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Model Theory of Fields and Heights

La Théorie des Modèles des Corps et des Hauteurs

Thèse de doctorat

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Abstract

In this thesis, we deal with the model theory of algebraically closed fields expanded by predicates to denote either elements of small height or multiplicative subgroups satisfying a diophantine condition. The questions we consider belong to the area of model theory and stability theory. In Chapter 2, we investigate an algebraically closed field with a distinguished multiplicative subgroup satisfying the Mann property. The model theory of this pair was first studied in the papers of B. Zilber and L. van den Dries - A. Günaydın respectively. In 1965, H. Mann showed that the set of complex roots of unity has the Mann Property. Later, it was proved that any multiplicative group of finite rank in any field of characteristic zero has the Mann property. In this chapter, we characterize the independence which enables us to characterize definable and interpretable groups. In Chapter 3, we study algebraically closed fields expanded by two unary predicates denoting an algebraically closed subfield and a multiplicative subgroup. This will be a proper extension of algebraically closed fields with a group satisfying the Mann property, and also pairs of algebraically closed fields. Then we characterize definable and interpretable groups in the triple. Another goal of this thesis is to study the field of algebraic numbers with elements of small height. In Chapter 4, we show that this theory is not simple and has the independence property. We also relate the simplicity of a certain pair with Lehmer's conjecture. In Chapter 5, we apply nonstandard analysis to prove the existence of certain height bounds on the complexity of the coefficients of some polynomials. This allows us to characterize the ideal membership of a given polynomial. Moreover, we obtain a bound for the logarithmic height function, which enables us to test the primality of an ideal.

Résumé

Dans cette thèse, nous traitons la théorie des modèles des corps algébriquement clos étendu par prédicats pour désigner soit des éléments de hauteur bornée, soit des sous-groupes multiplicatifs satisfaisant une condition diophantienne. Les questions que nous considérons appartiennent au domaine de la théorie des modèles et la théorie de la stabilité. Dans le Chapitre 2, nous examinons un corps algébriquement clos avec un sous-groupe multiplicatif distingué qui satisfait la propriété Mann. La théorie des modèles de cette paire était d'abord étudiée dans les articles de B. Zilber et L. van den Dries - A. Günaydın respectivement. En 1965, H. Mann a montré que l'ensemble des racines de l'unité a la propriété Mann. Plus tard, il était prouvé que tout groupe multiplicatif de rang fini dans tout corps de caractéristique zéro a la propriété Mann. Dans ce chapitre, nous caractérisons l'indépendance qui nous permet de caractériser les groupes définissables et les groupes interprétables. Dans le Chapitre 3, nous étudions les corps algébriquement clos étendu par deux prédicats unaires qui dénotent un sous-corps algébriquement clos et un sous-groupe multiplicatif. Ce sera une extension propre du corps algébriquement clos avec un groupe satisfaisant la propriété Mann, et aussi les paires des corps algébriquement clos. Ensuite, nous caractérisons les groupes définissables et interprétables dans le triple. Un autre but de cette thèse est d'étudier la théorie de corps des nombres algébriques avec des éléments de petite hauteur. Dans le Chapitre 4, nous montrons que cette théorie n'est pas simple et a la propriété d'indépendance. Nous nous rapportons aussi à la simplicité de la certaine paire avec la conjecture de Lehmer. Dans le Chapitre 5, nous appliquons l'analyse nonstandard pour prouver l'existence de certaines bornes de hauteur de la complexité des coefficients de certains polynômes. Cela nous permet de caractériser l'appartenance idéale d'un polynôme donné. De plus, nous obtenons une borne pour la fonction de la hauteur logarithmique, ce qui nous permet de tester la primalité d'un idéal.

"Our greatest glory is not in never falling, but in rising every time we fall."

Confucius

Dedication

To Ali and Züleyha Göral.

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Contents

Introduction

In this thesis, we study fields with a predicate to denote either elements of small height or multiplicative subgroups satisfying a certain diophantine condition. In particular, we are applying Nonstandard Analysis to find height bounds and Geometric Model Theory to analyse extensions of algebraically closed fields with a distinguished group.

In model theory, one of the main objectives is to understand the definable sets in a structure. Recall that a set is called *definable* if it is given by a first-order formula. Type-definable means an intersection of definable sets. Another important notion in model theory is interpretability. A set is called *interpretable* if it is a quotient of a definable set by a definable equivalence relation. To illustrate, in algebraically closed fields, definable sets are the constructible sets from algebraic geometry and definable groups correspond to the algebraic groups.

Classification theory is an extensive project in model theory, emerged from the work of M. Morley in the 1960's and S. Shelah in the 1970's, pursuing the categorization of first-order theories based on how much discrepancy there is among their models, and also to get back structural information about the models. The question of how many models a theory can have has been at the heart of the most fundamental progresses in the history of model theory. Stable theories are important to classify their models. If a theory is not stable then its models are too complicated and numerous to classify, and arithmetic fits in this case. A theory T is said to be *stable* if there is no first-order formula in T which defines an infinite linear order. Stable theories enjoy a notion of *independence*, namely the *forking* independence, with a well-defined set of properties, reminiscent of the algebraic independence in fields and the linear independence in vector spaces. A theory is *simple*, if in fact it is characterized by the presence of such an independence notion which is symmetric. A theory T is said to have *the independence property*, if there is a first-order formula in some model of T that can code any given subset of an arbitrarily large finite set. It is known that stable theories are simple and do not have the independence property. For instance, the theory of algebraically closed fields and abelian groups in the group language are stable. The theory of the real numbers is unstable since one can define the order, however it does not have the independence property.

In Chapter 2, we study algebraically closed fields with a distinguished multiplicative subgroup in terms of stability. Let K be an algebraically closed field, the field $\mathbb{F}$ its prime field and G be a multiplicative subgroup of $K^\times$. In this thesis, a fundamental notion about multiplicative groups is the *Mann property*. To define this property, consider an equation

$$a_1x_1 + \cdots + a_nx_n = 1 \tag{0.0.1}$$

with $n \geq 2$ and $a_i \in \mathbb{F}$. A solution $(g_1, ..., g_n)$ of this equation is called non-degenerate if for every non-empty subset I of $\{1, 2, ..., n\}$, the sum $\sum_{i\in I} a_ig_i$ is not zero. We say that G has the *Mann property* if every such equation (0.0.1) has only finitely many non-degenerate solutions in G. In 1965, H. Mann [26] showed that the set of complex roots of unity μ has the Mann property. Later, it was proved that any multiplicative group of finite rank in any field of characteristic zero, for instance $2^{\mathbb{Z}}$, has the Mann property; see [32, 13, 23]. In 1990, in an unpublished note, B. Zilber [41] showed that the pair $(\mathbb{C}, \mu)$ is stable and his pattern was based on the result of H. Mann [26].

Now we fix K and G as above. The model theory of the pair (K, G) was first studied in the paper of L. van den Dries and A. Günaydın [9]. The result of B. Zilber [41] was generalized by L. van den Dries and A. Günaydın [9] to (K, G) where G has the Mann property. They axiomatized (K, G) and proved that the theory of (K, G) is stable. Furthermore in [9], L. van den Dries and A. Günaydın showed that G has the Mann property over K, this means that if a_i is in K in (0.0.1), we still have only finitely many non-degenerate solutions in G. They also proved that every subset of G^n definable in (K, G) is definable in the abelian group G, in other words the induced structure on G is the pure group structure.

In Chapter 2, we first characterize the model-theoretic algebraic closure in (K, G). This allows us to characterize the independence in the pair (K, G) in terms of the algebraic independence in K with the help of the group G. In [31], B. Poizat characterized the independence for beautiful pairs, in particular for pairs of algebraically closed fields. It turns out that the independence in (K, G) is different from the beautiful pairs and it is simpler in a sense. Definable groups in stable theories is a recurrent topic in model theory and they play a significant role for the classification theory pioneered by B. Zilber in the 1970's. In Chapter 2, after characterizing the independence in the pair, we turn our attention to definable groups in the pair (K, G). Characterization of the model-theoretic algebraic closure and the independence in (K, G) enable us to characterize definable groups in (K, G) up to isogeny, in terms of definable and interpretable groups in K and G. The proof entails a well-known technique from geometric stability theory, namely the group configuration theorem [19]. We follow the

approach of [2], where T. Blossier and A. Martin-Pizarro characterized interpretable groups in pairs of proper extension of algebraically closed fields using a result of A. Pillay [28]. Precisely, we prove the following result:

Theorem A. *(Definable Groups* (2.30)*) Let K be an algebraically closed field and G be a multiplicative subgroup of $K^\times$ with the Mann property. Any type-definable group in (K, G) is isogenous to a subgroup of an algebraic group. Moreover any type-definable group is, up to isogeny, an extension of a type-interpretable group in G by an algebraic group.*

In the case where G is divisible, we can characterize interpretable groups in (K, G) in terms of definable groups in (K, G). Our method will combine the methods in [2] and [28] and we prove the following theorem:

Theorem B. *(Interpretable groups* (2.56)*) Let K be an algebraically closed field and G be a divisible multiplicative subgroup of $K^\times$ with the Mann property. Every interpretable group H in (K, G) is, up to isogeny, an extension of an interpretable abelian group in G by an interpretable group N, which is a quotient of an algebraic group V by a subgroup N_1, which is an abelian group interpretable in G.*

In Chapter 3, we study algebraically closed fields expanded by two unary predicates denoting an algebraically closed subfield and a multiplicative subgroup. This will be a proper extension of algebraically closed fields with a group satisfying the Mann property as in Chapter 2, and also pairs of algebraically closed fields. More precisely: let Ω be an algebraically closed field, the field k be a proper subfield of Ω which is also algebraically closed and Γ be a multiplicative subgroup of $\Omega^\times$. Consider an equation

$$k_1 x_1 + \cdots + k_n x_n = 1 \tag{0.0.2}$$

with $n \geq 1$ and $k_i \in k$.

We say that (k, Γ) is a *Mann pair* if for all n there is a finite subset $\Gamma(n)$ of Γ such that for all $k_1, ..., k_n$ in $k^\times$ all non-degenerate solutions of (0.0.2) in Γ lie in $\Gamma(n)$. In particular, the group Γ has the Mann property and the intersection $k \cap \Gamma$ is finite. This is a uniform version of the Mann property. For instance, the pair $(\overline{\mathbb{Q}}, \exp(\overline{\mathbb{Q}}))$ is a Mann pair by Lindemann's theorem.

Now we fix Ω, k and Γ as above. The model theory of the triple (Ω, k, Γ) was first studied by L. van den Dries and A. Günaydın [10, 11]. Among other things, they axiomatized the triple and characterized definable sets by a relative quantifier elimination. They also proved that the theory of (Ω, k, Γ) is stable. Moreover in [10], L. van den Dries and A. Günaydın showed that if the intersection $k \cap \Gamma$ is trivial and Γ has finite rank, then (k, Γ) is a Mann pair.

In Chapter 3, as an initial step we characterize the model-theoretic algebraic closure in the triple (Ω, k, Γ). This permits us to characterize the independence in the triple, which is given by the algebraic independence in Ω and k. After characterizing the independence in the triple, we focus on definable and interpretable groups in the triple. We pursue a similar procedure as in the Chapter 2. We first characterize definable groups, up to isogeny, in the triple in terms of definable and interpretable groups in Ω, k and Γ. Again the proof requires the group configuration theorem and the motivation comes from [2].

Theorem C. *(Definable Groups for the triple* (3.23)*) Let Ω be an algebraically closed field, the field k be a proper subfield of Ω which is also algebraically closed and Γ be a multiplicative subgroup of $\Omega^\times$ such that (k, Γ) is a Mann pair. Any type-definable group in (Ω, k, Γ) is isogenous to a subgroup of an algebraic group. Moreover any type-definable group is, up to isogeny, an extension of a direct sum of k-rational points of an algebraic group defined over k and a type-interpretable abelian group in Γ by an algebraic group.*

When Γ is divisible, the characterization of definable groups in the triple enables us to characterize interpretable groups. In particular, we conclude:

Theorem D. *(Interpretable groups for the triple* (3.33)*) Let Ω be an algebraically closed field, the field k be a proper subfield of Ω which is also algebraically closed and Γ be a divisible multiplicative subgroup of $\Omega^\times$ such that (k, Γ) is a Mann pair. Every interpretable group H in (Ω, k, Γ) is, up to isogeny, an extension of a direct sum of k-rational points of an algebraic group defined over k and an interpretable abelian group in Γ by an interpretable group N, which is a quotient of an algebraic group by a subgroup N_1 which is isogenous to a cartesian product of k-rational points of an algebraic group defined over k and an interpretable abelian group in Γ.*

Another objective of this work is to study the field of algebraic numbers with elements of small height. In Chapter 4, our concern will be mainly the model theory of the field of algebraic numbers with a certain predicate. Model theory of pairs have been studied for some time. More generally, stable theories with a predicate were studied in the paper of E. Casanovas and M. Ziegler [6]. Their result in [6] implies the result of B. Zilber [41] and also the stability of the theory of pairs of algebraically closed fields [31].

A height function is a function that measures the complexity of an element. This is a fundamental notion at the basis of diophantine geometry. The most significant example is the logarithmic height function on the field of algebraic numbers. In order to define the logarithmic height function, we first define the Mahler measure of a polynomial over $\mathbb{C}$. For a polynomial

$$f(x) = a_d(X - \alpha_1) \cdots (X - \alpha_d),$$

its *Mahler measure* is defined as the product

$$m(f) = |a_d| \prod_{|\alpha_j| \geq 1} |\alpha_j|.$$

Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. For α in $\overline{\mathbb{Q}}$ with minimal polynomial $f(x)$ in $\mathbb{Z}[X]$ of degree d, we define its *Mahler measure* as $m(\alpha) = m(f)$ and the *logarithmic height* of α is defined as

$$h(\alpha) = \frac{\log m(\alpha)}{d}.$$

Kronecker's theorem, which is a characterization of being a root of unity, states that $h(\alpha) = 0$ if and only if α is a root of unity or zero. *Lehmer's conjecture*, which is still open, states that there exists an absolute constant $c > 1$ such that if $m(\alpha) > 1$ then $m(\alpha) \geq c$. In other words, it asserts that the Mahler measure is bounded away from 1 except for the set of roots of unity. This question was posed by D. Lehmer [24] around 1933. The best known example of smallest Mahler measure greater than 1 so far was also given by Lehmer: if α is a root of the polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1,$$

then $m(\alpha) \approx 1.17628$. A real algebraic integer $\alpha > 1$ is called a *Salem number* if α and $1/\alpha$ are Galois conjugate and all others Galois conjugates of α lie on the unit circle. Clearly, for any Salem number α we have that $m(\alpha) = \alpha$. Another pertinent open question is whether 1 is a limit point of Salem numbers. Details can be found in in the surveys [35, 36] of C. Smyth.

The model-theoretic properties of $\overline{\mathbb{Q}}$ are well-known; it has quantifier elimination, which corresponds to the projection of a constructible set being constructible from algebraic geometry. Moreover it is stable. Let $P_b := \{a \in \overline{\mathbb{Q}} : m(a) \leq b\}$ where $b \geq 1$ and $S_\epsilon = \{a \in \overline{\mathbb{Q}} : h(a) \leq \epsilon\}$ where $\epsilon > 0$. Note that both P_b and S_ϵ contain the set of roots of unity μ. In other words, Lehmer's Conjecture states that there is $b > 1$ such that $P_b = P_1 = \mu$. The pairs $(\overline{\mathbb{Q}}, P_b)$ and $(\overline{\mathbb{Q}}, S_\epsilon)$ can be seen as an $L_m(U) = L_m \cup \{U\}$ structures where $L_m = \{1, \cdot\}$, the operation $\cdot$ is the usual multiplication and U is an unary relation symbol whose interpretations are P_b and S_ϵ respectively.

In Chapter 4, we study the model theory of $(\overline{\mathbb{Q}}, S_\epsilon)$ and we prove a result which shows that small perturbations of the property of being a root of unity changes drastically the stability properties of the ambient structure. We also relate the simplicity of the pair $(\overline{\mathbb{Q}}, P_b)$ with Lehmer's conjecture. In the same chapter, we prove the following theorem (see (4.11) and (4.16)):

Theorem E. *The theory of $(\overline{\mathbb{Q}}, S_\epsilon)$ is not simple and has the independence property in the language $L_m(U)$. Moreover, if the theory of $(\overline{\mathbb{Q}}, P_b)$ is simple for some $b > 1$ in*

$L_m(U)$, then Lehmer's conjecture holds for all Salem numbers.

In Chapter 5, the results are in the realm of nonstandard analysis which we apply to find certain height bounds. Nonstandard analysis was originated in the 1960's by the work of A. Robinson, which was arose as a rigorous and exhaustive way of studying infinitesimal calculus. We refer the reader to [15, 14] for a treatment of the topic. Given a field K, if $f_0, f_1, ..., f_s$ are in $K[X_1, ..., X_n]$ all have degree less than D and f_0 is in $\langle f_1, ..., f_s \rangle$, then $f_0 = \sum_{i=1}^{s} f_i h_i$ for certain h_i whose degrees are bounded by a constant $C = C(n, D)$ depending only on n and D. This result was first established in a paper of G. Hermann [16] using algorithmic tools. Then the same result was proved by L. van den Dries and K. Schmidt [8] using nonstandard methods, and they paved the way for how nonstandard methods can be used for such bounds. Let R be a commutative domain and $\theta : \mathbb{N} \to \mathbb{N}$ be a function. A function

$$h : R \to [0, \infty)$$

is said to be a *height* function of θ-type if for any x and y in R with $h(x) \leq n$ and $h(y) \leq n$, then both $h(x + y) \leq \theta(n)$ and $h(xy) \leq \theta(n)$. We say that h is a *height* function on R if h is a height function of θ-type for some $\theta : \mathbb{N} \to \mathbb{N}$. In plain words, a height function behaves well under certain arithmetic operations.

We can extend the height function h to the polynomial ring $R[X_1, ..., X_n]$ by setting

$$h\Big(\sum_{\alpha} a_\alpha X^\alpha\Big) = \max_{\alpha} h(a_\alpha).$$

Inspired by [8], using nonstandard methods, we prove the existence of certain height bounds on the complexity of the coefficients of some polynomials. This allows us to characterize the ideal membership of a given polynomial. Moreover, we obtain a bound for the logarithmic height function, which enables us to test the primality of an ideal. We say that an ideal I of $\overline{\mathbb{Q}}[X_1, ..., X_n]$ is a (D, H)-type ideal if the degree of all generators of I is bounded by D and the logarithmic height of all generators of I is bounded by H. For the following theorem, the degree bounds c_1 and B follow from [8]. We prove the existence of the constants c_2 and c_3 below, see (5.12) and (5.17):

Theorem F. *Let R be a commutative domain with a height function. For all $n \geq 1$, $D \geq 1$ and $H \geq 1$ there are two constants $c_1(n, D)$ and $c_2(n, D, H)$ such that if $f_1, ..., f_s$ in $R[X_1, ..., X_n]$ have no common zero in a field containing R with $\deg(f_i) \leq D$ and $h(f_i) \leq H$, then there exist nonzero a in R and $h_1, ..., h_s$ in $R[X_1, ..., X_n]$ such that*

(i) $a = f_1 h_1 + \cdots + f_s h_s$

(ii) $\deg(h_i) \leq c_1$

(iii) $h(a), h(h_i) \leq c_2$.

Moreover, if $R = \overline{\mathbb{Q}}$ *and* h *is the logarithmic height function, there are bounds* $B(n, D)$ *and* $c_3(n, D, H)$ *such that if* I *is a* (D, H)*-type ideal of* $\overline{\mathbb{Q}}[X_1, ..., X_n]$ *then* I *is prime if and only if* $1 \notin I$*, and for all* f*,* g *in* $\overline{\mathbb{Q}}[X_1, ..., X_n]$ *of degree less than* $B(n, D)$ *and height less than* $c_3(n, D, H)$*, if* $fg \in I$*, then either* f *or* g *is in* I*.*

Introduction en Français

Dans cette thèse, nous étudions les corps avec un prédicat pour dénoter soit des éléments de hauteur bornée soit des sous-groupes multiplicatifs qui satisfont une certaine condition diophantienne. En particulier, nous appliquons l'analyse nonstandard pour trouver des bornes de hauteur et la théorie des modèles géométrique pour analyser l'extension des corps algébriquement clos avec un groupe distingué.

En théorie des modèles, l'un des principaux objectifs est de comprendre les ensembles définissables dans une structure. Rappelons qu'un ensemble est appelé *définissable* s'il est donné par une formule du premier-ordre. Un type-définissable correspond à l'intersection des ensembles définissables. Une autre notion importante en théorie des modèles est l'interprétabilité. Un ensemble est dit *interprétable* s'il est un quotient d'un ensemble définissable par une relation d'équivalence définissable. Pour illustrer, dans un corps algébriquement clos, les ensembles définissables sont les ensembles constructibles de géométrie algébrique et les groupes définissables correspondent aux groupes algébriques.

La théorie de la classification est un projet vaste en théorie des modèles, issue des travaux de M. Morley dans les années 1960 et S. Shelah dans les années 1970, poursuivant la catégorisation des théories de premier-ordre basée sur combien de différence il y a entre leurs modèles, et également de reprendre des informations structurelles sur les modèles. La question de savoir combien de modèles une théorie peut avoir a été au cœur des développements les plus fondamentaux dans l'histoire de la théorie des modèles. Les théories stables sont importantes pour classifier leurs modèles. Si une théorie n'est pas stable alors ses modèles sont trop complexes et nombreux à classifier, et l'arithmétique s'adapte dans ce cas. Une théorie T est dite *stable* s'il n'y a aucune formule du premier-ordre dans T qui définit un ordre linéaire infini. Les théories stables jouissent d'une notion d'*indépendance*, nommé l'indépendance de *déviation*, avec un ensemble de propriétés bien définies, rappellant l'indépendance algébrique dans les corps et l'indépendance linéaire dans les espaces vectoriels. Une théorie est *simple*, si en effet elle est caractérisée par la présence d'une telle notion d'indépendance qui est symétrique. Une théorie T a la *propriété d'indépendance*, s'il y a une formule du premier-ordre dans un certain modèle de T qui peut coder tout sous-ensemble donné

d'un ensemble fini arbitrairement grand. Il est connu que les théories stables sont simples et n'ont pas la propriété d'indépendance. Par exemple, la théorie des corps algébriquement clos et les groupes abéliens dans le language de groupes sont stables. La théorie des nombres réels est instable puisque l'on peut définir l'ordre, mais elle n'a pas la propriété d'indépendance.

Dans le Chapitre 2, nous étudions les corps algébriquement clos avec un sous-groupe multiplicatif distingué en termes de stabilité. Soient K un corps algébriquement clos, le corps $\mathbb{F}$ son corps premier et G un sous-groupe multiplicatif de $K^\times$. Dans cette thèse, une notion fondamentale à propos des groupes multiplicatifs est la *propriété Mann*. Pour définir cette propriété, considérons une équation

$$a_1x_1 + \cdots + a_nx_n = 1 \tag{0.0.3}$$

avec $n \geq 2$ et $a_i \in \mathbb{F}$. Une solution $(g_1, ..., g_n)$ de cette équation est appelée non-dégénérée si pour tout sous-ensemble non-vide I de $\{1, 2, ..., n\}$, la somme $\sum_{i\in I} a_ig_i$ n'est pas zéro. On dit que G a la *propriété Mann* si chaque équation (0.0.3) a seulement un nombre fini non-dégénéré de solutions dans G. En 1965, H. Mann [26] a montré que l'ensemble des racines de l'unité μ a la propriété Mann. Plus tard, il était prouvé que tout groupe multiplicatif de rang fini dans tout corps de caractéristique zéro, par exemple $2^{\mathbb{Z}}$, a la propriété Mann; voir [32, 13, 23]. En 1990, dans une note non-publiée, B. Zilber [41] a démontré que la paire $(\mathbb{C}, \mu)$ est stable en utilisant un résultat de H. Mann [26].

Maintenant nous fixons K et G comme ci-dessus. La théorie des modèles de la paire (K, G) était d'abord étudiée dans l'article de L. van den Dries et A. Günaydın [9]. Le résultat de B. Zilber [41] était généralisé par L. van den Dries et A. Günaydın [9] à (K, G) où G a la propriété Mann. Ils avaient axiomatisé (K, G) et prouvé aussi que la théorie de (K, G) est stable. De plus dans [9], L. van den Dries et A. Günaydın ont montré que G a la propriété Mann sur K, cela siginifie que si a_i dans K dans (0.0.3), nous avons encore un nombre fini non-dégénéré de solutions dans G. Ils ont prouvé que chaque sous-ensemble de G^n définissable dans (K, G) est définissable dans le groupe abélien G, en d'autres termes la structure induite sur G est la structure pur groupe.

Dans le Chapitre 2, nous caractérisons premièrement la clôture algébrique dans (K, G). Cela nous permet de caractériser l'indépendance dans la paire (K, G) en termes d'indépendance algébrique de K avec l'aide du groupe G. Dans [31], B. Poizat a caractérisé l'indépendance pour des belles paires, en particulier pour les paires de corps algébriquement clos. Il s'avère que l'indépendance en (K, G) est différente des belles paires et elle est plus simple dans un sens. Les groupes définissables dans les théories

stables sont des sujets récurrents en théorie des modèles et ils jouent un rôle significatif pour la théorie de classification mis au point par B. Zilber dans les années 1970. Dans le Chapitre 2, après avoir caractérisé l'indépendance dans la paire, nous tournons notre attention vers les groupes définissables dans la paire (K, G). La caractérisation de la clôture algébrique du modèle théorique et l'indépendance dans (K, G) nous permet de caractériser les groupes définissables dans (K, G), à isogénie près, en termes de groupes définissables et interprétables dans K et G. La démonstration entraîne une technique bien connue de la théorie de stabilité géométrique, nommée le théorème de la configuration de groupe [19]. Nous suivons l'approche de [2], où T. Blossier et A. Martin-Pizarro ont caractérisé les groupes interprétables dans les paires d'extension propre de corps algébriquement clos en utilisant le résultat de A. Pillay [28] . Plus précisément, nous prouvons le résultat suivant:

Théorème A. *(Groupes définissables* (2.30)*) Soient K un corps algébriquement clos et G un sous-groupe multiplicatif de $K^\times$ avec la propriété Mann. Tout groupe de type-définissable dans (K, G) est isogène à un sous-groupe d'un groupe algébrique. De plus, un groupe type-définissable est, à isogénie près, une extension d'un groupe type-interprétable dans G par un groupe algébrique.*

Dans le cas où G est divisible, nous pouvons caractériser les groupes interprétables dans (K, G) en termes des groupes définissables de (K, G). Notre méthode sera de combiner les méthodes des [2] et [28] et nous prouvons le théorème suivant:

Théorème B. *(Groupes interprétables* (2.56)*) Soient K un corps algébriquement clos et G un sous-groupe multiplicatif divisible de $K^\times$ avec la propriété Mann. Chaque groupe interprétable H dans (K, G) est, à isogénie près, une extension d'un groupe commutatif interprétable dans G d'un T_P-groupe interprétable N, qui est un quotient d'un groupe algébrique V par un sous-groupe N_1 qui est un groupe commutatif interprétable dans G.*

Dans le Chapitre 3, nous étudions les corps algébriquement clos étendus par deux prédicats unaires qui dénotent un sous-corps algébriquement clos et un sous-groupe multiplicatif. Ce sera une extension propre du corps algébriquement clos avec un groupe satisfaisant la propriété Mann comme dans le Chapitre 2, et aussi les paires de corps algébriquement clos. Plus précisément: soient Ω un corps algébriquement clos, k un sous-corps de Ω qui est aussi algébriquement clos et Γ un sous-groupe multiplicatif de $\Omega^\times$. Considérons l'équation

$$k_1 x_1 + \cdots + k_n x_n = 1 \tag{0.0.4}$$

avec $n \geq 1$ et $k_i \in k$.

Nous disons que (k, Γ) est une *paire Mann* si pour tout n il y a un sous-ensemble fini $\Gamma(n)$ de Γ tel que pour tout $k_1, ..., k_n \in k^\times$ toutes les solutions non dégénérées de

(0.0.4) dans Γ se trouvent dans $\Gamma(n)$. En particulier, le groupe Γ a la propriété Mann et l'intersection $k \cap \Gamma$ est finie. C'est une version uniforme de la propriété Mann. Par exemple, la paire $(\overline{\mathbb{Q}}, \exp(\overline{\mathbb{Q}}))$ est une paire Mann par le théorème de Lindemann.

Maintenant nous fixons Ω, k et Γ comme ci-dessus. La théorie des modèles du triple (Ω, k, Γ) était premièrement étudiée par L. van den Dries et A. Günaydın [10, 11]. Entre autres, ils ont axiomatisé le triple et caractérisé les ensembles définissables par élimination d'un rapport de quantificateur. Ils ont aussi prouvé que la théorie de (Ω, k, Γ) est stable. De plus dans [10], L. van den Dries et A. Günaydın ont prouvé que si l'intersection $k \cap \Gamma$ est triviale et Γ a le rang fini, alors (k, Γ) est une paire Mann.

Dans le Chapitre 3, comme une première étape nous caractérisons la clôture algébrique du modèle théorique dans le triple (Ω, k, Γ). Cela nous permet de caractériser l'indépendance dans le triple. On obtient ainsi que l'indépendance est donnée par l'indépendance algébrique. Après avoir caractérisé l'indépendance dans le triple, nous nous concentrons sur les groupes définissables et interprétables dans le triple. Nous suivons le même chemin que dans le Chapitre 2. Nous caractérisons d'abord les groupes définissables, à isogénie près, dans le triple en termes de groupes définissables et interprétables dans Ω, k et Γ. Encore la preuve nécessite le théorème de la configuration du groupe et la motivation vient du [2].

Théorème C. *(Groupes définissables* (3.23)*) Soient Ω un corps algébriquement clos, le corps k un sous-corps propre de Ω qui est aussi algébriquement clos et Γ un sous-groupe multiplicatif de $\Omega^\times$ où (k, Γ) est une paire Mann. Tout groupe de type-définissable dans (Ω, k, Γ) est isogène à un sous-groupe d'un groupe algébrique. De plus, un groupe de type-définissable est, à isogénie près, l'extension d'une somme directe de k-rationnels points d'un groupe algébrique défini sur k et un groupe de type-interprétable commutatif dans Γ par un groupe algébrique.*

Quand Γ est divisible, la caractérisation des groupes définissables dans le triple nous permet de caractériser les groupes interprétables. En particulier, nous concluons:

Théorème D. *(Groupes interprétables* (3.33)*) Soient Ω un corps algébriquement clos, le corps k un sous-corps propre de Ω qui est aussi algébriquement clos et Γ un sous-groupe multiplicatif divisible de $\Omega^\times$ où (k, Γ) est une paire Mann. Tout groupe interprétable H dans (Ω, k, Γ) est, à isogénie près, une extension d'un produit cartésien de k-rationnels points d'un groupe algébrique défini sur k et un groupe commutatif interprétable dans Γ par un groupe interprétable N, qui est le quotient d'un groupe algébrique par un sous-groupe N_1 qui est isogène à un produit cartésien de k-rationnels points d'un groupe algébrique défini sur k et un groupe commutatif interprétable dans Γ.*

Un autre objectif de ces travaux est d'étudier le corps des nombres algébri-ques avec des éléments de petite hauteur. Dans le Chapitre 4, notre préoccupation sera princi-

palement la théorie des modèles des nombres algébriques avec un certain prédicat. La théorie des modèles des paires a été étudiée pendant un certain temps. Plus généralement, les théories stables avec un prédicat ont été étudiées dans l'article de E. Casanovas et M. Ziegler [6]. Leur résultat dans [6] implique le résultat de B. Zilber [41] et aussi la stabilité de la théorie des paires de corps algébriquement clos [31].

Une fonction hauteur est une fonction qui mesure la complexité d'un élément. Cette notion est fondamentale à la base de la géométrie diophantienne. L'exemple le plus significatif est la fonction de la hauteur logarithmique sur le corps des nombres algébriques. Avant de définir la fonction de la hauteur logarithmique, nous définissons d'abord la mesure de Mahler d'un polynôme sur $\mathbb{C}$. Pour un polynôme

$$f(x) = a_d(X - \alpha_1) \cdots (X - \alpha_d),$$

sa *mesure de Mahler* est définie par le produit

$$m(f) = |a_d| \prod_{|\alpha_j| \geq 1} |\alpha_j|.$$

Soit $\overline{\mathbb{Q}}$ le corps des nombres algébriques. Pour α dans $\overline{\mathbb{Q}}$ avec un polynôme minimal $f(x) \in \mathbb{Z}[X]$ de degré d, nous définissons sa *mesure de Mahler* par $m(\alpha) = m(f)$ et la *hauteur logarithmique* de α est définie par

$$h(\alpha) = \frac{\log m(\alpha)}{d}.$$

Le théorème de Kronecker, qui est une caractérisation d'être une racine de l'unité, indique que $h(\alpha) = 0$ si et seulement si α est une racine de l'unité ou zéro. La *conjecture de Lehmer*, qui est toujours ouverte, indique qu'il existe une constante absolue $c > 1$ telle que si $m(\alpha) > 1$, alors $m(\alpha) \geq c$. En d'autres termes, cela affirme que la mesure de Mahler est bornée loin de 1, sauf pour l'ensemble des racines de l'unité. Cette question a été posée par D. Lehmer [24] vers 1933. L'exemple le plus connu de la plus petite mesure de Mahler supérieure à 1 a également été donnée par Lehmer: si α est une racine du polynôme

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1,$$

alors $m(\alpha) \approx 1,17628$. Un entier algébrique réel $\alpha > 1$ est appelé un *nombre Salem* si α et $1/\alpha$ sont Galois conjugués et tous les autres Galois conjugués de α se trouvent sur le cercle unité. De toute évidence, pour tout nombre Salem α, nous avons $m(\alpha) = \alpha$. Une autre question ouverte pertinente est de savoir si 1 est un point limite des nombres Salem. Les détails peuvent être trouvés dans les expositions [35, 36] de C. Smyth.

Les propriétés modèles-théoriques de $\overline{\mathbb{Q}}$ sont bien connues; le corps $\overline{\mathbb{Q}}$ a l'élimination des quantificateurs, ce qui correspond à un résultat de géométrie algébrique: la projection d'un ensemble constructible étant constructible. De plus, il est stable. Soient $P_b = \{a \in \overline{\mathbb{Q}} : m(a) \leq b\}$ où $b \geq 1$ et $S_\epsilon = \{a \in \overline{\mathbb{Q}} : h(a) \leq \epsilon\}$ où $\epsilon > 0$. Notons que P_b et S_ϵ contiennent l'ensemble des racines de l'unité μ. Autrement dit, la conjecture de Lehmer indique qu'il y a $b > 1$ tel que $P_b = P_1 = \mu$. Les paires $(\overline{\mathbb{Q}}, P_b)$ et $(\overline{\mathbb{Q}}, S_\epsilon)$ peuvent être considérées comme $L_m(U) = L_m \cup \{U\}$ structures où $L_m = \{1, \cdot\}$, l'operation $\cdot$ est la multiplication usuelle et U est un symbole de relation unaire dont l'interprétations sont P_b et S_ϵ respectivement.

Dans le Chapitre 4, nous étudions la théorie des modèles de $(\overline{\mathbb{Q}}, S_\epsilon)$ et nous prouvons un résultat qui montre que de petites perturbations de la propriété d'être une racine de l'unité change radicalement les propriétés de stabilité de la structure ambiante. Nous nous rapportons aussi à la simplicité de la paire $(\overline{\mathbb{Q}}, P_b)$ avec la conjecture de Lehmer. Dans le même chapitre, nous prouvons le théorème suivant (voir (4.11) et (4.16)):

Théorème E. *La théorie de $(\overline{\mathbb{Q}}, S_\epsilon)$ n'est pas simple et a la propriété de l'indépendance dans le language $L_g(U)$. De plus, si la théorie de $(\overline{\mathbb{Q}}, P_b)$ est simple pour un certain $b > 1$ dans $L_g(U)$, alors la conjecture de Lehmer pour tous les nombres Salem est vraie.*

Dans le Chapitre 5, les résultats sont dans le domaine de l'analyse nonstandard que nous appliquons pour trouver certaines bornes hauteur. Dans les années 1960, l'analyse nonstandard est apparu avec les travaux de A. Robinson, qui a surgi de façon rigoureuse et exhaustive pour étudier le calcul infinitésimal. Nous renvoyons le lecteur à [15, 14] pour un traitement du sujet. Étant donné un corps K, si $f_0, f_1, ..., f_s$ sont dans $K[X_1, ..., X_n]$ et ont tous un degré inférieur à D et f_0 est dans $\langle f_1, ..., f_s \rangle$ alors $f_0 = \sum_{i=1}^{s} f_i h_i$ pour certains h_i dont les degrés sont délimités par $C = C(n, D)$ une constante ne dépendant que de n et D. Ce résultat a été établi dans un article de G. Hermann [19] en utilisant des outils algorithmiques. Ensuite, le même résultat a été prouvé par L. van den Dries et K. Schmidt [8] à l'aide des méthodes nonstandard, et ils ont ouvert la voie à la façon dont les méthodes nonstandard peuvent être utilisées pour de telles bornes. Soit $\theta : \mathbb{N} \to \mathbb{N}$ une fonction. Nous disons que

$$h : R \to [0, \infty)$$

est une fonction *hauteur* de type θ si pour tout x et y en R avec $h(x) \leq n$ et $h(y) \leq n$, puis les deux $h(x + y) \leq \theta(n)$ et $h(xy) \leq \theta(n)$. Nous disons que h est une fonction *hauteur* sur R si h est une fonction hauteur de type θ pour un certain $\theta : \mathbb{N} \to \mathbb{N}$.

Nous pouvons étendre la fonction hauteur h à l'anneau de polynômes $R[X_1, ...X_n]$

par

$$h\Big(\sum_{\alpha} a_\alpha X^\alpha\Big) = \max_{\alpha} h(a_\alpha).$$

Inspiré par [8], en utilisant les méthodes nonstandard, nous prouvons l'existence de certaine hauteur des bornes sur la complexité des coefficients de certains polynômes. Cela nous permet de caractériser l'appartenance idéale d'un polynôme donné. De plus, nous obtenons une borne pour la fonction de la hauteur logarithmique, ce qui nous permet de tester la primalité d'un idéal. On dit qu'un idéal I de $\overline{\mathbb{Q}}[X_1, ..., X_n]$ est un type (D, H) idéal si le degré de tous les générateurs de I est borné par D et la hauteur logarithmique de tous les générateurs de I est bornée par H. Pour le théorème suivant, les degrés bornés c_1 et B viennent d'après [8]. Nous démontrons l'existence des constantes c_2 et c_3 ci-dessous, voir (5.12) et (5.17):

Théorème F. *Soit R un anneau avec une fonction de la hauteur. Pour tous les $n \geq 1$, $D \geq 1$ et $H \geq 1$ il existe deux constantes $c_1 = c_1(n, D)$ et $c_2 = c_2(n, D, H)$ de telle sorte que si $f_1, ..., f_s$ dans $R[X_1, ..., X_n]$ n'ont pas de zéro en commun dans un corps qui contient R avec $\deg(f_i) \leq D$ et $h(f_i) \leq H$, alors il existe une valeur non nulle a dans R et $h_1, ..., h_s$ dans $R[X_1, ...X_n]$ tel que*

(i) $a = f_1h_1 + \cdots + f_sh_s$

(ii) $\deg(h_i) \leq c_1$

(iii) $h(a), h(h_i) \leq c_2$.

De plus, si $R = \overline{\mathbb{Q}}$ et h est la fonction de la hauteur logarithmique, il existe deux constantes $B(n, D)$ et $c_3(n, D, H)$ telles que, si I est un type (D, H) idéal de $\overline{\mathbb{Q}}[X_1, ..., X_n]$, alors I est premier ssi $1 \notin I$ et pour tout f, g dans $\overline{\mathbb{Q}}[X_1, ..., X_n]$ de degré inférieur à $B(n, D)$ et la hauteur inférieure à $c_3(n, D, H)$, si $fg \in I$ alors soit f ou g est dans I.

1

Preliminaries and Notations

This chapter provides an introduction to the tools and notations which will be used later on. We assume familiarity with algebraic number theory and basic model theory. Some of these will be given briefly, with no proofs, since the detailed arguments can be found in [4, 18, 29, 37].

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers. For a given field K, the algebraic closure of K will be denoted by K^{ac}.

1.1 The Logarithmic Height Function

In this section we define the logarithmic height function which will be needed in chapters 4 and 5 and we refer the reader to [4, Chapter 1] and [18, Chapter 5].

An *absolute value* on a field K is a map $|\cdot| : K \to [0, \infty)$ from K to positive real numbers including zero such that for every x, y in K,

- $|x| = 0$ if and only if $x = 0$,
- $|xy| = |x||y|$,
- There exists an absolute constant C such that $|x + y| \leq C \max(|x|, |y|)$.

Let $M_{\mathbb{Q}} = \{|\cdot|_p : p \text{ prime or } \infty\}$ be the set of representative of absolute values on $\mathbb{Q}$, where if $p = \infty$ then $|\cdot|_p$ is the ordinary absolute value on $\mathbb{Q}$, and if p is prime

then the absolute value is the p-adic absolute value on $\mathbb{Q}$ with $|p|_p = 1/p$.

Let K be a number field with the ring of integers $\mathcal{O}$. Now we define the set of standard absolute values M_K on K. Suppose that there are r_1 real embeddings and r_2 pairs of complex embeddings of K. Therefore we have $[K : \mathbb{Q}] = r_1 + 2r_2$. Every embedding σ from K into $\mathbb{C}$ provides an absolute value. Thus we have $r_1 + r_2$ absolute values defined by $|x|_\sigma = |\sigma(x)|$ if σ is real and $|x|_\sigma = |\sigma(x)|^2$ if σ is complex. These absolute values are the archimedean absolute values on K. Now we define the non-archimedean absolute values on K. For a non-zero ideal I of $\mathcal{O}$, we denote the norm of I by $N(I)$. If q is a prime number factoring by $q\mathcal{O} = {p_1}^{e_1} \cdots {p_m}^{e_m}$ with $N(p_i) = q^{f_i}$ and $\sum_{i \le m} e_i f_i = [K : \mathbb{Q}]$, then for each nonzero prime ideal p in $\mathcal{O}$, we define the absolute value $|a|_p$ on K by $|a|_p = N(p)^{-v_p(a)}$, where $v_p(a)$ is the exponent of p in the prime factorization of the ideal $a\mathcal{O}$ for nonzero $a \in \mathcal{O}$, and $v_p(a/b) = v_p(a) - v_p(b)$ for any nonzero $a, b \in \mathcal{O}$. As usual, we let $v_p(0) = \infty$. By the set M_K we mean these absolute values on K.

An important result for the absolute values on a number field is the product formula.

Proposition 1.1. *[4, 1.4.4](Product Formula) Let K be a number field and M_K be the set of absolute values on K extending the absolute values in $M_\mathbb{Q}$. For any nonzero $\alpha \in K$,*

$$\prod_{v \in M_K} |\alpha|_v = 1.$$

Since all but finitely many of the $|\alpha|_v$'s are 1, this infinite product is actually a finite product, so it is well-defined.

Before defining the logarithmic height function we first define the *Mahler measure* of a polynomial over $\mathbb{C}$. For a polynomial $f(x) = a_d(X - \alpha_1) \cdots (X - \alpha_d)$, its *Mahler measure* is defined as the product

$$m(f) = |a_d| \prod_{|\alpha_j| \ge 1} |\alpha_j|.$$

For α in $\overline{\mathbb{Q}}$ with minimal polynomial $f(x) \in \mathbb{Z}[X]$, we define its Mahler measure as $m(\alpha) = m(f)$. The absolute *non-logarithmic height* of α is defined as

$$H(\alpha) = m(\alpha)^{1/d}$$

where $d = \deg f$. Then *the logarithmic height* of α is defined as

$$h(\alpha) = \log H(\alpha) = \frac{\log m(\alpha)}{d}.$$

It is not known whether there exists an absolute constant $c > 1$ such that if $m(\alpha) > 1$ then $m(\alpha) \geq c$. This question was posed by D. Lehmer [24] around 1933, who claimed that the polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

has the smallest Mahler measure, which is approximately 1.17628. For a detailed exposition on Mahler measure and Lehmer's problem, see [35].

The logarithmic height function measures the arithmetic complexity of an algebraic number and it behaves well under arithmetic operations. However, using the definition above, it is not immediate to see. Here we give an equivalent definition based on the absolute values on a number field.

Let α be an algebraic number contained in a number field K. Then we set

$$h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} \max\{0, \log |\alpha|_v\}.$$

The definition is independent from the choice of the number field K, and now one can see the logarithmic height function behaves well under arithmetic operations:

- $h(0) = h(1) = 0$,
- For a rational number a/b where a and b are coprime,

$$h(a/b) = \max\{\log |a|, \log |b|\},$$

- For all α in $\overline{\mathbb{Q}}$, we have $h(\alpha^n) = nh(\alpha)$ where $n \in \mathbb{N}$,
- For all α and β in $\overline{\mathbb{Q}}$, we have $h(\alpha + \beta) \leq h(\alpha) + h(\beta) + \log 2$,
- For all α and β in $\overline{\mathbb{Q}}$, we have $h(\alpha\beta) \leq h(\alpha) + h(\beta)$,
- For all non-zero α in $\overline{\mathbb{Q}}$, we have $h(1/\alpha) = h(\alpha)$.

1.2 Model Theory and Stability

In this section, we recall briefly some definitions and theorems in model theory. Details can be found in [29, 30, 34, 37].

From now on, let T be a complete theory in a fixed language L and fix a sufficiently saturated and strongly homogeneous model $\mathcal{U}$ of T. An *imaginary element* is an equivalence class d/E for some tuple d in $\mathcal{U}$ and some definable equivalence relation E on $\mathcal{U}$. Let $(E_i)_{i \in I}$ be a list of all 0-definable equivalence relations on $\mathcal{U}$ with

n_i-tuples. Let $\mathcal{U}^{\text{eq}}$ be the many-sorted structure $(\mathcal{U}, \mathcal{U}^{n_i}/E_i)$ and T^{eq} be its complete theory in the appropriate many-sorted language L^{eq}. Note that the elements in the home sorts $\mathcal{U}^{n_i}/E_i$ are exactly the imaginary elements. Moreover, the theory T^{eq} and the language L^{eq} expand T and L respectively. The algebraic closure and the definable closure in $\mathcal{U}^{\text{eq}}$ are denoted by acl^{eq} and dcl^{eq} respectively. We say that T has *elimination of imaginaries*, EI for short, if every imaginary element is interdefinable with some real tuple, that is to say for every imaginary element e there is a real tuple c such that $e \in \text{dcl}^{\text{eq}}(c)$ and $c \in \text{dcl}(e)$. It is known that T^{eq} eliminates imaginaries. Elimination of imaginaries enables us to work with quotients in the original model. We say that T has *weak elimination of imaginaries*, WEI for short, if for every imaginary element e there is a real tuple c such that $e \in \text{dcl}^{\text{eq}}(c)$ and $c \in \text{acl}(e)$. The theory T has *geometric elimination of imaginaries*, GEI for short, if every imaginary element is interalgebraic with a real element.

Example 1.2. *Let ACF_p and DCF_p be the theories of algebraically closed and differentially closed fields with a fixed characteristic p. Then they both have QE and EI. Let μ be the set of complex roots of unity. Then $Th(\mu)$ has QE and WEI in the pure group language.*

Now we recall some definitions from stability. In model theory, *Morley rank* is a natural notion of dimension on the definable sets of $\mathcal{U}$. It generalizes the Krull dimension from algebraic geometry. Let X be a definable subset. We define the relation $\text{MR}(X) \geq \alpha$ by induction on the ordinal α.

- The Morley rank $\text{MR}(X) \geq 0$ if and only X is not empty,
- For a limit ordinal α, the Morley rank $\text{MR}(X) \geq \alpha$ if $\text{MR}(X) \geq \beta$ for all $\beta < \alpha$,
- We say that $\text{MR}(X) \geq \alpha + 1$ if there are pairwise disjoint definable sets $X_i \subset X$ for $i < \omega$ such that $\text{MR}(X_i) \geq \alpha$ for all i.

If X is empty, we define $\text{MR}(X) = -\infty$. If X is not empty, we say that $\text{MR}(X) = \alpha$ if $\text{MR}(X) \geq \alpha$ but $\text{MR}(X) \ngeq \alpha + 1$. If there is no such an ordinal, we let $\text{MR}(X) = \infty$ and we say that the Morley rank does not exist. The Morley rank of a theory is the Morley rank of the formula $x = x$.

Definition 1.3.

- *Let $\mathbb{M} \models T$ and $\phi(x)$ be a non-algebraic formula $L(M)$-formula. The set $\phi(\mathbb{M})$ is called minimal in $\mathbb{M}$ if for all $L(M)$-formulas $\psi(x)$, the intersection $\phi(\mathbb{M}) \cap \psi(\mathbb{M})$ is either finite or cofinite in $\phi(\mathbb{M})$.*
- *The formula $\phi(x)$ is strongly minimal if $\phi(x)$ defines a minimal set in any elementary extension of $\mathbb{M}$.*
- *A theory T is called strongly minimal if the formula $x = x$ is strongly minimal.*

- *A formula $\psi(x)$ is called almost strongly minimal if there is a strongly minimal formula $\varphi(x)$ defined over a set of parameters B such that, for every $\mathbb{M} \models T$ containing B, we have*

$$\psi(\mathbb{M}) \subseteq \mathrm{acl}(\varphi(\mathbb{M}), B).$$

Remark 1.4. *Morley rank is definable in almost strongly minimal formulas. More precisely, for any almost strongly minimal formula $\phi(x)$ and any formula $\psi(x_1, ..., x_n, y)$ which implies $\phi(x_i)$ for all i, the set $\{b : \mathrm{MR}(\psi(x_1, ..., x_n, b)) = k\}$ is definable for every k in $\mathbb{N}$.*

We now define stability.

Definition 1.5. *Let κ be an infinite cardinal. We say that T is κ-stable if in each model of T, over every parameter set of size at most κ, there are at most κ many n-types for any integer $n \geq 1$, i.e*

$$|A| \leq \kappa \implies |S_n(A)| \leq \kappa.$$

A theory T is said to be stable if it is κ-stable for some infinite cardinal κ.

In ω-stable theories, Morley rank always exists. Strongly minimal theories are ω-stable of Morley rank 1. By QE, one sees that ACF_p is strongly minimal. The theory DCF_0 is ω-stable with Morley rank ω. It is also known that abelian groups are stable in the pure group language. If T is stable, then so is T^{eq}.

Before defining the notion of independence, we need one more definition.

Definition 1.6. *Let I be a linear order, the set M be an L-structure and $A \subseteq M$ be a set of parameters. A family of elements $(a_i)_{i \in I}$ of M is called an indiscernible sequence over A if for all L-formulas $\phi(x_1, ..., x_n)$ over A and two increasing sequences $i_1 < ... < i_n$ and $j_1 < ... < j_n$ from I*

$$M \models \phi(a_{i_1}, ..., a_{i_n}) \leftrightarrow \phi(a_{j_1}, ..., a_{j_n}).$$

For instance, a transcendence basis in an algebraically closed field is an example of an indiscernible sequence over the empty set.

Next, we define dividing (forking) in stable theories.

Definition 1.7. *Let T be a stable theory.*
1- A formula $\varphi(x, a)$ is said to divide (fork) over A if there is an indiscernible sequence $(a_i)_i$ over A with $a_0 = a$ such that the set $\{\varphi(x, a_i)\}$ is inconsistent.
2- A type p divides over A if it contains some formula which divides over A.
3- As usual $a \underset{C}{\downarrow} b$ means that the type $\mathrm{tp}(a/bC)$ does not fork over C, and we read: a is independent from b over C.

4- A sequence $(a_i)_i$ is said to be independent over A if for every j we have that $a_j \underset{A}{\downarrow} \{a_i : i < j\}$. It is called a Morley sequence over A if it is indiscernible over A and independent over A.
5- A Morley sequence in a type $p \in S(A)$ is a Morley sequence over A consisting of realizations of p.

Example 1.8. *In algebraically closed fields, forking and algebraic independence agree. In vector spaces, forking is the linear independence. In pure sets, the independence $a \underset{C}{\downarrow} b$ means $\{a, C\} \cap \{b, C\} = C$.*

Fact (Kim): A formula $\varphi(x, a)$ does not divide over A if and only if there is a Morley sequence $(a_i)_i$ in $\mathrm{tp}(a/A)$ such that the set $\{\varphi(x, a_i)\}$ is consistent.

In a stable theory, forking independence has the following properties. Let a be a tuple and $A \subseteq B$ be parameter sets.

- (Existence of nonforking extensions) There exists an element b such that $\mathrm{tp}(b/A) = \mathrm{tp}(a/A)$ and $b \underset{A}{\downarrow} B$.
- (Transitivity) For $B \subseteq C$, we have $a \underset{A}{\downarrow} C$ if and only $a \underset{A}{\downarrow} B$ and $a \underset{B}{\downarrow} C$.
- (Symmetry) If b is another tuple then $a \underset{A}{\downarrow} b$ if and only if $b \underset{A}{\downarrow} a$.
- (Finite character) $a \underset{A}{\downarrow} B$ if and only if $a \underset{A}{\downarrow} B_0$ for all finite $B_0 \subseteq B$.

Remark 1.9.
- *If $a \notin \mathrm{acl}(A)$, then $\mathrm{tp}(a/aA)$ divides over A.*
- *We have always $a \underset{A}{\downarrow} \mathrm{acl}(A)$. So by transitivity, for any A, B and C we have $A \underset{C}{\downarrow} B$ if and only if $\mathrm{acl}(A) \underset{\mathrm{acl}(C)}{\downarrow} \mathrm{acl}(B)$.*

Another important notion concerning stable groups is the notion of generics which we define next. For the details, we refer the reader to [30] and [29, Chapter 1, Section 6].

Definition 1.10. *Let T be a stable theory and M be a model of T. Let G be a type-definable group in M.*

- *By a relatively definable subset A of G we mean a set of the form $\{a \in G : \varphi(a)\}$ where $\varphi(x)$ is some formula in M.*
- *A relatively definable subset $A \subseteq G$ is called a generic in G, if G is covered by finitely many right or left translates of A.*
- *Let $B \subseteq M$ be a set of parameters. A type $p \in S_1(B)$ which contains the type $x \in G$ is called generic, if it only contains generic formulas.*

- *The connected component of G, denoted by G^0, is the intersection of all definable (relatively) subgroups of G of finite index. We say that G is connected if $G = G^0$.*

Theorem 1.11. *[30] Let T be a stable theory and M be a model of T. Let G be a type-definable group in M without parameters and B be a set of parameters in M. There exists a generic type over B. A generic type over B does not fork over the empty set. Moreover G is connected if and only if there is a unique generic type of G over a given set of parameters.*

Remark 1.12. *[29, Chapter 1, Lemma 6.9] Let G and p be as in* (1.10)*. Then p is generic if and only if whenever $a \models p$ and g is in G with $a \downarrow_A g$, then $g \cdot a \downarrow_A g$.*

An element g in G is called a generic if it realizes a generic type. Every non-forking extension of a generic is also generic. Every element of G can be written as a product of two generics. The product of two independent generics over A is also a generic over A and also it is independent from the each factor over A.

Now we recall stationarity.

Definition 1.13. *Let T be a stable theory and p be a type over A. The type p is said to be stationary if there is a unique non-forking extension to every B containing A.*

Definition 1.14. *Let T be stable with $M \models T$ and $M \subset A$ be a set of parameters. Let p be a type over M and $q \in S(A)$ an extension of p to A. We call q an heir of p if for every $L(M)$-formula $\varphi(x, y)$ such that $\varphi(x, a) \in q$ for some $a \in A$ there is some $m \in M$ with $\varphi(x, m) \in p$.*

Let T be a stable theory. Then types over models are stationary. Moreover in T^{eq}, a type over an algebraically closed set $A = \mathrm{acl}^{\mathrm{eq}}(A)$ is stationary. Let p be a type over a model M of T. If $M \subset A$ then $q \in S(A)$ is the non-forking extension of p to A if and only if q is an heir of p.

Next, we define canonical bases in stable theories which generalizes the field of definition of a variety from algebraic geometry.

Definition 1.15. *A canonical base for a type $p \in S(\mathcal{U})$ is a set B which is pointwise fixed by the same automorphism which leave p invariant.*

In stable theories, canonical bases always exist, however they can be imaginary tuples (possibly infinite). If T is ω-stable, then one can choose a canonical base to be a finite tuple. Note that if B_1 and B_2 are both canonical bases of p, then they are interdefinable. For $p(x) \in S(A)$ a stationary type (it can be viewed as a global type over $\mathcal{U}$), the canonical base of p, denoted by $\mathrm{Cb}(p)$, means the definable closure of a canonical base. Moreover, the type p does not fork over its canonical base $\mathrm{Cb}(p)$.

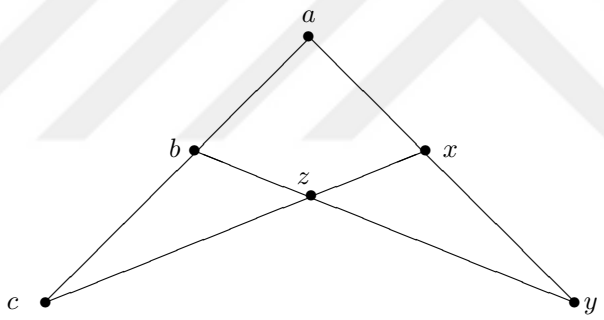
The next definition takes an important role in the classification of geometries.

Definition 1.16. *The theory T is one-based if for every real set A and every real tuple c the canonical base* $\mathrm{Cb}(c/A)$ *is algebraic over c.*

It is known that pure abelian groups are one-based. However ACF is not one-based; see [37, Chapter 10, Proposition 10.3.6].

Now, we give the group configuration theorem from geometric stability theory which is useful to construct a group from some independent elements. This theorem will be vital in Chapter 2 and Chapter 3. By a $*$-tuple, we mean a possibly infinite tuple $(a_i)_{i\in I}$ in $\mathcal{U}^{\mathrm{eq}}$ where the index set I has cardinality less than of saturation of $\mathcal{U}$. By a $*$-definable set, we mean a collection of $*$-tuples (each tuple being indexed by some fixed I), which is the set of realizations of a partial infinitary type. By a *$*$-definable group*, we mean a group G such that both G and the graph of multiplication are $*$-definable. A $*$-definable group G is called connected, if it has a unique generic in terms of the independence as given in remark (1.12).

Definition 1.17. *Let T be a stable theory and $\mathcal{U}$ its sufficiently saturated model. By a group configuration over A we mean 6-tuple of points (possibly infinite) (in $\mathcal{U}^{\mathrm{eq}}$) (a, b, c, x, y, z) such that*



- *any triple of non-collinear points are A-independent,*
- $\mathrm{acl}^{\mathrm{eq}}(A, a, b) = \mathrm{acl}^{\mathrm{eq}}(A, a, c) = \mathrm{acl}^{\mathrm{eq}}(A, b, c)$,
- *x and y are interalgebraic over Aa, the elements y and z are interalgebraic over Ab, the elements z and x are interalgebraic over Ac,*
- $a \in \mathrm{acl}^{\mathrm{eq}}(x, y, A)$, $b \in \mathrm{acl}^{\mathrm{eq}}(y, z, A)$ *and* $c \in \mathrm{acl}^{\mathrm{eq}}(x, z, A)$.

Now we give the group configuration theorem which states that every group configuration arises from a definable group with its independent generic elements.

Theorem 1.18. *[19] (The Group Configuration Theorem) Let T be a stable theory and $\mathcal{U}$ its sufficiently saturated model. Suppose $M \subset \mathcal{U}$ to be a $|T|^+$-saturated model of T, and suppose (a, b, c, x, y, z) is a group configuration over M. Then there is a $*$-definable group G in $\mathcal{U}^{\mathrm{eq}}$ over M, and there are elements a', b', c', x', y', z' of G which form a group configuration, each generic over M, such that the element a is*

interalgebraic with a' over M and the same holds for the other elements. Moreover, we have $a'x' = y'$, $b'y' = z'$, $c'x' = z'$ and $b'a' = c'$.

We recall some basic definitions which play a role in the classification theory which emerged from the work of Shelah [34].

Definition 1.19. *A formula $\phi(x, y)$ has the tree property (TP) if there is a set of parameters $(a_s : \emptyset \neq s \in \omega^{<\omega})$ such that*

(i) For all $s \in \omega^{<\omega}$, the set $(\phi(x, a_{si}) : i < \omega)$ is 2-inconsistent.

(ii) For all $\sigma \in \omega^{\omega}$, the set $(\phi(x, a_s) : \emptyset \neq s \subset \sigma)$ is consistent.

We say that T is simple if no formula has the tree property.

It is known that stable theories are simple. For more on simplicity we refer the reader to [37, 38].

Example 1.20. *The theory of dense linear orders without end points (DLO) is not simple because the formula $\phi(x, y, z) : y < x < z$ has tree property. Let I be the set of rational numbers in the interval $(0, 1)$ which is a model of DLO. We fix a tree of parameters $(q_s : \emptyset \neq s \in \omega^{<\omega})$ for this model which we will use in order to prove Theorem (4.11).*

Definition 1.21. *Let T be a complete L-theory where L is a language. An L-formula $\phi(x, y)$ is said to have the independence property (IP) if in every model M of T there is for each n a family of tuples $b_1, ..., b_n$, such that for each of the 2^n subsets I of $\{1, ..., n\}$ there is a tuple a_I in M for which*

$$M \models \phi(a_I, b_i) \iff i \in I.$$

A theory T is called NIP if no formula has IP.

It is known that stable theories are NIP. Moreover T is stable if and only if T is simple and NIP. For more on the subject; see [34].

Now we define the notion of smallness in a structure. For the following definitions, we follow [6, 40] and [9].

Given sets X and Y we write

$$f : X \xrightarrow{\text{n}} Y$$

to indicate that f is a map from X to the power set of Y such that $|f(x)| \leq n$ for all $x \in X$. For such a map f, its graph $Graph(f)$ is the set

$$\{(x, y) \in X \times Y : y \in f(x)\},$$

and for a subset Z of X, we set $f(Z) = \bigcup_{x\in Z} f(x)$. If X and Y are definable in a structure M, then such an f is called definable if $Graph(f)$ is.

Definition 1.22. *Let M be an L-structure and A be an infinite subset of M. We call A small if there is a pair (N, B) elementarily equivalent to (M, A) such that for every finite subset b of N every L–type over Bb is realized in N. We call A large if there exist natural numbers m and n and a definable function $f : M^m \xrightarrow{\text{n}} M$ such that $f(A^m) = M$.*

For the following lemma, the first part does not use strong minimality.

Lemma 1.23. *In strongly minimal theories, being large is equivalent to not being small.*

Proof. Let T be a strongly minimal theory, $M \models T$ and A be an infinite subset of M. Suppose that A is large in M. So there exist natural numbers m and n and a definable function $f : M^m \xrightarrow{\text{n}} M$ such that $f(A^m) = M$. Let (N, B) be elementarily equivalent to (M, A). So we also have that $f : N^m \xrightarrow{\text{n}} N$ is definable and $f(B^m) = N$. Thus the type

$$p(x) = \{x : (x, f(b_1, ..., b_m)) \notin Graph(f), b_i \in B\}$$

is not realized in N. Hence A is not small in M. Conversely suppose that A is not small in M. Since in strongly minimal theories there is a unique non-algebraic type over a given set of parameters, we conclude that for any (N, B) elementarily equivalent to (M, A), we have that $N = \text{acl}(B)$. By compactness, there is a formula $\phi(x, y)$ such that $\phi(x, b)$ is algebraic for all tuples $b \in B$ and $N = \bigcup_{b\in B} \phi(x, b)$. This gives the desired definable function and hence A is large in M. □

Finally, we define a nonstandard extension of a given structure. We apply nonstandard analysis in Chapters 4 and 5.

Definition 1.24. *(Nonstandard Extension of a Structure) Let $\mathbb{M}$ be a nonempty structure in a countable language L. A nonstandard extension ${}^*\mathbb{M}$ of $\mathbb{M}$ is an ultrapower of $\mathbb{M}$ with respect to a nonprincipal ultrafilter on $\mathbb{N}$.*

Now let ${}^*\mathbb{M}$ be a nonstandard extension of $\mathbb{M}$ with respect to a nonprincipal ultrafilter D on $\mathbb{N}$. Note that ${}^*\mathbb{M}$ is an elementary extension of $\mathbb{M}$ and the elements of ${}^*\mathbb{M}$ are of the form $(x_n)_n/D$ where $(x_n)_n$ is a sequence in $\mathbb{M}$. Ultraproduct of structures automatically become $\aleph_1$-saturated. For a subset A of $\mathbb{M}$, the set ${}^*A$ is defined to be the set

$$\{(a_n)_n/D : \{n : a_n \in A\} \in D\}.$$

Subsets of ${}^*\mathbb{M}$ of the form ${}^*A$ for some subset A of $\mathbb{M}$ are called internal. Not every subset of ${}^*\mathbb{M}$ need to be internal. The following sets ${}^*\mathbb{N}$, ${}^*\mathbb{Z}$, ${}^*\mathbb{Q}$, ${}^*\mathbb{R}$ are called hypernatural numbers, hyperintegers, hyperrational numbers and hyperreals respectively.

The elements ${}^*\mathbb{R} \setminus \mathbb{R}$ are called nonstandard real numbers. Let

$$\mathbb{R}_{fin} = \{x \in {}^*\mathbb{R} : |x| < n \text{ for some } n \in \mathbb{N}\}.$$

The elements in ${}^*\mathbb{R} \setminus \mathbb{R}_{fin}$ are called infinite.

Note that the notion of a nonstandard extension and its properties can be generalized to many-sorted structures. This will be significant for the definition of the height function which takes values in $\mathbb{R}$. For more detailed information about Nonstandard Analysis, the reader might consult [14, 15].

2

Algebraically Closed Field with a Group

2.1 Mann Property

In this chapter, we analyze algebraically closed fields with a distinguished multiplicative subgroup. Let K be an algebraically closed field, the field $\mathbb{F}$ its prime field and G be a multiplicative subgroup of $K^\times$. Consider an equation

$$a_1x_1 + \cdots + a_nx_n = 1 \tag{2.1.1}$$

with $n \geq 2$ and $a_i \in \mathbb{F}$. A solution $(g_1, ..., g_n)$ of this equation is called *non-degenerate* if for every non-empty subset I of $\{1, 2, ..., n\}$, the sum $\sum_{i\in I} a_ig_i$ is not zero. We say that G has *the Mann property* if every such equation (2.1.1) has only finitely many non-degenerate solutions in G. In [26], H. Mann showed that the set of complex roots of unity μ has the Mann property and his proof is effective. The rank of an abelian group G is the dimension of the $\mathbb{Q}$-vector space $G \otimes_{\mathbb{Z}} \mathbb{Q}$, where G is viewed as a $\mathbb{Z}$-module. In the 1980's, H. Mann's result was generalized and it was proved that any multiplicative group of finite rank (note that μ has rank 0) in any field of characteristic zero has the Mann property; see [32, 13, 23]. To illustrate, every finitely generated multiplicative subgroup of $\mathbb{C}^\times$ has the Mann property, such as $2^{\mathbb{Z}}3^{\mathbb{Z}}$. The result above is not true in the positive characteristic, for instance the multiplicative group of the algebraic closure of a finite field has rank 0, however it does not have the Mann property since the equation $x + y = 1$ has infinitely many non-degenerate solutions. More generally, the multiplicative group of an infinite field does not have the Mann

property. However, any cyclic group has the Mann property in all characteristics. The unit circle in $\mathbb{C}$ does not also have the Mann property. It was first B. Zilber in 1990, who considered the model theory of the pair $(\mathbb{C}, \mu)$. In his unpublished note, B. Zilber [41] showed that the pair $(\mathbb{C}, \mu)$ is ω-stable, using as a main tool the result of H. Mann.

Now fix K and G with the Mann property as above. By the pair (K, G), we mean the structure $(K, G, +, -, \cdot, 0, 1)$. So our language is $L(U) = \{+, -, \cdot, 0, 1, U\}$ where U is a unary relation whose interpretation in K is G. The model theory of the pair (K, G) was first studied in the paper of L. van den Dries and A. Günaydın [9], and it was proved that G is small in K as is defined in the previous chapter in the definition (1.22). So by changing the model if necessary, we may assume that K is $|G|^+$-saturated as a field. Moreover, we suppose that the pair (K, G) is κ-saturated for some big uncountable cardinal $\kappa \geq \omega_1$. In this chapter we will be working in this sufficiently saturated model. Among other things, in [9] an axiomatization of the theory of (K, G) was obtained by adding the constants to denote the collection of non-degenerate solutions of the equation (2.1.1). L. van den Dries and A. Günaydın [9] generalized B. Zilber's result to (K, G), where K is an algebraically closed field and G has the Mann property, that is to say, the theory of (K, G), denoted by T_P, is stable and if G is superstable (ω-stable) in the pure group language then so is the pair (K, G). In [9], it was also proved that the Mann property is global, which means we can choose a_i to be in K in (2.1.1) and this still gives finitely many non-degenerate solutions in G. Furthermore, L. van den Dries and A. Günaydın [9] showed that every subset of G^n definable in (K, G) is definable in the abelian group G, in other words the induced structure on G is just the pure abelian group structure.

Our results in this chapter will mainly concern the stability theoretical framework of the pair (K, G). In particular, we study the pair (K, G) in terms of geometric model theory. In the next two sections, we characterize algebraic closure and forking in the pair in terms of the algebraic closure and the independence in the pure field K. Then in 2.4, we characterize definable groups in (K, G) by applying the group configuration theorem [19] and the tools used in the article of T. Blossier and A. Martin-Pizarro [2]. It turns out that, up to isogeny, a type-definable group in (K, G) is an extension of a type-interpretable group in G by an algebraic group defined in K. Next, we study imaginaries in the pair via canonical bases as studied by A. Pillay [28] and we give a description in terms of real elements. Finally, we obtain characterization of interpretable groups in the pair.

Now we fix some more notations. For a substructure A in the sense of the pair, we denote $G_A = A \cap G$. By $\mathrm{acl}(A)$ we mean the algebraic closure of A in the field sense, and $\mathrm{acl}_P(A)$ signifies the algebraic closure of A in the pair (K, G). By $\downarrow$ we mean the independence in the pure field K, and $\overset{P}{\downarrow}$ denotes the independence in the pair.

If A is a subset of G, the algebraic closure of A in G will be denoted by $\text{acl}_G(A)$. By $\underset{G}{\downarrow}$ we represent the independence in the pure group G. Let a be a tuple in K and B be a set of parameters. Unless otherwise stated, the type $\text{tp}(a/B)$ denotes the type of a over B in the pure field sense and T indicates the theory of K, in other words ACF_p where p is the characteristic of K. By $\text{tp}^P(a/B)$, we mean the type of a over B in (K, G) and $\text{tp}_G(\gamma/C)$ indicates a type in the pure group G where γ and C lie in G. Finally given three fields E, F and $L \subseteq E \cap F$, the notation $E \underset{L}{\overset{ld}{\downarrow}} F$ represents that E is linearly disjoint from F over L.

The next remark will be subsequently used through the chapter.

Remark 2.1. *Let Nd denote the collection of all non-degenerate solutions in G of the equation* (2.1.1) *for every $n \geq 2$ and $a_1, ..., a_n$ in the prime subfield. Then for every natural number n and elements of the prime field $a_1, ..., a_n$, the set*

$$\{(g_1, ..., g_n) : a_1 g_1 + \cdots + a_n g_n = 0\} \subseteq G^n$$

is definable over Nd in the pure abelian group structure G. Now let C be a subset of G containing the set Nd. This in turn gives that every group automorphism of G over C extends to a ring automorphism of the ring $\mathbb{F}[G]$, which further extends to a field automorphism of the field $\mathbb{F}(G)$. In particular, since every algebraically closed structure A in the sense of the pair (K, G) contains Nd, every group automorphism of G over G_A extends to a field automorphism of the field $\mathbb{F}(G)$.

For the following lemma see [3]:

Lemma 2.2. *[3, Lemma 2.1] Let $T_1 \subset T_2$ be stable theories. Suppose that T_1 eliminates imaginaries. Let M be a model of T_2 and a, b be tuples in M. If C is an algebraically closed set in the sense of T_2, then $a \underset{C}{\overset{T_2}{\downarrow}} b$ implies $a \underset{C}{\overset{T_1}{\downarrow}} b$.*

In particular, if C is algebraically closed in (K, G) and $a, b \in K$, then the independence $a \underset{C}{\overset{P}{\downarrow}} b$ implies the algebraic independence $a \underset{C}{\downarrow} b$.

2.2 Characterization of Algebraic Closure

In this section, we characterize algebraically closed structures in the pair (K, G) which will be used frequently for all other proofs in this chapter. In order to characterize the algebraic closure, we need the stability of the pair (K, G) which we know by [9], and we apply lemma (2.2).

After characterizing algebraic closure, in the next section we will characterize independence. Thus, our method will be different from [1], though we use a similar technique for the characterization of forking. We begin with a definition.

Definition 2.3. *We say that a substructure A is G-independent if $A \underset{\mathbb{F}(G_A)}{\overset{ld}{\downarrow}} \mathbb{F}(G)$.*

Lemma 2.4. *Let A be algebraically closed in the sense of the pair. Then A is G-independent.*

Proof. Let $a_1, ..., a_n$ be in A and $\Sigma_1, \Sigma_2, ..., \Sigma_n$ be in $\mathbb{F}(G)$ such that

$$a_1\Sigma_1 + a_2\Sigma_2 + \cdots + a_n\Sigma_n = 0.$$

By multiplying with the denominators, we may assume that $\Sigma_i = k_{i1}g_1 + \cdots + k_{im}g_m$ for some natural number $m \geq 1$ where k_{ij} in $\mathbb{F}$ and g_j in G for $1 \leq i \leq n$ and $1 \leq j \leq m$. Thus we obtain that $c_1g_1 + \cdots + c_mg_m = 0$ where $c_i = a_1k_{1i} + a_2k_{2i} + \cdots + a_nk_{ni}$ which is in A. Without loss of the generality, we may assume that no proper subsum of $c_1g_1 + \cdots + c_mg_m$ is 0. Then we have

$$\frac{-c_2}{c_1}\frac{g_2}{g_1} + \cdots + \frac{-c_m}{c_1}\frac{g_m}{g_1} = 1.$$

Note that the tuple $(\frac{g_2}{g_1}, ..., \frac{g_m}{g_1})$ is a non-degenerate solution of the equation

$$\frac{-c_2}{c_1}x_2 + \cdots + \frac{-c_m}{c_1}x_m = 1.$$

As G has the Mann property over K, we know that this equation has only finitely many solutions in G. Since A is algebraically closed, we conclude that the tuple $(\frac{g_2}{g_1}, ..., \frac{g_m}{g_1})$ is in A and hence in G_A. Therefore we obtain that $c_1 + c_2\frac{g_2}{g_1} + \cdots + c_m\frac{g_m}{g_1} = 0$. Hence we conclude that

$$a_1\frac{\Sigma_1}{g_1} + \cdots + a_n\frac{\Sigma_n}{g_1} = 0$$

and $\frac{\Sigma_i}{g_1}$ in $\mathbb{F}(G_A)$ for $1 \leq i \leq n$. □

Next, we give the characterization of the algebraic closure in the pair. We apply a similar method as in the paper [9] where they apply a back-and-forth argument. We also benefit from the stability of the pair and lemma (2.2). In [9, Corollary 3.7], they prove that A is an elementary substructure of K in the sense of the pair if and only if A is an algebraically closed field, the group G_A is an elementary substructure of G and A is G-independent. The next lemma is inspired by their result.

Lemma 2.5. *(Algebraic Closure for pairs) Let $A \subset K$. Then A is algebraically closed in the sense of the pair if and only if A is an algebraically closed field, the group G_A is algebraically closed in G containing the set Nd and A is G-independent.*

Proof. Clearly if A is algebraically closed in the sense of the pair, then A is an algebraically closed field and G_A is an algebraically closed subgroup of G. Moreover by lemma (2.4), we know that A is G-independent. Conversely, suppose that A is an algebraically closed field, the group G_A is an algebraically closed subgroup of G and

A is G-independent. Let α be in K but not in A.

Case 1: Let $\alpha \in G \setminus G_A$. Then since G_A is algebraically closed, we know that α has infinitely many conjugates in G. Choose a conjugate $\beta \in G$ of α. Then, there is an automorphism $f \in Aut(G/G_A)$ sending α to β. Since G has the Mann property, by remark (2.1) f extends to a ring automorphism of $\mathbb{F}[G]$, which also extends to a field automorphism of $\mathbb{F}(G)$. Since A is G-independent, by linear disjointness the former automorphism extends to a field automorphism of $A(G)$ over A and this further extends to an automorphism of K over A which is actually an automorphism of the pair (K, G) over A. This yields that α is not in $\mathrm{acl}_P(A)$. In particular, we have $G_{\mathrm{acl}_P(A)} = G_A$.

Case 2: Let $\alpha \in \mathrm{acl}(A, G) \setminus A$. Then there exist $g_1, ..., g_n \in G$ such that α is in $A(g_1, ..., g_n)^{ac} \setminus A$. So there is a rational polynomial $r(x_0, x_1, ..., x_n)$ with coefficients from A such that $r(\alpha, g_1, ..., g_n) = 0$. We may assume that $g_1, ..., g_n$ are algebraically independent over A. Therefore by the first case, we know that g_i is not in $\mathrm{acl}_P(A)$ for $1 \leq i \leq n$. Thus the type $p = \mathrm{tp}^P(g_1, ..., g_n / \mathrm{acl}_P(A))$ is not algebraic. Now take $(h_1, ..., h_n) \models p$ such that $h_1, ..., h_n \underset{\mathrm{acl}_P(A)}{\overset{P}{\downarrow}} g_1, ..., g_n$. By lemma (2.2) we obtain that $h_1, ..., h_n \underset{\mathrm{acl}_P(A)}{\downarrow} g_1, ..., g_n$. Moreover since $\mathrm{acl}_P(A)$ is G-independent by lemma (2.4) and $G_{\mathrm{acl}_P(A)} = G_A$, by transitivity we get that

$$h_1, ..., h_n \underset{A}{\downarrow} g_1, ..., g_n.$$

Since there is a pair automorphism over A sending $(g_1, .., g_n)$ to $(h_1, ..., h_n)$, this gives a conjugate β of α owing to the polynomial equation $r = 0$. Observe that β is different from α thanks to the independence $h_1, ..., h_n \underset{A}{\downarrow} g_1, ..., g_n$. Choosing other independent conjugates, as a result we conclude that α has infinitely many conjugates over A and hence α is not in $\mathrm{acl}_P(A)$.

Case 3: The element α is not in $\mathrm{acl}(A, G)$. Note that every field automorphism fixing G is an automorphism of the pair (K, G). This shows that $\mathrm{acl}(A, G) = \mathrm{acl}_P(A, G)$, and hence α is not in $\mathrm{acl}_P(A)$. Thus we are done. □

The lemma (2.5) yields the following corollary.

Corollary 2.6. *For any subset D in K,*

$$\mathrm{acl}_P(D) = \mathrm{acl}(D, G_{acl_P(D)}).$$

In particular, if H is an algebraically closed substructure of G containing Nd in the sense of the pure group, then $\mathrm{acl}_P(H) = \mathrm{acl}(H)$.

Proof. As $\mathrm{acl}(D, G_{\mathrm{acl}_P(D)}) \subseteq \mathrm{acl}_P(D)$ and $\mathrm{acl}_P(D)$ is G-independent by lemma (2.4), we conclude by applying (2.5). The proof of lemma (2.5) case 1 shows that $G_{\mathrm{acl}_P(H)} = H$. Since $\mathrm{acl}_P(H)$ is G-independent by lemma (2.4), applying lemma (2.5) we deduce that $\mathrm{acl}_P(H) = \mathrm{acl}(H)$. □

2.3 Characterization of Forking

In this section, we characterize forking in the pair. Since we have the following equivalence $A \overset{P}{\underset{C}{\downarrow}} B$ if and only if

$$\mathrm{acl}_P(A) \overset{P}{\underset{\mathrm{acl}_P(C)}{\downarrow}} \mathrm{acl}_P(B),$$

we characterize forking for algebraically closed sets in terms of the algebraic independence in K.

First, we start with a lemma concerning just pure abelian groups. Let A be an abelian group written additively. An equation is a formula $\gamma(\bar{x})$ of the form

$$m_1x_1 + \cdots + m_nx_n = 0,$$

where $m_i \in \mathbb{Z}$. A positive primitive formula is of the form $\exists\bar{y}(\gamma_1 \wedge ... \wedge \gamma_k)$ where $\gamma_i(\bar{x}\bar{y})$ are equations. The next lemma requires that every formula in a pure abelian group is equivalent to a Boolean combinations of positive primitive formulas, see [39, Theorem 4.2.8].

Lemma 2.7. *(Algebraic Closure for abelian groups) Let A be an abelian group written additively and B, C be algebraically closed subgroups of A. Then $B + C$ is also algebraically closed in the sense of the pure abelian group A.*

Proof. Let $\varphi(x, b, c)$ be an algebraic formula over $B + C$ where b is a tuple in B with $|b| = s$ and c is a tuple in C with $|c| = t$. Since every formula in an abelian group is equivalent to a Boolean combinations of positive primitive formulas, we may assume that $\varphi(x, y, z) = \alpha + D$ where $\alpha \in \mathrm{acl}_A(\emptyset)$ and D is a subgroup of $A \times A^s \times A^t$. Note that $B_1 = \{g : (g, b, 0) \in \alpha + D\}$ is finite and so it lies in B. Similarly $C_1 = \{g : (g, 0, c) \in D\}$ is finite and hence it is a subset of C. As $B_1 + C_1 = \{g : (g, b, c) \in \alpha + D\} \subseteq B + C$ and it is finite, we are done. □

The next lemma affirms when two algebraically closed structures have the same type over a common algebraically closed substructure.

Lemma 2.8. *Let B_1, B_2 and $C \subseteq B_1 \cap B_2$ be three algebraically closed sets in the sense of the pair (K, G). Then $\mathrm{tp}^P(B_1/C) = \mathrm{tp}^P(B_2/C)$ if and only if there is a field automorphism over C sending B_1 to B_2 with G_{B_1} to G_{B_2}, and $\mathrm{tp}_G(G_{B_1}/G_C) = \mathrm{tp}_G(G_{B_2}/G_C)$.*

Proof. Suppose that there is a field automorphism f sending (B_1, G_{B_1}) to (B_2, G_{B_2}) over C and a group automorphism h of G sending G_{B_1} to G_{B_2} over G_C. Since G has the Mann property, by remark (2.1) the automorphism h extends to a field automorphism of $\mathbb{F}(G)$. As C is G-independent by lemma (2.4), the automorphism h further extends

to an automorphism of $C(G)$ over C. Since both B_1 and B_2 are G-independent, we obtain $B_i \overset{ld}{\underset{C(G_{B_i})}{\downarrow}} C(G)$ for each $i = 1, 2$. The automorphisms f and h now extend to an isomorphism between $B_1(G)$ and $B_2(G)$ over C which further extends to K. Thus we conclude that $\mathrm{tp}^P(B_1/C) = \mathrm{tp}^P(B_2/C)$.

□

Now we prove a lemma which will be crucial for the characterization of forking and the characterization of definable groups in this chapter.

Lemma 2.9. *Let $C = A \cap B$ and all be algebraically closed in the sense of the pair. If $A \underset{C,G}{\downarrow} B, G$, then $\mathrm{acl}_P(A, B) = \mathrm{acl}(A, B)$ and $G_{\mathrm{acl}_P(A,B)} = \mathrm{acl}_G(G_A, G_B)$.*

Proof. Since A is algebraically closed and it contains C, by lemma (2.4) we have $A \underset{C,G_A}{\downarrow} C, G$. Transitivity of the algebraic independence yields that $A \underset{C,G_A}{\downarrow} B, G$ and so $A \underset{B,G_A}{\downarrow} G$. As B is algebraically closed, similarly we have that $B \underset{G_B}{\downarrow} G$ and therefore $B \underset{G_AG_B}{\downarrow} G$. By transitivity of the algebraic independence again, we obtain that

$$A, B \underset{G_AG_B}{\downarrow} G$$

which in turn gives

$$\mathrm{acl}(A, B) \overset{ld}{\underset{\mathrm{acl}(G_AG_B)}{\downarrow}} \mathrm{acl}(G).$$

Note that by lemma (2.7), the group G_AG_B is algebraically closed in G. By corollary (2.6), we see that $\mathrm{acl}_P(G_AG_B) = \mathrm{acl}(G_AG_B)$. Since by lemma (2.4) $\mathrm{acl}(G_AG_B)$ is also G-independent, by transitivity and in terms of linear disjointness, we conclude that

$$\mathrm{acl}(A, B) \overset{ld}{\underset{\mathbb{F}(G_AG_B)}{\downarrow}} \mathbb{F}(G).$$

Hence by lemma (2.5), we deduce that $\mathrm{acl}_P(A, B) = \mathrm{acl}(A, B)$ and $G_{\mathrm{acl}_P(A,B)} = G_AG_B = \mathrm{acl}_G(G_A, G_B)$. □

The next lemma is somewhat surprising, however it holds in the pair (K, G) contrary to pairs of algebraically closed fields.

Lemma 2.10. *Let $C = A \cap B$ and all be algebraically closed in the sense of the pair. If $A \underset{C,G}{\downarrow} B, G$ then $A \underset{C}{\downarrow} B$.*

Proof. Since A is algebraically closed in the sense of the pair, by lemma (2.4) and transitivity we can replace $A \underset{C,G}{\downarrow} B, G$ by $A \underset{C,G_A}{\downarrow} B, G$. By transitivity, it is enough to

show that $B \overset{ld}{\underset{C}{\downarrow}} C(G_A)$. Let $b_1, ..., b_n \in B$ and $g_1, ..., g_n \in G_A$ be such that

$$b_1 g_1 + \cdots + b_n g_n = 0.$$

We may suppose that no proper subsum of this equation is 0. Thus similar to the proof of lemma (2.4), we obtain that $\frac{g_i}{g_1} \in B$ for $i = 1, 2, ..., n$. As they are also in G_A, we deduce that $\frac{g_i}{g_1} \in G_C$. Hence we are done. □

Remark 2.11. *The lemma* (2.10) *does not hold in pairs of algebraically closed fields. This is one of the points where the pair* (K, G) *and pairs of algebraically closed fields differ from each other. This will be the main reason why the independence in* (K, G) *is more plain than the independence in pairs of algebraically closed fields. More precisely, let* K *and* E *be two algebraically closed fields such that* E *is small in* K*, that is to say* E *is not* K*. Let* $C = A \cap B$ *and all be algebraically closed in the pair* (K, E)*. Note also that all the fields* E_A, E_B *and* E_C *are also algebraically closed in the pair* (K, E)*. Then we do not necessarily have* $B \overset{ld}{\underset{C}{\downarrow}} C(E_A)$*, since* C *is algebraically closed, by transitivity the previous independence is equivalent to* $B \overset{ld}{\underset{E_C}{\downarrow}} E_A$*. As* B *is algebraically closed, by transitivity the latter independence is equivalent to the independence* $E_B \underset{E_C}{\downarrow} E_A$*. Since as a field* E *is not one-based, the independence* $E_B \underset{E_C}{\downarrow} E_A$ *does not hold always. A counter example can be found in [37, Chapter 10, Proposition 10.3.6]. This is one of the differences between* (K, G) *and* (K, E)*, and therefore we have just one independence in the field sense in order to characterize forking in the next Theorem* (2.12)*. This results from the fact that* $E^\times$ *does not have the Mann property as it is infinite and the equation* $x + y = 1$ *has infinitely many non-degenerate solutions in* $E^\times$*. Recall that in* (K, E)*,*

$$A \overset{(K,E)}{\underset{C}{\downarrow}} B$$

if and only if $A \underset{C,E}{\downarrow} B, E$ *and* $A \underset{C}{\downarrow} B$*. For the details, we refer the reader to [1, 31].*

Now we are ready to characterize forking in the pair (K, G) by using the characterization of the algebraic closure (2.5), lemma (2.9) and lemma (2.10).

Theorem 2.12. *(Forking) Let* $C = A \cap B$ *and all be algebraically closed in the sense of the pair. Then the following are equivalent:*

(i) $A \overset{P}{\underset{C}{\downarrow}} B$,

(ii) $A \underset{C,G}{\downarrow} B, G$,

(iii) $A \overset{ld}{\underset{C(G_A)}{\downarrow}} B(G)$.

Proof. We first show that (i) and (ii) are equivalent and then we prove that (ii) and (iii) are equivalent.
Now suppose that $A \underset{C}{\overset{P}{\downarrow}} B$. Moreover, suppose for a contradiction that $A \underset{C,G}{\not\downarrow} B, G$. Let $q = \mathrm{tp}(B/C \cup G_B)$ and $\lambda \geq \omega_1$. By saturation, there exists a sequence $(B_i)_{i\leq\lambda}$ with $B = B_0$ such that $B_i \models q$ and $(B_i)_{i\leq\lambda}$ is independent over $C \cup G$ in the field sense, and in particular $B_i \underset{C,G_B}{\downarrow} C, G$ and $G_B \subseteq G_{B_i}$ for all i. On the other hand, by the independence $B_i \underset{C,G_B}{\downarrow} C, G$ we have that $G_{B_i} \subseteq \mathrm{acl}(C, G_B) \subseteq B$. Thus we obtain the equality $G_B = G_{B_i}$ for all i. As B is G-independent by lemma (2.4), we have that

$$C(G_B) \underset{\mathbb{F}(G_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G).$$

Thus by lemma (2.5), we deduce that $\mathrm{acl}_P(C, G_B) = \mathrm{acl}(C, G_B)$ and $G_B = G_{\mathrm{acl}(C,G_B)}$. So we see that $\mathrm{acl}(C, G_B) \underset{\mathbb{F}(G_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G)$. As $B_i \underset{C,G_B}{\downarrow} C, G$, by transitivity and in terms of linear disjointness, we obtain that

$$B_i \underset{\mathbb{F}(G_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G).$$

Therefore by lemma (2.5) again, this results in that B_i is algebraically closed in the sense of the pair for all i. Then by lemma (2.8), we conclude that $\mathrm{tp}^P(B_i/C) = \mathrm{tp}^P(B/C)$. By Erdős-Rado theorem, we may assume that $(B_i)_{i\leq\lambda}$ is C-indiscernible in the sense of T_P. Let $p_i = \mathrm{tp}^P(A/B_i)$. Since $A \underset{C}{\overset{P}{\downarrow}} B$, we know that $\bigcup_{i\leq\lambda} p_i(x, B_i)$ is consistent. So there exists A_1 such that $\mathrm{tp}^P(A_1 B_i) = \mathrm{tp}^P(AB)$ for all i. Now, the sequence $(B_i)_{i\leq\lambda}$ is independent over $C \cup G$ and $A_1 \underset{C,G}{\not\downarrow} B_i$ for each B_i. This contradicts the stability of the field K.

Conversely, assume that $A \underset{C,G}{\downarrow} B, G$. By lemma (2.10), we know that $A \underset{C}{\downarrow} B$. Let $(B_i)_i$ be a Morley sequence over C in the sense of the pair where $B_0 = B$. Note that the sequence $(B_i, G_{B_i})_i$ is also a Morley sequence over C but for simplicity we write $(B_i)_i$ instead. The independence $A \underset{C}{\downarrow} B$ yields that $G_A \underset{C}{\downarrow} B$. By stationarity over algebraically closed sets (they are models of ACF) and since K is ω-stable, we may assume that $(B_i)_i$ is a Morley sequence over $C \cup G_A$ in the field sense. Since $A \underset{C}{\downarrow} B$, we also have $A \underset{C,G_A}{\downarrow} B, G_A$. Let $p(x) = \mathrm{tp}(A/B \cup G_A)$ and $p_i(x)$ be the copy over B_i. Then by $A \underset{C,G_A}{\downarrow} B, G$ and by saturation, there exists an element $d \models \bigcup_i p_i(x)$ such that

$$d \underset{C,G_A}{\downarrow} B_i, G$$

for all i. Observe that $G_d = G_A$ and $\mathrm{tp}(dB_iG_A) = \mathrm{tp}(ABG_A)$ for all i. Moreover, since A is G-independent by lemma (2.4), we have that $C(G_A) \underset{\mathbb{F}(G_A)}{\overset{ld}{\downarrow}} \mathbb{F}(G)$. So by lemma (2.5), we see that $\mathrm{acl}_P(C, G_A) = \mathrm{acl}(C, G_A)$. As we also have that $d \underset{C,G_A}{\downarrow} G$, by transitivity and in terms of linear disjointness, we obtain that $d \underset{\mathbb{F}(G_A)}{\overset{ld}{\downarrow}} \mathbb{F}(G)$. By lemma (2.5) again, we deduce that d is algebraically closed in the sense of the pair. By lemma (2.9), we know that $\mathrm{acl}_P(A, B) = \mathrm{acl}(A, B)$ and $G_{\mathrm{acl}_P(A,B)} = G_AG_B$, and also

$$\mathrm{acl}(A, B) \underset{\mathbb{F}(G_AG_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G).$$

By the choice of d and lemma (2.9) once again, we also have that

$$\mathrm{acl}(d, B_i) \underset{\mathbb{F}(G_AG_{B_i})}{\overset{ld}{\downarrow}} \mathbb{F}(G).$$

Applying $\mathrm{tp}(dB_iG_A) = \mathrm{tp}(ABG_A)$ and the previous two linear disjointless, we conclude that $\mathrm{tp}^P(dB_iG_A) = \mathrm{tp}^P(ABG_A)$ for all i. Hence we obtain $A \underset{C}{\overset{P}{\downarrow}} B$.

Now we prove that (ii) and (iii) are equivalent. Clearly (iii) implies (ii). So suppose (ii). By lemma (2.9), we know that $\mathrm{acl}_P(A, B) = \mathrm{acl}(A, B)$ and $G_{\mathrm{acl}_P(A,B)} = G_AG_B$, and also $\mathrm{acl}(A, B) \underset{\mathbb{F}(G_AG_B)}{\overset{ld}{\downarrow}} \mathbb{F}(G)$. Therefore we obtain that $AB \underset{B(G_A)}{\overset{ld}{\downarrow}} B(G)$. Moreover since C is algebraically closed and $A \underset{C}{\downarrow} B$ by (2.10), we see that $A \underset{C}{\overset{ld}{\downarrow}} B$ and hence $A \underset{C(G_A)}{\overset{ld}{\downarrow}} B(G_A)$. By transitivity we conclude that $A \underset{C(G_A)}{\overset{ld}{\downarrow}} B(G)$.

□

Now we give some corollaries of Theorem (2.12).

Corollary 2.13. *Let a and b be finite tuples from K and C be a subset of K. If $a \underset{C}{\overset{P}{\downarrow}} b$ then $a \underset{C,G}{\downarrow} b$. Moreover, if we have*

$$\mathrm{acl}_P(a, C) \cap \mathrm{acl}_P(b, C) = \mathrm{acl}_P(C),$$

then $a \underset{C,G}{\downarrow} b$ also implies $a \underset{C}{\overset{P}{\downarrow}} b$.

Proof. Observe that for any subset D of K we have that

$$\mathrm{acl}(D, G) = \mathrm{acl}(\mathrm{acl}_P(D), G) = \mathrm{acl}_P(D, G).$$

Now we conclude by Theorem (2.12). □

Remark 2.14. *Note that if* $a \overset{P}{\underset{C}{\downarrow}} b$*, then we do not necessarily have* $a \underset{C}{\downarrow} b$ *unless* $C = \mathrm{acl}_P(C)$*. To see this, it is enough to take a subset* C *such that* $\mathrm{acl}_P(C)$ *is not equal to* C^{ac}*.*

Corollary 2.15. *For every* $a \in K$*, we have the independence* $a \overset{P}{\underset{G_{\mathrm{acl}_P(a)}}{\downarrow}} G$*.*

Proof. Since $\mathrm{acl}_P(a)$ is G-independent by lemma (2.4), we have the independence

$$\mathrm{acl}_P(a) \underset{\mathrm{acl}(G_{\mathrm{acl}_P(a)})}{\downarrow} \mathrm{acl}(G)$$

and $\mathrm{acl}_P(a) \cap \mathrm{acl}(G) = \mathrm{acl}(G_{\mathrm{acl}_P(a)})$. By (2.6), we see that $\mathrm{acl}(G_{\mathrm{acl}_P(a)}) = \mathrm{acl}_P(G_{\mathrm{acl}_P(a)})$ and $\mathrm{acl}(G) = \mathrm{acl}_P(G)$. Therefore we have

$$\mathrm{acl}_P(a) \underset{\mathrm{acl}_P(G_{\mathrm{acl}_P(a)})}{\downarrow} \mathrm{acl}_P(G).$$

With the help of Theorem (2.12) we finish the corollary. □

Next, we prove that the independence in the pair implies the independence in G.

Lemma 2.16. *Let* $C = A \cap B$ *and all be algebraically closed in the sense of the pair and* $A \overset{P}{\underset{C}{\downarrow}} B$*. Then we have the independence* $G_A \overset{G}{\underset{G_C}{\downarrow}} G_B$ *in* G*.*

Proof. As $A \overset{P}{\underset{C}{\downarrow}} B$, we have $G_A \overset{P}{\underset{C}{\downarrow}} G_B$. Corollary (2.15) and transitivity of the independence yield that $G_A \overset{P}{\underset{G_C}{\downarrow}} G_B$. Hence we conclude that $G_A \overset{G}{\underset{G_C}{\downarrow}} G_B$. □

2.3.1 Stationarity

In this subsection, we prove that types over algebraically closed sets are stationary if G has WEI. First, we need two basic lemmas.

Lemma 2.17. *(Shelah lemma) Let* T *be a stable theory. If the type* $\mathrm{tp}(A/C)$ *is stationary and* $B \underset{C}{\downarrow} A$*, given two* C*-elementary maps* $f : A \to A_1$ *and* $g : B \to B_1$ *such that* $A_1 \underset{C}{\downarrow} B_1$*, then* $f \cup g$ *is also* C*-elementary.*

Proof. Since g is C-elementary, we have that $g(A) \underset{C}{\downarrow} B_1$ and $g(A) \equiv_C A \equiv_C A_1$. By stationarity there is an automorphism h fixing B_1 and sending $g(A)$ to A_1. Now one can see that $h \circ g$ restricted to $A \cup B$ is $f \cup g$. □

The next lemma states that under WEI, types over algebraically closed sets are stationary in stable theories.

Lemma 2.18. *Suppose that T is stable and $M \models T$ has weak elimination of imaginaries (WEI). Then every type over an algebraically closed set is stationary.*

Proof. Let A be an algebraically closed set in M. Since T has WEI, we have that $\mathrm{acl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(A)$. Since in a stable theory types over $\mathrm{acl}^{\mathrm{eq}}(A)$ are stationary, we are done. □

The next remark does not require that the group G has the Mann property.

Remark 2.19. *The pair (K, G) does not have GEI.*

Proof. To see that (K, G) does not have GEI, we consider the quotient $K^\times/G$. Now let a be an element of K which is not in $\mathrm{acl}(G)$. Put $e = aG$ which is an imaginary element. Suppose there is a real element c such that $e \in \mathrm{acl}_P{}^{\mathrm{eq}}(c)$ and $c \in \mathrm{acl}_P(e)$. If we have $c \in \mathrm{acl}(G)$, then as a is not in $\mathrm{acl}(G)$, we can send a to any element b which is also not in $\mathrm{acl}(G)$. Since there are infinitely elements which are not in $\mathrm{acl}(G)$ as G is small in K, this contradicts the assumption $e \in \mathrm{acl}_P{}^{\mathrm{eq}}(c)$. This yields that c is not in $\mathrm{acl}(G)$. Since $e \in \mathrm{dcl}_P(a)$, we see that $c \in \mathrm{acl}_P(e) \subseteq \mathrm{acl}_P(a) \subseteq \mathrm{acl}(a, G)$. Thus there is a polynomial f such that $f(a, c, g) = 0$ where g is a tuple from G. So there are polynomials $f_0, ..., f_k$ such that a satisfies the polynomial $f_k(c, g)X^k + \cdots + f_1(c, g)X + f_0(c, g)$ and $f_k(c, g)$ is not 0. Since $c \in \mathrm{acl}_P(e)$, there are finitely may conjugates of c under e, say $c = c_1, ..., c_m$. Note also that any automorphism fixing G and sending a to ah fixes e, where $h \in G$. Therefore, every such automorphism gives finitely many polynomials with coefficients from g and $c_1, ..., c_m$, but infinitely many elements satisfying these polynomials, a contradiction. □

With the help of the lemmas (2.17), (2.18) and assuming that G has WEI, we are able to prove that the types in (K, G) over algebraically closed sets are stationary although (K, G) does not have GEI.

Corollary 2.20. *Suppose that G has WEI. Let A be algebraically closed in the sense of the pair and $a \in K$. Then the type $\mathrm{tp}^P(a/A)$ is stationary.*

Proof. Let B be a set containing A. We may suppose that B is algebraically closed in the sense of the pair. Let a_1 and a_2 be such that $\mathrm{tp}^P(a_1/A) = \mathrm{tp}^P(a_2/A)$, and we have the following two independence $a_1 \overset{P}{\underset{A}{\downarrow}} B$ and $a_2 \overset{P}{\underset{A}{\downarrow}} B$. Note that $\mathrm{acl}_P(a_1, A)$ and $\mathrm{acl}_P(a_2, A)$ have the same type over A in the sense of the pair. By the characterization of the independence (2.12), we see that $a_i, A \overset{ld}{\underset{A(G)}{\downarrow}} B(G)$ for $i = 1, 2$. By lemma (2.16), we also have $G_{\mathrm{acl}_P(a_i,A)} \overset{G}{\underset{G_A}{\downarrow}} G_B$ for $i = 1, 2$. Since $\mathrm{tp}_G(G_{\mathrm{acl}_P(a_1,A)}/G_A) = \mathrm{tp}_G(G_{\mathrm{acl}_P(a_2,A)}/G_A)$ and G has WEI, we obtain that

$$\mathrm{tp}_G(G_{\mathrm{acl}_P(a_1,A)}/G_B) = \mathrm{tp}_G(G_{\mathrm{acl}_P(a_2,A)}/G_B)$$

with the help of lemma (2.18). Let f be an automorphism of G over G_B sending $G_{\mathrm{acl}_P(a_1,A)}$ to $G_{\mathrm{acl}_P(a_2,A)}$. As G has the Mann property, by remark (2.1) the map f extends to a field automorphism of $\mathbb{F}(G)$ over $\mathbb{F}(G_B)$. Moreover since B is G-independent by lemma (2.4), the map f further extends to an automorphism of $B(G)$ over B. Now since $a_i, A \overset{ld}{\underset{A(G)}{\downarrow}} B(G)$ for $i = 1, 2$ and $\mathrm{tp}^P(a_1/A) = \mathrm{tp}^P(a_2/A)$, we have an isomorphism from $B(a_1, G)$ to $B(a_2, G)$ sending a_1 to a_2 over B. Since this also extends to an automorphism of K, we conclude that $\mathrm{tp}^P(a_1/B) = \mathrm{tp}^P(a_2/B)$.

□

2.4 Definable Groups for the Pair

In this section, we give the characterization of type-definable groups in the pair (K, G). We apply the group configuration theorem (1.18) and it turns out that any definable group, up to isogeny, is an extension of a type-interpretable abelian group in G by an algebraic group over K. In plain words, definable and interpretable groups in K and G give rise to all definable groups in the pair (K, G). For more on the applications of the group configuration theorem, we refer the reader to [22].

Remark 2.21. *In stable theories, every type-definable group is an intersection of definable groups in this theory.*

2.4.1 Generics and Isogeny

Generics of a group as defined in Chapter 1, are useful to construct a group in stable theories. They will play a significant role in the characterization of definable groups in the pair.

The following lemma is from [30]:

Lemma 2.22. *(Generic [30, 5.4]) Suppose that H is a stable group. Every formula $\varphi(x, y)$ can be associated with a natural number $n = n(\varphi)$ such that, if A is a generic subset of H defined by a formula $\varphi(x, a)$, then H is covered by n translates of A.*

The next lemma states when a type-definable group in the pair is actually an algebraic group and the motivation comes from [2, 2.1]

Lemma 2.23. *(The Group Lemma) Let H be a connected T_P-type-definable subgroup of an algebraic group V, all definable over an algebraically closed set A in the sense of the pair. Let a be the generic over A which lies in some translate of H which is also definable over A. If $G_{\mathrm{acl}_P(a,A)} = G_A$, then H is an algebraic group. In particular, H is definable.*

Proof. First we may assume that $a \in H$ as follows: Suppose that $a \in bH$. Let a' be such that $\mathrm{tp}^P(a'/A) = \mathrm{tp}^P(a/A)$ and $a' \overset{P}{\underset{A}{\downarrow}} a$. Then we have $a^{-1}a' \overset{P}{\underset{A}{\downarrow}} a$ and $a^{-1}a' \in H$ is the generic over A. As we have $a', A \overset{P}{\underset{A}{\downarrow}} a, A$, therefore by Theorem (2.12) and lemma (2.9), we see that

$$G_{\mathrm{acl}_P(a^{-1}a',A)} \subseteq G_{\mathrm{acl}_P(a',a,A)} \subseteq \mathrm{acl}_G(G_{\mathrm{acl}_P(a',A)}, G_{\mathrm{acl}_P(a,A)}).$$

Since $G_{\mathrm{acl}_P(a',A)} = G_{\mathrm{acl}_P(a,A)} = G_A$, we obtain that $G_{\mathrm{acl}_P(a^{-1}a',A)} = G_A$. So we may assume that $a \in H$.

Put $p = \mathrm{tp}^P(a/A)$ and p_0 its T-reduct. Let H_0 be the smallest algebraic group containing H which exists by the assumption and the ω-stability of K. Note that $H = stab_P(p) \subset stab_T(p_0)$. So $H_0 \subseteq stab_T(p_0)$. On the other hand since $p_0(x)$ implies that $x \in H_0$, we get that $stab_T(p_0) \subseteq H_0$. Thus we have the equality and moreover H_0 is T-connected.

To prove the lemma, it is enough to show that p is the unique generic of H_0 since this implies that H_0 is T_P-connected and $H_0 = stab_P(p) = H$. Let h be a generic of H_0 over A in the sense of the pair and put $q = \mathrm{tp}^P(h/A)$. Observe that $p_0 \subseteq q$ since H_0 is T-connected.

Claim: We have the following independence $h \overset{}{\underset{A}{\downarrow}} G$.

Proof of the claim: First, note that $a \in H_0$ and since the algebraic closure is G-independent and also by the assumption $G_{\mathrm{acl}_P(a,A)} = G_A$, we obtain that $a, A \overset{}{\underset{G_A}{\downarrow}} G$ and so $a \overset{}{\underset{A}{\downarrow}} G$. As a result, the element a is a generic over $A \cup G$. Now if $h \not\overset{}{\underset{A}{\downarrow}} G$, then there exists a T-formula $\varphi(x, g) \in \mathrm{tp}(h/A, G)$ with parameters from A which is not generic in H_0. Put $n = n(\varphi)$ as in lemma (2.22) and let

$$\theta(y) = \exists h_1 ... \exists h_n \in H_0 (\forall x \in H_0 \bigvee_{i \leq n} h_i \varphi(x, y))$$

and $\phi(x, y) = \neg\theta(y) \wedge \varphi(x, y)$. Observe that for any tuple b, the formula $\phi(x, b)$ is not generic in H_0. However the formula $\psi(x) = \exists y(U(y) \wedge \phi(x, y))$ with parameters from A is realized by h, and so it is generic in H_0. Thus a finite number of translates of $\psi(x)$ cover H_0, say $H_0 = \bigcup_{i \leq k} \alpha_i \psi(x)$. Take a' such that $\mathrm{tp}^P(a'/A) = \mathrm{tp}^P(a/A)$ and $a \overset{P}{\underset{A}{\downarrow}} \alpha_1, ... \alpha_k$. Thus for certain $\alpha \in H_0$, we may suppose that $a \in \alpha\psi(x)$ and $a \overset{P}{\underset{A}{\downarrow}} \alpha$. So $a \in \alpha\phi(x, g')$ for some $g' \in G$. By the characterization of the independence (2.12), we have that $a \overset{}{\underset{A,G}{\downarrow}} \alpha$ and by transitivity we get the independence $a \overset{}{\underset{A}{\downarrow}} \alpha, G$. This is a contradiction since the formula $\alpha\phi(x, g')$ is not generic in H_0. Hence we have the claim.

Now as A is G-independent by lemma (2.4), by transitivity of the independence, we

get that $h, A \underset{G_A}{\downarrow} G$. In terms of linear disjointness, this gives that $\mathrm{acl}(h, A) \overset{ld}{\underset{\mathbb{F}(G_A)^{ac}}{\downarrow}} \mathbb{F}(G)^{ac}$. By corollary (2.6), we see that $\mathrm{acl}_P(G_A) = \mathbb{F}(G_A)^{ac}$. Moreover, since the algebraic closure is G-independent, by transitivity we get that

$$\mathrm{acl}(h, A) \overset{ld}{\underset{\mathbb{F}(G_A)}{\downarrow}} \mathbb{F}(G).$$

Therefore by lemma (2.5), we deduce that $\mathrm{acl}_P(h, A) = \mathrm{acl}(h, A)$ and $G_{\mathrm{acl}_P(h,A)} = G_A$. Since there is a field automorphism sending a to h over A, by linear disjointness this in turn gives rise to a field automorphism over $A \cup G$. We conclude that $q = p$ which is determined by p_0. Hence $H = H_0$ as required.

□

Remark 2.24. *The group G does not satisfy the conditions of the previous lemma even it is connected, as for any a in G we do not have the equality $G_{\mathrm{acl}_P(a,A)} = G_A$. This is expected as G is not an algebraic group.*

Definition 2.25. *(Isogeny) Let G and H be two type-definable groups in a stable theory. We say that G and H are isogenous (or there is an isogeny between them) if there is a type-definable subgroup S of $G \times H$ such that*

- *The projection of S into G, denoted by G_S, has bounded index (the index is less than the saturation cardinal κ) in G,*
- *The projection of S into H, denoted by H_S, has bounded index in H,*
- *The kernel $ker(S) = \{g \in G : (g, 1) \in S\}$ and the co-kernel $coker(S) = \{h \in H : (1, h) \in S\}$ are finite.*

Note that if G and H are isogenous, then there is an isomorphism between $G_S/ker(S)$ and $H_S/coker(S)$.

Remark 2.26. *Note that the isogeny relation is an equivalence relation. Every group is isogenous to its connected component and every isogeny of the connected component gives rise to an isogeny of the group.*

The following lemma is in [2] and it enables us to construct an isogeny between two groups.

Lemma 2.27. *[2, 2.4 and 2.5] Let G_1 and G_2 be two groups type-definable (type-interpretable) in a stable theory. If there exist parameters $C = \mathrm{acl}^{\mathrm{eq}}(C)$ and elements a_1, b_1 of G_1 and a_2, b_2 of G_2 such that*

(1) a_1 and a_2, b_1 and b_2, a_1b_1 and a_2b_2 are C-interalgebraic

(2) a_1, b_1 and a_1b_1 are pairwise independent over C,

then the element a_1 (respectively a_2) is generic in a unique translate of a connected subgroup H_1 of G_1 (respectively H_2 of G_2), all definable over C and there is an isogeny between H_1 and H_2 given by the stabilizer of the type $\mathrm{tp}(a_1, a_2/C)$. If in the condition (1), if we just have a_2 is algebraic over C, a_1 (respectively for b_2 and $a_2 b_2$), then there is a type-interpretable projection from H_1 to a quotient of H_2 by a finite subgroup.

The above results can be generalized to the case when both groups G_1 and G_2 are $*$-interpretable. Furthermore, if G_1 is type-definable and G_2 is $*$-interpretable, by stability there is an isogeny between G_1 (respectively a projection with the same kernel) and a connected $*$-interpretable subgroup D of G_2 whose generic is C-interalgebraic with the generic of H_2.

Remark 2.28. *In the previous lemma* (2.27)*, if G_2 is $*$-interpretable and if the kernel of the projection is definable, then we can take D to be a type-interpretable group by compactness.*

2.4.2 Characterization of Definable Groups

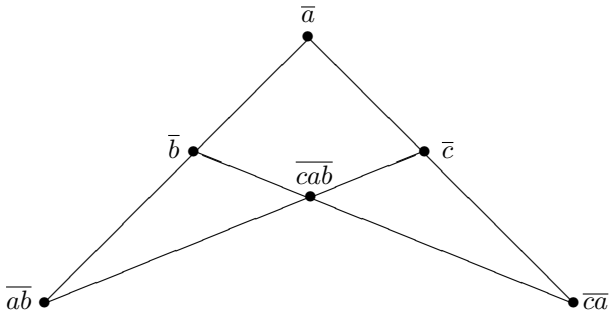
Now we are ready for the characterization of definable groups in the pair (K, G) up to isogeny. Our method is to apply group configuration from geometric stability theory as in [2]. So we need Theorem (2.12), lemma (2.16), lemma (2.23) and lemma (2.27). In this subsection, by T we mean the theory of K as an algebraically closed field, in other words $T = ACF_p$ where $p = char(K)$.

Recall the result proved by E. Hrushovski and A. Pillay [20]:

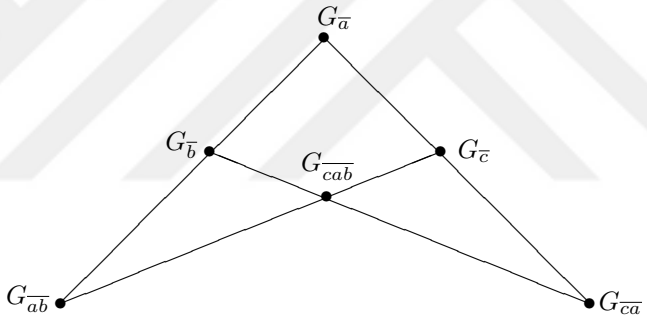
Theorem 2.29. *[20] Let A be an interpretable group in a one-based stable theory. Then the connected component of A is abelian.*

Theorem 2.30. *(Definable Groups) Let K be an algebraically closed field and G be a multiplicative subgroup of $K^\times$ with the Mann property. Any type-definable group in (K, G) is isogenous to a subgroup of an algebraic group. Moreover any type-definable group is, up to isogeny, an extension of a type-interpretable abelian group in G by an algebraic group.*

Proof. By remark (2.26), it is enough to assume that H is a connected type-definable group in (K, G) over some parameters. We will work over a model containing these parameters which we will omit. Given two independent generics a and b of H, we write $\overline{a}$, $\overline{b}$ and $\overline{ab}$ instead of their algebraic closures in the sense of the pair respectively. By lemma (2.9) and Theorem (2.12), the set $\overline{ab}$ is T-algebraic over $\overline{a} \cup \overline{b}$ since $\overline{a}$, $\overline{b}$ are two independent algebraically closed subsets. With the help of the third generic c which is independent from a, b, we obtain the following diagram:



Then by lemma (2.2), we have a T-group configuration. Therefore by the group configuration theorem (1.18) and lemma (2.27), there exists a $*$-interpretable group V in T, whose generic is T_P-interalgebraic with the generic of H. Thus by elimination of imaginaries in K and applying the lemma (2.27) again, we may assume that there exists an algebraic group in which H embeds up to isogeny. In other words, up to isogeny, we may suppose that H is a subgroup of an algebraic group. By lemma (2.9), the set $G_{\overline{ab}}$ is G-algebraic over $G_{\overline{a}} \cup G_{\overline{b}}$. Moreover by lemma (2.16), we have the following diagram in G:



So by the group configuration theorem (1.18) and lemma (2.27) again, we obtain a connected $*$-interpretable group H_1 in G and a projection π from H to H_1. Also, the generic h of H_1 is G-interalgebraic with $G_{\overline{a}}$. Furthermore, we may assume that the generic h of H_1 is T_P-interalgebraic with $G_{\overline{a}}$. Note that H_1 is abelian by Theorem (2.29) and remark (2.21). Finally, we show that the connected component N of the kernel $ker(\pi)$ is an algebraic group by the group lemma (2.23). Let n be a generic of N over a in the sense of the pair. So we have $n \overset{P}{\downarrow} a$ and $na \overset{P}{\downarrow} a$. Observe that $na \in Na$ is a generic also. Since the tuple $(n, 1)$ is in the stabilizer of $\mathrm{tp}^P(a, h)$, we have that the tuples (na, h) and (a, h) have the same P-type. Thus in particular, we see that $G_{\overline{na}} = G_{\overline{a}}$. Moreover by lemma (2.9) and Theorem (2.12), the group $G_{\mathrm{acl}_P(na,a)}$ is in the G-algebraic closure of $G_{\overline{na}}$ and $G_{\overline{a}}$. Therefore we obtain that $G_{\mathrm{acl}_P(na,a)} = G_{\overline{a}}$. Hence, the type $\mathrm{tp}^P(na/\overline{a})$ satisfies the hypothesis of the group lemma (2.23) and we conclude that N is an algebraic group. Now by (2.28), the group H_1 can be taken to be type-interpretable.

□

Remark 2.31. *Note that if G is ω-stable in the pure group language, then every type-interpretable group in G is interpretable in G. For example if G is an elementary extension of μ, then every definable group in the pair is an extension of an interpretable group in G by an algebraic group.*

Example 2.32. *(Some definable groups) The additive group of K, the multiplicative group $K^\times$, any algebraic group over K, the group G and its powers, the cartesian product $K \times G$ and*

$$SL(2, G, K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \in G \right\}$$

are all definable in the pair. Observe that they all satisfy the conclusion of Theorem (2.30).

2.5 Imaginaries and Interpretable Groups

In this section, we characterize interpretable groups in (K, G). To achieve this, we need (K, G) to be ω-stable since we will use the existence of the Morley rank in lemma (2.53). By [9, Corollary 6.2], it is enough and sufficient to suppose that G is ω-stable as a pure abelian group. Moreover by [25], we see that G is divisible by finite. This means $G = DF$ where D is divisible and F is a finite group and $D \cap F = 1$. Furthermore, the group D is the connected component of G, it is definable and strongly minimal. In other words, G is almost strongly minimal. Moreover, any infinite algebraically closed subset A of G contains F and it is of the form $D_A F$ where $D_A = A \cap D$ which is also divisible. Hence, any infinite algebraically closed subset of G is an elementary substructure.

In this section, we assume that G is ω-stable in the pure group language. As an example, we can take (K, G) to be an elementary extension of the pair $(\mathbb{C}, \mu)$. As noted in [9, 6.4], the pair (K, G) is ω-stable and $\mathrm{MR}(K, G) = \omega$. Thus the pair (K, G) cannot be uncountably categorical since uncountably categorical theories have finite Morley rank. Moreover if K and L are countable algebraically closed fields where L is a proper extension of K, then we have $(K, G) \preceq (L, G)$ by [9, Corollary 3.7]. Thus the pair (K, G) is not categorical for any infinite cardinal.

First, we compute the Morley rank of tuples from K and we show that Morley rank and U-rank coincide. This will be a generalization of the result $\mathrm{MR}(K, G) = \omega$ in [9]. We will also provide a description for imaginaries in (K, G), which allows us to characterize interpretable groups in (K, G). Our description of imaginaries will be via canonical bases as in [28]. Recall that (K, G) does not eliminate imaginaries which is expained in (2.19), since one can not eliminate the quotient $K^\times/G$.

Now we recall the notions of internality and orthogonality from geometric stability theory.

2.5.1 Internality and Orthogonality

Definition 2.33. *Let T be a stable theory. Let a be a tuple and A be set of parameters (possibly containing imaginary elements), and D be a definable subset. We say that the type* $\mathrm{tp}(a/A)$ *is almost D-internal if there exists a set of parameters B such that* $a \underset{A}{\downarrow} B$ *and* $a \in \mathrm{acl}(A, B, D)$. *We say that* $\mathrm{tp}(a/A)$ *is D-internal if we replace* acl *by* dcl.

Note that every extension of an almost D-internal type is almost D-internal and every non-forking restriction is also almost D-internal.

Remark 2.34. *For a stable theory T, if $p = \mathrm{tp}(a/A)$ is a type and D is a definable set, then it is known that p is almost D-internal if and only if there is some B such that if a' realizes p then $a' \in \mathrm{acl}(B, D)$. Details can be found in [30, Lemma 2.17].*

Lemma 2.35. *Let a be a real tuple from K. If* $\mathrm{tp}^P(a/A)$ *is almost G-internal over a real set of parameters A, then $a \in \mathrm{acl}(A, G)$.*

Proof. Take B containing A such that $a \overset{P}{\underset{A}{\downarrow}} B$ and $a \in \mathrm{acl}_P(B, G) = \mathrm{acl}(B, G)$. The characterization of the independence (2.12) yields that $a \underset{A,G}{\downarrow} B, G$ and therefore we obtain that $a \in \mathrm{acl}(A, G)$. □

Remark 2.36. *Note that lemma* (2.35) *may fail if the parameter set A contains imaginary elements.*

The following lemma uses the characterization of definable groups in the pair and it will be vital for the characterization of interpretable groups in the pair (K, G).

Lemma 2.37. *Let H be a definable group in the sense of the pair. If a generic of H is almost G-internal then H is isogenous to an interpretable group in G.*

Proof. By almost internality, we deduce that H is of finite Morley rank. Since infinite algebraic groups have infinite Morley rank in the pair, by Theorem (2.30) and ω-stability we conclude the lemma. □

The following lemma is from [28] which will be needed in lemma (2.51).

Lemma 2.38. *[28, 1.2] Let T be a stable theory and M be a model of T. Suppose that* $\mathrm{tp}(a/A)$ *is stationary and almost internal to the definable set Σ. Then there is $a' \in M^{\mathrm{eq}}$ such that $a' \in \mathrm{dcl}(a, A)$, $a \in \mathrm{acl}(a')$ and* $\mathrm{tp}(a'/A)$ *is internal to Σ. Moreover a' can be taken to be a code (canonical parameter) for a certain finite set of realizations of* $\mathrm{tp}(a/A)$.

Now, we define the U-rank, weight, orthogonality and regular types.

Definition 2.39. *Let T be a stable theory. For a type p, we define its* U*-rank* $\mathrm{U}(p)$ *as follows:*
1- The U*-rank of p is always* ≥ 0,
2- The U*-rank* $\mathrm{U}(p) \geq \alpha + 1$ *if and only if there is a forking extension q of p such that* $\mathrm{U}(q) \geq \alpha$,
3- For limit ordinal δ, the U*-rank* $\mathrm{U}(p) \geq \delta$ *if and only if* $\mathrm{U}(p) \geq \alpha$ *for all* $\alpha < \delta$.
We say that $\mathrm{U}(q) = \alpha$ *when the* $\mathrm{U}(q) \geq \alpha$ *but not* $\mathrm{U}(q) \geq \alpha + 1$. *If* $\mathrm{U}(q) \geq \alpha$ *for all ordinals α, we say the* U*-rank is unbounded.*

If $p = \mathrm{tp}(a/A)$, then $\mathrm{U}(p)$ is denoted by $\mathrm{U}(a/A)$. If T is an ω-stable theory then we always have that $\mathrm{U}(p) \leq \mathrm{MR}(p)$. Now we define the symmetric sum $\oplus$ for ordinals. If α is an ordinal, we can write α as a finite sum $\sum_{i=1}^{n} \omega^{\alpha_i} m_i$, where $\alpha_1 > ... > \alpha_n$ and $m_i \in \mathbb{N}$. If $\alpha = \sum_{i=1}^{n} \omega^{\alpha_i} m_i$ and $\beta = \sum_{i=1}^{n} \omega^{\alpha_i} n_i$, then $\alpha \oplus \beta$ is defined to be

$$\sum_{i=1}^{n} \omega^{\alpha_i}(m_i + n_i).$$

Note that $\alpha + \beta \leq \alpha \oplus \beta$. For example, the sum $1 + \omega = \omega$ while $1 \oplus \omega = \omega + 1 > \omega$. One of the properties of the U-rank is Lascar's inequalities [29, Chapter 1, 3.26]:

$$\mathrm{U}(a/bC) + \mathrm{U}(b/C) \leq \mathrm{U}(ab/C) \leq \mathrm{U}(a/bC) \oplus \mathrm{U}(b/C).$$

Definition 2.40. *Let T be a stable theory. Let a be a tuple and A be set of parameters. The preweight of a complete type* $\mathrm{tp}(a/A)$ *is defined to be the supremum of the cardinals κ such that there is some A-independent set $\{b_\lambda : \lambda < \kappa\}$ such that a forks with b_λ over A for every λ. We denote the preweight of a type as $prewt(p)$.*
If p is a stationary type, the weight of p is defined to be the largest preweight of any non-forking extension of p. We denote the weight of p by $wt(p)$.

Next we define orthogonality and regular types.

Definition 2.41. *1- If $p, q \in S(A)$, then p and q are said to be almost orthogonal if whenever a and b realize p and q respectively then a and b are independent over A.*
2- Two stationary types p and q are said to be orthogonal if all their non-forking extensions to common domains are almost orthogonal.
3- A stationary type p is called regular if it is non-algebraic and it is orthogonal to all its forking extensions.

It is known that if p is a regular type then $wt(p) = 1$, see [29, Chapter 1, 4.5.3]. Moreover, the type $p \in S(A)$ is regular if and only if the independence over A is a

pregeometry on the realizations of p; see [29, Chapter 7, Remark 1.1].

Proposition 2.42. *Let a be in K. Let $p = \mathrm{tp}^P(a/A)$ be a non-algebraic type such that $a \notin A(G)^{ac}$ and $A = \mathrm{acl}_P(A)$ be a parameter set. Then p is stationary and it is a regular type with the pregeometry $cl(C) = \mathrm{acl}(C, G)$ on the set of realizations of p. In particular, if $a \notin A(G)^{ac}$ then $wt(a/A)$ is 1. Furthermore, we have*

$$\mathrm{U}(p) = \mathrm{MR}(p) = \omega.$$

Proof. Let A' be a parameter set containing A. Let a_1 and a_2 be such that $\mathrm{tp}^P(a_1/A) = \mathrm{tp}^P(a_2/A) = \mathrm{tp}^P(a/A)$ and $a_i \overset{P}{\underset{A}{\downarrow}} A'$ for $i = 1, 2$. By Theorem (2.12), we see that a_1 and a_2 are not in $A'(G)^{ac}$. Since there is only one transcendental type over $A'(G)^{ac}$ in the pure field K and since every field automorphism fixing G is an automorphism of the pair (K, G), we conclude that the type p is stationary. Let R be the set of realizations of the type p in K. It is sufficient to show that (R, cl) is a pregeometry where

$$cl(B) = \{b \in R : b \overset{P}{\underset{A}{\not\downarrow}} B\}$$

for $B \subseteq R$. First observe that if $b \in R$ then we also have that $b \notin A(G)^{ac}$. Moreover as $b \underset{A}{\downarrow} G$ and A is G-independent, by transitivity we obtain that $b, A \underset{G_A}{\downarrow} G$. By corollary (2.6) and in terms of linear disjointness we see that

$$\mathrm{acl}(b, A) \overset{ld}{\underset{\mathbb{F}(G_A)}{\downarrow}} \mathbb{F}(G).$$

Therefore by lemma (2.5), we deduce that $\mathrm{acl}_P(b, A) = \mathrm{acl}(b, A)$ and $G_{\mathrm{acl}_P(b,A)} = G_A$. Now by applying corollary (2.13), we conclude that $b \overset{P}{\underset{A}{\not\downarrow}} B$ if and only if $b \underset{A,G}{\not\downarrow} B$ if and only if $b \in B(G)^{ac} = \mathrm{acl}(B, G)$. So we have a pregeometry since (K, acl) is a pregeometry. Hence the type p is regular and has weight 1.

Let C be a set containing A. Since $\mathrm{acl}_P(a, A) = \mathrm{acl}(a, A)$ and $G_{\mathrm{acl}_P(a,A)} = G_A$ as shown before, by Theorem (2.12) we see that $a \overset{P}{\underset{A}{\not\downarrow}} C$ if and only if $a \underset{A,G}{\not\downarrow} C$ iff $a \in \mathrm{acl}(C, G)$. Now take $d_1, ..., d_n$ which are algebraically independent over $A(G)$. Let $g_1, ..., g_n$ be independent generics of G over $D = Ad_1...d_n$. Put $a' = d_1g_1 + \cdots + d_ng_n$. Note that no proper subsum of $d_1g_1 + \cdots + d_ng_n$ is 0. Moreover, the elements a and a' have the same type over A and a' forks with D over A. Since G has the Mann property over K, we see that $g_1, ..., g_n \in \mathrm{acl}_P(a, D)$. This proves that $\mathrm{U}(a/D) \geq \mathrm{U}(g_1, ..., g_n/D) = n$. Thus we obtain that $\mathrm{U}(p) \geq \omega$. Finally, since the theory (K, G)

has Morley rank ω, we conclude that $\mathrm{U}(p) \leq \mathrm{MR}(p) \leq \omega$ and hence

$$\mathrm{U}(p) = \mathrm{MR}(p) = \omega.$$

□

Remark 2.43. *In [9, 6.4], it was shown that if $a \in A(G)^{ac}$ then $\mathrm{MR}(a/A)$ is finite. This shows that if b is not in $A(G)^{ac}$, then $\mathrm{MR}(b/A) \leq \omega$ since there is only one such type.*

Remark 2.44. *Note that if $a \in A(G)^{ac} \setminus A$, then the type $q = \mathrm{tp}^P(a/A)$ need not be regular. To see this, let g_1 and g_2 be two independent generics of G over A and put $a = g_1 + g_2$. Since G has the Mann property, we see that $g_1, g_2 \in \mathrm{acl}_P(a)$. However, each g_i forks with a over A since they are not in A. Therefore $wt(a/A) \geq 2$ and hence it cannot be regular.*

Now we recall a theorem which connects Morley rank and U-rank, and also the definability of Morley rank.

Theorem 2.45. *[39, Chapter 4, 4.7.10] Let T be an ω-stable theory. Suppose that there exists strongly minimal formulas $\phi_1, ..., \phi_k$ such that any type is non-orthogonal to a certain ϕ_i. Then T has finite Morley rank which coincides with U-rank and Morley rank is definable, that is to say for every formula $\theta(x, y)$ and every natural number $n < \omega$ the set $\{a : \mathrm{MR}(\theta(x, a)) = n\}$ is definable.*

Next we compute the rank of an element from K.

Lemma 2.46. *If $a \in A(G)^{ac}$ where $A = \mathrm{acl}_P(A)$, then*

$$\mathrm{U}(a/A) = \mathrm{MR}(a/A) = \mathrm{MR}_G(G_{\mathrm{acl}_P(a,A)}/G_A).$$

Proof. As G is almost strongly minimal and a is in $A(G)^{ac}$, by Theorem (2.45) we see that $\mathrm{U}(a/A) = \mathrm{MR}(a/A)$. Now we prove the other equality. Since $a \in A(G)^{ac}$, there are $g_1, ..., g_n$ from G such that $a \in A(g_1, ..., g_n)^{ac}$. We may assume that $g_1, ..., g_n$ are algebraically independent over A and n is minimal. So without loss of generality, we may suppose that $a = d_1h_1 + \cdots + d_mh_m$ where h_i is in G and d_i is in the field generated by a and A, the element a is algebraic over $d_1, ..., d_m, A$ and no proper subsum of $d_1h_1 + \cdots + d_mh_m$ is 0. Since G has the Mann property, this yields that $h_1, ..., h_n \in \mathrm{acl}_P(a, A)$. Thus $\mathrm{MR}(a/A) = \mathrm{MR}(h_1, ..., h_n/A) = \mathrm{MR}(G_{\mathrm{acl}_P(a,A)}/A)$. Since A is G-independent by lemma (2.4) and the induced structure on G is the pure group structure, we conclude that

$$\mathrm{MR}(G_{\mathrm{acl}_P(a,A)}/A) = \mathrm{MR}_G(G_{\mathrm{acl}_P(a,A)}/G_A).$$

□

Combining proposition (2.42) and lemma (2.46) we have the following formula for the ranks in the pair:

Corollary 2.47. *Let a be finite tuple from K and $A = \mathrm{acl}_P(A)$ be a set of parameters. Then*

$$\mathrm{U}(a/A) = \mathrm{MR}(a/A) = \omega \cdot tr.deg(a/A(G)) + \mathrm{MR}_G(G_{\mathrm{acl}_P(a,A)}/G_A).$$

Proof. We have already proved it if a is an element in K by (2.42) and (2.46). Now let a be a finite tuple from K. In this case, we split the tuple into two parts a_1 and b_1 where a_1 is algebraically independent over $A(G)$ and $b_1 \in \mathrm{acl}(a_1, A, G)$. Then we can replace b_1 by a tuple g in G as we did in (2.46). Thus we obtain that

$$\mathrm{U}(a/A) = \mathrm{U}(a_1b_1/A) = \mathrm{U}(a_1g/A)$$

and similarly for the Morley rank. Note also that a_1 is independent from g over A in the sense of the pair. Since also a_1 is independent over A and the U-rank is additive, we obtain the formula for the finite tuples for the U-rank. For the Morley rank, we proceed by induction and use a similar argument as in remark (2.43) to obtain that $\mathrm{MR}(a_1g/A) \leq \omega \cdot tr.deg(a_1/A(G)) + \mathrm{MR}(g/A)$. Since we know the formula for the U-rank and U-rank is always smaller than the Morley rank, this yields the formula for finite tuples. □

We end this subsection by characterizing definable subfields of K.

Proposition 2.48. *If F is an infinite definable subfield of K in the sense of the pair, then $F = K$.*

Proof. Note that F is an algebraically closed field since it is ω-stable. So its additive group and multiplicative group are connected. First, we show that Morley rank of F is infinite. If the Morley rank of F is finite, then by proposition (2.42), the generic of the additive group of F and the multiplicative group of F are G-internal. By lemma (2.37), they are isogeneous to a group interpretable in G. However, since G is an abelian group, it cannot interpret an infinite field. This indicates that $\mathrm{MR}(F)$ is infinite. Thus we obtain that $\mathrm{MR}(F) = \omega$ as $\mathrm{MR}(K) = \omega$. So the extension K/F cannot be infinite. Since F is algebraically closed, we conclude that $F = K$. □

2.5.2 Canonical Base Lemmas

In this subsection we prove several lemmas for the properties of canonical bases in (K, G). These lemmas will be analogous to the lemmas in [28] for the pair (K, G).

Lemma 2.49. *Let B be an elementary substructure of (K, G). Suppose that $d = \mathrm{Cb}(\mathrm{tp}(a/\mathrm{acl}(B, G)))$. Then $a \overset{P}{\underset{d}{\downarrow}} B, G$.*

Proof. By elimination of imaginaries in K, we may assume that d is in K. First observe that

$$\mathrm{acl}_P(B, G) = \mathrm{acl}(B, G)$$

and $a \underset{d}{\downarrow} B, G$. Note also that $\mathrm{acl}(d, G_{\mathrm{acl}_P(d)})$ is algebraically closed in the sense of the pair by corollary (2.6). For the same reason, the set $\mathrm{acl}(G_{\mathrm{acl}_P(d)})$ is also algebraically closed in (K, G). In particular, they are G-independent by lemma (2.4). The independence $a \underset{d}{\downarrow} B, G$ yields that

$$\mathrm{acl}(a, d, G_{\mathrm{acl}_P(d)}) \underset{d, G_{\mathrm{acl}_P(d)}}{\downarrow} G,$$

and since $\mathrm{acl}(d, G_{\mathrm{acl}_P(d)}) \underset{G_{\mathrm{acl}_P(d)}}{\downarrow} G$, by transitivity we obtain that

$$\mathrm{acl}(a, d, G_{\mathrm{acl}_P(d)}) \underset{G_{\mathrm{acl}_P(d)}}{\downarrow} G.$$

As $\mathrm{acl}(G_{\mathrm{acl}_P(d)}) = \mathrm{acl}_P(G_{\mathrm{acl}_P(d)})$, by transitivity and in terms of linear disjointness this gives us that

$$\mathrm{acl}(a, d, G_{\mathrm{acl}_P(d)}) \overset{ld}{\underset{\mathbb{F}(G_{\mathrm{acl}_P(d)})}{\downarrow}} \mathbb{F}(G).$$

Thus by lemma (2.5), we deduce that $\mathrm{acl}_P(a, d, G_{\mathrm{acl}_P(d)}) = \mathrm{acl}(a, d, G_{\mathrm{acl}_P(d)})$ and $G_{\mathrm{acl}_P(a,d)} = G_{\mathrm{acl}_P(d)}$. Now since $a \underset{d}{\downarrow} B, G$, we have that

$$\mathrm{acl}(a, d, G_{\mathrm{acl}_P(d)}) \underset{\mathrm{acl}(d, G_{\mathrm{acl}_P(d)})}{\downarrow} \mathrm{acl}(B, G).$$

By corollary (2.6) again, we see that $\mathrm{acl}(d, G_{\mathrm{acl}_P(d)}) = \mathrm{acl}_P(d)$ and we obtain that

$$\mathrm{acl}_P(a, d) \underset{\mathrm{acl}_P(d)}{\downarrow} \mathrm{acl}_P(B, G).$$

We finish the lemma by applying the characterization of the independence (2.12).

□

Corollary 2.50. *Let B be an elementary substructure of (K, G) and let a be a finite tuple from K. Set $d = \mathrm{Cb}(\mathrm{tp}(a/\mathrm{acl}(B, G)))$. Then $\mathrm{Cb}(\mathrm{tp}^P(a/B))$ is interalgebraic in (K, G) with $\mathrm{Cb}(\mathrm{tp}^P(d/B))$.*

Proof. Set $p_B = \mathrm{tp}^P(a/B)$ and $q_B = \mathrm{tp}^P(d/B)$. Let $e_1 = \mathrm{Cb}(p)$ and $e_2 = \mathrm{Cb}(q)$. Note that both e_1 and e_2 are in B^{eq}. By lemma (2.49), we know that $a \overset{P}{\underset{d}{\downarrow}} B, G$. So $a \overset{P}{\underset{d}{\downarrow}} B, d$ and $a \overset{P}{\underset{e_2, d}{\downarrow}} B, d$. As $d \overset{P}{\underset{e_2}{\downarrow}} B$, by transitivity we conclude that $a \overset{P}{\underset{e_2}{\downarrow}} B$. This yields that e_1

is algebraic over e_2.

Now we show the converse. Take B_1 such that $\mathrm{tp}^P(B_1/e_1) = \mathrm{tp}^P(B/e_1)$ and $B_1 \underset{e_1}{\overset{P}{\downarrow}} B$. Let p_{B_1} and q_{B_1} be the corresponding types. Choose an element $a_1 \models p_B \cup p_{B_1}$ such that $a_1 \underset{e_1}{\overset{P}{\downarrow}} B, B_1$. Put $d_1 = \mathrm{Cb}(a_1/\operatorname{acl}(B, B_1, G))$, the element $d_2 = \mathrm{Cb}(a_1/\operatorname{acl}(B, G))$ and $d_3 = \mathrm{Cb}(a_1/\operatorname{acl}(B_1, G))$. The independence $a_1 \underset{e_1}{\overset{P}{\downarrow}} B, B_1$ gives that $a_1 \underset{B}{\overset{P}{\downarrow}} B_1$ and by the characterization of the independence (2.12) we obtain that $a_1 \underset{B,G}{\downarrow} B_1$. As d_2 is in $\operatorname{acl}(B, G)$ and by transitivity, we see that $a_1 \underset{d_2}{\downarrow} B, B_1, G$. Thus we deduce that d_1 is algebraic over d_2 and in particular it is in $\operatorname{acl}(B, G)$. Similarly, the element d_1 is algebraic over d_3 and it is in $\operatorname{acl}(B_1, G)$. Moreover, the independence $a_1 \underset{d_1}{\downarrow} B, B_1, G$ and $d_1 \in \operatorname{acl}(B, G)$ yield that $a_1 \underset{d_1}{\downarrow} B, G$ and hence d_2 is algebraic over d_1. As a result, we conclude that $d_1 = d_2 = d_3$. By the choice of the element a_1, we see that $d_1 \models q_B \cup q_{B_1}$. Furthermore, the element d_1 is in $\operatorname{acl}_P(a_1, B) \cap \operatorname{acl}_P(a_1, B_1)$ as it is the canonical base of the types $\mathrm{tp}(a_1/\operatorname{acl}(B, G))$ and $\mathrm{tp}(a_1/\operatorname{acl}(B_1, G))$. Now, from $a_1 \underset{B_1}{\overset{P}{\downarrow}} B$ and $d_1 \in \operatorname{acl}_P(a_1, B_1)$, we obtain that $d_1 \underset{B_1}{\overset{P}{\downarrow}} B$. By the independence $B_1 \underset{e_1}{\overset{P}{\downarrow}} B$ and transitivity, we get that $d_1 \underset{e_1}{\overset{P}{\downarrow}} B$. Hence e_2 is algebraic over e_1. □

The next lemma states that, up to interalgebraicity, an imaginary element is a canonical base of a type over itself and this type is almost G-internal.

Lemma 2.51. *Let $e \in (K, G)^{\mathrm{eq}}$ be an imaginary element. Then there is $e' \in (K, G)^{\mathrm{eq}}$ interalgebraic with e, such that for some finite tuple d' from K we have $e' = \mathrm{Cb}(\mathrm{tp}^P(d'/e'))$ and $\mathrm{tp}^P(d'/e')$ is almost G-internal.*

Proof. Let a be a tuple in K such that $e = f(a)$ for some 0-definable function in $(K, G)^{\mathrm{eq}}$. Set $e_1 = \mathrm{Cb}(\mathrm{tp}^P(a/\operatorname{acl}_P{}^{\mathrm{eq}}(e)))$. Observe that e_1 is algebraic over e. As $e = f(a)$ and $a \underset{e_1}{\overset{P}{\downarrow}} e$, we obtain that $e \underset{e_1}{\overset{P}{\downarrow}} e$ and hence e and e_1 are interalgebraic. Now let (B, G_B) be an elementary substructure of (K, G) such that $e_1 \in (B, G_B)^{\mathrm{eq}}$ and $a \underset{e_1}{\overset{P}{\downarrow}} B$. Let $d = \mathrm{Cb}(\mathrm{tp}(a/\operatorname{acl}(B, G)))$. We may assume d to be a finite tuple in K owing to the ω-stability and elimination of imaginaries in K. Let $e_2 = \mathrm{Cb}(\mathrm{tp}^P(d/B))$. Then by corollary (2.50) and $a \underset{e_1}{\overset{P}{\downarrow}} B$, we see that e_1 and e_2 are interalgebraic. Note that the type $\mathrm{tp}^P(d/B)$ is almost G-internal. Thus as we have $d \underset{e_2}{\overset{P}{\downarrow}} B$, the type $\mathrm{tp}^P(d/e_2)$ is also almost G-internal. By lemma (2.38), there is an imaginary element $d' \in (K, G)^{\mathrm{eq}}$ such that $d' \in \operatorname{acl}_P(d, e)$ and $d \in \operatorname{acl}_P(d')$, and also the type $\mathrm{tp}^P(d'/e_2)$ is almost

G-internal. Let $e' = \mathrm{Cb}(\mathrm{tp}^P(d'/e_2))$. Then $e' \in \mathrm{acl}_P(e_2)$ and $d \overset{P}{\underset{e'}{\downarrow}} e_2$ as $d \in \mathrm{acl}_P(d')$. Hence $e_2 \in \mathrm{acl}_P(e')$. As a result, we conclude that e and e' are interalgebraic. □

From now on, we assume that G is ω-stable with $\mathrm{acl}_G(\emptyset)$ infinite in the pure group language.

Lemma 2.52. *(Coheir) Let $e \in (K,G)^{\mathrm{eq}}$ and $B = \mathrm{acl}_P(e) \cap G$. Let c be a tuple from G. Then $\mathrm{tp}^P(c/B,e)$ is finitely satisfiable in B.*

Proof. First observe that the type $p = \mathrm{tp}^P(e/\mathbb{F}(G)^{ac})$ is stationary. Let d be the canonical base of p. Thus, we see that d is in $\mathrm{acl}_P{}^{\mathrm{eq}}(G)$. Note also that, any automorphism in the sense of the pair fixes G setwise, and so fixes $\mathbb{F}(G)^{ac}$ setwise as well. This yields that d is contained in $\mathrm{acl}_P{}^{\mathrm{eq}}(e)$, and as a result we obtain that d is contained in $\mathrm{acl}_P{}^{\mathrm{eq}}(B)$. Therefore p is the non-forking extension of the type $\mathrm{tp}^P(e/B)$ and hence p is definable over B. Therefore for a given formula $\phi(x,y)$ of $L(U)^{\mathrm{eq}}$ over B, there is a formula $\psi(y)$ over B such that $\phi(x,\gamma) \in p$ if and only if $\psi(\gamma)$ holds. Since G has the Mann property and by stability, there exists a formula $f(y)$ over B in the language of pure groups such that, for all $\gamma \in G$ we have $\models \psi(\gamma)$ if and only if $\models f(\gamma)$. Since B is an elementary substructure of G (because it is infinite and algebraically closed in G), if $c \in G$ and $\models \phi(e,c)$ then $\models \psi(c)$ and so $\models f(c)$, therefore for some $c_1 \in B$ we have $\models f(c_1)$ and as a consequence $\models \phi(e,c_1)$.

□

Lemma 2.53. *Let $e \in (K,G)^{\mathrm{eq}}$ be an imaginary element. There is a tuple d from K, an $L(U)$-definable function $f(x)$ over $\emptyset$, an $L(U)$-formula $\psi(y)$ over e and an $L(U)$-definable function $h(y,z)$ over e such that*

(i) $f(d) = e$,

(ii) $\psi(y) \in \mathrm{tp}(d/e)$,

(iii) $(\forall y, y')(\psi(y) \wedge \psi(y') \implies \exists z(U(z) \wedge h(y,z) = y'))$

(iv) Furthermore, the element d is independent from G over e.

Proof. For (i), (ii) and (iii) we refer the reader to [28, 2.4]. Now we prove (iv). Choose d such that $\mathrm{MR}(\mathrm{tp}^P(d/e))$ is minimized. Moreover by lemma (2.51), we can assume that $q = \mathrm{tp}^P(d/e)$ is almost G-internal. Thus remark (2.34) yields that there is some set u such that if $d' \models q$ then $d' \in \mathrm{acl}(u,G)$. We will show that d is independent from G over e. Suppose not and choose $b \in G$ such that d forks with b over e. Note that by almost internality and as G has Morley rank 1, we deduce that $\mathrm{MR}(\mathrm{tp}^P(d/e))$ is finite. Let $m = \mathrm{MR}(\mathrm{tp}^P(d/e,b)) < \mathrm{MR}(\mathrm{tp}^P(d/e))$. Note that the Morley rank is definable in $\mathrm{acl}(u,G)$ by remark (1.4) or by Theorem (2.45) as G is almost strongly

minimal. Let $\chi(y,z)$ be a formula over e such that $\chi(d,b)$ holds and for any c, we have $\mathrm{MR}(\chi(y,c)) = m$ if it is consistent. Let $\Delta(z)$ be the formula

$$\exists y(f(y) = e \wedge \psi(y)) \wedge (\forall y, y')(\psi(y) \wedge \psi(y') \implies (h(y,z) = y' \wedge U(z))).$$

Therefore $\Delta(b)$ holds. Let $B = \mathrm{acl}_P(e) \cap G$. By lemma (2.52), there is $b_1 \in B$ such that $\Delta(b_1)$ holds. Then we find d_1 satisfying (i), (ii) and (iii) of the lemma with $\chi(d_1, b_1)$ holds. As b_1 is algebraic over e, we have that $\mathrm{MR}(\mathrm{tp}^P(d_1/e)) \leq m$, contradicting the choice of d.

□

Therefore combining lemmas (2.51) and (2.53), we have the following theorem describing imaginariy elements in the pair:

Theorem 2.54. *Let $e \in (K,G)^{\mathrm{eq}}$ be an imaginary element. There is a finite real tuple d such that e is algebraic over d, the type $\mathrm{tp}^P(d/e)$ is almost G-internal and d is independent from G over e in the sense of the pair.*

2.5.3 Characterization of Interpretable Groups

We start with a lemma from [2]:

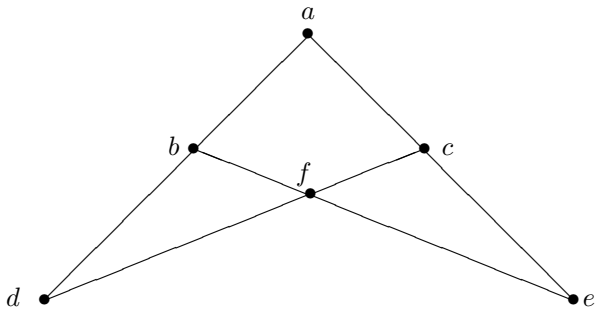
Lemma 2.55. *[2, Lemma 3.1] Let H be a connected interpretable group in a stable theory. Let α, β and γ be three independent generics of H and a_0 be a real element such that α is algebraic over a_0. Then there exist real tuples a, b, c, d, e and f such that*

$$(a,\alpha) \equiv (a,\beta) \equiv (c,\gamma) \equiv (d,\alpha\beta) \equiv (e,\gamma\alpha) \equiv (f,\gamma\alpha\beta) \equiv (a_0,\alpha)$$

and

$$a \underset{\alpha}{\downarrow} b, c, d, e, f$$

and the same for the other tuples. Moreover in the following diagram:



all non-linear triples are independent and each point is independent from the lines which do not contain it.

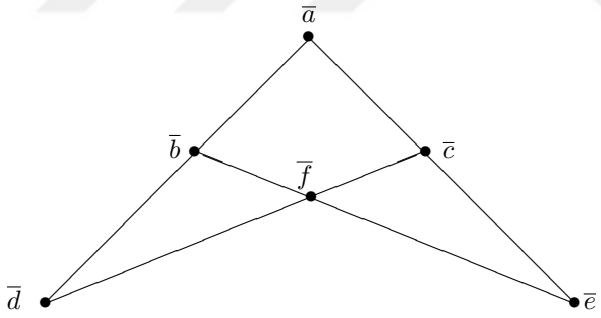
Now we are ready to characterize interpretable groups in the pair (K, G) which requires all the tools developed through the chapter.

Theorem 2.56. *(Interpretable groups in (K, G)) Let K be an algebraically closed field and G be an ω-stable multiplicative subgroup of $K^\times$ with $\mathrm{acl}_G(\emptyset)$ infinite and with the Mann property. Every interpretable group H in (K, G) is, up to isogeny, an extension of an interpretable abelian group in G by a T_P-interpretable group N, which is a quotient of an algebraic group V by a subgroup N_1 which is an interpretable abelian group in G.*

Proof. Let H be an interpretable group in (K, G). By remark (2.26), we may assume that H is connected. Again we work over a small model that we omit. Let α, β and γ be three independent generics of H in the sense of the pair. By Theorem (2.54), the generic α is algebraic over a real tuple a_0 which is independent from G over α in the sense of the pair, and the type $\mathrm{tp}^P(a_0/\alpha)$ is almost G-internal. Then by lemma (2.55), there are real tuples a, b, c, d, e and f such that

$$(a, \alpha) \equiv^P (a, \beta) \equiv^P (c, \gamma) \equiv^P (d, \alpha\beta) \equiv^P (e, \gamma\alpha) \equiv^P (f, \gamma\alpha\beta) \equiv^P (a_0, \alpha)$$

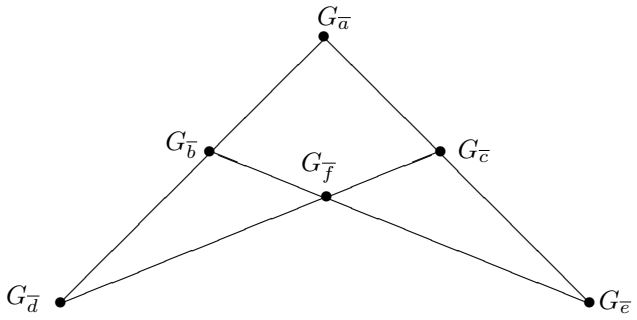
and if we put $\overline{a} = \mathrm{acl}_P(a)$ and the same for the others, we have the following diagram:



such that all non-linear triples are T_P-independent and each point is T_P-independent from the lines which do not contain it. Since $a \overset{P}{\underset{\alpha}{\downarrow}} G$, we see that $G_{\overline{a}} \subset \mathrm{acl}_P(\alpha) \subset \overline{a}$. Therefore, we obtain that $G_{\overline{a}} = \mathrm{acl}_P(\alpha) \cap G$. Moreover by lemma (2.9) and Theorem (2.12), we have that $G_{\mathrm{acl}_P(\overline{a},\overline{b})} = \mathrm{acl}_G(G_{\overline{a}}, G_{\overline{b}})$. Since

$$G_{\overline{d}} = \mathrm{acl}_P(\alpha\beta) \cap G \subset G_{\mathrm{acl}_P(\overline{a},\overline{b})},$$

we get that $G_{\overline{d}} \subset \mathrm{acl}_G(G_{\overline{a}}, G_{\overline{b}})$. This is true for all other tuples and by lemma (2.16), the set $G_{\overline{a}}$ is G-independent from $\mathrm{acl}_G(G_{\overline{b}}, G_{\overline{e}})$. The same holds for the others. So we have the following diagram:



and by the group configuration theorem (1.18), we have a connected $*$-interpretable group $H_1(G)$ in G whose generic h is G-algebraic with $G_{\overline{a}}$. By ω-stability and by lemma (2.27), we may assume that H_1 is interpretable and its generic h is T_P-interalgebraic with $G_{\overline{a}}$. Moreover, we have a type-definable surjection

$$\pi : H \to H_1(G).$$

Furthermore by Theorem (2.29), we see that H_1 is abelian.

Next we show that the points $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}$ give a T-group configuration with the help of the parameter set G. We know that any three non-colinear points among them are independent in the sense of the pair, and hence they are T-independent over G. As β is algebraic over b and $\alpha\beta$ is algebraic over d, we have that α is algebraic over b, d. Moreover by lemma (2.9), we know that $\operatorname{acl}(\overline{b}, \overline{d}) = \operatorname{acl}_P(\overline{b}, \overline{d})$. Since the type $\operatorname{tp}^P(a/\alpha)$ is almost G-internal and α is algebraic over b, d, we observe that the type $\operatorname{tp}^P(a/\operatorname{acl}(\overline{b}, \overline{d}))$ is also almost G-internal. Thus by lemma (2.35) we obtain that $a \in \operatorname{acl}(G, \overline{b}, \overline{d})$. The same holds for the other tuples.

Therefore we obtain a connected $*$-interpretable group V over $\operatorname{acl}(G)$ in the field sense and two independent generics a_1, b_1 of V such that a_1 is field interalgebraic with $\overline{a}$ over G, the element b_1 is field interalgebraic with $\overline{b}$ and $a_1 b_1$ is field interalgebraic with $\overline{d}$. Since the tuples α, β and γ are algebraic over the finite tuples a_1, b_1 and $a_1 b_1$ respectively and as V is a connected pro-algebraic group, there exists a connected algebraic group W over $\operatorname{acl}(G)$ and two independent generics a_2, b_2 such that α is algebraic over a_2 and the same for the others. Note that a_1 is field algebraic over $G, \overline{a}$ and the same holds for the others. Moreover, since $\overline{a}, \overline{b}, \overline{d}$ are pairwise T_P-independent over G, so are a_2, b_2 and $a_2 b_2$.

As α is algebraic over a and $a \overset{P}{\underset{G_{\overline{a}}}{\downarrow}} G$ by corollary (2.15), we see that $\alpha \overset{P}{\underset{G_{\overline{a}}}{\downarrow}} G$. Now let N be the connected component of $ker(\pi)$. Then α is generic in $N\alpha$ over $\operatorname{acl}_P(h) = \operatorname{acl}_P(G_{\overline{a}})$, therefore α is also generic over the group G.

Now we apply the lemma (2.27) to the tuples (a_2, α) and (b_2, β). This yields a

type-definable surjection ϕ from W to N, up to isogeny. Lastly, we prove that the connected component N_1 of $ker(\phi)$ is isogenous to an interpretable group in G. Let n_1 be a generic of N_1 over G, a_2. Then the point $(n_1, 1_N)$ is in the stabilizer of the type $\mathrm{tp}(a_2, \alpha/\mathrm{acl}^{\mathrm{eq}}(G))$ and so $\mathrm{tp}^P(n_1 a_2/\alpha) = \mathrm{tp}^P(a_2/\alpha)$. Since $\mathrm{tp}^P(a_2/\alpha)$ is almost G-internal and as a_2 is algebraic over G, a then the type $\mathrm{tp}^P(n_1 a_2/\alpha)$ is also almost G-internal. As α is algebraic over G, a_2, the type $\mathrm{tp}^P(n_1/G, a_2)$ is almost G-internal. Owing to the independence $n_1 \overset{P}{\underset{G}{\downarrow}} a_2$, we conclude that $\mathrm{tp}^P(n_1/G)$ is also almost G-internal. Then by lemma (2.37) we have that N_1 is isogenous to an interpretable group in G. Theorem (2.29) yields again that the group N_1 is abelian.

□

2.6 Remarks on Differentially Closed Field Case

In this section, we just give the analogous theorems for differentially closed fields with no proofs. Let (Ω, ∂) be a differentially closed field of characteristic 0 and let

$$C = \{x \in \Omega : \partial(x) = 0\}$$

be the constant field of Ω. Recall that (Ω, ∂) has QE and EI. Moreover, it is ω-stable. Let G be a multiplicative subgroup of $\Omega^\times$ with the Mann property. The pair (Ω, ∂, G) can be seen as an $L(U) = L \cup \{U\}$ structure where L is the usual language for differential fields and U is an unary predicate whose interpretation in Ω is G. We begin with a question:

Question: Is the theory of (Ω, ∂, G) stable?

In contrast to algebraically closed case, the question is not always affirmative. Even there is a possibility to define the ring of integers $\mathbb{Z}$ in (Ω, ∂, G). First assume that $\partial(G)$ is not zero. Let $g \in G$ be such that $\partial(g)$ is not zero. In particular, it is not a torsion element. Observe that $m = g\partial(g^m)/\partial(g)g^m$. Therefore even in the simplest case where G is cyclic and generated by the element g, the formula

$$\varphi(x, g) = \exists y \left(U(y) \wedge x = \frac{g\partial(y)}{\partial(g)y} \right)$$

defines $\mathbb{Z}$. Now we give some examples. Let Ω be a differentially closed field containing $\mathbb{C}((t))$ where $\partial(t) = 1$. Thus $(\Omega, \partial, t^{\mathbb{Z}})$ and $(\Omega, \partial, e^{t\mathbb{Z}})$ are not stable and we can define $\mathbb{Z}$ in $(\Omega, \partial, e^{t\mathbb{Z}})$ without parameters as $\partial(e^t) = e^t$.

Hence in order to prove the stability of (Ω, ∂, G), this leads us to assume that $\partial(G) = 0$, in other words G is a subset of the constant field C. From now on we suppose that $\partial(G) = 0$. This in return gives us that, the results 3.2, 3.4, 3.5 and 5.1

in [9] are valid also in the differential case, so similarly we conclude that the induced structure on G is itself, the pair (Ω, ∂, G) is stable and if G is ω-stable in the pure group language then so is (Ω, ∂, G).

Moreover one can similarly prove that the independence in (Ω, ∂, G) is given exactly by the independence (2.12). One can prove all the analogous results from this with similar methods. We just give definable and interpretable groups in (Ω, ∂, G). Thus one can obtain the following results:

Theorem 2.57. *(Definable Groups (Ω, ∂, G)) Any type-definable group in (Ω, ∂, G) is isogenous to a subgroup of a differential algebraic group. Moreover any type-definable group is, up to isogeny, an extension of a type-interpretable group in G by a differential algebraic group.*

In the case where G is divisible with $\mathrm{acl}_G(\emptyset)$ infinite, we can characterize interpretable groups in (Ω, ∂, G):

Theorem 2.58. *(Interpretable groups in (Ω, ∂, G))*

Every interpretable group H in (Ω, ∂, G) is, up to isogeny, an extension of an interpretable abelian group in G by an interpretable group N, which is a quotient of a differential algebraic group D by a subgroup N_1 which is interpretable in G.

3

Mann Pairs

In this chapter, let Ω be an algebraically closed ambient field, the field k be a proper subfield of Ω which is also algebraically closed and Γ be a multiplicative subgroup of $\Omega^\times$. Now we define a uniform version of the Mann property which was introduced in the previous chapter. Consider an equation

$$a_1x_1 + \cdots + a_nx_n = 1 \tag{3.0.1}$$

with $n \geq 1$ and $a_i \in k$.

We say that (k, Γ) is a *Mann pair* if for all n there is a finite subset $\Gamma(n)$ of Γ such that for all $a_1, ..., a_n$ in $k^\times$ all non-degenerate solutions of (3.0.1) in Γ lie in $\Gamma(n)$. In particular, the group Γ has the Mann property. Observe that if (k, Γ) is a Mann pair, then taking $n = 1$ in the definition, we see that $k \cap \Gamma$ is finite, thus the intersection is a finite subset of the group of roots of unity in Ω. Therefore, most of the elements in Γ are transcendental over the field k. To illustrate, the pair $(\overline{\mathbb{Q}}, \exp(\overline{\mathbb{Q}}))$ is a Mann pair by Lindemann's theorem. In [10, Theorem 1.1], L. van den Dries and A. Günaydın proved that if the intersection $k \cap \Gamma$ is trivial and if Γ is of finite rank, then (k, Γ) is a Mann pair. This provides substantial examples of Mann pairs, such as $(\mathbb{C}, t^{\mathbb{Z}})$ where t is an indeterminate.

Now fix Ω_0, k_0 and Γ_0 where (k_0, Γ_0) is a Mann pair as above. By the triple $(\Omega_0, k_0, \Gamma_0)$ we actually mean the structure $(\Omega_0, k_0, \Gamma_0, +, -, \cdot, 0, 1)$. Thus our language

is $\{+,-,\cdot,0,1,P_1,P_2\}$ where P_1 and P_2 are unary predicates whose interpretations in Ω_0 are k_0 and Γ_0 respectively. The model theory of the triple (Ω_0,k_0,Γ_0) was studied in [10, 11] by L. van den Dries and A. Günaydın, where they proved that the theory $Th(\Omega_0,k_0,\Gamma_0)$ is stable and it is ω-stable if Γ_0 is divisible. In order to study definable groups in the triple, we need to add the constants $k_0\cup\Gamma_0$ as they did in [10] in order to have certain sets definable with parameters from k_0 and Γ_0, see theorem (3.1). So our language L_t through the chapter is $\{+,-,\cdot,0,1,P_1,P_2\}$ together with the constants for each element of $k_0\cup\Gamma_0$. Let T_t be the complete theory of (Ω_0,k_0,Γ_0) in the language L_t. Therefore if (Ω,k,Γ) is a model of T_t then k contains k_0 and Γ contains the group Γ_0. Note also that, if (Ω,k,Γ) is a model of T_t then the triple (Ω,k_0,Γ_0) is an elementary substructure of (Ω,k,Γ) by [11, 4.4]. Moreover in [10], L. van den Dries and A. Günaydın proved that $k\cup\Gamma$ is small in Ω as defined in Chapter 1. Therefore, by changing the model we may assume that Ω is $|k\cup\Gamma|^+$-saturated as a field, and the triple is κ-saturated for some uncountable cardinal κ. Through the chapter, we will be working in this sufficiently saturated model.

In the previous chapter, we focused on the model theory of the pair (Ω,Γ) in terms of stability. As cited before, the model theory and definable groups in (Ω,k) were studied in [31, 1, 2]. In this chapter, our concern will be the triple (Ω,k,Γ) in the stability frame work and we bring present the connection between the triple (Ω,k,Γ) and the pairs (Ω,k) and (Ω,Γ). More precisely, we characterize the algebraic closure and forking in the triple. This allows us to characterize definable groups in the triple in terms of definable and interpretable groups in Ω, k and Γ. As the strongest result in this chapter, we characterize interpretable groups in a similar way to the previous chapter.

We stick to a similar notation as in Chapter 2. For a substructure A in the sense of the triple, we denote $k_A=A\cap k$ and $\Gamma_A=A\cap\Gamma$. By $\mathrm{acl}(A)$, we mean the algebraic closure of A in the field sense and $\mathrm{acl}_t(A)$ indicates the algebraic closure of A in the triple (Ω,k,Γ). By $\mathop{\smile\!\!\!\!\!\!|}$ we mean the algebraic independence in the pure field Ω and $\mathop{\smile\!\!\!\!\!\!|}^{t}$ signifies the independence in the triple. Similarly, $\mathop{\smile\!\!\!\!\!\!|}^{P_1}$ denotes the independence in the pair (Ω,k) and $\mathop{\smile\!\!\!\!\!\!|}^{P_2}$ indicates the independence in the pair (Ω,Γ). If A is a subset of Γ, the algebraic closure of A in Γ will be represented by $\mathrm{acl}_\Gamma(A)$.

Let a be a tuple in Ω and B be a set of parameters. Unless otherwise stated, the type $\mathrm{tp}(a/B)$ denotes the type of a over B in the pure field sense. By $\mathrm{tp}^t(a/B)$ we mean the type of a over B in the sense of the triple. We use similar notations for tp_k and tp_Γ to indicate the types in k and Γ respectively. Finally, for three fields E,F and $L\subseteq E\cap F$, the notation $E\mathop{\smile\!\!\!\!\!\!|}_{L}^{ld}F$ means that E is linearly disjoint from F over L.

Adding the constants for each element of k_0 and Γ_0 will be significant to control the parameters for definability, since we need algebraically closed structures to contain enough elements.

The following is in [10] and it states that k and Γ are orthogonal in model-theoretic sense.

Theorem 3.1. *[10, Theorem 1.2 and Remark in 8.3] For all $m, n \geq 1$, every definable subset of $k^m \times \Gamma^n$ definable in (Ω, k, Γ) is a finite union of sets $X \times Y$ with $X \subseteq k$ definable in the field k and $Y \subseteq \Gamma$ definable in the group Γ. Moreover, the set $\Sigma_n = \{(k_1, ..., k_n, g_1, ..., g_n) : k_1 g_1 + \cdots + k_n g_n = 0\} \subseteq k^n \times \Gamma^n$ is a finite union of sets $X \times Y$ with $X \subseteq k$ definable in the field k with parameters from k_0 and $Y \subseteq \Gamma$ definable in the group Γ with parameters from Γ_0. In other words, the induced structure on (k, Γ) is itself.*

Using the theorem above, the following lemma follows immediately.

Lemma 3.2. *Let f and g be automorphisms of $Aut(k/k_0)$ and $Aut(\Gamma/\Gamma_0)$ respectively. Then there is an automorphism of $k(\Gamma)$ which extends both f and g.*

Proof. Define $\Sigma_n = \{(k_1, ..., k_n, g_1, ..., g_n) : k_1 g_1 + \cdots + k_n g_n = 0\} \subseteq k^n \times \Gamma^n$. By Theorem (3.1), the set Σ_n is a finite union of sets $X \times Y$ with $X \subseteq k$ definable in the field k with parameters from k_0 and $Y \subseteq \Gamma$ definable in the group Γ with parameters from Γ_0. Therefore $(k_1, ..., k_n, \gamma_1, ..., \gamma_n) \in \Sigma_n$ if and only if $(f(k_1), ..., f(k_n), g(\gamma_1), ..., g(\gamma_n)) \in \Sigma_n$. This yields a ring automorphism h of the ring $k[\Gamma]$ given by

$$h(k_1\gamma_1 + \cdots + k_n\gamma_n) = f(k_1)g(\gamma_1) + \cdots + f(k_n)g(\gamma_n)$$

which further extends to the field $k(\Gamma)$. □

Remark 3.3. *Let k_1 be an algebraically closed subfield of k and Γ_1 be an elementary substructure of Γ. Then the pair (k_1, Γ_1) is an elementary substructure of (k, Γ).*

Proof. By quantifier elimination of algebraically closed fields, the field k_1 is an elementary substructure of k and Γ_1 is an elementary substructure of Γ. We conclude by Theorem (3.1). □

3.1 Characterization of Algebraic Closure for the Triple

In this section, we give the characterization of algebraically closed structures in the triple which will be a key tool for all other proofs through the chapter. In order to characterize the algebraic closure, we depend upon the stability of the triple (Ω, k, Γ) which we know by [10] and we apply lemma (2.2). We begin with a definition.

Definition 3.4. *We say that a substructure A of the triple (Ω, k, Γ) is (k, Γ)-independent if*

$$A \overset{ld}{\underset{k_A(\Gamma_A)}{\downarrow}} k(\Gamma).$$

Similarly A is k-independent if $A \overset{ld}{\underset{k_A}{\downarrow}} k$ and A is Γ-independent if

$$A \overset{ld}{\underset{\mathbb{F}(\Gamma_A)}{\downarrow}} \mathbb{F}(\Gamma).$$

Note that if A is algebraically closed in the sense of the triple, then it is k-independent and Γ-independent.

Recall that the determinant of a matrix $S = (s_{ij})_{i,j \leq n}$ is given by

$$\sum_{\sigma \in S_n} sgn(\sigma) \prod s_{i\sigma(i)}.$$

Lemma 3.5. *Let A be algebraically closed in the sense of the triple. Then A is (k, Γ)-independent.*

Proof. Since A is also algebraically closed in the sense of (Ω, k), this yields that $A \overset{ld}{\underset{k_A}{\downarrow}} k$ and so we have $A \overset{ld}{\underset{k_A(\Gamma_A)}{\downarrow}} k(\Gamma_A)$. By transitivity, it is enough to prove that $A \overset{ld}{\underset{k(\Gamma_A)}{\downarrow}} k(\Gamma)$. Let $a_1, ..., a_n$ be in A (not necessarily distinct), the elements $k_1, ..., k_n$ be in k and $g_1, ..., g_n$ be in Γ such that $a_1 k_1 g_1 + \cdots + a_n k_n g_n = 0$. If $g_1, ..., g_n \in A$ then we are done. So suppose that the tuple $g = (g_1, ..., g_n)$ is not in A. Thus g has infinitely many A-conjugates in the triple. Let $g_i = (g_{i1}, ..., g_{in})$ be some conjugates of g over A where $2 \leq i \leq n$ and $g = (g_{11}, ..., g_{1n})$. Then we have a system of linear equations:

$$a_1 k_{i1} g_{i1} + \cdots + a_n k_{in} g_{in} = 0.$$

Thus the determinant of this matrix is in the field $k(\Gamma)$. Moreover this determinant is zero and it is a polynomial q such that $q(k_1 g_1, ..., k_{nn} g_{nn}) = 0$. By the determinant formula, we have an homogeneous equation

$$\sum_{\sigma \in S_n} k_\sigma h_\sigma = 0$$

where $k_\sigma \in k$ and $h_\sigma = \prod g_{i\sigma(i)} \in \Gamma$. We may assume that no proper subsum of the equation $\sum_{\sigma \in S_n} k_\sigma h_\sigma = 0$ is zero. Dividing by the element $k_{(12)} h_{(12)}$, we obtain the inhomogeneous equation

$$\sum_{\sigma \in S_n \setminus \{(12)\}} \frac{-k_\sigma}{k_{(12)}} \frac{h_\sigma}{h_{(12)}} = 1.$$

Since (k, Γ) is a Mann pair and A is algebraically closed, this yields that the element

$$\frac{h_{(Id)}}{h_{(12)}} = \frac{g_1 g_{22}}{g_2 g_{21}} \in A.$$

However, if we put other conjugates to the determinant and since (k, Γ) is a Mann pair, by uniform finiteness $\Gamma(n! - 1)$ and cancellation we deduce that $g_{21}/g_{11} = g_2/g_1 \in A$. Choosing other transpositions, similarly we obtain that $g_i/g_1 \in A$ for all i. Hence we finish the proof by dividing the equation $a_1 k_1 g_1 + \cdots + a_n k_n g_n = 0$ by g_1.

□

Next, we give the characterization of the algebraic closure in the triple. We follow a similar method as in the previous chapter.

Lemma 3.6. *(Algebraic closure for triples) Let $A \subset \Omega$. Then A is algebraically closed in the sense of the triple if and only if A and k_A are algebraically closed fields, the group Γ_A is algebraically closed in Γ and A is (k, Γ)-independent.*

Proof. If A is algebraically closed in the sense of the triple then A and k_A are algebraically closed fields and Γ_A is algebraically closed in Γ. Moreover by lemma (3.5), A is (k, Γ)-independent. Now we prove the converse. Let α be in $\Omega \setminus A$.
Case 1: Let $\alpha \in \Gamma$. Then since Γ_A is algebraically closed, we know that α has infinitely many conjugates in Γ. Choose a conjugate $\beta \in \Gamma$ of α. Then there is an automorphism $f \in Aut(\Gamma/\Gamma_A)$ sending α to β. Since Γ_A contains Γ_0, by lemma (3.2) there is an automorphism h of $k(\Gamma)$ which is identity on k and f on Γ. Since A is (k, Γ)-independent, by linear disjointness the former automorphism extends to a field automorphism of $A(k, \Gamma)$ over A and this extends to an automorphism of Ω over A which is actually an automorphism of the triple (Ω, k, Γ) over A. Thus α is not in $\mathrm{acl}_t(A)$. In particular, we have $\Gamma_{\mathrm{acl}_t(A)} = \Gamma_A$.
Case 2: Let $\alpha \in k$. Then since k_A is an algebraically closed field, we know that α has infinitely many conjugates in k. Choose a conjugate $\beta \in k$ of α. Then there is an automorphism $f \in Aut(k/k_A)$ sending α to β. Since k_A contains k_0, by lemma (3.2) there is an automorphism h of $k(\Gamma)$ which is identity on Γ and f on k. Since A is (k, Γ)-independent, by linear disjointness h extends to a field automorphism of $A(k, \Gamma)$ over A and this extends to an automorphism of Ω over A which is an automorphism of the triple (Ω, k, Γ) over A. Thus α is not in $\mathrm{acl}_t(A)$. This indicates that $k_{\mathrm{acl}_t(A)} = k_A$.
Case 3: Let $\alpha \in A(k, \Gamma)^{ac} \setminus A$. Then there exist $k_1, ..., k_n \in k$ and $g_1, ..., g_n \in \Gamma$ such that $\alpha \in A(k_1, ..., k_n, g_1, ..., g_n)^{ac} \setminus A$. So for a rational polynomial

$$r(x_0, x_1, ..., x_n, y_1, ..., y_n)$$

with coefficients from A and we have that

$$r(\alpha, k_1, ..., k_n, g_1, ..., g_n) = 0.$$

Moreover, we may assume that $k_1, ..., k_n, g_1, ..., g_n$ are algebraically independent over A. Thus by the first two cases, we know that k_i and g_i are not in $\mathrm{acl}_t(A)$ for $1 \leq i \leq n$. Thus the type $p = \mathrm{tp}^t(k_1, ..., k_n, g_1, ..., g_n / \mathrm{acl}_t(A))$ is not algebraic. Now take $m_1, ..., m_n, h_1, ..., h_n$ such that $(m_1, ..., m_n, h_1, ..., h_n) \models p$ and

$$m_1, ..., m_n, h_1, ..., h_n \overset{t}{\underset{\mathrm{acl}_t(A)}{\downarrow}} k_1, ..., k_n, g_1, ..., g_n.$$

By lemma (2.2), we obtain that

$$m_1, ..., m_n, h_1, ..., h_n \underset{\mathrm{acl}_t(A)}{\downarrow} k_1, ..., k_n, g_1, ..., g_n.$$

Moreover since $\mathrm{acl}_t(A)$ is (k, Γ)-independent by lemma (3.5), as $k_{\mathrm{acl}_t(A)} = k_A$ and $\Gamma_{\mathrm{acl}_t(A)} = \Gamma_A$, by transitivity we get that

$$m_1, ..., m_n, h_1, ..., h_n \underset{A}{\downarrow} k_1, ..., k_n, g_1, ..., g_n.$$

Since there is a triple automorphism over A sending $(k_1, ..., k_n, g_1, .., g_n)$ to the tuple $(m_1, ..., m_n, h_1, ..., h_n)$, this gives a conjugate β of α with the help of the polynomial equation $r = 0$. Observe that β is different than α as we have

$$m_1, ..., m_n, h_1, ..., h_n \underset{A}{\downarrow} k_1, ..., k_n, g_1, ..., g_n$$

and α is not in A. Choosing other independent elements, as a result, we conclude that α has infinitely many conjugates over A and hence α is not in $\mathrm{acl}_t(A)$.
Case 4: The element α is not in $A(k, \Gamma)^{ac}$. Since any field automorphism of Ω fixing k and Γ is an automorphism of the triple, we deduce that $\mathrm{acl}(A, k, \Gamma) = \mathrm{acl}_t(A, k, \Gamma)$. This indicates that α is not in $\mathrm{acl}_t(A)$. Hence we are done.

□

Now we give two immediate corollaries of the previous lemma.

Corollary 3.7. *For any subset D in Ω,*

$$\mathrm{acl}_t(D) = \mathrm{acl}(D, k_{\mathrm{acl}_t(D)}, \Gamma_{\mathrm{acl}_t(D)}).$$

Moreover if $B = \mathrm{acl}_t(k_1, \Gamma_1)$ where k_1 and Γ_1 are algebraically closed in k and Γ respectively, then $B = \mathrm{acl}(k_1, \Gamma_1)$.

Proof. As $\mathrm{acl}(D, k_{\mathrm{acl}_t(D)}, \Gamma_{\mathrm{acl}_t(D)}) \subseteq \mathrm{acl}_t(D)$ and $\mathrm{acl}_t(D)$ is (k, Γ)-independent by lemma (3.5), we conclude by (3.6). In the proof of lemma (3.6) Case 1 and Case 2, we observe that $k_B = k_1$ and $\Gamma_B = \Gamma_1$. We finish by lemma (3.6) again.

□

Corollary 3.8. *Let B be algebraically closed in the sense of the triple. Then $\Gamma_B = \Gamma_{\mathrm{acl}_t(B,k)}$ and $k_B = k_{\mathrm{acl}_t(B,\Gamma)}$. In particular, we have $B(k)\cap\Gamma = \Gamma_B$ and $B(\Gamma)\cap k = k_B$.*

Proof. As B is (k,Γ)-independent by lemma (3.5), we obtain that $B(k) \overset{ld}{\underset{k(\Gamma_B)}{\downarrow}} k(\Gamma)$ and $B(\Gamma) \overset{ld}{\underset{k_B(\Gamma)}{\downarrow}} k(\Gamma)$. Therefore by lemma (3.6), we see that $\Gamma_B = \Gamma_{\mathrm{acl}_t(B,k)}$ and $k_B = k_{\mathrm{acl}_t(B,\Gamma)}$. In particular, we obtain that $B(k) \cap \Gamma = \Gamma_B$ and $B(\Gamma) \cap k = k_B$. □

3.2 Characterization of Forking

In this section, we characterize forking in the triple. First, we need several lemmas. The following lemma states when two algebraically closed structures in the sense of the triple have the same type over a common substructure.

Lemma 3.9. *Let B_1,B_2 and $C \subseteq B_1 \cap B_2$ be three algebraically closed sets in the sense of the triple. Then $\mathrm{tp}^t(B_1/C) = \mathrm{tp}^t(B_2/C)$ if and only if there is a field automorphism over C sending B_1 to B_2 with (k_{B_1}, Γ_{B_1}) to (k_{B_2}, Γ_{B_2}), and $\mathrm{tp}_k(k_{B_1}/k_C) = \mathrm{tp}_k(k_{B_2}/k_C)$ and $\mathrm{tp}_\Gamma(\Gamma_{B_1}/\Gamma_C) = \mathrm{tp}_\Gamma(\Gamma_{B_2}/\Gamma_C)$.*

Proof. $\Leftarrow$: By $\mathrm{tp}_k(k_{B_1}/k_C) = \mathrm{tp}_k(k_{B_2}/k_C)$ and $\mathrm{tp}_\Gamma(\Gamma_{B_1}/\Gamma_C) = \mathrm{tp}_\Gamma(\Gamma_{B_2}/\Gamma_C)$, there is an automorphism $f \in Aut(k/k_C)$ sending k_{B_1} to k_{B_2} and there is an automorphism $g \in Aut(\Gamma/\Gamma_C)$ sending Γ_{B_1} to Γ_{B_2}. As k_C contains k_0 and Γ_C contains Γ_0, by lemma (3.2) there is an automorphism h of $k(\Gamma)$ over $k_C(\Gamma_C)$ sending $k_{B_1}(\Gamma_{B_1})$ to $k_{B_2}(\Gamma_{B_2})$. Since C is (k,Γ)-independent by lemma (3.5), the map h further extends to an automorphism of $C(k,\Gamma)$ over C. Moreover since each B_i is (k,Γ)-independent by lemma (3.5), we deduce that

$$B_i \overset{ld}{\underset{C(k_{B_i},\Gamma_{B_i})}{\downarrow}} C(k,\Gamma).$$

As we also have $\mathrm{tp}((B_1, k_{B_1}, \Gamma_{B_1})/C) = \mathrm{tp}((B_2, k_{B_2}, \Gamma_{B_2})/C)$ and the linear disjointness above, the map h further extends to an isomorphism between $B_1(k,\Gamma)$ and $B_2(k,\Gamma)$ over C which extends to Ω. Hence $\mathrm{tp}^t(B_1/C) = \mathrm{tp}^t(B_2/C)$. The other direction is clear. □

Now we prove another lemma before characterizing the independence in $(\Omega, k.\Gamma)$.

Lemma 3.10. *Let $C \subseteq A \cap B$ be three algebraically closed structures in the sense of the triple and suppose that $A \underset{C,k,\Gamma}{\downarrow} B, k, \Gamma$. Then $\mathrm{acl}_t(A,B) = \mathrm{acl}(A,B)$. Moreover, we have $k_{\mathrm{acl}_t(A,B)} = \mathrm{acl}(k_A, k_B)$ and $\Gamma_{\mathrm{acl}_t(A,B)} = \mathrm{acl}_\Gamma(\Gamma_A, \Gamma_B)$.*

Proof. Since A is algebraically closed, lemma (3.5) and transitivity yield that

$$A \underset{C,k_A,\Gamma_A}{\downarrow} B,k,\Gamma$$

and so $A \underset{B,k_A,\Gamma_A}{\downarrow} k,\Gamma$. As B is algebraically closed, similarly we have that $B \underset{k_B,\Gamma_B}{\downarrow} k,\Gamma$ and thus

$$B \underset{\mathrm{acl}(k_A,k_B)(\Gamma_A\Gamma_B)}{\downarrow} k,\Gamma.$$

By transitivity, we obtain that

$$A,B \underset{\mathrm{acl}(k_A,k_B)(\Gamma_A\Gamma_B)}{\downarrow} k,\Gamma.$$

Note that by lemma (2.7), the group $\Gamma_A\Gamma_B$ is algebraically closed in Γ. By corollary (3.7), we see that $\mathrm{acl}_t(k_A,k_B,\Gamma_A,\Gamma_B) = \mathrm{acl}(k_A,k_B,\Gamma_A,\Gamma_B)$. Since $\mathrm{acl}(k_A,k_B,\Gamma_A,\Gamma_B)$ is also (k,Γ)-independent by lemma (3.5), by transitivity and in terms of linear disjointness we deduce that

$$\mathrm{acl}(A,B) \overset{ld}{\underset{\mathrm{acl}(k_A,k_B)(\Gamma_A\Gamma_B)}{\downarrow}} k(\Gamma).$$

Hence by lemma (3.6), we deduce that $\mathrm{acl}_t(A,B) = \mathrm{acl}(A,B)$ and also that $k_{\mathrm{acl}_t(A,B)} = \mathrm{acl}(k_A,k_B)$ and $\Gamma_{\mathrm{acl}_t(A,B)} = \mathrm{acl}_\Gamma(\Gamma_A,\Gamma_B) = \Gamma_A\Gamma_B$. □

Now we are ready to give the characterization of forking in the triple by applying lemmas (3.6), (3.9) and (3.10). We follow a similar method as in (2.12). It turns our that independence in the triple is given by the algebraic independence in Ω and k; see (*iii*) below.

Theorem 3.11. *(Characterization of Forking)*

Let $C = A \cap B$ and all be algebraically closed in the sense of the triple T_t. Then the following are equivalent:

(i) $A \overset{t}{\underset{C}{\downarrow}} B$,

(ii) $A \underset{C,k,\Gamma}{\downarrow} B,k,\Gamma$ *and* $A \underset{C}{\downarrow} B$

(iii) $A \underset{C,k,\Gamma}{\downarrow} B,k,\Gamma$ *and* $k_A \underset{k_C}{\downarrow} k_B$.

Proof. First suppose that $A \overset{t}{\underset{C}{\downarrow}} B$. By lemma (2.2), we have that $A \underset{C}{\downarrow} B$. In particular, we obtain $k_A \underset{C}{\downarrow} k_B$. Moreover since C is algebraically closed in the sense of (Ω,k), we also have that $C \underset{k_C}{\downarrow} k$ and so $C \underset{k_C}{\downarrow} k_B$. This two independence give us that $k_A \underset{k_C}{\downarrow} k_B$.

Now we prove that $A \underset{C,k,\Gamma}{\downarrow} B, k, \Gamma$. Suppose for a contradiction that

$$A \underset{C,k,\Gamma}{\not\downarrow} B, k, \Gamma.$$

Let $q = \mathrm{tp}(B/C \cup k_B \cup \Gamma_B)$ and $\lambda \geq \omega_1$. By saturation, there exists $(B_i)_{i\leq\lambda}$ with $B = B_0$ such that $B_i \models q$ and $(B_i)_{i\leq\lambda}$ is independent over $C \cup k \cup \Gamma$ in the field sense, and in particular $B_i \underset{C,k_B,\Gamma_B}{\downarrow} C, k, \Gamma$. So $k_B \subseteq k_{B_i}$ and $\Gamma_B \subseteq \Gamma_{B_i}$ for all i. On the other hand, by the independence $B_i \underset{C,k_B,\Gamma_B}{\downarrow} C, k, \Gamma$ we have that $k_{B_i}, \Gamma_{B_i} \subseteq \mathrm{acl}(C, k_B, \Gamma_B) \subseteq B$. Thus we obtain the equalities $k_B = k_{B_i}$ and $\Gamma_B = \Gamma_{B_i}$ for all i. As C is (k, Γ)-independent by lemma (3.5), we see that

$$C(k_B, \Gamma_B) \overset{ld}{\underset{k_B(\Gamma_B)}{\downarrow}} k(\Gamma).$$

Thus by lemma (3.6), we deduce that $\mathrm{acl}_t(C, k_B, \Gamma_B) = \mathrm{acl}(C, k_B, \Gamma_B)$, and also that $k_B = k_{\mathrm{acl}(C,k_B,\Gamma_B)}$ and $\Gamma_B = \Gamma_{\mathrm{acl}(C,k_B,\Gamma_B)}$. So we see that

$$\mathrm{acl}(C, k_B, \Gamma_B) \overset{ld}{\underset{k_B(\Gamma_B)}{\downarrow}} k(\Gamma).$$

As $B_i \underset{C,k_B,\Gamma_B}{\downarrow} k, \Gamma$, by transitivity and in terms of linear disjointness, we obtain that

$$B_i \overset{ld}{\underset{k_B(\Gamma_B)}{\downarrow}} k(\Gamma).$$

Therefore by lemma (3.6) again, we deduce that B_i is algebraically closed in the sense of the triple for all i. Then, lemma (3.9) yields that $\mathrm{tp}^t(B_i/C) = \mathrm{tp}^t(B/C)$. By Erdős-Rado theorem, we may assume that $(B_i)_{i\leq\lambda}$ is C-indiscernible in the sense of T_t. Let $p_i = \mathrm{tp}^t(A/B_i)$. Since $A \overset{t}{\underset{C}{\downarrow}} B$, we know that $\bigcup_{i\leq\lambda} p_i(x, B_i)$ is consistent. So there exists A_1 such that $\mathrm{tp}^t(A_1 B_i) = \mathrm{tp}^t(AB)$ for all i. Now $(B_i)_{i\leq\lambda}$ is independent over $C \cup k \cup \Gamma$ and $A_1 \underset{C,k,\Gamma}{\not\downarrow} B_i$ for each B_i. This contradicts the stability of the field Ω. Hence we proved that (i) implies (ii) and (iii).

Now we prove that (ii) and (iii) are equivalent. We already proved that (ii) implies (iii) in the beginning. Suppose that we have (iii). Since A is algebraically closed in the sense of the triple, it is k-independent. In particular, $C(k_A)$ is k-independent and hence we get that $C(k_A) \underset{k_A}{\downarrow} k_B$. As we also have $k_A \underset{k_C}{\downarrow} k_B$, by transitivity we obtain that $C(k_A) \underset{k_C}{\downarrow} k_B$ and hence $k_A \underset{C}{\downarrow} k_B$. As A and B are k-independent, by transitivity we conclude the independence $A \underset{C}{\downarrow} B$.

Lastly, we prove that (ii) implies (i). Let $(B_i)_i$ be a Morley sequence over C in the sense of the triple where $B_0 = B$. Note that $(B_i, k_{B_i}, \Gamma_{B_i})_i$ is also a Morley sequence over C in the sense of the triple but for simplicity we write $(B_i)_i$ instead. By (ii) we also have that $k_A, \Gamma_A \underset{C}{\downarrow} B$. By stationarity over algebraically closed sets in ACF and as Ω is ω-stable, we may assume that $(B_i)_i$ is a Morley sequence over $C \cup k_A \cup \Gamma_A$ in the field sense. Since $A \underset{C}{\downarrow} B$, we also have $A \underset{C,k_A,\Gamma_A}{\downarrow} B, k_A, \Gamma_A$. Let $p(x) = \mathrm{tp}(A/B \cup k_A \cup \Gamma_A)$ and $p_i(x)$ be the copy over B_i. Then by $A \underset{C,k_A,\Gamma_A}{\downarrow} B, k, \Gamma$ and saturation also, there exists an element $d \models \bigcup_i p_i(x)$ such that

$$d \underset{C,k_A,\Gamma_A}{\downarrow} B_i, k, \Gamma$$

for all i. Observe that $k_d = k_A$ and $\Gamma_d = \Gamma_A$, and also $\mathrm{tp}(dB_i k_A \Gamma_A) = \mathrm{tp}(ABk_A\Gamma_A)$ for all i. Moreover since A is (k, Γ)-independent by lemma (3.5), we have that

$$C(k_A, \Gamma_A) \overset{ld}{\underset{k_A(\Gamma_A)}{\downarrow}} k(\Gamma).$$

So by lemma (3.6), we obtain that $\mathrm{acl}_t(C, k_A, \Gamma_A) = \mathrm{acl}(C, k_A, \Gamma_A)$. In particular $\mathrm{acl}(C, k_A, \Gamma_A)$ is (k, Γ)-independent by lemma (3.5). Since also we have $d \underset{C,k_A,\Gamma_A}{\downarrow} k, \Gamma$, by transitivity and in terms of linear disjointness, we deduce that

$$d \overset{ld}{\underset{k_A(\Gamma_A)}{\downarrow}} k(\Gamma).$$

Therefore by lemma (3.6) again, we conclude that d is algebraically closed in the sense of the triple. By lemma (3.10), we see that $\mathrm{acl}_t(A, B) = \mathrm{acl}(A, B)$, and also $k_{\mathrm{acl}_t(A,B)} = \mathrm{acl}(k_A, k_B)$ and $\Gamma_{\mathrm{acl}_t(A,B)} = \mathrm{acl}_\Gamma(\Gamma_A, \Gamma_B)$. Moreover, we have

$$\mathrm{acl}(A, B) \overset{ld}{\underset{\mathrm{acl}(k_A,k_B)(\Gamma_A\Gamma_B)}{\downarrow}} k(\Gamma).$$

By the choice of d, we also have that

$$\mathrm{acl}(d, B_i) \overset{ld}{\underset{\mathrm{acl}(k_A,k_{B_i})(\Gamma_A\Gamma_{B_i})}{\downarrow}} k(\Gamma).$$

Using $\mathrm{tp}(dB_i k_A \Gamma_A) = \mathrm{tp}(ABk_A\Gamma_A)$ and the previous two linearly disjointness, we conclude that $\mathrm{tp}^t(dB_i k_A \Gamma_A) = \mathrm{tp}^t(ABk_A\Gamma_A)$ for all i. Hence we have (i). □

Corollary 3.12. *For every $a \in \Omega$, we have that*

$$a \overset{t}{\underset{k_{\mathrm{acl}_t(a)},\Gamma_{\mathrm{acl}_t(a)}}{\downarrow}} k, \Gamma.$$

Moreover we also have the independence $a \overset{t}{\underset{\Gamma_{\mathrm{acl}_t(a)}}{\downarrow}} \Gamma$.

Proof. Since $\mathrm{acl}_t(a)$ is (k,Γ)-independent by lemma (3.5), we have that

$$\mathrm{acl}_t(a) \underset{\mathrm{acl}(k_{\mathrm{acl}_t(a)},\Gamma_{\mathrm{acl}_t(a)})}{\downarrow} \mathrm{acl}(k,\Gamma)$$

and $\mathrm{acl}_t(a) \cap \mathrm{acl}(k,\Gamma) = \mathrm{acl}(k_{\mathrm{acl}_t(a)}, \Gamma_{\mathrm{acl}_t(a)})$. By corollary (3.7), we see that

$$\mathrm{acl}(k_{\mathrm{acl}_t(a)}, \Gamma_{\mathrm{acl}_t(a)}) = \mathrm{acl}_t(k_{\mathrm{acl}_t(a)}, \Gamma_{\mathrm{acl}_t(a)})$$

and also that $\mathrm{acl}_t(k,\Gamma) = \mathrm{acl}(k,\Gamma)$. Therefore, we deduce that

$$\mathrm{acl}_t(a) \underset{\mathrm{acl}_t(k_{\mathrm{acl}_t(a)},\Gamma_{\mathrm{acl}_t(a)})}{\downarrow} \mathrm{acl}_t(k,\Gamma).$$

Applying Theorem (3.11), we have the first part. For the second part, we have $\mathrm{acl}_t(\Gamma) = \mathrm{acl}(\Gamma, k_0)$ and $\mathrm{acl}_t(\Gamma_{\mathrm{acl}_t(a)}) = \mathrm{acl}(\Gamma_{\mathrm{acl}_t(a)}, k_0)$, and also that $k_{\mathrm{acl}_t(\Gamma)} = k_0$ by corollary (3.7). As $\mathrm{acl}_t(a)$ is Γ-independent by remark (**??**), we conclude by Theorem (3.11) similar to the first part. □

The next lemma states that the independence in the triple implies the independence in Γ as a pure group.

Lemma 3.13. *Let $C = A \cap B$ and all be algebraically closed in the sense of the triple and $A \overset{t}{\underset{C}{\downarrow}} B$. Then we have the independence $\Gamma_A \overset{\Gamma}{\underset{\Gamma_C}{\downarrow}} \Gamma_B$ in the abelian group Γ.*

Proof. As $A \overset{t}{\underset{C}{\downarrow}} B$, we have $\Gamma_A \overset{t}{\underset{C}{\downarrow}} \Gamma_B$. Corollary (3.12) and the transitivity of the independence yield that $\Gamma_A \overset{t}{\underset{\Gamma_C}{\downarrow}} \Gamma_B$. Hence we conclude that $\Gamma_A \overset{\Gamma}{\underset{\Gamma_C}{\downarrow}} \Gamma_B$. □

3.2.1 Independence over Models and Stationarity

In this subsection we study the independence over models. Then we investigate the relation between the independence in the triple and the independence in (Ω, k) and (Ω, Γ).

Proposition 3.14. *Let $M = A \cap B$ where A, B are algebraically closed in the sense of the triple T_t and M is a model of T_t. Then $A \overset{t}{\underset{M}{\downarrow}} B$ if and only if $A \overset{ld}{\underset{M(k_A,\Gamma_A)}{\downarrow}} B(k,\Gamma)$*

and $k_A \underset{k_M}{\overset{ld}{\downarrow}} k_B$.

Proof. By Theorem (3.11), it is enough to prove that $A \underset{M}{\overset{t}{\downarrow}} B$ implies $A \underset{M(k_A,\Gamma_A)}{\overset{ld}{\downarrow}} B(k,\Gamma)$. Now let $a_1,...,a_n$ be in A (not necessarily distinct), the elements $b_1,...,b_n$ be in B, the elements $k_1,...,k_n$ be in k and $g_1,...,g_n$ be in Γ such that

$$a_1b_1k_1g_1 + \cdots + a_nb_nk_ng_n = 0.$$

Let $f(\bar{x},\bar{y},\bar{z},\bar{t})$ be the formula $x_1y_1z_1t_1 + \cdots + x_ny_nz_nt_n = 0$. Let $\phi(\bar{x},\bar{y})$ be the formula

$$\exists \bar{z} \in P_1 \exists \bar{t} \in P_2 f(\bar{x},\bar{y},\bar{z},\bar{t}).$$

Then $\bar{a} \models \phi(\bar{x},\bar{b})$. As $A \underset{M}{\overset{t}{\downarrow}} B$, by stability the type $\mathrm{tp}^t(A/B)$ is an heir extension of $\mathrm{tp}^t(A/M)$. So there is $\bar{m} \in M$ such that $\bar{a} \models \phi(\bar{x},\bar{m})$. We finish the proof since A is (k,Γ)-independent by lemma (3.5).

□

Proposition 3.15. *Let $C = A \cap B$ and all be algebraically closed in the sense of the triple T_t. If $A \underset{C}{\overset{t}{\downarrow}} B$ then we have the independences $A \underset{C}{\overset{P_1}{\downarrow}} B$ and $A \underset{C}{\overset{P_2}{\downarrow}} B$ in the sense of the pairs (Ω,k) and (Ω,Γ) respectively.*

Proof. Suppose that $A \underset{C}{\overset{t}{\downarrow}} B$. As A is (k,Γ)-independent by lemma (3.5), by transitivity we have $A \underset{C,k_A,\Gamma_A}{\downarrow} B,k,\Gamma$ and $A \underset{C}{\downarrow} B$. Thus by remark (**??**), it is enough to show that $A \underset{C,k}{\downarrow} B,k$ and $A \underset{C,\Gamma}{\downarrow} B,\Gamma$. Note that by corollary (3.8), we have that $\Gamma_{\mathrm{acl}_t(B,k)} = \Gamma_B$ and $k_{\mathrm{acl}_t(B,\Gamma)} = k_B$. In order to show $A \underset{C,k}{\downarrow} B,k$, by transitivity it is enough to show that $B(k) \underset{C(k)}{\overset{ld}{\downarrow}} C(k,\Gamma_A)$. So let $b_1,...,b_n$ be in B, the elements $k_1,...,k_n$ be in k and $g_1,...,g_n$ be in Γ_A such that

$$b_1k_1g_1 + \cdots + b_nk_ng_n = 0.$$

We may suppose that no proper subsum of this equation is zero. Since Γ has the Mann property over Ω, we obtain that $\frac{g_i}{g_1} \in \mathrm{acl}_t(B,k)$ and so $\frac{g_i}{g_1} \in \Gamma_B$ for all i. As $\frac{g_i}{g_1} \in \Gamma_A$ also, we obtain that $\frac{g_i}{g_1} \in \Gamma_C$ for all i. Thus we have what we desired. Similarly, to prove $A \underset{C,\Gamma}{\downarrow} B,\Gamma$, we need to show that $B(\Gamma) \underset{C(\Gamma)}{\downarrow} k_A$. Since $k_{\mathrm{acl}_t(B,\Gamma)} = k_B$ and $\mathrm{acl}_t(B,\Gamma)$ is k-independent, in particular we obtain that $B(\Gamma) \underset{k_B}{\downarrow} k_A$. As we also have $k_B \underset{k_C}{\downarrow} k_A$ by $A \underset{C}{\overset{t}{\downarrow}} B$, we conclude by transitivity. Hence we have the proposition. □

Now we give more equivalences for the characterization of forking in the triple:

Corollary 3.16. *Let $C = A \cap B$ and all be algebraically closed in the sense of the triple T_t. Then the following are equivalent:*

(i) $A \overset{t}{\underset{C}{\downarrow}} B$,

(ii) $A \underset{C,k,\Gamma}{\downarrow} B, k, \Gamma$ *and* $A \underset{C}{\downarrow} B$

(iii) $A \underset{C,k,\Gamma}{\downarrow} B, k, \Gamma$ *and* $k_A \underset{k_C}{\downarrow} k_B$,

(iv) $A \underset{C,k,\Gamma}{\downarrow} B, k, \Gamma$ *and* $A \underset{C,\Gamma}{\downarrow} B, \Gamma$.

Proof. By Theorem (3.11) and proposition (3.15) we know that (i), (ii) and (iii) are equivalent and (i) implies (iv). Now we show that (iv) implies (ii). If we have $A \underset{C,\Gamma}{\downarrow} B, \Gamma$, then lemma (2.10) yields the desired independence $A \underset{C}{\downarrow} B$.

□

Remark 3.17. *Note that in corollary* (3.16)*(iv), we cannot replace* $A \underset{C,\Gamma}{\downarrow} B, \Gamma$ *by* $A \underset{C,k}{\downarrow} B, k$ *since the latter independence does not imply the independence* $A \underset{C}{\downarrow} B$.

Next we prove that the types over algebraically closed sets are stationary under Γ has WEI, even though we do not have WEI in the triple.

Proposition 3.18. *Suppose that Γ has WEI. Let C be algebraically closed in the sense of the triple T_t and b is a tuple (possibly infinite) from Ω. Then the type $\mathrm{tp}^t(b/C)$ is stationary.*

Proof. Suppose that Γ has WEI. Let $B = \mathrm{acl}_t(B)$ be a set containing C. Let b_1 and b_2 be such that $b_i \overset{t}{\underset{C}{\downarrow}} B$ for $i = 1, 2$ and $\mathrm{tp}^t(b_1/C) = \mathrm{tp}^t(b_2/C)$. Put $d_i = \mathrm{acl}_t(b_i, C)$ for $i = 1, 2$. By the characterization of the independence (3.11) and since k_C is an algebraically closed field, we see that $d_i \overset{ld}{\underset{C(k_{d_i},\Gamma_{d_i})^{ac}}{\downarrow}} B(k,\Gamma)^{ac}$ and $k_{d_i} \overset{ld}{\underset{k_C}{\downarrow}} k_B$ for $i = 1, 2$. Note that we also have that $\mathrm{tp}^t(d_1/C) = \mathrm{tp}^t(d_2/C)$. Now we will prove that $\mathrm{tp}^t(b_1/B) = \mathrm{tp}^t(b_2/B)$. By lemma (3.13), we obtain that $\Gamma_{d_i} \overset{\Gamma}{\underset{\Gamma_C}{\downarrow}} \Gamma_B$ for $i = 1, 2$. By WEI and since Γ_C is algebraically closed in Γ, the type $\mathrm{tp}_\Gamma(\Gamma_{d_1}/\Gamma_C)$ is stationary by lemma (2.18). So by lemma (2.17), there is an automorphism $g \in Aut(\Gamma/\Gamma_B)$ sending Γ_{d_1} to Γ_{d_2}. Similarly with the help of $k_{d_i} \overset{ld}{\underset{k_C}{\downarrow}} k_B$ and stationarity over k_C, there is an automorphism $f \in Aut(k/k_B)$ sending k_A to k_{A_1}. As k_B contains k_0 and Γ_B contains Γ_0, by lemma (3.2) there is an automorphism h of $k(\Gamma)$ over $k_B(\Gamma_B)$ sending $k_{d_1}(\Gamma_{d_1})$ to $k_{d_2}(\Gamma_{d_2})$. Moreover since B is (k,Γ)-independent, the map h further extends to an automorphism of $B(k,\Gamma)$ over B and this also extends to $B(k,\Gamma)^{ac}$.

Now since $b_i \overset{ld}{\underset{C(k_{b_i},\Gamma_{b_i})^{ac}}{\downarrow}} B(k,\Gamma)^{ac}$ and $\mathrm{tp}^t(b_1/C) = \mathrm{tp}^t(b_2/C)$, we have an isomorphism from $b_1B(k,\Gamma)^{ac}$ to $b_2B(k,\Gamma)^{ac}$ sending b_1 to b_2 over B. Since this also extends to an automorphism of Ω, we conclude that $\mathrm{tp}^t(b_1/B) = \mathrm{tp}^t(b_2/B)$. □

3.3 Definable Groups

In this section, we characterize definable groups in the triple (Ω, k, Γ) up to isogeny. It emerges that definable groups in the triple are given by definable and interpretable groups in Ω, k and Γ. We follow a similar method to the previous chapter.

3.3.1 Generics and Examples

The following lemma is analogous to the lemma (2.23). It states when a type-definable group in the triple is an algebraic group.

Lemma 3.19. *(The Group Lemma for the triple) Let H be a connected T_t-type-definable subgroup of an algebraic group V, all definable over an algebraically closed set A in the sense of the triple. Let a be the generic over A which lies in some translate of H which is also definable over A. If $k_{\mathrm{acl}_t(a,A)} = k_A$ and $\Gamma_{\mathrm{acl}_t(a,A)} = \Gamma_A$, then H is an algebraic group. In particular H is definable.*

Proof. First we may assume that $a \in H$: Suppose that $a \in bH$. Let a' be such that $\mathrm{tp}^t(a'/A) = \mathrm{tp}^t(a/A)$ and $a' \overset{t}{\underset{A}{\downarrow}} a$. Then we have $a^{-1}a' \overset{t}{\underset{A}{\downarrow}} a$ and $a^{-1}a' \in H$ is generic. Since $a' \overset{t}{\underset{A}{\downarrow}} a$, we have that $a', A \overset{t}{\underset{A}{\downarrow}} a, A$. So by lemma (3.10) and Theorem (3.11), we see that

$$k_{\mathrm{acl}_t(a^{-1}a',A)} \subseteq k_{\mathrm{acl}_t(a',a,A)} \subseteq \mathrm{acl}(k_{\mathrm{acl}_t(a',A)}, k_{\mathrm{acl}_t(a,A)}).$$

Since $k_{\mathrm{acl}_t(a',A)} = k_{\mathrm{acl}_t(a,A)} = k_A$, we deduce that $k_{\mathrm{acl}_t(a^{-1}a',A)} = k_{\mathrm{acl}_t(A)}$. Similarly we obtain that $\Gamma_{\mathrm{acl}_t(a^{-1}a',A)} = \Gamma_A$. Thus we may assume that $a \in H$. Put $p = \mathrm{tp}^t(a/A)$ and p_0 its T-reduct. Let H_0 be the smallest algebraic group containing H which exists by the assumption and the ω-stability of Ω. Note that $H = stab_t(p) \subset stab_T(p_0)$. So $H_0 \subseteq stab_T(p_0)$. On the other hand, since $p_0(x)$ implies that $x \in H_0$, we get that $stab_T(p_0) \subseteq H_0$. Thus we have the equality and moreover H_0 is T-connected. To prove the lemma it is enough to show that p is the unique generic of H_0. Let h be a generic of H_0 over A in the sense of the triple and put $q = \mathrm{tp}^t(h/A)$.
Claim: We have $h \underset{A}{\downarrow} k, \Gamma$. First note that $a \in H_0$ and as the algebraic closure is (k,Γ)-independent and by the assumptions $k_{\mathrm{acl}_t(a,A)} = k_A$ and $\Gamma_{\mathrm{acl}_t(a,A)} = \Gamma_A$, we have that $a, A \underset{k_A,\Gamma_A}{\downarrow} k, \Gamma$ and so $a \underset{A}{\downarrow} k, \Gamma$. As a consequence, we see that a is a generic over $A \cup k \cup \Gamma$. Now if $h \not\underset{A}{\downarrow} k, \Gamma$, then there exists a formula $\varphi(x, m, \gamma) \in \mathrm{tp}(h/A, k, \Gamma)$

which is not generic in H_0, with parameters from A, where $m \in k$ and $\gamma \in \Gamma$. Put $n = n(\varphi)$ as in lemma (2.22) and

$$\theta(y,z) = \exists h_1 ... \exists h_n \in H_0 (\forall x \in H_0 \bigvee_{i \leq n} h_i \varphi(x,y,z))$$

and $\phi(x,y,z) = \neg\theta(y,z) \wedge \varphi(x,y,z)$. Observe that for all tuples (b,c), the formula $\phi(x,b,c)$ is not generic in H_0. However the formula

$$\psi(x) = \exists y \in P_1 \ \exists z \in P_2 \ \phi(x,y,z)$$

whose parameters from A is realized by h and so it is generic in H_0. Therefore finite number of translates of $\psi(x)$ cover H_0, say $H_0 = \bigcup_{i \leq k} \alpha_i \psi(x)$. Take a' such that $\mathrm{tp}^t(a'/A) = \mathrm{tp}^t(a/A)$ and $a \overset{t}{\underset{A}{\downarrow}} \alpha_1, ... \alpha_k$. Thus for certain $\alpha \in H_0$ we may suppose that $a \in \alpha\psi(x)$ and $a \overset{t}{\underset{A}{\downarrow}} \alpha$. So $a \in \alpha\phi(x,m',g')$ for some $m' \in k$ and $\gamma' \in \Gamma$. By the characterization of the independence (3.11), we have that $a \underset{A,k,\Gamma}{\downarrow} \alpha$ and by transitivity we get $a \underset{A}{\downarrow} \alpha, k, \Gamma$. This is a contradiction since the formula $\alpha\phi(x,m',g')$ is not generic in H_0. So we have the claim.

Now since A is (k,Γ)-independent by lemma (3.5), by transitivity of the independence, we see that $h, A \underset{k_A,\Gamma_A}{\downarrow} k, \Gamma$. As $\mathrm{acl}_t(k_A,\Gamma_A) = \mathrm{acl}(k_A,\Gamma_A)$ by corollary (3.7), it is (k,Γ) independent. Therefore, by transitivity and in terms of linear disjointness, we obtain that

$$\mathrm{acl}(h,A) \overset{ld}{\underset{k_A(\Gamma_A)}{\downarrow}} k(\Gamma).$$

Thus by lemma (3.6), we conclude that $\mathrm{acl}_t(h,A) = \mathrm{acl}(h,A)$, and also that $k_{\mathrm{acl}_t(h,A)} = k_A$ and $\Gamma_{\mathrm{acl}_t(h,A)} = \Gamma_A$. Since there is a field automorphism over A sending a to h, linear disjointness yields a field automorphism over $A \cup k \cup \Gamma$ sending a to h. Hence we obtain that $q = p$ and we conclude that $H = H_0$.

□

Remark 3.20. *Observe that none of the groups $k^\times$ and Γ satisfy the conclusion of lemma* (3.19) *as they are not algebraic groups in* Ω.

Lemma 3.21. *Let v and h be generics over a small model M of T_t of an algebraic group V defined over k and a type-definable group H in Γ respectively. Then we have the independence $v \overset{t}{\underset{M}{\downarrow}} h$. If V and H are connected, then so is $V \times H$.*

Proof. Suppose that there is a formula $\varphi(x,h)$ over M such that $\models \varphi(v,h)$ and the formula $\varphi(x,h)$ forks. Let $\phi(x,y) = \varphi(x,y) \wedge (x \in P_1) \wedge (y \in P_2)$. Then by Theorem (3.1), we know that $\phi(x,y) = \bigcup_{i,j} \phi_i(x) \wedge \phi_j(y)$, where $\phi_i(x)$ is a formula defined in

k and $\phi_j(y)$ is a formula defined in Γ. Thus for some i or j, we obtain that $\phi_i(x)$ or $\phi_j(y)$ fork. However, a generic does not fork over the empty set, a contradiction. If V and H are connected, lemma (2.17) yields that the tuple (v, h) is the unique generic of $V \times H$ over M and hence the product is also connected. □

Before characterizing definable groups in the triple, we give some examples.

Example 3.22. *(Some definable groups in the triple) Algebraic groups over Ω, algebraic groups over k, the group Γ and its powers, the product $\Omega \times k \times \Gamma$ and*

$$SL(2, k, \Gamma, \Omega) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \in k^\times \right\} \times \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc \in \Gamma \right\}$$

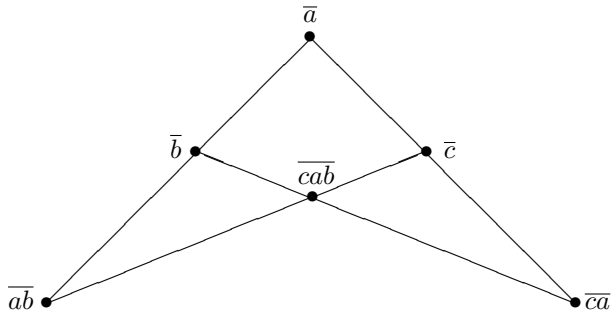
are all definable in the triple. One can see that each of them satisfy the hypothesis of the following Theorem (3.23).

3.3.2 Characterization of Definable Groups

Now we are ready to characterize definable groups in the triple (Ω, k, Γ) in terms of definable and interpretable groups in each sort. We use the group configuration theorem (1.18) together with lemma (2.27), Theorem (3.11), lemma (3.13), lemma (3.19) and lemma (3.21).

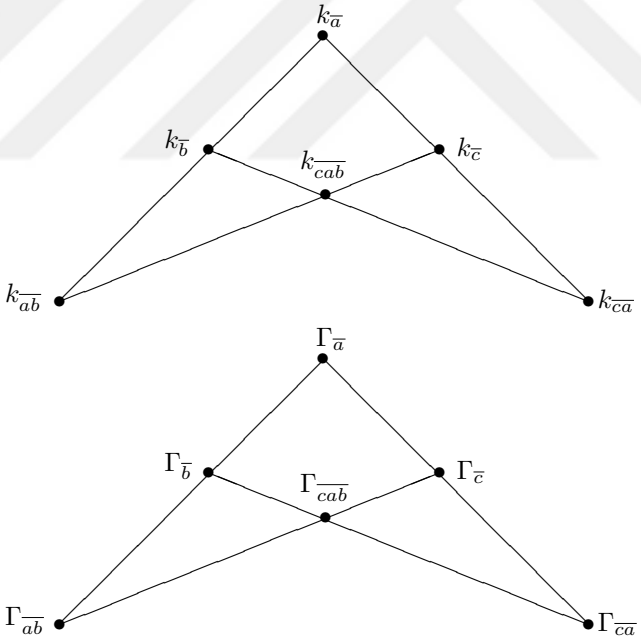
Theorem 3.23. *(Definable Groups for the triple) Let Ω be an algebraically closed field, the field k be a proper subfield of Ω which is also algebraically closed and Γ be a multiplicative subgroup of $\Omega^\times$ such that (k, Γ) is a Mann pair. Any type-definable group in (Ω, k, Γ) is isogenous to a subgroup of an algebraic group. Moreover any type-definable group is, up to isogeny, an extension of a direct sum of k-rational points of an algebraic group defined over k and a type-interpretable abelian group in Γ by an algebraic group.*

Proof. Let H be a type-definable group in (Ω, k, Γ) over some parameters. By remark (2.26), we may suppose that H is connected. We will work over a model containing the parameters defining H which we will omit. Given two independent generics a and b of H, we write $\overline{a}$, $\overline{b}$ and $\overline{ab}$ instead of their algebraic closures in the sense of the triple respectively. Observe that by lemma (3.10) and the characterization of the independence (3.11), the tuple $\overline{ab}$ is T-algebraic over $\overline{a} \cup \overline{b}$ since $\overline{a}$, $\overline{b}$ are two independent algebraically closed subsets. With the help of the third generic c which is independent from a and b, we obtain the following diagram:



Then by lemma (2.2), we have a T-group configuration. So by the group configuration theorem (1.18) and lemma (2.27), there exists a $*$-interpretable group in the pure field Ω whose generic is interalgebraic with the generic of H. So we conclude that there is an algebraic group in Ω which H embeds in up to isogeny. Thus up to isogeny, we may assume that H is a subgroup of an algebraic group.

By lemma (3.10) and Theorem (3.11), the set $k_{\overline{ab}}$ is k-algebraic over $k_{\overline{a}} \cup k_{\overline{b}}$. Similarly, the set $\Gamma_{\overline{ab}}$ is Γ-algebraic over $\Gamma_{\overline{a}} \cup \Gamma_{\overline{b}}$. Applying the characterization of independence (3.11) and lemma (3.13), we have the following diagrams:



So by the group configuration theorem (1.18), we obtain a connected $*$-interpretable group V_1 in k and a connected $*$-interpretable group H_1 in Γ. Moreover a generic v of V_1 is k-interalgebraic with $k_{\overline{a}}$. Similar thing holds for Γ. Now lemma (2.27) yields a projection π_1 from H to a V_1 and a projection π_2 from H to H_1. Furthermore the generic v of V_1 is T_t-interalgebraic with $k_{\overline{a}}$ and the generic h of H_1 is T_t-interalgebraic with $\Gamma_{\overline{a}}$. By lemma (3.21), the tuple (v, h) is a generic of $V_1 \times H_1$ which is T_t-interalgebraic with $(k_{\overline{a}}, \Gamma_{\overline{a}})$. Since V_1 and H_1 are connected, lemma (3.21) also yields that the tuple

(v, h) is the only generic of $V_1 \times H_1$ and this product is also connected. Thus by lemma (2.27) again, we have a projection π from H to $V_1 \times H_1$ in $k \times \Gamma$ which is given by the stabilizer of the type $\mathrm{tp}^t(a, v, h)$. Finally we show that the connected component N of the kernel $ker(\pi)$ is an algebraic group by the group lemma (3.19): Let n be a generic of N over a in the sense of the triple. So we have $n \overset{t}{\downarrow} a$ and $na \overset{t}{\downarrow} a$. Observe that $na \in Na$ is a generic also. Since the tuple $(n, 1, 1)$ is in the stabilizer of $\mathrm{tp}^t(a, v, h)$, we have that the tuples (na, v, h) and (a, v, h) have the same t-type. Thus in particular, we have $k_{\overline{na}} = k_{\overline{a}}$ and $\Gamma_{\overline{na}} = \Gamma_{\overline{a}}$. Moreover by lemma (3.10) and Theorem (3.11) we obtain that $k_{\overline{na,a}}$ is in the k-algebraic closure of $k_{\overline{na}}$ and $k_{\overline{a}}$. Therefore we see that $k_{\overline{na,a}} = k_{\overline{a}}$ and similarly $\Gamma_{\overline{na,a}} = \Gamma_{\overline{a}}$. Now the type $\mathrm{tp}^t(na/\overline{a})$ satisfies the hypothesis of the lemma (3.19) and we conclude that N is an algebraic group. So by elimination of imaginaries in k and by (2.28), we can take V_1 to be an algebraic group and H_1 to be type-interpretable. Note also that H_1 is abelian by Theorem (2.29) and remark (2.21).

□

3.4 Imaginaries and Interpretable Groups

Our goal in this section is to characterize interpretable groups in the triple (Ω, k, Γ). Through the section, we assume that Γ is divisible. As in the previous chapter, note that Γ is strongly minimal and every infinite algebraically closed subset of Γ is an elementary substructure. By [10], the triple is ω-stable has infinite Morley rank. In this section, we will give a description of imaginaries in the triple which enables us to characterize interpretable groups in (Ω, k, Γ). Observe that (Ω, k, Γ) does not eliminate imaginaries.

3.4.1 Canonical Base Lemmas

Our description of imaginaries will be by means of canonical bases as in the previous chapter and as in [28]. The next three results are analogues of lemma (2.49), corollary (2.50) and lemma (2.51) from Chapter 2 which are adapted to the triple (Ω, k, Γ).

Lemma 3.24. *Let B be an elementary substructure of (Ω, k, Γ). Suppose that $d = \mathrm{Cb}(\mathrm{tp}(a/\operatorname{acl}(B, k, \Gamma)))$. Then $a \overset{t}{\underset{d}{\downarrow}} B, k, \Gamma$.*

Proof. As Ω eliminates imaginaries, we might assume that d is contained in Ω. First of all, note that $\operatorname{acl}_t(B, k, \Gamma) = \operatorname{acl}(B, k, \Gamma)$ and $a \underset{d}{\downarrow} B, k, \Gamma$ as d is the canonical base. The independence $a \underset{d}{\downarrow} B, k, \Gamma$ yields that

$$\operatorname{acl}(a, d) \underset{d, k_{\operatorname{acl}_t(d)}, \Gamma_{\operatorname{acl}_t(d)}}{\downarrow} k, \Gamma.$$

Note that by corollary (3.7) we see that

$$\mathrm{acl}_t(d) = \mathrm{acl}(d, k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)})$$

and $\mathrm{acl}_t(k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}) = \mathrm{acl}(k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)})$. So in particular, they are (k, Γ)-independent by lemma (3.5). Since

$$d \underset{k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}}{\downarrow} k, \Gamma,$$

by transitivity we obtain that

$$\mathrm{acl}(a, d, k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}) \underset{k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}}{\downarrow} k, \Gamma.$$

In terms of linear disjointness, this gives us that

$$\mathrm{acl}(a, d, k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}) \overset{ld}{\underset{k_{\mathrm{acl}_t(a)}(\Gamma_{\mathrm{acl}_t(d)})}{\downarrow}} k(\Gamma).$$

Thus by corollary (3.7) again, we deduce that

$$\mathrm{acl}_t(a, d, k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}) = acl(a, d, k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}),$$

and also that $k_{\mathrm{acl}_t(a,d)} = k_{\mathrm{acl}_t(d)}$ and $\Gamma_{\mathrm{acl}_t(a,d)} = \Gamma_{\mathrm{acl}_t(d)}$. Now as $a \underset{d}{\downarrow} B, k, \Gamma$, we have that

$$\mathrm{acl}(a, d, k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)}) \underset{\mathrm{acl}(d, k_{\mathrm{acl}_t(d)}, \Gamma_{\mathrm{acl}_t(d)})}{\downarrow} \mathrm{acl}(B, k, \Gamma)$$

and this yields that

$$\mathrm{acl}_t(a, d) \underset{\mathrm{acl}_t(d)}{\downarrow} \mathrm{acl}_t(B, k, \Gamma).$$

We finish the lemma by the characterization of the independence (3.11). □

Corollary 3.25. *Let B be an elementary substructure of (Ω, k, Γ) and let a be a finite tuple from Ω. Put $d = \mathrm{Cb}(\mathrm{tp}(a/\mathrm{acl}(B, k, \Gamma)))$. Then $\mathrm{Cb}(\mathrm{tp}^t(a/B))$ is interalgebraic in (Ω, k, Γ) with $\mathrm{Cb}(\mathrm{tp}^t(d/B))$.*

Proof. Set $p = \mathrm{tp}^t(a/B)$ and $q = \mathrm{tp}^t(d/B)$. Let $e_1 = \mathrm{Cb}(p)$ and $e_2 = \mathrm{Cb}(q)$. Note that e_1 and e_2 are in B^{eq}. By lemma (3.24), we know that $a \overset{t}{\underset{d}{\downarrow}} B, k, \Gamma$. So $a \overset{t}{\underset{d}{\downarrow}} B, d$ and $a \overset{t}{\underset{e_2, d}{\downarrow}} B, d$. As $d \overset{t}{\underset{e_2}{\downarrow}} B$, by transitivity we conclude that $a \overset{t}{\underset{e_2}{\downarrow}} B$. This gives that e_1 is algebraic over e_2.
The converse follows from a similar proof of corollary (2.50). □

We say that a type in the triple (Ω, k, Γ) is almost (k, Γ)-internal if it is almost

$k \cup \Gamma$-internal. The next lemma asserts that, up to interalgebraicity, an imaginary element in the triple is a canonical base of a type over itself and this type is almost (k, Γ)-internal.

Lemma 3.26. *Let $e \in (\Omega, k, \Gamma)^{\mathrm{eq}}$ be an imaginary element. There is $e' \in (\Omega, k, \Gamma)^{\mathrm{eq}}$ interalgebraic with e, such that for some finite tuple d' from Ω we have $e' = \mathrm{Cb}(\mathrm{tp}^t(d'/e'))$ and $\mathrm{tp}^t(d'/e')$ is almost (k, Γ)-internal.*

Proof. Let a be a tuple in Ω such that $e = f(a)$ for some 0-definable function in $(\Omega, k, \Gamma)^{\mathrm{eq}}$. Set $e_1 = \mathrm{Cb}(\mathrm{tp}^t(a/\mathrm{acl}_t{}^{\mathrm{eq}}(e)))$. Observe that e_1 is algebraic over e. As $e = f(a)$ and $a \overset{t}{\underset{e_1}{\downarrow}} e$, we obtain that $e \overset{t}{\underset{e_1}{\downarrow}} e$ and hence e and e_1 are interalgebraic. Now let (B, k_B, Γ_B) be an elementary substructure of (Ω, k, Γ) such that $e_1 \in (B, k_B, \Gamma_B)^{\mathrm{eq}}$ and $a \overset{t}{\underset{e_1}{\downarrow}} B$. Let $d = \mathrm{Cb}(\mathrm{tp}(a/\operatorname{acl}(B, k, \Gamma)))$. We may assume d to be a finite tuple in Ω owing to ω-stability and elimination of imaginaries. Put $e_2 = \mathrm{Cb}(\mathrm{tp}^t(d/B))$. Then by corollary (3.25) and $a \overset{t}{\underset{e_1}{\downarrow}} B$, we see that e_1 and e_2 are interalgebraic. Note that the type $\mathrm{tp}^t(d/B)$ is almost (k, Γ)-internal. Thus since $d \overset{t}{\underset{e_2}{\downarrow}} B$, the type $\mathrm{tp}^t(d/e_2)$ is also almost (k, Γ)-internal. By lemma (2.38), there is an imaginary element $d' \in (\Omega, k, \Gamma)^{\mathrm{eq}}$ such that $d' \in \mathrm{acl}_t(d, e)$ and $d \in \mathrm{acl}_t(d')$, and also the type $\mathrm{tp}^t(d'/e_2)$ is almost (k, Γ)-internal. Let $e' = \mathrm{Cb}(\mathrm{tp}^t(d'/e_2))$. Then $e' \in \mathrm{acl}_t(e_2)$ and $d \overset{t}{\underset{e'}{\downarrow}} e_2$ as $d \in \mathrm{acl}_t(d')$. Hence $e_2 \in \mathrm{acl}_t(e')$. Thus, we conclude that e and e' are interalgebraic. □

Lemma 3.27. *(Coheir) Let $e \in (\Omega, k, \Gamma)^{\mathrm{eq}}$ and $B = \mathrm{acl}_t(e) \cap (k \cup \Gamma)$. Let c be a tuple from $k \cup \Gamma$. Then $\mathrm{tp}^t(c/B, e)$ is finitely satisfiable in B.*

Proof. First observe that the type $p = \mathrm{tp}^t(e/k(\Gamma)^{ac})$ is stationary. Let d be the canonical base of p. Therefore, the element d is in $\mathrm{acl}_t{}^{\mathrm{eq}}(k, \Gamma)$. Note that any automorphism in the triple fixes $k \cup \Gamma$ setwise, and so fixes $k(\Gamma)^{ac}$ setwise as well. Thus d is contained in $\mathrm{acl}_t(e)$. As a consequence, the element d is contained in $\mathrm{acl}_t{}^{\mathrm{eq}}(B)$. As a result, the type p is the non-forking extension of the type $\mathrm{tp}^t(e/B)$ and hence p is definable over B. Therefore for a given formula $\phi(x, y)$ of $L_t{}^{\mathrm{eq}}$ over B, there is a formula $\psi(y)$ over B such that $\phi(x, \beta) \in p$ if and only if $\psi(\beta)$ holds. By Theorem (3.1) we know that the induced structure on (k, Γ) is itself and by stability, there exists a formula $f(y)$ in the pure field k and a formula $g(y)$ in Γ in the language of pure groups such that, for all $\beta \in k \cup \Gamma$ we have $\models \psi(\beta)$ if and only if

$$\models (\beta \in k \wedge f(\beta)) \vee (\beta \in \Gamma \wedge g(\beta)).$$

Note that B contains $k_0 \cup \Gamma_0$. Since k_B is an elementary substructure of k and Γ_B is an elementary substructure of Γ, if $c \in k \cup \Gamma$ and $\models \phi(e, c)$ then $\models \psi(c)$, so $\models (c \in k \wedge f(c)) \vee (c \in \Gamma \wedge g(c))$, thus either for some $c_1 \in k_B$ we have $\models f(c_1)$ or some $c_2 \in \Gamma_B$ we have $\models g(c_2)$ and this yields that $\models \phi(e, m)$ for some $m \in B$.

□

We know that the Morley rank is definable in k and Γ as they are strongly minimal. The next remark states that the Morley rank is also definable in the pair (k, Γ) and in its algebraic closure.

Remark 3.28. *(Morley Rank is definable in (k,Γ)) Morley rank is definable in the pair (k,Γ) since they are both strongly minimal and orthogonal. Let B be a set of parameters from Ω. Furthermore the Morley rank is definable in $\mathrm{acl}(B,k,\Gamma)$ for every natural number m by Theorem (2.45).*

Lemma 3.29. *Let $e \in (\Omega,k,\Gamma)^{\mathrm{eq}}$ be an imaginary element. There is a tuple d from Ω, an L_t-definable function $f(x)$ over $\emptyset$, an L_t-formula $\psi(y)$ over e and an L_t-definable function $h(y,z)$ over e such that*

(i) $f(d) = e$,

(ii) $\psi(y) \in \mathrm{tp}(d/e)$,

(iii) $(\forall y, y')(\psi(y) \wedge \psi(y') \implies \exists z((P_1(z) \vee P_2(z)) \wedge h(y,z) = y'))$

(iv) Moreover d is T_t-independent from $k \cup \Gamma$ over e.

Proof. For (i), (ii) and (iii) we refer the reader to [28, 2.4] as before. Now we prove (iv). Choose d such that $\mathrm{MR}(\mathrm{tp}^t(d/e))$ is minimized. By lemma (3.26) we can assume that the type $q = \mathrm{tp}^t(d/e)$ is almost (k,Γ)-internal. Thus remark (2.34) yields that there is some set u such that if $d' \models q$ then $d' \in \mathrm{acl}(u,k,\Gamma)$. We will show that d is independent from k,Γ over e. Suppose not and choose $b \in k \cup \Gamma$ such that d forks with b over e. Note that by almost internality and as k and Γ are strongly minimal, we deduce that $\mathrm{MR}(\mathrm{tp}^t(d/e))$ is finite. Let $m = \mathrm{MR}(\mathrm{tp}^t(d/e,b)) < \mathrm{MR}(\mathrm{tp}^t(d/e))$. Note that the Morley rank is definable in $\mathrm{acl}(u,k,\Gamma)$ by remark (3.28). Let $\chi(y,z)$ be a formula over e such that $\chi(d,b)$ holds and for any c, we have $\mathrm{MR}(\chi(y,c)) = m$ if it is consistent. Let $\Delta(z)$ be the formula

$$\exists y(f(y) = e \wedge \psi(y)) \wedge (\forall y, y')(\psi(y) \wedge \psi(y') \implies (h(y,z) = y' \wedge (P_1(z) \vee P_2(z))).$$

Observe that $\Delta(b)$ holds. Let $B = \mathrm{acl}_t(e) \cap (k \cup \Gamma)$. By lemma (3.27), there is $b_1 \in B$ such that $\Delta(b_1)$ holds. Then we find d_1 satisfying (i), (ii) and (iii) of the lemma with $\chi(d_1,b_1)$ holds. Since b_1 is algebraic over e, we have that $\mathrm{MR}(\mathrm{tp}^t(d_1/e)) \leq m$, contradicting the choice of d. □

Combining lemmas (3.26) and (3.29), we obtain the following theorem which is a description of imaginaries in terms of real elements:

Theorem 3.30. *Let $e \in (\Omega,k,\Gamma)^{\mathrm{eq}}$ be an imaginary element. There is a finite real tuple d such that e is algebraic over d, the type $\mathrm{tp}^t(d/e)$ is almost (k,Γ)-internal and d is independent from $k \cup \Gamma$ over e in the sense of the triple.*

3.4.2 Characterization of Interpretable Groups

In this subsection, we characterize interpretable groups in the triple. We need two more lemmas.

Lemma 3.31. *If* $\mathrm{tp}^t(a/A)$ *is almost* (k,Γ)*-internal over a real set of parameters* A*, then* $a \in \mathrm{acl}(A,k,\Gamma)$.

Proof. Take $B = \mathrm{acl}_t(B)$ containing A such that $a \overset{t}{\underset{A}{\downarrow}} B$ and $a \in \mathrm{acl}_t(B,k,\Gamma) = \mathrm{acl}(B,k,\Gamma)$. The characterization of the independence (3.11) yields that $a \underset{A,k,\Gamma}{\downarrow} B$ and therefore we obtain that $a \in \mathrm{acl}(A,k,\Gamma)$. □

Lemma 3.32. *Let* H *be a definable group in the sense of the triple. If a generic of* H *is almost* (k,Γ)*-internal then* H *is isogenous to a cartesian product of* k*-rational points of an algebraic group defined over* k *and an interpretable group in* Γ.

Proof. By almost internality, we conclude that H is of finite Morley rank. Since infinite algebraic groups have infinite Morley rank in the triple, we conclude by Theorem (3.23) and ω-stability. □

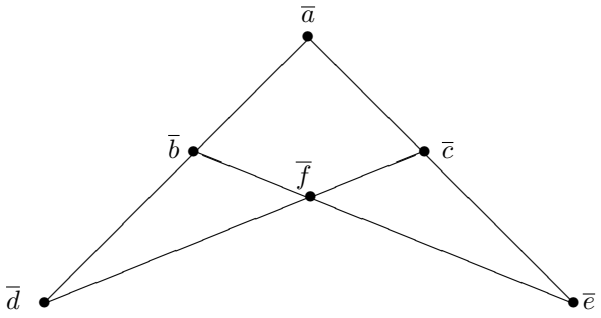
Now we are ready to characterize interpretable groups in the triple. This is the strongest result in this chapter and it demands all the tools we have proved through the chapter.

Theorem 3.33. *(Interpretable groups in* (Ω,k,Γ)*) Let* Ω *be an algebraically closed field, the field* k *be a proper subfield of* Ω *which is also algebraically closed and* Γ *be a divisible multiplicative subgroup of* $\Omega^\times$ *such that* (k,Γ) *is a Mann pair. Every interpretable group* H *in* (Ω,k,Γ) *is, up to isogeny, an extension of a direct sum of* k*-rational points of an algebraic group defined over* k *and an interpretable abelian group in* Γ *by an interpretable group* N*, which is the quotient of an algebraic group by a subgroup* N_1*, which is isogenous to a cartesian product of* k*-rational points of an algebraic group defined over* k *and an interpretable abelian group in* Γ.

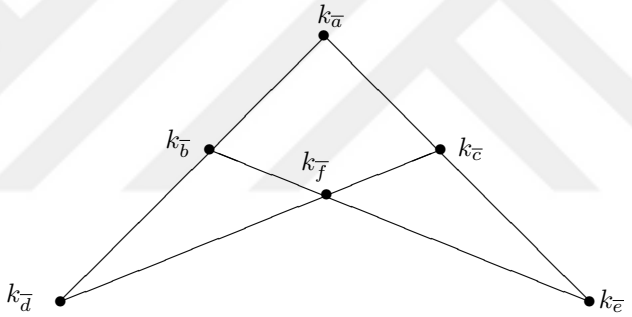
Proof. Let H be an interpretable group in (Ω,k,Γ). By remark (2.26), we may suppose that H is connected. Again we work over a small model that we omit. Let α, β and γ be three independent generics of H in the sense of the triple. By Theorem (3.30), the generic α is algebraic over a real tuple a_0 which is T_t-independent from k,Γ over α and the type $\mathrm{tp}^t(a_0/\alpha)$ is almost (k,Γ)-internal. Then by (2.55), there are real tuples a,b,c,d,e and f such that

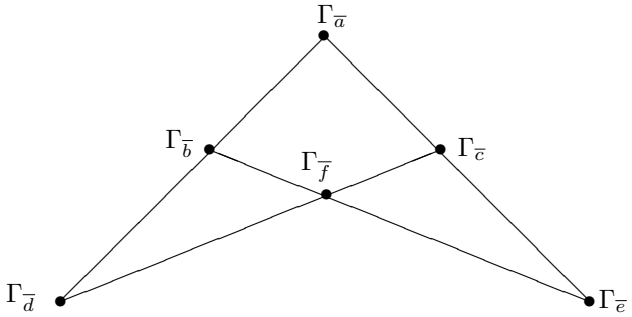
$$(a,\alpha)\equiv^t(a,\beta)\equiv^t(c,\gamma)\equiv^t(d,\alpha\beta)\equiv^t(e,\gamma\alpha)\equiv^t(f,\gamma\alpha\beta)\equiv^t(a_0,\alpha)$$

and if we put $\overline{a} = \mathrm{acl}_t(a)$ and the same for the others, we have the following diagram:



such that each non-colinear triples of them are T_t-independent and each set is T_t-independent from the lines which do not contain it. Since $a \underset{\alpha}{\overset{t}{\downarrow}} k, \Gamma$, we see that $k_{\overline{a}} \subset \mathrm{acl}_t(\alpha) \subset \overline{a}$. Therefore, we obtain that $k_{\overline{a}} = \mathrm{acl}_t(\alpha) \cap k$. Moreover by lemma (3.10) and Theorem (3.11), we have that $k_{\mathrm{acl}_t(\overline{a},\overline{b})} = \mathrm{acl}(k_{\overline{a}}, k_{\overline{b}})$. Since $k_{\overline{d}} = \mathrm{acl}_t(\alpha\beta) \cap k \subset k_{\mathrm{acl}_t(\overline{a},\overline{b})}$, we get that $k_{\overline{d}} \subset \mathrm{acl}(k_{\overline{a}}, k_{\overline{b}})$. This is true for all other tuples and by Theorem (3.11), the set $k_{\overline{a}}$ is independent in the field sense from $\mathrm{acl}(k_{\overline{b}}, k_{\overline{e}})$ and the same for the others. We have the same thing for the group Γ by lemma (3.13). So we have the following diagrams:





and by the group configuration theorem (1.18), there is a connected $*$-interpretable group $V_1(k)$ in k whose generic v is k-algebraic with $k_{\overline{a}}$. By ω-stability, lemma (2.27) and since k eliminates imaginaries, we may assume that V_1 is an algebraic group and its generic v is interalgebraic with $k_{\overline{a}}$ in the sense of the triple. Similarly, there exists a connected interpretable group $H_1(\Gamma)$ in Γ whose generic h is interalgebraic with $\Gamma_{\overline{a}}$ in T_t. Furthermore, by Theorem (2.29), the group H_1 is abelian. By lemma (3.21), the tuple (v, h) is the generic of $V_1 \times H_1$ which is T_t-interalgebraic with $(k_{\overline{a}}, \Gamma_{\overline{a}})$. Moreover by lemma (2.27), we have a projection π from H to the connected group $V_1 \times H_1$ in

$k \times \Gamma$ which is given by the stabilizer of the type $\mathrm{tp}^t(a, v, h)$.

Next we show that the points $\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}$ give a T-group configuration with the help of the parameter set k, Γ. We know that all three non-colinear of them are independent in the sense of the triple. As β is algebraic over b and $\alpha\beta$ is algebraic over d, we have that α is algebraic over b, d. Moreover by lemma (3.10), we know that $\mathrm{acl}(\overline{b}, \overline{d}) = \mathrm{acl}_t(\overline{b}, \overline{d})$. Since the type $\mathrm{tp}^t(a/\alpha)$ is almost (k, Γ)-internal and α is algebraic over b, d, we observe that the type $\mathrm{tp}^t(a/\mathrm{acl}(\overline{b}, \overline{d}))$ is also almost (k, Γ)-internal. Thus by lemma (3.31) we obtain that $a \in \mathrm{acl}(k, \Gamma, \overline{b}, \overline{d})$. The same holds for the other tuples.

Therefore, we obtain a connected $*$-interpretable group V over $\mathrm{acl}(k, \Gamma)$ in the field sense and two independent generics a_1, b_1 of V such that a_1 is field interalgebraic with $\overline{a}$ over k, Γ, the element b_1 is field interalgebraic with $\overline{b}$ and $a_1 b_1$ is field interalgebraic with $\overline{d}$. Since the tuples α, β and γ are algebraic over the finite tuples a_1, b_1 and $a_1 b_1$ respectively and as V is a connected pro-algebraic group, there exists a connected algebraic group W over $\mathrm{acl}(k, \Gamma)$ and two independent generics a_2, b_2 such that α is algebraic over a_2 and the same for the others. Note that a_1 is field algebraic over $k, \Gamma, \overline{a}$ and the same for the others. Moreover, since $\overline{a}, \overline{b}$ and $\overline{d}$ are pairwise T_t-independent over k, Γ, then so are a_2, b_2 and $a_2 b_2$.

As α is algebraic over a and $a \overset{t}{\underset{k_{\overline{a}}, \Gamma_{\overline{a}}}{\downarrow}} k, \Gamma$ by corollary (3.12), we see that $a \overset{t}{\underset{k_{\overline{a}}, \Gamma_{\overline{a}}}{\downarrow}} k, \Gamma$. Now let N be the connected component of $ker(\pi)$. Then α is generic in $N\alpha$ over $\mathrm{acl}_t(v, h) = \mathrm{acl}_t(k_{\overline{a}}, \Gamma_{\overline{a}})$, so α is also generic over $k \cup \Gamma$.

Now we apply the lemma (2.27) to the tuples (a_2, α) and (b_2, β). So this gives us a type-definable surjection ϕ from W to N, up to isogeny. Lastly, we show that the connected component N_1 of $ker(\phi)$ is isogenous to a cartesian product of k-rational points of an algebraic group defined over k and an interpretable group in Γ. Let n_1 be a generic of N_1 over k, Γ and a_2. Then the point $(n_1, 1_N)$ is in the stabilizer of the type $\mathrm{tp}^t(a_2, \alpha/\mathrm{acl}^{\mathrm{eq}}(k, \Gamma))$ and so $\mathrm{tp}^t(n_1 a_2/\alpha) = \mathrm{tp}^t(a_2/\alpha)$. Since $\mathrm{tp}^t(a_2/\alpha)$ is almost (k, Γ)-internal and as a_2 is algebraic over k, Γ, a then the type $\mathrm{tp}^t(n_1 a_2/\alpha)$ is also almost (k, Γ)-internal. As α is algebraic over k, Γ, a_2, the type $\mathrm{tp}^t(n_1/k, \Gamma, a_2)$ is almost (k, Γ)-internal. Owing to the independence $n_1 \overset{t}{\underset{k, \Gamma}{\downarrow}} a_2$, we conclude that $\mathrm{tp}^t(n_1/k, \Gamma)$ is also almost (k, Γ)-internal. Then by lemma (3.33) we have that N_1 is isogenous to a cartesian product of k-rational points of an algebraic group defined over k and an interpretable group in Γ which is abelian by Theorem (2.29).

□

4

Algebraic Numbers with low height elements

Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. The model theoretic properties of $\overline{\mathbb{Q}}$ are well understood; it is strongly minimal and has quantifier elimination in the language of rings $L_r = \{+, -, \cdot, 0, 1\}$. In this chapter, we mainly focus on the pair $\overline{\mathbb{Q}}$ together with a predicate, in particular our predicate will be the elements of height less than a given positive real number. Moreover, we study these pairs in terms of stability. As mentioned earlier in this thesis, B. Zilber [41] showed that the pair

$$(\mathbb{C}, \mu) \equiv (\overline{\mathbb{Q}}, \mu)$$

is ω-stable where μ is the set of complex roots of unity. In this chapter, our predicate, namely the elements of small height, contains the group μ, and the theory of $\overline{\mathbb{Q}}$ with this predicate will be a proper extension of the theory of $(\overline{\mathbb{Q}}, \mu)$. Model theory of pairs have attracted a lot of attention recently. More generally, stable theories with a predicate were analysed in the paper of E. Casanovas and M. Ziegler [6], where they gave criteria for a pair to be stable. If the based model is strongly minimal, this criterion coincides with the induced structure on the predicate is stable. Their result in [6] involves the result of B. Zilber [41] and also B. Poizat's result on the ω-stability of the theory of pairs of algebraically closed fields [31]. Before defining our notations in the chapter, we state Kronecker theorem from diophantine geometry with no proof, as the details can be found in the book [4].

Theorem 4.1. *(Kronecker [4, 1.5.9]) Let α in $\overline{\mathbb{Q}}$ be a non-zero algebraic number.*

Then $h(\alpha) = 0$ if and only if it is a root of unity.

Combining Zilber's result [41] with Kronecker theorem, we see that the pair

$$(\overline{\mathbb{Q}}, \{a \in \overline{\mathbb{Q}} : h(a) = 0\})$$

is ω-stable.

Through this chapter, the language L_m will denote the language $\{1, \cdot\}$ where the binary operation $\cdot$ is the usual multiplication. Let

$$S_\epsilon = \{a \in \overline{\mathbb{Q}} : h(a) \leq \epsilon\}$$

be the set of algebraic numbers whose heights are less than ϵ, where h is the absolute logarithmic height function and $\epsilon > 0$. We call S_ϵ the set of algebraic numbers of small height. Note that the Mahler measure of an algebraic number is again an algebraic number. Put also $S = S_1 = \{a \in \overline{\mathbb{Q}} : h(a) < 1\}$, as there is no algebraic number whose height is 1 by Lindemann's theorem.

The pair $(\overline{\mathbb{Q}}, S_\epsilon)$ can be seen as an $L_r(U) = L_r \cup \{U\}$ structure where U is an unary relation symbol whose interpretation is S_ϵ. Let T_ϵ be the theory $Th(\overline{\mathbb{Q}}, S_\epsilon)$. In this chapter, we focus on the model theory of $(\overline{\mathbb{Q}}, S_\epsilon)$ in the language $L_m(U) = L_m \cup \{U\}$ and we prove a result which shows that small perturbations of the property of being a root of unity modify immensely the stability properties of the ambient structure. We also relate the simplicity of a certain pair with Lehmer's conjecture.

4.1 Height Lemmas

In this section, we give the height inequality for the height function on the field of algebraic numbers. The following two lemmas, details and generalized versions can be found in [4, 17].

Lemma 4.2. *[4, 1.6.7] Let $f = a_0 + a_1X + \cdots + a_dX^d \in \mathbb{C}[X]$. Put $|f| = \max_i\{|a_i|\}$. Then we have $2^{-d}|f| \leq m(f) \leq 2^{2d+1}|f|$.*

Let $f = a_0 + a_1X + \cdots + a_dX^d \in K[X]$, where K is a number field. For any absolute value $v \in M_K$, we define $|f|_v = \max_i\{|a_i|_v\}$. The following lemma is the Gauss lemma.

Lemma 4.3. *(Gauss lemma [4, 1.6.3]) Let K be a number field and suppose f, g are in $K[X]$. For a non-archimedean absolute value v on K, we have $|fg|_v = |f|_v|g|_v$.*

There is a relation between height of a polynomial and height of its roots. For a polynomial $f = a_0 + a_1X + \cdots + a_dX^d$ over the field of algebraic numbers, define

$H(f) = \max_i H(a_i)$, where H is the non-logarithmic height function. Then if f is a polynomial over a number field K, we have that

$$H(f) = \prod_{v \in M_K} \max\{1, |f|_v\}^{\frac{1}{[K:\mathbb{Q}]}}.$$

Now we are ready to prove the height inequality.

Lemma 4.4. *For a polynomial*

$$f(x) = (X - \alpha_1) \cdots (X - \alpha_d) = a_0 + a_1 X + \cdots + X^d \in \overline{\mathbb{Q}}[X]$$

over $\overline{\mathbb{Q}}$, the non-logarithmic height (the logarithmic height respectively) $H(\alpha_i)$ is uniformly bounded by $H(f)$ and d i.e

$$2^{-d} H(f) \leq \prod_{i \leq d} H(\alpha_i) \leq 2^{2d+1} H(f).$$

Proof. Let K be a number field containing the elements α_i and a_j for $i, j \leq d$. By lemma (4.2), we see that

$$2^{-d}|f| \leq m(f) \leq 2^{2d+1}|f|.$$

For non-archimedean absolute value $v \in M_K$, the Gauss lemma (4.3) yields that $|f|_v = \prod_{i \leq d} \max\{1, |\alpha_i|_v\}$. As $m(f) \geq 1$, we obtain that

$$2^{-d} \prod_{v \in M_K} \max\{1, |f|_v\} \leq \prod_{\substack{i \leq d \\ v \in M_K}} \max\{1, |\alpha_i|_v\} \leq 2^{2d+1} \prod_{v \in M_K} \max\{1, |f|_v\}.$$

Hence we get the desired inequality

$$2^{-d} H(f) \leq \prod_i H(\alpha_i) \leq 2^{2d+1} H(f).$$

□

Now using the properties of the height function, we prove several lemmas which we will need in the next section.

Lemma 4.5. *For any non-zero algebraic number α which is not a root of unity, the set $A(\alpha) = \{h(\alpha^q) : q \in \mathbb{Q}\}$ is dense in the positive real numbers.*

Proof. Let α be a non-zero algebraic number which is not a root of unity. Then by Theorem (4.1), we know that $h(\alpha) > 0$. Furthermore, one sees that $h(\alpha^q) = |q|h(\alpha)$ for any $q \in \mathbb{Q}$. Given an interval (a, b) where $0 \leq a < b$; choose $r \in \mathbb{Q}$ such that $a/h(\alpha) < r < b/h(\alpha)$. Then we conclude that $a < h(\alpha^r) < b$. □

Lemma 4.6. *Let $p_1, ..., p_k$ and $m_1, ..., m_k$ be positive natural numbers. Then*

$$h({p_1}^{1/m_1} \cdots {p_k}^{1/m_k}) = h({p_1}^{1/m_1}) + \cdots + h({p_1}^{1/m_k}).$$

Proof. Put $m = m_1...m_k$ and $n_i = m/m_i$. Then by the properties of the logarithmic height function, we know that for any algebraic number α and rational number q we have that $h(\alpha^q) = |q|h(\alpha)$. Therefore, we see that

$$h({p_1}^{1/m_1} \cdots {p_k}^{1/m_k}) = \frac{h({p_1}^{n_1} \cdots {p_k}^{n_k})}{m}.$$

Since for a natural number $n \geq 1$ we have $h(n) = \log n$, we conclude that

$$\frac{h({p_1}^{n_1} \cdots {p_k}^{n_k})}{m} = \frac{h({p_1}^{n_1}) + \cdots + h({p_k}^{n_k})}{m} = h({p_1}^{1/m_1}) + \cdots + h({p_1}^{1/m_k}).$$

□

Lemma 4.7. *Let $p_1, ..., p_k, p_{k+1}$ and $m_1, ..., m_k, m_{k+1}$ be natural numbers such that $p_1...p_k$ and p_{k+1} are coprime. Then we have*

$$h\left(\frac{{p_1}^{1/m_1} \cdots {p_k}^{1/m_k}}{{p_{k+1}}^{1/m_{k+1}}}\right) = \max\{h({p_1}^{1/m_1} \cdots {p_k}^{1/m_k}), h({p_{k+1}}^{1/m_{k+1}})\}.$$

Proof. Put $m = m_1...m_km_{k+1}$ and $n_i = m/m_i$. Recall that for coprime integers a and b, we have $h(a/b) = \max\{\log|a|, \log|b|\} = \max\{h(a), h(b)\}$. Thus by the properties of the logarithmic height function, we have

$$h\left(\frac{{p_1}^{1/m_1} \cdots {p_k}^{1/m_k}}{{p_{k+1}}^{1/m_{k+1}}}\right) = \frac{1}{m} h\left(\frac{{p_1}^{n_1} \cdots {p_k}^{n_k}}{{p_{k+1}}^{n_{k+1}}}\right)$$

$$= \frac{1}{m} \max\{h({p_1}^{n_1} \cdots {p_k}^{n_k}), h({p_{k+1}}^{n_{k+1}})\},$$

and the lemma follows from the properties of the logarithmic height function again. □

Let μ be the set of complex roots of unity. Utilizing Kronecker's theorem, we can define μ uniformly in the pair $(\overline{\mathbb{Q}}, S_\epsilon)$ for any positive ϵ.

Lemma 4.8. *Let $\phi(x)$ be the formula $x \neq 0 \wedge \forall y(U(y) \to U(xy))$ in the language $L_m(U)$. Then for any positive ϵ, the formula $\phi(x)$ defines μ in the pair $(\overline{\mathbb{Q}}, S_\epsilon)$.*

Proof. Let ϵ be a positive real number. A root of the unity satisfies the formula since the height function satisfies $h(xy) \leq h(x) + h(y)$ and the height of a root of unity is 0. Now suppose α satisfies the formula. If we take $y = 1$ in the formula, we see that $h(\alpha) \leq \epsilon$. Then letting $y = \alpha$, we get $h(\alpha) \leq \epsilon/2$. So taking powers of α, we conclude that $h(\alpha) = 0$. The lemma follows from Kronecker's theorem (4.1).

□

The next lemma is in [9]. It states that in an algebraically closed field, the many-valued function in the definition of "large" can be replaced by an ordinary function:

Lemma 4.9. *[9, 2.4] Let K be an algebraically closed field and suppose $A \subseteq K$ is large in K. Then there is a definable function $F : K^l \to K$ such that $F(A^l) = K$.*

Since ACF_0 is strongly minimal, by lemma (1.23) and by the height inequality (4.4), we can prove that S_ϵ is small in $\overline{\mathbb{Q}}$.

Lemma 4.10. *The set S_ϵ is small in $\overline{\mathbb{Q}}$ for any $\epsilon \geq 0$.*

Proof. We can assume that ϵ is 1. Suppose that S is large in $\overline{\mathbb{Q}}$. So by lemma (4.9), there is a definable function $f : \overline{\mathbb{Q}}^m \to \overline{\mathbb{Q}}$ such that $f(S^m) = \overline{\mathbb{Q}}$. By quantifier elimination, we see that $Graph(f) = \{(x_1, ..., x_m, y) : f(x_1, ..., x_m) = y\}$ is a subset of union of varieties in $\overline{\mathbb{Q}}^{m+1}$. Without loss of generalities, we may assume that

$$Graph(f) = V = \{(x_1, ..., x_m, y) : p(x_1, ..., x_m, y) = 0\}$$

is a variety, where p is a polynomial of degree d. Then we see that y is algebraic over S^d having degree at most d, where S^d is the set of algebraic elements whose heights are bounded by d. Now by lemma (4.4) we obtain that $h(y)$ is also bounded. However, since h is unbounded on $\overline{\mathbb{Q}}$, we cannot have that $f(S^m) = \overline{\mathbb{Q}}$. By lemma (1.23), we conclude that S is small as desired.

□

4.2 Small Height Elements

In this section, we turn our attention to model-theoretic properties of algebraic numbers expanded by a predicate to denote elements of small heights. We have all the tools to work on these pairs.

4.2.1 Simplicity and Independence Property

Now we are ready to prove our main result in this chapter by using the lemmas (4.5), (4.6) and (4.7).

Theorem 4.11. *The theory of $(\overline{\mathbb{Q}}, S_\epsilon)$ is not simple and has the independence property (IP) in the language $L_m(U)$.*

Proof. Recall that the Mahler measure of an algebraic number is again an algebraic number. Therefore by Lindemann's theorem, the logarithmic height of an algebraic number α is transcendental if $h(\alpha) \neq 0$. First we prove that the theory is not simple. In order to show this, we exhibit a formula which has the tree property.

Case 1: The element ϵ is in the range of the logarithmic height function, that is to say $\epsilon = h(\alpha)$ for some $\alpha \in \overline{\mathbb{Q}}$. Now put

$$\phi(x,y,z,t) : U\left(\frac{ty}{x}\right) \wedge U\left(\frac{tx}{z}\right).$$

Observe that for any rational number $r, s \in (0,1)$,

$$h\left(\alpha\frac{\alpha^r}{\alpha^s}\right) < \epsilon \iff r < s.$$

So we can order rational numbers in this theory. Furthermore if (r_1, s_1) and (r_2, s_2) are disjoint intervals of $(0,1)$, where $r_1 < s_1 < r_2 < s_2 \in \mathbb{Q}$, then we cannot have

$$(\overline{\mathbb{Q}}, S_\epsilon) \models \exists x(\phi(x, \alpha^{r_1}, \alpha^{s_1}, \alpha) \wedge \phi(x, \alpha^{r_2}, \alpha^{s_2}, \alpha)),$$

otherwise we have

$$h\left(\alpha\frac{x}{\alpha^{s_1}}\right) \leq \epsilon \qquad \text{and} \qquad h\left(\alpha\frac{\alpha^{r_2}}{x}\right) \leq \epsilon.$$

Therefore, if we multiply these elements, we obtain that $h\left(\alpha^2\frac{\alpha^{r_2}}{\alpha^{s_1}}\right) \leq 2\epsilon$. This is a contradiction, since $s_1 < r_2$ and $h(\alpha^2) = 2\epsilon$. Thus the formula $\phi(x,y,z,t)$ has the tree property if we take the parameters (α^{q_s}, α) where the parameters $(q_s : \emptyset \neq s \in \omega^{<\omega})$ as given in (1.20).

Case 2: The element ϵ is not in the range of the logarithmic height function. Without loss of generality, we can assume that $\epsilon = 1$ i.e $S_\epsilon = S$ as 1 is not transcendental. This time we use the fact that the range of the logarithmic height function is dense in the positive reals. Again we set

$$\phi(x,y,z,t) : U\left(\frac{ty}{x}\right) \wedge U\left(\frac{tx}{z}\right).$$

We will show that this formula has the tree property by finding some parameters in some model of T.

Let ${}^*\mathbb{M}$ be a nonstandard extension of the many-sorted structure

$$\mathbb{M} = (\overline{\mathbb{Q}}, +, -, \cdot, 0, 1, h, \mathbb{R}_{\geq 0}).$$

Then the logarithmic height function extends to ${}^*\overline{\mathbb{Q}}$ and it takes values in positive hyperreal numbers. We also denote this extension as h. Then the pair $({}^*\overline{\mathbb{Q}}, {}^*S)$ is an elementary extension of $(\overline{\mathbb{Q}}, S)$ in $L_m(U)$. Note that ${}^*S$ is the set of hyperalgebraic numbers whose heights are less than 1. Let $st(a)$ denote the standard part of a finite hyperreal number. By lemma (4.5), we know that the sequence $\{h(2^k)\}$ is dense in positive real numbers where k is a rational number. In particular, there is a

hyperrational number q such that $st(h(2^q)) = 1$. Moreover $q > 1$ in the sense of usual ordering of hyperreal numbers and it is infinitely close to the real number $1/\log 2$. Observe again that for any rational number $r, s \in (0, 1)$,

$$h\left(2^q \frac{2^r}{2^s}\right) < 1 \iff r < s.$$

Furthermore if (r_1, s_1) and (r_2, s_2) are disjoint intervals of $(0, 1)$, where $r_1 < s_1 < r_2 < s_2 \in \mathbb{Q}$, then we cannot have

$$(^*\overline{\mathbb{Q}}, {}^*S) \models \exists x(\phi(x, 2^{r_1}, 2^{s_1}, 2^q) \wedge \phi(x, 2^{r_2}, 2^{s_2}, 2^q)),$$

otherwise we have

$$h\left(2^q \frac{x}{2^{s_1}}\right) \leq 1 \qquad \text{and} \qquad h\left(2^q \frac{2^{r_2}}{x}\right) \leq 1.$$

Therefore, if we multiply these elements, we obtain that

$$h\left(2^{2q} \frac{2^{r_2}}{2^{s_1}}\right) \leq 2.$$

This is a contradiction, since $s_1 < r_2$ and $st(h(2^{2q}) = 2$, and also

$$h\left(2^{2q} \frac{2^{r_2}}{2^{s_1}}\right) > 2.$$

Thus the formula $\phi(x, y, z, t)$ has the tree property if we take the parameters $(2^{q_s}, 2^q)$ where the parameters $(q_s : \emptyset \neq s \in \omega^{<\omega})$ as given in (1.20). Hence T is not simple.

Now we show that T has the independence property. Let $\phi(x, y)$ be the formula

$$U(x/y).$$

We will show that this formula has IP. Let $n \geq 1$ be given. Let $p_1, ..., p_n$ be distinct prime numbers. Put $b_m = {p_m}^k$ where $k \in \mathbb{Q}$ will be chosen. Now let I be a subset of $\{1, ..., n\}$ and r be the size of I. For $i \in I$, by lemma (4.5) we can choose $e_i \in \mathbb{Q}$ such that

$$\frac{1}{r} < h({p_i}^{e_i}) \leq \frac{1}{r} + \frac{1}{(n+1)^2}.$$

Set

$$a_I = \prod_{i \in I} {p_i}^{e_i}.$$

By lemma (4.6), we have

$$1 < h(a_I) \leq 1 + \frac{1}{n+1}.$$

Now by lemma (4.5), choose k such that

$$1 - \frac{1}{n+1} < h(b_m) < 1.$$

Then by lemma (4.7) and the properties of the height function, we get $\phi(a_I, b_i)$ if and only if $i \in I$. This yields that $\phi(x, y)$ has the independence property and hence T is not NIP. □

Let $\overline{\mathbb{R}}$ be the real algebraic numbers and put $S(\overline{\mathbb{R}}) = S \cap \overline{\mathbb{R}}$. For real algebraic numbers, we have the same result:

Corollary 4.12. *The theory of $(\overline{\mathbb{R}}, S(\overline{\mathbb{R}}))$ is not simple and has IP in the language $L_m(U)$.*

Remark 4.13. *The proof of Theorem* (4.11) *indicates that the theory T_ϵ has the tree property of the first kind TP_1.*

4.2.2 Elliptic Curve Case

In this short subsection, we give an analogous result of Theorem (4.11) for elliptic curves with the canonical height function on it. For more on the subject; see [33].

An elliptic curve over $\overline{\mathbb{Q}}$ is the solution set of the equation

$$y^2 = x^3 + ax + b$$

in $\overline{\mathbb{Q}}$ with an additional point at the infinity O, where $a, b \in \overline{\mathbb{Q}}$ and $4a^3 + 27b^2$ is not zero. An elliptic over $\overline{\mathbb{Q}}$ will be denoted by $E = E(\overline{\mathbb{Q}})$. An elliptic curve $(E(\overline{\mathbb{Q}}), O)$ is an abelian group such that O is the identity element. Since $\overline{\mathbb{Q}}$ is algebraically closed, the group $E(\overline{\mathbb{Q}})$ is divisible.

Analogous to the logarithmic height function on the set of algebraic numbers, there is a canonical height function on elliptic curves.

Theorem 4.14. *[18, Chapter 5, 2.2.2] Let (E, O) be an elliptic curve over $\overline{\mathbb{Q}}$. We define the canonical height $\hat{h}$ by the formula $\hat{h}(O) = 0$ and if P is not O then*

$$\hat{h}(P) = \lim_{n\to\infty} \frac{h(x(2^n P))}{4^n}.$$

This height over E satisfies the parallelogram law:

$$\hat{h}(P + Q) + \hat{h}(P - Q) = 2\hat{h}(P) + 2\hat{h}(Q).$$

In particular $\hat{h}(mP) = m^2\hat{h}(P)$. Finally, $\hat{h}(P) = 0$ if and only if P is a torsion point.

Let $Tors$ be the elements in E whose canonical height is zero and S be the elements in E whose canonical height ≤ 1. Note that for every integer $n \geq 1$, there are finitely many n-torsion points in E. Similar to the main theorem in this chapter, we have the following theorem:

Proposition 4.15. *Let E be an elliptic curve over $\overline{\mathbb{Q}}$ and $\hat{h}$ be the canonical height on it. Then the pair $(E(\overline{\mathbb{Q}}), Tors)$ is stable, however the pair $(E(\overline{\mathbb{Q}}), S)$ is not simple in the language $L_m(U)$.*

Proof. The proof is similar to the proof of the previous theorem, so we will be brief. Note that since E is divisible with finitely many n-torsions for every $n \geq 1$, it has QE and it is ω-stable. Similar to lemma (4.10), one can show that $Tors$ is small in E. By Theorem (4.14), we know that $Tors$ is exactly the torsion elements in E. Thus $Tors$ is a divisible group. Hence $Tors$ is an elementary substructure of E. This indicates the stability of the pair $(E, Tors)$ by [6]. On the other hand by Theorem (4.14), for all q in $\mathbb{Q}$ and $P \in E$, we have that $\hat{h}(qP) = q^2\hat{h}(P)$. Moreover, since $\mathbb{Q}^2$ is dense in the positive real line, by choosing a non-torsion element, similar to the proof of Theorem (4.11), one can show that the formula

$$\phi(x, y, z, t) : U\left(\frac{ty}{x}\right) \wedge U\left(\frac{tx}{z}\right)$$

has the tree property. □

4.3 Salem Numbers

Lehmer's Conjecture (1933): There exists an absolute constant $c > 1$ such that if α is not a root of unity then $m(\alpha) \geq c$. This conjecture is still open.

A real algebraic integer $\alpha > 1$ is called a *Salem number* if α and $1/\alpha$ are Galois conjugate and all others Galois conjugates of α lie on the unit circle. Observe that for a Salem number α and a positive integer n, we have

$$m(\alpha^n) = m(\alpha)^n = \alpha^n,$$

and in general this is not true for an arbitrary algebraic number. It is an open question whether 1 is a limit point of Salem numbers. This is a special case of Lehmer's Conjecture. The smallest known Salem number α was given by D. Lehmer [24] which is the root of the polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

with $\alpha \approx 1.17628$ and not many Salem numbers known in the interval $(1, 1.3)$. For more about Lehmer's conjecture and Salem numbers, we direct the reader to [35, 36]. Let

$P_b := \{a \in \overline{\mathbb{Q}} : m(a) \leq b\}$ where $b \geq 1$. Note that P_b contains the set of roots of unity μ. Lehmer's Conjecture is equivalent of there exists $b > 1$ such that $P_b = P_1 = \mu$. So if Lehmer's Conjecture is true, then we know that the pair $(\overline{\mathbb{Q}}, P_b)$ is ω-stable for some $b > 1$ by a theorem of B. Zilber [41]. The next proposition relates the simplicity of the pair $(\overline{\mathbb{Q}}, P_b)$ with Lehmer's Conjecture. It states that if the pair $(\overline{\mathbb{Q}}, P_b)$ is simple for some $b > 1$, which is a weaker statement than ω-stability, then Lehmer's Conjecture is true for Salem numbers.

Proposition 4.16. *If the theory of $(\overline{\mathbb{Q}}, P_b)$ is simple for some $b > 1$ in $L_m(U)$, then Lehmer's Conjecture is true for Salem numbers.*

Proof. Suppose that $(\overline{\mathbb{Q}}, P_b)$ is simple for some $b > 1$. Assume that 1 is a limit point of Salem numbers in order to get a contradiction. Then we can choose a Salem number α and a positive integer n such that α^n is very close to b. Let ${}^*\mathbb{M} = ({}^*\overline{\mathbb{Q}}, +, -, \cdot, 0, 1, m, {}^*\mathbb{R}_{\geq 1})$ be a nonstandard extension of the many-sorted structure $\mathbb{M} = (\overline{\mathbb{Q}}, +, -, \cdot, 0, 1, m, \mathbb{R}_{\geq 1})$. Then the Mahler measure m extends to ${}^*\overline{\mathbb{Q}}$ and it takes values in positive hyperreal numbers ≥ 1. We also denote this extension as m. Then the pair $({}^*\overline{\mathbb{Q}}, {}^*P_b)$ is an elementary extension of $(\overline{\mathbb{Q}}, P_b)$ in $L_m(U)$. Note that ${}^*P_b$ is the set of hyper algebraic numbers whose Mahler measure is less than b. Then there is a nonstandard Salem number $\beta > 1$ which is infinitely close to 1, and an infinite nonstandard natural number N such that $st(\beta^N) = b$. Put

$$\phi(x, y, z, t) : U\left(\frac{ty}{x}\right) \wedge U\left(\frac{tx}{z}\right)$$

as before. Denote the integer part of a as $[a]$. Observe that for any rational numbers r and s in $(0, 1)$, we have

$$m\left(\beta^N \frac{\beta^{[Nr]}}{\beta^{[Ns]}}\right) < b \iff r < s.$$

Therefore, we can order rational numbers in this pair. Then similar to the proof of (4.11), the formula $\phi(x, y, z, t)$ has the tree property by taking the parameters $a_s = (\beta^{[Nq_s]}, \beta^N)$. Hence $(\overline{\mathbb{Q}}, P_b)$ is not simple, a contradiction. □

Remark 4.17. *Let C_b be the set of Salem numbers less than b, where $b > 1$. The proof of proposition* (4.16) *indicates that the simplicity of the pair $(\overline{\mathbb{Q}}, C_b)$ for some $b > 1$ in $L_m(U)$ is equivalent to Lehmer's conjecture for Salem numbers. We call an algebraic number α multiplicative if $m(\alpha^n) = m(\alpha)^n$ for all natural numbers $n \geq 1$. Let D_b be set of multiplicative algebraic numbers whose Mahler measure is less than b. Note that D_b contains C_b. Then the same proof of proposition* (4.16) *shows that the simplicity of the pair $(\overline{\mathbb{Q}}, D_b)$ for some $b > 1$ in $L_m(U)$ is equivalent to Lehmer's conjecture for multiplicative algebraic numbers.*

Remark 4.18. *One way of showing the simplicity it to find a notion of independence which is symmetric and satisfying the axioms of non-forking.*

We end this chapter by posing the following question and the conjecture which are related to Lehmer's conjecture.

- Is μ definable in $(\overline{\mathbb{Q}}, P_b)$ for some $b > 1$?

Conjecture 4.19. *The theory of $(\overline{\mathbb{Q}}, P_b)$ is stable for some $b > 1$ in $L_m(U)$. Moreover the stability of the pair $(\overline{\mathbb{Q}}, P_b)$ in $L_m(U)$ for some $b > 1$ implies Lehmer's conjecture.*

5

Nonstandard Analysis and its applications to heights

The arithmetic version of the Nullstellensatz states that if $f_1, ..., f_s$ belong to the ring $\mathbb{Z}[X_1, ..., X_n]$ without a common zero in $\mathbb{C}$, then there exist a in $\mathbb{Z} \setminus \{0\}$ and $g_1, ..., g_s$ in $\mathbb{Z}[X_1, ..., X_n]$ such that $a = f_1 g_1 + \cdots + f_s g_s$. Finding degree and height bounds for a and $g_1, ..., g_s$ has received continuous attention using computational methods. By $\deg f$, we mean the total degree of the polynomial f in several variables. T. Krick, L. M. Pardo and M. Sombra [21] proved degree and height bounds for a and $g_1, ..., g_s$ which are sharp and effective.

On the other hand, finding bounds in mathematics using nonstandard extensions has been studied often, for example: Given a field K, if $f_0, f_1, ..., f_s$ in $K[X_1, ..., X_n]$ all have degree less than D and f_0 is in $\langle f_1, ..., f_s \rangle$, then $f_0 = \sum_{i=1}^{s} f_i h_i$ for certain h_i whose degrees are bounded by a constant $C = C(n, D)$ depending only on n and D. This result was first validated in a paper of G. Hermann [16], where his pattern was based on linear algebra and computational methods. Then the same result was proved by L. van den Dries and K. Schmidt [8] using nonstandard methods, and their technique smoothed the way how nonstandard methods can be used for such bounds. In this chapter, we apply nonstandard methods in order to prove the existence of bounds for the complexity of the coefficients of h_i as above by taking $f_0 = 1$. In plain words, using nonstandard methods, we prove the existence of certain height bounds on the

complexity of the coefficients of some polynomials. This enables us to characterize the ideal membership of a given polynomial. Moreover, we obtain a bound for the logarithmic height function, which entitles us to test the primality of an ideal.

In this chapter, we also define an abstract height function on a ring R, which generalizes the absolute value function and the logarithmic height function, and it measures the complexity of the coefficients of polynomials over $R[X_1, ..., X_n]$. We will generalize the result of [21] to any integral domain and height function, and furthermore our constant for the height function does not depend on R or s, however it is ineffective. We assume that all rings are commutative with unity through the chapter. Moreover throughout this chapter, the ring R stands for an integral domain and K for its field of fractions. The symbol h denotes a height function on a ring R, which will be defined in the next section.

5.1 Generalized Height Function

Let $\theta : \mathbb{N} \to \mathbb{N}$ be a function. We say that

$$h : R \to [0, \infty)$$

is a *height function of θ-type* if for any x and y in R with $h(x) \leq n$ and $h(y) \leq n$, then both $h(x + y) \leq \theta(n)$ and $h(xy) \leq \theta(n)$. We say that h is a *height function* on R if h is a height function of θ-type for some $\theta : \mathbb{N} \to \mathbb{N}$.

We can extend the height function h to the polynomial ring $R[X_1, ..., X_n]$ by

$$h\left(\sum_{\alpha} a_\alpha X^\alpha\right) = \max_{\alpha} h(a_\alpha).$$

Note that this extension does not have to be a height function, it is just an extension of functions. Now we give some examples of height functions.

Example 5.1. *For the following examples of height functions, one can take $\theta(n) = (n + 1)^2$.*

- *If $(R, |\cdot|)$ is an absolute valued ring then $h(x) = |x|$ is a height function. Moreover $h(x) = |x| + 1$ and $h(x) = \max(1, |x|)$ are also height functions on R.*
- *The degree function on $R[X_1, ..., X_n]$ is a height function.*
- *Let λ be a positive real number. On $\mathbb{Z}[X]$, define*

$$h(a_0 + a_1 X + \cdots + a_k X^k) = \sum_{i=0}^{k} |a_i| \lambda^i.$$

Then this is a height function on $\mathbb{Z}[X]$.

- *Let $h : R \to [0, \infty)$ be a function such that the sets*

$$A_n = \{x \in R : h(x) \leq n\}$$

are all finite for all $n \geq 1$. Then h is a height function of θ-type where $\theta(n) = \max_{x,y \in A_n} \{h(x + y) + h(xy)\}$.

- *The p-adic valuation on $\mathbb{Z}$ is not a height function. Note that 1 and $p^n - 1$ are not divisible by p, however their sum is divisible by p^n.*

5.1.1 Nonstandard Extensions and Height Function

Let ${}^*\mathbb{K}$ be a nonstandard extension of the many-sorted structure

$$\mathbb{K} = (K[X_1, ..., X_n], +, -, \cdot, 0, 1, h, \theta, \deg, R[X_1, ..., X_n], \mathbb{R}_{\geq 0}, \mathbb{N}),$$

where h is a function from $R[X_1, ..., X_n]$ to $\mathbb{R}_{\geq 0}$, the function θ is a function from $\mathbb{N}$ to $\mathbb{N}$ and deg is the degree function on $K[X_1, ..., X_n]$ which takes values in $\mathbb{N}$. As usual, the functions deg and θ extend to ${}^*\mathbb{N}$ and they take values in ${}^*\mathbb{N}$. Note that ${}^*K[X_1, ..., X_n] \subsetneq {}^*(K[X_1, ..., X_n])$ and ${}^*K[X_1, ..., X_n] = \{f \in {}^*(K[X_1, ..., X_n]) : \deg f \in \mathbb{N}\}$. If h is a height function on R of θ-type, then it extends to ${}^*(R[X_1, ..., X_n])$ which takes values in ${}^*\mathbb{R}_{\geq 0}$ though this extension is no longer a height function if h is unbounded. Moreover it satisfies the same first-order properties as h. In particular if x, y are in ${}^*R$ with $h(x) \leq n$ and $h(y) \leq n$, where $n \in {}^*\mathbb{N}$, then we have both $h(x + y) \leq \theta(n)$ and $h(xy) \leq \theta(n)$. Define

$$R_{fin} = \{x \in {}^*R : h(x) \in \mathbb{R}_{fin}\}$$

where $\mathbb{R}_{fin} = \{x \in {}^*\mathbb{R} : |x| < n \text{ for some } n \in \mathbb{N}\}$ and ${}^*\mathbb{R}$ is a nonstandard extension of $\mathbb{R}$. The elements in ${}^*\mathbb{R} \setminus \mathbb{R}_{fin}$ are called infinite.

By the properties of a height function, if there is a height function on R, we see that R_{fin} is a subring of ${}^*R$ and it contains R. Note that ${}^*(R[X_1, ..., X_n])$, ${}^*R$ and ${}^*K$ are internal sets. The next lemma shows when R_{fin} is internal.

Lemma 5.2. *The set R_{fin} is an internal subset of ${}^*R$ if and only if the height function on R is bounded.*

Proof. Suppose $R_{fin} = {}^*A$ for some subset A of R. First we show that the height function on A must be bounded. To see this, if there is a sequence $(a_n)_n$ in A such that $\lim_{n \to \infty} h(a_n) = \infty$, then there is an element in ${}^*A$ whose height is infinite. This contradicts the fact that all the elements in R_{fin} have bounded height. So the height function on A is bounded. Therefore the height function on ${}^*A$ is also bounded. However since R_{fin} contains R, the height function on R must be bounded. Conversely

if the height function on R is bounded, then we have $R_{fin} = {}^*R$ and so R_{fin} is internal. □

Now we fix some more notations. Put $L = Frac(R_{fin})$ which is a subfield of ${}^*K$. Note that ${}^*K$ is the fraction field of ${}^*R$. In fact, of being a height function is very related to the set R_{fin}. The following proposition is the nonstandard point of view definition of a height function. However it is ineffective, i.e. it does not provide the θ-type of the height function.

Proposition 5.3. *A function $h : R \to [0, \infty)$ is a height function on R if and only if R_{fin} is a subring of ${}^*R$.*

Proof. We have seen that if h is a height function then R_{fin} is a subring. Conversely suppose R_{fin} is a subring and h is not a height function. This means there is some $N \in \mathbb{N}$ such that we have two sequences (r_n) and (s_n) in R with $h(r_n) \leq N$ and $h(s_n) \leq N$, however $\lim_{n\to\infty} h(r_n \star s_n) = \infty$, where the binary operation $\star$ means either addition or multiplication. Thus we obtain two elements r and s in ${}^*R$ such that $h(r) \leq N$, $h(s) \leq N$, but $h(r \star s)$ is infinite. This contradicts the fact that R_{fin} is a subring. □

5.1.2 Degree Bounds and Primality

In this subsection, we prove some results from commutative algebra and give the results in [8] that lead to the existence of the constant c_1 in Theorem (5.8).

Lemma 5.4. *Let F be a field and $f_1, ..., f_s \in F[X_1, ..., X_n]$. Then $1 \in \langle f_1, ..., f_s \rangle$ if and only if $f_1, ..., f_s$ have no common zeros in F^{ac}.*

Proof. $\Rightarrow$: Clear.
$\Leftarrow$: By Hilbert's Nullstellensatz, there are $g_1, ..., g_s \in F^{ac}[X_1, ..., X_n]$ such that $1 = f_1g_1 + \cdots + f_sg_s$. This is a system of linear equations when we consider the coefficients of all the polynomials. Therefore $1 = f_1Y_1 + \cdots + f_sY_s$ has a solution in F^{ac}. Now by the Gauss-Jordan Theorem, this linear system has a solution in F. So there are $h_1, ..., h_s \in F[X_1, ..., X_n]$ such that

$$1 = f_1h_1 + \cdots + f_sh_s.$$

□

Remark 5.5. *Let $F \subseteq F_1$ be a field extension and $I \subset F[X_1, ..., X_n]$ be a proper ideal. Then the ideal $IF_1[X_1, ..., X_n] \subset F_1[X_1, ..., X_n]$ is also proper.*

Proof. Let $I \subset F[X_1, ..., X_n]$ be a proper ideal. Then since I is finitely generated, the ideal $I = \langle f_1, ..., f_s \rangle$ for some $f_1, ..., f_s \in F[X_1, ..., X_n]$. By lemma (5.4), the polynomials $f_1, ..., f_s$ have a common zero in F^{ac}. Since we may assume $F^{ac} \subseteq {F_1}^{ac}$,

there is a common zero of $f_1, ..., f_s$ in ${F_1}^{ac}$. So by lemma (5.4) again, $IF_1[X_1, ..., X_n] \neq F_1[X_1, ..., X_n]$. □

Recall that the theory of algebraically closed fields is model complete since it has QE. Next we prove that of being an irreducible variety does not change if we go to an extension.

Lemma 5.6. *Let $F_1 \subset F_2$ be a field extension such that both are algebraically closed. Let V be an irreducible variety in ${F_1}^n$. Then the Zariski closure of V in ${F_2}^n$ (with respect to the Zariski topology on ${F_2}^n$) is an irreducible variety in ${F_2}^n$.*

Proof. Since V is a variety in ${F_1}^n$, there are some polynomials $p_1, ..., p_s$ such that V is the zero set of $p_1, ..., p_s$. Then clearly the Zariski closure of V in ${F_2}^n$ is the zero set of $p_1, ..., p_s$ in ${F_2}^n$. Call this closure $cl(V)$. Thus both V and $cl(V)$ are defined by the formula

$$\phi(x) = \bigwedge_{i \leq s} p_i(x).$$

Now suppose that $cl(V)$ is not irreducible, so there are two proper subvarieties V_1 and V_2 of $cl(V)$ such that $cl(V) = V_1 \cup V_2$. Then since the theory of algebraically closed fields is model complete, we deduce that V is also reducible. □

Corollary 5.7. *Let $F_1 \subset F_2$ be a field extension such that F_1 is algebraically closed. Then I is a prime ideal in $F_1[X_1, ..., X_n]$ if and only if $IF_2[X_1, ..., X_n]$ is a prime ideal in $F_2[X_1, ..., X_n]$.*

Proof. Suppose $I = \langle f_1, ..., f_s \rangle$ is a prime ideal in $F_1[X_1, ..., X_n]$. Let $V = V(I)$ be the variety given by I. Then by Nullstellensatz V is irreducible. So by lemma (5.6), the variety $cl(V)$ is also irreducible in ${F_2}^n$. Again by Nullstellensatz, the ideal $IF_2[X_1, ..., X_n]$ is prime. Conversely, the equality

$$(IF_2[X_1, ..., X_n]) \cap F_1[X_1, ..., X_n] = I$$

follows from a similar proof of lemma (5.4), since we need to solve a system of linear equations. □

The following theorem yields the existence of the constant c_1 in Theorem (5.8).

Theorem 5.8. *[8, 1.11] If $f_0, f_1, ..., f_s$ in $K[X_1, ..., X_n]$ all have degree less than D and f_0 is in $\langle f_1, ..., f_s \rangle$, then $f_0 = \sum_{i=1}^{s} f_i h_i$ for certain h_i whose degrees are bounded by a constant $c_1 = c_1(n, D)$ depending only on n and D.*

The following is also from [8]:

Theorem 5.9. *[8, 2.5] I is a prime ideal in ${}^*K[X_1, ..., X_n]$ if and only if $I^*(K[X_1, ..., X_n])$ is a prime ideal in ${}^*(K[X_1, ..., X_n])$.*

5.1.3 UFD with the p-property

Definition 5.10. *We say that R is a UFD with the p-property if R is a unique factorization domain endowed with an absolute value such that every unit has absolute value 1 and if there are primes p and q satisfying*

$$|p| < 1 < |q|,$$

then there is another prime r non-associated to p with $|r| < 1$.

Examples

- The ring of integers $\mathbb{Z}$ is a UFD with the p-property whose primes have absolute value bigger than 1.
- The p-adic integers $\mathbb{Z}_p$ is a UFD with the p-property whose only prime has absolute value $1/p$.
- Let $\gamma \in (0,1)$ be a transcendental number. Then the ring $S = \mathbb{Z}[\gamma]$ is a unique factorization domain since it is isomorphic to $\mathbb{Z}[X]$ and its units are only 1 and -1. We put the usual absolute value on S. Then S has infinitely many primes p with $|p| < 1$ and infinitely many primes q with $|q| > 1$. So S is a UFD with the p-property.

Lemma 5.11. *Suppose R is a UFD with the p-property. If there are primes p and q with $|p| < 1 < |q|$, then there are infinitely many non-associated primes with absolute value strictly less than 1 and infinitely many non-associated primes with absolute value strictly bigger than 1.*

Proof. We know there are at least two non-associated primes with absolute value less than 1. Let $p_1, ..., p_k$ (for $k \geq 2$) be non-associated primes with absolute value less than 1. Put $A = p_1 ... p_k$. Now choose m large enough such that $\left| \sum_{i=1}^{k} (A/p_i)^m \right| < 1$. Since this element is not a unit, as it does not have absolute value 1, it must be divisible by a prime whose absolute value is strictly less than 1. This yields a new prime. For the second part, given $q_1, ..., q_k$ primes of absolute value larger than 1, for large n the element ${q_1}^n q_2 ... q_k + 1$ provides a new prime that has absolute value greater than 1. □

5.2 Height Bounds

In this section, we will give our main results of this chapter.

Theorem 5.12. *Let R be a ring with a height function of θ-type. For all $n \geq 1$, $D \geq 1$ and $H \geq 1$ there are two constants $c_1(n, D)$ and $c_2(n, D, H, \theta)$ such that if $f_1, ..., f_s$ in $R[X_1, ..., X_n]$ have no common zero in K^{ac} with $\deg(f_i) \leq D$ and $h(f_i) \leq H$, then there exist nonzero a in R and $h_1, ..., h_s$ in $R[X_1, ..., X_n]$ such that*

(i) $a = f_1h_1 + \cdots + f_sh_s$

(ii) $\deg(h_i) \leq c_1$

(iii) $h(a), h(h_i) \leq c_2$

(iv) Furthermore, if R is a UFD with the p-property and $h(x) = |x|$ is the absolute value on R, then we can choose a such that $\gcd(a, a_1, ..., a_m) = 1$ where $a_1, ..., a_m$ are all elements that occur as some coefficient of some h_i.

Remark 5.13. *The constant c_1 does not depend on s because the vector space*

$$V(n, D) = \{f \in K[X_1, ..., X_n] : \deg(f) \leq D\}$$

is finite dimensional over K. In fact the dimension is $q(n, D) = \binom{n+D}{n}$. Given $1 = f_1h_1 + \cdots + f_sh_s$, we may always assume $s \leq q = q(n, D)$ because if $s > q$ then $f_1, ..., f_s \in V(n, D)$ are linearly dependent over K. Assume first that $r \leq q$ many of them are linearly independent. Therefore the other terms $f_{r+1}, ..., f_s$ can be written as a linear combination of $f_1, ..., f_r$ over K. Thus the equation $1 = f_1h_1+\cdots+f_sh_s$ may be transformed into another equation $1 = f_1g_1+\cdots+f_rg_r$. Consequently if $1 \in \langle f_1, ..., f_s\rangle$, then $1 \in \langle f_{i_1}, ..., f_{i_r}\rangle$ where $r \leq q$ and $i_j \in \{1, ..., s\}$. Hence, we can always assume $s = q$. Similarly the constant c_2 does not depend on s. Moreover, none of the constants depend on R.

Remark 5.14. *There is also a direct proof of* (5.12) *as follows: Using the degree bound $B(n, D)$ for the polynomials $g_1, ..., g_s$ in a Bezout expression $1 = f_1g_1 + \cdots + g_sf_s$, we can derive a height bound since the degree bound allows to translate the problem to solving a linear system of equations with precise number of unknowns equations and the height function satisfies some additive and multiplicative properties. However, this computational method is also complicated since the bounds for the height function depend on θ which is implicitly given. Thus in practice, this method is ineffective. For this reason and to show how the problem is related to model theory, we prefer nonstandard methods as in [8].*

Remark 5.15. *If R is a ring with absolute value which has arbitrarily small nonzero elements, then we can multiply both sides of the equation*

$$a = f_1h_1 + \cdots + f_sh_s$$

by some small $\epsilon \in R$. Therefore the height bound c_2 can be taken 1 and the result becomes trivial. Note that (iv) in (5.12) *prevents us from doing this if there are no small units in R. However, if there is a unit u with $|u| < 1$, then multiplying both sides of the equation with powers of u the height can be made small again. So the interesting case is when there are no small units which is equivalent to all the units having absolute value 1. Note also that if $|ab| < 1$ then $|a|$ can be very big and $|b|$ can*

be very small. So cancellation can make the height larger if there are sufficiently small and big elements in the ring. Thus for the equation

$$a = f_1h_1 + \cdots + f_sh_s,$$

simply dividing by $\gcd(a, a_1, ..., a_m)$ *may not work in order to obtain* (iv) *in Theorem* (5.12).

Proof. If $s = 1$ then by Nullstellensatz, f_1 must be a nonzero constant. Thus we may assume that $s \geq 2$ and f_1f_2 is not 0. By Theorem (5.8), the constant c_1 exists and it only depends on n and D. Now we prove the existence of the constant c_2. Our language is the many-sorted language which was given in the subsection (5.1.1). Assume n, D and H are given and there is no bound c_2. Therefore for every $m \geq 1$ there exists an integral domain R_m with a height function ht_m of θ-type and $f_1, ..., f_s$ in $R_m[X_1, ..., X_n]$ with $\deg f_i \leq D$ and $ht_m(f_i) \leq H$ witnessing to this. Thus in the field of fractions K_m of R_m, there exist $g_1, ..., g_s$ in $K_m[X_1, ..., X_m]$ with $\deg g_i \leq c_1$ and

$$1 = f_1g_1 + \cdots + f_sg_s,$$

however for all $h_1, ..., h_s \in K_m[X_1, ..., X_n]$ with $\deg h_i \leq c_1$, the sum

$$1 = f_1h_1 + \cdots + f_sh_s$$

implies $\max_j ht_m(a_j) > m$ where $a_j \in R_m$ is an element that occurs as a numerator or denominator of some h_i. By compactness there is an integral domain R with a height function h_R of θ-type and polynomials $f_1, ..., f_s$ in $R_{fin}[X_1, ..., X_n]$ of degrees less than D such that the linear system

$$f_1Y_1 + \cdots + f_sY_s = 1$$

has a solution in ${}^*K[X_1, ..., X_n]$ but not in $L[X_1, ..., X_n]$, where K is the field of fractions of R and L is field of fractions of R_{fin}. This contradicts remark (5.5) since the ideal $\langle f_1, ..., f_s\rangle$ is proper in $L[X_1, ..., X_n]$.

Hence we know that given $f_1, ..., f_s \in R[X_1, .., X_n]$ with no common zeros in K^{ac} with $\deg(f_i) \leq D$ and $h(f_i) \leq H$, there are $h_1, ..., h_s$ in $K[X_1, ..., X_n]$ such that $1 = f_1h_1+\cdots+f_sh_s$ and $\deg(h_i) \leq c_1(n, D)$. Moreover $s \leq q(n, D)$ and $h(e) \leq c_3(n, D, H, \theta)$ for some c_3, where $e \in R$ is an element which occurs as a numerator or denominator for some coefficient of some h_i. Let $b_1, ..., b_m$ be all the elements in R that occur as a denominator for some coefficient of some h_i. Note that $m = m(n, D) \leq q^2$ depends on n and D only. Also we know that $h(b_i) \leq c_3$. Put

$$a = b_1...b_m.$$

By the multiplicative properties of the height function, we get $h(a) \leq c_4(n, D, H, \theta)$ for some c_4. Now we see that

$$a = \sum_{i=1}^{s} f_i \cdot (ah_i),$$

f_i and ah_i are in $R[X_1, .., X_n]$ and $\deg(ah_i) = \deg(h_i) \leq c_1$. Moreover, again by the multiplicative properties of the height function, we have $h(ah_i) \leq c_5(n, D, H, \theta)$ for some c_5. Now take $c_2 = \max(c_4, c_5)$. Therefore, we obtain (i), (ii) and (iii).

Now we prove (iv). Assume R is a UFD with the p-property. We need to choose a such that $\gcd(a, a_1, ..., a_m) = 1$ where $a_1, ..., a_m$ are all elements that occur as some coefficient of some h_i. If all the primes in R have absolute value bigger than 1 or smaller than 1, then we can divide both sides of the equation

$$a = f_1h_1 + f_2h_2 + \cdots + f_sh_s$$

by $\gcd(a, a_1, ..., a_m)$ and get the result, because if all the primes in R have absolute value bigger than 1, then cancellation makes the height smaller and if all the primes in R have absolute value less than 1 then height is bounded by 1. The remaining case is when there are primes of absolute value bigger than 1 and primes of absolute value smaller than 1. By lemma (5.11), there are infinitely many primes with absolute value strictly less than 1. Now choose a prime p such that $|p| < 1$ and p does not divide a. Let d be the greatest common divisor of all coefficients of f_1 and f_2. Then, the coefficients of f_1/d and f_2/d have no common divisor. On the other hand, since there are both small and large elements in the ring, the element d can be very small and so f_1/d and f_2/d may have very large absolute values. Thus choose a natural number k such that $p^k f_1/d$ and $p^k f_2/d$ have absolute value less than 1. Put $v = c_1(n, D) + 1$. Then we have

$$0 = f_1({X_1}^v p^k f_2/d) + f_2(-{X_1}^v p^k f_1/d).$$

Therefore, we obtain that

$$a = f_1(h_1 + {X_1}^v p^k f_2/d) + f_2(h_2 - {X_1}^v p^k f_1/d) + \cdots + f_sh_s$$

$$= f_1g_1 + f_2g_2 + \cdots + f_sg_s$$

where $\deg g_i \leq D(c_1 + 1) = c(n, D)$ and $h(g_i) \leq c_2$. Observe that

$$\gcd(a, a_1, ..., a_m) = 1$$

where $a_1, ..., a_m$ are all elements that occur as some coefficient of some g_i. □

Let F be a field and I an ideal of $F[X_1, ..., X_n]$. We say that I is a D-type ideal if the degree of all the generators of I is bounded by D. By [8] it is known that there is

a bound $B(n, D)$ such that if I is a D-type ideal then I is prime if and only if $1 \notin I$, and for all f, g in $F[X_1, ..., X_n]$ of degree less than $B(n, D)$, if $fg \in I$ then f or g is in I. Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. We say that an ideal I of $\overline{\mathbb{Q}}[X_1, ..., X_n]$ is a (D, H)-type ideal if it is a D-type ideal and the logarithmic height of all generators of I is bounded by H. Next, we show that it is enough to check the primality up to a certain height bound.

From now on, the function h denotes the logarithmic height function on the set of algebraic numbers $\overline{\mathbb{Q}}$. Set

$$\overline{\mathbb{Q}}_{fin} = \{x \in {}^*\overline{\mathbb{Q}} : h(x) \in \mathbb{R}_{fin}\}.$$

Lemma 5.16. *The ring $\overline{\mathbb{Q}}_{fin}$ is an algebraically closed field.*

Proof. Since the logarithmic height function behaves well under algebraic operations and inverse, we obtain that $\overline{\mathbb{Q}}_{fin}$ is a field. By the height inequality (4.4), we see that $\overline{\mathbb{Q}}_{fin}$ is algebraically closed. □

Theorem 5.17. *Let h be the logarithmic height function. There are bounds $B(n, D)$ and $C(n, D, H)$ such that if I is a (D, H)-type ideal of $\overline{\mathbb{Q}}[X_1, ..., X_n]$ then I is prime if and only if $1 \notin I$, and for all f, g in $\overline{\mathbb{Q}}[X_1, ..., X_n]$ of degree less than $B(n, D)$ and height less than $C(n, D, H)$, if $fg \in I$, then either f or g is in I.*

Proof. First note that if $J = (f_1, ..., f_s)$ is an ideal of D-type then the number of generators of J can be taken less than

$$q = q(n, D) = \dim_K\{f \in K[X_1, ..., X_n] : \deg f \leq D\}.$$

So we can always assume that $s \leq q$. We know the existence of the bound B=$B(n, D)$ by [8]. Now we prove the existence of the bound $C(n, D, H)$. Suppose there is no such bound. This means for all $m > 0$ there is an ideal I_m of (D, H)-type of $\overline{\mathbb{Q}}[X_1, ..., X_n]$ which is not prime such that for all f, g with $\deg f$ and $\deg g$ less than B, and $h(f)$, $h(g)$ less than m, if $fg \in I_m$ then either f or g is in I_m. Then by compactness there is an ideal I of (D, H)-type of ${}^*(\overline{\mathbb{Q}}[X_1, ..., X_n])$ such that the ideal I is not prime but for all $m > 0$, if f, g are of degree less than B and are of height less than m, if $fg \in I$ then either f or g is in I. Now, we see that the ideal I is prime in $\overline{\mathbb{Q}}_{fin}[X_1, ..., X_n]$. However, it is not prime in ${}^*\overline{\mathbb{Q}}[X_1, ..., X_n]$ by (5.9). This contradicts Theorem (5.7) since $\overline{\mathbb{Q}}_{fin}$ is algebraically closed by lemma (5.16).

□

Question: Can we compute $C(n, D, H)$ in (5.17) effectively?

Next, we prove the existence of a height bound similar to the height bound in Theorem (5.17). For the details, we direct the reader to [12, 27]. Let R be a commutative

Noetherian ring with 1 and M be an R-module. For a prime ideal p of R, we say that p is an associated prime of M, if p is the annihilator of some x in M. For an ideal J of R, the associated prime ideals containing J coincides with $Ass_R(R/J)$, which in fact is the set of prime ideals which are the radicals of the primary ideals occurring in the primary decomposition of J. First, we recall the followings facts from commutative algebra.

Remark 5.18.
- *An ideal J is a primary ideal if and only if $Ass_R(R/J)$ has exactly one element.*
- *Every ideal J (through primary decomposition) is expressible as a finite intersection of primary ideals. The radical of each of these ideals is a prime ideal and these primes are exactly the elements of $Ass_R(R/J)$.*
- *Any prime ideal minimal with respect to containing an ideal J is in $Ass_R(R/J)$. These primes are precisely the isolated primes.*

Corollary 5.19. *Let $n \in \mathbb{N}$, $X = (X_1, ..., X_n)$ and I be an ideal of $\overline{\mathbb{Q}}_{fin}[X]$.*

(1) If $p_k, ..., p_m$ are the distinct minimal primes of I then

$$p_1{}^*\overline{\mathbb{Q}}[X], ..., p_m{}^*\overline{\mathbb{Q}}[X]$$

are the distinct minimal primes of $I^*\overline{\mathbb{Q}}[X_1, ..., X_n]$.

(2) $\sqrt{I^*\overline{\mathbb{Q}}[X]} = \sqrt{I}^*\overline{\mathbb{Q}}[X]$.

(3) If M is a $\overline{\mathbb{Q}}_{fin}[X]$-module, then

$$Ass_{{}^*\overline{\mathbb{Q}}[X]}(M \otimes_{\overline{\mathbb{Q}}_{fin}[X]} {}^*\overline{\mathbb{Q}}[X]) = \{p^*\overline{\mathbb{Q}}[X] : p \in Ass_{\overline{\mathbb{Q}}_{fin}[X]}(M)\}.$$

(4) The ideal I is a primary ideal if and only if $I^*\overline{\mathbb{Q}}[X]$ is a primary ideal of ${}^*\overline{\mathbb{Q}}[X]$.

(5) Let $I = I_1 \cap ... \cap I_m$ be a reduced primary decomposition, I_k being a p_k-primary ideal. Then

$$I^*\overline{\mathbb{Q}}[X] = I_1{}^*\overline{\mathbb{Q}}[X] \cap ... \cap I_m{}^*\overline{\mathbb{Q}}[X]$$

is a reduced primary decomposition of $I^*\overline{\mathbb{Q}}[X]$, and $I_k{}^*\overline{\mathbb{Q}}[X]$ is a $p_k{}^*\overline{\mathbb{Q}}[X]$-primary ideal.

Proof. (1) is an immediate consequence of corollary (5.7) and lemma (5.16). (2) follows from (1), since radical of an ideal is the intersection of minimal prime ideals which contain the ideal. Since $\overline{\mathbb{Q}}_{fin}[X]$ is Noetherian, (3) follows from [5, Chapter 4, 2.6, Theorem 2] and remark (5.5). To prove (4), suppose that I is a p-primary ideal. So we get $Ass_{\overline{\mathbb{Q}}_{fin}[X]}(\overline{\mathbb{Q}}_{fin}[X]/I) = \{p\}$. Applying (3) with $M = \overline{\mathbb{Q}}_{fin}[X]/I$, we obtain that $Ass_{{}^*\overline{\mathbb{Q}}[X]}({}^*\overline{\mathbb{Q}}[X]/I) = \{p^*\overline{\mathbb{Q}}[X]\}$ and this yields (4) with the help of remark (5.18). The converse of (4) can be seen by remark (5.5). (5) follows from (4).

□

Now we give the standard corollaries. For the following corollary, the existence of the constant $E(n, D, H)$ is new.

Corollary 5.20. *There are constants $B(n, D)$, $C(n, D)$ and $E(n, D, H)$ such that if I is an ideal of (D, H)-type, then*

(1) $\sqrt{I}$ is generated by polynomials of degree less than B and height less than E, if $f \in \sqrt{I}$ then $f^C \in I$.

(2) There are at most B associated primes of I and each generated by polynomials of degree less than B and height less than E.

(3) I is primary if and only if $1 \notin I$, and for all f, g of degree less than B and height less than E, if $fg \in I$ then $f \in I$ or $g^C \in I$.

(4) There is a reduced primary decomposition of I consisting of at most B primary ideals, each of which is generated by polynomials of degree at most B and height at most E.

Proof. We know the existence of $B(n, D)$ and $C(n, D)$ by [8]. The existence of $E(n, D, H)$ follows from the previous corollary. Proofs are similar to the proof of Theorem (5.17). □

Question: Can we compute $E(n, D, H)$ effectively in corollary (5.20)?

5.3 Concluding Remarks

In this section, we discuss Theorem (5.12) in terms of unique factorization domains, valuations and some arithmetical functions. Also, we give some counter examples for (5.12) for non-height functions.

Definition 5.21. *We say that R is a UFD with the 1-property if R is a unique factorization domain endowed with an absolute value such that every unit has absolute value 1 and there is only one prime p of absolute value less than 1 and infinitely many primes q of absolute value greater than 1.*

Example 5.22. *Let R be a unique factorization domain and p be a prime in R. Put the p-adic absolute value on R with $|p|_p = 1/2$. Let $c > 1$ be any real number. On $R[X]$ we define*

$$|a_0 + a_1 X + \cdots + a_k X^k| = \max_i c^i |a_i|_p.$$

Then $R[X]$ is a UFD with the 1-property whose only small prime is p.

We proved Theorem (5.12) for UFD with the p-property. Thus the remaining case is when R is a UFD with the 1-property. Now we show that Theorem (5.12) is not true for a UFD with the 1-property. The reason behind this is the fact that an element has small absolute value if and only if its p-adic valuation is very large where p is the unique prime of absolute value less than 1.

Remark 5.23. *Let R be a UFD with the 1-property. Then we cannot ensure the correctness of (iii) and (iv) simultaneously in Theorem* (5.12).

Proof. Let p be the unique small prime in R of absolute value less than 1. Let B be an element in R of absolute value very big which is coprime to p. Choose m minimal such that $|p^m B| \leq 1$. Similarly choose k minimal such that $|p^k B| \leq c_2$. Note that as B is very large then so are m and k. Set $f_1 = p^{2m+1} + p^{2m}X$ and $f_2 = p^m B - p^m BX$. Clearly f_1 and f_2 have no common zero since

$$p^{2m}B(p+1) = Bf_1 + p^m f_2$$

and p is not -1. Whenever we write $a = f_1h_1 + f_2h_2$, we observe that p^m divides h_2 and B divides h_1. Also we have that $p^{2m}B$ divides a. Now suppose $|h_i| \leq c_2$ for $i = 1, 2$. Since B divides h_1, we see that p^k divides h_1 since p is the unique small prime in R. Thus p^k divides a, h_1 and h_2. Furthermore we may assume that the only prime divisor of a, h_1 and h_2 is p, because if there is q dividing all of them which is coprime to p, then there is $l \geq k$ such that p^l divides h_1 in order to make the absolute value of h_1 less than c_2. Similar observation shows that p^l also divides h_2 and a. Therefore, in order to satisfy (iv) in Theorem (5.12), we need to divide a, h_1 and h_2 by p^k. So the absolute value of h_1/p^k becomes very large.

□

Definition 5.24. *A valuation v on an integral domain R is a function*

$$v : R \to \Gamma \cup \{\infty\}$$

from R into an ordered abelian group Γ that satisfies the followings:

(i) $v(a) = \infty$ *if and only if* $a = 0$

(ii) $v(xy) = v(x) + v(y)$

(iii) $v(x + y) \geq \min(v(x), v(y))$.

Here ∞ is some element that is bigger than every element in Γ.

For a nonzero polynomial in n-variable we define its valuation as follows:

$$v\left(\sum_{\alpha} a_\alpha X^\alpha\right) = \max_{\alpha}\{v(a_\alpha) : a_\alpha \neq 0\}.$$

Note that this may not be a valuation that satisfies the three conditions above. Take $R = \mathbb{Z}$ and as a valuation we put a p-adic valuation for some prime p. Set $f_1 = 1 + X + (1 - p^m)X^2$ and $f_2 = X^3$ where m is some large integer. Then the valuations of f_1 and f_2 are 0 and clearly they have no common zero in $\mathbb{C}$. One can see that 1 is a linear combination of f_1 and f_2 and so every integer is. However, whenever we write $a = f_1h_1 + f_2h_2$ where a is nonzero, then h_1 must have degree bigger than 2 and the first three coefficients of h_1 are uniquely determined: if $h_1(x) = b_0 + b_1X + b_2X^2 + \cdots + b_kX^k$ then automatically we have $b_0 = a$, $b_1 = -a$ and $b_2 = ap^m$. So the valuation of b_2 can be very large. The main nonstandard reason behind this is the fact that

$$R_{vfin} = \{x \in {}^*R : v(x) \in \mathbb{R}_{fin}\} \cup \{0\}$$

is not a ring, because for nonstandard $N \in {}^*\mathbb{N}$ the elements $p^N - 1$ and 1 is in R_{vfin} but not their sum. Therefore by proposition (5.3), we know that the p-adic valuation on $\mathbb{Z}$ is not a height function.

If we take $g_1 = p^m - 1 + X$ and $g_2 = 1 - X$ then they have no common zero and whenever we write $a = g_1h_1 + g_2h_2$, then h_1 and h_2 must have the same degree and same leading coefficient. This implies that p^m divides a which means that valuation of a can be very big even if the valuations of g_1 and g_2 are 0.

A valuation is called trivial if for all nonzero x we have $v(x) = 0$. We say that a valuation is a height function if the set R_{vfin} is a subring. In fact we can determine when a valuation is a height function.

Remark 5.25. *A valuation v on R is a height function if and only if it is trivial.*

Proof. If the valuation is trivial then clearly it is a height function. Conversely is v is not trivial, then it is unbounded. So by saturation there is an element a in ${}^*R$ whose valuation is infinite. Then

$$v(a - 1) = 0$$

because if two elements have different valuation then the valuation of their sum is the minimum of their valuations. So the elements $a - 1$ and 1 are in R_{vfin}, but not their sum. □

Now we discuss some arithmetical functions and which of them are height functions and relate them to Theorem (5.12).

Definition 5.26. *A function $g : \{1, 2, 3, ...\} \to \mathbb{C}$ is called an arithmetical function.*

Every arithmetical function g extends to $\mathbb{Z}$ by defining $g(n) = g(-n)$ and $g(0) = 0$. Such a function is called an arithmetical function on $\mathbb{Z}$. Similarly for an arithmetical

function g on $\mathbb{Z}$, we extend it to $\mathbb{Z}[X]$ by

$$g(a_0 + a_1X + \cdots + a_kX^k) = \max_i g(a_i).$$

Let ${}^*\mathbb{Z}$ be a proper nonstandard extension of $\mathbb{Z}$. Note that

$$\mathbb{Z}_{fin} = \{x \in {}^*\mathbb{Z} : |x| < n \text{ for some } n \in \mathbb{N}\} = \mathbb{Z}.$$

For an arithmetical function g, we define

$$\mathbb{Z}_{gfin} = \{x \in {}^*\mathbb{Z} : |g(x)| < n \text{ for some } n \in \mathbb{N}\}.$$

By proposition (5.3), observe that $|g|$ is a height function if and only if $\mathbb{Z}_{gfin}$ is a subring. Now we give some examples of arithmetical functions.

Example 5.27.

- $\varphi(n) = |\{1 \leq k \leq n : (k, n) = 1\}|$
- $d(n) =$ *number of divisors of* n
- $\omega(n) =$ *number of distinct prime factors of* n.

Lemma 5.28. *Let g be an arithmetical function and assume that*

$$\lim_{n\to\infty} g(n) = \infty.$$

Then $|g|$ is a height function.

Proof. If N is an infinite number in ${}^*\mathbb{Z}$ then $g(N)$ is also infinite. This shows that $\mathbb{Z}_{gfin} = \mathbb{Z}_{fin} = \mathbb{Z}$ which is a subring of ${}^*\mathbb{Z}$. Hence by proposition (5.3), the function $|g|$ is a height function on $\mathbb{Z}$. □

Lemma 5.29. *For all $n \geq 1$, we have $\frac{\sqrt{n}}{2} \leq \varphi(n)$.*

Proof. Since $\varphi(n) = \prod_{p|n} n(1 - \frac{1}{p})$, we get $\varphi(n) \geq \frac{n}{2^{\omega(n)}} \geq \frac{n}{d(n)}$. Finally since $d(n) \leq 2\sqrt{n}$, we get the result. □

Corollary 5.30. *The function $\varphi(n)$ is a height function.*

Proof. Owing to the inequality $\frac{\sqrt{n}}{2} \leq \varphi(n)$ and lemma (5.28), we conclude the corollary. □

Fact: Every sufficiently large odd integer can be written as a sum of three primes. This was proved by I. M. Vinogradov. For more about this theorem, we direct the reader to [7].

Lemma 5.31. *The divisor function $d(n)$ is not a height function.*

Proof. By three primes theorem and compactness, there is an odd infinite N in ${}^*\mathbb{Z}$ which can be written as a sum of three primes in ${}^*\mathbb{P}$ where $\mathbb{P}$ is the set of all primes. Furthermore we can choose N such that $\omega(N)$ is infinite. This shows that the set $\mathbb{Z}_{dfin}$ is not closed under addition. So by proposition (5.3), it cannot be a height function on $\mathbb{Z}$. □

The next corollary is also true for the function $\omega(n)$. For simplicity, we just give the proofs for the divisor function.

Corollary 5.32. *There exist a natural number A and two sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{N}$ such that $d(a_n) \leq A$ and $d(b_n) \leq A$ but*

$$\lim_{n\to\infty} d(a_n + b_n) = \infty.$$

Remark 5.33. *The result* (5.12) *is not true for the divisor function $d(n)$.*

Proof. Set $f_1 = a_n + X + {b_n}^2 X^2$ and $f_2 = X^3$ where a_n and b_n are as in (5.32). Then $d(f_1)$ and $d(f_2)$ are bounden by A^2 and they have no common zero in $\mathbb{C}$. However, whenever we write $a = f_1 h_1 + f_2 h_2$ where a is nonzero, then h_1 must have degree bigger than 2 and the first three coefficients of h_1 are uniquely determined: if $h_1(x) = c_0 + c_1 X + c_2 X^2 + \cdots + c_k X^k$ then automatically we have $c_0 = a$, $c_1 = -a_n a$ and $c_2 = a(a_n - b_n)(a_n + b_n)$. Hence $d(c_2)$ can be very large. Moreover if we put $g_1 = a_n + X$ and $g_2 = b_n - X$ then they have no common zero. However, whenever we write $a = g_1 h_1 + g_2 h_2$, then $d(a) \geq d(a_n + b_n)$. Thus a has many divisors although $d(g_1)$ and $d(g_2)$ are bounded by A.

□

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