \\ &= \tilde{E}_1^{\tilde{A}} \tilde{E}_A^2 - \tilde{E}_{\tilde{A}}^1 \tilde{E}_2^{\tilde{A}}. \end{aligned} \quad (2.36)$$

### Special Kähler geometry

Using some of the notation introduced above we come to the description of the structure underlying the vector multiplet sector of the 4 dimensional lagrangian (2.29).

It follows from the dimensional reduction of IIA that the kinetic term of the vector multiplet scalars is given in terms of the sigma model metric [48, 49]

$$G_{AB} = \frac{1}{4\mathcal{V}} \int_X D_A \wedge \star D_B = -\frac{3}{2} \left( \frac{J_{AB}}{J^3} - \frac{3}{2} \frac{J_A^2 J_B^2}{J^6} \right) = -\partial_A \bar{\partial}_B \left( \ln \frac{4}{3} J^3 \right). \quad (2.37)$$

Let us first explain the shorthand notation used in this formula:  $\partial_A = \frac{\partial}{\partial t^A}$ ,  $\bar{\partial}_A = \frac{\partial}{\partial \bar{t}^A}$  and

$$\begin{aligned} J^3 &:= \int_X J \wedge J \wedge J = D_{ABC} J^A J^B J^C = 6\mathcal{V}, \\ J_A^2 &:= \int_X D_A \wedge J \wedge J = D_{ABC} J^B J^C, \\ J_{AB} &:= \int_X D_A \wedge D_B \wedge J = D_{ABC} J^C, \end{aligned} \quad (2.38)$$

here  $J = J^A D_A$  is a Kähler form on  $X$  and  $\mathcal{V}$  is the volume of  $X$ . Coming back to (2.37), the second equality is proven by using the relation between the Kähler form and the Kähler metric and is detailed in e.g. [54, 55]. The last equality in (2.37) can easily be checked and shows that  $G_{AB}$  is itself a Kähler metric with Kähler potential  $\mathcal{K} = -\ln \frac{4}{3} J^3$ . Apart from being Kähler the metric (2.37) has additional structure as the Kähler potential itself can be derived from a *prepotential*, making it what is called a *special Kähler* metric [52, 53].

To make this explicit we introduce projective coordinates  $X^{\tilde{A}}$  such that  $t^A = \frac{X^A}{X^0}$ . In terms of these coordinates we can write the Kähler potential as

$$\mathcal{K} = -\ln i \left[ \bar{X}^{\tilde{A}} \partial_{\tilde{A}} \mathcal{F} - X^{\tilde{A}} \bar{\partial}_{\tilde{A}} \bar{\mathcal{F}} \right], \quad (2.39)$$

where the prepotential  $\mathcal{F}$  is in this case given by<sup>4</sup>

$$\mathcal{F} = -\frac{1}{6} D_{ABC} \frac{X^A X^B X^C}{X^0}. \quad (2.40)$$

<sup>4</sup> This expression for the prepotential doesn't take into account instanton corrections due to Euclidean branes wrapping cycles in the internal manifold see e.g. [56, 57]. This approximation is justified in the limit where the volume of the internal manifold is large in string units.

Note that the arbitrariness introduced by adding an additional projective coordinate is directly translated into the fact that the Kähler potential is only determined up to Kähler transformations; making a choice for  $X^0(t^A)$  corresponds to fixing this gauge freedom.

In fact the prepotential determines the complete vector multiplet part of the lagrangian (2.29), as also the vectorfield couplings  $\mathcal{N}_{\tilde{A}\tilde{B}}$  can be determined from it [51]:

$$\mathcal{N}_{\tilde{A}\tilde{B}} = \bar{\mathcal{F}}_{\tilde{A}\tilde{B}} + 2i \frac{\text{Im}(\mathcal{F}_{\tilde{A}\tilde{C}})X^{\tilde{C}}\text{Im}(\mathcal{F}_{\tilde{B}\tilde{D}})X^{\tilde{D}}}{\text{Im}(\mathcal{F})_{\tilde{C}\tilde{D}}X^{\tilde{C}}X^{\tilde{D}}}, \quad (2.41)$$

where the shorthand  $\mathcal{F}_{\tilde{A}} = \partial_{\tilde{A}}\mathcal{F}$  and  $\mathcal{F}_{\tilde{A}\tilde{B}} = \partial_{\tilde{A}}\partial_{\tilde{B}}\mathcal{F}$  was used. In our case of the prepotential (2.40), we have more explicitly [48]

$$\begin{aligned} \text{Re}\mathcal{N} &= \begin{pmatrix} -\frac{1}{3}B^3 & \frac{1}{2}B_A^2 \\ \frac{1}{2}B_B^2 & -B_{AB} \end{pmatrix}, \\ \text{Im}\mathcal{N} &= -\frac{J^3}{6} \begin{pmatrix} 1 + 4G_{AB}B^AB^B & -4G_{AB}B^B \\ -4G_{AB}B^A & 4G_{AB} \end{pmatrix}, \end{aligned} \quad (2.42)$$

using again the notation of (2.38).

One can organize the special geometry data introduced above into a single element of  $H^{2*}(X)$  by defining

$$\Omega_{\text{hol}} = X^{\tilde{A}}D_{\tilde{A}} - \mathcal{F}_{\tilde{A}}D^{\tilde{A}}. \quad (2.43)$$

In the more abstract formulations  $\Omega_{\text{hol}}$ , called the *period vector*, is often taken as the starting point for defining special Kähler geometry. The bundle  $H^{2*}(X)$  of our specific case is in general an arbitrary holomorphic  $Sp(2n+2)$  vector bundle of which  $\Omega_{\text{hol}}$  is a section [52, 53, 58].

We will remain very pragmatic however and from here on often forget about the projective coordinates by fixing  $X^0 = -1$ , in this case we find that

$$\Omega_{\text{hol}} = -e^{t^A D_A}. \quad (2.44)$$

It will be useful to define a normalized version of  $\Omega_{\text{hol}}$  that satisfies  $\langle \Omega, \bar{\Omega} \rangle = -i$ , a straightforward calculation shows:

$$\Omega = e^{\mathcal{K}/2}\Omega_{\text{hol}} = -\frac{1}{\sqrt{\frac{4}{3}J^3}} e^{(B^A + iJ^A)D_A}. \quad (2.45)$$

Using this normalized period vector one can associate a *central charge* to every element  $E \in H^{2*}(X)$  as

$$Z(E) = \langle E, \Omega \rangle = \frac{1}{\sqrt{\frac{4}{3}J^3}} \left( E^0 \frac{t^3}{6} - \frac{E^A t_A^2}{2} + E_A t^A - E_0 \right). \quad (2.46)$$

The name ‘central charge’ will be justified later when we will see that this expression becomes a central charge in the relevant 4 dimensional supersymmetry algebra.

### The even cohomology revisited

From the point of view of the underlying special geometry the most natural basis of  $H^{2*}(X)$  is not the one introduced in (2.31). Given a period vector  $\Omega$  one defines its covariant derivative as

$$\mathcal{D}_A \Omega = \partial_A \Omega + \frac{1}{2} \partial_A \mathcal{K}. \quad (2.47)$$

The need of a covariant derivative springs from the fact that  $\Omega$  transforms under Kähler transformations and we want its derivative  $\mathcal{D}_A \Omega$  to transform in the same way. One can check that we have the equalities

$$\langle \Omega, \bar{\Omega} \rangle = -i, \quad \langle \mathcal{D}_A \Omega, \bar{\mathcal{D}}_B \bar{\Omega} \rangle = G_{AB}, \quad \langle \mathcal{D}_A \Omega, \Omega \rangle = 0. \quad (2.48)$$

The set  $\{\Omega, \mathcal{D}_A \Omega, \bar{\mathcal{D}}_A \bar{\Omega}, \bar{\Omega}\}$  furnish a basis of  $H^{2*}(X)$  and the relations (2.48) show it is 'orthonormal' with respect to the intersection product. In this new basis a real element  $E \in H^{2*}(X)$  is decomposed as

$$\begin{aligned} E &= i\bar{Z}(E)\Omega - iG^{AB}\bar{\mathcal{D}}_A \bar{Z}(E)\mathcal{D}_B \Omega + iG^{AB}\mathcal{D}_A Z(E)\bar{\mathcal{D}}_B \bar{\Omega} - iZ(E)\bar{\Omega}, \\ &= 2\text{Im} [\bar{Z}(E)\Omega - G^{AB}\bar{\mathcal{D}}_A \bar{Z}(E)\mathcal{D}_B \Omega]. \end{aligned} \quad (2.49)$$

Finally let us introduce an operation  $\diamond$ , defined on the basis elements as

$$\diamond \Omega = -i\Omega, \quad \diamond \mathcal{D}_A \Omega = i\mathcal{D}_A \Omega, \quad \diamond \bar{\mathcal{D}}_A \bar{\Omega} = -i\bar{\mathcal{D}}_A \bar{\Omega} \quad \text{and} \quad \diamond \bar{\Omega} = i\bar{\Omega}. \quad (2.50)$$

The definition is extended to a general element of  $H^{2*}(X)$  by linearity. It is clear by its definition that  $\diamond$  is a real operator, i.e.  $\diamond \bar{E} = \overline{\diamond E}$  or thus  $\diamond = \bar{\diamond}$ , and one can also immediately infer that  $\diamond^2 = -1$ . On the basis (2.33) the operator acts as

$$\begin{aligned} \diamond D_{\tilde{A}} &= [\text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1}]_{\tilde{A}}^{\tilde{B}} D_{\tilde{B}} - [\text{Im}\mathcal{N} + \text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1}\text{Re}\mathcal{N}]_{\tilde{A}\tilde{B}} D^{\tilde{B}}, \\ \diamond D^{\tilde{A}} &= [(\text{Im}\mathcal{N})^{-1}]^{\tilde{A}\tilde{B}} D_{\tilde{B}} - [(\text{Im}\mathcal{N})^{-1}\text{Re}\mathcal{N}]^{\tilde{A}}_{\tilde{B}} D^{\tilde{B}}. \end{aligned} \quad (2.51)$$

This last form makes the dependence of  $\diamond$  on the moduli  $t^A$  explicit. Using our new operator  $\diamond$  we can introduce a symmetric moduli dependent bilinear form on  $H^{2*}(X)$ :

$$(E_1, E_2) = \langle E_1, \diamond E_2 \rangle. \quad (2.52)$$

Using the decomposition (2.49) and the definition of  $\diamond$  one can calculate explicitly (for  $E_1, E_2$  real)

$$(E_1, E_2) = 2\text{Re} [Z(E_1)\bar{Z}(E_2) + G^{AB}\mathcal{D}_A Z(E_1)\bar{\mathcal{D}}_B \bar{Z}(E_2)]. \quad (2.53)$$

This form shows that we can define a positive non-degenerate<sup>5</sup> norm on  $H^{2*}(X, \mathbb{R})$  as

$$|E| = \sqrt{(E, E)}. \quad (2.54)$$

<sup>5</sup> Note that the scalar product is moduli dependent, so at singular points in moduli space the norm might degenerate. In this thesis we will silently neglect these subtleties as we will always restrict ourselves to smooth parts of moduli space.

In the literature this norm often goes under the name of the first symplectic invariant  $I_1$ , see e.g. [59, 60].

The operator  $\diamond$  might look rather ad hoc and not very familiar, note however that in case  $B^A = 0$  its action (2.51) simplifies quite a bit. In this case one can check using (2.37) that the scalar product (2.52) is nothing but the familiar Hodge product:

$$\langle E_1, \diamond E_2 \rangle_{B=0} = \int_X E_1 \wedge \star E_2. \quad (2.55)$$

We end this subsection with a formula from [27] that will be of use in later derivations:

$$(\partial_\tau + i\text{Im}(\partial_A \mathcal{K} \dot{t}^A) + i\dot{\alpha}) (e^{-i\alpha} \Omega) = e^{-i\alpha} \mathcal{D}_A \Omega \dot{t}^A. \quad (2.56)$$

This equation follows quite directly from using the chain rule and observing that  $\bar{\partial}_A \Omega = \frac{1}{2} \bar{\partial}_A \mathcal{K} \Omega$ , note that  $\alpha$  is an arbitrary real function of  $\tau$  and that the dot denotes derivation with respect to  $\tau$ .

### 2.2.3 Supersymmetric particles from wrapped D-branes

In the next section we will start our discussion of black hole solutions to the  $\mathcal{N} = 2$  supergravity we introduced above. These black holes can be seen as configurations of the supergravity fields sourced by massive charged particles. Before we look at this *backreacted* system it is interesting to have a look at the coupling of point particles to the fields of the theory without considering the dynamics of the fields themselves, the so called *probe* or *test particle* approximation.

In this subsection we will first introduce the action of such a probe particle in a supergravity background under the assumption that the particle preserves some of the supersymmetries of that background. We will then show how these particles in the 4 dimensional picture are naturally interpreted as D-branes from the ten dimensional IIA point of view. To make this subsection somehow self contained we will also give a (very) short review of D-branes.

#### BPS particles

Let us in the spirit of section 2.2.2 define a generalized electro-magnetic field-strength

$$F = F^0 + F^A D_A - G_A D^A - G_0 dV, \quad (2.57)$$

with

$$\begin{aligned} G_A &= \text{Re} \mathcal{N}_{A\bar{B}} F^{\bar{B}} + \text{Im} \mathcal{N}_{A\bar{B}} \star F^{\bar{B}}, \\ G_0 &= -\text{Re} \mathcal{N}_{0\bar{B}} F^{\bar{B}} - \text{Im} \mathcal{N}_{0\bar{B}} \star F^{\bar{B}}. \end{aligned} \quad (2.58)$$

The equations of motion and Bianchi identities of the action (2.29) now imply the conservation of the following charges:

$$\int_{S^2} F = 4\pi\Gamma. \quad (2.59)$$

Here again we conveniently summarized a number of formulas by working with  $H^{2*}(X)$  valued objects, i.e. we can expand

$$\Gamma = p^0 + p^A D_A + q_A D^A + q_0 dV. \quad (2.60)$$

Here  $\Gamma$  contains both the electric charges  $q_{\tilde{A}}$  and the magnetic charges  $p_{\tilde{A}}$  of the respective U(1) Maxwell fields  $F^{\tilde{A}}$ . Note that with this notation  $\langle \Gamma_1, \Gamma_2 \rangle$  is the Dirac-Schwinger-Zwanziger duality invariant product between magnetic and electric charges. Consistency of the quantum theory demands this product to be integral.

We can now consider point particles carrying these various electric and magnetic charges. We will restrict our attention to particles whose worldvolume action preserves some of the supersymmetries of a given  $\mathcal{N} = 2$  background. For a test particle of charge  $\Gamma$  the worldvolume action can be uniquely determined by supersymmetry, see e.g. [61], and the result is

$$S_{\text{WV}} = - \int |Z(\Gamma, t)| ds + \frac{1}{2} \int \langle \Gamma, A \rangle, \quad (2.61)$$

with  $dA = F$ . The second term in the action should be familiar as it is the standard coupling of a charged particle to the corresponding gaugefield. The first term contains two types of couplings. First the line element  $ds$  describes the coupling of the particle to gravity. Secondly it encodes the coupling of the particle to the scalar fields through what we called the *central charge* in the previous section:

$$Z(\Gamma, t) = \langle \Gamma, \Omega(t) \rangle. \quad (2.62)$$

This name refers to the fact that  $Z$  appears as a central element in the 4 dimensional  $\mathcal{N} = 2$  supersymmetry algebra. From this algebra it also follows [62] that all states with charge  $\Gamma$  satisfy the *BPS bound*  $M \geq Z(\Gamma)$ . In case the state preserves some of the supersymmetries of the theory it follows that this bound is saturated, i.e.  $M = Z$ , and such states are therefore called *BPS states*.

We can check explicitly that the action (2.61) describes such BPS particles. By construction particles described by (2.61) preserve half of the 8 supercharges of the maximally supersymmetric Minkowski background. In this background the scalars  $t^A$  take constant values and all gaugefields are trivial, the action (2.61) thus simply becomes

$$S_{\text{WV}} = -|Z(\Gamma, t)| \int ds. \quad (2.63)$$

So we see indeed that for these particles  $M = Z(\Gamma, t)$ .

The fact that the mass of the particles is given by the scalar dependent central charge implies that the same particle of charge  $\Gamma$  has different mass when put into a different background. This is related to the fact that the constant value of the scalar field selects a certain vacuum and that the energy of the particle is measured with respect to this vacuum.

Later we will consider probe particles in backgrounds where the scalar fields are not constant, the coupling of the particle to the scalar fields can then be thought of as making the particle behave as if it had a variable mass.

### D-branes

The particles introduced above are sources for the different  $U(1)$  gaugefields of the 4d theory. As these gaugefields originate from the RR fields in ten dimensional IIA it is clear the point particles should be related to the sources of the RR fields. From string theory it is by now well known that these fields are sourced by fundamental solitons called D-branes [63]. Such  $Dp$ -branes have a  $p + 1$  dimensional worldvolume and at small  $g_s$  they are very massive hypersurfaces on which open strings can end.

Quantizing the open strings in the presence of such a D-brane shows that their dynamics can be described by a set of worldvolume scalars  $\phi_\mu$  and a worldvolume gaugefield  $A_\alpha$ . Since loop diagrams for these open strings can be interpreted as closed strings propagating at tree level, the D-branes interact with the closed string modes or, in our low energy description, the background supergravity fields. The action for a D-brane in such a supergravity background is given by <sup>6</sup>[15]

$$S_{Dp} = -T_p \int d^{p+1}\sigma e^\phi \sqrt{-\det(G_{\alpha\beta} + \mathcal{F}_{\alpha\beta})} + \mu_p \int e^{\frac{\mathcal{F}}{2\pi}} \wedge C. \quad (2.64)$$

Here the  $\sigma^\alpha$  are  $p + 1$  worldvolume coordinates. There are scalar fields  $\phi^\mu(\sigma)$  describing the embedding of the brane in space-time. The metric  $G_{\alpha\beta} = \partial_\alpha \phi^\mu \partial_\beta \phi^\nu g_{\mu\nu}$  is the space-time metric pulled back to the D-brane worldvolume. Furthermore there is a worldvolume two-form  $\mathcal{F} = \partial_\alpha \phi^\mu \partial_\beta \phi^\nu B_{\mu\nu} + 2\pi\alpha' F_{\alpha\beta}$ , which exists out of the pullback of the background  $B$ -field and the fieldstrength of a worldvolume gaugefield:  $F = dA$ . The gaugefields  $C$  and the dilaton  $\phi$  are those of the background Type II supergravity, again pulled back to the worldvolume.

The coupling constants in the action (2.64) are given by

$$T_p = \frac{2\pi}{g_s l_s^{p+1}}, \quad \mu_p = \frac{1}{2}. \quad (2.65)$$

Note that the tension  $T_p$  is inversely proportional to the string coupling constant  $g_s$ . This makes the D-branes very massive objects in the perturbative string

<sup>6</sup> We have neglected certain curvature couplings in this action as they will not be relevant in the regimes discussed in the thesis.

regime, but it also implies they actually become the dominant light degrees of freedom in the strong coupling regime. Furthermore it is so that in Type IIA string theory only branes with even  $p$  are stable while in Type IIB only the ones with odd  $p$  are. This is of course very much related to difference in supersymmetry and RR-sector between these two theories.

From the action (2.64) we see that the worldvolume scalars  $\phi^\mu$  play the role of maps from the worldvolume to space-time and thus describe the embedding of the brane in space-time, the metric  $G_{\alpha\beta}$  is then the pullback to the worldvolume of the space-time metric. If we neglect the two form  $\mathcal{F}$  for a moment we see that the first term in the action, called the Dirac-Born-Infeld term, is the volume form on the brane. Without background  $B$ -field and worldvolume flux  $F$ , branes can thus intuitively be thought of as rubber surfaces trying to minimize their volume.

The second term of (2.64), called the Wess-Zumino (WZ) or Chern-Simons (CS) term, gives the coupling of the brane to the RR gaugefields, in the same way a particle, or 0-brane, couples naturally to a one-form gaugefield a  $p$ -brane couples to a  $p + 1$  gaugefield. We also see by expanding the exponential in the WZ term that in case there is some non-vanishing worldvolume flux higher dimensional branes can carry lower dimensional brane charge.

### BPS particles from supersymmetrically wrapped branes

In case we compactify space-time on an internal manifold with non-trivial topology we can wrap D-branes around the various non-trivial cycles, see figure 2.1 for an artists impression. Wrapping a  $Dp$ -brane around a non-trivial  $p$ -cycle it will effectively appear in the external dimensions as a point particle.

Let us make this connection between Type IIA D-branes wrapped on non trivial cycles of the internal Calabi-Yau manifold  $X$  and the charged particles of  $\mathcal{N} = 2$  supergravity a bit more concrete. The non-trivial even dimensional cycles of  $X$  are given by (the Poincaré duals to) the basis 1,  $D_A$ ,  $D^A$  and  $dV$  introduced in (2.31). Assume for the moment that the worldvolume flux  $\mathcal{F} = 0$  and let us first look to the WZ coupling.

In case we for example wrap a D2 brane around the 2-cycle  $\Gamma = q_A D^A$ , the WZ part of the action (2.64) can be written as:

$$\int_{\gamma \times \Gamma} C_3 = \int_{\gamma} \int_X C_3 \wedge \Gamma = q_A \int_{\gamma} A^A, \quad (2.66)$$

where  $\gamma$  is the worldline in the four external dimensions and we used the decomposition of the 3-form gaugefield  $C_3 = A^A D_A$  as defined in (2.23). So one sees that such a D2-brane can be effectively described in the four external dimensions as a point particle carrying electric charge  $q_A$  under the gaugefields  $A^A$ .

A generic set of various even dimensional D-branes that all coincide in a single point in the external dimensions can be conveniently identified by an element of  $H_2(X, \mathbb{Z})$  as

$$\Gamma = (p^0, p^A, q_A, q_0) = p^0 + p^A D_A + q_A D^A + q_0 dV, \quad (2.67)$$

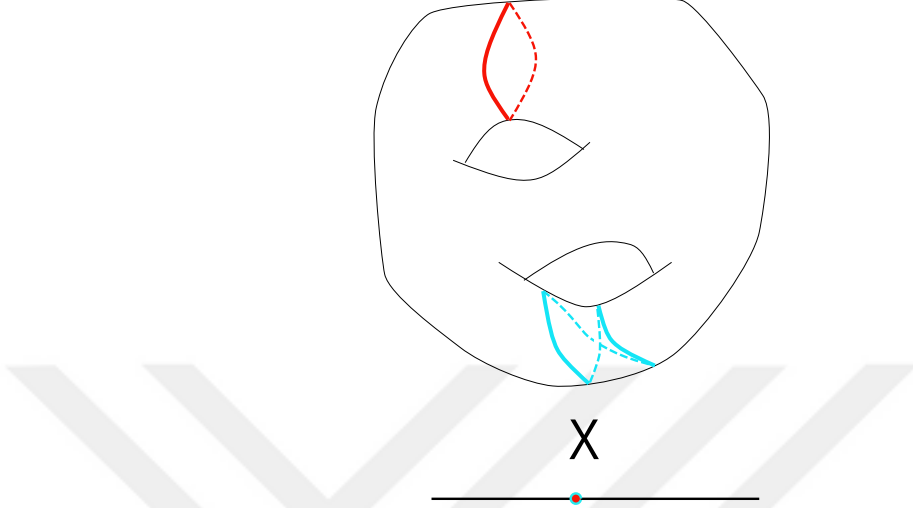


**Figure 2.1** A sketch of two  $D$ -branes wrapping topologically distinct non-trivial cycles. In an analogy with a real Calabi-Yau manifold one could imagine that the red cycle would be represented in a basis of homology as  $D^1$  and the cycle wrapped by the blue  $D$ -brane as  $D_1$ . This  $D$ -brane configuration would then correspond to the charge  $\Gamma = (0, 1, 2, 0) = 0 + D^1 + 2D_1 + 0dV$  since the cycle  $D^1$  is wrapped once and  $D_2$  twice. Furthermore, in this example, there are no pointlike branes in the internal manifold and no branes wrapping the complete manifold. The straight line below represents the external dimensions in which the  $D$ -branes appear as a single point charge.

where we imply with this notation  $p^0$  D6 branes wrapped around the full  $CY_3$ ,  $p^A$  D4 branes wrapped around the four cycles  $D_A$ ,  $q_A$  D2 branes wrapped around the two-cycles  $D^A$  and finally  $-q_0$  anti-D0's floating around as points inside  $X$ . A straightforward generalization of the argument (2.66) then shows that such a brane configuration  $\Gamma$  corresponds in four dimensions to a charged particle with charge  $\Gamma$ , as defined in (2.60). By this correspondence we will mostly refer to the charges  $(p^0, p^A, q_A, q_0)$  as the  $(D6, D4, D2, D0)$ -brane charges respectively.

It is not difficult to calculate the effect on the four dimensional charges induced by turning on a worldvolume flux  $F$  on the branes. For example in case of a single D6-brane with a worldvolume flux  $F = p^A D_A$  the four dimensional charge vector becomes

$$\Gamma = e^p = (1, p, \frac{p^2}{2}, \frac{p^3}{6}). \quad (2.68)$$

We will refer to brane configurations consisting of branes of a single type (possibly with worldvolume flux) as pure (or purely fluxed) branes.

As argued before we will in this thesis be interested in particles that preserve some supersymmetry of the  $\mathcal{N} = 2$  supergravity. These particles then correspond to D-branes that are wrapped in such a way that they preserve some supersymmetry of the compactification. This puts restrictions on the embedding of the D-branes in the Calabi-Yau, selecting specific cycles in a given homology class, and on the allowed worldvolume flux.

The condition that an even dimensional D-brane wrapped on an even cycle preserves half of the supersymmetry of a Calabi-Yau compactification amounts to the condition that the wrapped cycle is holomorphic, see [64, 65]. In these works and e.g. [61] it was also shown that the BI term of the action for such supersymmetric D-branes wrapping a cycle  $\Gamma$  reduces in four dimensions to

$$\int_{\gamma} |Z(\Gamma)| ds, \quad (2.69)$$

where  $\gamma$  is the worldline in the external dimensions and  $Z(\Gamma) = \int_{\Gamma} \Omega$ .

**Summary:** We saw that there is a one to one correspondence between the supersymmetric charged particles in 4 dimensions and holomorphically wrapped D-branes of the ten dimensional IIA theory, mapping the action (2.64) to (2.61).

This identification will play an important role in the rest of the thesis. For certain supersymmetric black holes it leads to the celebrated understanding of the black hole microstates as D-brane states, see section 2.6.

## 2.3 Supersymmetric Black holes

The four dimensional  $\mathcal{N} = 2$  supergravity action (2.29) provides us with an effective theory to study supersymmetric black holes in string theory. As mentioned before we will not need the hyperscalar sector of the theory as we can consistently decouple it by taking those scalars to be constant. Let us for convenience repeat the vector multiplet part of the action here

$$\begin{aligned} S = & \frac{2\pi}{l_4^2} \int \sqrt{-g} \mathcal{R} - \frac{64\pi^2}{l_4^2} \int G_{AB} dt^A \wedge \star d\bar{t}^B \\ & + \frac{1}{16\pi} \int \left( \text{Im} \mathcal{N}_{\tilde{A}\tilde{B}}(t) F^{\tilde{A}} \wedge \star F^{\tilde{B}} + \text{Re} \mathcal{N}_{\tilde{A}\tilde{B}}(t) F^{\tilde{A}} \wedge F^{\tilde{B}} \right). \end{aligned} \quad (2.70)$$

If we compare this action to the Einstein-Maxwell action (2.6), we see that as a 'price' for introducing supersymmetry we had to complicate the theory in two ways. First the supersymmetric theory has additional electro-magnetic fields and second it contains interactions with scalar fields.

The minimum number of Maxwell gaugefields is two. This is the case when we take our internal manifold  $X$  to have<sup>7</sup>  $h^{1,1} = 1$ , as is e.g. the case for the

<sup>7</sup> Note that  $h^{1,1} \neq 0$  as there is always a Kähler class.

quintic Calabi-Yau introduced in (2.22). Having multiple Maxwell fields doesn't really complicate the theory however and we will leave their number,  $h^{1,1}+1 \geq 2$ , arbitrary in most of this thesis.

The second and more essential new feature with respect to (2.6) is the appearance of the complex scalar fields  $t^A$ . These play an important role, contrary to the hypermultiplet fields, as they are dynamically coupled to the gaugefields through their appearance in the coupling matrix  $\mathcal{N}$ , see (2.42). This implies that if we excite the gaugefields also the scalar fields will vary in space-time. In other words, charged objects will also couple to the scalar fields.

The presence of the scalar fields brings another new consequence for the theory. There where the gaugefield and metric both vanish at spatial infinity, if we restrict ourselves to asymptotically flat spaces, the scalar fields can take arbitrary constant values at infinity. The most natural way to see the geometry of space-time at infinity is as the vacuum with respect to which the full geometry is an 'excitation'. The introduction of the scalar fields brings us from Einstein-Maxwell theory with a single vacuum to the  $\mathcal{N} = 2$  theory with an infinite number of vacua. In general the set of vacua will be a submanifold of the scalar target space. In our rather simple case where there is no potential term for the scalars the vacuum manifold is the full target space.

Many physical properties, as e.g. the mass and the preserved supersymmetries, will depend on the choice of vacuum. We already discussed this for a probe particle around (2.63) and as we will see in this section it also holds for the full solutions to the theory (2.70). As we will see in the next chapter there are BPS states that exist in certain vacua but not in others.

Given these 'complications' with respect to ordinary Einstein-Maxwell theory it might seem like a daunting task to look for black hole solutions. As we will see below however, the complications are actually very minute in case we restrict ourselves to supersymmetric black holes. These black holes that preserve half of the supersymmetries of the Lagrangian are very much constrained and in this section we will construct the explicit form for their metric, and the electromagnetic and scalar fields they excite. Along the way we will gain many insights into their physics and learn some techniques that help understand their multicentered cousins introduced in the next chapter.

### 2.3.1 The attractor mechanism

First we will show that the supersymmetric black holes of our  $\mathcal{N} = 2$  theory (2.70) can be found as solutions to a set of first order equations. These equations, called the *attractor flow equations* are determined in terms of the black hole charge  $\Gamma$  and the special geometry data of the  $\mathcal{N} = 2$  theory. As we will see in the last part of this subsection they define a one parameter flow on moduli space which minimizes the central charge, itself a functional on moduli space.

### Constraints by symmetry

The black hole solutions we are looking for are static, spherically symmetric and should preserve one half of the supersymmetries. These constraints are very strong and it can be shown, see e.g. [66, 67], that the most generic metric with these symmetries is of the form:

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dx^i dx^i. \quad (2.71)$$

Spherical symmetry implies that the function  $U$  depends on the radial coordinate only, the same goes for the scalar fields:

$$U = U(r), \quad t^A = t^A(r). \quad (2.72)$$

We will also often use the coordinate  $\tau = \frac{1}{r}$  as a shorthand. The fact that we want the metric (2.71) to be asymptotically flat imposes the constraint  $U_{r=\infty} = U_{\tau=0} = 0$ . As we shall see later the points at  $r = 0$ ,  $\tau = \infty$  form a horizon.

Given that we are looking for a solution corresponding to a charge vector  $\Gamma$ , see (2.60), the normalization (2.59) and the demand of spherical symmetry fix the magnetic components of all the Maxwell fields completely:

$$F_m = \Gamma \sin \theta d\theta \wedge d\phi. \quad (2.73)$$

The various components of the generalized fieldstrength  $F$  are however related by construction, see (2.57, 2.58), and the magnetic components determine the electric part  $F_e$  uniquely. The total fieldstrength is:

$$F = F_m + F_e = \Gamma \sin \theta d\theta \wedge d\phi + \diamond \Gamma e^{2U} dt \wedge d\tau, \quad (2.74)$$

where the operation  $\diamond$  was defined in (2.50) in the previous section.

So the symmetries already fix the solution quite a bit, leaving only the functions (2.72) as dynamical variables. The equations of motion of the Lagrangian (2.70) are a set of coupled second order differential equations and still quite non-trivial. The requirement of some supersymmetry to be preserved however corresponds to satisfying the Killing spinor equations, coming from the vanishing of the fermionic supersymmetry variations. These Killing spinor equations are first order equations and very often, as in our case, a solution to them automatically solves the equations of motion. The most straightforward way to find a set of first order equations for the fields (2.72) is thus by analyzing the Killing spinor equations, as was done in the original work of [68, 69, 70]. Here we however opt for an alternative derivation of these equations following [27].

### Derivation of the attractor flow equations

Let us proceed by putting the expressions (2.71, 2.72, 2.74) obtained by symmetry, back into the supergravity action (2.29). After some algebra this leads

to the following one dimensional action<sup>8</sup> for the remaining fields (neglecting a boundary term proportional to  $\dot{U}$ ):

$$S_{\text{eff}} = -\frac{1}{2} \int_0^\infty d\tau \left( \dot{U} + G_{AB} \dot{t}^A \dot{t}^B + e^{2U} V(t) \right) - |Z|_{\tau=\infty}. \quad (2.75)$$

We used the shorthand notation  $Z = Z(\Gamma, t)$  and the potential is given by

$$V(t^A) = \frac{1}{2} |\Gamma|^2 = |Z|^2 + 4G^{AB} \partial_A |Z| \bar{\partial}_B |Z|. \quad (2.76)$$

The definition of  $|\Gamma|$  can be found in (2.52, 2.53). Using the special geometry result (2.56), the effective action can be rewritten as a sum of positive terms:

$$S_{\text{eff}} = -\frac{1}{4} \int_0^\infty d\tau e^{2U} \left| 2\text{Im} \left[ (\partial_\tau + i\text{Im}(\partial_A \mathcal{K} \dot{t}^A) + i\dot{\alpha}) (e^{-U} e^{-i\alpha} \Omega) \right] + \Gamma \right|^2 - |Z|_{\tau=0}, \quad (2.77)$$

where  $\alpha = \arg Z$ . As for the time-independent solutions we are considering the Lagrangian is equal to (minus) the Hamiltonian, the form (2.77) of the effective action immediately reveals a BPS bound for these solutions:

$$M \geq |Z|_{\tau=0}. \quad (2.78)$$

This bound is saturated and the action maximized in case the first term in (2.77) vanishes, leading to the first order equations

$$2\text{Im} \left[ (\partial_\tau + i\text{Im}(\partial_A \mathcal{K} \dot{t}^A) + i\dot{\alpha}) (e^{-U} e^{-i\alpha} \Omega) \right] = -\Gamma. \quad (2.79)$$

These equations can be simplified by taking the intersection product with  $\Gamma$  on both sides, leading to  $\text{Im}(\partial_A \mathcal{K} \dot{t}^A) + \dot{\alpha} = 0$ . So (2.79) simplifies to

$$2\partial_\tau (e^{-U} \text{Im} [e^{-i\alpha} \Omega]) = -\Gamma. \quad (2.80)$$

These are the first order differential equations referred to above and they are equivalent to the Killing spinor equations. In this form the BPS equations are simple enough to be integrated explicitly:

$$2e^{-U} \text{Im} [e^{-i\alpha} \Omega] = -\Gamma\tau + 2\text{Im} [e^{-i\alpha} \Omega]_{\tau=0}. \quad (2.81)$$

This rather simple formula encodes  $1 + 2h^{1,1}$  independent<sup>9</sup> real algebraic equations for the  $1 + 2h^{1,1}$  real variables  $U, B^A$  and  $J^A$ . They determine the radial dependence of  $U$  and hence of the full black hole metric, furthermore these equations encode a one parameter flow on moduli space by determining the scalar fields  $t^A$  in terms of  $\tau$  earning them the name of the *attractor flow equations*. In the next section we will construct the explicit solutions to (2.81) but first it will be interesting to study some of the properties of the flow on moduli space defined by (2.81) and explain its attractor behaviour.

<sup>8</sup> To be precise: this is not really an action but action per time. Furthermore after simplifying an action by inserting a certain ansatz the equations of motion from this reduced action have to be supplemented by additional constraints. In this case these constraints are automatically implied by the equations of motion of the reduced action, as is shown in [27].

<sup>9</sup> Note that (2.81) has  $2 + 2h^{1,1}$  components but that these are not independent as the intersection product of (2.81) with  $\Gamma$  is trivially zero.

### Properties of the attractor flow

To get a better understanding of the one-parameter flow on moduli space given by (2.81) let us take a step back to (2.80). By taking the intersection product of (2.80) with  $\Omega$  one can extract that

$$\dot{U} = -e^U |Z|. \quad (2.82)$$

Plugging this back into (2.80) and then taking the intersection product with  $\diamond\Gamma$  leads to the equation

$$\partial_\tau |Z| = -4e^U G^{AB} \partial_A |Z| \bar{\partial}_B |Z| \leq 0. \quad (2.83)$$

The inequality shows that when radially evolving inward the size of the central charge decreases. This implies that if we can solve the equations all the way up to  $\tau = \infty$ , the central charge thus reaches a minimum  $|Z|_{\tau=\infty} = |Z(\Gamma)|_{\min}$  there.

As we discussed before the values of the scalar moduli at radial infinity,  $t^A(\tau = 0)$ , are free boundary conditions. The fact that  $|Z|$  is at a minimum at  $\tau = \infty$  implies that there the moduli are at fixed values determined in terms of the charge  $\Gamma$ , independent of variations of the boundary conditions at  $\tau = 0$ . This important phenomenon goes under the name of the *attractor mechanism* [70, 59]. Note that in principle one could worry about multiple basins of attraction where for some values of the moduli the flow runs towards one local minimum and for other values to another local minimum. This was investigated in e.g. [71, 72] where it was found that this can indeed be the case but only in singular regions of moduli space. In this thesis we will restrict ourselves to regular parts of moduli space where for every charge  $\Gamma$  there is a unique minimum of the central charge.

The fixed values of the scalars at  $\tau$  can be found in terms of  $\Gamma$  by evaluating (2.81) in the near horizon region where  $\tau \rightarrow \infty$ . Equation (2.82) implies that in this regime  $e^{-U} \rightarrow |Z|_\infty \tau$ , this shows that  $\tau = \frac{1}{r} = \infty$  is a horizon. Plugging this into the flow equation (2.81) gives an algebraic equation for the values of the scalars at the horizon in terms of the charges  $\Gamma$ , these equations go under the name of the *attractor equations*:

$$2\text{Im} [\bar{Z}(\Gamma, t_\infty) \Omega(t_\infty)] = -\Gamma. \quad (2.84)$$

Because the moduli flow to a fixed value at the horizon also the near horizon geometry is completely determined in terms of the charges:

$$ds^2 = \frac{r^2}{|Z|_{\min}^2} dt^2 + \frac{|Z|_{\min}^2}{r^2} dr^2 + |Z|_\infty^2 d\Omega_2^2. \quad (2.85)$$

This metric is the  $\text{AdS}_2 \times \text{S}^2$  near horizon geometry shared by all four dimensional extremal black holes. The horizon is thus a sphere of radius  $|Z|_{\min}$  and so the Bekenstein-Hawking entropy is

$$S(\Gamma) = \pi |Z(\Gamma)|_{\min}^2. \quad (2.86)$$

It is important to stress that due to the attractor mechanism the entropy does not depend on the arbitrary values of the moduli at infinity and is completely determined in terms of the electromagnetic charges only. It is quite striking that the more complicated  $\mathcal{N} = 2$  theory (2.70) gives rise to the same entropy and near horizon geometry as a black hole of the same charges considered in the Einstein-Maxwell theory (2.6) without the dynamical scalar fields. Formally we can see Einstein-Maxwell theory as the  $\mathcal{N} = 2$  theory with the boundary conditions for the scalar fields fixed at the attractor point.

### Existence of solutions

Although formally one can always integrate (2.80) to (2.81) this does not guaranty a physically acceptable solution for all values of  $\tau$ . It is useful to think of (2.80) as defining a dynamics for the moduli. One can take the boundary values  $t^A(0)$  as initial conditions and then let the system evolve towards  $\tau > 0$  by the equations (2.80). In case  $|Z|_{\min} \neq 0$  one can show, see e.g. [71], that the moduli rapidly converge to their attractor values reaching these at  $\tau = \infty$ . This implies we have a well defined metric all the way from infinity to the near-horizon and that furthermore the near-horizon geometry is non singular, see (2.85).

Furthermore [71] showed that when  $|Z|_{\min} = 0$  the attractor values are reached after a finite 'time'  $\tau_*$  and one cannot extend the solution to infinite  $\tau$ . In this case there is no well defined corresponding space-time geometry and the solution should be discarded as unphysical. Note however that this argument is evaded in case the minimum  $|Z|_{\min} = 0$  is reached at a singularity or a boundary of moduli space. As we will later illustrate in section 2.5.2 this is for example the case for a pure D6 black hole where the attractor point lies on the boundary  $J = 0$  of moduli space. In that example the zero of the central charge is reached at  $\tau = \infty$  however, meaning we have a well defined space-time geometry all the way to the horizon. The fact that at the horizon  $|Z| = 0$  implies that the near horizon geometry is mildly singular, see (2.85), but this can possibly be circumvented by including higher curvature corrections from string theory. To summarize we have three different situations

- When  $|Z(\Gamma)|_{\min} \neq 0$  a physical black hole solution of charge  $\Gamma$  exists,
- In case  $|Z(\Gamma)|_{\min} = 0$  in a regular point of moduli space a physical black hole solution of charge  $\Gamma$  doesn't exist,
- If  $|Z(\Gamma)|_{\min} = 0$  in a singular or boundary point of moduli space a physical black hole solution of charge  $\Gamma$  may exist.

### 2.3.2 Solving the attractor flow equations

After having discussed some general properties it is time to return to equation (2.81) and show how it can be solved explicitly in terms of the various supergravity fields. The explicit solution will be found in two steps.

First, following [73] and [27], we will make a great step forward by determining the expression for all the fields in terms of a single real function  $\Sigma$ . As will follow from the calculation,  $\Sigma$  is nothing but the entropy as a function of the various charges.

Although the function  $\Sigma$  is completely determined by the attractor equations it takes some more work to explicitly find it as this depends strongly on the special geometry of the  $\mathcal{N} = 2$  theory of interest. For Calabi-Yau compactifications an explicit form for the entropy function was found in [74] and we will use it to come to an (almost) fully explicit solution of all the relevant supergravity fields in terms of the charge  $\Gamma$  and the boundary values of the scalar fields.

### Towards a single function

Our starting point will be the implicit solution (2.80) of the attractor flow equations. Let us rewrite it here indicating more precisely its dependence on the relevant variables<sup>10</sup>:

$$2e^{-U} \text{Im} \left[ e^{-i\alpha(H,t)} \Omega(t) \right] = -H, \quad (2.87)$$

where we defined a harmonic function  $H(\tau) = \Gamma\tau - 2\text{Im} \left[ e^{-i\alpha} \Omega \right]_{\tau=0}$ .

We are looking for an explicit form of both the scalars  $t^A(\tau)$  and the warping  $U(\tau)$  in terms of  $\tau = \frac{1}{r}$ . As  $H$  is a very simple function in terms of  $\tau$  we will first rephrase our problem to that of solving for  $t^A(H)$  and  $U(H)$ . First we will solve for  $U$ , this can simply be done taking the intersection product of (2.87) with  $\Omega(t)$ . The resulting equation is

$$e^{-2U(H)} = |Z(H, t(H))|^2 =: \Sigma(H). \quad (2.88)$$

At first sight this might not be much new information as  $\Sigma$  is essentially unknown, but as we will see it is also possible to determine the  $t^A$  in terms of  $\Sigma$ .

One can now plug this first 'result' (2.88) back into (2.87) to find

$$2\text{Im} \left[ \bar{Z}(H, t) \Omega(t) \right] = -H. \quad (2.89)$$

This essentially gives us the imaginary part of the  $t^A$  as  $\langle \Omega, D^A \rangle \sim -t^A$ , more precisely we can extract out of this equation that

$$\begin{aligned} 2\text{Im} \left[ \bar{Z}(H, t) t^A \right] &= \frac{1}{2} \sqrt{3J^3} H^A, \\ 2\text{Im} \left[ \bar{Z}(H, t) \right] &= \frac{1}{2} \sqrt{3J^3} H^0. \end{aligned} \quad (2.90)$$

So if we can find expressions for the real parts of  $\bar{Z}(H, t) t^A$  and  $\bar{Z}(H, t)$  we can solve for the  $t^A$ . Taking the intersection product of (2.87) with  $\mathcal{D}_A \Omega$  and using

<sup>10</sup> We have used that  $\alpha(H, t) = \alpha(\Gamma, t)$ , an equality which follows from (2.80) after taking the intersection product with  $H$ .

some tricks allows one to exactly do that, see [73] for the details, and the result is

$$\begin{aligned} 2\text{Re} [\bar{Z}(H, t)] &= -\frac{1}{2}\sqrt{3J^3} \frac{\partial \Sigma(H)}{\partial H_0}, \\ 2\text{Re} [\bar{Z}(H, t)t^A] &= \frac{1}{2}\sqrt{3J^3} \frac{\partial \Sigma(H)}{\partial H_A}. \end{aligned} \quad (2.91)$$

Combining this with (2.90) now straightforwardly leads to

$$t^A = \frac{H^A - i\partial_{H_A}\Sigma}{H^0 + i\partial_{H_0}\Sigma}. \quad (2.92)$$

An important property that follows from the derivation in [73] we referred to, is that  $\Sigma$  is homogeneous of degree two, i.e.

$$\Sigma(\lambda H) = \lambda^2 \Sigma(H). \quad (2.93)$$

We will shortly say more about the interpretation of the function  $\Sigma$  but let us first give a more explicit form for the Maxwell fields. Still implicit in (2.74) is the fact that  $\diamond\Gamma$  depends on the moduli  $t^A$  and thus on  $\tau$ . One can rewrite  $\diamond\Gamma$  by expanding it in the basis  $\Omega, \mathcal{D}_A\Omega, \bar{\mathcal{D}}_A\bar{\Omega}$  and  $\bar{\Omega}$  and then using the formula (2.56) and the fact that  $\text{Im}(\partial_A\mathcal{K}t^A) + \dot{\alpha} = 0$  for solutions to the attractor equations, the result is

$$\diamond\Gamma = -2e^{-2U} \text{Re} [\partial_\tau (e^U e^{-i\alpha}\Omega)]. \quad (2.94)$$

Plugging this into (2.74) and using (2.91) we find rather simple expressions for the gaugefields in terms of  $\Sigma$ :

$$A^0 = -\partial_{H_0} \log \Sigma \frac{dt}{\sqrt{G_4}} - p^0 \cos \theta d\phi, \quad (2.95)$$

$$A^A = \partial_{H_A} \log \Sigma \frac{dt}{\sqrt{G_4}} - p^A \cos \theta d\phi. \quad (2.96)$$

We can now summarize the solution to the attractor flow equations in terms of a single function  $\Sigma(H)$  that is still to be determined:

$$\begin{aligned} ds^2 &= -\frac{dt^2}{\Sigma} + \Sigma dx^i dx^i, \\ t^A &= \frac{H^A - i\partial_{H_A}\Sigma}{H^0 + i\partial_{H_0}\Sigma}, \\ A^0 &= -\partial_{H_0} \log \Sigma \frac{dt}{\sqrt{G_4}} - p^0 \cos \theta d\phi, \\ A^A &= \partial_{H_A} \log \Sigma \frac{dt}{\sqrt{G_4}} - p^A \cos \theta d\phi, \\ H &= \frac{\sqrt{G_4}\Gamma}{r} - 2\text{Im} (e^{-i\alpha}\Omega)_{r=\infty}. \end{aligned} \quad (2.97)$$

### The entropy function

Now let us have a closer look at the function  $\Sigma$ . In the near horizon limit we know the expression for  $U$  and as such also for  $\Sigma$ :

$$\Sigma(\Gamma\tau) = |Z(\Gamma, t(\Gamma))|^2 \tau^2 = \frac{1}{\pi} S(\Gamma) \tau^2. \quad (2.98)$$

Due to the fact that  $\Sigma$  is homogeneous of degree 2, see (2.93), we see that

$$\Sigma(\Gamma) = \frac{1}{\pi} S(\Gamma). \quad (2.99)$$

So  $\Sigma(H)$  is nothing but the entropy of the black hole in terms of the charges, with the charges replaced by their corresponding harmonic functions.

Note that in 'solving' the attractor equations above we didn't use all of the  $2h^{1,1} + 1$  equations, the remaining ones can be used to determine an explicit expression for  $\Sigma$ . This is however quite complicated and depends strongly on the form of the prepotential. In case of a Calabi-Yau compactification the solution was found in [74], the result reads in our conventions

$$\begin{aligned} \Sigma(H) &= \sqrt{\frac{Q^3(H) - L^2(H)}{(H^0)^2}}, \\ L(H) &= H_0(H^0)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0, \\ Q^3(H) &= \left( \frac{1}{3} D_{ABC} y^A(H) y^B(H) y^C(H) \right)^2. \end{aligned} \quad (2.100)$$

Here the  $y^A(H)$  are the solutions to an equation determined by the Calabi-Yau intersection products:

$$D_{ABC} y^A y^B = -2H_C H^0 + D_{ABC} H^A H^B. \quad (2.101)$$

So up to this last algebraic equation we have by a combination of (2.100) and (2.97) an explicit form of the metric, gaugefields and scalars as functions of the radial coordinate  $r$  in terms of the charge  $\Gamma$  of the black hole and a choice  $t_\infty^A$  for the asymptotic scalars. Note that equation (2.101) can be solved trivially in the one modulus case  $h^{1,1}$ , but otherwise has to be analysed for each Calabi-Yau separately.

### Black holes in type IIA on a $CY_3$

Finally for later convenience and reference we summarize the combination of the explicit expression for  $\Sigma$  (2.100) and the black hole solution (2.97). In this overview we choose to rewrite the magnetic part of  $A^0$  and  $A^A$  in a slightly more abstract way than in (2.97) to make the parallel to the multicenter solutions of the next chapter more transparent. In the equations below  $G_4$  is the four dimensional Newton's constant and  $\star$  the Hodge dual on the flat  $\mathbb{R}^3$  given by spatial coordinates  $x^i$ .

$$\begin{aligned} ds^2 &= -\frac{1}{\Sigma} dt^2 + \Sigma dx^i dx^i, \\ A^0 &= \frac{-L}{\Sigma^2} \frac{dt}{\sqrt{G_4}} + \omega_0 \\ A^A &= \frac{H^A L - Q^{3/2} y^A}{H^0 \Sigma^2} \frac{dt}{\sqrt{G_4}} + \mathcal{A}_d^A, \\ t^A &= \frac{H^A}{H^0} + \frac{y^A}{Q^{\frac{3}{2}}} \left( i\Sigma - \frac{L}{H^0} \right), \end{aligned} \quad (2.102)$$

$$H = \frac{\sqrt{G_4} \Gamma}{r} - 2\text{Im} \left( e^{-i\alpha} \Omega \right)_{r=\infty}. \quad (2.103)$$

$$\begin{aligned} \Sigma &= \sqrt{\frac{Q^3 - L^2}{(H^0)^2}}, \\ L &= H_0 (H^0)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0, \\ Q^3 &= \left( \frac{1}{3} D_{ABC} y^A y^B y^C \right)^2, \\ D_{ABC} y^A y^B &= -2H_C H^0 + D_{ABC} H^A H^B. \end{aligned} \quad (2.104)$$

$$\begin{aligned} d\omega_0 &= \frac{1}{\sqrt{G_4}} \star dH^0, \\ d\mathcal{A}_d^A &= \frac{1}{\sqrt{G_4}} \star dH^A. \end{aligned}$$

## 2.4 A detour to eleven/five dimensions

One of the seminal breakthroughs in string duality was the insight [75] that at strong coupling Type IIA string theory is best described as an 11 dimensional theory, called M-theory. The low energy limit of this theory is 11-dimensional

supergravity, a unique theory with 32 supercharges. This relation between IIA in ten dimensions and M-theory in 11 dimensions translates itself under a Calabi-Yau compactification as a relation between our  $\mathcal{N} = 2$  supergravity (2.29) in 4 dimensions and a  $\mathcal{N} = 1$  supergravity in 5 dimensions [76].

It will turn out useful to study various properties of the supersymmetric black holes of the previous section by mapping them to five dimensions. This will especially be the case when we will define a decoupling limit for these black holes and their multicenter generalizations in chapter 4, where as we will argue the five dimensional description is the natural one. This relation to M-theory and five dimensions is also an essential ingredient in the microstate counting we will discuss at the end of this chapter. So leaving a more detailed motivation for later we introduce some aspects of this map of four dimensional black holes to five/eleven dimensions.

### Type IIA and M-theory

We will remain very brief here referring for more insights and details to textbooks as [15, 16]. For the readers convenience we have summarized many relations between the physical parameters of M-theory, IIA and four and five dimensions in appendix A. So let it suffice here to just stress the most important features we will need later on.

The bosonic sector of eleven dimensional supergravity consists of a metric  $g$  and a three-form gaugefield  $A_3$ , the action is given by [77]

$$S_M = \frac{2\pi}{l_{11}^9} \int \sqrt{-\det g} \mathcal{R} - \frac{\pi}{l_{11}^3} \int G \wedge \star G + \frac{1}{6} \int G \wedge G \wedge A_3, \quad (2.105)$$

where  $G = dA_3$ .

Eleven dimensional supergravity is equivalent at low energies to type IIA supergravity when the eleventh dimension is assumed to be a small circle. We write the eleven dimensional metric in terms of the ten dimensional one (in string frame) as [75]

$$ds_{11d}^2 = R^2 e^{4\phi/3} (d\theta + 2\pi A_1)^2 + e^{-2\phi/3} ds_{10d}^2, \quad (2.106)$$

with  $\theta \cong \theta + 2\pi$  the coordinate along the M-theory circle. If one performs the Kaluza Klein reduction,  $\phi$  can be identified with the dilaton of IIA and  $A_1$  with the RR one form  $C_1$ . Under this reduction the three form  $A_3$  decomposes as  $A_{3\mu\nu\rho} = C_{3\mu\nu\rho}$  and  $A_{3\mu\nu 11} = B_{2\mu\nu}$ ,  $\mu, \nu = 1 \dots 10$ , where  $C_3$  and  $B_2$  are the IIA fields as in (2.15).

As M-theory has only a single gaugefield there are only two types of conserved U(1) charges in the theory. The fundamental objects carrying these charges are an M2-brane that couples electrically to  $A_3$  and an M5 brane that is magnetically charged.

The M2 brane action is very simple as the only worldvolume degrees of freedom are the scalar fields  $X^\rho$  defining the embedding of the 2-brane in space-

time. The M2 brane action is given by [16]

$$S_{M2} = -T_{M2} \int d^3\sigma \sqrt{g_{\rho\sigma} \partial_\alpha X^\rho \partial_\beta X^\sigma} + \mu_{M2} \int A_3. \quad (2.107)$$

The M5 brane has a slightly more complicated dynamics as apart from the embedding scalars it also has a worldvolume two form-from gaugefield  $b$ . Furthermore its three-form fieldstrength  $h = db$  is self-dual making the formulation of an action slightly more involved. The action is given for example in [78], we will not repeat it here as we won't need its explicit expression. What will be of some importance later is to know that the M5-brane worldvolume action has (0,2) supersymmetry. The fact that the theory is chiral is of course related to the self duality constraint. The appearance of the worldvolume gaugefield is due to the fact that M2 branes can end on an M5-brane in analogy with open strings inducing a worldvolume one-form on D-branes. Note that here the situation is less well understood though, as a way to quantize M2 branes remains unknown for the moment.

Let us end by giving the relation between the various fundamental objects in IIA and those in M-theory:

- a D0 brane lifts to a gravitational shockwave, its conserved charge becomes momentum around the M-theory circle
- a IIA string becomes an M2 brane wrapped around the circle
- a D2 brane corresponds to an M2 brane stretched in the 10 external dimensions
- a D4 brane is an M5 brane wrapped around the M-theory circle
- a NS5 brane is identified with an M5 brane stretched in the 10 external dimensions
- a D6 brane relates to a gravitational monopole, its conserved charge becomes a topological invariant of the space-time manifold

We denote the M-theory charges introduced above as (P,M2,M5,KK).

### Supersymmetric black holes/rings in five dimensions

As we reviewed in section 2.2.1, type IIA supergravity gives rise to a four dimensional  $\mathcal{N} = 2$  supergravity theory when compactified on a Calabi-Yau manifold  $X$ . In case we compactify M-theory on the same Calabi-Yau manifold  $X$  it is thus described by a five dimensional  $\mathcal{N} = 1$  supergravity. This five dimensional theory is then related to the four dimensional one by further compactification along the eleventh dimensional circle.

We won't go into the details of these relations and the precise form of the five dimensional theory but will immediately jump to the description of our black

holes (2.97) from the five dimensional point of view. We will follow [79, 80] where furthermore details on the five dimensional supergravity can be found.

Given a solution to the four dimensional  $\mathcal{N} = 2$  lagrangian (2.70) the five dimensional solution is given in terms of the 4d solution as

$$\begin{aligned} ds_{5d}^2 &= \tilde{V}_{\text{IIA}}^{2/3} \ell_5^2 (d\psi + A^0)^2 + \tilde{V}_{\text{IIA}}^{-1/3} \frac{\hat{R}}{2} ds_{4d}^2, \\ A_{5d}^A &= A^A + B^A (d\psi + A^0), \\ Y^A &= \tilde{V}_{\text{IIA}}^{-1/3} J^A, \quad \tilde{V}_{\text{IIA}} = \frac{D_{ABC}}{6} J^A J^B J^C = \frac{1}{2} \left( \frac{\Sigma}{Q} \right)^3. \end{aligned} \quad (2.108)$$

Here  $\psi$  parameterizes the M-theory circle with periodicity  $4\pi$ . We defined, in terms of the 11d Planck length  $l_{11}$  and the physical asymptotic M-theory circle radius  $R$ ,

$$\ell_5 = \frac{l_{11}}{4\pi \tilde{V}_M^{1/3}}, \quad \hat{R} = \frac{R}{\ell_5}, \quad (2.109)$$

where  $\tilde{V}_M = V_M/l_{11}^6$  is the M-theory volume of  $X$  in 11d Planck units. The reduced 5d Planck length  $\ell_5$  is related to the 4d Newton constant  $G_4$  by

$$\ell_5^3 = R G_4 \quad (2.110)$$

and we also have the relation  $\hat{R} = 2 \tilde{V}_{\text{IIA}}^{1/3}|_\infty$ , see appendix A for more details.

We see from (2.108) that the real part of the four dimensional vectormultiplet scalar,  $B^A$ , is absorbed into the five dimensional gaugefield. Furthermore in five dimensions there are  $h^{1,1}$  real vectormultiplet scalars  $Y^A$  given in terms of the four dimensional Kähler moduli  $J^A$  by (2.108). Note that only  $h^{1,1} - 1$  of the  $Y^A$  are independent since by construction they satisfy the constraint  $D_{ABC} Y^A Y^B Y^C = 6$ . This is closely related to the special geometry of the 5 dimensional  $\mathcal{N} = 1$  theory. In 5 dimensions the vector multiplet has so called *very special real Kähler geometry* while in four dimensions it is of special Kähler type.

The five dimensional vector multiplet scalars  $Y^A$  are related to the M-theory Kähler moduli by  $J_M^A = \tilde{V}_M^{1/3} Y^A$ . Here  $\tilde{V}_M = \frac{V_M}{l_{11}^6}$  is the volume of the internal Calabi-Yau  $X$  as measured with the M-theory metric. Note that this differs from volume of  $X$  as measured in IIA, the relation between the two volumes follows from (2.106). As a result, see appendix A for some details, the volume of the Calabi-Yau  $X$  as measured in M-theory is a hypermultiplet scalar. In the black hole solutions of our interest  $\tilde{V}_M$  is thus an arbitrary constant as all the other hypermultiplet fields. Remember that the volume of  $X$  as measured in IIA is determined by the 4 dimensional vectormultiplet scalars:  $\tilde{V}_{\text{IIA}} = V_{\text{IIA}}/l_s^6 = \mathcal{V} = \frac{J^3}{6}$ , see also (2.38).

The rather general formula (2.108) can now be applied to the solutions (2.102) to give the map of our black holes in 5d dimensions. We will not do this here

explicitly as it involves only trivial algebra and the result isn't very enlightening. The different types of charges in five dimensions have a natural interpretation in M-theory. We can summarize the various relations between the different branes discussed above as  $(D6, D4, D2, D0) \rightarrow (KK, M5, M2, P)$ .

When adding the additional dimension we effectively smear the four dimensional solution along the circle. This has as a result that a black hole in four dimensions will generically become a *black ring* in 5 dimensions. A black hole of charge  $(0, p^A, q_A, q_0)$  will become a black string wrapped around the M-theory circle. Furthermore it rotates with momentum  $P = q_0$  around that circle.

In case the four dimensional black hole carries D6 charge however the M-theory circle pinches off at the position of the black hole. In this case the four dimensional black hole will remain a pointlike black hole in 5 dimensions. This is made more explicit in the following examples section, so we will leave further discussion for there.

## 2.5 Examples

Having spent quite some time on the general theory it is time to illustrate some points by two examples.

### 2.5.1 The D4D2D0 black hole

This first 'example' still contains a whole set of different black holes, but they share many similar properties. We will discuss the solution in four and five dimensions mainly focussing on the near horizon region.

A black hole formed by wrapped D4, D2 and D0 branes has the 4 dimensional charges

$$\Gamma = (0, p^A, q_A, q_0). \quad (2.111)$$

We will assume  $p = p^A D_A$  to lie in the Kähler cone. We will work in a basis where all the  $D_A$  lie in the Kähler cone themselves, which then implies  $p^A > 0$ .

There won't exist a black hole for all possible values of the charges (2.111). When discussing the properties of the attractor flow equations we saw that a regular black hole solution exists if the minimum of  $|Z|$  is non-zero. By a short argument as in [29] we can show that there is a zero for the central charge in case  $\hat{q}_0 = q_0 - \frac{1}{2} D^{AB} q_A q_B > 0$ , where we used the notation  $D^{AB} = (D_{AB})^{-1} = (D_{ABC} p^C)^{-1}$ . To show this, rewrite the central charge for  $\Gamma$ , see (2.46), as

$$Z(\Gamma) = -\frac{1}{\sqrt{\frac{4J^3}{3}}} \left( D_{AB} (\tilde{B}^A + iJ^A) (\tilde{B}^B + iJ^B) + \hat{q}_0 \right), \quad (2.112)$$

where  $\tilde{B}^A = B^A - D^{AB} q_B$ . Now one directly sees that  $Z = 0$  iff

$$D_{AB} \tilde{B}^A J^B = 0 \quad \text{and} \quad \frac{1}{2} (D_{AB} \tilde{B}^A \tilde{B}^B - D_{AB} J^A J^B) = \hat{q}_0. \quad (2.113)$$

By using a mathematical result called the Hodge index theorem [81]  $D_{AB}$  is a metric of signature  $(1, h^{11} - 1)$  where the one dimensional positive definite subspace is spanned by  $J$  and the directions orthogonal to it are negative definite. So  $D_{AB}J^AJ^B > 0$  and the first equation of (2.113) tells us that  $D_{AB}\tilde{B}^A\tilde{B}^B < 0$ . This implies that there is a point where  $Z = 0$  in moduli space iff  $\hat{q}_0 \geq 0$ .

By our discussion at the end of section 2.3.1 a well defined attractor flow in this example only exists when  $\hat{q}_0 < 0$ . In this case we can use our previous formulas, summarized in (2.102), to calculate the attractor values of the moduli<sup>11</sup>

$$t^A(\Gamma) = D^{AB}q_B + ip^A \sqrt{\frac{-6\hat{q}_0}{p^3}}. \quad (2.114)$$

Similarly we can calculate the entropy of the black hole in terms of the charges from (2.102):

$$S = \pi\Sigma(\Gamma) = 2\pi\sqrt{-\hat{q}_0 \frac{p^3}{6}} \quad (2.115)$$

One can than easily check that the near horizon geometry of the black hole is  $\text{AdS}_2 \times \text{S}^2$  with radius  $R_2 = (-\frac{2}{3}p^3\hat{q}_0)^{1/4}\sqrt{G_4}$ ,

$$ds_\star^2 = -\frac{r^2}{R_2^2}dt^2 + \frac{R_2^2}{r^2}dr^2 + R_2^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (2.116)$$

In this near horizon region the  $U(1)$  fields become purely magnetic and wrap the  $\text{S}^2$ :

$$F^A = p^A \sin\theta d\theta \wedge d\phi. \quad (2.117)$$

As by assumption  $p > 0$  we see that the near horizon geometry and the entropy (2.115) are only well defined in case  $\hat{q}_0 < 0$ , in agreement with our analysis of the central charge.

Let us now lift the black hole to five dimensions via (2.108). The prefactor of the  $d\psi^2$  term in (2.108) indicates that the M-theory circle grows like  $\frac{1}{r}$  near the horizon. So in five dimensions the horizon has a topology of  $\text{S}^2 \times \text{S}^1$  and thus the four dimensional black hole lifts to a five dimensional black ring wrapped around the M-theory circle. The near horizon geometry lifts to a quotient of  $\text{AdS}_3 \times \text{S}^2$  [82]:

$$ds^2 = \frac{u^2}{4U^2} \left( -dt d\psi + \frac{U^{-1}S^2}{\pi^2 r} d\psi^2 \right) + \frac{4U^2}{u^2} du^2 + U^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (2.118)$$

where the radius of the  $\text{S}^2$ ,  $U = \left(\frac{p^3}{6}\right)^{1/3}$ , is half the radius of the  $\text{AdS}_3$ .

<sup>11</sup> Although the formulas of (2.102) look singular for the case  $p^0 = 0$  it is easily verified that they have a well defined limit  $p^0 \rightarrow 0$ . It is this limit, which we haven't explicitly written down but that can be straightforwardly calculated, that we use here.

### 2.5.2 Purely fluxed D6 vs Taub-NUT

In this second example we take a closer look at a black hole formed by a purely fluxed D6-brane. In case we wrap the internal Calabi-Yau with a D6 brane which has some worldvolume flux  $F = p = p^A D_A$ , it gives rise to a four dimensional black hole with charges

$$\Gamma = e^p = (1, p^A, \frac{p_A^2}{2}, \frac{p^3}{6}). \quad (2.119)$$

Calculating the attractor values for the moduli from (2.102) we find

$$B_*^A = p^A \quad \text{and} \quad J_*^A = 0. \quad (2.120)$$

In this case the attractor point lies on the boundary of moduli space (the Kähler moduli need to be in the Kähler cone  $J^A \geq 0$ ). Furthermore it is easy to check that the central charge  $Z(\Gamma)$  is zero at this attractor point.

This example is thus one of the boundary cases for the existence of the solution, as discussed at the end of section 2.3.1. Let us look at the metric for the charge  $\Gamma$ , for convenience we take the asymptotic moduli to be

$$B_\infty^A = p^A \quad \text{and} \quad J_\infty^A = j_\infty p^A, \quad (2.121)$$

with  $j_\infty$  some positive constant. We make this choice as it simplifies the full metric tremendously. Do note that all our comments on the near horizon region remain general however as they are independent of the boundary conditions for the scalars. Under the assumption (2.121) the 4d metric given by (2.102) becomes

$$ds^2 = \left(1 + \frac{a}{r}\right)^{-1} dt^2 + \left(1 + \frac{a}{r}\right) dx^i dx^i, \quad (2.122)$$

here  $a = \sqrt{\frac{J_\infty^3}{3}} G_4$ . This metric is regular everywhere but has a naked curvature singularity at  $r = 0$ , essentially due to the fact that the central charge vanishes at  $r = 0$ .

Calculating the entropy from the general expression (2.104) gives  $S(\Gamma) = 0$ . Taking into account higher curvature corrections due to internal instantons the singularity will be resolved and the solution becomes an *empty hole* [27].

#### Taub-NUT

When we lift the singular geometry (2.122) to five dimensions the metric becomes completely smooth and there is no need for  $\alpha'$  corrections [83]. Calculating the 5d metric from (2.108) gives

$$\begin{aligned} ds_{5d}^2 = & -dt^2 + \left(1 + \frac{R}{2r}\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) \\ & + \frac{R}{4} \left(1 + \frac{R}{2r}\right)^{-1} (d\psi^2 - \cos \theta d\phi)^2, \end{aligned} \quad (2.123)$$

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where  $R$  is the radius of the M-theory circle as in (2.106), furthermore we used  $R = 2a$ , see appendix A. The space-like part of this metric is the Euclidean version of the well known *Taub-NUT* metric, see e.g. [84]. Being a bit sloppy we will refer to the complete metric (2.123) as Taub-NUT, another common name for it is *Kaluza-Klein monopole* or *gravitational monopole*. The last name is due to the similarity to magnetic monopoles; the  $U(1)$  parameterized by  $\psi$  is non-trivially fibred over the  $\mathbb{R}^3$  base.

At infinity the metric becomes that of flat space and the topology of the angular manifold  $(\theta, \phi, \psi)$  is  $S^2 \times S^1$ . In the 'near horizon' region where  $r \rightarrow 0$  however the topology becomes that of  $S^3$  as the  $U(1)$  fibration becomes the well-known Hopf fibration. Furthermore this  $S^3$  shrinks exactly to zero as happens for the angular variables of  $\mathbb{R}^4$  in standard spherical coordinates, showing that the apparent singularity of (2.123) at  $r = 0$  is a mere coordinate singularity.

An intuitive way to think about Taub-Nut is as a 'cigar' as near  $r = 0$  space looks like a hemisphere of an  $S^4$  while for large  $r$  it takes the shape of a cylinder  $\mathbb{R}^3 \times S^1$ .

Adding additional D4D2D0 charge to the purely fluxed D6 black hole results in five dimensions to a black hole in the center of this Taub-Nut [79].

## 2.6 A microscopic derivation of the Bekenstein-Hawking entropy

As we shortly discussed in section 2.2.3, at weak coupling the supersymmetric black holes we introduced can be described by D-brane states. In [22] it was shown for a certain set of 5-dimensional black holes that the degeneracy of the D-brane groundstates exactly reproduces the black hole entropy in the large charge limit. This result was the first example of a black hole where a statistical derivation of the black hole entropy could be made. In the wake of [22] many other examples were found, most of them supersymmetric black holes but also some generalizations to the non-BPS case were made, see [23] for a review. The discussion in this section focusses on a microscopic derivation of the entropy of the supersymmetric D4D0 black holes introduced in this chapter. The first such derivation was made in [36] in a way analogous to [22]. As we will discuss in this section, when mapped to 5 dimensions the D4D0 black hole becomes an M5 black string. In the appropriate regime the worldvolume theory of this string is expected to become a 2 dimensional conformal field theory (CFT). In our specific case the field content of that CFT is given by a reduction of the M5 worldvolume theory, we will refer to this CFT as the MSW CFT. One can then use the well known result of Cardy [85] that the number of states at large momentum is determined solely by the central charge of the CFT. Counting the number of states with Cardy's formula one recovers the entropy of the black hole given by the Bekenstein-Hawking formula. Note that the agreement of the number of states in the microscopic regime is guaranteed to be the same as in

the supergravity regime by supersymmetry. This because for large charges the number of states is essentially given by an index, the elliptic genus [86], which is protected by supersymmetry from changes under variations of the coupling.

In this section we will try to be brief and focus on the main points leading to the derivation of the entropy. We will revisit the MSW CFT in some more detail in section 4.5.3, where we propose some possible refinements to the model.

As a final note before we start this section let us point out that there are other (probably equivalent) microscopic pictures in which the D4D0 black hole entropy can be derived. In [87] the microscopic entropy was calculated directly in four dimensions by studying quantummechanics on the D-brane moduli space. Another more recent derivation was made in [33] where near horizon D-brane states are counted. We will come back to this last proposal in some detail in chapter 5.

### 2.6.1 The microscopic degrees of freedom

As we saw in section 2.4 when lifted to 5 dimensions a D4D0 black hole of charge  $(0, p^4, 0, q_0)$  is described by a black string which has an  $\text{AdS}_3 \times \text{S}^2$  near-horizon. Microscopically, at small  $g_s$ , this string can be interpreted as an M5-brane wrapped around the 4-cycle  $p$  in the internal Calabi-Yau with momentum  $q_0$  around the M-theory circle. In case we take the length of the M-theory circle, along which the string is wrapped, much larger than the Calabi-Yau volume the M5-worldvolume theory effectively reduces to a 2 dimensional theory on the worldvolume of the string. At low energies the theory is expected to reach a conformal fixed point [36]. In this CFT it is enough to know the bosonic and fermionic field content to calculate the number of groundstates, due to a result of Cardy [85].

The different fields on the string come from the reduction of the M5 worldvolume theory over the 4-cycle  $p$ . We already shortly mentioned this theory in section 2.4. The M5 worldvolume theory consists of scalar fields, fermions and a self-dual 3-form fieldstrength. The M5 theory has  $(0,2)$ -supersymmetry and as in the case of our interest the M5 is wrapped supersymmetrically this will result in a  $(0,4)$ -supersymmetric theory on the string [88].

#### The embedding scalars

The first set of degrees of freedom on the string come from the reduction of the M5 worldvolume scalars. These scalars represent the embedding of the M5 in the 11-dimensional space-time and become scalar fields on the string. Of these embedding scalars three denote the embedding of the string into the external dimensions, we denote them with  $x^i$ . To find the other scalar degrees of freedom we have to work a bit harder and first understand better the embedding of the M5 in the Calabi-Yau manifold.

As we discussed before in section 2.2.3, to be supersymmetric the M5 has to wrap a holomorphic four cycle. Such a holomorphic surface is of complex

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codimension 1 in our  $CY_3$  and is called a *divisor* [89]. The homology class of this divisor is specified by the charge  $p = p^A D_A$  in four dimensions. However there are many different divisors (i.e. holomorphic surfaces) sharing this same homology class. There is thus a whole moduli space of possible holomorphic embeddings that give rise to the same charge in 4/5-dimensions and the remaining worldvolume scalar fields of the M5 can thus be understood as coordinates on this moduli space. Let us make this a little bit more precise. Take a line bundle  $\mathcal{L}$  on the Calabi-Yau  $X$ , the zero locus of a section of this bundle then defines a divisor. So  $H^0(X, \mathcal{L})$ , the set of sections of  $\mathcal{L}$ , can be identified with a set of divisors. To be more precise we should really consider the projectivization  $PH^0(X, \mathcal{L})$ , as two sections  $s$  and  $s' = \lambda s$  define the same divisor. One can show that all the divisors defined in this way have the same representation in cohomology, namely  $c_1(\mathcal{L})$ . It is one of the more basic facts in complex geometry that this construction is one to one, i.e. all divisors with cohomology class  $p$  are the zero's of a given section of a line bundle  $\mathcal{L}$  with  $c_1(\mathcal{L}) = p$ , see e.g. [89, 81]. So what this mathematical digression teaches us physically, is that we can understand the remaining scalars on the M5 worldvolume as taking values in  $PH^0(X, \mathcal{L})$ . The dimension of this space thus gives the number of such independent scalars on the string. Under the assumption that  $p$  is very ample, or, expressed in a basis  $D_A$  that lies in the Kähler cone,  $p^A \gg 1$ , one can calculate this dimension using index theory, see [36]. The result is (neglecting terms subleading in  $p$ )

$$\dim PH^0(X, \mathcal{L}) = \frac{p^3}{3} \quad \text{when} \quad c_1(\mathcal{L}) = p \gg 1. \quad (2.124)$$

So to summarize the physical result of this discussion: The scalar fields on the M5 worldvolume reduce to the following scalar fields on the effective string

- three real scalars  $x^i$  parameterizing the embedding of the string in the 3 large spatial dimensions,
- $\frac{p^3}{3}$  complex scalars  $z^a$  interpreted as divisor moduli.

As will be important later on, these scalar fields have both left and right moving components along the string.

### The worldvolume gaugefield

As mentioned before there is also a worldvolume gaugefield  $b$  on the M5, which has a self-dual fieldstrength  $h = db$ . The following reduction ansatz can be made for the two-form  $b$  on the four cycle  $p$ :

$$b = b^\alpha \wedge \Sigma_\alpha \quad (2.125)$$

where  $\Sigma_\alpha$  is an integral harmonic basis of  $H^2(p)$  and the  $b^\alpha$  are scalar fields on the string. Note that in principle one could also expect a one form  $\beta^n$  in this expansion through a term  $\beta^n \omega_n$  with  $\omega_n$  in  $H^1(p)$ . In the case of  $p$  very ample

we are considering here, there is the property [36] that  $b^1(p) = b^1(X)$  and as can be read from table 2.1 this is zero. So  $\omega_n = 0$  and we see that we only need to take into account the scalars  $b^\alpha$  on the effective string.

The basis  $\Sigma_\alpha$  can be split into the direct product of the self-dual  $\Sigma_{\hat{\alpha}} = \star \Sigma_{\check{\alpha}}$  and anti-self dual forms  $\Sigma_{\check{\alpha}} = -\star \Sigma_{\hat{\alpha}}$ ,  $\alpha = (\hat{\alpha}, \check{\alpha})$ . By the self-duality of  $h$  we see from (2.125) that the scalars split into purely left- and rightmoving degrees of freedom:

$$b^{\hat{\alpha}}(\tau, \sigma) = b_+^{\hat{\alpha}}(\tau + \sigma) \quad \text{and} \quad b^{\check{\alpha}}(\tau, \sigma) = b_-^{\check{\alpha}}(\tau - \sigma) \quad (2.126)$$

The upshot of the discussion is that we can count the number of right/left moving scalars, coming from the two-form gaugefield, from the dimensions of  $H_+^2(p)$  and  $H_-^2(p)$  respectively. These dimensions can be calculated in terms of some topological invariants of the divisor and the result is [36]

$$\dim H_-^2(p) = \frac{2p^3}{3} \quad \text{and} \quad \dim H_+^2(p) = \frac{p^3}{3} \quad (2.127)$$

So to summarize, we find in this sector of the string worldvolume theory

- $\frac{2p^3}{3}$  leftmoving scalars  $b_-^{\check{\alpha}}$ ,
- $\frac{p^3}{3}$  rightmoving scalars  $b_+^{\hat{\alpha}}$ .

### The fermions

As was discussed in the introduction of this section the effective theory on the string has  $(0, 4)$  supersymmetry, meaning there will be no fermions in the left moving sector and in the right moving sector there should be an equal amount of bosons and fermions, that furthermore come in multiplets of four real bosons and four fermions.

The fermions on the string come from the worldvolume fermions of the M5. The divisor the M5 wraps is a Kähler manifold and so we can use the standard representation of fermions as holomorphic forms, see e.g. [12], to do the reduction. So the number of fermions on the string is  $h^{0,0}(p) + h^{0,1}(p) + h^{0,2}(p)$ . The even forms correspond to rightmoving and the odd forms to leftmoving fermions. As we assumed  $p$  very ample there is the identity  $h^{0,1}(p) = h^{0,1}(X)$  which is zero as one can find in Table 2.1. So this absence of leftmoving fermions indeed coincides with our general expectations from supersymmetry.

In the rightmoving sector  $h^{0,0} = 1$  gives rise to a single spinor with four real worldvolume fermionic degrees of freedom. These four fermions can be interpreted as the four Goldstinos  $\chi^l$  associated to the breaking of 4 of the eight background supersymmetries by the presence of the string. These four Goldstinos form a single  $\mathcal{N} = 4$  multiplet together with the three scalars  $x^i$  and  $b^0$  [88]. This field  $b^0$  is the single rightmoving scalar coming from flux pulled back from the Calabi-Yau. This multiplet is always present independent of the  $CY_3$  we consider and is thus referred to as the *universal multiplet*.

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|          | Left moving                            | Right moving                                     |
|----------|----------------------------------------|--------------------------------------------------|
| Bosons   | $p^3 = \frac{p^3}{3} + \frac{2p^3}{3}$ | $\frac{2p^3}{3} = \frac{p^3}{3} + \frac{p^3}{3}$ |
| Fermions | 0                                      | $\frac{2p^3}{3}$                                 |

**Table 2.3** *A summary of the field content of the MSW CFT, keeping only track of terms leading in  $p$ . In the bosonic sector we have two contributions: there are scalars originating from the divisor moduli and those from the reduction of the worldvolume gaugefield. Their separate contributions are the first respectively second terms in the sums above.*

The remaining  $4h^{0,2}$  rightmoving fermions  $\psi^\kappa$  pair up with the other  $b^\alpha$ . One can calculate  $h^{2,0}$  by using some additional algebraic geometry to derive  $h^{0,2}(p) = \frac{p^3}{6}$  or just use the knowledge that this has to equal the number of rightmoving bosonic fields as required by supersymmetry. So we summarize the different fermionic fields on the string:

- no leftmoving fermions ,
- 4 rightmoving fermions  $\chi^l$  interpreted as Goldstinos ,
- $\frac{2p^3}{3}$  rightmoving fermions  $\psi^\kappa$  represented as two forms on the divisor  $p$ .

### 2.6.2 The central charges and the entropy

We can naively calculate the left and right central charge of the theory by counting the number of bosons and half the number of fermions in the respective sector. For the readers convenience we have summarized the field content once more, ordered by their 'parity' and fermion number in table 2.3. Doing the simple sum gives us the result

$$c_L = p^3, \quad c_R = p^3. \quad (2.128)$$

It is important to remark that if one includes subleading corrections  $c_L$  and  $c_R$  do differ from each other, see e.g. [36].

We have now assembled all the ingredients to come to a derivation of the entropy of a D4D0 black hole/string in the large  $p$  limit. As we explained the D4 charge  $p$  corresponds to the divisor the M5 wraps. The D0 charge translates in the 5 dimensional picture to momentum along the string. So the number of states we would naturally associate to a given black hole of charge  $(p, q_0)$  are the supersymmetric states on an M5 wrapping  $p$  that carry total momentum  $q_0$ . The first requirement, that of supersymmetry, is only of importance in the right moving sector. It implies that all the right moving modes have to be in their groundstates. In the left moving sector the states can be excited as there is no constraint by supersymmetry, so we see that all the contribution to the momentum  $q_0$  comes from the left-moving sector.

So the conclusion is that the number of states corresponding to a given D4D0 black hole is that of all possible excited left-moving states with fixed total momentum  $q_0$ . Now we can make use of a well known result in conformal field theory, the Cardy formula [85], that for large momentum expresses this number in terms of the central charge of the theory, in our case:

$$N_{q_0} \sim e^{2\pi\sqrt{\frac{q_0 c_L}{6}}} . \quad (2.129)$$

Using the expression (2.128) for the central charge one thus finds an entropy of

$$S = 2\pi\sqrt{\frac{q_0 p^3}{6}} . \quad (2.130)$$

When first confronted with this result one can only be stunned by the perfect agreement with the black hole entropy (2.115). Note that this agreement actually continues holds when one includes corrections in  $p$ , see [36]. Furthermore it is not very difficult to extend the analysis to D4D2D0 black holes and also there the agreement holds. We won't spend time discussing this extension here as it would take us too far, some more details can be found in the discussion in section 4.5.3.

### 2.6.3 Some comments on $\text{AdS}_3/\text{CFT}_2$

From a more modern point of view the derivation of MSW is but a single application of one of the many examples of a duality known as AdS/CFT. In [90], Maldacena was the first to realize that string theory on an AdS space-time has a dual description as a conformal field theory. The most famous and best understood example of this duality is that of string theory on  $\text{AdS}_5 \times \text{S}^5$  which is dual to  $\mathcal{N} = 4$  super Yang-Mills. AdS/CFT realizes the principle of holography; that a gravitational theory in  $d$  dimensions can be equivalently described as a  $d - 1$  dimensional theory without gravity. Some background on AdS/CFT can be found in e.g. [91, 16, 92].

In the case of string theory on  $\text{AdS}_3$  the most studied example is that of  $\text{AdS}_3 \times \text{S}^3 \times M_4$  with  $M_4 = T_4$  or  $K3$ . This is the case of the D1D5 system of [22] which has (4,4)-supersymmetry. The case of our interest  $\text{AdS}_3 \times \text{S}^2 \times \text{CY}_3$ , which has only (0,4)-supersymmetry, is less understood, especially on the CFT side. In chapter 4 we will study the role played by the multicenter black holes in this duality. As we will show there, their existence points to some interpretational puzzles but also interesting possible applications.

## Multicentered black holes

In this chapter we review the main objects of study in this thesis, the multicentered black holes. Building on [93] it was shown in [27] and [28] that there exist supersymmetric bound states of the BPS black holes of the previous chapter. These bound states are made up from a number of charged black hole centers  $\Gamma_a$  of which the relative distance in the spatial  $\mathbb{R}^3$  is constrained. Contrary to single center black holes the multicenter variants are only stable in a certain region of moduli space and decay when leaving this region. Originally they were discovered as the solution to the puzzle that certain BPS states that are known to exist microscopically, i.e. as D-brane states, do not exist as single center supergravity solutions. It was then realized by [27] that these BPS states do exist in supergravity, but as a multicentered configuration. As we will see, there are certain charges that exist both as single center black holes and as multicentered black holes but thus also charges that are only realized as multicenter black holes. This already points at one interesting reason to study these objects; in certain cases they are the only supergravity realizations of a given BPS state. In the case where both a single center and multicenter realization exist this naturally leads to many interesting questions as it makes the map between the microscopic picture and the macroscopic picture much more subtle, some of these questions will be central in the next chapter. Here we will summarize the supergravity solutions for these bound black holes and discuss some of their physics, we mainly follow [27, 94, 95, 73] but also present some of the authors research done in [31] and [96].

In the first section we will show that the multicentered solutions can be seen as bound states of single center black holes by studying the potential for a probe particle in a black hole background and investigating when it has supersymmetric minima.

In the second section we review the derivation of the precise supergravity solutions by solving the BPS equations for stationary supergravity solutions.

The third section discusses some properties of the black holes such as their angular momentum and lift to five dimensions. We will also spend some time investigating what we call the solution space of a given multicentered configuration. As we will show there is some freedom to move the centers without changing the energy of the system and those different possible positions form what we call solution space. We introduce this space and analyse certain properties of it. We end the section by the powerful method to classify the multicentered solutions by split attractor flow trees, a specific type of graphs on moduli space.

The last section of this chapter is devoted to some examples and contains some original work first presented in [31]. We detail the concept of threshold stability in a first example and in a second example we show the existence of D4D2D0 bound states.

### 3.1 Probing a black hole

The first insights in the dynamics behind bound states of black holes can most easily be gained by considering a charged test particle in a single black hole background. Following [97] we show here that such a test particle feels a potential and that a supersymmetric minimum of this potential can exist. When one takes the backreaction of a test particle at that minimum into account the result is expected to be a stationary bound states of two black holes. We will show the explicit supergravity solutions for these multicentered black holes in section 3.2.3.

#### The probe potential

The action of a probe particle with charge  $\Gamma_p$  in an arbitrary  $\mathcal{N} = 2$  background was given in (2.61). We will analyse this action on the background of one of the supersymmetric black holes introduced in the previous chapter, see (2.102) for the explicit form of these supergravity solutions.

Let us for clarity repeat the probe action (2.61) here:

$$S_{\Gamma_p} = - \int |Z(\Gamma_p, t^A)| ds + \frac{1}{2} \int \langle \Gamma_p, A \rangle. \quad (3.1)$$

The line element  $ds$  is given by the pull-back of the background metric:  $ds = \sqrt{-g_{\mu\nu} \frac{dx^\mu dx^\nu}{d\sigma^2}}$  for some worldline coordinate  $\sigma$ . It is simplest to work in the so called static gauge, where we take the worldline coordinate  $\sigma$  for the probe to correspond to the timelike coordinate of the background, i.e.  $\sigma = t$ .

We will restrict our attention to static probes as we are interested in stable bound states, i.e. we want a worldline such that the probe stays fixed in the spatial dimensions, i.e.  $\partial_t r = \partial_t \phi = \partial_t \theta = 0$ . In this case the worldvolume line-element is simply  $ds = e^U dt$ .

For such a static particle all kinetic terms are zero and the only contribution to the action is a potential term. If there are spatial positions where the potential

is extremized then a probe particle sitting at this position is stable and our static ansatz is consistent.

To calculate the potential more explicitly we use (2.74, 2.94) from the last chapter to rewrite the background gaugefield as:

$$A = -\Gamma \cos \theta d\phi + \frac{2}{\sqrt{\Sigma}} \operatorname{Re} [e^{-i\alpha} \Omega] dt. \quad (3.2)$$

Here  $\Gamma$  is the charge of the black hole in whose background we analyse the probe,  $\alpha$  is the argument of the central charge  $Z = Z(\Gamma, t^A)$ , function of the background charge and moduli. With a little algebra one can now calculate the probe lagrangian (3.1) in case of our static probe. The result is the following radially dependent potential

$$V(r) = \frac{2}{\sqrt{\Sigma}} |Z(\Gamma_p, t^A)| \sin^2 \left( \frac{\alpha - \alpha_p}{2} \right). \quad (3.3)$$

Hidden in the notation is radial dependence through the background moduli  $t^A(r)$ , note that these also appear in  $\alpha$  and  $\alpha_p$  as  $\alpha_a = \arg(Z(\Gamma_a, t^A))$ , and the function  $\Sigma(H(r))$ .

### Supersymmetric minimum

Given the potential (3.3) we can now search for extrema. In this thesis we will restrict ourselves to supersymmetric minima. Although in certain cases non-supersymmetric minima can exist no non-supersymmetric bound black hole configurations are known, finding these backreacted solutions is an interesting open problem.

As the potential (3.3) is everywhere positive the only supersymmetric minima are the zero's of the potential. A first rather trivial such supersymmetric minimum is when the probe coincides with the black hole, this can be checked by studying the potential in the near horizon region. Using the near horizon expansion of  $\Sigma$ , which can be found from (2.104), the potential takes the following form at small  $r$ :

$$V(r) \sim \frac{2}{\sqrt{S}} |Z(\Gamma_p, t_*)| \sin^2 \left( \frac{\alpha_* - \alpha_{p*}}{2} \right) r + \mathcal{O}(r), \quad (3.4)$$

So we see that the potential decreases linearly with the radial distance, vanishing at the horizon where the probe coincides with the black hole.

At infinity, i.e.  $r \rightarrow \infty$ , the potential goes to a positive constant

$$V(\infty) = 2 |Z(\Gamma_p, t_\infty)| \sin^2 \left( \frac{\alpha_\infty - \alpha_{p\infty}}{2} \right). \quad (3.5)$$

One can now wonder if there is an additional zero of the potential somewhere at a finite radius. Clearly this is the case iff there exists some radius  $r_{\text{ms}}$  such that

$$\alpha(r_{\text{ms}}) = \alpha_p(r_{\text{ms}}). \quad (3.6)$$

For such a radius to exist the attractor flow has to pass through a point in moduli space where the two central charges  $Z(\Gamma)$  and  $Z(\Gamma_p)$  obtain the same phase. This condition is called *marginal stability* and is actually rather generic. As the constraint  $\alpha_p = \alpha$  imposes a single real equation on the moduli it will generically be satisfied on a hypersurface in moduli space, we will call such a hypersurface a *wall of marginal stability*. We will come back later to the physical meaning and consequences of marginal stability in 3.3.2.

Using the attractor flow equations (2.81) we derived in the previous chapter we can calculate  $r_{\text{ms}}$ . Taking the intersection product of equation (2.81) with  $\Gamma_p$  we see that  $\alpha = \alpha_p$  implies that

$$r_{\text{ms}} = \frac{\langle \Gamma, \Gamma_p \rangle}{2\text{Im}[e^{-i\alpha} Z_p]_\infty}. \quad (3.7)$$

This is only a physically reasonable solution in case the RHS is positive, as  $r$  is a radial coordinate. So we find a condition on the asymptotic moduli:

$$\langle \Gamma_p, \Gamma \rangle \text{Im}[e^{-i\alpha} Z_p]_\infty > 0. \quad (3.8)$$

The CY moduli space is thus divided into two parts, namely one where (3.8) holds and one where it doesn't, the boundary between these regions is the wall of marginal stability (or a wall of anti-marginal stability, where the phases anti-align). In case we take the asymptotic moduli for the background in the part where (3.8) holds we see there exists a bound state of the probe and the background black hole where the probe is stable at a radius  $r_{\text{ms}}$  (3.7), while in case the asymptotic moduli are in the conjugate part of moduli space no bound state exists.

The decay of the bound state can be understood quite intuitively. When we start with the asymptotic values of the moduli in the region where the bound state exists and start changing them towards the wall of marginal stability the radius  $r_{\text{ms}}$  will keep on increasing till it becomes infinite exactly at marginal stability. Once the asymptotic scalars cross the wall of marginal stability the bound state no longer exists. The corresponding behaviour for the potential under changes of the asymptotic moduli is shown in figure 3.1.

As a final remark note that we have assumed that  $\langle \Gamma, \Gamma_p \rangle \neq 0$ . In case  $\langle \Gamma, \Gamma_p \rangle = 0$  the situation is much more trivial and there are no macroscopic 'bound' states. When the intersection product vanishes the potential is either completely flat, when the asymptotic moduli are at marginal stability, or there exist no supersymmetric minimum other than the trivial one at  $r = 0$ , when the asymptotic moduli are not at marginal stability. In the rest of the thesis we will mainly restrict our attention to the more interesting case of  $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$ .

## 3.2 Multicentered black holes

In this section we will show that supersymmetric bound states of black holes exist as solutions to the  $\mathcal{N} = 2$  supergravity Lagrangian (2.70), note that as in

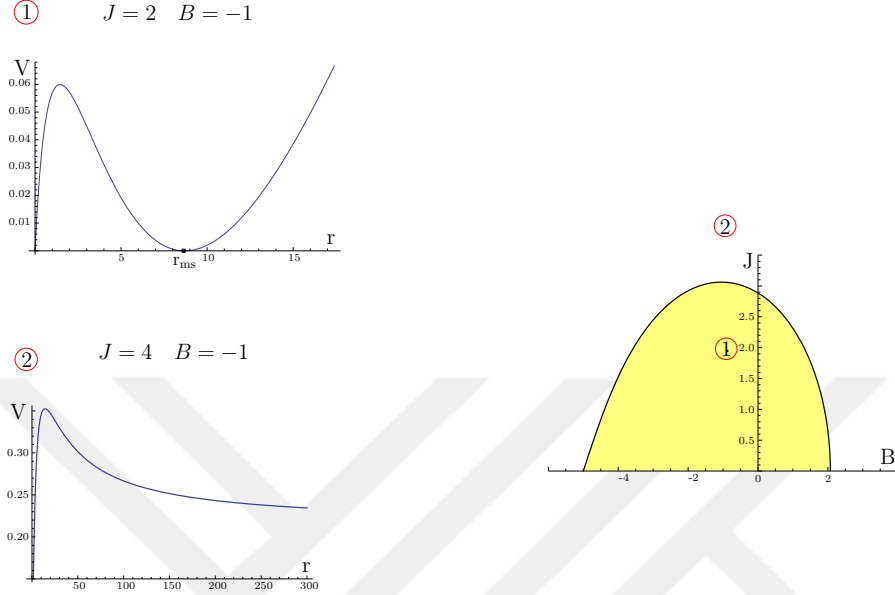


**Figure 3.1** In this figure the probe potential is plotted in a one modulus example with  $\Gamma = (1, 5, \frac{75}{2}, \frac{125}{6})$  and  $\Gamma_p = (-1, 5, -\frac{25}{2}, \frac{125}{6})$ . On the left the potential is shown for two different values of the asymptotic moduli. Only in case 1 there is a stable minimum corresponding to a bound state. Note the difference in scale in the plot of case 2. For 2 the potential asymptotes to a constant without a minimum apart from  $r = 0$ . In the plot on the right the moduli space is shown and the wall of marginal stability for this particular example is plotted. The yellow region under the wall of marginal stability corresponds to stable bound states while no such configurations exist above the wall of marginal stability. This is illustrated by the two specific cases 1 and 2, for which we showed the potentials on the left.

the case of a single center black hole we restrict ourselves to the vector multiplet sector of the theory. The solutions can be found by making an ansatz for all the fields inspired by symmetry and then derive the corresponding BPS equations [27]. Then one can solve these BPS equations, which are first order equations, to come to the full supergravity solution [73].

### 3.2.1 The BPS equations

As we did for the single center case in the last chapter we will first write down the most general ansatz for all the fields that is consistent with the symmetries of the system.

Where a single center clearly preserves spherical symmetry this will be broken

for multiple centers. In case of two centers an axial symmetry is preserved but for three or more generically placed centers all of the spherical symmetry is broken.

As we will be interested in stable, non evolving bound states the fields should still be time independent. The multicentered solutions will however not be static but stationary. This can be understood as there are spatially separated electric and magnetic charges which will give rise to an electro-magnetic angular momentum, more on this in section 3.3.1.

So the only bosonic symmetry we have left is that of time independence, luckily the additional demand of supersymmetry constrains the metric drastically. The most general stationary supersymmetric metric in four dimensions is given by [66, 67]:

$$ds^2 = e^U (dt + \omega_i dx^i)^2 + e^{-U} dx^i dx^i. \quad (3.9)$$

In this more general case where there is no spherical symmetry the function  $U$  and the scalars  $t^A$  depend on all spatial coordinates  $x^i$ . The new off-diagonal term in the metric is given by  $\omega = \omega_i dx^i$  which can be thought of as a one-form on the spatial  $\mathbb{R}^3$ . The U(1) field strengths grouped in  $F$ , see (2.57), are left arbitrary for the moment.

As in the single center case one can look to the reduced action for this ansatz and perform the BPS squaring trick. The presence of  $\omega$  and the more general dependence on the spatial coordinates makes this technically much more involved than in the spherically symmetric static case and as such we will just refer to [27] for the details. In a real 'tour de force' [27] showed that the resulting BPS equations take the following simple form:

$$\begin{aligned} 2e^{-U} \text{Im} [e^{-i\alpha} \Omega] &= -H, \\ A &= A_d + 2e^U \text{Re} [e^{-i\alpha} \Omega] \frac{dt}{\sqrt{G_4}}, \\ \star dA_d &= dH, \\ \star d\omega &= \langle dH, H \rangle, \end{aligned} \quad (3.10)$$

where  $H$  is a generic Harmonic function. Note that these equations nicely reduce to the BPS equations (2.80), (2.74) and (2.94) we found in the single center case, if we take our harmonic function to be that of a single source:  $H = \frac{\Gamma}{r} - 2\text{Im} [e^{-i\alpha} \Omega]_\infty$ . In case we take multiple sources of charge  $\Gamma_a$  at positions  $x_a$  the above equations (3.10) and the definitions of the charges (2.59) determine the corresponding harmonic function to be

$$H = \sum_a \frac{\Gamma_a}{r_a} - 2\text{Im} [e^{-i\alpha} \Omega]_\infty, \quad (3.11)$$

where we write  $r_a$  for  $|x - x_a|$ .

From our probe analysis in the previous section we learned that BPS bound states of different charged centers indeed exist but only for certain equilibrium distances between the centers. Here this constraint on the relative distances

comes about as an integrability condition for the last equation of (3.10). Applying the operator  $d\star$  on both sides of that equation results in the constraint

$$\langle \Delta H, H \rangle = 0. \quad (3.12)$$

In the presence of sources, i.e. with  $H$  of the form (3.11), we have that  $\Delta H = \sum_a \Gamma_a \delta^3(x - x_a)$ . The constraint (3.12) then becomes<sup>1</sup>

$$\langle \Gamma_a, H \rangle|_{x=x_a} = 0 \quad \forall a \quad \Leftrightarrow \quad \sum_{b \neq a} \frac{\langle \Gamma_a, \Gamma_b \rangle}{r_{ab}} = \langle h, \Gamma_a \rangle \quad \forall a \quad (3.13)$$

where  $r_{ab} = |x_a - x_b|$  and  $h = -2\text{Im}(e^{-i\alpha}\Omega)|_\infty$ .

So indeed we find that the equilibrium distances between the different centers are constrained and as such the black holes really form *bound* states. It is pleasing to see that in case of just two charged centers the integrability condition (3.13) reduces to the dynamical constraint (3.7) of the probe analysis. We will return later in this chapter to the structure of the constraints (3.13) but let us first go back to the original BPS equations (3.10).

### 3.2.2 Solving the BPS equations

The only thing that remains to be done to obtain the explicit supergravity solutions is to solve the BPS equations (3.10), we follow [73].

#### The scalars and gaugefields

The first equation of (3.10) is exactly the attractor flow equation (2.87) of the single center version in the last chapter, but with  $H$  being a more generic harmonic function. As the solution for these equations for  $U$  and  $t^A$  in terms of a single function  $\Sigma(H)$  is purely algebraic the method used for single centers directly applies to the multicentered case. So the expressions for  $t^A$  and  $U$  written in terms of  $\Sigma(H)$  is as in the single center case. The relation between  $\Sigma$  and the entropy is also as in the single center case, but now for each center separately:

$$\lim_{r_a \rightarrow 0} \Sigma(H) = \pi S(\Gamma_a) r_a^2. \quad (3.14)$$

It is not difficult to check that also the equation for the gaugefield in (3.10) in terms of  $H$  is the same as in the single center case. The electric part is again algebraic in  $H$  and the solution in terms of the function  $\Sigma(H)$  can thus be directly read from the single center case (2.97). The equation for the magnetic part  $A_d$  is not algebraic however and the solution is a bit more involved. Implicitly however

<sup>1</sup> For brevity we use unconventional notation here: by  $\sum_{s \neq r}$  we mean a sum over all  $s$  different from  $r$  whereas  $\sum_{s \neq r}$  denotes a doubles sum over all  $s$  and  $r$  such that  $s$  and  $r$  are different.

it is clear how to solve it. This because the equation is linear and we know the contribution for a given charge  $\Gamma_a$  in the origin:

$$A_{da}^{\tilde{A}} = -p^{\tilde{A}} \cos \theta_a d\phi_a, \quad (3.15)$$

where the subscript  $a$  indicates that we use spherical coordinates with the origin such that  $x_a = 0$ . The full solution for  $A_d$  is thus

$$A_d^{\tilde{A}} = \sum_a A_{da}^{\tilde{A}}. \quad (3.16)$$

In principle one can now try to write this out in a single coordinate system but that is a rather involved calculation that we will spare ourselves from.

### The solution for $\omega$

The remaining equation in (3.10) is the one for  $\omega$ . We first consider the two center case as will then show how to implicitly generalize that solution to an arbitrary number of centers.

Given two centers of charge  $\Gamma_1$  and  $\Gamma_2$  we can always choose coordinates such that the  $z$ -axis runs through the two centers and such that the origin lies exactly in the middle of the two centers, i.e. the centers have the coordinates  $x_1 = (0, 0, l)$  and  $x_2 = (0, 0, -l)$ . To be able to find a solution to the equations for  $\omega$  the integrability constraint (3.13) needs to be fulfilled. Here it implies that the intercenter distance  $2l$  has to be

$$l = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{4\text{Im}(e^{-i\alpha}\Omega)_\infty}. \quad (3.17)$$

Writing the harmonic functions as  $H = \frac{\Gamma_1}{r_1} + \frac{\Gamma_2}{r_2} + h$  the last BPS equation of (3.10) then becomes

$$\begin{aligned} d\omega &= \langle \Gamma_1, \Gamma_2 \rangle \star \left[ \frac{1}{r_2} d\frac{1}{r_1} - \frac{1}{r_2} d\frac{1}{r_1} + \frac{1}{2l} \left( \frac{dr_1}{r_1^2} - \frac{dr_2}{r_2^2} \right) \right], \\ &= \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2l} d \left( \frac{r^2 - l^2}{\sqrt{r^4 + l^4 - 2r^2 l^2 \cos 2\theta}} + 1 - \cos \theta_1 + \cos \theta_2 \right) \wedge d\phi. \end{aligned} \quad (3.18)$$

The last equality takes some calculation to verify, to derive it we introduced three sets of spherical coordinates  $(r, \theta, \phi)$ ,  $(r_1, \theta_1, \phi)$  and  $(r_2, \theta_2, \phi)$ . These three sets of coordinates are centered respectively around the origin,  $x_1$  and  $x_2$  (note we chose the origin such that  $x_1 + x_2 = 0$ ) and they all share a common  $z$ -axis oriented along  $x_1$ , see figure 3.2. Note that the expression for  $\omega$  can be written purely in the first set of spherical coordinates by using the formulae

$$\begin{aligned} r_1 \cos \theta_1 &= r \cos \theta - l, & r_1^2 &= r^2 + l^2 - 2rl \cos \theta, \\ r_2 \cos \theta_1 &= r \cos \theta + l, & r_2^2 &= r^2 + l^2 + 2rl \cos \theta. \end{aligned} \quad (3.19)$$

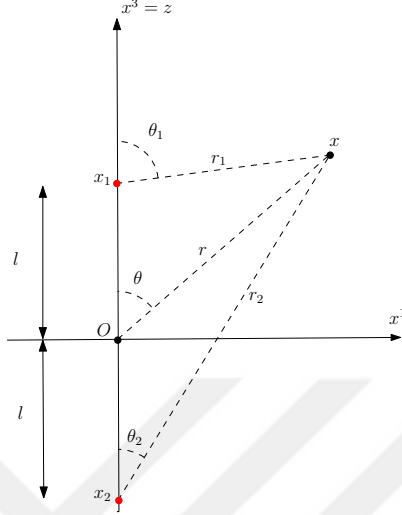


**Figure 3.2** Here our choice for the different sets of spherical coordinates is illustrated. The  $x_1$  and  $x_2$  denote the position of the two centers.

So we find that for two centers  $\omega$  takes the rather simple form<sup>2</sup>

$$\omega = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2l} \left( \frac{r^2 - l^2}{\sqrt{r^4 + l^4 - 2r^2 l^2 \cos 2\theta}} + 1 - \cos \theta_1 + \cos \theta_2 \right) d\phi. \quad (3.20)$$

In the case of more than two centers we generically lose axial symmetry, this makes an explicit expression for  $\omega$  much more involved. As for the magnetic part of the gaugefields it is however not so hard to give an implicit solution. For this note that for an arbitrary bunch of centers  $(\Gamma_a, x_a)$  the BPS equation for  $\omega$  in (3.10) reads

$$\star d\omega = \sum_{a < b} \langle \Gamma_a, \Gamma_b \rangle \left[ \frac{1}{r_a} d\frac{1}{r_b} - \frac{1}{r_b} d\frac{1}{r_a} + \frac{1}{2r_{ab}} \left( \frac{dr_a}{r_b^2} - \frac{dr_b}{r_a^2} \right) \right]. \quad (3.21)$$

Comparing with (3.18) shows that we can see  $\omega$  as the sum of all two center contributions:

$$\omega = \sum_{a < b} \omega_{ab}, \quad (3.22)$$

where  $\omega_{ab}$  is the solution (3.20) for the two centers  $\Gamma_a$  and  $\Gamma_b$ . Do note that (3.22) is not a completely explicit solution as we only have an expression for  $\omega_{ab}$

<sup>2</sup> Actually the expression for  $\omega$  is even simpler in prolate spheroidal coordinates, see e.g. section 5.2.3. These coordinates have the disadvantage that generically the metric is less transparent.

in a coordinate system where both centers lie on the  $z$ -axis. In more generic cases where not all centers are aligned giving an explicit expression for  $\omega$  then amounts to relating the different sets of spherical coordinates, this however is a technical matter and for us it will suffice that (3.22) shows that at least implicitly there exists a well defined solution.



### 3.2.3 The supergravity solution

With the results of the previous subsection and using some formulae of the previous chapter we summarize the multicentered black hole solutions here. It is interesting to compare them to the single center solutions (2.102) and see they are a natural generalization. In its full glory a multicentered solution is given by<sup>3</sup>

$$\begin{aligned}
 ds^2 &= -\frac{1}{\Sigma}(dt + \sqrt{G_4}\omega)^2 + \Sigma dx^i dx^i, \\
 A^0 &= \frac{-L}{\Sigma^2} \left( \frac{dt}{\sqrt{G_4}} + \omega \right) + \omega_0 \\
 A^A &= \frac{H^A L - Q^{3/2} y^A}{H^0 \Sigma^2} \left( \frac{dt}{\sqrt{G_4}} + \omega \right) + \mathcal{A}_d^A, \\
 t^A &= \frac{H^A}{H^0} + \frac{y^A}{Q^{\frac{3}{2}}} \left( i\Sigma - \frac{L}{H^0} \right),
 \end{aligned} \tag{3.23}$$

$$H = \sum_a \frac{\sqrt{G_4} \Gamma_a}{r_a} - 2\text{Im} (e^{-i\alpha} \Omega)_{r=\infty}. \tag{3.24}$$

$$\sqrt{G_4} \sum_{b \neq a} \frac{\langle \Gamma_a, \Gamma_b \rangle}{r_{ab}} = \langle h, \Gamma_a \rangle, \tag{3.25}$$

$$\begin{aligned}
 \Sigma &= \sqrt{\frac{Q^3 - L^2}{(H^0)^2}}, \\
 L &= H_0 (H^0)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0,
 \end{aligned} \tag{3.26}$$

$$\begin{aligned}
 Q^3 &= \left( \frac{1}{3} D_{ABC} y^A y^B y^C \right)^2, \\
 D_{ABC} y^A y^B &= -2H_C H^0 + D_{ABC} H^A H^B.
 \end{aligned}$$

$$\begin{aligned}
 d\omega_0 &= \frac{1}{\sqrt{G_4}} \star dH^0, \\
 d\omega &= \frac{1}{\sqrt{G_4}} \star \langle dH, H \rangle, \\
 dA_d^A &= \frac{1}{\sqrt{G_4}} \star dH^A.
 \end{aligned} \tag{3.27}$$

<sup>3</sup> We reinstated the factors of  $G_4$  as they will be of importance in the next chapter.

### 3.3 Physical properties

In this section we take some time to discuss some physical properties characteristic to multicentered black holes. First we point out the solution has angular momentum and is thus no longer static. Furthermore for a given set of charged centers there is a whole manifold of solutions parameterized by the different possible positions of the centers. We will discuss the general structure of this solution space and two specific cases of interest, when the solution space is 'non-compact' and when it includes a scaling point. We also comment on the lift of the multicentered solutions to five dimensions which will be of use in the next chapter. Then we come to a conjecture on the existence criterium for the multicentered black holes. This split attractor flow conjecture relates the existence of the multicentered black holes to the existences of a graph on moduli space. Such split flows will prove very useful tools in analyzing multicentered solutions.

#### 3.3.1 Angular momentum

A new feature of the bound black holes with respect to the single black holes is that they are stationary. This is encoded in the one-form  $\omega$  that mixes the spatial coordinates with time in the metric (3.23). As is standard in general relativity the angular momentum  $\vec{J}$  can be read from the asymptotic behaviour of these off-diagonal terms when  $r \rightarrow \infty$ :

$$\omega = 2\epsilon_{ijk} \frac{J^i x^j dx^k}{r^3} + \mathcal{O}(r^{-2}). \quad (3.28)$$

The explicit expression (3.20) for  $\omega$  in the case of two centers then shows that

$$\vec{J} = \frac{1}{2} \langle \Gamma_1, \Gamma_2 \rangle \frac{\vec{x}_1 - \vec{x}_2}{|x_1 - x_2|}. \quad (3.29)$$

The angular momentum is 'topological' as its size is only determined by the intersection product of the charges and doesn't depend on the equilibrium distance  $r_{12} = |x_1 - x_2|$  and hence neither on the CY moduli. Note that none of the centers themselves are spinning, as was for example the case for the Kerr-black holes (2.3) from chapter 2. Here the angular momentum is completely due to the electro-magnetic interaction of electric charges with their corresponding magnetic charges. Indeed, in case of a single  $U(1)$  field the intersection product  $\langle \Gamma_1, \Gamma_2 \rangle$  would just reduce to the familiar *eg* term known from the electron-monopole interaction. Due to the Dirac quantization of the charges, the angular momentum (3.29) is quantized in half integer units.

In the general case of more than two centers we can use the additivity (3.22) of  $\omega$ , and thus that of the angular momentum, to see that the total angular momentum is

$$\vec{J} = \frac{1}{2} \sum_{a < b} \frac{\langle \Gamma_a, \Gamma_b \rangle \vec{x}_{ab}}{r_{ab}}. \quad (3.30)$$

The presence of this angular momentum has some interesting consequences, specifically when one looks at the quantum properties of the system [97, 29].

### 3.3.2 Solution space

One of the most important properties of the multicentered black holes is that they generically are bound states. There are specially trivial configurations where all centers carry the same (or parallel) charge. In this case the centers can be positioned arbitrarily with respect to each other, as there is an exact cancelation of all forces, and those are thus not really 'bound'. The less trivial and more interesting cases are those where the different centers carry charges which have non-vanishing intersection product with each other:  $\langle \Gamma_a, \Gamma_b \rangle \neq 0$ . As we saw before the integrability condition (3.12) now impose constraints on the intercenter distances implying the centers are bound to each other. In case there are more than two centers the constraints do not fix the positions of the centers completely and there is actually a whole manifold of possible positions. As each point in this manifold corresponds to a different supergravity solution we will call it *solution space*, see below for a more precise definition.

In this subsection we will first discuss some general properties of the constraint equations and the corresponding solution space. Then we study two cases in more detail, the conditions under which centers can move off to infinity and the conditions under which some centers can come arbitrarily close together.

#### The constraint equations and solution space

A multicentered black hole is determined by  $n$  positions  $x_a$  in the spatial  $\mathbb{R}^3$  base and associating to each center a charge  $\Gamma_a$ . As discussed in section 3.2.1, the relative coordinate distances  $r_{ab} = |x_a - x_b|$  between the different centers have to satisfy an integrability condition for the supergravity solution to be consistent. For convenience let us repeat these constraint equations (3.25) coming from the integrability condition here:

$$\forall a \quad \sqrt{G_4} \sum_{b \neq a} \frac{\langle \Gamma_a, \Gamma_b \rangle}{r_{ab}} = 2\text{Im}(e^{-i\alpha} Z_a)_\infty. \quad (3.31)$$

In the RHS  $Z_a = Z(\Gamma_a)$  and  $\alpha$  is the phase of the total central charge  $\alpha = \arg Z = \arg \sum_a Z_a$ . Through these terms the constraints thus depend on the asymptotic values of the moduli. As was remarked before, in the case of two centers the distance found from the equation (3.31) corresponds to the equilibrium distance (3.7) we found from a probe analysis before. Also in the case of more centers the positions that satisfy these constraints (3.31) are those for which the interaction potential between the different centers vanishes. In this precise sense the multicenter black holes are genuine black hole bound states.

The first thing to note about the equations (3.31) is that they are not all independent, this because their sum is trivial:  $\sum_{a \neq b} \frac{\langle \Gamma_a, \Gamma_b \rangle}{|r_{ab}|} = 0$  by antisymmetry and  $\sum_a \text{Im}(e^{-i\alpha} Z_a) = \text{Im}(e^{-i\alpha} Z) = 0$  as well. So for  $N$  centers there

are  $N - 1$  constraints on the  $3N - 3$  coordinate variables (neglecting the three overall center of mass coordinates) so there is in general a  $2N - 2$  dimensional moduli space of solutions for fixed charges and asymptotics. This space may or may not be connected and it may even have interesting topology, e.g. in some cases it can be shown to be a toric variety [96]. We will refer to this  $2N - 2$  dimensional manifold as the moduli space of solutions or *solution space*, the latter terminology will be preferred as it is less likely to be confused with the moduli space of the Calabi-Yau, in which the scalar fields  $t^A$  take value.

For two centers the distance between them is uniquely determined by the constraint (3.31) and the two degrees of freedom remaining are the two angles parameterizing the overall rotation of the system and the solution space is thus  $S^2$ . From three centers onward there is also some freedom in the relative distances between the centers and the precise geometry of the moduli space becomes more complicated. The fact that for more than two centers we have these extra degrees of freedom is due to the interaction potential having flat directions. Note however that this freedom in moving the centers around is very much constrained and the centers are thus still bound to each other. There exist interesting border cases where certain centers are free to move off to infinity, such barely bound states or threshold states will be separately discussed below. Another special case is when certain centers can freely move arbitrarily close to each other and also this case deserves a special discussion below.

We will come back to the interpretation of the different connected components of the solution space for a given set of charges when we discuss split attractor trees in the next section.

### Decay at marginal stability

As we already stressed the solution space depends on the values of the moduli at infinity and critically so near marginal stability. Before we come to the analysis of the solution space near marginal stability let us first say a few general words on decay at marginal stability.

In theories with extended supersymmetry BPS states are normally protected against decay as their representation is a short multiplet [62]. The only way the state can decay is into other BPS states as all non-BPS states are represented in larger multiplets. In our case of interest, multicentered black holes in  $\mathcal{N}=2$  supergravity, the BPS states are characterized by their charge  $\Gamma$  and their central charge is determined in terms of this charge and the scalar moduli  $t$  by  $Z(\Gamma, t) = \langle \Gamma, \Omega(t) \rangle$ . The length of the central charge vector,  $|Z|$ , corresponds to the mass, as we are considering BPS states, and its phase,  $\alpha$ , characterizes the supersymmetries left unbroken by this state.

For generic values of the moduli the BPS states are absolutely stable as they can only decay into a set of other BPS states but this is forbidden by conservation of energy:

$$|Z_{1+2}| \leq |Z_1| + |Z_2|. \quad (3.32)$$

There is a boundary case however, when the moduli are such that for two BPS states  $\Gamma_1$  and  $\Gamma_2$  the phase aligns, the two BPS particles preserve the same supersymmetries and the binding energy of a BPS bounds state of them vanishes, as  $|Z_{1+2}| = |Z_1| + |Z_2|$ . In this case the particle is not forced to decay but can decay, hence the term *marginal stability*. This is equivalent to the condition

$$\text{Im}(\bar{Z}_1 Z_2) = 0 \quad \text{and} \quad \text{Re}(\bar{Z}_1 Z_2) > 0. \quad (3.33)$$

The second inequality is needed to ensure that the central charges not only align but also point in the same direction, when they point in opposite direction one speaks of *anti-marginal stability*. As the condition (3.33) is a single real equation it will, in general, be satisfied on a codimension one surface in moduli space referred to as a *wall of marginal stability*.

Let us now come back to our BPS supergravity solutions and see how decay at marginal stability is realized here. First thing to note is that when we talk about varying the moduli in this context, we mean variations of the asymptotic boundary moduli. These of course change when radially evolving inward but the discussion here is of comparing the different solutions corresponding to different asymptotic values of the moduli. The boundary values are the ones to consider as e.g. the mass of our supergravity solutions is given by  $M = |Z|_\infty$ .

Due to the dependence of the equations (3.31) on the asymptotic moduli the solution space will change under variations of these moduli. For most variations of the moduli the space will change smoothly but for a variation across a wall of marginal stability the solution space will cease to exist. We already discussed the feature of decay at marginal stability in the probe analysis in section 3.1, the physics is exactly the same for the full supergravity solutions. But because it is such a crucial property of the multicentered black holes let us discuss it here again. Let us first focus on two centers, this will essentially capture the full picture as we will later argue that the generic decay of multiple centers can be seen as the decay of two clusters of centers.

In case of just two centers, there is only a single constraint equation (3.31) that furthermore uniquely fixes the relative position of the two centers:

$$r_{12} = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2\text{Im}(e^{-i\alpha} Z_1)_\infty}. \quad (3.34)$$

Let us once more point out that through the factor  $\text{Im}(e^{-i\alpha} Z_1)_\infty$  this equilibrium distance depends critically on the values of the moduli at infinity. Clearly for there to be a solution the RHS of (3.34) has to be positive, so we see that there are two regions in moduli space: one where the two centers are bound and sit at a fixed distance and one where  $\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(e^{-i\alpha} Z_1)_\infty < 0$  where no bound state exists. The boundary between these two regions is given by moduli such that

$$\text{Im}(e^{-i\alpha} Z_1) = 0. \quad (3.35)$$

This condition is equivalent to the moduli being at marginal stability, see (3.33), or anti-marginal stability. We will concentrate on the case of marginal stability here.

Equation (3.34) then shows that if we approach the wall of marginal stability from the stable side the two bound centers will be forced further and further apart till they are infinitely separated exactly at marginal stability, this is the physical picture of the decay of the BPS bound state in the supergravity regime. The important point that should be stressed here is that if a single center black hole exists at a certain value of the asymptotic moduli it will exist at all values of those moduli, this is no longer true for a generic multicenter configuration. Such a bound state will only exist in a certain region of moduli space and can be made to decay by tuning the asymptotic moduli towards a wall of marginal stability.

Let us finally have a look at an arbitrary number of centers. Group the centers into two clusters  $C_1 = \{\Gamma_i\}$  and  $C_2 = \{\Gamma_m\}$  such that a solution to the constraint equations exists near marginal stability for the two clusters:  $\text{Im}(\bar{Z}_2 Z_1) = 0$  with  $Z_1 = \sum_i Z_i$  and  $Z_2 = \sum_m Z_m$ . By summing the constraint equations (3.31) over  $i$  or equivalently over  $m$  one finds that

$$\sum_{i,m} \frac{\langle \Gamma_i, \Gamma_m \rangle}{r_{im}} = 2\text{Im}(e^{-i\alpha} Z_1). \quad (3.36)$$

When approaching marginal stability for the two clusters the RHS goes to zero and thus a solution to the constraint equations is given by a diverging distance between centers in the first cluster and centers in the second cluster. So we see that generically for multiple centers the system decays by splitting up into two clusters that then are forced infinitely far apart.

For three or more clusters being forced infinitely far apart at the same point in moduli space three or more central charges have to align simultaneously. This happens only on the intersection of different walls of marginal stability and such loci are thus generically of codimension two. These are thus much less generic and can furthermore be evaded when varying the moduli appropriately, this is not the case for walls of marginal stability which are codimension one and which have to be crossed if one wants to move from one region in moduli space to the other. So we see that the most natural decay of multicentered solutions is the decay into two subclusters. We will come back to this phenomenon of clustering when discussing split flow trees in subsection 3.3.4.

### Scaling solutions

Let us discuss here in some more detail the case where the solution space includes points where different centers coincide. As we will show in a moment at such a point the solution gains an additional scaling symmetry, hence we will name such a point of coincidence a *scaling point*. Multicentered black holes of which the solution space includes such a scaling point are called *scaling solutions*. Scaling solutions seem to be very much related to single center black holes and have some properties not shared by generic multicenter black holes, they are for example absolutely stable in all regions of moduli space and cannot be made to decay.

To investigate the conditions under which the solution space has a scaling point take a number of charged centers  $\Gamma_m$  and divide the centers into two groups  $m = (a, i)$ . Let us assume the centers with index  $a$  can approach each other arbitrarily close while the other centers  $\Gamma_i$  are forced to stay at some finite distance from the centers  $\Gamma_a$ . The point in space-time where the centers  $\Gamma_a$  approach each other we denote with  $x_0$ , so by our assumptions we can then write the positions of the centers as:

$$x_a(\lambda) = x_0 + \lambda w_a + \lambda^2 v_a + \mathcal{O}(\lambda^3), \quad (3.37)$$

$$x_i(\lambda) = x_i + \lambda z_i + \mathcal{O}(\lambda^2). \quad (3.38)$$

$$(3.39)$$

We now plug this into the constraint equations (3.25) and expand this for small  $\lambda$ . By equating the different terms of order  $\lambda^{-1}$  and  $\mathcal{O}(0)$  we get the equations

$$\begin{aligned} \sum_{a \neq b} \frac{\langle \Gamma_a, \Gamma_b \rangle}{|w_a - w_b|} &= 0, \\ \sum_{a \neq b} \frac{\langle \Gamma_a, \Gamma_b \rangle |v_a - v_b|}{|w_a - w_b|^2} + \sum_i \frac{\langle \Gamma_i, \Gamma_b \rangle}{|x_i - x_0|} &= h_b, \\ \frac{\langle \Gamma_0, \Gamma_j \rangle}{|x_0 - x_j|} + \sum_{i \neq j} \frac{\langle \Gamma_i, \Gamma_b \rangle}{|x_i - x_0|} &= h_j. \end{aligned} \quad (3.40)$$

Here we used the notation  $\Gamma_0 = \sum_a \Gamma_a$  and  $h_m = \langle \Gamma_m, h \rangle = 2\text{Im}(e^{-i\alpha} Z_a)$ .

These equations have two important implications. First for a group of centers to be able to coincide the charges  $\Gamma_a$  must be such that there exists a solution to the first equation<sup>4</sup> of (3.40), this is a condition on the charges only and independent of the moduli. This has some important consequences and we will discuss this equation in some more detail for two and three centers below. The second, maybe more trivial consequence of (3.40) is that very close to coincidence the centers not involved in the collapse do not influence the centers that collapse to a single point. Furthermore we can rewrite the second and third equation of (3.40) as (we sum the second equation over  $b$ )

$$\sum_{\tilde{i} \neq \tilde{j}} \frac{\langle \Gamma_{\tilde{i}}, \Gamma_{\tilde{j}} \rangle}{|x_{\tilde{i}} - x_{\tilde{j}}|} = h_{\tilde{j}}, \quad (3.41)$$

where the index  $\tilde{i}$  runs over  $(0, i)$  and furthermore by linearity  $\sum_a h_a = h_0$ . This last equation is nothing more than the stability equation for the non-collapsing centers together with a single center carrying the total charge of the collapsed

<sup>4</sup> Note that it is not trivial that a solution exists as the different  $r_{ab}$  are not independent variables for three centers they have to satisfy e.g. the triangle inequalities. In general they are  $\frac{(N)(N-1)}{2}$  functions of  $2N - 2$  variables and so it is clear they are highly interrelated for large  $N$ .

centers. So we see that for the centers remaining at a distance the coinciding centers behave exactly as a single center.

From the above analysis one might be tempted to think that when  $\lambda \rightarrow 0$  and the coordinates of the different centers approach the point  $x_0$  closer and closer, the centers also approach each other arbitrarily close in physical space. This is however not the case! When the centers start approaching each other a gravitational throat encapsulating them develops. Due to this increasing warp factor the smaller and smaller relative coordinate distances converge to a finite spatial distance. As  $\lambda$  becomes zero this throat becomes infinitely deep and the centers behave as in an  $\text{AdS}_2$  space and the solutions gain a scaling symmetry.

To make this precise let us take a number of centers  $\Gamma_a$  that can approach each other arbitrarily closely in coordinate distance. We rescale there relative distances as  $r_{ab} = \lambda w_{ab}$  and are interested in the limit where  $\lambda \rightarrow 0$ . To see how the metric behaves in this limit first note that

$$H(x, r_{ab}, h) = \frac{1}{\lambda} H(w, w_{ab}, \lambda h), \quad (3.42)$$

$$\Sigma(\lambda^{-1} H) = \lambda^{-2} \Sigma(H). \quad (3.43)$$

This then implies that the physical distance inside the throat, where the asymptotic constants  $h$  can be neglected, does not scale with  $\lambda$  and stays finite<sup>5</sup>:

$$ds^2 = \Sigma(x) d\vec{x}^2 = \Sigma(w) d\vec{w}^2. \quad (3.44)$$

These rescaled intercenter coordinate distances  $w_{ab}$  are the ones that naturally appear in the constraints in the  $\lambda \rightarrow 0$  limit as we saw in (3.40), those constraints take the form:

$$\sum_{a \neq b} \frac{\langle \Gamma_a, \Gamma_b \rangle}{|w_a - w_b|} = 0. \quad (3.45)$$

The solutions of this equation at  $\lambda = 0$  are only determined up to an overall scaling of the coordinate distances, hence we refer to them as scaling solutions. An important property of these equations (3.45) is that all dependence on the asymptotic moduli has disappeared, this implies that although the full solution space can change under the variation of moduli the component connected to the scaling point can not disappear as the scaling point can not be made to decay. This is reflected in the supergravity solution by the fact that the constant terms in the harmonic functions have been dropped. As we will see below this also implies that scaling solutions cannot be classified by attractor flow trees.

<sup>5</sup> For this argument to hold it is necessary that the total charge,  $\sum_a \Gamma_a$ , corresponds to a BPS black hole with non-vanishing horizon. In case this is not true, e.g. when two identical charges of the form  $\Gamma = (1, 0, 0, 0)$  approach each other, only a finite throat develops in the  $\lambda \rightarrow 0$  limit. In this case one can explicitly check that the two centers do coincide also in physical space, the above argument breaks down as the constants  $h$  cannot be neglected in this case.

In case of two centers the condition (3.45) is rather trivial and only has a solution when  $\langle \Gamma_1, \Gamma_2 \rangle = 0$ . With three centers the situation is more interesting. Now (3.45) is solved by taking

$$w_{12} = \langle \Gamma_1, \Gamma_2 \rangle \rho, \quad w_{23} = \langle \Gamma_2, \Gamma_3 \rangle \rho, \quad w_{31} = \langle \Gamma_3, \Gamma_1 \rangle \rho, \quad (3.46)$$

with  $\rho$  an arbitrary positive/negative constant. So the first condition on the charges is that the above intersection products should all have the same sign to have all the  $w_{ij}$  positive. A more subtle constraint is that the  $w_{ij}$  are relative distances that satisfy the triangle inequalities. In this case this directly translates to the intersection products having to satisfy the triangle inequalities. In case we order the centers such that all intersection products are positive the scaling equations (3.45) can only be solved for three centers such that

$$\langle \Gamma_1, \Gamma_2 \rangle + \langle \Gamma_2, \Gamma_3 \rangle \geq \langle \Gamma_3, \Gamma_1 \rangle \quad \text{and cyclic.} \quad (3.47)$$

For more than three centers the scaling equations (3.45) become harder to analyze in general and such configurations have only been studied in certain specific cases [98, 99]

### Barely bound states and threshold stability

For most charges and values of the moduli the solution space to (3.31) has no scaling point and is furthermore compact. For these cases it is clear that the multicentered black holes are bound states that can approach each other only to a minimal distance and can also move away only up to a certain maximal distance from the other centers. There exist border cases however, where some subset of the centers can sit at a finite distance to the other centers but can also freely be taken off to infinity. These configurations are thus not really ‘bound’ in the purest sense of the word but at the same time the centers are not completely free to move around either as there is some non-trivial dynamics between them. We will therefore refer to these configurations as *barely bound states*, in the literature similar situations are often referred to as *bound states at threshold* see e.g. [100]. As we will show in this section this non-compactness<sup>6</sup> of solution space can only appear when the moduli are at marginal stability for certain specific types of charges. As the physics is however completely different from the more typical cases of marginal stability that correspond to forced decay of the multicentered solutions, we will refine the notion of marginal stability at the end of the section and define the special case of what we call *threshold stability*.

Let us introduce some convenient notation to study when the solution space can become non-compact. We group those centers that stay at a finite distance to each other in clusters while the clusters are allowed to move infinitely far

<sup>6</sup> Note that the solution space is noncompact as embedded in  $\mathbb{R}^{3N-3}$ . In [96] it is shown however that the volume of this solution space with respect to its natural symplectic form is finite.

apart from each other. As such we can label the coordinates of the centers as:

$$x_{A\alpha} = \lambda x_A + x_\alpha + v_\alpha \lambda^{-1} + \mathcal{O}(\lambda^{-2}), \quad (3.48)$$

in this notation we have split the index  $a$  into two indices  $A$  and  $\alpha$  where  $A$  runs over different clusters and  $\alpha$  over the different centers in cluster  $A$ , furthermore we take by convention that  $x_A \neq x_B$  if  $A \neq B$ . We will be interested in the limit  $\lambda \rightarrow \infty$  and so we see that (3.48) indeed corresponds to a number of groups of centers moving far apart while the centers in a given cluster stay close to each other.

Let us see under which conditions there exist solutions of the form (3.48) to the stability constraints (3.31) for an arbitrary large separation  $\lambda$ . One can in this case calculate that (3.31) are only fulfilled if

$$\begin{aligned} \sum_{\alpha \neq \beta} \frac{\langle \Gamma_{A\alpha}, \Gamma_{A\beta} \rangle}{r_{\alpha\beta}} &= h_{A\beta}, \\ \sum_{A \neq B} \frac{\langle \Gamma_A, \Gamma_B \rangle}{r_{AB}} &= 0, \end{aligned} \quad (3.49)$$

where we defined  $\Gamma_A = \sum_\alpha \Gamma_{A\alpha}$ . The second equation is familiar to the one we found for scaling solutions in the previous paragraph, note however that here it is a constraint on the charges of the clusters, not of the individual centers! The first equation looks like the standard constraint equations and tells us that we can only form clusters of centers that form bound states on themselves, i.e. in the absence of the centers in the other clusters.

There is however an additional and much more important constraint hidden in these equations. To see this constraint appear sum the first equation of (3.49) over the index  $\beta$ . This implies  $h_A = \sum_\beta h_{A\beta} = 0$ , or when written out:

$$\text{Im}(e^{-i\alpha} Z_A) = 0. \quad (3.50)$$

This condition means the clusters have to be at marginal stability<sup>7</sup>! This specific case of marginal stability for charges that satisfy the "scaling condition", i.e. the second equation of (3.49), is actually very different from the more typical case of marginal stability for generic charges. For generic charges moving the moduli towards marginal stability forces the centers infinitely far apart and the state thus decays when crossing the wall of marginal stability. In the more special case of marginal stability between charges of 'scaling' type, i.e. those for which a solution to (3.49) exists, the centers are free to move off to infinity or stay at a finite distance. Furthermore in this case there is no decay when moving across the wall.

<sup>7</sup> To be a bit more precise: in case of more than two clusters there is always an additional solution to (3.50) apart from all the central charges aligning. This solution is a generalization of anti-marginal stability and we will not consider it here.

We will distinguish between the two different types of marginal stability by referring to this new more constrained version as threshold stability while still referring to the old notion as marginal stability. We will restrict ourselves in this thesis to the case of threshold stability for two clusters where our precise definitions are:

**Marginal stability:**  $\text{Im}(\bar{Z}_1 Z_2) = 0$ ,  $\text{Re}(\bar{Z}_1 Z_2) > 0$  and  $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$

**Threshold stability:**  $\text{Im}(\bar{Z}_1 Z_2) = 0$ ,  $\text{Re}(\bar{Z}_1 Z_2) > 0$  and  $\langle \Gamma_1, \Gamma_2 \rangle = 0$

Furthermore we will refer to the codimension one hypersurfaces on which this condition is satisfied as walls of marginal/threshold stability respectively.

Note that in a generalization of the definition of threshold stability to more than two clusters the constraint on the charges of the clusters should be that a solution to (3.49) exist. For more than three centers a simple understanding of these constraints eludes us however and furthermore the condition of more than two central charges aligning in a single point is of codimension two and thus of less relevance.

The discussion here might have been a little abstract and therefore we have worked out the difference between typical marginal stability and threshold stability in much detail in an example in section 3.4.

### 3.3.3 The lift to five dimensions

In [80] (see also [79, 101, 102, 103, 104, 105, 106, 107]) the multicentered black hole solutions were lifted to five dimensions via the connection between IIA and M-theory on a circle, see section 2.4. It is interesting to note that the 5 dimensional metric, obtained by plugging the four dimensional multicentered solution (3.23) into the general formula (2.108) for the lift, can be rewritten in the following form:

$$\begin{aligned}
 ds^2 = & -2^{-2/3} \left( \frac{Q}{H^0} \right)^{-2} \ell_5^2 \left( \sqrt{\frac{R}{\ell_5^3}} dt + \omega + L(d\psi + \omega_0) \right)^2 \\
 & + 2^{-2/3} \ell_5^2 \frac{Q}{H^0} \left( H^0 \frac{R}{\ell_5^3} dx^i dx^i + \frac{1}{H^0} (d\psi + \omega_0)^2 \right). \quad (3.51)
 \end{aligned}$$

This form of the metric is useful as it shows the special role played by the D6 charge. The second term in (3.51) is a 4 dimensional Euclidean hyper-Kähler manifold known as a Gibbons-Hawking metric and is a multi-centered generalization of the Taub-Nut metric. As we discussed in an example in section 2.5.2 the lift of a center with D6-charge gives rise to a Taub-Nut center. In case there are centers with both positive and negative D6-charge the five dimensional space has non-contractible two-spheres extending between those D6-charged centers. This  $S^2$  can be pictured by gluing the two 'cigars' made up by the Taub-NUT centers. In case one has very many centers with various D6-charges the resulting 5d geometry has a very complicated topology consisting of many non-contractible

spheres and is called a *bubbling solution* by [32], where they were independently discovered from the 5-dimensional point of view.

As was also discussed in the single center case in section 2.5.1, a D4 charged center will lift to a black string unless it also carries D6-charge in which case it lifts to what locally looks like a BMPV black hole at the center of 5 dimensional Taub-NUT [79]. The 4-dimensional multicentered black holes thus lift to bound states of the various objects, black rings, BMPV black holes or Taub-Nut centers in 5 dimensions. The different possibilities can be summarized as follows: the topology of the horizon at a given center is that of an  $S^1$ -bundle over  $S^2$  of degree  $p_a^0$ , i.e.  $S^1 \times S^2$  for  $p_a^0 = 0$  and  $S^3/\mathbb{Z}_{|p_a^0|}$  otherwise. Many more properties of these 5 dimensional geometries have been discussed in the literature, e.g. [32, 102, 106, 80] and we refer the reader to those works for more details. Note that although the lifts of multicentered black holes give rise to a lot of different BPS objects in five dimensions, by construction all these five dimensional solutions have a  $U(1)$  isometry along the  $\psi$  direction. They are therefore not the complete set of five dimensional BPS solutions.

For practical computations and later reference we express the metric (3.51) in a form where the radius of the M-theory circle can be manifestly read off to be  $2^{-1/3} \ell_5 \frac{\Sigma}{Q}$ :

$$\begin{aligned}
 ds_{5d}^2 = & 2^{-2/3} Q^{-2} \left[ -\ell_5^2 (H^0)^2 \left( \sqrt{\frac{R}{\ell_5^3}} dt + \omega \right)^2 - 2\ell_5^2 L \left( \sqrt{\frac{R}{\ell_5^3}} dt + \omega \right) (d\psi + \omega_0) \right. \\
 & \left. + \ell_5^2 \Sigma^2 (d\psi + \omega_0)^2 \right] + 2^{-2/3} \frac{R}{\ell_5} Q dx^i dx^i. \tag{3.52}
 \end{aligned}$$

### 3.3.4 The split attractor flow conjecture

We found the multicentered solutions (3.23) by solving the BPS equations in section 3.2.1. As in the single center case we have to be careful however that the solutions we found make physical sense. The bound black hole solutions (3.23) are relatively complicated and it is non-trivial to determine if they are well-behaved everywhere. In particular one should be concerned about the appearance of closed timelike curves or singularities. If the entropy function,  $\Sigma$ , which involves a square root, becomes zero or takes imaginary values in some regions the 4d solution is clearly ill behaved; this is equivalent to closed timelike curves in the 5d metric as discussed in [102] and [32]. One can on the other hand show that if  $\Sigma^2 > \omega_i \omega^i$  everywhere then there can be no closed timelike curves [106]. This is a rather complicated condition to check for a generic multicenter solution however and furthermore it is sufficient but not necessary; the condition could be violated without closed timelike curves appearing. In [27] and [29] a simplified criteria was proposed for the existence of (well-behaved) solutions which we will relate in this subsection. First we will state the conjecture and explain its many uses, afterwards we will present the physical motivation behind this conjecture.

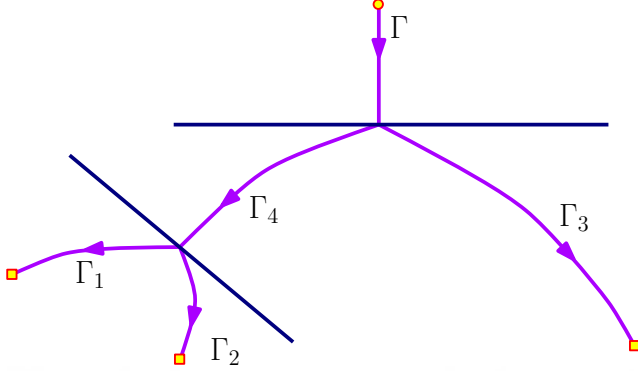


**Figure 3.3** Sketch of an attractor flow tree. The dark blue lines are lines of marginal stability, the purple lines are single center attractor flows. The tree starts at the yellow circle and flows towards the attractor points indicated by the yellow boxes.

### Split attractor flows and existence of bound black holes

A first condition necessary for the existence of a multicentered solution is that the positions should satisfy the constraint equations (3.25), which we discussed at length in subsection 3.3.2. This condition is not sufficient however as the function  $\Sigma$ , which involves a square, can still become imaginary. We already saw this in the single center case, in that case there are no constraint equations to be satisfied but for certain charges no black hole solution existed as the entropy function was ill-defined for these charges. In the case of a single black hole we could relate this to the attractor flow on moduli space being well defined, equivalent to the central charge having no zeros on regular points of moduli space. Inspired by the single center case and by some motivations we will explain later, also for multicentered solutions we can define an attractor flow and the solutions are conjectured to exist iff the attractor flow is well behaved.

The generalization of the single center attractor flow to more centers is called a *split attractor flow tree*. This is a graph on the Calabi-Yau moduli space beginning at the moduli at infinity,  $t^A|_\infty$ , and ending at the attractor points for each center, see e.g. figure 3.3. The edges correspond to single center flows towards the attractor point for the sum of charges further down the tree. Vertices can occur where single center flows for a charge, e.g.  $\Gamma = \Gamma_4 + \Gamma_3$ , cross walls of marginal stability where the central charges are all aligned ( $|Z(\Gamma)| = |Z(\Gamma_3)| + |Z(\Gamma_4)|$ ). Each leg can then later split again if another wall of marginal stability is encountered, as e.g. the split  $\Gamma_4 \rightarrow \Gamma_1 + \Gamma_2$  in figure 3.3.

The data of a split flow tree can also be more abstractly summarized in a nested list denoting all the centers and the order in which they split from the total charge, for the configuration given in figure 3.3 this is e.g.  $\{\Gamma_3, \{\Gamma_1, \Gamma_2\}\}$ . So let us summarize the upshot: for a two center flow tree  $\{\Gamma_1, \Gamma_2\}$  starting from

the moduli  $t_\infty^A$  to be well defined we need the following

- The single center flow for the total charge  $\Gamma = \Gamma_1 + \Gamma_2$  starting at  $t_\infty^A$  must cross a wall of marginal stability where  $|Z| = |Z_1| + |Z_2|$ , call the point of intersection  $t_\times^A$ .
- There should exist single center flows from  $t_\times^A$  to the attractor points of both the charges  $\Gamma_1$  and  $\Gamma_2$ .
- Note that a necessary and sufficient condition for all three of the above single center flows to exist is that they stay in the physical part of moduli space and do not reach a zero of the corresponding central charge, i.e.  $Z, Z_1$  and  $Z_2$  respectively.

This definition is then iteratively generalized to more than two centers by allowing further splits of the single center flows further down the tree under the same conditions, see e.g. figure 3.3.

#### The many uses of flow trees

The simple flow tree picture captures surprisingly many features of the multicentered black holes. First note that the flow tree is characterized by the moduli at infinity, i.e. the starting point of the tree, and the charges of the centers, that determine the endpoints of the tree. The precise positions of the centers are however not determined by a flow tree, a single flow tree actually corresponds to a single connected component of solution space. Different tree topologies for the same starting and end points, see e.g. figure 3.4, correspond to the different connected components. Some more motivation for this picture will be given below.

Another physical property that is encoded in the flow tree is that of decay at marginal stability and the fact that multicenter solutions are only stable in a certain region of moduli space. A flow tree is clearly only defined when the starting point is separated from the attractor point of the total charge by a wall of marginal stability. Once the starting point is on the same side of the wall of marginal stability as the attractor point a flow tree no longer exists. This as the first leg of the flow won't encounter the wall of marginal stability anymore and thus can't split. So we see that the decay of a multicentered solution once a wall of marginal stability is crossed is translated in the flow tree picture as the tree ceasing to exist. For a more complicated tree with many vertices and branches we see that the decay naturally happens between two clusters of centers where the clusters are formed by the centers that split further down the tree, a phenomenon we saw from the analysis of solution space before. As was also discussed in section 3.3.2, in principle one can have decays into more than two clusters/centers, this would correspond in the tree picture to three-pronged splits. We mentioned before that this can only happen on the intersection of two walls of marginal stability and that the process is thus much less likely and can furthermore always be evaded by choosing the moduli appropriately.

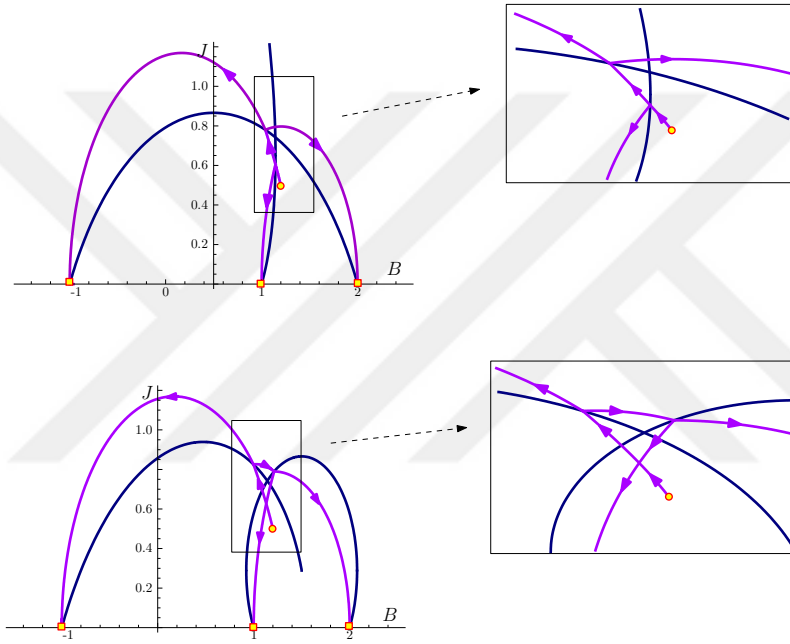


**Figure 3.4** Two flow trees with the same initial point and the same end points but different topology are shown. From left to right the attractor points correspond to the respective charges  $\Gamma_1 = e^{-D_1}$ ,  $\Gamma_2 = -e^{D_1}$  and  $\Gamma_3 = e^{2D_1}$  and the starting point is at  $(J, B) = (\frac{1}{2}, \frac{6}{5})D_1$ . The upper flow tree has topology  $\{\Gamma_2, \{\Gamma_1, \Gamma_3\}\}$  while the lower flow tree corresponds to  $\{\Gamma_1, \{\Gamma_2, \Gamma_3\}\}$ .

The *attractor flow conjecture* that a multicentered black hole solution exists iff the corresponding flow tree exists is a very practical condition as it is given in terms of local data: the starting moduli and the charges of the centers. From this data one can check if the tree exists and thus if the supergravity solution should be physical. Checking if a supergravity solution is well-defined is much harder as it is a non-local condition, e.g.  $\Sigma$  should be real all over  $\mathbb{R}^3$ . So when trying to find solutions of a certain type one can first check if a flow tree exists, rather simple, and then later verify that the supergravity solution corresponding to the flow is well defined, a bit harder but feasible numerically. Up till now no counterexamples have been found to the split flow conjecture and in a moment we will argue why this is to be expected from physical grounds. In section 3.4.2 we illustrate the use of the split attractor flow conjecture by using it to uncover the existence of an interesting set of multicentered solutions.

As a final remark we should point out a certain caveat. The split flow conjecture only holds for those components of the solution space that do not have a scaling point. As was mentioned in the discussion on scaling solutions in subsection 3.3.2 those components that have a scaling point can not be made to decay and exist for all values of the moduli. So clearly they cannot correspond to a flow tree, which can always be made to decay. A good local criterion for the existence of general scaling solutions remains unknown for the moment, for two center scaling solutions such a necessary and sufficient criterion was in [108] shown to be  $\Sigma(\Gamma) \leq \Sigma(\Gamma_1) + \Sigma(\Gamma_2)$ .

### Motivation of the conjecture

As we saw in chapter 2 on single center black holes, for a single supersymmetric black hole the attractor flow is nothing but the radial flow of the moduli. In the multicenter case the moduli are no longer uniquely determined by the radius  $r$ , but because spherical symmetry is broken they also depend on the angular variables  $\theta$  and  $\phi$ . So the region in moduli space that is covered in a given multicenter solution is no longer a one-dimensional flow, rather it is some kind of 'pant'-shaped thickening of the attractor tree we introduced before, see figure 3.5 for an example. It is not difficult to see that the attractor flow tree corresponding to a solution is always part of the region traced out by the moduli of the solution. This is most easily illustrated in the case of two centers. Let us place the two centers symmetrically on the  $z$ -axis, i.e.  $x_1 = (0, 0, l/2)$  and  $x_2 = (0, 0, -l/2)$  with  $l$  the equilibrium distance. For two centers we still have some axial symmetry and the moduli are thus only function of  $r$  and  $\theta$ :  $t^A = t^A(r, \theta)$ . Then the first leg of the flow tree is given by  $\gamma_0(\tau) = t^A(\frac{1}{\tau}, \frac{\pi}{2})$  where  $\tau$  goes from zero to  $\sqrt{\frac{4}{3}}$ , see figure 3.6. The points  $(r, \theta) = (\frac{\sqrt{3}}{2}, \frac{\pi}{2})$  are those for which the distance to both centers is equal to  $l$ , it is easy to check by the BPS equation (3.10) that the moduli are indeed at marginal stability at these points. The second and third leg are given by the values of the moduli along arcs of the circles of radius  $l$  around both centers, see figure 3.6, more

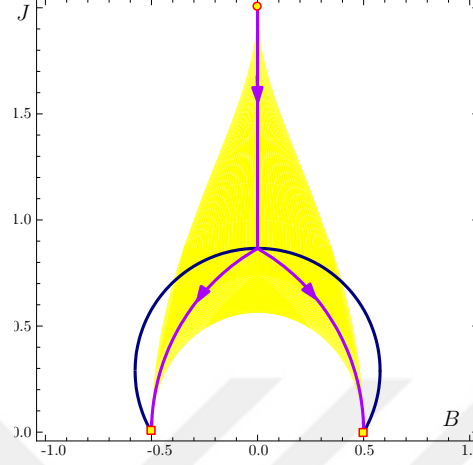


**Figure 3.5** In this figure the split flow tree for a pair of purely fluxed D6 and purely fluxed  $\overline{D6}$  is shown. In yellow the image of the scalar maps from space-time to moduli space is shown.

precisely one has that  $\gamma_1(\alpha) = t^A(r(\alpha), \frac{\pi}{2} - \alpha)$  and  $\gamma_2(\alpha) = t^A(r(\alpha), \frac{\pi}{2} + \alpha)$  where  $\alpha$  runs from zero to  $\pi/2$  and

$$r(\alpha) = \frac{l}{2} \left( \sqrt{\frac{7 - \cos 2\alpha}{2}} - \sin \alpha \right). \quad (3.53)$$

So in one direction the attractor flow conjecture is thus certainly true; if a valid multicentered supergravity solution exist also the split attractor flow is well defined.

The other implication of the conjecture, namely that if the attractor flow tree exists then the multicentered bound state exists, is harder to prove and a solid mathematical derivation hasn't yet been found. On physical grounds there however exists a rather elegant argument for this part of the conjecture. To make this argument we first need to point out the following fact. A single center attractor flow for a charge  $\Gamma$  can only cross a wall of marginal stability for a given split  $\Gamma \rightarrow \Gamma_1 + \Gamma_2$  once<sup>8</sup>. This can be easily checked by taking the intersection product of the single center attractor flow (2.81) with  $\Gamma_1$  and noting that the LHS is zero at marginal stability and that the RHS is zero for a unique value of the parameter  $\tau$  along the flow.

Let us now argue for the existence of a multicentered solution when corresponding flow tree exists. Let us consider two charged centers  $\Gamma_1$  and  $\Gamma_2$  for which an attractor flow tree starting at  $t_o^A$  exists. Clearly that flow tree also

<sup>8</sup> We assume here that  $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$ , in the other case the flow can actually run *along* the wall of marginal stability [95].

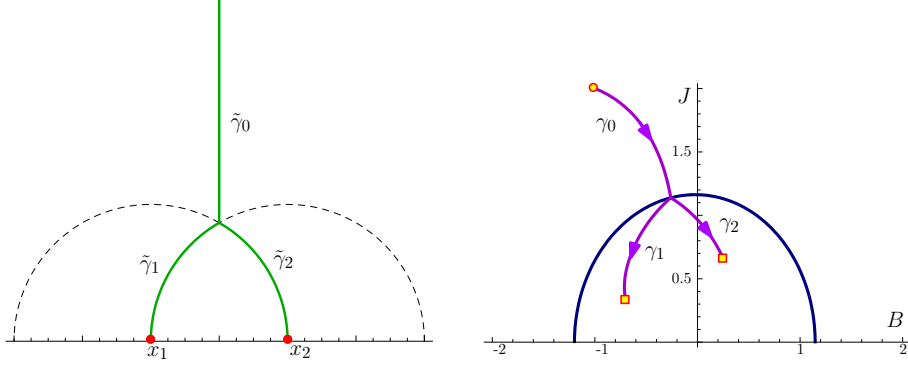


**Figure 3.6** In the left picture the trajectory in space on which the moduli map to the attractor flow tree is shown in green. To illustrate this point an example flow tree is shown in the right. The the legs  $\gamma_i$  are the images of the  $\tilde{\gamma}_i$  under the scalar moduli, i.e.  $\gamma_i = (J(\tilde{\gamma}_i), B(\tilde{\gamma}_i))$ .

exists if we would take the starting point somewhere further down the flow for  $\Gamma$  as long as it is before the intersection with the wall of marginal stability. If we take this 'new' starting point  $\theta_\epsilon^A$  close enough to the wall of marginal stability we know from the constraint equations that the two centers are separated extremely far apart, actually we can place them as far apart as we want by choosing the moduli  $\theta_\epsilon^A$  close enough to the wall of marginal stability. We know that the two separate centers  $\Gamma_1$  and  $\Gamma_2$  exist as good single black hole geometries as the single center flows for those centers are part of the split flow tree. So for the moduli  $t_\epsilon^A$  close enough to marginal stability the geometry of the bound system becomes that of two very far separated and almost independent single black hole geometries that we know are well defined, so the full geometry should be well defined as well. Now we can start to change the moduli  $t_\epsilon^A$  continuously along the first leg of the three, the single flow for  $\Gamma$ , back to the original starting point  $t_o^A$ . As we argued above, along this trajectory the moduli cannot cross a wall for the split in  $\Gamma_1$  and  $\Gamma_2$ . So as a BPS state can only decay at marginal stability the state which we argued existed for the moduli  $t_\epsilon^A$  still has to exist at the original moduli  $t_o^A$ , which completes the argument. This reasoning made for two centers can now be iteratively applied to a more complicated tree.

### 3.4 Examples

It is clear that there is an extremely rich zoo of all kind of possible bound configurations. All of them share certain physical properties but then there are also properties that will differ from configuration to configuration. For the moment the best guiding principle we have is the split attractor flow conjecture.

We already discussed many general physical properties of the bound black holes in the previous sections, but to really get a feel of how these multicenter solutions behave, some well chosen examples might be helpful. With the split attractor principle as our guide we can start exploring the multicenter jungle. In this section we will take the reader to some of the interesting beasts that are of further relevance in the thesis. Further examples and their physics can be found in [29, 27, 94, 109], but also in e.g. [32, 110, 99, 102] where essentially similar systems have been explored from the five dimensional point of view.

The examples in this section were first studied by the author in a collaboration with J. de Boer, F. Denef, S. El-Showk and I. Messamah in [31]. We will first discuss a simple three center example that allows us to illustrate the dynamics of the solution space under variation of the asymptotic moduli. Furthermore we will show that in another charge regime the same three center solution becomes scaling and this scaling solution will be studied in more detail in chapter 5. The second example we will work out shows that sometimes it can be too restrictive to focus on a one modulus Calabi-Yau. Once we consider Calabi-Yau manifolds with more moduli bound states of centers without D6 charge become possible and we will give an example of those. These last solutions will be of relevance in the next chapter when we take a decoupling limit, as in this limit they behave completely different than the other types of bound states.

### 3.4.1 A simple three center solution

As a first example we consider a bound state of three centers. The main reason why this deserves a separate discussion is that it is the simplest case where we have a non-trivial solution space, as for two centers the degrees of freedom of the solution space just correspond to the possibility to rotate the bound state. For three centers there is also some freedom to deform the system however, i.e. one can change the distances between the centers without changing the energy of it. Especially interesting to see is how the solution space, and hence the possible ways to deform the bound state, changes under variations of the asymptotic moduli. This we will illustrate in some detail here. Let us first introduce the system we will take as our example. Consider the following three charges:

$$\Gamma_1 = (1, \frac{p}{2}, \frac{p^2}{8}, \frac{p^3}{48}), \quad \Gamma_2 = (-1, \frac{p}{2}, -\frac{p^2}{8}, \frac{p^3}{48}) \quad \text{and} \quad \Gamma_3 = (0, 0, 0, -n). \quad (3.54)$$

Here  $\Gamma_1$  corresponds microscopically to a single D6 brane with some flux  $p$  on its worldvolume. The second charge  $\Gamma_2$  is an anti D6 with a worldvolume flux opposite to that of the D6. As the third center we take a number  $|n|$  anti-D0 branes. The first step to see if a bound state of these charges makes physical sense at all is to consider the constraint equations (3.25). For our charges (3.54)

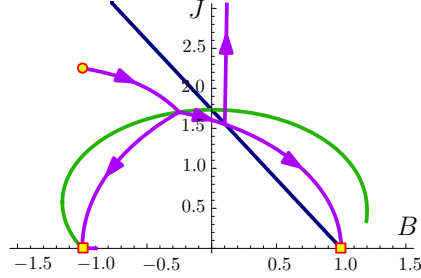


**Figure 3.7** Attractor flow for the charges  $\Gamma_1 = (1, 1, 1/2, 1/6)$ ,  $\Gamma_2 = (-1, 1, -1/2, 1/6)$  and  $\Gamma_3 = (0, 0, 0, -1/100)$ . The attractor point for  $\Gamma_1$  is the box on the  $B$ -axis on the left, that for  $\Gamma_2$  the one on the right. The attractor point for  $\Gamma_3$  lies at infinite  $J$ .

they become

$$\frac{p^3}{6} \frac{1}{r_{12}} - \frac{n}{r_{13}} = -h_1, \quad (3.55)$$

$$n \left( \frac{1}{r_{13}} - \frac{1}{r_{23}} \right) = -h_3. \quad (3.56)$$

The solution space of these equations can be parameterized by  $\lambda$  as

$$r_{13} = \lambda n, \quad r_{23} = \frac{\lambda n}{1 + \lambda h_3} \quad \text{and} \quad r_{12} = \frac{\lambda p^3}{6(1 - h_1)}. \quad (3.57)$$

As we know from our general discussion in section 3.3.2, if the intersection products satisfy the triangle inequalities, then the solution space has a scaling point and exists for all asymptotic moduli. In this example this is the case if  $n > \frac{p^3}{12}$ , and indeed under exactly this condition the solution space includes the scaling point  $\lambda = 0$ . We will come back to this scaling solution in chapter 5 and restrict ourselves in this section to the non-scaling regime  $n < \frac{p^3}{12}$ .

In this non-scaling case the bound state of the three centers doesn't exist for all values of the moduli and one has to check where the various walls of marginal stability lie, what their stable sides are and ultimately if a split attractor flow exists. For the example of our current interest this can be done with some effort and the result is summarized in figure 3.7, where an attractor flow tree is shown starting at the moduli  $B = -1$  and  $J = 9/4$ . We will now use this example to illustrate the behaviour of the bound state under change of the asymptotic moduli. This allows us to show more concretely the difference between marginal and threshold stability introduced in 3.3.2.

In figure 3.7 the green line is a wall of marginal stability for the charges  $\Gamma_2$  and  $\Gamma_1 + \Gamma_3$ . More precisely on this line  $\text{Im}(\bar{Z}_2 Z_{1+3}) = 0$ . As the intersection product  $\langle \Gamma_2, \Gamma_1 + \Gamma_3 \rangle = \frac{p^3}{6} - n$  is non-vanishing this is thus a wall of marginal stability in the refined sense of section 3.3.2. In this same example the  $J$ -axis,

i.e.  $B = 0$ , is a wall of threshold stability for the charges  $\Gamma_1 + \Gamma_2$  and  $\Gamma_3$ , i.e.  $\text{Im}(\bar{Z}_{1+2}Z_3) = 0$  at  $B = 0$  but this time  $\langle \Gamma_1 + \Gamma_2, \Gamma_3 \rangle = 0$ . We will now look at the behavior of the split flow and the solution space in approaching this wall of marginal or threshold stability respectively. In both cases we start from the attractor flow depicted in figure 3.7, which has its starting point at  $B = -1$  and  $J = 9/4$ . First we will discuss what happens while we keep  $B$  fixed and lower  $J$  thus approaching the wall of marginal stability discussed above. Secondly we will see what happens when one keeps  $J$  fixed but moves  $B$  towards positive values thus crossing the wall of threshold stability at  $B = 0$  pointed to above.

Starting at a negative value for the  $B$ -field modulus and a large enough Kähler modulus a split flow  $(\Gamma_2, (\Gamma_1, \Gamma_3))$  exists and in spacetime this corresponds to a supergravity solution corresponding to the  $\overline{\text{D0}}$  “orbiting” the D6 which then together bind to the  $\overline{\text{D6}}$ , see figure 3.8 A. We can now see what happens in case we start moving the starting point of the attractor flow tree. We keep the value of the B-field fixed and lower the Kähler modulus towards zero. In this way we will approach the wall of marginal stability for the charges  $\Gamma_2$  and  $\Gamma_1 + \Gamma_3$ , the green line in figure 3.7. Approaching this wall corresponds to the  $(\Gamma_1, \Gamma_3)$  cluster being forced further and further away from the  $\Gamma_2$  center. A plot of the solution space for values of the moduli closer and closer to marginal stability is given in figure 3.8 A through C. The centers are forced infinitely far apart and decay the moment the starting point coincides with the wall of marginal stability and the solution ceases to exist once the wall has been crossed. This is the familiar decay of multicenter bound states when crossing a wall of marginal stability. Also microscopically the bound state disappears out of the spectrum and the BPS index makes a jump. The way this is manifested in the split flow picture is by the fact that the split flow tree only exists on one side of the wall of marginal stability.

In case of crossing a wall of threshold stability the physics is different. We can start from the same initial configuration but now deform it in a different way. We now move the starting point in moduli space towards the  $J$ -axis along a trajectory of constant  $J$ . We have plotted the solution space along this trajectory in figure 3.9 A through E. Approaching the wall of threshold stability  $B = 0$ , the orbit of the D0 around the D6 becomes more and more deformed and it expands. Once we reach threshold stability the D0 is equally bound to the D6 as to the  $\overline{\text{D6}}$  and can sit anywhere on the equidistant plane between D6 and  $\overline{\text{D6}}$ . Note that this plane is non-compact, i.e. the D0 can move arbitrarily far away along this plane, while the orbits before were large but always compact. Continuing further to positive values for  $B$  the orbit of the D0 becomes compact again but has now become an orbit around the  $\overline{\text{D6}}$ . This corresponds to the fact that the split flow has now changed topology from  $(\Gamma_2, (\Gamma_1, \Gamma_3))$  to  $(\Gamma_1, (\Gamma_2, \Gamma_3))$ . In this process no states have decayed and no solutions have been lost.

This example illustrates the general behavior that we can summarize as follows:

- A wall of marginal stability (in the refined sense) corresponds to a bound-

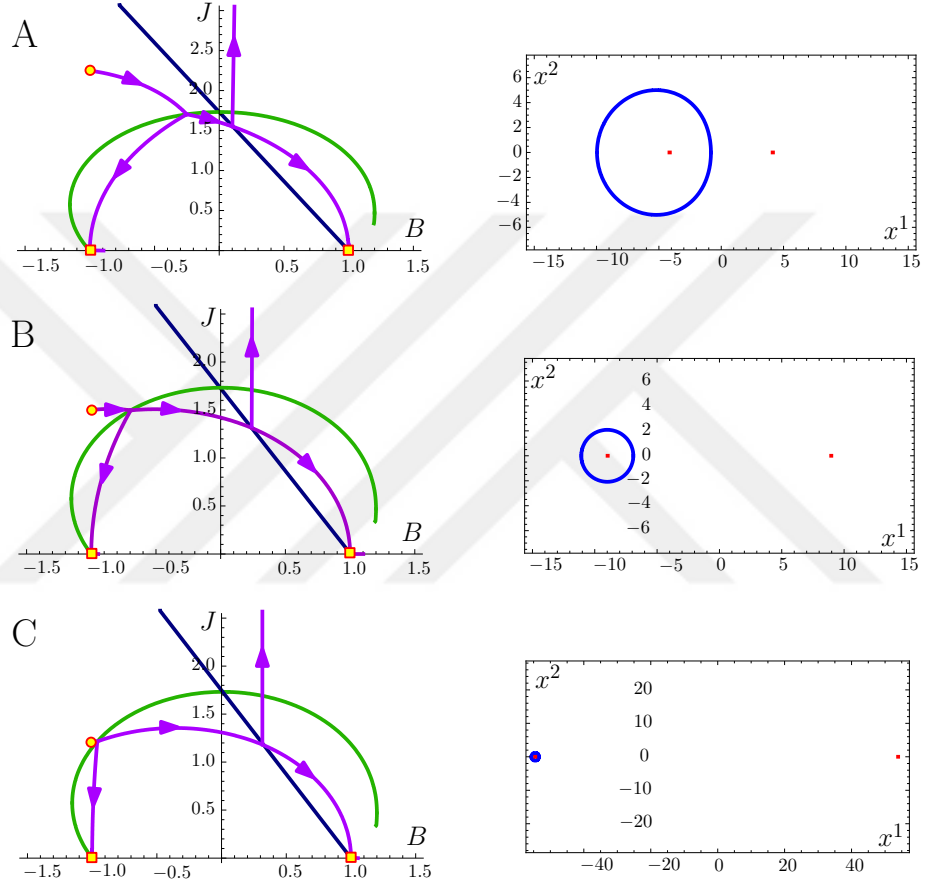


**Figure 3.8** On the left the attractor flows for the charges of fig. 3.7 are shown for different values of the starting moduli. On the right the corresponding solution moduli space is plotted. The red points are the positions of  $\Gamma_1$  (right) and  $\Gamma_2$  (left). In blue are the possible positions of  $\Gamma_3$ . Note the difference in the scale in the last plot, this as once we approach marginal stability the relative position of the centers diverges.

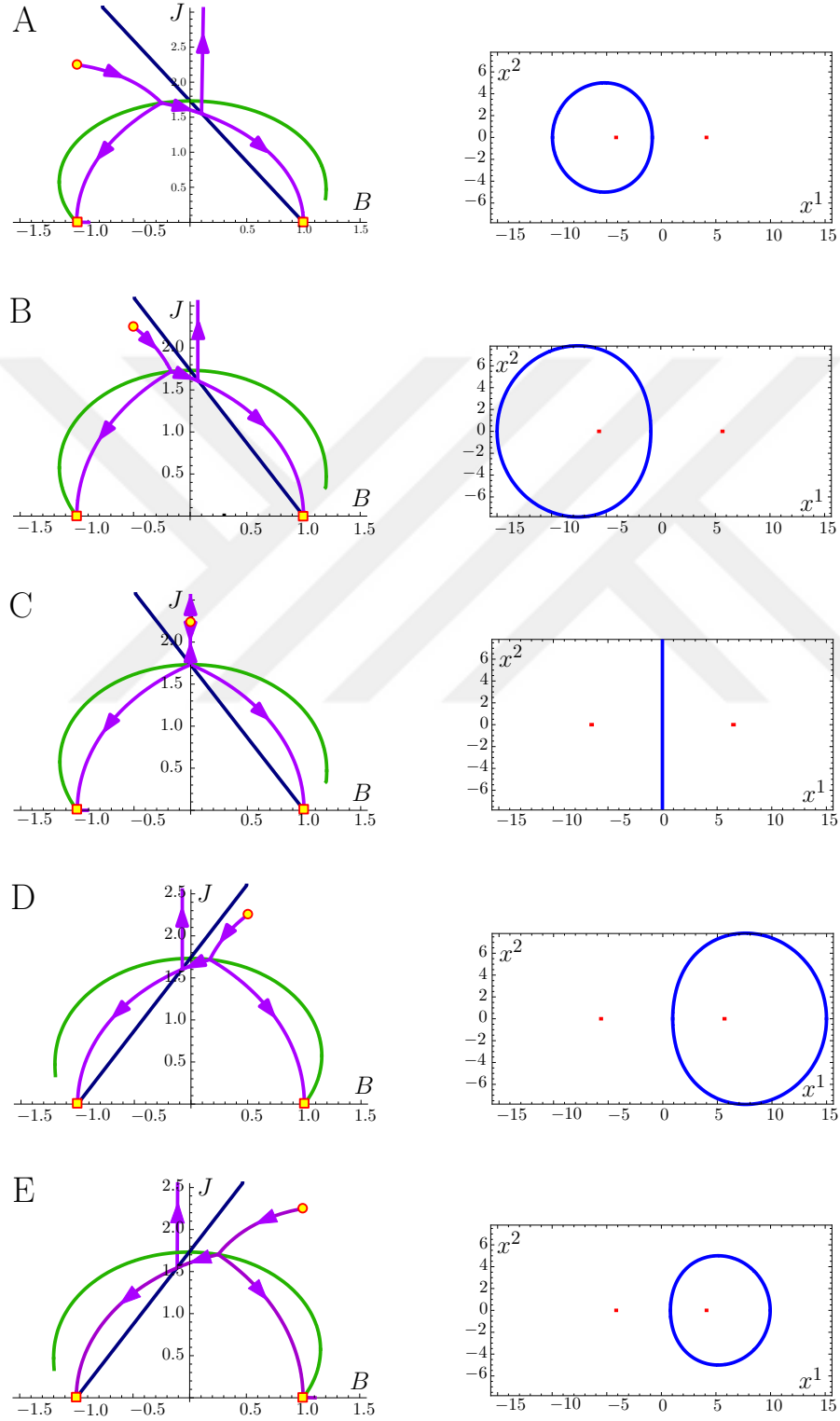


**Figure 3.9** Here we show the same type of plots as in fig. 3.8, but now taking the starting point through a wall of threshold stability, in this case the  $J$ -axis.

ary between a region in moduli space where a certain multicenter solution exists and a region where it no longer exists. In the supergravity picture the disappearance of the bound state happens as a number of centers separate further and further towards infinite separation at marginal stability. Crossing a wall of marginal stability corresponds to a decay of states and a jump in the index counting these states.

- A wall of threshold stability corresponds to a boundary between two regions of different 'topology'. This holds both on the level of the flow tree that changes topology, i.e. the type and order of splits changes, as on the level of the solution space that changes topology as a manifold. This change of topology of the solution manifold can happen as exactly at threshold stability the solution space becomes non compact. Note that when crossing a wall of threshold stability no states decay, they only change character.

So at threshold stability some centers are allowed to move off to infinity but it is also possible for them to sit at finite distance to the other centers; they are not forced to infinite separation as is the case for marginal stability. Although the solution space is non-compact, it turns out to have finite symplectic volume when considered as a phase space [96]. One can check explicitly that this number of states equals that on both sides of the wall of threshold stability and so crossing a wall of threshold stability does *not* correspond to a decay of states — rather, at the wall, the BPS states exist as bound states *at threshold* (hence the name), similar to D0-branes in type IIA string theory in flat space, see e.g. [100].

### 3.4.2 D4D2D0 bound states

In this subsection we show that bound states of two D4D2D0 centers exist for arbitrarily large asymptotic Kähler moduli (i.e.  $J_\infty^3 \gg 1$ ). The existence of these states will have some important consequences for the decoupling limit studied in the next chapter.

It was noticed in [29] that such a bound state of charges  $\Gamma_1 = (0, p_1^A, q_1^1, q_0^1)$  and  $\Gamma_2 = (0, p_2^A, q_2^2, q_0^2)$  doesn't exist in case  $J_\infty^A = p^A \lambda$  with  $\lambda \gg 1$  and  $B^A|_\infty = D^{AB} q_B$ . This can be understood by computing the following quantity:

$$\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(Z_1 \bar{Z}_2) = -\frac{3}{8} (p_1^A q_2^2 - p_2^A q_1^1)^2 + \mathcal{O}(\lambda^{-1}) < 0. \quad (3.58)$$

As the distance between the two centers is given by  $\frac{\langle \Gamma_1, \Gamma_2 \rangle}{2\text{Im}(Z_1 \bar{Z}_2)}$  this implies no such solutions can exist for these asymptotic moduli. So bound states of D4D2D0 centers don't exist in the large volume region of moduli space if the asymptotic Kähler moduli are proportional to the D4- charge,  $J_\infty^A \sim p^A$ , and the asymptotic B-field moduli are at the attractor point for the total charge:  $B^A|_\infty = D^{AB} q_B$ . But a priori nothing forbids to consider asymptotic moduli of a more general form. Indeed, as we will show here, D4D2D0 bound states turn out to exist for more general moduli. Although we expect such bound states to exist quite

generically, a thorough analysis of these type of multicenters is outside the scope of this work. We will only present a simple class of such solutions for a given compactification to show that they do indeed exist. Related examples where discussed recently in [109].

### A class of solutions

The simplest setting one can consider to find these D4D2D0 bound states is in case of a two dimensional moduli space, as otherwise the moduli are trivially proportional to the D4 charge. As an example we take the resolution of the hypersurface  $x_1^8 + x_2^8 + x_3^4 + x_4^4 + x_5^4 = 0$  in  $\mathbb{P}^{(1,1,2,2,2)}$  as our Calabi-Yau threefold (see e.g. [111]). This Calabi-Yau has two Kähler moduli and its intersection numbers are  $D_{11A} = 0, D_{122} = 4, D_{222} = 8$  (and permutations). We will often parametrize the two Kähler moduli as

$$J^A = \tilde{V}_{\text{IIA}}^{1/3} Y^A, \quad (3.59)$$

with  $\tilde{V}_{\text{IIA}}^{1/3} = (\frac{1}{6} D_{ABC} J^A J^B J^C)^{1/3}$  and  $2\tilde{V}_{\text{IIA}}^{1/3}|_\infty = \hat{R}$ . In this specific case this implies the constraint

$$\frac{4(Y^1)^3}{3} + 2Y^1(Y^2)^2 = 1. \quad (3.60)$$

Note furthermore that the  $Y^A$  are related to the M-theory Kähler moduli as  $J_M^A = \tilde{V}_M^{1/3} Y^A$ .

We will now show that there exist bound states of charges

$$\begin{aligned} \Gamma_1 &= \left(0, \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}, \begin{pmatrix} q \\ -\lambda q \end{pmatrix}, q_0\right), \\ \Gamma_2 &= \left(0, \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}, \begin{pmatrix} -q \\ \lambda q \end{pmatrix}, q_0\right), \end{aligned} \quad (3.61)$$

in case  $q_0 < 0$ . The total charge is thus

$$\Gamma = \left(0, \begin{pmatrix} 2p^1 \\ 2p^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2q_0\right). \quad (3.62)$$

Note that as  $q_0 < 0$  this total charge can also exist as a single center BPS black hole [29]. We show the existence of the two center bound state by proving that an attractor flow exists if certain conditions on  $q$  and  $\lambda$  are met. Furthermore we check the existence of the solution explicitly for a numerical example.

To show that a well defined split attractor flow exists it is enough to show the following:

- There exist moduli at infinity for which the stability constraint

$$\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(\bar{Z} Z_1)|_\infty > 0 \quad (3.63)$$

is satisfied.

- The single center attractor flow for the total charge, starting at these moduli at infinity, crosses a wall of marginal stability for the split.
- The two centers exist separately as single centers.

We will show that these three conditions can all be fulfilled for a split of any given total D4D0 charge with negative D0 charge, in charges of the form (3.61), if one chooses  $q, \lambda$  and the asymptotic moduli  $t^A|_\infty$  appropriately.

### Stability condition

Take the moduli at infinity to be

$$\hat{R} \gg 1, \quad (3.64)$$

$$Y_\infty^1 = Y_\infty^2 = \left(\frac{3}{10}\right)^{1/3}, \quad (3.65)$$

$$B_\infty^A = 0. \quad (3.66)$$

It is not difficult to verify that for the charges (3.61) and on the Calabi-Yau we consider

$$\begin{aligned} \langle \Gamma_1, \Gamma_2 \rangle &= -2(p^1 - \lambda p^2)q, \\ \text{Im}(\bar{Z} Z_1)|_\infty &= -\frac{3}{20}(\lambda - 1)(p^1 + 4p^2)q + \mathcal{O}(\hat{R}^{-1}). \end{aligned} \quad (3.67)$$

The stability constraint (3.63) is thus satisfied if  $(\lambda - 1)(p^1 - \lambda p^2) > 0$  i.e.

$$1 < \lambda < \frac{p^1}{p^2} \quad \text{or} \quad \frac{p^1}{p^2} < \lambda < 1. \quad (3.68)$$

### Flow and marginal stability

One can easily calculate that the attractor moduli for the total charge (3.62) are given by

$$B_*^A = 0, \quad (3.69)$$

$$J_*^1 = -\frac{8p^1 q_0}{S(\Gamma_1 + \Gamma_2)}, \quad (3.70)$$

$$J_*^2 = -\frac{8p^2 q_0}{S(\Gamma_1 + \Gamma_2)}, \quad (3.71)$$

where

$$S(\Gamma_1 + \Gamma_2) = 8\sqrt{-\frac{2(p^2)^2}{3}q_0(3p^1 + 2p^2)} \quad (3.72)$$

is the entropy corresponding to the total charge. We will now show that the single center attractor flow for this charge has to cross a wall of marginal stability for

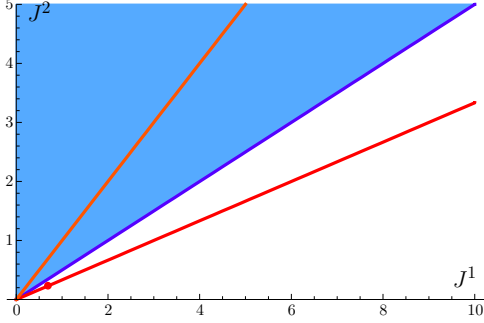


**Figure 3.10** In this figure the  $(J^2, J^1)$ -plane is shown. The red dot is the single center attractor point that lies on the (red) line  $J^1 = \frac{p^1}{p^2} J^2$ . The solid blue line is a line of marginal stability  $J^1 = \lambda J^2$ ,  $1 > \lambda > \frac{p^1}{p^2}$ . It is clear that if one takes the boundary moduli in the shaded blue area, the flow to the attractor point always has to cross the wall of marginal stability. This is thus e.g. the case if we choose our moduli at infinity to lie on the orange line  $J^1 = J^2$ . The numerical values have been taken from the example below.

a split of the form (3.61). For this note first that the B-field moduli are  $B^A = 0$ , both at infinity and at the attractor point. One can check that indeed the B-field stays constant under the attractor flow and that the moduli only flow in the  $J^1, J^2$  plane. Secondly, note that the moduli at infinity lie far away from the origin on the line  $J^1 = J^2$  and that at the attractor point (for the total charge) they lie at some finite distance from the origin on the line  $J^1 = \frac{p^1}{p^2} J^2$ . One can verify however that there exists a line of marginal stability  $J^1 = \lambda J^2$  for a split in charges (3.61). In the case that  $p^1 < p^2$  we can choose  $\lambda$  such that  $\frac{p^1}{p^2} < \lambda < 1$  and in this case the single center flow has to cross the line of marginal stability somewhere on its flow from infinity to the attractor point. The situation is illustrated in figure 3.10. The case  $p^1 > p^2$  is analogous, with now  $1 < \lambda < \frac{p^1}{p^2}$ . So we can always find charges of the form (3.61) such that the single center attractor flow for the total charge has to cross a wall of marginal stability.

### Existence of the separate centers

As was shown in section 2.5.1, to check if a single center D4D2D0 exists it is enough to verify that

$$\hat{q}_0 = -\frac{1}{2} D^{AB} q_A q_B + q_0 < 0. \quad (3.73)$$

Evaluating this constraint for two charges of the form (3.61) gives a single constraint:

$$\frac{p^1 + 2\lambda p^2 + 2p^2}{8(p^2)^2} q^2 + q_0 < 0, \quad (3.74)$$

this thus gives a constraint on the size of  $q$  i.e.

$$q^2 < \frac{-8q_0(p^2)^2}{p^1 + 2\lambda p^2 + 2p^2}. \quad (3.75)$$

### Combining the results

Putting the results of the three separate discussions above together, shows that a valid split attractor flow exists for two-centers of the form

$$\begin{aligned} \Gamma_1 &= (0, \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}, \begin{pmatrix} q \\ -\lambda q \end{pmatrix}, q_0), \\ \Gamma_2 &= (0, \begin{pmatrix} p^1 \\ p^2 \end{pmatrix}, \begin{pmatrix} -q \\ lq \end{pmatrix}, q_0), \end{aligned} \quad (3.76)$$

in case

$$p^A > 0, \quad (3.77)$$

$$q_0 < 0, \quad (3.78)$$

$$1 < \lambda < \frac{p^1}{p^2} \quad \text{or} \quad \frac{p^1}{p^2} < \lambda < 1, \quad (3.79)$$

$$q^2 < \frac{-8q_0(p^2)^2}{p^1 + 2\lambda p^2 + 2p^2}. \quad (3.80)$$

It is clear that any total D4D0 state with negative D0 can be split in such a way.

Note that the equilibrium distance between the centers is

$$|x_1 - x_2| = \frac{2(p^1 - \lambda p^2)}{(\frac{3}{10})^{\frac{1}{3}}(\lambda - 1)} \ell_5 + \mathcal{O}(\ell_5^2), \quad (3.81)$$

where we used the relations between  $R, \ell_5$  and  $G_4$ , see appendix A. Note that this distance scales as  $\ell_5$  while the distance between bound centers carrying D6 charge scales as  $\ell_5^3/R^2$ . This has some important consequences when considering the decoupling limit as discussed in section 4.2.5.

### Numerical example

Here we will numerically compute the region in moduli space covered by the scalars in a specific example of the class of solutions presented in the previous subsection. Take the following charges<sup>9</sup>

$$\begin{aligned}\Gamma_1 &= (0, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}, -3), \\ \Gamma_2 &= (0, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, -3).\end{aligned}\quad (3.82)$$

The total charge is thus

$$\Gamma = (0, \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, -6).\quad (3.83)$$

There is a wall of marginal stability (at  $B^A = 0$ )

$$J^1 = \frac{1}{2}J^2.\quad (3.84)$$

The attractor point for the total charge is

$$B_{*t}^A = 0, \quad J_{*t}^1 = \frac{1}{3\sqrt{2}}, \quad J_{*t}^2 = \frac{1}{\sqrt{2}}.\quad (3.85)$$

Note that indeed in the  $(J^1, J^2)$ -plane, the wall of marginal stability  $(1, 2)s$  separates this attractor point  $\frac{1}{3\sqrt{2}}(1, 3)$  from the starting point at infinity:  $\hat{R}(\frac{3}{10})^{1/3}(1, 1)$ . Figure 3.10 shows the attractor point and the line of marginal stability in the  $(J^1, J^2)$ -plane for this example. As discussed in the previous subsection, if we take the moduli at infinity to be

$$\hat{R} = 200, \quad Y_\infty^1 = Y_\infty^2 = (\frac{3}{10})^{1/3}, \quad B_\infty^A = 0,\quad (3.86)$$

the single center flow crosses the wall of marginal stability and the split flow corresponding to the charges (3.82) exists. The two centers of the split have the following attractor points:

$$B_{*1}^1 = -\frac{17}{36}, \quad B_{*1}^2 = \frac{1}{6}, \quad J_{*1}^1 = \frac{\sqrt{\frac{11}{3}}}{9}, \quad J_{*1}^2 = \frac{\sqrt{\frac{11}{3}}}{3},\quad (3.87)$$

and

$$B_{*1}^1 = \frac{17}{36}, \quad B_{*1}^2 = -\frac{1}{6}, \quad J_{*1}^1 = \frac{\sqrt{\frac{11}{3}}}{9}, \quad J_{*1}^2 = \frac{\sqrt{\frac{11}{3}}}{3},\quad (3.88)$$

<sup>9</sup> In principle the charges need to be very large to satisfy all kind of assumptions silently made. This can be easily obtained by using a scaling symmetry [29] to scale the charges uniformly to some big value.

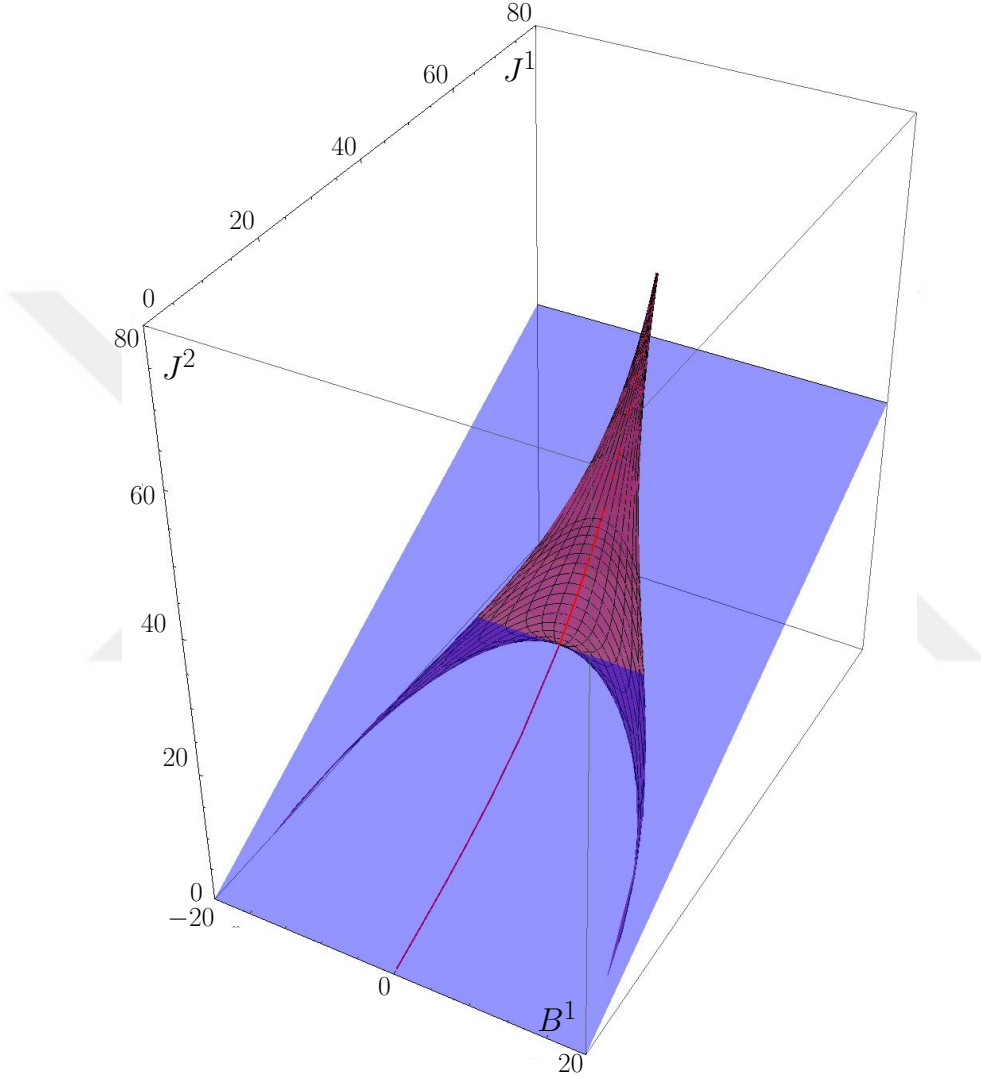


**Figure 3.11** In this figure the  $(50B^1, J^1, J^2)$  subspace in moduli space is shown. The blue surface is a wall of marginal stability for the split (3.82). The central red line is the attractor flow for a single center solution with the same total charge as the two center. The pink surface shows the values the moduli take in the two center solution.

note that they both exist as a single center and the central charges at the attractor points are  $Z_{*1} = Z_{*2} = \frac{88}{9} > 0$ . A numerical computation of the moduli is shown in figure 3.11. One can explicitly check that furthermore the function  $\Sigma$  stays positive everywhere.

## 3.5 Microscopic picture

In the first chapter we shortly commented on the microscopic picture of the supersymmetric black holes in  $\mathcal{N} = 2$  supergravity. At small  $g_s$  the black holes can be identified with D-branes wrapping non-trivial cycles in the Calabi-Yau and localized in a single point in the external space-like dimensions. Now that we know of the existence of the multicentered black holes in supergravity it is only natural to wonder about their microscopic description. One's first intuition, namely that these will probably look like a set of D-branes sitting at some equilibrium separation in the external dimensions turns out to be too naive. A simple dimensional argument [97] already shows that such a configuration that is multicentered at finite  $g_s$  will collapse to a single point at  $g_s = 0$ . The key point is that the equilibrium distance between the bound black holes is proportional to the natural length scale of supergravity, in our case the four dimensional planck length  $l_4$ . The natural length scale in the microscopic string theory picture is the string length  $l_s$  and as detailed in e.g. appendix A the two are related as

$$l_4 = \frac{g_s l_s}{\sqrt{\tilde{V}_{\text{IIA}}}}. \quad (3.89)$$

This relation implies that the equilibrium distance (3.25) vanishes in the  $g_s \rightarrow 0$  limit. More physically one can think of the process of dialing down  $g_s$  as weakening the interaction strength of the various supergravity forces. The equilibrium distance is determined by the radius where the different forces cancel. The above dimensionality argument then shows that lowering the interaction strength corresponds to a linear decrease of the stable distance, ultimately vanishing when  $g_s = 0$ .

As we will discuss now, the transition between the well separated black holes and the picture of D-branes at a single point is very rich and involves some interesting phases in between. The complete story was beautifully worked out in [97]. Sadly we can not digress too much in this thesis and so we will only be able to roughly sketch the main insights of [97], the reader is encouraged to look up the background and details there.

### Description as an intersecting D-brane system

The picture we want to sketch is most naturally understood in IIB where all D-branes in the compactification are different intersecting D3-branes. So in this section we will make an exception to the rest of the thesis and work in the IIB frame. Note that all the results apply to IIA as well through mirror-symmetry

but to derive them in that picture from first principles is much more involved and at the cost of a big loss of intuition.

Let us consider the case of two D3-branes that wrap the three cycle  $\Gamma_1, \Gamma_2$  in the Calabi-Yau respectively. In the IIB case the intersection product  $\langle \Gamma_1, \Gamma_2 \rangle$  is nothing but the number of (oriented) intersections of the two 3-cycles (hence its name). There are three types of strings stretching between the branes. Those that begin and end on the same brane, let us call them of type 1-1 and 2-2 respectively. The third set of strings are those that stretch between  $\Gamma_1$  and  $\Gamma_2$ , we denote them by 1-2. At low energies we can focus on the light modes of these strings. For our purposes it will furthermore be sufficient to consider only the scalar modes.

The light scalar fields coming from the 1-1 and 2-2 types of strings parameterize the position of that brane in the external dimensions<sup>10</sup> and there are six of them. We neglect the overall translational degrees of freedom of the system and concentrate on those scalars that describe the 'center of mass' position of the branes, we denote those three scalar fields by  $x^i$ . These three real scalars  $x^i$  are massless and measure the separation<sup>11</sup> between the two branes in the external dimensions.

As the mass of stretched strings is proportional to their length the 1-2 strings will only give rise to light modes near the points where the two D-branes  $\Gamma_1$  and  $\Gamma_2$  intersect. In the IIB picture there are exactly  $\kappa = \langle \Gamma_1, \Gamma_2 \rangle$  such intersection points. So in addition to the three scalars  $x^i$  there are  $\kappa$  additional light complex scalars we denote with  $\phi^m$ .

Apart from the scalars we discussed, there are additional light degrees of freedom which come in the form of two gaugefields and some fermions. As shown in [97] in the case of a general multicentered configuration with  $n$  centers  $\Gamma_a$  one can reduce the theory describing all the light degrees of freedom to a one dimensional theory on the worldline in the external dimensions. The theory takes the form of what is known as a *quiver gauge theory*. We won't go any deeper into the beautiful subject of quivers in this thesis and once again point to [97] for further reference.

In our simple example of two rigid branes one can show [97] that there is a potential for the scalar fields that has the form

$$V(\phi, x) = \frac{1}{2\mu} (|\phi|^2 l_s^{-2} + \theta)^2 + |x|^2 |\phi|^2 l_s^{-4}, \quad (3.90)$$

with  $|\phi|^2 = \sum_m |\phi^m|^2$ . The constant  $\mu$  is given by the 'reduced mass' of the two branes:  $\mu = \frac{|Z_1||Z_2|}{(|Z_1|+|Z_2|)l_4}$ . The constant  $\theta$  is a so called *Fayet-Iliopoulos*

<sup>10</sup> In general there are also such scalars describing 'internal' degrees of freedom of the brane, like e.g. the divisor moduli we encountered in the IIA picture in section 2.6. We will neglect those internal degrees of freedom in the present discussion, a fact that can be justified in case the D-branes wrap a *rigid* cycle.

<sup>11</sup> Note that most naturally scalar fields in one dimension have dimensions square root of length. Here we make an exception however as it is more natural to give the  $x^i$  the dimension of length due to their physical interpretation as a distance.

parameter of the theory. In this case it is given in terms of the phase difference between the central charges of the two branes:  $\theta = \mu(\alpha_2 - \alpha_1)$ ,  $\alpha_a = \arg(Z_a)$ .

We can read the mass of the complex scalars  $\phi^m$  from the quadratic part of the Lagrangian:

$$m_\phi^2 = |x|^2 l_s^{-4} + (\alpha_2 - \alpha_1) l_s^{-2}. \quad (3.91)$$

The first term is easily understood; the mass of the stretched string modes is proportional to the separation of the branes. The second term is a bit less intuitive but is familiar from the theory of intersecting branes [112], it is valid in the regime  $\Delta\alpha = \alpha_2 - \alpha_1 \ll 1$ . Note that for small enough separations  $|x|$ , the scalars  $\phi^m$  can become tachyonic, this will have some interesting consequences.

The local minima of the potential (3.90) form the classical moduli space of groundstates of the system. There are two cases to distinguish:

- The configurations with  $\phi^m = 0$  are minima in case  $\mu x^2 > -\theta$ . This solution exists independent of the sign of  $\theta$  and is called the *Coulomb-branch*. Classically it is  $\mathbb{R}^3$  with a ball of radius  $\frac{-\theta}{\mu}$  removed or just  $\mathbb{R}^3$  in case  $\theta$  is positive.
- In case  $\theta < 0$  there exists a second branch, the *Higgs-branch*, given by  $|\phi|^2 = -\theta$ ,  $x^i = 0$ . Note that this branch ceases to exist in case  $\theta$  becomes positive. Topologically the Higgs-branch is<sup>12</sup>  $\mathbb{CP}^{k-1}$ .

Let us focus on the case  $\theta < 0$ . In the Coulomb-branch the scalars  $x^i$  are non-zero and one can thus interpret the Coulomb-branch as describing two separated branes. Note that the energy in the Coulomb branch is strictly positive and the groundstates are thus not supersymmetric, we will see that this changes drastically once we take into account quantum corrections.

Once the separation  $|x|$  becomes smaller than  $\sqrt{-\Delta\alpha}$  the extremum at  $\phi^m = 0$  becomes a maximum and the scalars  $\phi^m$  tachyonic. This shows that the system of two separate branes becomes unstable if they are too close together. The natural interpretation of the Higgs-branch is then as the end result of this dynamic instability which is described by the condensation of the tachyonic string modes. The Higgs-branch is the description of the single D-brane formed out of the decay of the two original branes. This might all seem a bit ad hoc in this short review, arguments to support this picture can be found in [97].

Note that classically the Coulomb branch is flat, i.e. outside of the tachyonic regime all possible separations are degenerate (non-supersymmetric) groundstates. Quantummechanically the Coulomb branch energy gets corrections however and the groundstate degeneracy is (partially) lifted. The effective potential induced by quantum corrections is no longer flat and has zero energy (supersymmetric) groundstates that, as we will see, are in natural correspondence to the bound states of the supergravity regime. The origin of the quantum corrections

<sup>12</sup> Note that the space is  $\mathbb{CP}^{k-1}$  instead of  $S^{2k-1}$  because the scalars  $\phi^m$  are charged under a  $U(1)$ , and different groundstates should be identified under this gauge symmetry.

is the following: as we saw in (3.91), the mass of the scalars  $\phi^m$  depends on the separation  $x^i$ . For sufficient separation the  $\phi^m$  are massive enough that they can be effectively integrated out. As is shown in [97] this gives rise to an effective potential on the Coulomb branch. Due to the dependence of the mass of the  $\phi^m$  on the  $x^i$ , this potential will depend on the separation  $|x|$ . Doing the detailed calculation, as in [97], one finds that the effective potential has a supersymmetric minimum at

$$|x|_\star = -\frac{\kappa}{2\theta}. \quad (3.92)$$

So we see that including quantum corrections the Coulomb branch is reduced to an  $S^2$  of radius (3.92), and consists of zero energy, supersymmetric, groundstates. This corrected description of the Coulomb branch of the D-brane system now allows the very natural interpretation of the two D-branes being bound at a certain equilibrium distance. To make the correspondence with the equilibrium distance in the supergravity regime even more clear we put the expression (3.92) in a slightly more familiar form. Use the identification  $\theta = \mu(\alpha_1 - \alpha_2)$  and consider the 'probe limit' where one center becomes infinitely more massive than the other, i.e.  $\mu \rightarrow m_p = |Z_p|l_4^{-1}$ . Furthermore take into account that (3.92) is only valid in the regime of small phase difference where  $\alpha_p - \alpha \sim \sin(\alpha_p - \alpha)$ . Using these formulas it follows that

$$|x|_\star = \frac{\langle \Gamma, \Gamma_p \rangle}{2\text{Im}(e^{-i\alpha} Z_p)}. \quad (3.93)$$

Comparing this formula at small  $g_s$  to that found in the probe analysis in the supergravity regime (3.7) one finds exact agreement. One might have expected some qualitative similarities between the supergravity and D-brane regime, but the exact quantitative agreement might come as a surprise. As shown in [97] this is actually guaranteed by supersymmetry. There exists a non-renormalization theorem that protects the expression for the equilibrium distance from changing in the different regimes.

### Overview of the different phases

To summarize, let us give an overview of the description of the multicentered BPS states in the different  $g_s$  regimes. For  $g_s\kappa$  large we have the description as bound black holes in supergravity. As discussed in the beginning of this section, when lowering  $g_s$  the intercenter distance starts decreasing. The supergravity description breaks down when the intercenter distance becomes of the order of the string length:  $|x_\star| \sim l_s$ , this can equivalently be rewritten in terms of  $g_s$  as

$$\kappa g_s \sim |\Delta\alpha|^{1/2} Z_p V_{\text{IIA}}^{1/2}. \quad (3.94)$$

Below this value the D-brane description in terms of quiver gauge theory becomes appropriate. We saw that for a large enough separation the supersymmetric

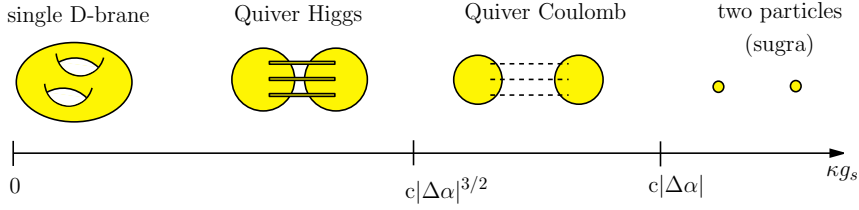


**Figure 3.12** Here the different phases under variations of  $g_s$  are depicted. This figure is borrowed from [97].

groundstates of the quiver gauge theory are in the (quantum corrected) Coulomb-branch. For the quantum corrected Coulomb branch description we introduced to be valid, the mass of the  $\phi^m$ , that are effectively integrated out, should be large enough. This mass of the  $\phi^m$  is directly linked to the separation  $|x|$  between the branes which in turn is set by the effective potential. So for the picture to be self consistent the equilibrium separation should be large enough. More precisely we want  $|x|_* \gg \sqrt{-\Delta\alpha} l_s$  which in terms of  $g_s$  reads

$$\kappa g_s \gg |\Delta\alpha|^{3/2} Z_p V_{\text{IIA}}^{-1/2}. \quad (3.95)$$

We also saw that there is a transition from the Coulomb to the Higgs-branch when the stretched string modes become tachyonic. This happens when

$$|x|_* \sim \sqrt{-\Delta\alpha} l_s, \quad (3.96)$$

which, as one can check, coincides with the lower boundary of (3.95). So we see that roughly when the effective potential description on the Coulomb-branch breaks down, tachyonic modes appear which make the system decay into the Higgs branch. Once in the Higgs-branch lowering  $g_s$  even further till  $g_s = 0$  the quiver description of the D-branes becomes trivial and the branes have become infinitely thin classical geometric objects. A summary of the above summary can be found in picture 3.12.

### Microscopic picture for flow trees

We end this section with a short comment on the interpretation of the split flow trees in the microscopic regime. Till now we mainly considered the case  $\Delta\alpha < 0$  and analysed what happened under changes of  $g_s$ . Alternatively one can also consider  $g_s = 0$  fixed and study the system under changes of  $\Delta\alpha$ , i.e changes of the moduli of the internal Calabi-Yau. We then see that (at  $|x| = 0$ ) for  $\Delta\alpha < 0$  there are tachyonic states and the system will undergo tachyon condensation. In this case there exists a BPS state of total charge  $\Gamma$ . On the other side of the wall of marginal stability where  $\Delta\alpha > 0$  there are no tachyons present and the Higgs-branch no longer exists. At  $g_s = 0$  there are no supersymmetric groundstates in the Coulomb-branch, showing that indeed the BPS state decays

when crossing the wall. This picture of tachyon condensation also provides a nice microscopic interpretation of the split attractor flow. One can think of the flow tree determining how the BPS state  $\Gamma$  is glued together out of different subcharges  $\Gamma_a$ . Going up a tree, every time one encounters a vertex on a wall of marginal stability one can glue the two single branes, represented by the two converging legs of the tree, together through the tachyons present near the wall and continue on the other side of the wall with a single leg representing the single D-brane that is the result of the tachyon condensation.



# Chapter 4

## Black hole bound states in $\text{AdS}_3 \times S^2$

This chapter contains essentially the authors work in collaboration with Frederik Denef, Jan de Boer, Sheer El-Showk and Ilies Messamah, that was reported in [31].

### 4.1 Introduction and summary

Consider M-theory compactified on the product of a Calabi-Yau  $X$  of volume  $V_X$  and a circle of radius  $R$  in the limit  $R/l_{11} \rightarrow \infty$ ,  $V_X/l_{11}^6$  fixed. In this limit the worldvolume excitations of M5-branes wrapped on 4-cycles in  $X$  and the  $S^1$  which are of finite energy in  $1/R$  units decouple from the bulk. Their dynamics is described by a (0,4) supersymmetric 1+1 dimensional nonlinear sigma model with target space given by the classical M5 moduli space. This is the MSW CFT [36, 88] which we already shortly introduced in section 2.6.

The holographic dual to this CFT is thought to be quantum M-theory with  $\text{AdS}_3 \times S^2 \times X$  boundary asymptotics. The scales of  $\text{AdS}_3$  and  $S^2$  are set by the central charge  $c = p^3$  of the CFT, where  $p^3 = D_{ABC}p^A p^B p^C$  denotes the triple self-intersection product of the 4-cycle homology class  $p$  wrapped by the M5 [36].<sup>1</sup> When this is large, semiclassical supergravity becomes reliable.

In this chapter we will study systematically the BPS sector on the gravity side, and uncover some surprises.

A first one is based on the following observations. A priori we can choose to embed the M5 in a Calabi-Yau  $X$  with arbitrary moduli. In particular, we

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<sup>1</sup> In this chapter we will drop subleading contributions linear in  $p$  to the central charge, so  $c_L = c_R \equiv c$ .

are free to pick any value for the normalized<sup>2</sup> Kähler moduli  $Y^A$ . On the other hand, the  $\text{AdS}_3 \times S^2 \times X$  solution freezes  $Y^A = p^A/U$ ,  $U := (p^3/6)^{1/3}$ . This presents a puzzle: As we will show in an explicit example, the spectrum of BPS states depends in general on the moduli  $Y^A$ ; there are walls of marginal stability in  $Y$ -space where certain M5 states split into two different M5 constituents. The decay always happens in the direction towards the attractor point  $Y^A = p^A/U$ . Moreover, congruent with this, when taking the decoupling limit on the gravity side with the  $Y^A$  on the side of a wall of marginal stability where the constituents are bound together, we will see that we do not end up with a single decoupled asymptotically  $\text{AdS}_3 \times S^2 \times X|_p$  space, but with several mutually decoupled  $\text{AdS}_3 \times S^2 \times X|_{p_i}$  spaces (embedded in asymptotically  $\mathbb{R}^5 \times X|_Y$  space), each with its own attractor point  $Y_i^A = p_i^A/U_i$ . This implies that the MSW CFT is not capable of capturing the entire moduli space of M5 bound states, as in the latter case sectors corresponding to M5-M5 bound state constituents decouple from each other in the IR, with each sector flowing to its own fixed point CFT. Some puzzles and questions related to this will be discussed in section 4.5.3.

In the bulk of this chapter, we focus on a single asymptotically  $\text{AdS}_3 \times S^2$  sector. Suitable density matrices of CFT states will be dual in the semi-classical sense to BPS black hole solutions. The simplest BPS black hole solution with the correct charges and asymptotics is the extremal rotating BTZ black hole [113] times  $S^2$ . Its entropy agrees with the Cardy formula for the asymptotic degeneracy of BPS states in the  $(0, 4)$  CFT for  $L_0 \gg \frac{c}{24}$ :  $S_{\text{BTZ}} = 4\pi\sqrt{\frac{c}{24}(L_0 - \frac{c}{24})}$ .

However, for  $h \equiv (L_0 - \frac{c}{24})/c$  sufficiently small (below to a critical value  $h_c$ ), when the BTZ black hole radius drops below the  $\text{AdS}_3$  ( $\sim S^2$ ) radius, a thermodynamic instability occurs: at the given value of  $L_0$  it becomes entropically favorable for the black hole to *localize* on the sphere; this more entropic solution looks locally like a 5d BMPV black hole [114], with M2 charge and  $S^3$  horizon, sitting at some point of the sphere and the center of  $\text{AdS}_3$ , the sphere being supported still by flux. Once  $L_0$  drops below zero, the BTZ black hole ceases to exist altogether as a regular black hole solution; instead one finds a conical defect singularity.

This instability can be viewed as a supersymmetric version of the instability pointed out for Schwarzschild-AdS black holes by Banks, Douglas, Horowitz and Martinec [37]. Related thermodynamical as well as dynamical instabilities were studied among others in [115, 116, 117, 118, 119, 120, 121, 122, 123, 124]. The nonsupersymmetric nature of these systems makes them difficult to study. In contrast, here, supersymmetry allows us to obtain completely explicit solutions.

In fact, the single sphere localized black hole is but one of a very intricate set of supersymmetric multi black hole bound states in  $\text{AdS}_3 \times S^2$  we will construct in general. We will do this by lifting four dimensional type IIA multi black hole bound states [27, 28, 73, 29] with D4, D2 and D0 total charge but no net D6 charge to five dimensions using the 4d-5d correspondence [79, 80, 101, 102,

<sup>2</sup> Normalized such that  $\frac{Y^3}{6} \equiv 1$ . The overall scale (volume) of  $X$  in 11d Planck units is in a hypermultiplet, while the relative scales are in vector multiplets.

103, 104, 105, 106, 107], and then carefully taking the decoupling limit. The black hole localization instability in  $\text{AdS}_3 \times S^2$ , it turns out, is then nothing but the uplift of the four dimensional “Entropy Enigma” of [29, 30]! The Entropy Enigma is the observation that in the regime in which the total D4-D2-D0 charge is scaled up uniformly, for sufficiently large background type IIA CY volume, multicentered black holes dominate the entropy. Since (for zero D2 charge)  $h = (L_0 - \frac{c}{24})/c = -q_0/p^3$ , with  $q_0$  the D0 charge and  $p$  the D4 charge, this regime indeed corresponds to  $h \rightarrow 0$ .

General two black hole configurations in  $\text{AdS}_3 \times S^2$  can be viewed as fat, backreacting versions of the M2 and anti-M2 probe particles sitting at the north and south poles of the  $S^2$  which were considered in [125] in a derivation of the OSV conjecture.

Although many multicentered black hole configurations exist, it appears that the entropically most dominant one (or at least the most entropic one we have been able to find) is the configuration which in four dimensions consist of one pure  $\overline{D6}$  particle with zero entropy and one large D6-D4-D2-D0 black hole; this lifts in  $\text{AdS}_3 \times S^2$  to a single BMPV black hole localized on the sphere — the localized black hole referred to above.

Unlike the BTZ black hole, the localized black holes in general have macroscopic  $S^2$  angular momentum, up to values of order  $p^3$ . (The maximal angular momentum is reached for the  $L_0 = 0$  ground state, rotating global  $\text{AdS}_3 \times S^2$ , obtained by uplifting a  $D6 - \overline{D6}$  2-particle state.) Assuming the BTZ alone dominates the entropy for  $h > h_c$  and the single localized black hole alone for  $h < h_c$ , we thus find that in the  $c = p^3 \rightarrow \infty$  limit, a first order phase transition occurs at  $h = h_c$ , with order parameter given by the  $S^2$  angular momentum.

We also argue that in the canonical ensemble, trading fixed  $L_0$  for fixed left-moving temperature  $T$ , this localization transition is not visible; instead the small BTZ destabilizes due to a supersymmetric version of the Hawking-Page phase transition, going from BTZ to rotating  $\text{AdS}_3 \times S^2$  at  $T_c = \frac{1}{2\pi}$ . Such a phase transition was observed already in [126] for  $\text{AdS}_3 \times S^3$  by studying the Fareytail expansion of the elliptic genus. Interestingly, this critical temperature can also be obtained as the temperature of the smallest supersymmetric BTZ black hole that can be made adiabatically as the limit of a certain class of “scaling” solutions. These zero entropy scaling solutions can be viewed as part of the microstates (in the sense of the fuzzball proposal) that make up the BTZ black hole above  $T_c$ . Whether there are other microstates which do contribute to the unstable BTZ black hole below the critical temperature remains to be investigated.

Finally, we initiate a discussion of the CFT interpretation of all this, with particular attention paid to the  $h \rightarrow 0$  regime. To this end we refine and improve the original analysis of [36], pointing out, in particular, the importance of including all  $c = p^3$  winding modes, which freeze 4-cycle deformation moduli at special supersymmetric points and contribute dominantly to the entropy at small  $h$ . We explain some of the qualitative features observed on the gravity

side, including the decrease in  $SU(2)_R$  charge expectation value with increasing  $L_0$ , but we leave a more complete analysis for future work.

The largest part of the paper is devoted to laying the necessary groundwork: constructing multicentered solutions in  $\text{AdS}_3 \times \text{S}^2$ , finding ways to establish their existence without having to construct them explicitly, classifying them, and identifying their dual CFT quantum numbers.

As mentioned above, our strategy for finding the configurations surviving the decoupling limit will consist of lifting four dimensional type IIA multi black hole bound states to five dimensions, and then carefully taking the decoupling limit. This turns out to *not* be the same as naively dropping the constant terms in the defining harmonic functions.

The decoupling procedure is not entirely straightforward, since from the type IIA point of view, it sends the CY volume in string units  $V_X/l_s^6 \sim (R^3/l_{11}^3)(V_X/l_{11}^6)$  to infinity, while keeping the 4d string coupling  $g_{4d} \sim (V_X/l_{11}^6)^{-1/2}$  finite. (The 10d string coupling goes to infinity too, of course, as it should for eleven dimensional supergravity to become the proper low energy effective theory.) Since the volume in string units is in a vector multiplet (unlike the 4d string coupling, which belongs to a hypermultiplet), a IIA multicentered solution existing at some finite value of  $V_X/l_s^6$  could be destroyed when taking the limit, as for instance a wall of marginal stability may be encountered at some value of  $V_X/l_s^6$ , where the solution decays.

The asymptotically  $\text{AdS}_3 \times \text{S}^2$  solutions thus obtained (after a suitable rescaling of coordinates<sup>3</sup>) can be written explicitly, and are completely determined by  $2 \dim H^2(X) + 2$  harmonic functions (we put  $R \equiv 1$  in what follows):

$$\begin{aligned} H^0 &= \sum_a \frac{p_a^0}{|x - x_a|}, & H^A &= \sum_a \frac{p_a^A}{|x - x_a|}, \\ H_A &= \sum_a \frac{q_A^a}{|x - x_a|}, & H_0 &= \sum_a \frac{q_0^a}{|x - x_a|} - \frac{1}{4}. \end{aligned}$$

Here the coordinate vector  $x_a$  gives the position in the spatial  $\mathbb{R}^3$  of the  $a$ th center with charge  $\Gamma_a = (p_a^0, p_a^A, q_A^a, q_0^a)$ . Furthermore  $\sum_a p_a^0 = 0$  and  $\sum_a p_a^A$  is positive, i.e. lies within the Kähler cone.<sup>4</sup> The IIA interpretation of these charges is (D6,D4,D2,D0); the M-theory one is (KK,M5,M2,P). The positions  $x_a$  have to satisfy the integrability constraints

$$\sum_b \frac{\langle \Gamma_a, \Gamma_b \rangle}{|x_a - x_b|} = -\frac{p_a^0}{4}. \quad (4.1)$$

<sup>3</sup> Namely  $\vec{x} \rightarrow \ell_5^3 \vec{x}$ , where  $\ell_5 := l_{11}/4\pi \tilde{V}_X^{1/3}$ ,  $\tilde{V}_X := V_X/l_{11}^6$ .

<sup>4</sup> This is necessary because the asymptotic Kähler moduli are  $Y^A = p^A/U$ . By far not all holomorphic 4-cycles have positive charge  $p$ . On the other hand, wrapping an M5 on such 4-cycles does seem to give rise to a sensible decoupled MSW sigma-model. This presents a puzzle similar to the one caused by M5-M5 bound states discussed above. If, as seems plausible, these charges are realized as M5-M5 2-centered bound states in gravity, the resolution would be the same as there.

The metric, gauge field and Kähler scalars of the solution are given by

$$\begin{aligned}
ds_{5d}^2 &= 2^{-2/3} Q^{-2} [-(H^0)^2(dt + \omega)^2 - 2L(dt + \omega)(d\psi + \omega_0) + \Sigma^2(d\psi + \omega_0)^2] \\
&\quad + 2^{-2/3} Q dx^i dx^i, \\
A_{5d}^A &= \frac{-H^0 y^A}{Q^{3/2}}(dt + \omega) + \frac{1}{H^0} \left( H^A - \frac{L y^A}{Q^{3/2}} \right) (d\psi + \omega_0) + \mathcal{A}_d^A, \\
Y^A &= \frac{2^{1/3} y^A}{\sqrt{Q}},
\end{aligned} \tag{4.2}$$

where  $x^i \in \mathbb{R}^3$  and  $\psi$  is an angular coordinate with period  $4\pi$ , and the functions appearing are given by

$$\begin{aligned}
d\omega_0 &= \star dH^0, \\
d\mathcal{A}_d^A &= \star dH^A, \\
\star d\omega &= \langle dH, H \rangle, \\
\Sigma &= \sqrt{\frac{Q^3 - L^2}{(H^0)^2}}, \\
L &= H_0(H^0)^2 + \frac{1}{3} D_{ABC} H^A H^B H^C - H^A H_A H^0, \\
Q &= \left( \frac{1}{3} D_{ABC} y^A y^B y^C \right)^{2/3}, \\
D_{ABC} y^A y^B &= -2H_C H^0 + D_{ABC} H^A H^B.
\end{aligned} \tag{4.3}$$

Asymptotically, the geometry is not quite  $\text{AdS}_3 \times S^2$ , but an  $S^2$  bundle over  $\text{AdS}_3$ :

$$ds^2 \approx d\eta^2 + e^{\eta/U} (-d\tau^2 + d\sigma^2) + U^2 (d\theta^2 + \sin^2 \theta (d\phi + \tilde{A})^2), \tag{4.4}$$

$$\tilde{A} = \frac{J}{J_{\max}} d(\tau - \sigma) \tag{4.5}$$

$$A_{5d}^A \approx -p^A \cos \theta (d\phi + \tilde{A}) + 2D^{AB} q_B d(\sigma + \tau), \tag{4.6}$$

$$Y^A \approx \frac{p^A}{U}. \tag{4.7}$$

where  $U := (\frac{1}{6} p^3)^{1/3}$ ,  $D^{AB} = (D_{ABC} p^C)^{-1}$ , and we made the change of coordinates<sup>5</sup>  $(r, t, \psi) \rightarrow (\eta, \tau, \sigma)$  to leading order given by:

$$\eta := U \log \frac{r}{U}, \quad \tau := t, \quad \sigma := \frac{\psi}{2} - t. \tag{4.8}$$

The flat connection  $\tilde{A}$  determines the twisting of the  $S^2$  over the  $\text{AdS}_3$  base.  $J$  is the  $S^2$ -angular momentum of the solution and  $J_{\max} := \frac{p^3}{12}$  is its maximal value for given  $p$ .

<sup>5</sup> Here, the coordinates  $(r, \theta, \phi)$  are standard spherical coordinates for  $\vec{x}$ .

A solution to the integrability constraints (4.1) does not automatically imply a well-behaved full solution — the formally obtained metric may still have various unacceptable pathologies such as closed timelike curves. As we discussed in section 3.3.4, in asymptotically flat space this can up to a certain extent<sup>6</sup> be circumvented by making use of the “split attractor flow” conjecture [29]. In the decoupled asymptotically  $\text{AdS}_3 \times \text{S}^2$  limit there are no moduli to tune — all relevant asymptotic moduli are frozen to their attractor values. Nevertheless, our uplift and decoupling procedure combined with the split attractor flow conjecture allows to conclude that there is a one to one correspondence between connected components of the solution space of multicentered asymptotic  $\text{AdS}_3 \times \text{S}^2$  solutions of total (M5,M2) charge  $(p^A, q_A)$  and IIA attractor flow trees which persist in the limit in which we take the starting point of the flow to

$$B^A + iJ^A = D^{AB}q_B + i\Lambda p^A, \quad \Lambda \rightarrow \infty. \quad (4.9)$$

Here  $B^A$  and  $J^A$  are the components of the IIA B-field and Kähler form. This limiting point is essentially the attractor point associated to the  $\text{AdS}_3 \times \text{S}^2 \times X$  geometry for the given charges, translated to type IIA variables. Split flows which disappear when going from  $J^A = \infty Y^A$  to  $J^A = \infty p^A$  due to wall crossing encode the fragmentation into multiple decoupled  $\text{AdS}_3 \times \text{S}^2$  geometries dual to the different CFT’s that appear in the low energy limit of wrapped M5 branes in a background with Kähler moduli  $Y^A$ .

The flow tree picture is also useful to understand certain possible degenerations of solution spaces in the decoupling limit. (Degenerations are more an issue now than in the asymptotically flat case, precisely because we no longer have asymptotic moduli we can use to tune away accidental degenerations, and because in the  $\Lambda \rightarrow \infty$  limit, central charges are prone to line up as they become dominated by their leading terms.) To this end, we distinguish the notions of marginal and threshold stability walls. Both are associated to central charges lining up, but in the former case the charges have intersection product nonzero, in the latter case zero<sup>7</sup>. When crossing the former, flow trees and solutions disappear and BPS indices jump, whereas when crossing the latter, flow trees merely change topology, solution spaces expand, hit infinite extent and contract again, and BPS indices remain invariant. If the limiting value of  $B + iJ$  given in (4.9) lies on a threshold stability wall, the corresponding solution exists in the decoupling limit, but the solution space will be noncompact in the sense that some centers can reach the boundary of  $\text{AdS}_3$ .<sup>8</sup>

<sup>6</sup> Multicentered “scaling” or “abyss” solutions [29, 97, 99], for which the centers’ coordinates can approach each other arbitrarily closely, are viewed as being continuously connected to single centered solutions, hence the split flow conjecture unfortunately does not say anything about the existence of such solutions for a given charge partitioning.

<sup>7</sup> This is the criterion for threshold stability for two charges, a precise definition for more than two charges is more involved see section 3.4.1 for a short discussion.

<sup>8</sup> However, as will be shown in the companion paper [96], the solution space, viewed as a BPS phase space, still has finite symplectic volume.

### Outline

The outline of this chapter is as follows.

In section 4.2, we take the decoupling limit of the multicentered solutions introduced in chapter 3. We furthermore study its asymptotics and determine the dual CFT quantum numbers of the solutions. For a single 5d black ring, the decoupling limit agrees with the decoupling limits considered in [127, 128]. We are not aware though of any systematic discussion in the literature of the decoupling limit in the case of multicentered solutions. We end the section by formulating the existence criterion based on the split flow conjecture, discussing various possible behaviors of the solutions in the decoupling limit in this picture.

In section 4.3, we give some examples. This includes the uplift of a 4d D6-anti-D6 dipole, which becomes global (twisted)  $\text{AdS}_3 \times S^2$  in the decoupling limit, as well as configurations giving rise to the 4d Entropy Enigma.

Section 4.4 is devoted to demystifying the entropy enigma by identifying it as a supersymmetric version of the Banks-Douglas-Horowitz-Martinec localization instability. We discuss the corresponding phase transitions in the microcanonical and canonical ensembles.

In section 4.5, we initiate interpretations of the observed gravitational phenomena in the MSW CFT.

We conclude in section 4.6 with a discussion of open problems.

## 4.2 Decoupling limit

As outlined in the above introduction, we want to study the geometries dual to states of M5-branes wrapped on 4-cycles with total homology class  $p^A D_A$ , in the decoupling limit  $R/l_{11} \rightarrow \infty$ ,  $V_M/l_{11}^6$  fixed. A convenient way to take the limit is to adapt units such that  $R$  remains finite — for example  $R \equiv 1$  — while  $\ell_5 \rightarrow 0$  (note that  $l_{11}/\ell_5$  is fixed because  $V_M/l_{11}^6$  is fixed). Then the dynamics of finite energy excitations of the M5 are described by a (0,4) supersymmetric 1+1 dimensional nonlinear sigma model with target space naively<sup>9</sup> given by the classical M5 moduli space, the MSW string [36, 88], decoupled from bulk and KK modes. For example, Kaluza-Klein excitations along the 4-cycle decouple as their mass is of order  $V_M^{-1/6}$ , which scales to infinity.

We wish to find out how multicentered solutions with total charge  $(0, p^A, q_A, q_0)$  behave when we take this limit. The IIA Kähler moduli  $J^A$  are related to the normalized scalars  $Y^A$  as  $J^A \sim \frac{R}{\ell_5} Y^A$ , hence  $J \rightarrow \infty$ . For two centered solutions involving D6-charges, the equilibrium separation following from the integrability condition (3.25) asymptotes to

$$|\vec{x}_1 - \vec{x}_2| = \frac{\langle \Gamma_1, \Gamma_2 \rangle}{2 \text{Im}(e^{-i\alpha} Z_1)|_\infty} \frac{\ell_5^{3/2}}{\sqrt{R}} \sim \frac{\langle \Gamma_1, \Gamma_2 \rangle}{R^2} \ell_5^3, \quad (4.10)$$

<sup>9</sup> As discussed in the introduction and further in section 4.5.3, the precise M5-brane interpretation of the decoupling limit is rather mysterious and still poses various puzzles.

where we used that for total D6-charge zero,  $\alpha \rightarrow 0$  when  $J \rightarrow \infty$ , while  $Z_1 \sim iJ^{3/2} \sim i(R/\ell_5)^{3/2}$ .

To keep the coordinate separation finite in the limit  $\ell_5 \rightarrow 0$ , we should therefore rescale all coordinates as

$$\vec{x} = \ell_5^3 \vec{\tilde{x}}. \quad (4.11)$$

The finite  $\vec{\tilde{x}}$  region then has the expected properties for a decoupling limit. First, as we will see, at finite values of  $\vec{\tilde{x}}$ , the metric converges to an expression of the form  $ds^2 = \ell_5^2 d\mathbf{s}^2$  with  $d\mathbf{s}^2$  finite. Finite fluctuations of  $d\mathbf{s}^2$  thus give rise to finite action fluctuations; the  $\ell_5^2$  metric prefactor cancels the  $\ell_5^{-3}$  in front of the Einstein-Hilbert action [90]. Similarly, M2-branes wrapping the M-theory circle and stretched over finite  $\vec{\tilde{x}}$  intervals have finite energy. Finally, the geometry becomes asymptotically an  $S^2$  bundle over  $\text{AdS}_3$  at large  $r = |\vec{\tilde{x}}| \rightarrow \infty$ :

$$d\mathbf{s}^2 \approx d\eta^2 + e^{\eta/U} (-d\tau^2 + d\sigma^2) + U^2 (d\theta^2 + \sin^2 \theta (d\phi + \tilde{A})^2), \quad (4.12)$$

$$\tilde{A} = \frac{J}{J_{\max}} d(\tau - \sigma) \quad (4.13)$$

$$A_{5d}^A \approx -p^A \cos \theta (d\phi + \tilde{A}) + 2D^{AB} q_B d(\sigma + \tau), \quad (4.14)$$

$$Y^A \approx \frac{p^A}{U}. \quad (4.15)$$

where  $U := (\frac{1}{6} D_{ABC} p^A p^B p^C)^{1/3}$  and we made the change of coordinates  $(r, t, \psi) \rightarrow (\eta, \tau, \sigma)$  to leading order given by:

$$\eta := U \log \frac{R^2 r}{U}, \quad \tau := \frac{t}{R}, \quad \sigma := \frac{\psi}{2} - \frac{t}{R}. \quad (4.16)$$

Notice that the normalized Kähler moduli  $Y^A$  and the  $U(1)$  vectors  $A^A$  are fixed at attractor values determined by the M5 and M2 charges. The flat connection  $\tilde{A}$  determines the twisting of the  $S^2$  over the  $\text{AdS}_3$  base;  $J$  is the  $S^2$ -angular momentum of the solution and  $J_{\max} := \frac{U^3}{2}$  is its maximal value for given  $p$ . Note that going around the M-theory circle in the new coordinates corresponds to

$$\sigma \rightarrow \sigma + 2\pi, \quad (4.17)$$

with all other coordinates fixed. Parallel transport of the  $S^2$  along this circle produces a rotation  $\Delta\phi = \frac{J}{J_{\max}} 2\pi$  around its  $z$ -axis (which is the axis determined by the direction of the four dimensional angular momentum). Because  $\tilde{A} \sim d(\sigma - \tau)$ , the sphere similarly gets rotated in time, resulting in angular momentum proportional to the amount of twisting around the  $S^1$ . Since the  $S^2$  descends from the spatial sphere at infinity in four dimensions, this equals the 4d angular momentum of the 4d multicentered solution. In the dual CFT, it translates to  $SU(2)_R$  charge.

The  $\sigma \rightarrow \sigma + 2\pi$  circle smoothly connects to the asymptotic M-theory circle in the original asymptotically flat geometry. Fermions must be periodic around

this circle, as antiperiodic fermions would produce a nonzero vacuum energy. Therefore we have periodic boundary conditions for the fermions on the  $\text{AdS}_3$  boundary circle, so the supersymmetric black hole configurations we are describing must correspond to supersymmetric states in the Ramond sector of the boundary CFT.

It is not true, however, that *all* multicentered solutions with total charge  $(0, p^A, q_A, q_0)$  give rise to such asymptotic  $\text{AdS}_3 \times S^2$  attractor geometries<sup>10</sup> in the decoupling limit. For example D4-D4 2-centered solutions (i.e.  $p_1^0 = p_2^0 = 0$ ), of which explicit examples are given in section 3.4.2, turn out to have equilibrium separations in the original coordinates scaling as  $|\vec{x}_1 - \vec{x}_2| \sim \langle \Gamma_1, \Gamma_2 \rangle \ell_5$ . The different scaling compared to the case with nonzero D6-charges is due to the fact that now  $\arg Z_1 \rightarrow 0$  in the decoupling limit. In the rescaled coordinates (4.11) the separation diverges, so these multicentered solutions therefore do not fit in the asymptotic  $\text{AdS}_3 \times S^2 \times X$  attractor geometry associated to the total M5 charge  $p^A$ . Rather, they give rise to two mutually decoupled  $\text{AdS}_3 \times S^2 \times X$  attractor geometries associated to the two individual centers. More elaborate configurations of this kind are possible too, for instance consisting of two clusters each with zero net D6-charge, but containing themselves more centers with nonzero D6-charge. The centers within each cluster will have rescaled coordinate separations of order 1, while the mutual separation between the clusters diverges like  $\ell_5^{-2}$  in these coordinates.

These D4-D4 type BPS bound states exist in regions of Kähler moduli space separated from the overall M5 attractor point  $Y^A = p^A/U$  by a wall of marginal stability. They correspond to ensembles of BPS states of the MSW string which exist at certain values of the  $Y^A$  but not at the attractor point. Their interpretation in the AdS-CFT context is therefore less clear — we will return to this in section 4.5.3.

In the following we wish to focus on solutions which do correspond to a single asymptotic  $\text{AdS}_3 \times S^2$  in the decoupling limit, and in particular find practical criteria to determine when this will be the case. We will proceed by rescaling coordinates as in (4.11) and carefully studying the behavior of the solutions when  $\ell_5 \rightarrow 0$ . As the explicit form of the multicenter solutions is rather complicated we will first make the dependence on  $\ell_5$  more clear by pulling it out through a rescaling of the variables in section 4.2.1. After this rescaling the dependence on  $\ell_5$  will simply be an overall factor in the metric as described above and a dependence left in the equilibrium distance between the centers and the constant terms of the harmonic functions. Once we have this simple form we will take the decoupling limit by sending  $\ell_5 \rightarrow 0$ . We calculate the asymptotics and some quantum numbers in sections 4.2.3 and 4.2.4 and finally we will discuss when the decoupling limit is well defined (in the sense that we do get a single asymptotically  $\text{AdS}_3 \times S^2$  geometry when  $\ell_5 \rightarrow 0$ ) in section 4.2.5.

<sup>10</sup> Despite the nontrivial twist of the  $S^2$ , we will still loosely refer to the asymptotic geometry as  $\text{AdS}_3 \times S^2$ .

### 4.2.1 Rescaling

As discussed above, to take the decoupling limit we want to work with the rescaled coordinates,  $\mathbf{x}^i$ ,

$$x^i = \ell_5^3 \mathbf{x}^i. \quad (4.18)$$

Furthermore we want to extract a factor of  $\ell_5$  out of the 5d metric. As the multicenter solutions are rather complicated we will here first simplify the dependence on  $\ell_5$  by redefining various quantities. In the rescaled coordinates it is natural to define rescaled harmonic functions,  $\mathbf{H}$ ,

$$\mathbf{H} = \ell_5^{3/2} H = \sum_a \frac{\Gamma_a}{\sqrt{R} |\mathbf{x} - \mathbf{x}_a|} - 2\ell_5^{3/2} \text{Im}(e^{-i\alpha} \Omega)|_\infty. \quad (4.19)$$

It is not difficult to verify that all functions appearing in (3.26) are actually homogenous under the rescaling of the coordinates and harmonic functions given above. For instance

$$\begin{aligned} y^A(H) &= \ell_5^{-3/2} y^A(\mathbf{H}), \\ Q(H) &= \ell_5^{-3} Q(\mathbf{H}), \\ L(H) &= \ell_5^{-9/2} L(\mathbf{H}), \\ \Sigma(H) &= \ell_5^{-3} \Sigma(\mathbf{H}). \end{aligned} \quad (4.20)$$

The scaling of  $\omega$  is a little more subtle. Here one has to take into account that the  $\star$  scales as well since the flat 3d metric scales as  $\ell_5^{-6}$  under the coordinate rescaling. This implies

$$\star_x = \ell_5^{3(3-2p)} \star_{\mathbf{x}}, \quad (4.21)$$

for the  $\star$  acting on a  $p$ -form. So from its equation of motion (3.26) we see that

$$\omega(H, dx, \sqrt{G_4}) = \ell_5^{-3/2} \omega(\mathbf{H}, d\mathbf{x}, R^{-1/2}), \quad (4.22)$$

where the factor  $\ell_5^{3/2}$  out of  $\sqrt{G_4} = \frac{\ell_5^{3/2}}{\sqrt{R}}$  is essential.

Note that the 4d metric from (3.23) scales as

$$ds_{4d}^2(H, dx, \sqrt{G_4}) = \ell_5^{-3} ds_{4d}^2(\mathbf{H}, d\mathbf{x}, R^{-1/2}). \quad (4.23)$$

Finally there are also some fields that remain invariant under the rescaling:

$$t^A(H) = t^A(\mathbf{H}) \quad (4.24)$$

$$\omega_0(H, dx, \sqrt{G_4}) = \omega_0(\mathbf{H}, d\mathbf{x}, R^{-1/2}) \quad (4.25)$$

$$\mathcal{A}(H, dx, \sqrt{G_4}) = \mathcal{A}(\mathbf{H}, d\mathbf{x}, R^{-1/2}). \quad (4.26)$$

It is clear from the discussion above that the whole solution transforms homogeneously under the rescaling of the coordinates and the redefinition of the

harmonic functions. In fact our solutions in rescaled coordinates take exactly the same form as the original solutions of chapter 3, with the only changes being the replacement of  $\sqrt{G_4}$  with  $R^{-1/2}$  and  $H$  with  $\mathbf{H}$  everywhere. Readers interested in the explicit rescaled form of the solutions can find it in appendix C of [31].

The 5d metric in these coordinates now has a prefactor  $\ell_5^2$

$$\begin{aligned} \frac{1}{\ell_5^2} ds_{5d}^2 = & 2^{-2/3} Q^{-2} \left[ -(\mathbf{H}^0)^2 (\sqrt{R} dt + \omega)^2 - 2L(\sqrt{R} dt + \omega)(d\psi + \omega_0) \right. \\ & \left. + \Sigma^2 (d\psi + \omega_0)^2 \right] + 2^{-2/3} R Q dx^i dx^i. \end{aligned} \quad (4.27)$$

Otherwise, the only appearance of  $\ell_5$  is through the harmonic functions  $\mathbf{H}$  in (4.19). It enters there in two ways. First through the constant terms

$$-2\ell_5^{3/2} \text{Im}(e^{-i\alpha}\Omega)|_\infty, \quad (4.28)$$

where it is important to recall that  $\Omega|_\infty$  also depends on  $\ell_5$  as  $J_\infty^A$  is related to  $\ell_5$  by  $\frac{4J_\infty^3}{3} = \left(\frac{R}{\ell_5}\right)^3$ . Secondly, the equilibrium positions  $x_i$  of the charged centers are determined by the consistency condition

$$\langle \Gamma_a, \mathbf{H} \rangle|_{x_a} = 0. \quad (4.29)$$

By this equation they depend on the constant part of the harmonics and thus  $\ell_5$ . We will elaborate in detail on this dependence in the next subsection when we consider the  $\ell_5 \rightarrow 0$  limit.

From this point onwards we will always be working with rescaled coordinates (unless we explicitly state otherwise). Hence, for notational simplicity **we will revert to original notation** (e.g.  $\Sigma, ds_{4d}^2, x, H$ ) though we will be referring to the *rescaled* expressions (e.g.  $\Sigma(\mathbf{H}), ds_{4d}^2(\mathbf{H}, dt, dx, R^{-1/2}), x, \mathbf{H}$ ). Hopefully this will not lead to excessive confusion.

### 4.2.2 Decoupling

Having rewritten our solutions in a rescaled form (4.27) where the  $\ell_5$  dependence is transparent we can consistently take the decoupling limit,  $\ell_5 \rightarrow 0$ , while keeping  $R, t, x^i, \psi, \tilde{V}_M$  and  $\Gamma_i$  fixed. As mentioned before, in the rescaled variables  $\ell_5$  only appears through the constants in the harmonic functions so taking the limit  $\ell_5 \rightarrow 0$  will leave the whole structure of the solution invariant except for replacing the harmonic functions by their limiting form. Changing  $\ell_5$  also effects the equilibrium distances of the centers,  $x_a$ , in the solution due to the appearance of the constant terms in the constraint equation (3.25). In general the equilibrium distances will vary in a rather complicated (and not unique) way, some interesting examples will be discussed explicitly in section 4.3.

Let us now examine the dependence on  $\ell_5$  in the small  $\ell_5$  regime. The constant terms of the rescaled harmonic functions are

$$h = -2\ell_5^{3/2} \text{Im}(e^{-i\alpha}\Omega)|_\infty, \quad (4.30)$$

where  $\Omega = -\frac{e^{B+iJ}}{\sqrt{\frac{4J^3}{3}}}$  and  $J^A|_\infty = \frac{R}{2l_5} Y^A|_\infty$ . We can write those constant terms in an expansion for small  $\ell_5$  as

$$\begin{aligned} h^0 &= h_{(4)}^0 \frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6), \\ h^A &= h_{(2)}^A \frac{\ell_5^2}{\sqrt{R}} + h_{(4)}^A \frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6), \\ h_A &= h_A^{(2)} \frac{\ell_5^2}{\sqrt{R}} + h_A^{(4)} \frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6), \\ h_0 &= -\frac{R^{3/2}}{4} + h_0^{(2)} \frac{\ell_5^2}{\sqrt{R}} + h_0^{(4)} \frac{\ell_5^4}{R^{5/2}} + \mathcal{O}(\ell_5^6), \end{aligned} \quad (4.31)$$

where the leading terms are<sup>11</sup>

$$\begin{aligned} h_{(4)}^0 &= 8 \frac{pYB - qY}{pYY}|_\infty, \\ h_{(2)}^A &= Y_\infty^A, \\ h_A^{(2)} &= (YB)_A|_\infty + \frac{Y_A^2}{pY^2} (qY - pYB)|_\infty, \\ h_0^{(2)} &= \frac{1}{2} YB^2|_\infty + \frac{BY^2}{pY^2} (qY - pYB)|_\infty + 2 \frac{(qY - pYB)^2}{(pY^2)^2}|_\infty. \end{aligned} \quad (4.32)$$

So in the limit  $\ell_5 \rightarrow 0$  all the constants in harmonics are sent to zero except for the one in the D0 harmonic  $H_0$  which reads

$$h_0 \rightarrow -\frac{R^{3/2}}{4}. \quad (4.33)$$

The equilibrium distances also depend on the asymptotic moduli through the integrability constraints (3.25). These constraints can be written in the form

$$\sum_b \frac{\langle \Gamma_a, \Gamma_b \rangle}{\sqrt{R} |x_a - x_b|} = -\langle \Gamma_a, h \rangle. \quad (4.34)$$

So from the behavior (4.33) we see that in the decoupling limit  $\ell_5 \rightarrow 0$  the consistency conditions (3.25) become

$$\sum_b \frac{\langle \Gamma_a, \Gamma_b \rangle}{|x_a - x_b|} = -\frac{p_a^0}{4} R^2. \quad (4.35)$$

<sup>11</sup> To keep the formulas in (4.31) readable we suppressed the various indices and contractions, these formulas should all be read as e.g.  $XYZ = D_{ABC} X^A Y^B Z^C$ ,  $(XY)_A = D_{ABC} X^B Y^C$ ,  $XY = X_A Y^A$ .

Summarized, the decoupling limit corresponds to replacing the harmonic functions by

$$\begin{aligned}
H^0 &= \sum_a \frac{p_a^0}{\sqrt{R}|x-x_a|}, \\
H^A &= \sum_a \frac{p_a^A}{\sqrt{R}|x-x_a|}, \\
H_A &= \sum_a \frac{q_A^a}{\sqrt{R}|x-x_a|}, \\
H_0 &= \sum_a \frac{q_0^a}{\sqrt{R}|x-x_a|} - \frac{R^{3/2}}{4}.
\end{aligned} \tag{4.36}$$

Furthermore the equilibrium distances are now determined by the equations (4.35).

Note that this limit is similar to the usual near horizon limit, but not quite the same, since we are not simply dropping all constant terms from the harmonic functions. A similar situation was encountered for instance in [127], where a similar decoupling limit is defined for the three charge super tubes.

It is useful to note that although under the decoupling limit the D0 constant goes to a fixed non vanishing value, this constant can, however, be removed by the following formal transformations

$$\begin{aligned}
H_0 &\rightarrow H_0 + \frac{R^{3/2}}{4} \\
L &\rightarrow L + \frac{R^{3/2}}{4} (H^0)^2 \\
t &\rightarrow v = t - \frac{R}{4} \psi
\end{aligned} \tag{4.37}$$

As this is the only effect of the constant term in the D0-brane harmonic function, we can set it to zero while replacing  $t$  by  $v = t - R/4 \psi$  and making a shift in  $L$  at the same time. This is sometimes technically convenient.

### 4.2.3 Asymptotics

Now that we have implemented the decoupling limit we want to study the new asymptotics of these solutions. This is completely determined by the asymptotics of the harmonic functions. For  $r \rightarrow \infty$  the harmonic functions (4.36) can be

expanded as

$$\begin{aligned}
 H^0 &\rightarrow R^{-1/2} \frac{e \cdot d^0}{r^2}, \\
 H^A &\rightarrow R^{-1/2} \left( \frac{p^A}{r} + \frac{e \cdot d^A}{r^2} \right), \\
 H_A &\rightarrow R^{-1/2} \left( \frac{q_A}{r} + \frac{e \cdot d_A}{r^2} \right), \\
 H_0 &\rightarrow R^{-1/2} \left( \frac{q_0}{r} + \frac{e \cdot d_0}{r^2} \right),
 \end{aligned} \tag{4.38}$$

where we have put the constant in  $H_0$  to zero by the procedure explained at the end of the last subsection. In our notation

$$d := \sum_a \Gamma_a x_a \tag{4.39}$$

is the dipole moment and  $\vec{e} = \frac{\vec{x}}{r}$ ,  $r = |x|$ , is the normalized position vector that gives the direction on the  $\text{S}^2$  at infinity. Note that for  $H^0$  the dipole term is leading as we take the overall D6 charge zero; the same is true for  $H_A$  if the total D2 charge is zero. As we will only consider cases of non-vanishing overall D4 charge here the dipole term is always subleading.

In studying the asymptotics of the physical fields it will be most straightforward to work in a coordinate system where  $d^0$  lies along the  $z$ -axis. In this case

$$e \cdot d^0 = \cos \theta |d^0|, \tag{4.40}$$

with the standard spherical coordinates  $(\theta, \phi)$ . To simplify the notation we will often write just  $d^0$  for  $|d^0|$ ; it should be clear from the context when the vectorial quantity is intended and when the scalar. Note that the different dipole moments don't have to align so in general there is no simple expression for e.g.  $e \cdot d^A$  in this coordinate system.

In the decoupled geometry  $d^0$  plays a distinguished role as it is proportional to the total angular momentum of the system. To see this we start from the stability condition in the decoupled theory, (4.35), multiply by  $x_b$  and sum over  $b$  (note that this still is a vector identity):

$$J = \frac{1}{2} \sum_{a \neq b} \frac{\langle \Gamma_a, \Gamma_b \rangle x_b}{|x_a - x_b|} = \frac{R^2}{8} \sum_a p_a^0 x_a = \frac{R^2}{8} d^0. \tag{4.41}$$

From the above asymptotic expansion of the harmonics (4.38), we can determine the asymptotic behavior of all the fields and functions appearing in our solution. First, let us determine the large  $r$  expansion of the functions  $y^A$ . These are given in the form of a quadratic equation which can be solved in a  $1/r$  expansion as

$$y^A = H^A - H^0 D^{AB} H_B - \frac{1}{2} (H^0)^2 D^{FA} D_{FBC} D^{BD} H_D D^{CE} H_E + \mathcal{O}\left(\frac{1}{r^4}\right), \tag{4.42}$$

where we defined

$$D^{AB} = (D_{ABC}H^D)^{-1}. \quad (4.43)$$

Armed with this expression for  $y^A$  we compute

$$D_{ABC}y^Ay^By^C = D_{ABC}H^AH^BH^C - 3H^0H^AH_A + \frac{3}{2}(H^0)^2H_AD^{AB}H_B + \mathcal{O}\left(\frac{1}{r^6}\right). \quad (4.44)$$

We can now evaluate the  $1/r$  expansion of the coefficient  $\frac{\Sigma^2}{Q^2}$  appearing in front of  $d\psi^2$  in the metric

$$\frac{\Sigma^2}{Q^2} = (H_AD^{AB}H_B - 2H_0) \left( \frac{D_{ABC}H^AH^BH^C}{3} \right)^{-1/3} + \mathcal{O}\left(\frac{1}{r}\right). \quad (4.45)$$

The expansion of  $L$  is straightforward, and the the expansion for  $Q$  follows directly from (4.44). The last non-trivial expansions to be calculated are those of  $\omega$  and  $\omega_0$ . For those the following result is convenient: for any vector  $n^i \in \mathbb{R}^3$  one has

$$d\left(\frac{\epsilon_{ijk}n^ir^jd r^k}{r^3}\right) = -*_3 d\left(\frac{n^ir^i}{r^3}\right). \quad (4.46)$$

In particular we find that

$$\omega_0 = -\epsilon_{ijk}\frac{(d^0)^ir^jd r^k}{r^3} + \mathcal{O}\left(\frac{1}{r^2}\right) = -\frac{\sin^2\theta d^0}{r}d\phi + \mathcal{O}\left(\frac{1}{r}\right), \quad (4.47)$$

where in the last equality we used our choice to take the  $z$  axis to be along the D6 dipole moment  $d^0$ . We will not need the explicit form of  $\omega$  because its leading term goes like  $\mathcal{O}(r^{-2})$ . This follows from the asymptotic form of the equations of motion

$$d\omega = \sqrt{R} \star \left( -h_0 dH^0 + \mathcal{O}\left(\frac{1}{r^4}\right) \right). \quad (4.48)$$

We are now ready to spell out the asymptotic expansion of the metric. We start from (4.27), use the expansions computed above and replace  $t$  by  $v$  to compensate for shifting the D0 constant  $h_0$ . The result one gets up to terms of order<sup>12</sup>  $\mathcal{O}(\frac{1}{r})$  is

$$\begin{aligned} ds_{5d}^2 = & -r\frac{R}{U}dv d\psi + \frac{U^{-4}}{4}[-R^2(d^0)^2dv^2 \\ & + 2R\left(\frac{e \cdot d^A D_{ABC}p^B p^C}{3} - \frac{p^A q_A d^0 \cos\theta}{3}\right)dv d\psi + \mathcal{D}d\psi^2] \\ & + U^2\frac{dr^2}{r^2} + U^2\left(d\theta^2 + \sin^2\theta(d\phi + \tilde{A})^2\right) + \mathcal{O}\left(\frac{1}{r}\right). \end{aligned} \quad (4.49)$$

<sup>12</sup> In this power counting we consider  $\mathcal{O}(dr) = \mathcal{O}(r)$ .

Here we introduced the notation

$$v = t - R/4\psi, \quad U^3 = \frac{p^3}{6}, \quad \mathcal{D} = \frac{p^3}{3} (D^{AB} q_A q_B - 2q_0) \text{ and } \tilde{A} = \frac{J}{J_{\max}} \frac{2v}{R}. \quad (4.50)$$

We used the relation between the D6-dipole moment  $d^0$  and the angular momentum  $J$  given by (4.41) and the fact that there is a maximal angular momentum  $J_{\max} = \frac{U^3}{2}$ . Note that  $\pi^2 \mathcal{D} = S(\Gamma_t)^2$ , so  $\mathcal{D}$  is the discriminant of the total charge. With a coordinate transformation to a new radial variable  $\rho$  one can show that the angular dependent part in the second term of (4.49) is really further subleading. The coordinate  $\rho$  is given by

$$\frac{\rho^2}{4U^2} = -\frac{U^{-4}}{2} R \left( \frac{e \cdot d^A D_{ABC} p^B p^C}{3} - \frac{p^A q_A d^0 \cos \theta}{3} \right) + \frac{R}{U} r. \quad (4.51)$$

In this new radial coordinate the expansion in large  $\rho$  takes the following form

$$\begin{aligned} ds_{5d}^2 = & -\frac{\rho^2}{4U^2} dv d\psi + \frac{U^{-4}}{4} [-R^2 (d^0)^2 dv^2 + \mathcal{D} d\psi^2] + 4U^2 \frac{d\rho^2}{\rho^2} \\ & + U^2 \left( d\theta^2 + \sin^2 \theta (d\phi + \tilde{A})^2 \right) + \mathcal{O}\left(\frac{1}{\rho^2}\right). \end{aligned} \quad (4.52)$$

Using the expansion formulas derived above it is straightforward to calculate the asymptotics of the gauge field and the scalars. Putting everything together we see that the solution asymptotes to

$$\begin{aligned} ds_{5d}^2 = & -\frac{\rho^2}{4U^2} dv d\psi + \frac{U^{-4}}{4} [-R^2 (d^0)^2 dv^2 + \mathcal{D} d\psi^2] \\ & + 4U^2 \frac{d\rho^2}{\rho^2} + U^2 \left( d\theta^2 + \sin^2 \theta (d\phi + \tilde{A})^2 \right) + \mathcal{O}\left(\frac{1}{\rho^2}\right), \end{aligned} \quad (4.53)$$

$$A_{5d}^A = -p^A \cos \theta d\alpha + D^{AB} q_B d\psi + \mathcal{O}\left(\frac{1}{\rho^2}\right), \quad (4.54)$$

$$Y^A = \frac{p^A}{U} + \mathcal{O}\left(\frac{1}{r^2}\right). \quad (4.55)$$

It is clear that the metric is locally asymptotically  $\text{AdS}_3 \times \text{S}^2$  with  $R_{\text{AdS}} = 2R_{\text{S}^2} = 2U$ . We have kept track of some subleading terms as they will be important in reading off quantum numbers in the next section. Note that we have in fact a nontrivial  $S^2$  fibration over  $\text{AdS}_3$  described by the flat connection  $\tilde{A} = \frac{J}{J_{\max}} \left( \frac{2dt}{R} - \frac{d\psi}{2} \right)$ . As  $\tilde{A}$  depends on the time coordinate we see that as time progresses the sphere rotates, implying the solution has angular momentum as expected. In the same way, going once around the M-theory circle, i.e.  $\psi \rightarrow \psi + 4\pi$ , induces a rotation of  $\frac{2\pi J}{J_{\max}}$  along the equator<sup>13</sup> of the  $S^2$ . The explicit coordinate transformation bringing the above metric in the form (4.12) after dropping the subleading terms will be given below.

<sup>13</sup> Remember we chose the canonical 'z-axis' of our spherical coordinates along the total angular momentum of the solution.

### 4.2.4 CFT quantum numbers

In this subsection we will do an analysis of the asymptotic conserved charges of the decoupled solutions. As we now have an asymptotic AdS geometry we can use the well developed technology for these spaces. In our case of AdS<sub>3</sub> a nice review can be found in [92]. The asymptotic charges as determined from the supergravity side can later be compared to various quantum numbers in the boundary CFT.

To proceed we first rewrite everything asymptotically in terms of a three dimensional theory on AdS<sub>3</sub> by reducing over the asymptotic sphere spanned by  $(\theta, \phi)$ . Reducing five dimensional  $\mathcal{N}=1$  supergravity over the  $S^2$  will result in a three dimensional theory with an  $SU(2)$  gauge group in addition to gravity (in an AdS<sub>3</sub> background) and the  $U(1)$  vector multiplet fields that descend from five dimensions. The metric of the reduced theory is

$$ds_{3d}^2 = -\frac{\rho^2}{4U^2} dv d\psi + \frac{U^{-4}}{4} [-R^2(d^0)^2 dv^2 + \mathcal{D}d\psi^2] + 4U^2 \frac{d\rho^2}{\rho^2}. \quad (4.56)$$

This can be put into a standard form for the asymptotic expansion around AdS<sub>3</sub> by the coordinate transformations

$$\rho^2 = \frac{e^{\frac{\eta}{U}} 4U^2}{R}, \quad dv = -\frac{R}{2} d\bar{w}, \quad d\psi = 2dw. \quad (4.57)$$

These are related to the coordinates  $\tau, \sigma$  we used in (4.12)-(4.15) by  $w = \sigma + \tau$ ,  $\bar{w} = \sigma - \tau$ . After Wick rotating  $\tau \rightarrow i\tau$ , these become the standard conjugate holomorphic coordinates on the boundary cylinder, with periodicity  $2\pi$ . The metric reads

$$ds_{3d}^2 = d\eta^2 + e^{\frac{\eta}{U}} dw d\bar{w} + \frac{1}{U^4} \left( \mathcal{D}dw^2 - \frac{R^4(d^0)^2}{16} d\bar{w}^2 \right), \quad (4.58)$$

which has the standard form  $ds_{3d}^2 = d\eta^2 + (e^{\frac{2\eta}{R_{\text{AdS}}}} g_{ij}^{(0)} + g_{ij}^{(2)}) du^i du^j$ . The boundary energy momentum tensor is then given by the formulas [92]:

$$\begin{aligned} T_{ww}^{\text{grav}} &= \frac{1}{8\pi G_3 R_{\text{AdS}}} g_{ww}^{(2)}, \\ T_{\bar{w}\bar{w}}^{\text{grav}} &= \frac{1}{8\pi G_3 R_{\text{AdS}}} g_{\bar{w}\bar{w}}^{(2)}. \end{aligned} \quad (4.59)$$

In our case this becomes<sup>14</sup>

$$\begin{aligned} T_{ww}^{\text{grav}} &= \frac{\mathcal{D}}{8\pi U^3}, \\ T_{\bar{w}\bar{w}}^{\text{grav}} &= \frac{-R^4(d^0)^2}{8\pi 16U^3}. \end{aligned} \quad (4.60)$$

<sup>14</sup> We used  $G_3 = \frac{\ell_5^3}{2R_{S^2}^2}$ . Note furthermore that the definitions (4.59) are given in unrescaled variables so that both  $R_{\text{AdS}}$  and  $R_{S^2}$  carry a factor  $\ell_5$ . Thus when rescaling  $g_{ij} \rightarrow \ell_5^2 g_{ij}$  all factors of  $\ell_5$  drop out of the energy momentum tensor. This is as expected since we defined our limit in such a way as to ensure that these energies stay finite as  $\ell_5 \rightarrow 0$ .

Apart from the metric, there are also gauge fields: the  $\text{SU}(2)$  gauge field coming from the reduction of the metric on  $\text{S}^2$  and the  $\text{U}(1)$  vectors of the 5d supergravity. These gauge fields do contribute to the asymptotic energy momentum tensor because the 5-dimensional action contains a Chern-Simons term involving them. Here we will just present the results of the derivation that is detailed in appendix B. The contribution of all the different gauge fields to the energy momentum is given by

$$T_{ww}^{\text{gauge}} = \frac{1}{4\pi} \left[ \frac{(p^A q_A)^2}{p^3} - (q_A D^{AB} q_B) \right], \quad T_{\bar{w}\bar{w}}^{\text{gauge}} = \frac{1}{4\pi} \frac{(p^A q_A)^2}{p^3} + \frac{R^4}{8\pi} \frac{(d^0)^2}{16U^3}. \quad (4.61)$$

So by combining (4.60) and (4.61), we see that the total energy momentum tensor is:

$$T_{ww} = \frac{1}{4\pi} \left( \frac{(p^A q_A)^2}{p^3} - 2q_0 \right), \quad T_{w\bar{w}} = 0, \quad T_{\bar{w}\bar{w}} = \frac{1}{4\pi} \frac{(p^A q_A)^2}{p^3} \quad (4.62)$$

The Virasoro charges  $(L_0)_{\text{cyl}}$  and  $(\tilde{L}_0)_{\text{cyl}}$  on the cylinder are obtained from the energy-momentum tensor as

$$\begin{aligned} (L_0)_{\text{cyl}} &= \oint dw T_{ww} = \frac{(p^A q_A)^2}{2p^3} - q_0, \\ (\tilde{L}_0)_{\text{cyl}} &= \oint d\bar{w} T_{\bar{w}\bar{w}} = \frac{(p^A q_A)^2}{2p^3}, \end{aligned} \quad (4.63)$$

where the contour integral is taken along a contour wrapped once around the asymptotic cylinder, i.e.  $w \rightarrow w + 2\pi$ . These are related to the standard Virasoro charges on the  $z = e^{iw}$ -plane by the transformations

$$L_0 = (L_0)_{\text{cyl}} + \frac{c}{24}, \quad \tilde{L}_0 = (\tilde{L}_0)_{\text{cyl}} + \frac{c}{24}, \quad (4.64)$$

with  $c$  the Brown-Henneaux central charge:

$$c = \frac{3R_{\text{AdS}}}{2G_3} = p^3. \quad (4.65)$$

These are exactly the quantum numbers of the BPS states of the dual CFT in the Ramond sector as determined in [129, 130], confirming our earlier assertion under (4.17). Naively one might have thought that the BPS condition would require  $\tilde{L}_0 = c/24$ . That this is not so follows from the particular structure of the  $(0, 4)$  theory under consideration. It has, besides the usual  $(0, 4)$  superconformal algebra, several additional  $\text{U}(1)$  currents, as well as additional right-moving fermions — these are superpartners of the center of mass degrees of freedom of the original wrapped M5-brane description. As was analyzed in [129, 130], the BPS conditions involve the right-moving fermions in a non-trivial way, and

this modifies the BPS bound into  $\tilde{L}_0 \geq \frac{(p^A q_A)^2}{2p^3} + \frac{p^3}{24}$ , consistent with our result above.

Often, it is more convenient to work with different but closely related quantum numbers,  $L'_0$  and  $\tilde{L}'_0$ , and similarly  $(L'_0)_{\text{cyl}}$  and  $(\tilde{L}'_0)_{\text{cyl}}$ , which are obtained from the original ones by subtracting out the contributions of the zero modes of the additional currents, so only the oscillator contributions remain. In our case they are given by [129]:

$$\begin{aligned} L'_0 - \frac{c}{24} = (L'_0)_{\text{cyl}} &= -\hat{q}_0 := -(q_0 - \frac{1}{2} D^{AB} q_A q_B), \\ \tilde{L}'_0 - \frac{c}{24} = (\tilde{L}'_0)_{\text{cyl}} &= 0. \end{aligned} \quad (4.66)$$

These reduced quantum numbers are in many cases more convenient. They are spectral flow invariant, and when we want to use Cardy's formula to compute the number of states with given  $U(1)$  charges, we can simply use the standard Cardy formula with  $L_0, \tilde{L}_0$  replaced by  $L'_0, \tilde{L}'_0$ . The reduced quantum numbers also have a simple interpretation in the AdS/CFT correspondence. They represent the contributions to  $L_0, \tilde{L}_0$  from the gravitational sector, ignoring the additional contributions from the gauge fields.

The total energy and momentum, in units of  $1/R$ , are given by

$$H = (L_0)_{\text{cyl}} + (\tilde{L}_0)_{\text{cyl}} = \frac{(p^A q_A)^2}{p^3} - q_0, \quad P = (L_0)_{\text{cyl}} - (\tilde{L}_0)_{\text{cyl}} = -q_0, \quad (4.67)$$

and the reduced energy and momentum by

$$H' = (L'_0)_{\text{cyl}} + (\tilde{L}'_0)_{\text{cyl}} = -\hat{q}_0, \quad P' = (L'_0)_{\text{cyl}} - (\tilde{L}'_0)_{\text{cyl}} = -\hat{q}_0 = H'. \quad (4.68)$$

The energy  $H$  can be seen to equal the BPS energy  $E = \frac{|Z|}{\sqrt{G_4}}$  of a D4-D2-D0 particle in a 4d asymptotically flat background with  $J^A \rightarrow \infty p^A$ ,  $B^A = 0$ , with the diverging part subtracted off. The reduced energy is the same but now at  $B^A = D^{AB} q_B$ .

Finally, the  $SU(2)_R$  charge can be read off from the sphere reduction connection appearing in the metric (4.12). In general it is given by

$$J_0^I = \oint \frac{d\bar{w}}{2\pi} J_{\bar{w}}^I = \frac{c}{12} \oint \frac{d\bar{w}}{2\pi} A_{\bar{w}}^I. \quad (4.69)$$

Details are given in appendix B. Thus the  $SU(2)_R$  charge equals the four dimensional angular momentum:

$$J_0 = \frac{R^2 d^0}{8} = J, \quad (4.70)$$

where we used (4.41). This is as expected, since the  $S^2$  descends from the spatial sphere at infinity in four dimensions.

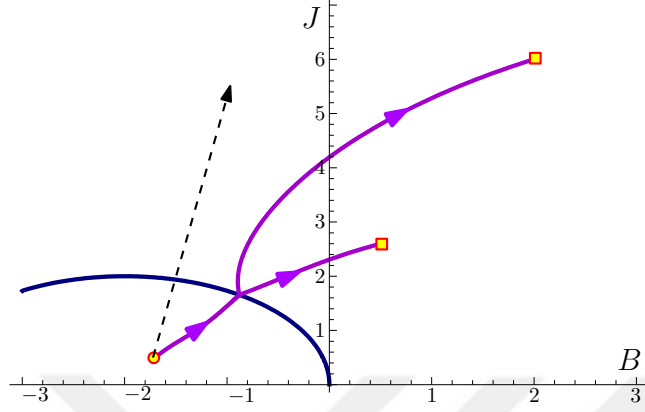


**Figure 4.1** This figure is an example of an attractor tree that exists in flat space but that will not survive in the decoupling limit, because the starting point of the split flow will move towards  $J = \infty$  and hence cross a wall of marginal stability and decay.

#### 4.2.5 Existence and attractor flow trees

Not all choices of charges  $\Gamma_a$  give rise to multicentered solutions in asymptotically flat space at finite  $R/\ell_5$ . Of those which do, not all survive the decoupling limit  $R/\ell_5 \rightarrow \infty$ . And of those which survive, not all give rise to a single  $\text{AdS}_3 \times S^2$  throat.

As reviewed in section 3.3.4, in four dimensional asymptotically flat space the well supported split attractor flow conjecture states there is a one to one correspondence between attractor flow trees and components of the moduli space of multicentered solutions. In particular, the existence of flow trees implies the existence of corresponding multicentered configurations, which can be assembled or disassembled adiabatically by dialing the asymptotic moduli according to the flow tree diagram. By the uplift procedure we followed, the same correspondence holds for five dimensional solutions asymptotic to  $\mathbb{R}^{1,3} \times S^1$  with a  $U(1)$  isometry corresponding to the extra  $S^1$ .

The 4d Kähler moduli scalars  $J^A$  are related to the five dimensional normalized Kähler scalars  $Y^A$  and the radius  $R$  of the circle by

$$J^A = \frac{R}{2\ell_5} Y^A \quad (4.71)$$

and the four dimensional  $B$ -field moduli  $B^A$  equal the Wilson lines around the  $S^1$  of the five dimensional gauge fields. Asymptotically  $\mathbb{R}^{1,3} \times S^1$  solutions surviving the  $R/\ell_5 \rightarrow \infty$  limit thus correspond to 4d flow trees surviving the  $J^A \rightarrow \infty$  limit. Figure 4.1 gives an example of a class of flow trees not surviving in this limit. Another example is given by figure 3.11 in section 3.4.2, which survives

when the limiting direction in the  $(J_1, J_2)$ -plane in figure 3.10 is taken above the marginal stability line, but not when it is taken below.

Now, not all asymptotically  $\mathbb{R}^{1,3} \times S^1$  configurations surviving in the limit fit into a single  $\text{AdS}_3 \times S^2$  throat. For example the D4-D4 bound states studied in section 3.4.2 have center separations of order  $p^3 \ell_5$  in the original coordinates, whereas multicentered configurations which do fit into an asymptotic  $\text{AdS}_3 \times S^2$  throat have separations of order  $p^3 \ell_5^3 / R^2$ . The diverging hierarchy between these distance scales in the decoupling limit  $R/\ell_5 \rightarrow \infty$  is manifest in the rescaled consistency condition (4.35) in the decoupling limit: for two D4 centers (or more generally clusters) with nonvanishing mutual intersection product, the (rescaled) equilibrium separation is infinite.

Looking at the asymptotics (4.12)–(4.15) of the decoupled solutions, we see that the value of  $Y^A$  at the boundary of AdS is proportional to  $p^A$ , and that the  $\theta$ -averaged Wilson line  $\frac{1}{4\pi} \oint A_{5d}^A$ , equals  $D^{AB} q_B$ . This suggests asymptotic  $\text{AdS}_3 \times S^2$  solutions correspond to 4d attractor flow trees with starting point at the “asymptotic  $\text{AdS}_3$  attractor point”

$$B^A + iJ^A = D^{AB} q_B + i\infty p^A. \quad (4.72)$$

As a test of this suggestion, note that, as pointed out in [29], this eliminates flow trees initially splitting into two flows carrying only D4-D2-D0 charges, and therefore configurations of two D4 clusters with nonvanishing intersection product, which as we just recalled indeed do not fit in a single  $\text{AdS}_3 \times S^2$  throat in the decoupling limit. To see this, it suffices to compute for  $\Gamma_a = (0, p_a^A, q_a^A, q_a^0)$  at  $B^A = D^{AB} q_B$ ,  $J^A = \Lambda p^A$ ,  $\Lambda \rightarrow \infty$ :

$$\langle \Gamma_1, \Gamma_2 \rangle \text{Im}(Z_1 \bar{Z}_2) = -\frac{3}{8} (p_1^A q_A^2 - p_2^A q_A^1)^2 + \mathcal{O}(\Lambda^{-1}) < 0. \quad (4.73)$$

This inequality (valid when  $\langle \Gamma_1, \Gamma_2 \rangle \neq 0$ ) implies that the initial point can never be on the stable side of a wall of marginal stability, and hence a flow tree with this initial split cannot exist. Initial splits involving nonzero D6-charge on the other hand are not excluded in this way, consistent with expectations.

Thus, we arrive at the following

**Conjecture:** *There is a one to one correspondence between (i) components of the moduli space of multicentered asymptotically  $\text{AdS}_3 \times S^2$  solutions with a  $U(1)$  isometry and (ii) attractor flow trees starting at  $J^A = p^A \infty$  and  $B^A = D^{AB} q_B$ .*

In what follows we will refer to this special point in moduli space as the *AdS point*. It is worth pointing out that the AdS point may lie on a wall of *threshold* stability,<sup>15</sup> as defined in section 3.3.2, for which the inequality (4.73) may become

<sup>15</sup> Note that it cannot lie on a wall of *marginal* stability  $\Gamma \rightarrow \Gamma_1 + \Gamma_2$ : if the constituents have nonzero D6-charge, these D6-charges have to be opposite in sign, so in the  $J \rightarrow \infty$  limit, the central charges cannot possibly align; if the constituents have zero D6-charge, (4.73) shows that their central charges cannot align either if their intersection product is nonvanishing.

an equality. As discussed there, the solution space becomes non-compact in this case, in the sense that constituents can be moved off to infinity — in this case to the boundary of AdS. An example is given by (3.54): since the overall D2-charge vanishes, the AdS point lies on the line  $B = 0$ , which is a line of threshold stability for splitting off the D0. The flow tree becomes degenerate as well, as it splits in a trivalent vertex. Keeping this in mind, the flow tree picture remains valid.

Finally, we should spend a few more words on our choice of  $B$ -field value for the AdS point. In general, the actual value of  $B^A$  at the boundary of AdS depends on the angle  $\theta$  with the direction of the total angular momentum:

$$B^A|_{\partial \text{AdS}} = \frac{1}{4\pi} \oint A_{5d} = D^{AB} q_B - \cos \theta \frac{J}{2J_{\max}} p^A. \quad (4.74)$$

Hence there is a significant spread of the actual asymptotic value of the  $B$ -field, proportional to the total angular momentum, which moreover grows with  $p$ . Although natural, it is therefore not immediately obvious that picking the average value (or equivalently the value at  $\theta = \frac{\pi}{2}$ ) as starting point is the right thing to do, and this is why our conjecture above is not an immediate consequence of the split attractor flow conjecture.

### 4.3 Some examples

In this section we will briefly describe the decoupling limit for some simple, but interesting multicentered configurations. The first example is rather straightforward as we show how the well known case of a single centered black hole/string fits in our more general story. Afterwards we discuss two 2-center systems of interest. First, we show that the decoupling limit of a purely fluxed  $\text{D6} - \overline{\text{D6}}$  bound state is nothing but global  $\text{AdS}_3 \times \text{S}^2$  and we discuss the link of this interpretation with spectral flow in the CFT. Second we analyze configurations leading to the Entropy Enigma of [29] in asymptotic AdS space. In the next section we will show how the Entropy Enigma translated to 5d coincides with a well known instability of small AdS black holes.

Note that from here on we put  $R \equiv 1$ .

#### 4.3.1 One center: BTZ

In the case of a single black string we expect to reproduce the standard BTZ black hole (times  $S^2$ ) as the decoupled geometry [82]. As a check on our results we show that this is indeed the case and that the entropy of the BTZ black hole corresponds to the one of the 4d black hole/5d black string we took the decoupling limit of. Given an M5M2P black string of charge  $(0, p^A, q_A, q_0)$  one can easily calculate that the metric (4.27) in the decoupling limit is

$$ds^2 = \frac{r}{U} \left[ -dt d\psi + \frac{1}{4} \left( 1 + \frac{1}{rU^3} \left( \frac{S}{\pi} \right)^2 \right) d\psi^2 \right] + \frac{U^2 dr^2}{r^2} + U^2 d\Omega_2^2, \quad (4.75)$$

where

$$S = 4\pi\sqrt{\frac{-\hat{q}_0 p^3}{24}}, \quad \hat{q}_0 = q_0 - \frac{1}{2}D^{AB}q_A q_B, \quad (4.76)$$

is the entropy of the 4d black hole. It is clear that this is indeed of the asymptotically local  $\text{AdS}_3 \times \text{S}^2$  form as found above. But in this case the full geometry, including the interior, is actually locally  $\text{AdS}_3 \times \text{S}^2$ . To see this perform the coordinate transformation

$$\psi = 2(t + \alpha), \quad r = U(\rho^2 - \rho_*^2) \quad \text{and} \quad \rho_* = \frac{S}{\pi U^2}, \quad (4.77)$$

to put this metric (4.75) into its well known BTZ form:

$$ds^2 = -\frac{(\rho^2 - \rho_*^2)^2}{\rho^2} dt^2 + \frac{4\rho^2 U^2}{(\rho^2 - \rho_*^2)^2} d\rho^2 + \rho^2 (d\alpha + \frac{\rho_*^2}{\rho^2} dt)^2 + U^2 d\Omega_2^2. \quad (4.78)$$

This is the geometry of a sphere times an extremal rotating BTZ black hole and as is well known [113], this can be viewed as a quotient of  $\text{AdS}_3 \times \text{S}^2$ . Calculating the Bekenstein-Hawking entropy of this BTZ black hole we find:

$$S_{\text{BH}} = \frac{2\pi\rho_*}{4G_3} = S, \quad (4.79)$$

in agreement with our expectations.

Note that the horizon topology is  $S^1 \times S^2$ , so from the 5d point of view we have a black ring.

### 4.3.2 Two centers: $\text{D6} - \overline{\text{D6}}$ and spinning $\text{AdS}_3 \times \text{S}^2$

The first new configurations appear by taking the decoupling limit of 2-center bound states. As follows from the constraint (4.35), only 2-centered solutions where the centers carry (opposite) non-vanishing D6 charge exist in asymptotic  $\text{AdS}_3 \times \text{S}^2$  space. Such centers sit at a fixed distance completely determined by their charges:

$$r_{12} = \frac{-4\langle\Gamma_1, \Gamma_2\rangle}{p_1^0}. \quad (4.80)$$

In general in the bulk the solution is now fully five-dimensional, mixing up the asymptotic sphere and AdS geometries in a complicated way.

The simplest two centered configuration is that of a bound state of a pure D6 and  $\overline{\text{D6}}$  carrying only  $U(1)$  flux, say  $F = \pm \frac{p}{2}$ . The two charges are then:

$$\begin{aligned} \Gamma_1 &= e^{\frac{p}{2}} = [1, \frac{1}{2}, \frac{1}{8}, \frac{1}{48}], \\ \Gamma_2 &= -e^{-\frac{p}{2}} = [-1, \frac{1}{2}, -\frac{1}{8}, \frac{1}{48}], \end{aligned} \quad (4.81)$$

where we introduced the following notation for (D6,D4,D2,D0)-charges:

$$[a, b, c, d] := (a, b p^A, c D_{ABC} p^B p^C, d D_{ABC} p^A p^B p^C). \quad (4.82)$$

We now show that the lift of such a 2-centered configuration in the decoupling limit yields rotating global  $\text{AdS}_3 \times \mathbf{S}^2$ . In this limit the harmonic functions are:

$$\begin{aligned} H^0 &= \frac{1}{|x - x_1|} - \frac{1}{|x - x_2|}, \\ H^A &= \frac{p^A}{2} \left( \frac{1}{|x - x_1|} + \frac{1}{|x - x_2|} \right), \\ H_A &= \frac{D_{ABC} p^B p^C}{8} \left( \frac{1}{|x - x_1|} - \frac{1}{|x - x_2|} \right), \\ H_0 &= \frac{p^3}{48} \left( \frac{1}{|x - x_1|} + \frac{1}{|x - x_2|} \right) - \frac{1}{4}. \end{aligned} \quad (4.83)$$

The equilibrium distance, solution to (4.35), is given by:

$$|x_1 - x_2| = \frac{2p^3}{3} =: 4U^3. \quad (4.84)$$

After a change of coordinates (see also [34]):

$$\begin{aligned} |x - x_1| &= 2U^3 (\cosh 2\xi + \cos \tilde{\theta}) \\ |x - x_2| &= 2U^3 (\cosh 2\xi - \cos \tilde{\theta}) \end{aligned} \quad (4.85)$$

$$t = \tau \quad (4.86)$$

$$\psi = 2(\tau + \sigma),$$

and letting  $\phi$  be the angular coordinate around the axis through the centers (so the coordinates  $(2\xi, \theta, \phi)$  are standard prolate spheroidal coordinates), the metric takes the simple form:

$$ds^2 = (2U)^2 (-\cosh^2 \xi d\tau^2 + \sinh^2 \xi d\sigma^2 + d\xi^2) + U^2 (\sin^2 \tilde{\theta} (d\phi + \tilde{A})^2 + d\tilde{\theta}^2), \quad (4.87)$$

where

$$\tilde{A} = d(\sigma - \tau). \quad (4.88)$$

The general asymptotic form (4.12) is obtained from this by the coordinate transformation  $\xi = \frac{\eta}{2U} - \ln U$ ,  $\tilde{\theta} = \theta$  and taking  $\eta \rightarrow \infty$ .

This metric describes an  $S^2$  fibration over *global*  $\text{AdS}_3$ , with connection  $\tilde{A}$ . The connection is flat except at the origin, where it has a delta function curvature singularity. Hence this is essentially a particular case of the geometries considered in [131, 132].<sup>16</sup> The twist of the sphere around the  $\text{AdS}_3$  boundary

<sup>16</sup> For the case of  $\text{AdS}_3 \times S^3 \times Z$ , i.e. the (4,4) D1-D5 CFT, these geometries were further studied in detail in [133, 134].

circle  $\sigma \rightarrow \sigma + 2\pi$  is given by the Wilson line  $\oint \tilde{A}$ . In this case the twist equals a  $2\pi$  rotation, in accordance with our general considerations under (4.17) and the fact that the angular momentum  $J = p^3/12$  is maximal. Translated to the CFT, this means we have maximal  $SU(2)_R$  charge. Moreover, as explained under (4.17), fermions are periodic around the  $\text{AdS}_3$  boundary circle  $\sigma \rightarrow \sigma + 2\pi$ , so this geometry corresponds, in a semi-classical sense, to a maximally charged R-sector supersymmetric ground state.<sup>17</sup>

Since the twist amounts to a full  $2\pi$  rotation of the sphere, the Wilson line can be removed by a large gauge transformation, that is, a coordinate transformation on the  $S^2$ ,

$$\phi \rightarrow \phi' = \phi + \sigma - \tau, \quad (4.89)$$

which brings the metric to trivial  $\text{AdS}_3 \times S^2$  direct product form, with  $\tilde{A}' = 0$ . In general, large gauge transformations of the bulk act as symmetries (or “spectral flows”) of the boundary theory — in general they map states to physically different states. Here in particular this large gauge transformation will affect the periodicity of the fermions, since a  $2\pi$  rotation of the sphere will flip the sign of the fermion fields. The fermions are then no longer periodic, but antiperiodic around  $\sigma \rightarrow \sigma + 2\pi$  — we are now in the NS sector vacuum of the theory, consistent with the symmetries of global  $\text{AdS}_3$  with  $\tilde{A} = 0$ .<sup>18</sup>

In the dual  $(0, 4)$  CFT, this transformation acts as spectral flow generated by the  $SU(2)_R$  charge  $J_0^3$ . The charges discussed in section 4.2.4 transform under this symmetry as [135]:

$$\begin{aligned} L_0 &\rightarrow L_0, \\ \tilde{L}_0 &\rightarrow \tilde{L}_0 + 2\epsilon J_0^3 + \frac{c}{6}\epsilon^2, \\ J_0^3 &\rightarrow J_0^3 + \frac{c}{6}\epsilon, \end{aligned} \quad (4.90)$$

with  $\epsilon = 1/2$  and  $c = p^3$ . According to our general results (4.63) and (4.70), we get for the original geometry  $L_0 = 0$ ,  $\tilde{L}_0 = p^3/24$ ,  $J_0^3 = -p^3/12$ . Applying the above spectral flow, we obtain  $L_0 = 0$ ,  $\tilde{L}_0 = 0$ ,  $J_0^3 = 0$ , as expected for the NS vacuum.

More general geometries corresponding to states in the NS sector, at least in the case of axially symmetric solutions, can be obtained by applying the spectral flow coordinate transformation (4.89) to the R sector solutions we have constructed.

<sup>17</sup> There is of course a  $2J + 1$  dimensional space of such ground states in the CFT. Correspondingly, on the gravity side, a spin  $J$  multiplet is obtained by quantizing the 2-particle  $D6 - \overline{D6}$  system [97, 29], or equivalently the solution moduli space.

<sup>18</sup> Spelled out in more detail, for a fermion field  $\psi$ , we have in the old coordinates  $\psi(\sigma, \phi, \dots) = \psi(\sigma + 2\pi, \phi, \dots)$ . Expressed in the new coordinates, this boundary condition is  $\psi(\sigma, \phi, \dots) = \psi(\sigma + 2\pi, \phi' + 2\pi, \dots) = -\psi(\sigma + 2\pi, \phi', \dots)$ , where in the last equality we used the fact that  $\phi'$  parametrizes rotations of the sphere.

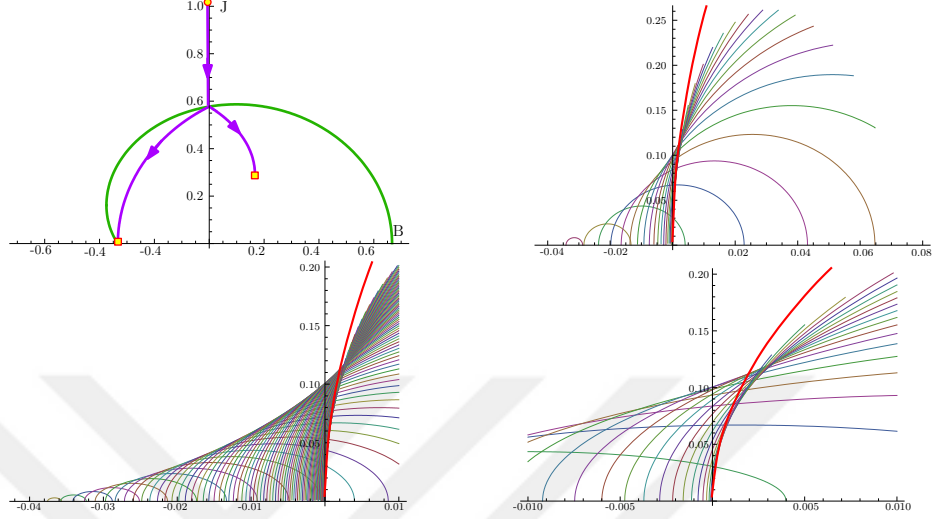


**Figure 4.2** In the upper left figure, the flow tree for the maximally entropic 2-centered configuration at  $h = 0$  is shown (i.e.  $u = 1/3$ ). The other three figures show the total entropy as a function of  $h$  for a number of uniformly spaced values of  $u$  between 0 and  $1/2$ , at three different zoom levels (and different  $u$ -spacings). The fat red line is the entropy of the BTZ black hole with the same total charge.

### 4.3.3 Enigmatic configurations

In [29] it was shown that there are some regions in charge space where the entropy corresponding to given total charges (with zero total D6 charge) is dominated not by single centered black holes, but by multicentered ones. This phenomenon was called the Entropy Enigma. For a short summary see [30].

Interestingly, these enigmatic configurations always survive the decoupling limit, because their walls of marginal stability are compact, with the stable side on the large type IIA volume side. This is to be contrasted with the 4d asymptotically flat case at fixed values of the asymptotic Kähler moduli; in this case, because the unstable region in Kähler moduli space grows with  $p$ , the enigmatic configurations always disappear when  $p \rightarrow \infty$  as the asymptotic moduli will eventually become enclosed by the wall of marginal stability. In this sense, they are most naturally at home in the decoupled  $\text{AdS}_3 \times \text{S}^2$  setup under consideration, where they persist for all  $p$ .

In [29] section 3.4, a simple class of examples was given, consisting of 2-centered bound states with centers of equal entropy. However, this configuration is not the most entropic one for the given total charge: the total entropy can be increased by moving charge from one center to the other. The maximal entropic configuration is obtained when all entropy is carried by one center only; this can be traced back to the fact that the Hessian of the entropy function of a single

black hole has some positive eigenvalues, making multi-black hole configurations generically thermodynamically unstable as soon as charges are allowed to be transported between the centers.

We have not been able to find other, more complicated configurations, involving more centers, with more entropy.

Thus we consider two charges  $\Gamma_i = (p^0, p^A, q_A, q_0)_i$  of the form

$$\Gamma_1 = -e^{-up} = \left[ -1, u, -\frac{u^2}{2}, \frac{u^3}{6} \right] \quad (4.91)$$

$$\Gamma_2 = p - h p^3 - \Gamma_1 = \left[ 1, (1-u), \frac{u^2}{2}, -h - \frac{u^3}{6} \right], \quad (4.92)$$

where we used the notation (4.82). The total charge of this system is

$$\Gamma = (0, p^A, 0, -h p^3) = [0, 1, 0, -h]. \quad (4.93)$$

If the bound state exists, the angular momentum (3.30) and rescaled equilibrium separation (4.35) between the centers are, respectively

$$J = \frac{1}{4}(u^2 - 2h)p^3, \quad |\vec{x}_1 - \vec{x}_2| = 2(u^2 - 2h)p^3. \quad (4.94)$$

The entropy is given by

$$S_{2c} = S_1 + S_2, \quad S_1 = 0, \quad S_2 = \frac{\pi}{3} p^3 \sqrt{8(\frac{1}{2} - u)^3 - 9(\frac{1}{3} - h - u + \frac{u^2}{2})^2}. \quad (4.95)$$

To get a bound state in the decoupling limit, the equilibrium separation in (4.94) must of course be positive and the expression under the square root in (4.95) must be nonnegative. A more detailed analysis using attractor flow trees shows that if we also require  $u \geq 0$ , these conditions are necessary and sufficient. (The latter condition is necessary to prevent the wall of marginal stability to be enclosed by a wall of anti-marginal stability.)

The minimal possible value of  $h$  is  $-\frac{1}{24}$ , reached at  $u = \frac{1}{2}$ , where  $\Gamma_2 = e^{up}$ . This corresponds to the pure fluxed D6– $\bar{D}6$  of section 4.3.2. The maximal value of  $h$  attainable by the configurations under consideration is  $9/128 \approx 0.07$ .

The entropy for a single center of the same total charge (the BTZ black hole of section 4.3.1) is given by

$$S_{1c} = 4\pi \sqrt{\frac{-q_0 p^3}{24}} = \frac{\pi \sqrt{2h} p^3}{\sqrt{3}}. \quad (4.96)$$

One way of phrasing the Entropy Enigma is that in the limit  $p \rightarrow \infty$  keeping  $q_0$  fixed, the 2-centered entropy is always parametrically larger than the 1-centered one,<sup>19</sup> as the former scales as  $p^3$ , while the latter scales as  $p^{3/2}$ . More generally

<sup>19</sup> Note that if  $q_0 > 0$ , there is no single centered black hole, so then this statement is trivially true.

this 2-centered parametric dominance will occur whenever  $h = -q_0/p^3 \rightarrow 0$ . A short computation starting from (4.95) shows that in this limit, the maximal 2-centered entropy is reached at  $u = 1/3$ , with entropy and angular momentum

$$S_{2c} = \frac{\pi p^3}{18\sqrt{3}} \approx 0.100767 p^3, \quad J = \frac{p^3}{36} = \frac{J_{\max}}{3}. \quad (4.97)$$

Indeed this entropy is manifestly parametrically larger than  $S_{1c}$  when  $h \rightarrow 0$ . More precisely the crossover point between one and two-centered dominance is at  $h_c \approx 0.00190622$ . This is illustrated in fig. 4.2. The phase transition this crossover suggests will be discussed further in section 4.4.2.

We should note that we have only analyzed a particular family of 2-centered solutions here. A slight generalization would be to let both centers have nonzero entropy. However this turns out to give a lower total entropy for the same total charge — for example in the symmetric 2-centered case described in [29], the maximal attainable entropy is  $S = \frac{\pi p^3}{48}$ . Similarly for other generalizations such as sun-earth-moon systems, we were unable to find configurations with higher entropy. We cannot exclude however that they exist. If so, this would affect the precise value of the crossover point  $h_c$ , but not its existence.

All of these 2-centered solutions have nonvanishing angular momentum, except in the degenerate limit of coalescing centers, when  $u^2 = 2h$ . In this case the entropy is always less than the single centered one, as it should not to violate the holographic principle. One might therefore suspect that the Entropy Enigma disappears when restricting to configurations with zero angular momentum. This is not the case though. A simple example of a multicentered solution with zero angular momentum but entropy  $S \sim p^3$  is obtained as follows. Instead of one particle of charge  $\Gamma_1 = -e^{-up}$  orbiting around a black hole of charge  $\Gamma_2 = \Gamma - \Gamma_1$ , consider  $k > 1$  particles of charge  $\Gamma_1(u) = -e^{-up}$  orbiting on a halo around a black hole of charge  $\Gamma'_2(k, u, h) = \Gamma(h) - k\Gamma_1(u)$ . Then by positioning the particles symmetrically on their equilibrium sphere around the black hole, we get configurations of zero angular momentum, but with entropy still of order  $p^3$  at large  $p$ . This can be extended quantum mechanically: quantizing the halos as in [97, 29], we get a number of spin zero singlets from tensoring  $k$  spin  $j$  single particle ground states.

Note that the entropy of the  $k$ -particle configuration at given  $u$  and  $h$  can be related to that of our original  $k = 1$  solution by

$$S(k, u, h) = S(\Gamma'_2(k, u, h)) = \frac{1}{k} S(\Gamma'_2(1, ku, k^2 h)) = \frac{1}{k} S(\Gamma_2(ku, k^2 h)). \quad (4.98)$$

The equilibrium separation between a  $\Gamma_1(u)$  particle and the  $\Gamma'_2$  core, for given  $h$  and  $u$ , does not depend on  $k$ , so

$$x_{12}(k, u, h) = x_{12}(1, u, h) = \frac{1}{k^2} x_{12}(1, ku, k^2 h). \quad (4.99)$$

From these relations, we can immediately deduce the existence conditions and maximal entropy configuration for  $k > 1$  particles using the results for the  $k = 1$

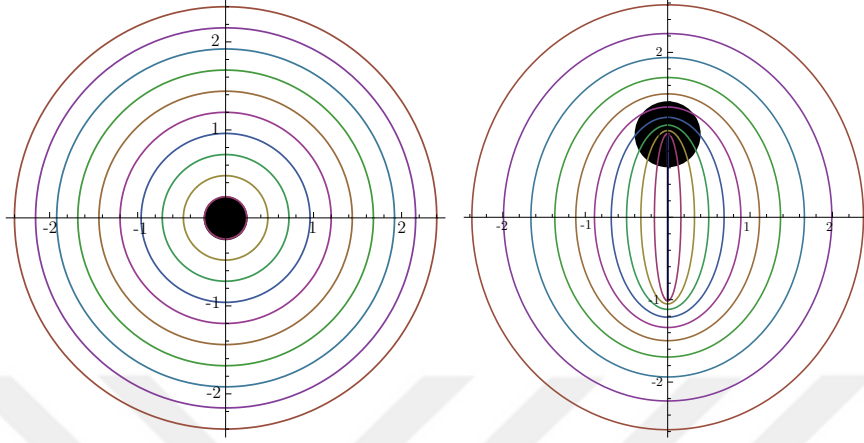


**Figure 4.3** On the left a representation is shown of the single centered 4d black hole; this lifts to the BTZ black hole (times  $S^2$ ) at the center of  $AdS_3$ . Surfaces of constant spherical coordinate  $r$  in  $\mathbb{R}^3$  are indicated — these become the  $S^2$  fibers of  $AdS_3 \times S^2$ . On the right one of the 2-centered 4d configurations of section 4.3.3 is depicted; this lifts to a BMPV-like black hole roughly localized on the north pole of the  $S^2$  and at the center of  $AdS_3$ . Surfaces of constant prolate spheroidal coordinate  $\xi$  are indicated. As is clear from (4.87), these are the  $S^2$  fibers of  $AdS_3 \times S^2$  in the zero size limit of the black hole at the north pole, i.e. the  $R$  vacuum. When the black hole has finite size, the metric near it will be deformed to that of a BMPV black hole in 5 dimensions.

case derived above. In particular we see that the entropy is maximized at  $u = 1/3k$ , and for e.g.  $h = 0$  equal to

$$S_{(1+k)c} = \frac{1}{k} \cdot \frac{\pi p^3}{18\sqrt{3}}. \quad (4.100)$$

Note that due to the factor  $k$  in the denominator, the  $k \geq 2$  (possible spin zero) configurations are thermodynamically disfavored compared to the  $k = 1$  (necessarily spinning) configurations.

## 4.4 Demystifying the Entropy Enigma

### 4.4.1 Interpretation as black hole localization on the sphere

From the discussion in section 4.3.3, it transpires that the entropy “enigma” is in fact nothing but a supersymmetric version of a well known general instability phenomenon in the (nonsupersymmetric) microcanonical ensemble on  $AdS_p \times S^q$ , first pointed out in [37]: Schwarzschild-AdS black holes become

thermodynamically unstable once their horizon radius shrinks below a critical value of the order of the AdS radius — at this point it becomes entropically favorable at the given energy to form a Schwarzschild black hole localized on the  $S^q$ . Related thermodynamical as well as dynamical instabilities were studied in [115, 116, 117, 118, 119, 120, 121, 122, 123, 124] and other works.

We see something very similar here: when the BTZ black hole radius is lowered below a critical value of the order of the AdS radius, it becomes thermodynamically unstable — at this point it is entropically favorable at the given energy and total charge to form a BMPV-type BPS black hole [114] localized on the  $S^2$ , which is precisely the “enigmatic” configuration studied in the previous subsection. This is illustrated in fig. 4.3.

In particular, we see now that the statement that multicentered black holes dominate the entropy in the small  $h$  regime is somewhat misleading. From the 4d point of view, the (presumably) dominant solution described in section 4.3.3 is two centered, with one zero entropy, pure fluxed D6 center; a naked timelike singularity. But from the 5d point of view, there is really only one black hole, since the 4d D6 singularity lifts to smooth geometry. Thus, the dominant configuration remains a single black hole — just one that is localized on the sphere.

To the best of our knowledge, this is the first instance of such an instability in a supersymmetric setting. As we spelled out in this article, the presence of supersymmetry makes it possible to write down completely explicit solutions, which is not possible in general nonsupersymmetric cases studied before. This might make explorations of this phenomenon as well as its dual CFT description more tractable.

#### 4.4.2 Phase transitions

As suggested by figure 4.2 and the discussion in the previous subsection, the microcanonical ensemble exhibits a phase transition in the  $p \rightarrow \infty$  limit. By microcanonical ensemble we mean more precisely here the statistical ensemble at fixed total charge  $\Gamma = (0, p^A, q_A, q_0)$  and fixed total energy saturating the BPS bound, but variable  $S^2$  angular momentum. Thus we introduce a potential  $\omega$  dual to say the 3-component  $J^3$ . For concreteness we further specialize to the situation of section 4.3.3, putting  $q_A = 0$  and  $q_0 = -hp^3$ .

Let us assume that, as our analysis suggest, the entropy below a critical value  $h = h_c$  is indeed dominated by the black hole localized on the  $S^2$ , while for  $h > h_c$  it is dominated by the BTZ black hole. Since the localized black holes have macroscopic angular momentum, we see that in the limit  $p \rightarrow \infty$  keeping  $\omega$  fixed, we get

$$\langle J^3 \rangle = \pm j_*(h) \quad (h < h_c), \quad \langle J^3 \rangle = 0 \quad (h > h_c), \quad (4.101)$$

where  $j_*(h)$  is the angular momentum of the most entropic configuration and the sign is determined by the sign of  $\omega$ . This is illustrated in fig. 4.4. If we assume

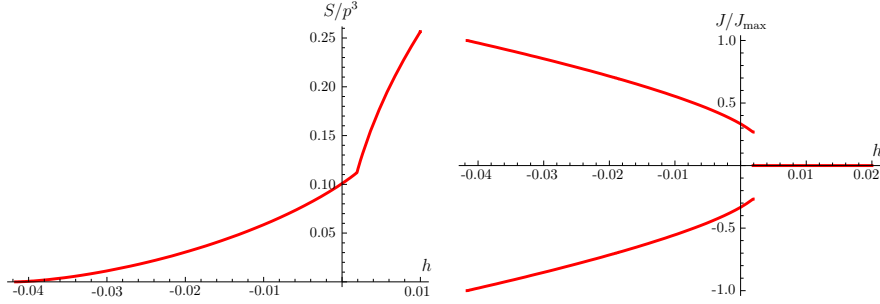


**Figure 4.4** Left: Entropy as a function of  $h$  in the limit  $p \rightarrow \infty$ . Right:  $J^3/J_{\max}$  as a function of  $h$  (the branch depending on the sign of  $\omega$ ), for  $p \rightarrow \infty$ .

either BTZ or single sphere localized black holes dominate, the critical value is  $h_c \approx 0.00190622$ , and in the large  $p$  limit, we have a sharp first order phase transition, with order parameter given by the angular momentum. However as we mentioned before, although we were unable to find any, we cannot exclude the existence of more complicated, more entropic multi-black hole / particle gas configurations which would push up  $h_c$ , and perhaps even smoothen the entropy and angular momentum as a function of  $h$ , changing the order of the phase transition.

We can also consider the “canonical” ensemble, trading  $-q_0$  for its dual potential  $\beta = 1/T$  while still keeping the  $q_A$  fixed (say  $q_A \equiv 0$ , which for simplicity of exposition we assume from now on), and keeping the total energy at BPS saturation.<sup>20</sup> As we will see below, in the dual CFT,  $T$  has an interpretation as the “left-moving temperature”, conjugate to  $(L_0)_{\text{cyl}} = H = hp^3 = -q_0$  (see section 4.5.1), while the constraint of BPS saturation can be enforced by taking the right-moving temperature  $\tilde{T} \rightarrow 0$ . Although  $T$  is strictly speaking not a real temperature, we will use terminology as if it were. The relation between  $h$  and  $T$  and the free energy are given by the Legendre transform

$$\frac{1}{T} = \frac{\partial S}{\partial H} = -\frac{\partial S}{\partial q_0}, \quad F = H - TS. \quad (4.102)$$

For the BTZ black hole, (4.96) thus gives

$$h(T) = \frac{(2\pi T)^2}{24}, \quad F(T) = -\frac{\pi^2 T^2}{6} p^3. \quad (4.103)$$

This means the BTZ black hole charge at thermal equilibrium is

$$\Gamma(T) = (0, p, 0, -h(T)p^3). \quad (4.104)$$

<sup>20</sup> As in the microcanonical ensemble we still allow the angular momentum  $J$  to vary and work at fixed  $\omega$ , but we will suppress this in the explicit formulae below — its only effect in the end at  $p \rightarrow \infty$  is to select a low temperature ground state.

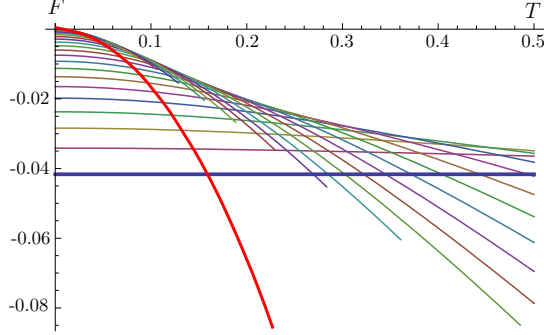


**Figure 4.5** Free energy  $F$  as a function of  $T$  in the limit  $p \rightarrow \infty$  for BTZ (the curved fat line) and sphere localized black holes at different values of  $u$  ranging from 0 to  $1/2$ . The bottom fat line corresponds to  $u = 1/2$ , that is,  $\text{AdS}_3 \times \text{S}^2$  without black holes. The end points of the black hole lines correspond to the 4d equilibrium separation and angular momentum becoming zero, i.e. becoming indistinguishable from BTZ in the asymptotic region.

For the localized black holes of section 4.3.3 we get more complicated expressions. The localized black hole charge and entropy in thermal equilibrium are, using the notation (4.82):

$$\Gamma_2 = \left[ 1, (1-u), \frac{u^2}{2}, \frac{(1-2u)^{3/2}}{3(\pi^2 T^2 + 1)^{1/2}} - \frac{u^3}{6} - \frac{u^2}{2} + u - \frac{1}{3} \right], \quad (4.105)$$

$$S_2 = \pi^2 T \frac{(1-2u)^{3/2} p^3}{3(\pi^2 T^2 + 1)^{1/2}}. \quad (4.106)$$

The resulting free energies as a function of  $T$  are shown in fig. 4.5. Again we see a phase transition in the large  $p$  limit: above a certain temperature  $T_c$ , the BTZ black hole minimizes the free energy due to its large entropy; below it the spinning global  $\text{AdS}_3 \times \text{S}^2$  vacuum (4.87) (with  $J^3 = \pm \frac{p^3}{12}$ ) takes over, as dumping energy into the reservoir becomes entropically favorable. (Both phases will also contain a thermal gas of particles, since we have coupled the system to a heat bath.) The free energy of the vacuum ( $u = 1/2$ ) is easy to compute as it has zero entropy:  $F_{\text{vac}} = H_{\text{vac}} = -\frac{p^3}{24}$ . By equating this with the BTZ free energy we get the critical temperature:

$$T_c = \frac{1}{2\pi}, \quad (\text{in units of } \frac{1}{R}). \quad (4.107)$$

This phase transition is nothing but (a BPS version of) the Hawking-Page transition [136]. Its existence in a supersymmetric context was observed already in [126], by examining the elliptic genus of the Hilbert scheme of  $k$  points on  $K3$  and its  $\text{AdS}_3 \times \text{S}^3 \times K3$  dual. Here we see its physical origin more directly.

Note that again the angular momentum jumps: from  $\langle J^3 \rangle = 0$  at  $T > T_c$  to  $\langle J \rangle = \pm J_{\max} = \pm \frac{p^3}{12}$  at  $T < T_c$ . The AdS-CFT correspondence therefore implies a phase transition in the dual 1+1 dimensional CFT breaking the continuous  $SU(2)_R$  symmetry. This is not in contradiction with the Coleman-Mermin-Wagner theorem [137, 138], since there is only a true phase transition in the strict limit  $p \rightarrow \infty$ . At any finite  $p$ , the combined free energy is smooth.

In any case, we are led to conclude that BTZ black holes much smaller than the AdS radius in fact do not provide stable classical ( $p \rightarrow \infty$ ) backgrounds representing macroscopic (thermodynamic) states in the CFT. This is just as well, as the opposite situation would lead to various paradoxes. For example, according to the philosophy of e.g. [25, 139, 140, 26], the BTZ black hole, when it exists as a proper classical geometry, should be obtained by coarse graining over all microstates of given energy or temperature, consistent with its interpretation as a purely thermal state [82]. However, when the BTZ black hole is small, it is hard to see how it could be the result of coarse graining over the ensembles of multicentered configurations, which typically extend far beyond the BTZ horizon size.

We end this subsection by giving an alternative way to arrive at the critical temperature (4.107). Let us start from the pure fluxed D6 –  $\overline{D6}$  system studied in section 4.3.2. Now add a number  $N$  of D0-branes (which according to (4.35) have to lie on the plane equidistant from the D6 and  $\overline{D6}$ ). This is essentially the setup of [34], which will be discussed in some detail in section 5.2. It was shown there that the D0-branes together with the D6 and anti-D6 can adiabatically<sup>21</sup> collapse into a scaling solution (or abyss) which approaches the single centered D4-D0 black hole arbitrarily closely, if and only if

$$N \geq \frac{p^3}{12} \quad , \quad \text{i.e.} \quad h = \frac{N - \frac{p^3}{24}}{p^3} \geq \frac{1}{24} .$$

This is in fact a direct consequence of the equilibrium constraints (4.35). In the  $\text{AdS}_3 \times S^2$  picture, what we have is a gas of gravitons and other massless modes orbiting at constant radius in  $\text{AdS}_3$  and at fixed  $\phi$  on the equator of  $S^2$ , which can adiabatically collapse into a BTZ black hole if  $h > \frac{1}{24}$ . From the relation (4.103) between  $T$  and  $h$ , this is equivalent to  $T > \frac{1}{2\pi}$ , coinciding with the critical temperature (4.107).

Thus, below the critical temperature  $T_c$ , there is a potential barrier preventing adiabatic gravitational collapse into a BTZ black hole, above  $T_c$ , there isn't. This is quite satisfying, and suggests that it might be impossible altogether to adiabatically create a BTZ black hole at temperatures below  $T_c$ , or in other words that such black holes cannot be “deconstructed” as scaling solutions. Since scaling solutions are natural candidates for the fuzzballs of the fuzzball proposal (for reviews see [25, 26]), this suggests there might be no D4-D0 fuzzballs for  $T < T_c$ . Of course, the observation we made here is not conclusive,

<sup>21</sup> By adiabatic we mean here by a evolution process with energy arbitrarily close to the BPS bound.

as there might well be other scaling solutions which do deconstruct the BTZ black hole below the Hawking page phase transition temperature  $T_c$ . It would be interesting to investigate this further.

## 4.5 Interpretation in the $(0, 4)$ CFT

We will now discuss the interpretation in the dual CFT of the Entropy Enigma and other phenomena we observed.

### 4.5.1 Translation to CFT

The quantum numbers of the decoupled solutions were given in section 4.2.4. In particular,  $L_0$  and  $\tilde{L}_0$  were given in (4.63), and we also defined reduced quantum numbers  $L'_0$  and  $\tilde{L}'_0$  in (4.66). In the regime  $L'_0 \gg \frac{c}{24}$ , the Cardy formula gives the microcanonical entropy of the CFT:

$$S_{\text{Cardy}} = 4\pi \sqrt{\frac{c}{24}(L'_0 - \frac{c}{24})} = 4\pi \sqrt{-\frac{\hat{q}_0 p^3}{24}} = S_{\text{BTZ}}, \quad (4.108)$$

where  $c = p^3$ , reproducing precisely the BTZ black hole entropy. Note that the regime where sphere localized black holes come to dominate is at  $(L'_0 - \frac{c}{24})/c \ll 1$ ; this is the opposite of the Cardy regime.

In both the microcanonical and the canonical ensembles we consider in the previous section, we kept the M2 charge  $q_A$  fixed and for simplicity we chose

$$q_A = 0. \quad (4.109)$$

We will do this here too. In this case the distinction between reduced and original Virasoro charges disappears, and we have the identifications

$$(L_0)_{\text{cyl}} = L_0 - \frac{c}{24} = -q_0 = hc, \quad (\tilde{L}_0)_{\text{cyl}} = \tilde{L}_0 - \frac{c}{24} = 0. \quad (4.110)$$

This implies furthermore  $H = hc$ , explaining our notation  $q_0 = -hc$  used in (4.93) and in the definition of the canonical ensemble in section 4.4.2.

The regime of particular interest to us is  $h$  small and positive, which is where the phase transitions are expected to occur based on the black hole picture.

### 4.5.2 Entropy for $L_0 \sim \frac{c}{24}$

There are not too many tools available to determine the number of states in a CFT for  $h = (L_0 - \frac{c}{24})/c \rightarrow 0$ . There is certainly no universal answer to this question, and in addition the answer may depend on moduli and other parameters — after all, it is not a protected quantity. In order for the  $\mathcal{N} = (0, 4)$  CFT, which is dual to the geometries we have been studying, to accommodate the sphere localized / multicenter solutions with entropy  $S \sim p^3 = c$  near  $h \rightarrow 0$ ,

the number of states at small  $h$  in the CFT should grow accordingly. One can view this as a prediction of AdS/CFT for the (presumably strongly coupled)  $\mathcal{N} = (0, 4)$  CFT.

The simplest possible model where one could investigate this question is in the CFT of  $c$  free bosons, which has partition function  $Z := \text{Tr } q^{L_0 - \frac{c}{24}} = Z_1^c$  where

$$Z_1 = q^{-\frac{1}{24}} \prod_{i>0} \frac{1}{(1 - q^i)} = \frac{1}{\eta(q)}. \quad (4.111)$$

Then the coefficient of  $q^0$  can be estimated at large  $c$  by a saddle point approximation. Parametrizing  $q = e^{2\pi i \tau}$ :

$$d(0) = \oint e^{c \log Z_1} d\tau \approx e^{c \log Z_1(\tau_*)}, \quad \frac{\partial \log Z_1}{\partial \tau} \Big|_{\tau_*} = 0. \quad (4.112)$$

The numerical solution to this is

$$\tau_* \approx 0.523524, \quad \log d(0) \approx 0.176491 c, \quad (4.113)$$

so this indeed gives an entropy of order  $c = p^3$  at  $h = 0$ . Comparing to (4.97), we see that the coefficient is different; of course there was no reason to expect it to be the same, since the coefficient is model dependent. For example, replacing  $Z_1$  with a more general weight  $w$  modular form

$$Z_1(q) = a_0 q^b + a_1 q^{b+1} + \dots, \quad (4.114)$$

we can estimate (4.112) by writing  $Z_1(\tau) = a_0 (-i\tau)^{-w} e^{-\frac{2\pi i b}{\tau}} + \dots$  which leads to

$$\tau_* \approx \frac{2\pi i b}{w}, \quad \log d(0) \approx (\log a_0 - w(1 + \log(2\pi b/w))) c. \quad (4.115)$$

For this to be a good approximation we need  $e^{-\frac{2\pi i}{\tau_*}} = e^{-w/b} \ll 1$ . For the free boson, we have  $w = -1/2$  and  $b = -1/24$ , so this is satisfied and indeed plugging in the numbers gives  $\log d(0) \approx \frac{1}{2}(1 + \log \frac{\pi}{6})c$ , reproducing (4.113) to very good accuracy.

In addition to similar saddle point approximations, a more refined analysis of the large  $c$  growth of  $d(0)$  for various modular forms, using the Fareytail expansion, was done in [141], and in agreement with the simple estimates given here.

Of course, since  $c$  is a measure for the number of degrees of freedom, it is hardly a surprise that the entropy for a fixed nonzero amount of energy per degree of freedom  $L_0/c = 1/24$  grows linearly in the number of degrees of freedom  $c$ . More interesting would be to compute the actual proportionality constant. Despite the model dependence of this number, (4.97) nevertheless suggests a universal number for all CFTs dual to  $\text{AdS}_3 \times S^2 \times \text{CY}_3$  in the large  $c$  limit:

$$\log d(0) = \frac{\pi}{18\sqrt{3}} c. \quad (4.116)$$

As mentioned earlier, this universality might however be an artifact of our lack of imagination in finding more entropic configurations.

In theories in which a “long string” picture exists, we can count the number of states in the long string CFT, which typically has reduced central charge  $\hat{c} = c/k$  and increased excitation energy  $\hat{L}_0 = kL_0$ . For  $k$  sufficiently large, we can then use Cardy even if the original  $L_0$  was of the order of  $c/24$ , and we find

$$\log d(0) = \frac{\pi}{6} c. \quad (4.117)$$

This does not agree with (4.116), but clearly our analyses on both sides are far from conclusive at this point.

To make further progress, it is necessary to delve into the intricacies of the actual dual CFTs. We will initiate this in the next subsection, improving the analysis of [36] by more carefully identifying entropic modes important at small  $h$ .

### 4.5.3 The MSW string

#### Content and supersymmetry conditions

As we already reviewed in section 2.6, the MSW (0,4) 1+1 dimensional sigma model on  $W = \mathbb{R} \times S^1$  arising from wrapping an M5 brane on  $W \times P$  with  $P$  a very ample divisor has the following massless field content [36, 88, 142]:

- $h^{0,2}(P) \approx p^3/6$  complex non-chiral scalars  $z^i$  arising from holomorphic deformations of  $P$ .<sup>22</sup>
- 3 real scalars  $\vec{x}$ , the position in  $\mathbb{R}^3$
- $b^2(P) \approx p^3$  real scalars from the reduction on  $P$  of the self-dual 2-form field  $b$  on the M5:

$$b = b^\alpha \Sigma_\alpha, \quad (4.118)$$

where  $\{\Sigma_\alpha\}$  is an integral basis of the space of harmonic 2-forms  $H^2(P)$ . In such a basis the scalars are periodic:  $b^\alpha \simeq b^\alpha + n^\alpha$ ,  $n^\alpha \in \mathbb{Z}$ . Furthermore they have to satisfy the self-duality constraint

$$db^\alpha \wedge \Sigma_\alpha = *_W db^\alpha \wedge *_P \Sigma_\alpha, \quad (4.119)$$

which implies there are  $b_+^2(P) = 2h^{2,0}(P) + 1 \approx p^3/3$  right-moving ( $*_W = +1$ ) degrees of freedom and  $b_-^2 = h^{1,1}(P) - 1 \approx 2p^3/3$  left-moving ( $*_W = -1$ ). The left-right split depends on the deformation moduli  $z^i$  and the background complex and Kähler moduli.

<sup>22</sup> Consistent with our practice in the rest of the paper, we suppress (large  $p$ ) subleading corrections to various Hodge numbers.

- $4h^{2,0}(P) + 4 \approx 2p^3/3$  real right-moving fermions  $\psi^\kappa$ . These pair up with the in total  $4h^{0,2}(P) + 4$  real right-moving scalars, as required by (0, 4) supersymmetry.

Motion of the string is supersymmetric if and only if it is (almost) purely left-moving<sup>23</sup> :

$$z^i(\tau, \sigma) = z^i(\tau + \sigma), \quad b^\alpha(\tau, \sigma) = b^\alpha(\tau + \sigma) - 2(q \cdot \tilde{J})\tilde{J}^\alpha \tau. \quad (4.120)$$

Here  $q \cdot \tilde{J} = q_A \tilde{J}^A$  with  $q_A$  the M2-charge and  $\tilde{J} = \tilde{J}^A D_A$  proportional to the Kähler form of  $X$ , normalized such that  $\int_P \tilde{J}^2 \equiv 1$ . Furthermore the components  $\tilde{J}^\alpha$  are defined by decomposing  $\tilde{J}$  pulled back to  $P$ :  $\tilde{J} = \tilde{J}^\alpha \Sigma_\alpha$ . The reason for the presence of the  $\tau$ -dependent term on the right hand side is the fact that supersymmetry is nonlinearly realized when  $q \cdot \tilde{J}$  is nonvanishing [36], which is related to the fact that  $q \cdot \tilde{J}$  is proportional to the imaginary part of the central charge  $Z$ , and therefore that a different subset of four supercharges out of the original eight is preserved for different  $q \cdot \tilde{J}$ . It is also closely related to the difference between  $L_0$  and  $L'_0$  as discussed at the end of section 4.2.4.

In addition (4.120) is a solution to the equations of motion if and only if the selfduality constraint (4.119) is satisfied. On the profile  $(z^i(s), b^\alpha(s))$ ,  $s \in S^1$  introduced in (4.120) this constraint becomes the anti-selfduality condition

$$\dot{b}^\alpha(s) \Sigma_\alpha - (q \cdot \tilde{J})\tilde{J} = - * [(\dot{b}^\alpha(s) \Sigma_\alpha) - (q \cdot \tilde{J})\tilde{J}]. \quad (4.121)$$

The dot denotes derivation with respect to  $s$ , and we used the fact that the right-moving contribution in (4.120) automatically obeys the self-duality constraint (4.119). Harmonic 2-forms on  $P$  are anti-selfdual if and only if they are of type  $(1, 1)$  and orthogonal to  $\tilde{J}$ . Following appendix G of [29], the first condition can be written as

$$\dot{b}^\alpha(s) \partial_i \Pi_\alpha(z(s)) = 0, \quad (4.122)$$

while the second one is

$$\dot{b}^\alpha(s) \tilde{J}_\alpha = q \cdot \tilde{J}. \quad (4.123)$$

Here  $\Pi_\alpha(z)$  is the period of the holomorphic 3-form on a 3-chain with one boundary on the 2-cycle in  $P(z)$  Poincaré dual to  $\Sigma_\alpha$ , and  $J_\alpha$  is the integral of the Kähler form  $J$  over the same 2-cycle.

### Supersymmetric solutions

One could now try to get the BPS spectrum by quantizing this moduli space of supersymmetric configurations. In general however this is a complicated system of coupled equations.

<sup>23</sup> As usual, the the extra winding term in  $b^\alpha$  can be written, using  $\tau = \frac{1}{2}(\tau + \sigma) + \frac{1}{2}(\tau - \sigma)$  as the sum of left-movers and right-movers, and the left-moving contribution can be absorbed in  $b^\alpha(\tau + \sigma)$ . We chose for convenience a convention in which the winding term depends on  $\tau$  only.

Things simplify when we consider linearized oscillations around some arbitrary fixed point  $(z_*^i, b_*^\alpha)$ . Because there are about  $p^3 b^\alpha$  and  $p^3/3 z^i$  real degrees of freedom, (4.122)-(4.123) will to lowest order just constrain the  $b^\alpha$  to lie on a  $2p^3/3$ -dimensional plane, while  $\delta z^i$  can oscillate freely. Hence we can think of this as in total  $p^3$  free bosonic modes. At large  $L_0$ , these oscillator modes will dominate the entropy, approximately reproducing the BTZ entropy.

In addition, since they are periodic, we can allow the scalars  $b^\alpha$  to have nonzero winding number  $k^\alpha$  in  $H^2(P)$ ; this corresponds to turning on world-volume flux on the M5 (and in particular these modes can therefore carry M2 charge). Still at fixed  $z_*$ , integrating (4.122) over the  $S^1$  then gives the constraint

$$\partial_i W(z_*) = 0, \quad W(z) := k^\alpha \Pi_\alpha(z). \quad (4.124)$$

For generic  $z_*$  and generic integral  $k^\alpha$ , this will not be satisfied. Only for  $k^\alpha$  in the sublattice  $L_X$  of  $H^2(P, \mathbb{Z})$  pulled back from the ambient Calabi-Yau  $X$ , this will be automatic (because these forms are always integral (1,1)).

Based on this and the fact that in the full M-theory, M2 instantons can interpolate between winding numbers except those in  $L_X$ , [36] rejected the possibility of turning on winding numbers except for those in  $L_X$ . However, at *special* points  $z_*$ , (4.124) *will* have solutions. Indeed these equations can be viewed as a superpotential critical point condition for  $z^i$  (formally identical to the one obtained for D4 flux vacua in appendix G of [29]), and as such it will have isolated critical points for sufficiently generic  $k^\alpha$ ; all  $z^i$  have become effectively massive. Integrating 4.123 over  $S^1$  gives the constraint  $k^\alpha J_\alpha = q \cdot J$ . This is automatically satisfied, because the winding modes are exactly the origin of the M2 charge, as they correspond to M5 worldvolume flux; in general one can read off from the WZ terms in the M5-brane effective action that  $q_A = \int_P D_A \wedge k^\alpha \Sigma_\alpha$ .

So, once we specify a winding vector  $k^\alpha$ , the string will still be supersymmetric when located at a critical point  $z_*(k)$ , and some or all of the  $z^i$  zeromodes will be lifted. At the semiclassical level, these are definitely valid supersymmetric ground states — and in fact there is a huge number of them, not quite unlike the landscape of string flux vacua. Instantons might tunnel between them and mix the states quantum mechanically, but this does not mean that they should not be considered; in particular when computing the Witten index, all these semiclassical vacua must be summed over (with signs).

The contribution of these winding modes to  $-q_0 = P = (L_0)_{\text{cyl}} - (\tilde{L}_0)_{\text{cyl}}$  is half the topological intersection product:

$$\Delta P = -\frac{1}{2} Q_{\alpha\beta} k^\alpha k^\beta, \quad Q_{\alpha\beta} := \int_P \Sigma_\alpha \wedge \Sigma_\beta. \quad (4.125)$$

If in addition to (4.124) we also set  $q \cdot J = 0$  (for example by restricting to the  $q_A = 0$  sector), then  $k^\alpha \Sigma_\alpha$  is anti-selfdual, and therefore  $\Delta L_0 = \Delta P \geq 0$ . Moreover, in the notation of section 4.2.4, we have  $\Delta L'_0 \geq 0$ .

There are more complicated solutions to (4.122) possible, for example when we let the string loop around a nontrivial closed path  $z(s)$  in the divisor moduli space and at the same time on some loop in the  $b^\alpha$ -torus. This can give

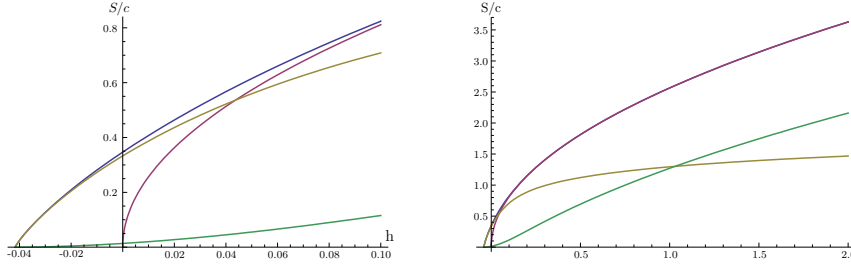


**Figure 4.6** Various entropies as a function of  $h$ , for  $h$  near 0 (left), and for a larger range of  $h$  (right). The blue line is the total entropy derived from (4.126), the yellow line is the entropy in the winding modes, the green line is the entropy in the oscillator modes, and the red line is the BTZ entropy.

rise to complicated twisted sectors. As stressed in [88], there will in general be monodromies  $b^\alpha \rightarrow M^\alpha_\beta b^\beta$  acting on the  $b$ -torus when circling around the discriminant locus in the divisor moduli space. Hence we should think of the target space of the string as a quotient of the total space of the  $b$ -torus fibration over Teichmüller space by the monodromy group. Closed strings can begin and end on different points identified by this group, leading to twisted sectors and possibly long strings.

Finally, we can form bound states of the localized winding strings described above. For example we can form a bound state of a closed string winding  $k_1$  at some  $z_*(k_1)$  and one winding  $k_2$  at  $z_*(k_2)$ , by connecting them with two interpolating pieces of string. Note though that now the constraint (4.123) becomes important: indeed generically  $k_1^\alpha J_\alpha \neq k_2^\alpha J_\alpha$ , so the string we just described cannot have constant  $\dot{b}^\alpha J_\alpha$  and we do not get a proper supersymmetric solution. It is conceivable however that in some cases at least the string will be able to relax down to a BPS configuration for which  $\dot{b}^\alpha J_\alpha$  is constant everywhere.

This is reminiscent of brane recombination. Moreover, note that the condition of having  $k_1^\alpha J_\alpha = k_2^\alpha J_\alpha$  corresponds to being on a wall of marginal stability for the two M5-branes represented by the two strings. Hence there is an obvious candidate for the gravitational interpretation of such configurations: they should correspond to the M5-M5 2-centered bound states of section 3.4.2. It would be interesting to make this more precise, but this is beyond the scope of the paper.

### Statistical mechanics

In this subsection we will give a rudimentary analysis of the statistical mechanics of the BPS sector of the MSW string, to see if we can reproduce some of the features we found on the black hole side.

We can roughly model the ensemble of winding and oscillator modes ignoring

nonlinearities, say in the  $q_A = 0$  sector, by the partition function

$$Z(q) = \text{Tr } q^{L_0 - \frac{c}{24}} = \left( \frac{\vartheta_3(q)}{\eta(q)} \right)^c \quad (4.126)$$

with  $c = p^3$ . Here the theta function models the winding mode contributions and the eta function the oscillator contributions.<sup>24</sup> By numerical saddle point evaluation, the total entropy and the (entropy maximizing) distribution of it over the oscillator and winding modes at given  $h = (L_0 - \frac{c}{24})/c$  can be straightforwardly computed. The result is shown in fig. 4.6. The inclusion of winding modes actually improves the match to the BTZ entropy compared to the most naive model with only free oscillators; it is almost perfect already slightly above the threshold. This can also be checked analytically: Because  $Z(q)$  has weight 0, the total entropy computed by saddle point evaluation is exactly  $S = 4\pi\sqrt{\frac{h}{24}}c = S_{\text{BTZ}}$ ; for the free oscillator model, there are corrections.

We also see that at  $h = 0$ , there is still an entropy of order  $c = p^3$ , and almost all of it is in the winding modes. There are still no phase transitions in this model of course, since the system is noninteracting.

Let us turn our attention now to the  $SU(2)_R$   $R$ -charge  $J^3$ ; the  $S^2$  angular momentum on the gravity side, which appeared as an order parameter  $J^3/p^3$  for the phase transition we discussed. The only fields transforming nontrivially under  $SU(2)_R$  are (i) the fermions, transforming in the **2**, but they are all rightmoving so cannot be excited except for their zeromodes, and (ii) the position  $\vec{x}$  transforming in the **3**, but this represents only three oscillators out of order  $c = p^3$ , so one expects their contribution to the total  $R$ -charge to be negligible in the thermodynamic limit  $p \rightarrow \infty$  (in the sense of their  $J^3$  having an expectation value growing slower than  $p^3$ ).

So, where does the large angular momentum,  $J = \frac{p^3}{12}$ , of the  $L_0 = 0$  gravity solution come from then? The answer is from the center of mass zero modes of the string. Since shifting the  $b^\alpha$  by constants independent of the string coordinate  $s$  corresponds to a gauge transformation, the only physical zero mode space is the deformation moduli space  $\mathcal{M}_P$  of  $P$ . These bosonic zero modes together with the fermionic ones (which we can have since they are independent of  $s$ ) will give ground state wave functions in one to one correspondence with harmonic differential forms on  $\mathcal{M}_P$ . The fermion number corresponds to fermion number and therefore to  $R$ -charge — or in other words the  $SU(2)_R$  is identified with Lefschetz  $SU(2)_R$  on cohomology (see for example [97] for a pedagogical explanation). This is analogous to how angular momentum is produced in the D4-brane model [29]. Since the moduli space  $\mathcal{M}_P = \mathbb{C}P^{p^3/6}$  (where as before

<sup>24</sup> Note that despite the fact that turning on winding modes is generically lifting *zeromodes* of  $z^i$ , it is not true that it also lifts the oscillator modes; in the presence of winding, it remains true that (4.122) reduces the number of *local* fluctuation (oscillator) degrees of freedom by  $p^3/3$ , so at our level of approximation the oscillator mode counting is essentially unaffected by winding: the number of oscillating degrees of freedom remains  $p^3/3 + p^3 - p^3/3 = p^3 = c$ .

we are dropping terms subleading to  $p^3$ ), this means the  $L_0 = 0$  ground states assemble into a spin  $J = \frac{p^3}{12}$  multiplet, exactly as expected from the gravity side.

Now, when we turn on some small  $L_0$ , we expect from what we observed on the gravity side that  $J$  will go down somewhat (see fig. 4.4). We propose the following picture of how this happens on the CFT side. At very small  $L_0$ , a small number of winding modes will get turned on. This will typically freeze a small number of the moduli  $z^i$ , reducing the moduli space  $\mathcal{M}_P$  to a lower dimensional space. The maximal Lefschetz spin always equals half the complex dimension  $n$  (this is the spin of the multiplet created by starting with 1 and subsequently wedging with the Kähler form on the moduli space till the volume is reached). Therefore the maximal  $J$  will go down. The higher  $L_0$ , the more winding modes get turned on, the smaller the dimension of the residual moduli spaces, and the smaller  $J$ . Eventually when  $L_0$  becomes sufficiently large, so many winding modes will be turned on that all moduli will generically be frozen, and the expectation value of  $J$  becomes zero. This is in agreement with what we observe on the gravity side.

Again, this is only a rudimentary qualitative picture, and in particular too rough to be able to address how phase transitions could arise. Perhaps a variant of the toy models of [143] would be of help to make further progress. A more in depth analysis is left for future work.

### The field theory description of the MSW string

One puzzle we have encountered several times in the paper has to do with the nature of the MSW sigma model which describes the low-energy excitations of the wrapped M5-brane. This sigma model is obtained from a suitable KK reduction of the M5-brane theory over the four-cycle over which the M5-brane is wrapped. Classically, this sigma model is a  $(0, 4)$  superconformal field theory, and the target space of the sigma model is the entire moduli space of supersymmetric four-cycles in the Calabi-Yau manifold.

The puzzle is that on the one hand, field theory arguments suggest that this sigma model also describes a quantum  $(0, 4)$  superconformal field theory which still probes the entire moduli space of supersymmetric four-cycles, whereas the bulk analysis shows that not all M5-brane bound states fit into a single asymptotically  $\text{AdS}_3 \times S^2$  geometry, which strongly suggests that a quantum SCFT which captures the entire moduli space does not exist.

The field theory arguments are based on claims in the literature that, unlike  $(2, 2)$  sigma models,  $(0, 4)$  sigma models are always finite [144, 145], in the sense that all renormalizations can be absorbed in finite field redefinitions, so that in particular the beta functions vanish and the theory is conformal also quantum mechanically. However there are potential caveats [146], to which in turn some counterarguments have been given in [147]; see also [148]. To the best of our knowledge, this issue remains not fully settled.

Perhaps our results shed some new light on this. As we observed in section 4.2, M5-M5 bound states of the type constructed in appendix 3.4.2 will not

fit in a single asymptotically  $\text{AdS}_3 \times S^2$  geometry, but split in two (or more) separated  $\text{AdS}_3 \times S^2$  throats. At values of the normalized Kähler moduli  $Y^A$  sufficiently far away from the AdS attractor point  $Y^A = p^A/U$ , they do exist as supersymmetric states of the MSW string, and we suggested a possible explicit MSW string realization of them above. When moving the  $Y^A$  to the attractor point, all of these states decay. Hence they cannot be part of the CFT which is dual to a single  $\text{AdS}_3 \times S^2$  geometry.

There are therefore, in our view, two possibilities:

1. The MSW sigma model is a quantum SCFT for all values of the Kähler moduli  $Y^A$ . If so, it is not equivalent to quantum gravity in asymptotic  $\text{AdS}_3 \times S^2 \times X$ , and therefore presents a situation very different from the usual AdS-CFT lore. It is not clear to us what the precise new prescription for a correspondence would be in this case.
2. The beta function in fact does not vanish for  $Y^A$  different from the attractor point and the  $Y^A$  undergo RG flow till they reach the attractor point, an IR fixed point. Along the flow, the constituents of M5-M5 bound states (whose gravity description is of the type studied in appendix 3.4.2) decouple from each other; each of them has its own IR fixed point corresponding to an  $\text{AdS}_3 \times S^2$ .

The second possibility seems much more attractive to us, but would imply that the MSW  $(0, 4)$  model does undergo RG flow. This need not be in contradiction with the finiteness of  $(0, 4)$  models, since the relevant non-renormalization theorems assume that the sigma model is weakly coupled and non-singular, and both assumptions are almost certainly violated for the MSW  $(0, 4)$  model. The latter can become strongly coupled whenever two-cycles in the moduli space shrink to zero volume (similar to what happens in the D1-D5 CFT), and is most likely singular when the four-cycle self-intersects: intersecting M5-branes support extra light degrees of freedom, coming from stretched M2-branes, and these need to be taken into account in a proper low-energy description. The classical MSW CFT, however, does not take these additional light degrees of freedom into account, and usually this gives rise to singularities in the incomplete low-energy theory. Finally, the nontrivial interaction between the  $b^\alpha$  and  $z^i$  modes leading to (4.122), will further complicate the RG flow.

It would be interesting to study this further.

## 4.6 Conclusions and discussion

In this chapter we described a large number of supersymmetric bound states of black holes and black rings in a space-time which is asymptotic to  $\text{AdS}_3 \times S^2 \times \text{CY}_3$ . M-theory on the latter space is supposedly a well-defined theory of quantum gravity and is equivalent to a particular  $\mathcal{N} = (0, 4)$  CFT in two dimensions, the MSW CFT at the attractor point. Therefore, one can hopefully

ask more precise questions (and provide more precise answers) about these black hole bound states than one can do in asymptotically flat space. In particular, no states will come in from or move out to infinity, and the number of states with given quantum numbers should be unambiguous. Indeed, the moduli of the CY are completely fixed at the attractor point at the boundary of AdS. Exactly how and when the low-energy description of wrapped M5-branes, which exists for all values of the moduli, flows to this  $(0, 4)$  SCFT at the attractor point remains puzzling. We already discussed this in section 4.5.3 and here we will simply take it as a given fact.

So given the  $(0, 4)$  CFT dual of  $\text{AdS}_3 \times S^2 \times \text{CY}_3$ , a first obvious question that arises is to determine the dual description of the black hole bound states in the CFT. This was one of the original motivations of this project, but clearly more work remains to be done to be able answer this question. Black hole/ring bound states should be described by suitable density matrices that include several generalized chemical potentials. The simplest example where this can be made explicit is the black ring, which has an extra chemical potential multiplying a dipole moment operator, both in its thermodynamic description [149, 150] as well as in the dual density matrix description [151]. The black hole bound states we have been considering should clearly involve many more chemical potentials. Each of the centers is described by  $n = 2(b_2 + 1)$  different charges, and the total entropy depends on  $kn$  quantum numbers, with  $k$  the number of centers. Therefore, in the first law of thermodynamics for the black hole bound states, we expect to see at least  $kn$  different chemical potentials appearing. It would be very interesting to find a nice basis for these chemical potentials and to determine to which CFT operators they couple. Though some insight can in principle be obtained by studying the subleading behavior of the supergravity solutions near the AdS boundary, lack of detailed knowledge of the  $\mathcal{N} = (0, 4)$  CFT makes it difficult to proceed in this direction. Qualitatively we expect that as we turn on more and more chemical potentials we can describe increasingly more complicated black hole bound states. In the limit where the number of chemical potentials goes to infinity, we can resolve individual microstates in the CFT and find the corresponding microstate geometries.

An alternative approach to understanding the CFT duals of the multi-centered black holes is to use their description in terms of attractor flows. As we have discussed, we expect to be able to associate a unique flow tree to a given supergravity solution. There are several subtleties which may invalidate this statement:

- There exist multicentered “scaling” or “abyss” solutions [29, 97, 99], for which the centers’ coordinates can be partitioned in groups in such a way that within each group the centers can approach each other arbitrarily closely. These are most naturally viewed as being continuously connected to a solution where each group is collapsed into a single center, and as such should be associated to the attractor flow tree of the latter solution. Unfortunately, the split attractor flow conjecture only addresses the existence

of the collapsed solution, but does not say anything about the existence of a particular scaling solution.

- It is not completely clear what the right starting point for the attractor flow tree should be, since the moduli vary on the boundary of  $\text{AdS}_3$ . More precisely, it is the  $B$ -field that varies, see equation (4.74).
- As discussed in section 3.4.1, for special values of the moduli charges can lie on a wall of “threshold” stability. If this happens, attractor flow trees can be continuously deformed into each other.

If we nevertheless assume the split attractor flow conjecture to hold, then it naturally leads to a partitioning of the Hilbert space of the CFT. The most obvious guess for the CFT dual to a particular black hole bound state would then be a density matrix that involves all the states that live in the sector of the CFT corresponding to the associated flow tree. However, we have not yet been able to make either this description or the description in terms of chemical potentials very explicit, nor have we been able to understand the phase transitions we encountered directly in the CFT. It would clearly be very interesting to make progress in any of these directions.

We would also like to understand in more detail the connection between the multi-centered solutions and the fuzzball proposal (for reviews see e.g. [25, 26]). Roughly, the idea is that the space of smooth BPS solutions of the supergravity equations of motion with given charges forms a phase space<sup>25</sup>, and that quantization of this phase space precisely reproduces the BPS states of the dual CFT. For 4d multicenter solutions such a proposal was put forward in [107]. If this works, one can establish a precise connection between density matrices in the CFT and coarse grained bulk geometries, and in particular black holes are the result of coarse graining over a large number of underlying microstate geometries [139, 140]. Before any of these ideas can be tested, we need to be able to quantize the moduli space of solutions. This is a problem of independent interest and will be discussed in [96]. As we will show there, quantization of the moduli space of solutions does lead to results that are in agreement with the wall-crossing formula of [29], at least in the two- and three-center cases, which is suggestive but certainly not enough to establish the validity of the fuzzball proposal. An obvious problem is that it is not clear that we can avoid the use of excitations that non-trivially involve the Calabi-Yau manifold (for the simpler case of  $1/2$  BPS states in  $\text{AdS}_3 \times \text{S}^3 \times K3/T^4$  one already needs excitations that involve  $K3/T^4$  to account for the total number of states, but they are under partial technical control, see e.g. [152]) nor is it clear that one can avoid the use of stringy excitations. In fact, we do not even know whether we have here the full set of half-BPS solutions of 5d  $N = 1$  supergravity. All our solutions

<sup>25</sup> The notion of smoothness is observer dependent, but for multi-centered solutions a minimal criterion is that each of the centers should represent a single state and not carry any entropy. Typically, the centers will therefore have to be single branes that carry fluxes only.

have a  $U(1)$  isometry, and there may well exist many more solutions that have no spatial isometries at all.

The quantization of the moduli space in [96] will also enable us to discuss issues related to the possible non-compactness of the space of solutions and to the aforementioned multicentered “scaling” or “abyss” solutions.

One of the issues that our work sheds some light on is the issue of giant gravitons in  $\text{AdS}_3$ . Originally, these were thought not to give rise to BPS states [153] because, unlike in other dimensions, their radial position is a free parameter. However, in [154, 155] this issue was re-analyzed and it was found that there are bound giant gravitons in global  $\text{AdS}_3$  but not in Poincaré coordinates. This analysis was done for  $\text{AdS}_3 \times S^3$ , but a similar result can be seen to hold for  $\text{AdS}_3 \times S^2$ : a giant graviton in a BTZ black hole background is described by a two-center solution, each of which carries D4D2D0 charge only. If the inner product of the two charges does not vanish, there are no supersymmetric solutions, and if it does vanish, the two centers are mutually BPS and do not form a bound state. Therefore, we never obtain BPS states in this way. A giant graviton in global  $\text{AdS}$ , however, is in our setup described by a three center solution, consisting of a pure fluxed D6 brane, a pure fluxed  $\bar{\text{D}}6$  brane, and a center with D4D2D0 brane charge only. As we reviewed in section 4.3.2 following [34], the pure fluxed D6 and  $\bar{\text{D}}6$  solution is equivalent to global  $\text{AdS}_3 \times S^2$ . Thus after a suitable change of coordinates this three center solution does describe a single giant graviton in global  $\text{AdS}_3$ . It is indeed a bound state, as the original three centers do form non-trivial bound states. Thus, we can understand bound states in global  $\text{AdS}_3$  by adding a pure fluxed D6 brane and a pure fluxed  $\bar{\text{D}}6$  brane and by considering this extended configuration in this framework. It would be interesting to explore this further and to understand in more detail the precise action of spectral flow on multiple bound states.

We also found the need to improve the notion of walls of marginal stability to distinguish walls where the number states jumps (still called walls of marginal stability) from walls where the topology of the flow changes but the number of states does not jump (called walls of threshold stability). This distinction is of particular importance in  $\text{AdS}_3$  as the moduli at infinity are fixed and can in fact lie on a wall of threshold stability. A detailed understanding of these two types of walls and their applications to state counting problems and stability questions of bound states of branes is clearly desirable but left to future work.

Among the many other open problems that remain we would like to mention the applications of our results to refine the computations of the elliptic genus of the  $\mathcal{N} = (0, 4)$  CFT [125, 129]. To leading order this partition function roughly looks like  $|Z_{\text{top}}|^2$ , but this contribution is entirely coming from two-center configurations, and by including three and more centers we should be able to make a more detailed study of the corrections that arise. These results would then carry over to the OSV conjecture restricted to infinite Kähler moduli and configurations without D6-brane charge since in this limit the BPS index reduces to the elliptic genus of the CFT.

Coming back to the description of the multi-centered black holes in the dual CFT, we would like to raise a few more points. First, one may ask the following question: which geometry is dual to the density matrix that consists of all states with fixed  $L_0$ ? If this is a single geometry, then it has to be spherically symmetric (because the density matrix is rotationally invariant), and the only spherically symmetric solution is the BTZ black hole. For large values of  $L_0$  this seems fine, since the entropy of the BTZ black hole agrees with the entropy of the CFT computed using the Cardy formula in this regime. However, for smaller values of  $L_0$  this is no longer true, as the two-centered solutions start to dominate the entropy. This shows that for small values of  $L_0$ , there cannot be a single, semiclassically reliable geometric dual of the density matrix consisting of all states. This is quite surprising, and it shows that for small  $L_0$  the appropriate bulk dual description of this very simple density matrix should be in terms of a sum over geometries. The small BTZ black hole is then dual to a density matrix which contains only a small subset of the total number of states. In [29] various arguments for and against were presented that the number of states in this small subset is equal to the index of the total number of states. This would require a large amount of cancellation in the index (some preliminary numerical evidence for this was presented in [156]) and it would be interesting to explore this further. We have not found any obvious mechanism for this cancellation in the space of multi-centered solutions.

It is also worth pointing out that a better understanding of the CFT description of the multi-centered solutions would probably allow us to give a CFT explanation of the entropy of the BMPV black hole: we would simply count the number of CFT states dual to the two-centered solution described in section 4.3.3, consisting of a BMPV black hole and an (entropyless) purely fluxed  $\overline{\text{D6}}$ .

Finally, it is well-known that moving into the interior of  $\text{AdS}_3$ , and ignoring the outside region, is like RG flow in the dual field theory. The moduli in the interior of  $\text{AdS}_3$  will approximately follow the attractor flow tree. Therefore, we should be able to understand the attractor flow tree, and the corresponding rearrangement of the degrees of freedom in seemingly disconnected pieces, from the point of view of the RG flow. In other words, as we lower the scale, we should encounter phase transitions whenever we cross a wall of marginal stability in which the degrees of freedom of the CFT split up into a tensor product of decoupled sectors. The mechanism responsible for this could be quite similar to tachyon condensation, as it is a tachyonic degree of freedom which is responsible for the decay of BPS states [97]. If we could make such a "split RG flow" picture more precise, we would in particular be able to explain the entropy of all 4d black holes in terms of a 2d CFT. We hope to come back to this in the near future.

# Chapter 5

## Black Hole Deconstruction

In this last chapter the authors work in collaboration with Frederik Denef, Davide Gaiotto, Andrew Strominger and Xi Yin [34] is presented. Before we cover the content of that paper we will first review some of the physics underlying and leading to it. The main motivation for [34] comes from an alternative derivation of the D4D0 entropy in [33].

As we reviewed in chapter 2, the first microscopic calculation of the D4D0 black hole entropy was made by MSW in [36]. Here the authors made use of the connection between IIA/M-theory and 4d/5d black holes/strings to relate the microstates of the black hole to states in a 2 dimensional (0,4)-supersymmetric CFT. The regime in which the CFT description is valid is however opposite to the gravitational regime. The 'number' of states, or to be more precise an index called the elliptic genus, is guaranteed to be independent of the regime it is computed in and so the CFT counting remains valid in the supergravity regime. But although we know the number of states in the supergravity regime by this duality the precise form or interpretation of these states remains elusive in the gravitational regime.

In the D1D5 system of [22], the original analogue of MSW with 16 supercharges, there exists the so called *fuzzball proposal* for an interpretation of the microstates in the supergravity regime, see e.g. [25, 26] for a review. The fuzzball idea amounts to associating to the microstates a smooth, i.e horizonless, supergravity solution that carries the same conserved charges as the black hole. In case of the D1D5 system this approach has proven to be partially successful.

In the system of our interest, the D4D0 black hole, the fuzzball proposal has never been precisely formulated and it is an open problem to construct and count the precise supergravity microstates. A natural guess would be to consider multicentered configurations of zero entropy centers with the same charges as a single centered black hole. Some progress along these lines is made in e.g. [32].

A first important step and very nice alternative interpretation of the mi-

crostates on its own was made in [33]. In this paper Gaiotto, Strominger and Yin (GSY) reproduced the D4D0 black hole entropy by counting probe states in the near-horizon geometry. This idea of interpreting the black hole microstates as states in some near horizon quantummechanics go back to [157, 158] but was never completely successful till [33, 159]. GSY rederive the D4D0 black hole entropy by interpreting the black hole as a fixed background D4-charge in whose near horizon region the D0's occupy a degenerate set of groundstates, this groundstate degeneracy then reproduces the black hole entropy. The idea of keeping the magnetic charges fixed and interpreting the entropy as the degeneracy of the possible electrically charged states is essentially based on the insights of [35].

The crucial ingredient in [33, 160] is to conjecture that due to a generalization of the Myers effect [161] the largest part of the D0 groundstates are of a non-abelian nature and can be represented by D2-branes wrapping the horizon. These puffed up D0's are then electrically charged particles in the magnetic field sourced by the D4 branes. Analogous to a simple electron in a magnetic field there is a large Landau-level groundstate degeneracy. This groundstate degeneracy combined with the different ways the D0's can blow up into D2's then leads to a total degeneracy that exactly reproduces the entropy of the macroscopic D4D0 black hole.

Despite the very intuitive picture it provides and the great success of the exact matching of the entropy, the GSY approach also involves some less understood arguments and leads to some unanswered questions. The near horizon states that are counted are only supersymmetric in the near horizon region of the black hole and do not preserve any of the supersymmetries of asymptotically flat space, contrary to the macroscopic D4D0 black hole. The resolution of this apparent paradox is not clear, for a more detailed analysis of this 'problem' we refer to [162]. Related to this fact is that the wrapped D2-branes carry a central charge absent for single D0's. The appearance of this central charge implies they have a non-vanishing groundstate energy while the abelian D0's have zero energy in the same background.

There are two other less understood steps in the GSY derivation that are probably less problematic from a conceptual point of view. The first is of course the conjectured Myers effect which is not explicitly understood in this setup, a better understanding could be directly linked to a solution of the above paradoxes. The second subtlety is that the groundstates GSY count are groundstates of the Hamiltonian related to evolution with respect to global time of the near horizon region. This is counterintuitive as the 'natural time' of the near horizon region, that becomes the standard time coordinate in the asymptotic flat region, is the Poincaré time coordinate of the near horizon  $\text{AdS}_2$ . As we will see below this idea of using the generator of global time rather than that of Poincaré time as Hamiltonian is very natural from the conformal quantummechanics point of view. Still an argument from first principles justifying this assumption is needed.

This last point is one of the issues addressed in the second section of this

chapter. In [34] a slightly refined version of [33] is proposed. The idea is to replace the D4 'core' by a bound state of a purely fluxed D6 and a purely fluxed anti-D6. The total system is then a multicentered configuration consisting of the D6,  $\overline{\text{D6}}$  and D0's. As we will show, for the charges of interest this system has a scaling point. At this scaling point the multicentered configuration becomes indistinguishable from a single centered black hole for an outside observer. Inside the infinitely deep throat the system retains its multicentered features however. This geometry inside the throat interpolates between a Poincaré patch of  $\text{AdS}_3$  at infinity but global  $\text{AdS}_3$  in the interior. So in this setup the asymptotic flat time does connect to global time in the interior AdS geometry, making the ad hoc choice of Hamiltonian the natural one. Furthermore we show that there exist supersymmetrically wrapped 2-branes in this geometry giving all the ingredients to repeat the original counting of GSY in this new setup.

As a bonus the system now consists completely out of parts that each have zero entropy on their own and so there are no more horizons in the geometry. This feature relates this refined way of counting the black hole entropy very much to the fuzzball proposal. A difference that remains however is that in our setup the black hole microstates are only considered as probes, to really make a precise connection to the fuzzball proposal a backreacted version of these probe states would be needed. Finding these backreacted solutions remains an interesting open problem. Some more recent work building on the deconstruction can be found in [163, 164].

In the first section of this chapter we will review the original GSY derivation of the entropy. We will try to balance once more between providing the necessary details while trying to keep brief. As a result we will only point out some of the more important features and refer to the original literature for more details.

In the second section we then present the authors work [34] which further develops the GSY proposal.

## 5.1 Derivation of the D4D0 entropy ala GSY

The main idea behind the GSY proposal [33] was already sketched above. Summarized GSY reproduce the entropy of the D4D0 black hole by counting the different possible quantum mechanical groundstates corresponding to D2-branes wrapping a D4-black hole. The D0 charge of the black hole is then reproduced by worldvolume flux on these D2's that induces D0-charge. Such D2's can probably be interpreted as an effective description of non-Abelian D0's through a version of the Myers effect [161].

What is crucial in the calculation is that the quantummechanics of the D2-branes obtains a conformal symmetry as we assume them to be in the near-horizon region [165]. So in the first subsection we will spend some time reviewing some aspects of superconformal quantummechanics that will be relevant later.

In the second subsection we come to the calculation itself. We start with constructing the bosonic part of the D2-quantummechanics. Using this bosonic

information and some knowledge about the supersymmetry of the system we will then explain the counting of the groundstates. These 'groundstates' are more precisely chiral primaries of the superconformal quantummechanics and can be shown to be classified by harmonic forms on the Calabi-Yau. Using some index theory we can calculate their degeneracy. The last step is then to count the possible ways of partitioning the total D0-charge of the black hole amongst the different chiral primaries. This partitioning leads to an exponential degeneracy that reproduces the entropy of the black hole exactly.

### 5.1.1 Some sightseeing in (super)conformal quantummechanics

In this subsection we review some of the tools from conformal quantummechanics that will be of use when we will study a D2 probe in the next subsection. We start by defining what is precisely meant by a conformal quantummechanics and point out a subtlety about the dynamics. This subtlety involving the 'naive' Hamiltonian having a continuous spectrum was already noticed in the original work of [166] and will be of particular relevance to us later. After having introduced conformal quantummechanics we discuss the specific example of a particle moving in an AdS space-time. We end by presenting D(2,1;0), a specific supersymmetric extension of the conformal algebra that will be relevant in the next subsection.

#### Groundstates in conformal quantummechanics

Let us start with the simplest possible model of a conformal quantummechanics. Consider the following Hamiltonian

$$H = \frac{p^2}{2m} + \frac{g}{2x^2}. \quad (5.1)$$

In the original paper on conformal quantummechanics [166] it was noted that one can construct the following additional two operators

$$D = \frac{1}{2}\{p, x\}, \quad K = \frac{mx^2}{2}. \quad (5.2)$$

Together the three above operators form an  $\text{SL}(2, \mathbb{R})$  algebra:

$$\begin{aligned} [D, H] &= 2iH, \\ [D, K] &= -2iK, \\ [H, K] &= -iD. \end{aligned} \quad (5.3)$$

More generally any quantummechanical system where apart from the Hamiltonian  $H$  there exists two additional operators  $K$  and  $D$  such that those three operators satisfy the algebra (5.3), is called a *conformal quantummechanics*.

As noted in [166] such a conformal quantummechanics is rather odd from a physical point of view when formulated in the above way. This is due to the fact that the Hamiltonian  $H$  has a continuous spectrum of eigenstates. One can easily verify this by checking that for every eigenstate  $|E\rangle$  there exists another state  $|\lambda\rangle = e^{i\lambda D}|E\rangle$  which is an eigenstate of energy  $e^{2\lambda}E$ . More physically this continuum of states is due the fact that the potential term in (5.1), i.e.  $V(x) = \frac{g}{2x^2}$ , has runaway behaviour.

The authors of [166] also suggested a remedy to the above features. Rather than  $H$  one could consider the operator  $L_0$  as the generator of quantummechanical evolution, with

$$L_0 = \frac{1}{2} \left( aH + \frac{1}{a}K \right), \quad (5.4)$$

where  $a$  is some constant length. This new 'Hamiltonian'  $L_0$  has the desired discrete spectrum as the new 'potential' is now given by  $\tilde{V}(x) = aV + K/a = \frac{ga}{2x^2} + \frac{mx^2}{2a}$ , which has a stable groundstate. The discreteness of the spectrum of  $L_0$  can be shown explicitly by defining the operators

$$L_{\pm 1} = \frac{1}{2} \left( aH - \frac{1}{a}K \mp iD \right). \quad (5.5)$$

In terms of these new operators  $L_m$  the  $\text{SL}(2, \mathbb{R})$  algebra (5.3) takes the so called Virasoro form:

$$[L_m, L_n] = (m - n)L_{m+n}, \quad (5.6)$$

for  $m \in \{-1, 0, 1\}$ . From these commutation relations (5.6) one can read that the operators  $L_{-1} = L_1^\dagger$  and  $L_1$  act as respectively creation and annihilation operators on the eigenstates of  $L_0$ .

Adopting this approach we will search for the smallest eigenvalue eigenstates of  $L_0$  when looking for 'groundstates'.

### Interpretation on AdS space

The above change of the relevant time generator might appear rather ad hoc. There is however a very natural interpretation of this choice in case the conformal quantummechanics is that of a particle in AdS space [165, 167].

In Poincaré coordinates the metric of  $\text{AdS}_2$  has the form [84]

$$ds^2 = -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2. \quad (5.7)$$

The lagrangian for a particle moving freely in this  $\text{AdS}_2$  space-time is then

$$L = -m\sqrt{\frac{r^2}{R^2} - \frac{R^2}{r^2}\dot{r}^2}, \quad (5.8)$$

Note that we made a gauge choice by taking the worldvolume coordinate  $\sigma$  to be  $\sigma = t$ , this is the static gauge in Poincaré coordinates. For simplicity we make a non-relativistic approximation  $\dot{r} \ll \frac{r^2}{R^2}$  and use the coordinate  $r = \frac{R^3}{\xi^2}$ . The Hamiltonian for (5.8) then becomes

$$H = \frac{P_\xi^2}{2m} + \frac{mR^2}{\xi^2}. \quad (5.9)$$

A first thing to note is that this Hamiltonian is exactly that of (5.1), furthermore it is the conserved charge corresponding to the symmetry generated by  $\partial_t$ . Because  $\text{AdS}_2$  has an  $\text{SL}(2, \mathbb{R})$  isometry group there are two other Noether charges  $K$  and  $D$ , those are<sup>1</sup>

$$K = \frac{m\xi^2}{2R^2}, \quad D = \frac{1}{2}(P_\xi \xi + \xi P_\xi). \quad (5.10)$$

After canonically quantizing this system one can simply check that the operators  $H, K$  and  $D$  satisfy the  $\text{SL}(2, \mathbb{R})$  algebra (5.3). So we see that the quantummechanics of a particle on  $\text{AdS}_2$  is a conformal quantummechanics.

As was argued above the more physically well behaved Hamiltonian for such a conformal quantummechanics is actually  $L_0 = RH + K/R$ . In this specific example of quantummechanics on an  $\text{AdS}$  space the operator  $L_0$  has an interesting and very natural interpretation. Instead of working in the Poincaré coordinates (5.7) let us write the  $\text{AdS}_2$  metric in so called global coordinates [84]

$$ds^2 = -R^2 \cosh^2 \rho d\tau^2 + R^2 d\rho^2. \quad (5.11)$$

Contrary to Poincaré coordinates, which only cover half of  $\text{AdS}_2$ , the global coordinates cover all of  $\text{AdS}_2$ . By calculating the conserved charge corresponding to the isometry generated by  $\partial_\tau$  one sees that this is nothing but the operator  $L_0$  defined above<sup>2</sup>. So we see that in this example changing the Hamiltonian from  $H$  to  $L_0$  corresponds to considering time evolution with respect to the global time  $\tau$  instead of Poincaré time  $t$ . So the choice of  $L_0$  over  $H$  as the more 'natural' Hamiltonian has now the interpretation of global time  $\tau$  being more 'natural' than Poincaré time. This indeed makes sense as Poincaré time is only defined on part of  $\text{AdS}_2$  where  $\tau$  covers all of space-time.

Physics with time evolution in  $\tau$  or  $t$  respectively is quite different. For some intuition we refer to figure 5.1.1, where the world lines corresponding to a particle static in respectively  $t$  and  $\tau$  are shown. If one furthermore interprets the  $\text{AdS}_2$  as the near-horizon geometry of an extremal black hole then one sees that particles static with respect to  $t$  stay at a fixed distance from the horizon  $r = 0$ . The particles static with respect to  $\tau$  however continuously oscillate in and out of the 'horizon'.

<sup>1</sup> Note that the two additional Noether charges are actually time dependent, it is their time independent part which we refer to with  $K$  and  $D$  respectively.

<sup>2</sup> Again we ignore some subtleties involving the time dependence of the various Noether charges.

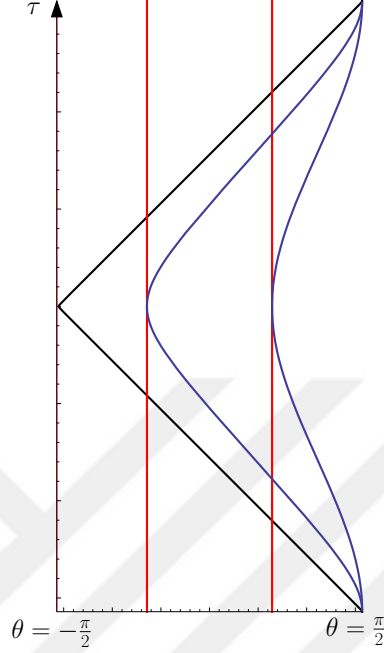


**Figure 5.1** Here a conformal compactification of  $AdS_2$  is shown. The coordinate  $\theta$  is related to  $\rho$  by  $\tan \theta = \sinh \rho$ , for more info see [84]. The red vertical lines are curves of constant  $\rho$  and are the trajectories of particles static with respect to  $\tau$ . The black lines correspond to  $r = 0$ , i.e. the horizon in case we interpret  $AdS_2$  as a near horizon metric, they bound the Poincaré patch. The blue curves are lines of constant  $r$  and thus worldlines static with respect to the Poincaré time  $t$ . As this picture shows particles static in  $t$  sit at a fixed distance from the 'horizon' while particles static in  $\tau$  leave and enter the 'horizon'.

### The $D(2,1;0)$ superconformal algebra

Apart from conformal symmetry the situation of our interest will have also some supersymmetry. The algebras that combine conformal symmetry and supersymmetry are called *superconformal* algebras. One of their features is that the supercharges always come in pairs under the conformal symmetry. We will stick to just introducing the algebra that will be of relevance to us, more information can be found in e.g. [168].

As will be motivated later, the superconformal algebra we will encounter is the  $D(2,1;0)$  algebra. This superconformal algebra contains of course the conformal algebra

$$[L_m, L_n] = (m - n)L_{m+n} \quad m, n \in \{-1, 0, 1\}. \quad (5.12)$$

As other bosonic symmetries it has two  $SU(2)$  algebras which we call left and right respectively:

$$[J_{R/L}^+, J_{R/L}^-] = 2J_{R/L}^3, \quad (5.13)$$

$$[J_{R/L}^3, J_{R/L}^\pm] = \pm J_{R/L}^\pm. \quad (5.14)$$

The  $D(2, 1; 0)$  algebra has  $N = 4$  superconformal symmetry, i.e. it has four regular supercharges and four conformal supercharges. Those in total eight supercharges can be conveniently labeled as  $G_r^{\alpha A}$  where  $\alpha, A \in \{+, -\}$  and  $r \in \{-\frac{1}{2}, \frac{1}{2}\}$ . Here the index  $\alpha, A$  corresponds to respectively the left or right  $SU(2)$  spin of the supercharge  $G_r^{\alpha A}$  and the index  $r$  is its conformal weight:

$$\begin{aligned} [J_R^3, G_r^{\pm A}] &= \pm \frac{1}{2} G_r^{\pm A}, \\ [J_R^\alpha, G_r^{\beta A}] &= |\epsilon^{\alpha\beta}| G_r^{\alpha A}, \\ [J_L^3, G_r^{\alpha\pm}] &= \pm \frac{1}{2} G_r^{\alpha\pm}, \\ [J_L^A, G_r^{\alpha B}] &= |\epsilon^{AB}| G_r^{\alpha A}, \\ [L_m, G_r^{\alpha A}] &= (r + \frac{m}{2}) G_{m-r}^{\alpha A}. \end{aligned} \quad (5.15)$$

Finally the anticommutation relations among the fermionic generators are

$$\begin{aligned} \{G_{\pm\frac{1}{2}}^{\alpha A}, G_{\pm\frac{1}{2}}^{\beta B}\} &= 2\epsilon^{\alpha\beta}\epsilon^{AB} L_{\mp 1}, \\ \{G_{\frac{1}{2}}^{\alpha A}, G_{-\frac{1}{2}}^{\beta B}\} &= 2\epsilon^{\alpha\beta}\epsilon^{AB} L_0 + 2\epsilon^{AB} T^{\alpha\beta} + \epsilon^{\alpha\beta} |\epsilon^{AB}| Z. \end{aligned} \quad (5.16)$$

Here we introduced the notation

$$T^{++} = J_R^+, \quad T^{--} = -J_R^-, \quad T^{+-} = T^{-+} = -J_R^3. \quad (5.17)$$

To be precise the anti-commutation relations above are those for a central extension of the standard  $D(2, 1; 0)$  algebra, the central charge is given by the positive constant  $Z$ .

Those readers familiar with superconformal algebras can check that the above (anti-)commutation relations imply that  $D(2, 1; 0) = SU(1, 1|2) \times SU(2)_L$ . In case of a non-vanishing central charge  $Z$  the  $SU(2)_L$  subalgebra is broken and the algebra of preserved symmetries is  $SU(1, 1|2)_Z$ .

### 5.1.2 D2 superconformal quantummechanics

In this subsection we summarize the results leading to the microscopic calculation of the entropy by GSY in [33]. To come to their beautiful conclusion quite some physical (and mathematical) results are needed. We will sketch the different steps here and refer for additional results, details and subtleties to the original literature. Most of this information and additional references can be found in [33, 160].

As was explained in the introduction to this chapter [33] reproduce the macroscopic entropy by counting groundstates of a D2-brane that has induced D0 charge and wraps a D4-black hole. This counting is possible by first noting that due to the  $SL(2, \mathbb{R})$  isometries of the near-horizon geometry the D2-brane quantummechanics is a conformal quantummechanics. Keeping also track of the supersymmetries the quantummechanics actually has  $\mathcal{N} = 4$  superconformal symmetry. The groundstates of such superconformal theories are so called *chiral primaries*. These chiral primaries are analogues to BPS states in more familiar supersymmetric theories; these states are annihilated by a number of supersymmetries implying they are represented in a short multiplet.

Using some general knowledge of the relevant superconformal algebra, the fermionic spectrum and the explicit form of the bosonic Hamiltonian, [33] were able to construct the chiral primary wavefunctions explicitly. This then shows that the chiral primaries are highly degenerate due to the presence of the magnetic field induced by the D4-brane in the background. The D2 is an electrically charged particle in that magnetic field and the degenerate groundstates are known from standard electro-magnetism as lowest Landau levels. The number of these lowest Landau levels can be explicitly calculated using some cohomological results as this is essentially a topological property of the magnetic field. The final step in computing the entropy is then counting the number of ways a D4D0 black hole can be represented by D2-branes that each sit in such lowest Landau levels. As we will see this counting coincides with the entropy of the D4D0 black hole.

### A D2-brane in a D4D0 attractor geometry

We will start our review by deriving the bosonic Hamiltonian for a D2-brane wrapping the horizon of a D4D0 black hole.

The supergravity solution corresponding to such a D4D0 black hole of charge  $\Gamma = (0, p^A, 0, q_0)$  is a special case of the supersymmetric black holes given by (2.102). In case we choose the moduli at infinity to satisfy  $B_\infty^A = 0$  the harmonic functions take the form

$$H^A = \frac{p^A}{r} + h^A, \quad H_0 = \frac{q_0}{r} + h_0. \quad (5.18)$$

In terms of these harmonic functions the full supergravity solution reads

$$\begin{aligned} ds_{4d}^2 &= - \left( -\frac{2}{3} D_{ABC} H^A H^B H^C H_0 \right)^{-\frac{1}{2}} dt^2 + \left( -\frac{2}{3} D_{ABC} H^A H^B H^C H_0 \right)^{\frac{1}{2}} dx^i dx^i, \\ A^0 &= \frac{dt}{2H_0}, \\ A^A &= p^A \cos \theta d\phi, \\ J^A &= \sqrt{\frac{6H_0}{D_{ABC} H^A H^B H^C}} H^A. \end{aligned}$$

From this expression one can easily find the four dimensional attractor geometry. For later convenience we give the full ten-dimensional attractor solution, including the internal Calabi-Yau manifold:

$$\begin{aligned} ds_\star^2 &= -\frac{r^2}{R^2}dt^2 + \frac{R^2}{r^2}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) + g_{a\bar{b}}dz^a d\bar{z}^{\bar{b}}, \quad (5.19) \\ A_\star^0 &= \frac{r}{2q_0 G_4}dt \\ A_\star^{(3)} &= p^A \cos\theta d\phi \wedge D_A, \\ J_\star &= \sqrt{\frac{-6q_0}{p^3}} p^A D_A. \end{aligned}$$

Here  $R = (-\frac{2}{3}p^3 q_0)^{1/4} \sqrt{G_4}$  and  $p^3 = D_{ABC} p^A p^B p^C$  and  $g_{a\bar{b}}$  is the Ricci Flat metric on the  $CY_3$  with Kähler class  $J_\star$ .

We will now analyse a probe D2-brane wrapping the  $S^2$  in this attractor background. The action for such a D2-brane was reviewed in (2.64). In our case of interest we want the D2-brane to carry D0-charge  $N$ , this can be achieved by a worldvolume flux

$$F_{\text{wv}} = \frac{N}{4\pi} \sin\theta d\theta d\phi. \quad (5.20)$$

The effective Lagrangian for a D2 with the above worldvolume flux in the attractor geometry (5.19) is

$$\begin{aligned} L_2 &= -\sqrt{(4\pi T_2 R^2)^2 + \left(\frac{N}{g_s l_s}\right)^2} \sqrt{\frac{r^2}{R^2} - \frac{R^2}{r^2} \dot{r}^2 - g_{a\bar{b}} \dot{z}^a \dot{\bar{z}}^{\bar{b}}} \\ &\quad + 2\pi p^A \left( A_{Aa}^1 \dot{z}^a + \bar{A}_{A\bar{b}}^1 \dot{\bar{z}}^{\bar{b}} \right) + \frac{N}{g_s l_s} \frac{r}{R}. \end{aligned}$$

To obtain this results we integrated the action (2.64) over the  $S^2$ , chose the static gauge  $\sigma = t$  and introduced a gauge potential  $A^1$  such that  $dA_A^1 = D_A$ .

To simplify the expression (5.21) we make a non-relativistic approximation  $\dot{r} \ll \frac{R^2}{r^2}$  and an expansion in large  $N$ . Note that none of those approximations are necessary but they make our life a bit easier. Furthermore it is useful to do a coordinate transformation  $r = \frac{4R^3}{\xi^2}$  and use some more condensed notation for the gaugefield  $A^1$ , i.e.:  $A_a = 2\pi p^A A_{Aa}^1$   $A_{\bar{a}} = 2\pi p^A \bar{A}_{A\bar{a}}^1$ . Going to the Hamiltonian formalism one then finds

$$H = \frac{P_\xi^2}{2m_e} + \frac{8R^2}{m_e \xi^2} (P_a - A_a) g^{a\bar{b}} (\bar{P}_{\bar{b}} - A_{\bar{b}}) + 2 \frac{Z^2}{m_e \xi^2}, \quad (5.21)$$

where we defined an effective mass  $m_e = \frac{N}{g_s l_s}$  and introduced the notation  $Z = 4\pi T_2 R^3$  with the foresight that  $Z$  will play the role of central charge later.

Neglecting the part dependent on the  $CY_3$  in (5.21) for a moment, one can see the similarity to the Hamiltonian for a particle on  $AdS_2$  given in (5.9). It

is not difficult to check that the full Hamiltonian (5.21) is part of an  $SL(2, \mathbb{R})$  algebra of the form (5.3) if we define

$$D = \frac{1}{2}(\xi P_\xi + P_\xi \xi) \quad \text{and} \quad K = \frac{m_e}{2}\xi^2. \quad (5.22)$$

This then shows that the quantummechanics of these D2-branes is a conformal quantummechanics as defined in the previous subsection.

### The worldvolume fermions

Apart from the bosonic part of the action that was described in detail above, the D2-brane also has fermionic worldvolume fields. Although we will not need the explicit form of the fermionic action to proceed some knowledge of the fermionic content of the theory is essential.

The worldvolume fermions are given by the fermions of the background IIA theory, pulled back to the D2 worldvolume and projected by kappa-symmetry, see e.g. [169]. There are four fermions that can be associated to the bosonic fields  $r$  and  $t$  of  $AdS_2$  and these can be grouped in  $SU(2)_L \times SU(2)_R$  doublets as  $\lambda^{\alpha A}$ . Furthermore each coordinate  $z^a$ ,  $z^{\bar{a}}$  on the  $CY_3$  has the superpartners  $\eta^{\alpha+a}$  and  $\eta^{\alpha-\bar{a}}$  respectively. The worldvolume fermions satisfy the following anti-commutation relations

$$\{\lambda^{\alpha\pm}, \lambda^{\beta\mp}\} = \pm \epsilon^{\alpha\beta} \quad \text{and} \quad \{\eta^{\alpha+a}, \eta^{\beta-\bar{b}}\} = \epsilon^{\alpha\beta} g^{a\bar{b}}. \quad (5.23)$$

The  $SU(2)_R$  acts as an R-symmetry on the states and takes the form

$$T^{\alpha\beta} = T_\eta^{\alpha\beta} + T_\lambda^{\alpha\beta}, \quad (5.24)$$

where

$$T_\eta^{\alpha\beta} = \frac{1}{2} g_{a\bar{b}} \left( \eta^{\alpha+a} \eta^{\beta-\bar{b}} + \eta^{\beta+a} \eta^{\alpha-\bar{b}} \right), \quad T_\lambda^{\alpha\beta} = \frac{1}{2} (\lambda^{\alpha+} \lambda^{\beta-} + \lambda^{\beta+} \lambda^{\alpha-}). \quad (5.25)$$

On a Kähler manifold there is a well known representation of the fermionic operators as acting on differential forms, see e.g. [12]. It will be useful later to use this representation of the fermions  $\eta$  as wedge and contraction operators:

$$\begin{aligned} \eta^{-+a} &= -g^{a\bar{b}} \frac{\partial}{\partial(dz^{\bar{b}})}, & \eta^{++a} &= dz^a \wedge, \\ \eta^{--\bar{a}} &= g^{\bar{a}b} \frac{\partial}{\partial(dz^b)}, & \eta^{+-\bar{a}} &= dz^{\bar{a}} \wedge. \end{aligned} \quad (5.26)$$

It follows that under this representation of states as forms, odd forms are fermionic while even forms correspond to the bosonic states. Furthermore the

$SU(2)_R$  action (5.25) is then identified with the Lefschetz  $SU(2)$  [81]:

$$T_\eta^{++} = g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}} \wedge iJ \wedge, \quad T_\eta^{--} = g^{a\bar{b}} \frac{\partial}{\partial(dz^a)} \frac{\partial}{\partial(d\bar{z}^{\bar{b}})}, \quad (5.27)$$

$$T_\eta^{+-} = \frac{1}{2} \left( dz^a \frac{\partial}{\partial(dz^a)} + d\bar{z}^{\bar{b}} \frac{\partial}{\partial(d\bar{z}^{\bar{b}})} - 3 \right) = \frac{1}{2} (n - 3), \quad (5.28)$$

where  $n$  is the form number. Finally let us also already remark that in this representation

$$\begin{aligned} \eta^{++a}(P_a - A_a) &= \mathcal{D} = \partial - A, & \eta^{+-\bar{a}}(P_{\bar{a}} - A_{\bar{a}}) &= \bar{\mathcal{D}} = \bar{\partial} - \bar{A}, \\ \eta^{-+a}(P_a - A_a) &= -\bar{\mathcal{D}}^\dagger, & \eta^{--\bar{a}}(P_{\bar{a}} - A_{\bar{a}}) &= \mathcal{D}^\dagger. \end{aligned} \quad (5.29)$$

This will be the basis for relating the groundstates to harmonic forms as will be explained below.

### Superconformal groundstates

The fact that the D2 probe has conformal symmetry is directly related to the attractor geometry (5.19) having an  $SL(2, \mathbb{R})$  isometry group. This attractor geometry furthermore preserves 8 of the 32 supercharges of the Type IIA theory it is a solution to. Using the classification of superconformal groups see, e.g. [168], one identifies the superisometry group of (5.19) as  $D(2,1;0)$ , see [160]. We introduced the algebra of  $D(2,1;0)$  in subsection 5.1.1.

The superconformal symmetry of the background implies that the full D2-brane action including fermions gives rise to a  $D(2,1;0)$  superconformal quantummechanics. It would be however rather involved to derive the full supersymmetric action explicitly. Furthermore these details are not really of interest to us as [33, 160] showed that knowing the superconformal algebra and the bosonic action is enough to perform the counting that is our aim.

As argued before we would like to count the 'groundstates' in the D2 superconformal quantummechanics, let us make this more precise. We explained in the previous subsection that it doesn't make sense to count groundstates of  $H$  as it has a continuous spectrum. Taking  $L_0$  as the natural Hamiltonian there is a discrete spectrum and so the notion of groundstate makes sense. As was also discussed above, for particles in AdS this choice of Hamiltonian corresponds to evolution with respect to global time instead of Poincaré time. This is rather odd from the black hole microstate point of view as the natural time for the black hole is defined as that of asymptotic flat infinity. This time corresponds to Poincaré time at the horizon. We will neglect this oddity for the moment but will come back to it in the next section.

From the definition  $L_0 = \frac{1}{2}(aH + K/a)$  and the expressions (5.21, 5.22) one can calculate that the minimum energy for  $L_0$  is given by  $Z$ . This already hints at the interpretation of  $Z$  as a central charge and a corresponding BPS bound.

As is shown in [170], a D2 brane wrapping the  $S^2$  of the attractor geometry (5.19) is BPS. In superconformal theories eigenstates of  $L_0$  that are in a short

multiplet are called *chiral primaries*. Some discussion on chiral primaries in various superconformal theories can be found in e.g. [160, 158], one of the features of relevance to us is that they are automatically  $L_0$  groundstates. In our case of interest, the central extension of  $D(2,1;0)$ , the chiral primaries  $|\chi\rangle$  are those states that are annihilated by five of the eight supercharges [160]:

$$G_{-\frac{1}{2}}^{\alpha A}|\chi\rangle = 0, \quad G_{\frac{1}{2}}^{++}|\chi\rangle = 0. \quad (5.30)$$

One can now check from the algebra (5.16) that these chiral primaries  $|\chi\rangle$  indeed saturate a BPS bound, i.e.

$$L_0|\chi\rangle = l|\chi\rangle, \quad J_R^3|\chi\rangle = j|\chi\rangle \quad \Rightarrow \quad l = j + Z. \quad (5.31)$$

The D2 branes of our interest have no angular momentum on the sphere, i.e.  $j = 0$  and so the minimal value of  $L_0$  is  $Z$ . This justifies our definition of  $Z$  as the central charge, see the remark just below 5.21.

### Counting chiral primaries

To identify the chiral primaries in the D2-brane quantummechanics we need an explicit expression for the supercharges. In principle those can be found from studying the full supersymmetric action and its supersymmetries, but as already said a shortcut can be made. Using the known form of the algebra and the explicit expression for the bosonic generators, in [160] GSY were able to reconstruct the supercharges.

Making use of the various notations and results we introduced above we are ready to present the supercharges in all their glory. By an impressive calculation the authors of [160] found those to be

$$\begin{aligned} G_{\frac{1}{2}}^{\alpha\pm} &= \sqrt{\frac{a}{m_e}} \left[ \lambda^{\alpha\pm} \left( P_\xi \pm \frac{(1 \mp 1 - 2Z)i}{\xi} \right) - \frac{i\lambda^{\beta\pm}}{\xi} (4T^\alpha{}_\beta \mp \lambda^{\alpha\mp} \lambda^{\gamma\pm} \epsilon_{\gamma\beta}) \right] \\ &\quad + \sqrt{\frac{a}{m_e}} \frac{4R}{\xi} \eta^{\alpha\pm} \cdot (P - A) + i\sqrt{\frac{m_e}{a}} \xi \lambda^{\alpha\pm} \\ G_{-\frac{1}{2}}^{\alpha\pm} &= \sqrt{\frac{a}{m_e}} \left[ \lambda^{\alpha\pm} \left( P_\xi \pm \frac{(1 \mp 1 - 2Z)i}{\xi} \right) - \frac{i\lambda^{\beta\pm}}{\xi} (4T^\alpha{}_\beta \mp \lambda^{\alpha\mp} \lambda^{\gamma\pm} \epsilon_{\gamma\beta}) \right] \\ &\quad + \sqrt{\frac{a}{m_e}} \frac{4R}{\xi} \eta^{\alpha\pm} \cdot (P - A) - i\sqrt{\frac{m_e}{a}} \xi \lambda^{\alpha\pm} \end{aligned}$$

here  $\eta^{\alpha+} \cdot (P - A) = \eta^{\alpha+a}(P_a - A_a)$  and  $\eta^{\alpha-} \cdot (P - A) = \eta^{\alpha-\bar{a}}(P_{\bar{a}} - A_{\bar{a}})$ .

With the above expressions for the supercharges one can then proceed and solve for the chiral primaries, these are states  $|\chi\rangle$  that satisfy  $G_{-\frac{1}{2}}^{\alpha A}|\chi\rangle = 0$  and  $G_{\frac{1}{2}}^{++}|\chi\rangle = 0$ . We will not go into the details of constructing the chiral primaries explicitly, this can be found in [33, 160], but we will sketch the crucial points.

A first step is to split the chiral primaries as  $|\chi\rangle = |\psi\rangle \otimes \omega$  where  $\psi$  only depends on the  $\text{AdS}_2$  coordinates and as discussed above the CY part  $\omega$  can be

represented as a differential form on the  $CY_3$ . Using the identification (5.29) one can show that for  $|\chi\rangle$  to be a chiral primary  $\omega$  needs to be a harmonic with respect to those differential operators (5.29). Furthermore it is shown in [33] that for each such harmonic  $\omega$  a unique normalizable wavefunction  $|\psi\rangle$  on  $AdS_2$  exists such that  $|\chi\rangle = |\psi\rangle \otimes \omega$  gives rise to a chiral primary. This argument maps the counting of chiral primaries to the counting of harmonic forms, which in turn boils down to calculating the dimensions of certain cohomology groups. Due to the presence of the magnetic field  $F = dA$  on the Calabi-Yau the differential operators (5.29) have an additional  $U(1)$  connection. This implies the precise cohomology the harmonic forms take value in is  $H^q(X, \mathcal{L} \otimes \Omega^p)$ , where the line bundle  $\mathcal{L}$  is related to the magnetic field by its first Chern class:  $c_1(\mathcal{L}) = F$ .

For  $c_1$  big enough, i.e.  $p^4 \gg 1$ , there exists a vanishing theorem, see e.g. [89], that implies that  $H^q(X, \mathcal{L} \otimes \Omega^p) = 0$  when  $q > 0$ . This fact then allows to calculate the relevant Hodge numbers by using the Riemann-Roch index theorem, see [33]:

$$\begin{aligned} h^0 &= H^0(X, \mathcal{L}) = \frac{p^3}{6}, \\ h^1 &= H^0(X, \mathcal{L} \otimes \Omega^1) = \frac{p^3}{2}, \\ h^2 &= H^0(X, \mathcal{L} \otimes \Omega^2) = \frac{p^3}{2}, \\ h^3 &= H^0(X, \mathcal{L} \otimes \Omega^3) = \frac{p^3}{6}, \end{aligned} \tag{5.32}$$

here corrections subleading in  $p^4$  are neglected.

So the final upshot of this long discussion: a D2-brane wrapping a D4D0 black hole gives rise to  $h^0 + h^2$  bosonic and  $h^1 + h^3$  fermionic chiral primaries. The Hodge numbers are determined in terms of the D4 charge  $p^4$  via (5.32).

### Counting microstates

By calculating the number of groundstates for a single D2-brane we have done the hard work in deriving the number of states corresponding to a macroscopic black hole. Let us now come to the final step.

An important feature of the above result is that the number of chiral primaries only depends on the D4-charge and is independent both of the D0-charge  $q_0$  of the background as well as the D0-charge  $N$  of the probe. Let us now think of a macroscopic black hole with D4D0 charge as formed microscopically out of a D4-brane surrounded by D2-branes that give rise to the D0-charge of the black hole. In that scenario the total number of states such D2-branes could be in, corresponds to a partitioning of the total D0-charge  $N$  over all different D2's that each are in one of the chiral primary states. Note that due to their statistics a given fermionic brane state can only appear once in this partitioning.

The number of such partitionings is denoted by  $P(N, b, f)$ , see e.g. [171], with in our case  $b = h_0 + h_2$  and  $f = h_1 + h_3$ . For large  $N$  the partition  $P(N, b, f)$  has the famous leading behaviour

$$P(N, b, f) \sim e^{2\pi\sqrt{\frac{N}{6}(b+\frac{1}{2}f)}}. \quad (5.33)$$

Taking the logarithm of this total number of states that give rise to a black hole of charge  $(0, p^A, 0, N)$  the gravitational entropy is exactly reproduced:

$$S(p, N) = \log P(N, \frac{2p^3}{3}, \frac{2p^3}{3}) = 2\pi\sqrt{\frac{p^3 N}{6}}. \quad (5.34)$$

## 5.2 Black hole deconstruction

This section is based on the authors work in reported in [34].

### 5.2.1 Introduction

In this section we will consider the deconstruction of a single center charge  $(p, q_0)$  D4D0 black hole into a multicenter system consisting of a D6 brane with flux  $F = p/2$  on the worldvolume, a  $\bar{D}6$  with flux  $F = -p/2$ , and  $\tilde{q}_0 = q_0 + \frac{p^3}{24}$  D0 branes, where  $p \gg 1$ . For simplicity we take the dimension of the D4 and D2 charge spaces to be one, but generalization is straightforward. This system appeared in [29] in the context of a derivation of the OSV conjecture, in which mainly the case  $\tilde{q}_0 \ll p^3$  was important. In this regime, a single centered black hole of charge  $(p, q_0)$  does not exist, and the centers stay well separated from each other. However when  $\tilde{q}_0$  becomes greater than  $p^3/12$ , a phase transition occurs: the BPS configuration moduli space develops a branch in which the centers can approach each other arbitrarily close in coordinate distance, leading to a capped off  $\text{AdS}_2$  throat of arbitrary depth. Such configurations are asymptotically indistinguishable from, and perhaps should be physically identified with, those of a single centered black hole. In this scaling limit there is a superconformal symmetry acting on the moduli space quantum mechanics, which one hopes is related to or identified with the  $\text{CFT}_1$  dual of the the  $\text{AdS}_2$  throat.

In subsection 2 we review the full asymptotically flat D0D6 $\bar{D}6$  classical solution in terms of harmonic functions on  $\mathbb{R}^3$ . In subsection 3 we analyze the scaling solution, which is effectively obtained by simply dropping the constants in the harmonic functions (this is consistent only when certain charge constraints are satisfied, see section 3.3.2). In the far region, the scaling solution looks like  $\text{AdS}_2 \times S^2$  and lifts to  $\text{AdS}_3 \times S^2$  in  $M$ -theory. In the interior there are pointlike D0 brane singularities which are free to move as long as they stay on an 'accretion disk' which lies on the plane equidistant to the D6 and  $\bar{D}6$ . In the central 'near region', deep inside the accretion disk, one finds a geometry whose  $M$ -lift is again  $\text{AdS}_3 \times S^2$ , but now with *global*  $\text{AdS}_3$ . The D6 and  $\bar{D}6$  are the north

and south poles of the  $S^2$ .<sup>3</sup> In subsection 4 we analyze the (super)symmetries of the solution, and find a nontrivial relation between the generators of the near and far  $AdS_3$ s. In particular the twisted global time generator  $\bar{L}_0^{\text{near}} - J^{\text{near}}$  of the near region is the Poincaré time generator of the far  $AdS_3$  as well as the Hamiltonian of the full asymptotically flat geometry. This surprising fact may provide a derivation of the conjecture [157, 158] that the black hole entropy is the degeneracy of quantum eigenstates with respect to the global time generator of the near horizon  $AdS_2$ . In section 5 we analyze supersymmetric branes which wrap the  $S^2$  'horizon' and correspond to D0 branes blown up via the Myers effect. In subsection 6, following earlier work [33], we show that such wrapped branes have a huge degeneracy, and can account for the Bekenstein-Hawking entropy. This is a positive indication that the superconformal quantum mechanics of the deconstruction given here may give a good picture of the D4D0 quantum black hole.

### 5.2.2 The multicentered solution

We will consider  $\tilde{q}_0 = q_0 + \frac{p^3}{24}$  D0's bound to a pair of a purely fluxed D6 and  $\bar{D}6$  and. We already introduced this system in the first example of section 2.5. Let us for convenience repeat some formulas here. For simplicity we assume our Calabi-Yau to have a single Kähler modulus, the harmonic functions are then

$$\begin{aligned} H^0 &= \frac{1}{r_6} - \frac{1}{r_{\bar{6}}} + h^0, \\ H^1 &= \frac{p}{2r_6} + \frac{p}{2r_{\bar{6}}} + h^1, \\ H_1 &= \frac{p^2}{8r_6} - \frac{p^2}{8r_{\bar{6}}} + h_1, \\ H_0 &= \frac{p^3}{48r_6} + \frac{p^3}{48r_{\bar{6}}} + \sum_{i=1}^{-\tilde{q}_0} \frac{1}{r_i} + h_0. \end{aligned} \tag{5.35}$$

The integrability constraints on the positions of the centers take the form

$$h^0 + \frac{1}{r_{6i}} = \frac{1}{r_{\bar{6}i}}, \quad \frac{p^3}{6r_{6\bar{6}}} + \sum_{i=1}^{-\tilde{q}_0} \frac{1}{r_{6i}} = -h_0 - \frac{p^2}{8}h^1 \tag{5.36}$$

The four dimensional metric is then given by

$$ds^2 = -\frac{1}{\Sigma}(dt + \omega)^2 + \Sigma dx^i dx^i \tag{5.37}$$

<sup>3</sup> This lift of  $D6 - \bar{D}6$  to  $AdS_3 \times S^2$  also relates the derivations of OSV given in [29], where the two factors of  $Z_{top}$  come from a  $D6$  and a  $\bar{D}6$ , to that of [125] where they come from the north and south poles of an  $S^2$ .

the general expressions (2.104) simplify a bit in this one modulus case and we have

$$\Sigma = \sqrt{\frac{Q^3 - L^2}{(H^0)^2}} \quad (5.38)$$

$$Q = -2H_1H^0 + (H^1)^2, \quad L = H_0(H^0)^2 + \frac{(H^1)^3}{3} - H_1H^1H^0.$$

The five dimensional lift is given by 2.108

$$\begin{aligned} ds_{5d}^2 = & 2^{-2/3}Q^{-2} \left[ -(H^0)^2(dt + \omega)^2 - 2L(dt + \omega)(d\psi + \omega_0) \right. \\ & \left. + \Sigma^2(d\psi + \omega_0)^2 \right] + 2^{-2/3}Q dx^i dx^i. \end{aligned} \quad (5.39)$$

### 5.2.3 The scaling solution

As was also discussed in section 3.3.2 in case the different intersection products satisfy the triangle inequalities the solution space of the above multicentered configuration has a scaling point. In this section we describe the 'far-horizon' and 'near-horizon' regions of the scaling solution, both of which M-lift to (quotients of)  $\text{AdS}_3 \times \text{S}^2$ .

We are looking for 'scaling solutions' obtained by rescaling all coordinates  $x \rightarrow \lambda \tilde{x}$  and taking  $\lambda \rightarrow 0$  while keeping  $\tilde{x}$  finite. This effectively amounts to dropping the constant terms  $h$  in the harmonic functions. The charges are then sitting at different locations far down a very long throat.

In this scaling limit, the integrability conditions (5.36) at the D0 location force the D0-branes to be equidistant from the D6 and the  $\overline{\text{D6}}$ . Hence all  $\tilde{q}_0$  of them lie in a plane, say  $x^3 = 0$ , resembling an accretion disk or ring surrounding the  $\text{D6}\overline{\text{D6}}$  pair. The integrability condition at the D6 location fixes the distance  $R_6 = \frac{1}{2}r_{6\bar{6}}$  between the D6 and this plane. Hence the moduli space of solutions is the  $2\tilde{q}_0$  positions of the D0s on  $\mathbb{R}^2$ .

Since the distance between any D0 and the D6 is at least  $R_6$ , the constraint implies  $\tilde{q}_0 \geq p^3/12$ , so we only have scaling solutions when this constraint on the charges is met. We are in particular interested in the regime

$$\tilde{q}_0 \gg p^3/12. \quad (5.40)$$

In this case generically the D6 and  $\overline{\text{D6}}$  will be much closer to the plane than to the D0s. For example if the D0s all lie on a ring of radius  $R_0 = r_{6i}$  the distance  $R_6$  between the D6 and the plane satisfies

$$\frac{p^3}{12} \frac{1}{R_6} = \frac{\tilde{q}_0}{\sqrt{R_0^2 + R_6^2}} \quad (5.41)$$

and  $R_0 \gg R_6$ . While the full geometry is pretty complicated, it simplifies in the far region  $|x| \gg R_0$  and near region  $|x| \ll R_0$ .

### Far region

Far outside the D0 ring, the fine structure of the pointlike branes disappears down the throat and the geometry following from (5.39) with  $h = 0$  is simply the near horizon geometry of a D4D0 black hole. The M-theory lift is

$$\frac{(24\tilde{q}_0 - P^3)}{8P} dx_5^2 + \frac{3r}{P} dt dx_5 + \frac{P^2}{2r^2} dr^2 + \frac{P^2}{2} (d\eta^2 + \sin^2 \eta d\psi^2), \quad (5.42)$$

where we have traded  $\vec{x}$  for  $r, \eta, \psi$ . The periodic identification  $x^5 \sim x^5 + 4\pi$  makes this an  $SL(2, \mathbb{R})_L$  quotient of  $AdS_3 \times S^2$ . On the boundary  $r \rightarrow \infty$ , the conformal metric is simply

$$ds_2^2 = dt dx^5, \quad (5.43)$$

where  $SL(2, \mathbb{R})_L$  ( $SL(2, \mathbb{R})_R$ ) acts in the standard fashion on the left (right) null coordinate  $x^5$  ( $t$ ). In particular we see that the Hamiltonian  $\partial_t$  of the original asymptotically flat region generates null translations of the unbroken  $SL(2, \mathbb{R})_R$ .

$AdS_3$  quotients of this type were analyzed in [82]. While  $t$  is a standard null coordinate on the Poincaré diamond of the boundary of  $AdS_3$ ,  $x^5$  in fact turns out to be a Rindler coordinate. The  $SL(2, \mathbb{R})_L$ -invariant  $AdS_3$  vacuum then gives, after  $x^5$  identifications, a thermal state for the left movers of the boundary CFT. The entropy of this thermal state agrees with that of the D4D0 black hole.

### Near region

Now we consider the near region, far inside the accretion disk. It is convenient to work in prolate spheroidal coordinates  $\rho, \eta, \phi$  (plus  $x^5$  and  $t$ ):

$$|x - x_{D6}| = R_6 (\cosh \rho + \cos \eta), \quad (5.44)$$

$$|x - x_{\bar{D}6}| = R_6 (\cosh \rho - \cos \eta). \quad (5.45)$$

Lines of constant  $\rho$  are ellipsoids with foci at the  $D6$  and  $\bar{D}6$  locations, while  $\eta$  is an angle around the ellipsoid, see figure 5.2. In this coordinate system, the functions appearing in the metric are

$$Q = -\frac{P^2}{4R_6 \cos \eta}, \quad (5.46)$$

$$J = \frac{P^3}{24R_6} \frac{\cosh \rho}{\cos^2 \eta} - \frac{1}{2} \sum_{i=1}^{\tilde{q}_0} \frac{1}{|x - x_{D0}^i|}. \quad (5.47)$$

The near region is characterized by

$$|x| \ll |x_{D0}^i| \sim R_0. \quad (5.48)$$

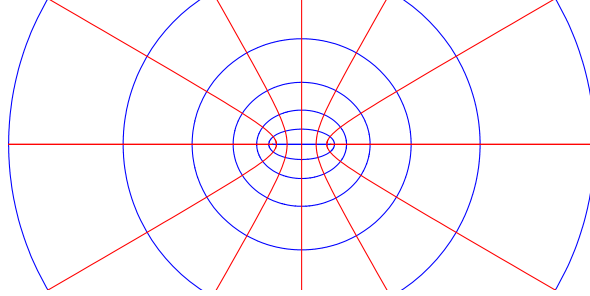


**Figure 5.2** Here the prolate spheroidal coordinate system is illustrated. The ellipses are curves of constant coordinate  $\rho$ . The hyperbola are lines of constant  $\eta$ .

This allows us to replace the summation in the RHS of the last harmonic function of (5.35) by a constant. This constant is fixed by the integrability condition (5.36). One then has

$$J = \frac{P^3}{24R_6} \left( \frac{\cosh \rho}{\cos^2 \eta} - 1 \right). \quad (5.49)$$

The last term in  $H_0$  reduces to the constant  $\frac{P^3}{12R_6}$ , and  $\omega$  can via (3.23) be calculated to be

$$\omega = \frac{P^3}{3R_6} \frac{\sinh^2(\frac{\rho}{2}) \sin^2 \eta d\phi}{(\cosh^2 \rho - \cos^2 \eta)}. \quad (5.50)$$

After a linear change of coordinates

$$\begin{aligned} t &= \frac{P^3}{3R_6} \tau, \\ x_5 &= 2(\tau + \theta), \\ \phi &= \psi + \tau - \theta, \end{aligned} \quad (5.51)$$

the metric ref takes the global  $\text{AdS}_3 \times \text{S}^2$  form

$$\frac{P^2}{2} (d\eta^2 + \sin^2 \eta d\psi^2) + 2P^2 \left( -\cosh^2 \left( \frac{\rho}{2} \right) d\tau^2 + \sinh^2 \frac{\rho}{2} d\theta^2 + \frac{1}{4} d\rho^2 \right) \quad (5.52)$$

with no identifications except the standard  $2\pi$  identifications of  $\psi$  and  $\theta$ . This near region metric has an enhanced  $\text{SL}(2, \mathbb{R})_L \times \text{SU}(1, 1|2)$  superisometry group.

#### 5.2.4 Supersymmetry

Calabi-Yau compactification of Type IIA or M theory preserves eight supersymmetries. Of these, a general multicenter solution preserves four which we

denote

$$Q^{\pm\pm} \quad (5.53)$$

where the right superscript denotes the charges under shifts of  $\phi$  and the left distinguishes between the chirality. Commutators of these preserved charges give the Hamiltonian ( $i\partial_t$ )

$$\{Q^{\alpha A}, Q^{\beta B}\} = \epsilon^{AB} \epsilon^{\alpha\beta} H + \text{central terms}, \quad (5.54)$$

where  $\alpha, A = \pm$ . The supercharges commute with  $H$ :

$$[Q^{\pm\pm}, H] = 0. \quad (5.55)$$

In the near and far AdS regions (5.52) and (5.42), the four supercharges are locally enhanced to eight  $SU(1,1|2)$  superconformal symmetries. We wish to understand which four of the eight supersymmetries in each of the near and far regions are globally preserved.

### Near region

Let us denote the eight supercharges of the near region in a standard convention<sup>4</sup>

$$G_{\bar{L}_0}^{\pm J^3} = G_{\pm \frac{1}{2}}^{\pm\pm}. \quad (5.56)$$

The supercharges  $G$  have eigenvalues  $\pm \frac{1}{2}$  under the action of

$$\begin{aligned} \bar{L}_0^{\text{near}} &= \frac{1}{2}(\partial_\tau - \partial_\theta) \\ J_{\text{near}}^3 &= \partial_\psi. \end{aligned} \quad (5.57)$$

Using the coordinate transformation (5.51) we find

$$\frac{P^3}{3R_6} H = 2(\bar{L}_0^{\text{near}} - J_{\text{near}}^3). \quad (5.58)$$

The four linear combinations of the eight  $G_{\pm \frac{1}{2}}^{\pm\pm}$  which correspond to  $Q^{\pm\pm}$  can then be deduced by demanding that they commute with  $\bar{L}_0^{\text{near}} - J_{\text{near}}^3$  as in (5.55). One finds

$$Q^{\pm+} = G_{\frac{1}{2}}^{\pm+}, \quad Q^{\pm-} = G_{-\frac{1}{2}}^{\pm-}. \quad (5.59)$$

<sup>4</sup> Note that this notation is equivalent to the one we used for the supercharges of the  $D(2,1;0) = SU(1,1|2) \times SU(2)_L$  algebra in section 5.1.1. The only difference is that here the  $SU(2)_L$  is replaced by an  $SL(2, \mathbb{R})_L$  algebra.

### Far Region

In the far region we similarly denote the supercharges by

$$F_{L_0}^{\pm J^3} = F_{\pm \frac{1}{2}}^{\pm \pm}. \quad (5.60)$$

These have eigenvalues  $\pm \frac{1}{2}$  under the action of

$$J_{\text{far}}^3 = \partial_\phi. \quad (5.61)$$

Employing the transformation (5.51) to global coordinates for the  $\text{AdS}_3$  one finds

$$\frac{P^3}{3a} H = \bar{L}_0^{\text{far}} - \frac{1}{2}(\bar{L}_1^{\text{far}} + \bar{L}_{-1}^{\text{far}}). \quad (5.62)$$

It then follows using (5.55) that

$$Q^{\pm \pm} = F_{\frac{1}{2}}^{\pm \pm} - F_{-\frac{1}{2}}^{\pm \pm}. \quad (5.63)$$

Comparing (5.57) and (5.61) with the help of the coordinate transformations (5.51) one finds

$$\begin{aligned} 2L_0^{\text{near}} &= \bar{L}_0^{\text{far}} - \frac{1}{2}(\bar{L}_1^{\text{far}} + \bar{L}_{-1}^{\text{far}}) + 2L_0^{\text{far}}, \\ \bar{L}_0^{\text{near}} - L_0^{\text{near}} &= J_{\text{far}}^{3\text{far}} - L_0^{\text{far}}, \\ J_{\text{near}}^3 &= J_{\text{far}}^3. \end{aligned} \quad (5.64)$$

### 5.2.5 Wrapped branes

In this section we describe some supersymmetric wrapped branes, whose microstates will be argued in the next section to dominate the entropy for large D0-charges.

#### Probe supersymmetries

Consider an M2 brane wrapped on the  $S^2$  in the near geometry, and sitting at the center  $\rho = 0$  of the  $\text{AdS}_3$ . As shown in [170] this breaks half the supersymmetry. Clearly this leaves unbroken both  $\text{SU}(2)$  rotations of the  $S^2$  and  $\text{U}(1)$  rotations of the  $\text{AdS}_3$ . Using the relations of the  $\text{SU}(1,1|2)$  algebra of section 5.1.1 it follows that the unbroken supercharges of the near region must be

$$G_{\frac{1}{2}}^{+\pm}, \quad G_{-\frac{1}{2}}^{-\pm}. \quad (5.65)$$

Comparing with (5.59) we see that there are two globally unbroken supercharges

$$Q^{++}, \quad Q^{--}. \quad (5.66)$$

The existence of two unbroken global supercharges is a generic feature of a supersymmetric probe configuration in backgrounds with four supercharges. Configurations preserving the same supersymmetries also exist which carry angular momentum in the  $\text{AdS}_3$  [170]. These are given by  $\rho = \rho_0$ ,  $x^5 = x_0^5$ , i.e.  $\theta = -\tau + \frac{x_0^5}{2}$ .

### Explicit solutions

A supersymmetric probe solution exists for M2 branes in the full geometry including the D0's. The Killing spinors of the metric (5.39) are given in terms of the covariantly constant spinors of the  $4d$  hyper-Kähler base space [172]

$$ds^2 = H^0 d\vec{x}^2 + (H^0)^{-1} (dx^5 + \omega^0)^2. \quad (5.67)$$

The kappa-symmetry constraint for M2 branes static with respect to the time  $t$  requires the M2 brane to wrap a supersymmetric cycle in the  $4d$  base space, i.e. a cycle holomorphic for one of the complex structures of the base space.

The geometry of the base space only depends on  $H^0$  and is the same for the full geometry and for the  $\text{AdS}_3 \times \text{S}^2$  near geometry. Hence the same equation  $\rho = \rho_0$ ,  $x^5 = x_0^5$  that defines a supersymmetric M2 brane that wraps the sphere in  $\text{AdS}_3 \times \text{S}^2$  also defines a supersymmetric M2 brane in the full geometry. This fact can be confirmed by looking at the complex structures of the base space. A natural set of complex coordinates is given by

$$U = e^{-i\frac{x^5}{2}} \tanh \frac{\rho}{2}, \quad V = e^{i\phi} \sinh \rho \sin \eta. \quad (5.68)$$

The M2 brane probe sits at  $U = U_0$ . Those are the only compact holomorphic cycles in the geometry.

In the near geometry (5.52), these probes move on a circle of constant radius:  $\rho = \rho_0$ ,  $\theta = \frac{x_0^5}{2} - \tau$ , and have angular momentum

$$J_\theta = -\frac{1}{4\pi} \frac{P^3}{3} \sinh^2 \frac{\rho}{2} \quad (5.69)$$

### Translation to 4d

In the original four dimensional picture the M2 that wraps the near region  $\text{S}^2$  becomes an elliptically shaped D2 brane with the D6 and  $\overline{\text{D6}}$  at the foci. In 5-dimensions the M2 is stabilized at some non-zero  $\rho_0$  its angular velocity. This translates in 4 dimensions to a non-zero worldvolume flux on the D2-brane that induces D0-charge which stabilizes the ellipsoidal shape.

Using the explicit relation between M2 and D2 from [169] one can calculate the flux  $F$  on the D2 in terms of M-theory quantities:

$$F = \sqrt{\frac{2\mathcal{V}^3}{1 + 2\mathcal{V}^3|Y|^2}} \star_{D2} Y. \quad (5.70)$$

Here  $F$  is normalized such that  $q_0 = \frac{1}{2\pi} \int_{D2} F$ ,

$$\mathcal{V} = \frac{9^{\frac{1}{3}} S}{2^{\frac{4}{3}} H^0 Q} = 2^{-\frac{4}{3}} 3^{-\frac{1}{3}} p \sqrt{2 \cosh \rho - \cos^2 \eta - 1}, \quad (5.71)$$

and

$$Y = dx^5 + A^0 \quad (5.72)$$

where  $A^0$  can be calculated from (3.23) to be

$$A^0 = \frac{2 \sin^2 \eta d\phi - 4(\cosh \rho - \cos^2 \eta) d\tau}{2 \cosh \rho - \cos^2 \eta - 1}. \quad (5.73)$$

Using that for the supersymmetric M2  $x^5 = \text{cst.}$  we find that

$$F = -\frac{J_\theta}{2} \sin \eta d\eta (d\phi - 2d\tau) - \frac{1}{8\pi} \frac{p^3}{3} \sin \eta d\eta d\tau \quad (5.74)$$

with the angular momentum  $J_\theta$  given in (5.69).

From the expression of the worldvolume flux above we can calculate the induced D0 brane on the D2. We see that  $q_0 = -J_\theta (= -2J_{x^5})$ , as expected.

Since the D2 carries both electric and magnetic flux, we have some sort of ellipsoidal *supertube* [173], and one might expect induced fundamental string charge as well as (4d) angular momentum carried by the D2-brane. Moreover, the induced D0-brane charge in the nontrivial  $A^0$  background will also contribute to the angular momentum.

We have electric and magnetic fields  $E_\eta$  and  $B_{\eta\phi}$  given by

$$F = E_\eta d\eta d\tau + E_\phi d\phi d\tau + B_{\eta\phi} d\eta d\phi \quad (5.75)$$

In the case at hand  $E_\phi = 0$ . The “electric displacement” is

$$\Pi^\eta = \frac{\partial \mathcal{L}(E, B)}{\partial E_\eta}, \quad \Pi^\phi = \frac{\partial \mathcal{L}(E, B)}{\partial E_\phi} \quad (5.76)$$

with

$$\mathcal{L} = \mathcal{L}_{\text{BI}} + \mathcal{L}_{\text{WZ}} = -\sqrt{\frac{-1}{2\mathcal{V}^3} \det(g + 8\pi F)} + F \wedge A^0. \quad (5.77)$$

The Poynting vector density  $\mathcal{P}_\phi = \Pi^\eta B_{\eta\phi}$  gives the 4d  $\phi$  angular momentum density around the  $z$ -axis.

A short calculation reveals:

$$(\Pi^\eta)_{\text{BI}} = -(\Pi^\eta)_{\text{WZ}} = -\frac{16\pi \sin^2 \eta}{2 \cosh \rho - \cos^2 \eta - 1} \quad (5.78)$$

so

$$\mathcal{P}_\phi = B_{\eta\phi}((\Pi^\eta)_{\text{BI}} + (\Pi^\eta)_{\text{WZ}}) = 0. \quad (5.79)$$

Thus we see that the D0-angular momentum in the  $\text{D6}\overline{\text{D6}}$  background (which is nonzero when we have individual D0 particles) gets completely canceled by

the induced electric flux on the D2! This interesting feature is a further hint that these wrapped D2's can indeed correspond single centered black hole microstates.

Of course, in hindsight this is not so surprising, since the  $\phi$  angular momentum in 4d equals the  $\psi$  angular momentum in 5d (i.e. around the  $S^2$  equator), which is (classically) zero for our wrapped M2 branes.

It can be checked that apart from  $\Pi^\eta = 0$ , we also have  $\Pi^\phi \sim E_\phi = 0$ . As a result, despite appearances, the actual induced fundamental string charge density on the D2 is also everywhere zero, since this is given precisely by  $\Pi$ .

So in the end this is a supertube which isn't quite a supertube. Do to their ellipsoidal shape one is tempted to call them super-eggs.

Finally, note that as in [33] these ellipsoidal D2 branes move in the Calabi Yau as particles in a magnetic field proportional to  $P$ , because of the background  $F^{(4)}$  RR-field.

This analysis shows the existence of a new, larger moduli space of multicenter configurations carrying D4D0 charges: the moduli space of a system formed by the  $D6\overline{D6}$  pair, a certain number of D0 branes and several ellipsoidal D2 branes carrying the extra D0 charge.

### 5.2.6 Black hole entropy

In this section we sketch how the Bekenstein-Hawking entropy emerges from the counting of microstates for a system that includes the ellipsoidal D2s.

The fastest way to understand the state-counting for this system is to reduce it to a problem previously analyzed in [33]. In that work, the superconformal chiral primary bound states of D0 branes in the near-horizon  $AdS_2 \times S^2$  geometry of a D4D0 black hole were counted. A large degeneracy arises from non-abelian D0 configurations which puff up via the Myers effect and form a D2D0 bound state wrapping the horizon. This carries no D2 charge as measured at infinity. Such a D2D0 configuration couples to the background  $F^{(4)}$  RR field and hence sees an effective magnetic field on the Calabi-Yau. Hence the supersymmetric ground states are highly degenerate lowest Landau levels. Partitioning the D0 charge among such configurations produces an exponential growth of states which exactly matches the Bekenstein-Hawking entropy.

If we lift the calculation of [33] to M-theory, the near horizon geometry becomes  $AdS_3 \times S^2$ , the horizon-wrapped D2 becomes a horizon wrapped M2, and the D0 charge becomes orbital angular momentum. Hence in M-theory the supersymmetric states of a horizon wrapped M2 are highly degenerate lowest Landau levels on the Calabi-Yau and reproduce the D4D0 entropy.

To summarize, in this last section we have deconstructed the D4D0 black hole into zero-entropy  $D0D6\overline{D6}$  bits, which have a classical moduli space. It is natural to try to identify the black hole microstates with the supersymmetric ground states of the moduli space quantum mechanics (including the non-abelian interactions of near-coincident D0s which are crucial for the Myers effect)). In the  $AdS_3 \times S^2$  centered between the  $D6\overline{D6}$  pair, the D0-branes have all been taken off to infinity. The entropy should appear as the number of ways of reincorporating

these D0s in this  $\text{AdS}_3 \times S^2$ . Using the wrapped M2 this counting proceeds exactly as above and gives the desired result.

In addition to reproducing all the successes of [33], the current analysis potentially has several important advantages. The main observation of [33] was that the horizon-wrapped bound states could account for the entropy. Reference [33] did not attempt either to provide a systematic framework in which higher order corrections to the entropy could be computed, or to derive or justify the conjecture of [157, 158] that states should be counted in the global, rather than the Poincaré, time of the near horizon  $\text{AdS}_2$ . Perhaps these shortcomings can both be overcome in the context of the deconstructed black hole quantum mechanics discussed here.





# Appendix A

## Conventions and notation

For various common definitions we refer to appendix A of [29], whose notation we follow.

In this appendix we give some more details on the conventions we take for various physical quantities. We work in units in which  $c = \hbar = k_B = 1$  but we will keep dimensions of length explicitly in most part of the paper. The coordinates  $x, t$  we take to have dimension of length. Angular coordinates, most of the time denoted by Greek letters as  $\alpha, \theta, \psi$  etc will be dimensionless however. Furthermore we will take forms to be dimensionless. As e.g.  $\omega = \omega_i dx^i$  is dimensionless this implies the components of forms have dimensions of inverse length, i.e.  $[dx^i] = L$ ,  $[\omega_i] = L^{-1}$  and  $[\omega] = 1$ . This convention implies that the Hodge star is dimensionful:  $[\star] = L^{d-2p}$  when acting on a  $p$ -form.

In each dimension we define a natural Planck length  $l_d$  ( $[l_d] = L$  of course) by normalising the Einstein-Hilbert action as

$$S_d^{\text{EH}} = \frac{2\pi}{(l_d)^{d-2}} \int \sqrt{-g_d} \mathcal{R}_d, \quad (\text{A.1})$$

and a reduced planck length by

$$\ell_d = \frac{l_d}{4\pi}. \quad (\text{A.2})$$

### A.0.7 M-theory vs IIA conventions

We start from the following 11d M-theory metric:

$$ds_{11d}^2 = R^2 e^{4\phi/3} \Theta^2 + e^{-2\phi/3} ds_{10d}^2, \quad (\text{A.3})$$

where  $ds_{10d}^2$  is a ten-dimensional metric and  $R$  is a constant with dimensions of length. The one form  $\Theta = d\theta + 2\pi A$ , with  $\theta = \theta + 2\pi$  and  $A$  is a one form

on the ten dimensional space. Furthermore  $\phi$  is normalized in such a way that  $\phi(\infty) = 0$ . The M2-branes of this theory have a tension

$$T_{M2} = \frac{2\pi}{l_M^3}, \quad (\text{A.4})$$

with  $l_M = l_{11}$  is the 11 dimensional Planck length.

We can relate these to IIA quantities by reduction on the  $\theta$  circle. As an M2 wrapped around this circle is a fundamental string we find:

$$T_{F1} = \frac{2\pi}{l_s^2} = 2\pi R T_{M2} = \frac{4\pi^2}{l_M^3} R \Rightarrow l_M^3 = 2\pi R l_s^2. \quad (\text{A.5})$$

From the relation between M2 and D2 one easily infers

$$l_M^3 = g_s l_s^3, \quad (\text{A.6})$$

where in our conventions  $T_{Dp} = \frac{2\pi}{g_s l_s^{p+1}}$ . The constants  $g_s$  and  $l_s$  are respectively the string coupling constant (at infinity) and the string length. They are arbitrary constants related to the 10 dimensional Planck length  $l_{10}$  by

$$l_{10}^4 = g_s l_s^4. \quad (\text{A.7})$$

We can now reduce both the 11d and 10d theory on the same Calabi-Yau manifold  $X$ . Since

$$ds_{11d}^2 = R^2 e^{4\phi/3} \Theta^2 + e^{-2\phi/3} (ds_{4d}^2 + ds_{CY\text{ IIA}}^2), \quad (\text{A.8})$$

we can relate the effective 5d and 4d metrics:

$$ds_{5d}^2 = R^2 \left( \frac{V_{\text{IIA}}}{V_M} \right)^{2/3} \Theta^2 + \left( \frac{V_{\text{IIA}}}{V_M} \right)^{-1/3} ds_{4d}^2, \quad (\text{A.9})$$

where we used that

$$e^{2\phi} = \frac{V_{\text{IIA}}}{V_M}. \quad (\text{A.10})$$

In a slightly more transparent form this is

$$ds_{5d}^2 = \tilde{V}_{\text{IIA}}^{2/3} \ell_5^2 (2\Theta)^2 + \tilde{V}_{\text{IIA}}^{-1/3} \frac{\hat{R}}{2} ds_{4d}^2, \quad (\text{A.11})$$

with  $\ell_5$  the reduced 5 dimensional Planck length, we will also use the notation  $2\Theta = d\psi + A_{4d}^0$ . We use various dimensionless objects:

$$\tilde{V}_M = \frac{V_M}{l_M^6}, \quad \tilde{V}_{\text{IIA}} = \frac{V_{\text{IIA}}}{l_s^6}, \quad \hat{R} = \frac{R}{\ell_5}. \quad (\text{A.12})$$

The Calabi-Yau reduction relates all the different parameters at infinity. We will give the relations that will be of most importance to us. The relation between the 4d Planck length  $l_4$  and the string length is

$$l_4 = g_{4d} l_s, \quad (\text{A.13})$$

where

$$g_{4d}^2 = \frac{g_s^2}{\tilde{V}_{\text{IIA}}^\infty} = \frac{1}{\tilde{V}_M}. \quad (\text{A.14})$$

The effective 4d string coupling  $g_{4d}$  is in a hypermultiplet and thus constant in the solutions we will consider. Note that the same is true for  $\tilde{V}_M$ . The 4d and 5d Planck lengths are related by the size of the M-theory circle:

$$\ell_5 = \sqrt{\frac{\hat{R}}{2}} \ell_4. \quad (\text{A.15})$$

Furthermore this size of the circle is immediately related to the size of the Calabi-Yau at infinity and thus to the value of the Kähler moduli at infinity, i.e.

$$\frac{\hat{R}^3}{8} = \tilde{V}_{\text{IIA}}^\infty = \frac{1}{6} (J_\infty)^3. \quad (\text{A.16})$$

Finally let us relate the reduced 4d plank length  $\ell_4$  to Newton's constant that appears in front of the 4d Einstein-Hilbert action as

$$S_4^{\text{EH}} = \frac{1}{16\pi G_4} \int \sqrt{-g_4} \mathcal{R}_4. \quad (\text{A.17})$$

This gives the relation

$$\ell_4 = \sqrt{2 G_4}, \quad (\text{A.18})$$

and by (A.15) this implies

$$\sqrt{G_4} = \frac{\ell_5^{3/2}}{\sqrt{R}}. \quad (\text{A.19})$$



# Appendix B

## Gauge field contribution to conserved charges

In this appendix we give some more detail concerning the contribution of the various gauge fields to the conserved boundary charges. The five dimensional  $\mathcal{N}=1$  supergravity theory of which our asymptotic  $\text{AdS}_3 \times \text{S}^2$  configurations are solutions has  $b_2$   $\text{U}(1)$  vectorfields, where  $b_2$  is the second Betti number of the Calabi-Yau we compactified on. After reduction over the asymptotic two-sphere we end up with an additional  $\text{SU}(2)$  gauge field as will be explained in some detail below. Analyzing how the action on the boundary of AdS varies with respect to these gauge fields and the metric gives the conserved currents of the boundary theory that can be identified with a 2d CFT.

Before doing this analysis explicitly one can save some work by considering the behavior of the theory near that asymptotic boundary. As in (4.58) we can in general write an asymptotic  $\text{AdS}_3$  metric as

$$ds_{3d}^2 = d\eta^2 + (e^{\frac{2\eta}{R_{\text{AdS}}}} g_{ij}^{(0)} + g_{ij}^{(2)}) du^i du^j, \quad (\text{B.1})$$

where the boundary is at  $\eta = \infty$  and  $g_{ij}^{(0)}$  is the metric on the boundary. A generic action for a gauge field in 3 dimensions has the following form

$$S = a \int \text{Tr}(F \wedge \star F) + b \int \text{Tr}(A \wedge F + \frac{2}{3} A \wedge A \wedge A), \quad (\text{B.2})$$

with  $a$  and  $b$  some coupling constants. Now one should remark that due to the appearance of the metric in the first term, this term decreases as  $e^{-\eta}$  near the boundary while the second term is purely topological and will thus dominate near the boundary. This implies that to calculate the boundary charges we only need to keep track of the topological Chern-Simons part of the gauge field action. In the following subsection B.1 we calculate these 3d Chern-Simons terms explicitly

for the case of our concern. In subsection B.2 we shortly review the general idea behind the calculation of the boundary charges from Chern-Simons theory and in B.3 and B.4 we calculate these for our solutions.

## B.1 Reduction of the Chern-Simons term

The three dimensional Chern-Simons term is a reduction over the sphere of the Chern-Simons term of five dimensional  $\mathcal{N}=1$  supergravity, which itself has its origin in such a topological term in the M-theory action. Starting from the Chern-Simons part of the 11-dimensional supergravity action and reducing over a  $CY_3$ , one gets the following action in 5-dimensions (where we went to Euclidean signature)

$$I_{CS} = \frac{i}{192\pi^2} \int D_{ABC} A^A \wedge F^B \wedge F^C. \quad (B.3)$$

The ansatz for the gauge field  $A^A$  to further reduce to 3 dimensions is given by the asymptotic form found in (4.54). So we propose as our general reduction ansatz a field strength of the form

$$F^A = \frac{1}{2} p^A e_2 + dC^A, \quad (B.4)$$

where  $C^A$  is a one-form on  $AdS_3$ . The two-form  $e_2$  is known in the literature as the global angular 2-form [174], [175], [176], it is the generalisation of the standard volume of the sphere to an  $S^2$  fibration and is defined as follows:

$$\begin{aligned} e_2 &= \epsilon_{ijk} (Dy^i \wedge Dy^j - \tilde{F}^{ij}) y^k, \\ ds^2 &= ds_{AdS_3}^2 + \frac{1}{l^2} (dy^i - \tilde{A}^{ij} y^j) (dy^i - \tilde{A}^{ik} y^k), \\ Dy^i &= dy^i - \tilde{A}^{ij} y^j, \\ \tilde{F}^{ij} &= d\tilde{A}^{ij} - \tilde{A}^{ik} \wedge \tilde{A}^{kj}. \end{aligned} \quad (B.5)$$

Summation over repeated indices is assumed and the  $y^i$  are the embedding coordinates of  $S^2$  in flat  $\mathbb{R}^3$ , i.e.  $y^i y^i = 1$ . The  $\tilde{A}$  are the  $SU(2)$  gauge fields coming from the reduction of the metric over the  $S^2$ . Keep in mind that  $\tilde{A}$  depends only on the  $AdS_3$  coordinates.

To make the reduction a bit more tractable we introduce the following quantities

$$\tilde{A}^{ij} = \epsilon_{ijk} A^k, \quad F^{ij} = \epsilon_{ijk} F^k. \quad (B.6)$$

To get compact expressions, we will associate to every quantity with an  $SU(2)$  index  $i, j, \dots$  the following notation

$$\mathcal{O} = \frac{i}{2} O^j \sigma_j, \quad (B.7)$$

where  $\sigma_j$  are the Pauli matrices which satisfy

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \text{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}, \quad \text{Tr}(\sigma_i\sigma_j\sigma_k) = 2i\epsilon_{ijk}. \quad (\text{B.8})$$

$$(\text{B.9})$$

For example, one has

$$\begin{aligned} D\mathcal{Y} &= d\mathcal{Y} + [\mathcal{Y}, \mathcal{A}], \quad \mathcal{F} = d\mathcal{A} - \mathcal{A} \wedge \mathcal{A}, \\ e_2 &= 4\text{Tr}[\mathcal{Y}d\mathcal{Y} \wedge d\mathcal{Y} + d(\mathcal{Y}\mathcal{A})] = 2[\sin\theta d\theta \wedge d\phi - d(y^i A^i)], \end{aligned} \quad (\text{B.10})$$

where in the last equation, we used spherical coordinates.

Plugging in eqn (B.3) and reducing over  $S^2$  keeping in mind that the only dependence on  $S^2$  resides in  $e_2$ , one ends up with the following Chern-Simons term on  $\text{AdS}^3$ :

$$I_{\text{gauge}} = -\frac{i}{4\pi} \frac{p^3}{6} \int \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} - \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) + \frac{i}{16\pi} D_{AB} \int C^A \wedge dC^B. \quad (\text{B.11})$$

As one sees the  $\mathcal{A}$  and  $C$  fields don't interact with each other, this is as expected from  $\text{SU}(2)$  gauge invariance. The  $\text{SU}(2)$  gauge field  $\mathcal{A}$  does change under such a gauge transformation but  $C$  does not. So one needs two  $\mathcal{A}$  and one  $C$  for a consistent interaction term. But  $\text{Tr}\mathcal{A} \wedge \mathcal{A} = 0$ . So there is no interaction term between  $\mathcal{A}$  and  $C$ .

## B.2 Boundary charges: lightning review

How in general the presence of Chern-Simons terms leads to contributions to both the boundary  $\text{SU}(2)$  and  $\text{U}(1)$  currents and the boundary energy momentum tensor is very nicely reviewed in [92] and so we will restrict ourselves to a short recapitulation here. Essential in the derivation is the addition of extra boundary terms to the bulk Chern-Simons action. Let's take the simple example of single  $\text{U}(1)$  field:

$$S = ik \int_{\text{AdS}} A \wedge dA. \quad (\text{B.12})$$

We can make the gaugechoice  $A_\eta = 0$  and furthermore the equations of motion imply that  $A$  is a flat connection. As argued in [92] there are two reasons to include an additional boundary term to this bulk action. The first is that imposing Dirichlet conditions for both components of the gauge fields, i.e.  $\delta A|_{\partial\text{AdS}} = 0$ , is too strict. Second is that one wants the current associated to the gauge field to be purely left or rightmoving. This last argument is natural from the canonical quantization of Chern-Simons theory [177]. Without the boundary term one has  $\delta S \sim \int_{\partial\text{AdS}} p\delta q + q\delta p$ , where  $p$  and  $q$  are both a component of the boundary gauge field. Adding the correct boundary term cancels the second term and gives

the natural interpretation to  $p$  as the momentum conjugate to  $q$ . The boundary term that does this is

$$S_{bd} = -\frac{|k|}{2} \int_{\partial AdS} A \wedge \star A. \quad (\text{B.13})$$

The absolute value of  $k$  is needed to have positive energy as we will see shortly. Introducing the standard complex coordinates  $w, \bar{w}$  on the boundary cylinder and noting that  $\star dw = idw$ ,  $\star d\bar{w} = -id\bar{w}$  one can verify that that once one adds the boundary term indeed

$$\delta S = \begin{cases} 2i \int_{\partial AdS} (\delta A_w) A_{\bar{w}} & \text{if } k > 0, \\ 2i \int_{\partial AdS} (\delta A_{\bar{w}}) A_w & \text{if } k < 0, \end{cases} \quad (\text{B.14})$$

where we have assumed the bulk fields to be on shell. Now we can impose the Dirichlet boundary conditions  $\delta A_w = 0$  and leave  $\delta A_{\bar{w}}$  arbitrary in case  $k > 0$  and vice versa if  $k < 0$ . The addition of this boundary term influences the boundary currents, these are defined as

$$\delta S = \int_{\partial AdS} \sqrt{g^{(0)}} \left( \frac{i}{2\pi} J^i \delta A_i + \frac{1}{2} T_{ij} \delta g^{ij}_{(0)} \right), \quad (\text{B.15})$$

so one sees that e.g. the contribution to the energy momentum tensor comes completely from the boundary term as the bulk term is purely topological. It is now easy to calculate these currents:

$$\begin{aligned} T_{ww} &= \frac{|k|}{2} A_w A_w, & T_{w\bar{w}} &= 0, & T_{\bar{w}\bar{w}} &= \frac{|k|}{2} A_{\bar{w}} A_{\bar{w}}, \\ J_w &= \begin{cases} 0 & \text{if } k > 0, \\ 2\pi A_w & \text{if } k < 0, \end{cases} \\ J_{\bar{w}} &= \begin{cases} 2\pi A_{\bar{w}} & \text{if } k > 0, \\ 0 & \text{if } k < 0. \end{cases} \end{aligned} \quad (\text{B.16})$$

Having reviewed the general philosophy we can now calculate the charges in our case of interest, note that the story generalizes straightforward to the non-abelian case [92].

### B.3 The U(1) part

The U(1) part of the Chern-Simons term (B.11) is given by

$$\frac{i}{16\pi} D_{AB} \int C^A \wedge dC^B. \quad (\text{B.17})$$

Due to the fact that  $D_{AB}$  as a metric on  $H^2(X)$  has a single positive eigenvector and  $b_2 - 1$  negative ones (see e.g. [36]) we have to treat the two cases slightly

differently, see the discussion above. The projectors to the positive and negative eigenspaces are

$$(P^+)_B^A = \frac{1}{p^3} p^A D_{BC} p^C, \quad (P^-)_B^A = \delta_B^A - (P^+)_B^A, \quad (\text{B.18})$$

which gives

$$\begin{aligned} C^{A+} &= \frac{1}{p^3} p^A D_{BC} p^B C^C = \frac{1}{p^3} (p^B q_B) p^A d\psi = \frac{2}{p^3} (p^B q_B) p^A dw, \\ C^{A-} &= \left( D^{AB} - \frac{1}{p^3} p^B p^A \right) q_B d\psi = 2 \left( D^{AB} - \frac{1}{p^3} p^B p^A \right) q_B dw, \end{aligned} \quad (\text{B.19})$$

where we used the asymptotic form of our gauge field, eqn. (4.54).

As explained in the previous subsection, once we add the correct boundary term we have the following boundary conditions left  $\delta C_w^{+A} = 0$  and  $\delta C_{\bar{w}}^{-A} = 0$ . So we have to choose a fixed value for those gauge fields at the boundary. It turns out that the correct choice is  $C_w^{+A} = 0$  and  $C_{\bar{w}}^{-A} = 0$ .

However our asymptotic gauge field (4.54) doesn't satisfy this boundary condition as one can see from (B.19). This is however easily cured by the following gauge transformation

$$C^A \longrightarrow C^A - \frac{4}{R} \frac{p^B q_B}{p^3} p^A dt,$$

which gives

$$C^{A+} = 2 \frac{1}{p^3} p^B q_B p^A d\bar{w}. \quad (\text{B.20})$$

Given this split into positive and negative modes one can now apply the general procedure as reviewed in the previous subsection to find

$$\begin{aligned} T_{\bar{w}\bar{w}} &= \frac{1}{4\pi} \frac{1}{p^3} (p^A q_A)^2, \\ T_{ww} &= \frac{1}{4\pi} \frac{1}{p^3} [(p^A q_A)^2 - p^3 (q_A D^{AB} q_B)], \\ J_{\bar{w}}^+ &= \frac{1}{4} \frac{1}{p^3} p^B q_B p^A, \\ J_w^- &= \frac{1}{4} \left( D^{AB} - \frac{1}{p^3} p^B p^A \right) q_B. \end{aligned} \quad (\text{B.21})$$

## B.4 The SU(2) part

The SU(2) part of (B.11) is

$$-\frac{i}{4\pi} \frac{p^3}{6} \int \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} - \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \quad (\text{B.22})$$

here  $k = \frac{1}{4\pi}U^3 > 0$ . Let's look at the value the  $SU(2)$  gauge field  $\mathcal{A}$  takes in our solution. The general sphere reduction Ansatz has the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{\alpha\beta} (dx^\alpha + A_\mu^I X_I^\alpha dx^\mu) (dx^\beta + A_\nu^I X_I^\beta dx^\nu) , \quad (B.23)$$

where the  $x^\mu$  are in our case the coordinates on  $AdS_3$  and the  $x^\alpha$  coordinates on the  $S^2$ , the  $X_I^\alpha \partial_\alpha$  are the killing vectors of the sphere. So using the form of the asymptotic metric (4.53), we have

$$d\theta^2 + \sin^2 \theta (d\phi + \frac{Rd^0}{2U^3} dv)^2 = (d\theta + A_\mu^I X_I^\theta dx^\mu)^2 + \sin^2 \theta (d\phi + A_\mu^I X_I^\phi dx^\mu)^2 \quad (B.24)$$

which then implies that the only non vanishing component of the gauge field is

$$A_v^3 = \frac{Rd^0}{2U^3} , \quad (B.25)$$

or in the complex coordinates we introduced

$$A_{\bar{w}}^3 = -\frac{R^2 d^0}{4U^3} . \quad (B.26)$$

And following the by now standard procedure its contribution to the energy momentum tensor is

$$T_{\bar{w}\bar{w}} = \frac{R^4}{8\pi} \frac{(d^0)^2}{16U^3} . \quad (B.27)$$

## Appendix C

### Nederlandse samenvatting

Beste lezer,  
welkom in de Nederlandse samenvatting. Alvorens we een overzicht van de inhoud van de thesis geven, wordt kort de achterliggende motivatie besproken. In deze Nederlandse versie is geprobeerd het gebruik van al te veel fysisch jargon te vermijden en zoveel mogelijk de achterliggende intuïtie te schetsen. Dit om mensen die de auteur beter kennen dan zijn werk in de fysica toch een kans te geven een glimp op te vangen van zijn bezigheden en drijfveren de laatste vier jaar. Andere lezers die op zoek zijn naar een meer gefundeerde en preciezere versie van deze discussie zouden het Engels machtig moeten zijn en worden dan ook doorverwezen naar de inleiding in hoofdstuk 1.

#### C.1 Zwaartekracht en zwarte gaten

De wereld zoals we hem waarnemen bestaat uit dingen, materie, die met elkaar interageren door middel van krachten. Enkele voorbeelden zijn een bal die naar de aarde valt door middel van de zwaartekracht of een magneet die aan de ijskast plakt door de elektromagnetische kracht. Als we de zaken nauwkeurig genoeg onderzoeken blijkt dat alle materie bestaat uit slechts een beperkt aantal minuscule deeltjes, onder andere quarks en elektronen, en dat de interacties tussen hen verklaard kunnen worden aan de hand van slechts vier krachten: de zwaartekracht, het elektromagnetisme, de sterke en de zwakke kernkracht.

##### **Zwaartekracht**

Onder de vier fundamentele natuurkrachten is de zwaartekracht, of gravitatie, een buitenbeentje. Ze is bijvoorbeeld veel zwakker dan de andere drie krachten. Dit kan mooi geïllustreerd worden met volgend huis-, tuin- en keuken-experiment: neem het magneetje van uw ijskast en gebruik het om een een vork van de tafel

te heffen. Dit simpele experiment toont aan dat de magnetische kracht uitgeoefend op de vork door het kleine magneetje sterker is dan de zwaartekracht uitgeoefend op de vork door de hele aarde!! Als we het in cijfers uitdrukken blijkt dat de elektromagnetische kracht 10 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 000 keer sterker is dan de zwaartekracht. Dat de zwaartekracht toch zoveel effect heeft, komt doordat alle fundamentele deeltjes interageren met de zwaartekracht. Alle massa's of vormen van energie trekken elkaar aan via de zwaartekracht. Daardoor is deze kracht, die heel zwak is voor elk fundamenteel deeltje afzonderlijk, toch best sterk is voor objecten van onze grootte. Om het metaforisch uit te drukken: eendracht maakt macht. Op deze grote schalen van meters tot lichtjaren is de zwaartekracht goed begrepen. Al in de 17e eeuw had Newton een precieze theorie van de gravitatie waarmee hij bijvoorbeeld de beweging van de planeten rond de Zon kon verklaren. In het begin van de vorige eeuw werd de theorie van Newton verder bijgeschaafd door Einstein, in een theorie die bekend staat als algemene relativiteitstheorie. Er stelt zich echter een probleem wanneer we de zwaartekracht willen beschrijven op een meer microscopische schaal, bv. die van de elementaire deeltjes. Voor afstanden kleiner dan de grootte van atomen voldoen de wetten van de natuur aan een aantal principes die kwantummechanica genoemd worden. De drie andere krachten, elektromagnetisme en de zwakke en sterke kern kracht, hebben een microscopische of kwantummechanische beschrijving in een theorie genaamd het standaard model. De manier waarop deze drie krachten kwantummechanisch beschreven worden werkt echter niet voor de zwaartekracht. We worden dus geconfronteerd met een fundamenteel probleem. We weten dat alle dingen opgebouwd zijn uit kleine deeltjes en we zien en begrijpen hoe 'grote' dingen interageren onder de zwaartekracht. Maar een microscopische beschrijving, namelijk hoe die kleine deeltjes zelf interageren met de zwaartekracht, kunnen we niet begrijpen zoals voor de andere krachten. Er is dus nood aan een nieuw idee voor de beschrijving van de zwaartekracht op zeer kleine schalen. Snaartheorie is zo'n idee voor wat men kwantumgravitatie noemt, en we zullen er in sectie C.2 dieper op ingaan.

### C.1.1 Zwarte gaten

Laten we eerst kort het bestaan en de relevantie van zwarte gaten bespreken. Het is best ironisch dat gravitatie, als zwakste kracht van de vier, toch aanleiding geeft tot de sterkste effecten. Einsteins algemene relativiteitstheorie voorspelt namelijk dat als we genoeg massa verzamelen binnen een bepaalde straal de zwaartekracht van deze massa zo sterk is dat alle mogelijke signalen erdoor gevangen worden. Zelfs het snelste aller signalen, een lichtstraal, kan niet ontsnappen aan de gravitatie van zo'n opeenhoping van massa. Daardoor ziet de sfeer waarin deze massa zich bevindt er van buiten uit als een volledig zwart gat, vandaar de naam. Zulke zwarte gaten zijn dus plekken in de ruimte waar dingen in kunnen vallen maar er niet weer uit kunnen gehaald worden. Recente waarnemingen tonen aan dat er zich zo'n zwart gat, Sagittarius A\*, in het centrum van onze melkweg bevindt.

Uit een eerste microscopische benadering van deeltjes in de buurt van een zwart gat volgt echter dat het zwarte gat toch niet zo zwart is als men eerst vermoedde. Hawking toonde immers aan dat een zwart gat een bepaalde temperatuur heeft en straling uitzendt, zoals elk warm object. Denk bijvoorbeeld aan een roodgloeiende pook. Het bestaan van die straling heeft echter enkele belangrijke gevolgen. Ten eerste schijnt er informatie verloren te gaan wanneer dingen aan een zwart gat ontsnappen als thermische straling. Het is namelijk zo dat een stier of een auto van dezelfde massa, aanleiding geven tot exact dezelfde straling nadat ze in een zwart gat gevallen zijn. Dit lijkt in tegenspraak met de principes van de kwantummechanica, volgens dewelke informatie behouden moet blijven. Het bovenstaande probleem is gekend als de informatie paradox. Het tweede gevolg van Hawkings berekening is dat zwarte gaten een bepaalde hoeveelheid entropie hebben. Men kan entropie zien als het aantal verschillende configuraties die 'van op een afstand' dezelfde eigenschappen hebben. Denk bijvoorbeeld aan twee ongeopende flessen water in de koelkast, die dezelfde temperatuur hebben en er identiek uitzien. Op microscopische schaal zijn deze flessen echter verre van identiek en kunnen ze uit een heleboel verschillende microscopische configuraties van bewegende moleculen zijn opgebouwd. Entropie is een getal dat uitdrukt over hoeveel van dergelijke configuraties het gaat. Het feit dat ook zwarte gaten entropie hebben, leidt opnieuw tot een uitdagend vraagstuk. Binnen Einsteins theorie van gravitatie lijkt er namelijk geen beschrijving mogelijk van verschillende microscopische configuraties voor zwarte gaten. De vraag is dus: waaruit zijn zwarte gaten microscopisch opgebouwd?

Aangezien het resultaat van Hawking steunt op sommige aspecten van kwantummechanica moeten we het antwoord op de informatie paradox en microtoestanden van zwarte gaten zoeken binnen een theorie van kwantumgravitatie. Dat is waarom zwarte gaten zo'n belangrijke rol spelen binnen snaartheorie of andersom, waarom snaartheorie van belang is voor het verstaan van zwarte gaten.

## C.2 Snaartheorie

Zoals we reeds opmerkten bestaan alle dingen om ons heen uit een heel beperkt aantal elementaire deeltjes. Dingen zijn opgebouwd uit kleine bolletjes die we atomen noemen en die zijn op hun beurt weer opgebouwd uit nog kleinere bolletjes die we quarks en electronen noemen. In het standaard model, tot op heden onze meest nauwkeurige microscopische theorie, worden de elementaire deeltjes gezien als zeer kleine bolletjes of punten. Snaartheorie wijkt af van dat idee en veronderstelt dat op een nog kleinere schaal, de deeltjes eigenlijk kleine trillende snaartjes zijn. De elementaire deeltjes zijn niet langer puntvormig maar uitgestrekt langs een lijn. Verschillende elementaire deeltjes zijn in deze theorie niets anders dan verschillende trillingswijzen van een uniek type snaar. Belangrijk is dat de trilling van gesloten snaartjes steeds aanleiding geeft tot een trillingswijze die zich gedraagt als een graviton, het deeltje dat de zwaartekracht veroorzaakt. Op deze manier geeft snaartheorie aanleiding tot een theorie van quantum-

gravitatie. Ook de andere deeltjes die we waarnemen kunnen beschreven worden als trillingen van dezelfde snaar en daardoor heeft snaartheorie het potentieel een microscopische theorie van alle gekende fysica te zijn.

Een speciale eigenschap van snaartheorie is dat het de dimensie van de ruimte-tijd voorspelt, wat geen enkele andere fysische theorie doet. Dat snaartheorie enkel werkt in tien ruimte-tijd dimensies, kan op het eerste zicht tegenstrijdig lijken met onze ervaringen van drie ruimtelijke dimensies en één tijd. Maar als men er iets langer over nadenkt, beseft men dat er geen enkele waarneming is die ons vertelt dat de ruimte op hele kleine schalen niet 9 dimensionaal kan zijn. De klassieke vergelijking is een touw: voor een acrobaat die erover loopt is dit 1-dimensionaal, hij kan alleen vooruit of achteruit, voor een mier is het touw echter 2-dimensionaal, zij kan vooruit en achteruit maar ook links of rechts omheen het touw. Het zou dus best kunnen dat er ook in onze wereld extra kleine 'opgerolde' dimensies zijn.

Ondanks haar naam bevat snaartheorie naast snaren ook meer uitgestrekte objecten zoals membranen of versies met nog meer dimensies die men D-branen noemt. Bepaalde types zwarte gaten worden in snaartheorie beschreven als opgebouwd uit die D-branen. Op deze manier is men er via snaartheorie in geslaagd het mysterie van de zwarte gat microtoestanden (in sommige gevallen) op te lossen. Het is namelijk zo dat de verschillende manieren om eenzelfde zwart gat op te bouwen uit D-branen, resulteren in precies de juiste entropie die Hawking voorspeld heeft. Dat is één van de belangrijke theoretische successen van snaartheorie.

Snaartheorie bestaat ondertussen al meer dan dertig jaar, maar blijft voorlopig spijtig genoeg beperkt tot een tak binnen de theoretische fysica. Zoals iedereen weet is een wiskundige theorie binnen de fysica pas relevant als zij ook experimenteel getest is. Dat is tot nu toe nog niet het geval voor snaartheorie. Gravitatie is namelijk zo zwak dat effecten van kwantumgravitatie pas rechtstreeks meetbaar zijn op enorm kleine afstanden, veel kleiner dan wat de huidige deeltjesversnellers, de moderne microscopen, aankunnen. Men kan echter toch hopen op experimentele bevestiging in de nabije toekomst door minder directe observaties zoals bv. deze in kosmologie. Verder zou de nieuwste deeltjesversneller, LHC, die binnenkort in werking wordt gesteld in CERN, over enkele jaren bewijs kunnen leveren voor de theorie van supersymmetrie. Dat is nog lang geen bewijs van de realisatie van snaartheorie, maar het zou toch een serieuze hint in de juiste richting zijn.

Hoewel elke rechtgeaarde fysicus het ultieme oordeel aan het experiment laat, heeft snaartheorie toch al gezorgd voor nieuwe inzichten in theoretische problemen, zoals bijvoorbeeld die van zwarte gaten. Daarnaast heeft de theorie geleid allerlei nieuwe ideeën binnen de wiskunde. Snaartheorie blijft tot op heden een hele rijke theorie met nog vele open problemen en mogelijke nieuwe toepassingen. Deze thesis is slechts een heel klein stukje in de grote snaartheorie-puzzel maar smaakt naar meer.

## C.3 Inhoud van de thesis

In deze thesis behandelen we verschillende eigenschappen en toepassingen van *multicenter* zwarte gaten. Dat zijn configuraties van verschillende sferische zwarte gaten die op een vaste evenwichtsafstand aan elkaar gebonden zijn. Ze komen voor in bepaalde veralgemeningen van algemene relativiteitstheorie die men supergravitatie noemt. Die supergravitatie theorieën geven een goede beschrijving van snaartheorie bij lage energie.

Multicenter zwarte gaten zijn interessant op zichzelf aangezien ze zeer rijke fysica vertonen die ons nieuwe dingen kan leren over snaartheorie. Verder lijken ze een belangrijke rol te spelen in het microscopisch begrijpen van eenvoudigere zwarte gaten.

In het eerste hoofdstuk van deze thesis geven we een meer technisch gefundeerde inleiding en geven we ook een meer gedetailleerd overzicht van de inhoud van de thesis.

In hoofdstuk 2 herhalen we wat resultaten omtrent het thermodynamisch gedrag van zwarte gaten en presenteren de voor ons relevante supergravitatie theorieën binnen snaartheorie. We gaan verder met een beschrijving van supersymmetrische zwarte gaten, geven enkele voorbeelden, en presenteren een afleiding van de entropie aan de hand van D-branen.

Het derde hoofdstuk introduceert dan de multicenter zwarte gaten en behandelt een aantal van hun eigenschappen. Het hoofdstuk eindigt met twee voorbeelden waarvan de relevantie tot voor kort niet volledig geapprecieerd werd. Deze voorbeelden werden uitgewerkt in een samenwerking van de auteur met J. de Boer, F. Denef, S. El-Showk en I. Messamah en kunnen teruggevonden worden in [31].

Hoofdstuk 4 presenteert het grootste gedeelte van de resultaten uit het bovenvermelde werk [31] van de auteur. Hier wijzigen we de multicenter zwarte gaten zodanig dat ze in een asymptotisch AdS ruimte zitten. Dat laat dan een beschrijving toe in een twee dimensionale conforme velden theorie (CFT) via de algemene principes van AdS/CFT. We starten een eerste analyse van implicaties en open problemen in deze CFT.

Het laatste hoofdstuk bevat de bijdrage van de auteur aan een meer recente D-braan interpretatie van zwarte gat microtoestanden. We beginnen dit hoofdstuk met een samenvatting van het model van Gaiotto Strominger en Yin voor zwarte gat microtoestanden. In het tweede deel wordt de verfijning van dat model met behulp van multicenter zwarte gaten gepresenteerd. Dat werk voerde de auteur uit in samenwerking met F. Denef, D. Gaiotto, A. Strominger en X. Yin.

Er zijn ook drie appendices. De eerste bevat conventies en de tweede enkele technische resultaten omtrent ladingen afkomstig van Chern-Simons termen. U bevindt zich in de derde.

**Nota:** Buiten het werk gepresenteerd in deze thesis heeft de auteur ook nog ander onderzoek binnen snaartheorie verricht tijdens zijn doctoraat. In [38] vindt

men de resultaten omtrent fermionische acties voor D-branen, bekomen door de auteur in samenwerking met L. Martucci, J. Rosseel en A. Van Proeyen. Een andere reeks resultaten over de interpretatie van pseudo-supersymmetrie in de context van  $*$ -supergravitaties kan men vinden in [39]. Dit was een samenwerking van de auteur met E. Bergshoeff, J. Hartong, A. Ploegh en J. Rosseel.



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