THE EFFECT OF DISTRIBUTION OF INFORMATION ON RECOVERY OF PROPAGATING SIGNALS

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR

THE DEGREE OF

MASTER OF SCIENCE

IN

ELECTRICAL AND ELECTRONICS ENGINEERING

By

Özgecan Karabulut

September, 2015

THE EFFECT OF DISTRIBUTION OF INFORMATION ON RECOVERY OF PROPAGATING SIGNALS
By Özgecan Karabulut
September, 2015

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Haldun M. Özaktaş (Advisor)

Assoc. Prof. Dr. Sinan Gezici

Assist. Prof. Dr. Sevinç Figen Öktem

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

THE EFFECT OF DISTRIBUTION OF INFORMATION ON RECOVERY OF PROPAGATING SIGNALS

Özgecan Karabulut
Ms. in Electrical and Electronics Engineering
Advisor: Prof. Dr. Haldun M. Özaktaş
September, 2015

Interpolation is one of the fundamental concepts in signal processing. The analysis of the difficulty of interpolation of propagating waves is the subject of this thesis. It is known that the information contained in a propagating wave field can be fully described by its uniform samples taken on a planar surface transversal to the propagation direction, so the field can be found anywhere in space by using the wave propagation equations. However in some cases, the sample locations may be irregular and/or nonuniform. We are concerned with interpolation from such samples. To be able to reduce the problem to a pure mathematical form, the fractional Fourier transform is used thanks to the direct analogy between wave propagation and fractional Fourier transformation. The linear relationship between each sample and the unknown field distribution is established this way. These relationships, which constitute a signal recovery problem based on multiple partial fractional Fourier transform information, are analyzed. Recoverability of the field is examined by comparing the condition numbers of the constructed matrices corresponding to different distributions of the available samples.

Keywords: fractional Fourier transform, wave propagation, signal recovery, sampling, nonuniform sampling, interpolation, partial information, redundancy, condition number.

ÖZET

BİLGİ DAĞILIMININ YAYILAN SİNYALLERİN GERİ ÇATILMASI ÜSTÜNDEKİ ETKİSİ

Özgecan Karabulut
Elektrik Elektronik Mühendisliği, Master
Tez Danışmanı: Prof. Dr. Haldun M. Özaktaş
Eylül 2015

Aradeğer bulma, sinyal işlemenin temel kavramlarındandır. Bu tezin konusu, yayılan dalgaların aradeğerlerinin bulunmasındaki zorluğun incelenmesidir. Yayılan bir dalganın içerdiği bilginin, yayılım eksenine dik düzlemsel bir yüzeyde alınan örneklemlerle tamamen karakterize edilebildiği ve böylece yayılan alanın, dalga denklemleri kullanılarak bütün uzayda bulunabileceği biliniyor. Fakat bazı durumlarda, örneklemler düzensiz ve/veya düzgün olmayan şekilde alınmış olabilir. Biz bu gibi örneklemler kullanılarak oluşturulan aradeğer bulma problemleriyle ilgileniyoruz. Bu çalışmada, aradeğer bulma problemlerini tamamen matematiksel bir biçimde ifade edebilmek için, dalga yayılımı ile olan direk ilişkisinden faydalanılarak, kesirli Fourier dönüşümü kullanıldı. Her bir örneklem ve bilinmeyen yayılan alan dağılımı arasındaki doğrusal ilişki bu şekilde kuruldu. Birçok kısmi kesirli Fourier dönüşüm bilgisi kullanarak ara değer bulma problemini oluşturan bu doğrusal ilişkiler incelendi. Sinyallerin geri çatılma yetisi, verilen örneklemlerin farklı dağılımlarına karşılık gelecek şekilde oluşturulan matrislerin kararsızlık oranları karşılaştırılarak incelendi.

Anahtar sözcükler: kesirli Fourier dönüşümü, dalganın yayılımı, sinyalin geri çatılması, örnekleme, düzgün olmayan örnekleme, aradeğer bulma, kısmi bilgi, artıklık, kararsızlık oranı.

Acknowledgement

I would like to express my gratitude to my supervisor, Prof. Dr. Haldun M. Özaktaş for his inestimable guidance and understanding throughout my MS studies. I am also thankful for his excellent teaching which helped me a lot to comprehend these conceptually challenging topics.

I am grateful to the members of my thesis committee, Assoc. Prof. Dr. Sinan Gezici and Asst. Prof. S. Figen Öktem for reading and commenting on this thesis. I would like to express my appreciation and thanks to Asst. Prof. S. Figen Öktem for the informative discussions.

I am thankful to Department of Electrical and Electronics Engineering at Bilkent University for the great educational environment and to TUBITAK, The Scientific and Technological Research Council of Turkey, for providing financial support during my MS studies.

I would like to thank Roketsan for the encouragement to finish my thesis.

Finally, I would like to thank to my dear family who always supported me at the cost of being apart for long years. Their trust and love always gave me courage and strength to struggle with all the challenges in my life. Last but not least, I want to thank to my friends İsmail and Erdem for all the support, motivation, long study hours and most importantly for being the essentials of the journey.

Contents

1 Introduction **1**

2 Background **5**

2.1 Sampling Theorem 5

2.2 Fractional Fourier Transform 8

2.2.1 FRT and Wave Propagation Relation 9

2.2.2 Fractional Fourier Domains 10

2.2.3 The Discrete Fractional Fourier Transform 12

2.2.4 Sampling in Fractional Fourier Transform Domain 13

2.3 Condition Number 14

3 Analysis of The Signal Recovery Problem Under Arbitrarily Distributed Information **16**

3.1 Basic Assumptions and Dimensional Normalizations 16

3.2 Methodology and Problem Setup 18

3.3 Simulation Results 25

3.3.1 Lines with Varying Slope 26

3.3.2 Lines with Varying Length 30

3.3.3 Line with Varying Locations 36

3.3.4 Samples on Two Planes with Different Distributions 40

3.3.5 Circular Sampling Surface 47

3.3.6 Sample Distributed All Over the u-a Coordinate System . 49

4 Discussion **55**

4.1 Why the horizontal line is unsolvable? 55

4.2 Examination of Different Possible Methodologies 56

4.2.1 A Problem Definition Based on Ordinary Interpolation Equation 57

4.2.2 Problem Definition Based on Interpolation Equation Written in FRT Domain 61

4.2.3 Problem Definition Based on LCT Interpolation Equation 62

4.2.4 Using Discrete FRT Orders 67

4.3 Physical Correspondence of the Problem 67

4.3.1 Graded Index Medium 67

4.3.2 Free Space Propagation 68

4.3.3 Quadratic Phase Systems 69

5 Conclusion and Future Work **71**

5.1 Conclusion 71

5.2 Future Work 74

List of Figures

2.1 Phase space and the ath order FRT domain 11

2.2 The u-a coordinate system and the FRT domains 12

3.1 The u-a coordinate system and sample locations 19

3.2 The cones originating from the unknown samples on reference domain 23

3.3 Horizontally sheared line sample locations 26

3.4 Condition number of horizontally sheared line 28

3.5 Rank of horizontally sheared line 28

3.6 Condition number of horizontally sheared line with higher resolution 29

3.7 Vertically sheared line sample locations 29

3.8 Condition number of vertically sheared line 31

3.9 Rank of vertically sheared line 31

3.10 Vertically sheared line rank with increased tolerance 32

3.11 Changing length horizontal line sample locations 33

3.12 Condition number of changing length horizontal lines 33

3.13 Rank of changing length horizontal lines 34

3.14 Changing length tilted line sample locations 35

3.15 Condition number of changing length tilted line 35

3.16 Rank of changing length tilted line 36

3.17 Changing location horizontal line sample locations 37

3.18 Condition number of changing location horizontal line 38

3.19 Rank of changing location horizontal line 38

3.20 Changing location vertical line sample locations 39

3.21 Condition number of changing location vertical line 39

3.22 Rank of changing location vertical line 40

3.23 Complementary sample locations for N=16 42

3.24 Different depths 43

3.25 Condition number of complementary planes 43

3.26 Rank of complementary planes versus depth 44

3.27 Rank of complementary planes versus period 44

3.28 Overlapping sample locations for N=16 45

3.29 Condition number of overlapping planes 46

3.30 Rank of overlapping planes 46

3.31 Rank of overlapping planes versus period 46

3.32 Circular Surface Sample Locations 47

3.33 Condition of circular sampling surface 48

3.34 Rank of circular sampling surface 48

3.35 Randomly distributed sample locations 50

3.36 Condition number versus the number of trials 50

3.37 Rank versus the number of trials 51

3.38 Condition number versus the number of trials 51

3.39 Shifted up uniform grid 54

3.40 Shifted down uniform grid 54

3.41 Uniform grid with one irregular point 54

3.42 Systematically irregular samples 54

3.43 Irregular grid with upper-half shifted up and lower-half shifted down 54

3.44 Irregular grid with upper-half shifted down and lower-half shifted up 54

4.1 Condition number versus the inverse slope of the horizontally sheared line by using method in section 4.2.1 60

4.2 Condition number versus the inverse slope of the horizontally sheared line by using method in section 4.2.2 62

5.1 The u-a coordinate system and sample locations for 2D case . . . 74

List of Tables

3.1 The dFrFT matrix for $a = 0.01$ 22

3.2 The dFrFT matrix for $a = 1$ 22

3.3 Known samples versus the unknown samples that affect them . . 24

3.4 Samples taken on the rectangular grid with different configurations 53

Chapter 1

Introduction

Consider a finite energy signal with most of its energy confined to a particular region in the phase space. The area of this region gives us the number of degrees of freedom of the signal. At least this many samples are necessary to uniquely characterize the signal. Nyquist theorem tells us how to choose those samples in order to recover the signal completely. However, in the case of wave propagation, it is possible to gather information of the propagating wave field both along and transversal to the propagation direction. Yet, the field is fully described on a transversal planar surface such that the knowledge of the propagating wave field distribution on a transverse plane along the optical axis is adequate to propagate it through any medium. Therefore, knowing the field on one transversal plane is equivalent to knowing the field on any point in space. Let $\hat{f}(x)$ and $\hat{g}(x)$ represent the input and output plane distribution of a one-dimensional propagation problem respectively. Then, the relationship between $\hat{f}(x)$ and $\hat{g}(x)$ is described by the Fresnel integral [1], given in eq. 1.1:

$$\hat{g}(x) = e^{i2\pi d/\lambda} e^{-i\pi/4} \sqrt{\frac{1}{\lambda d}} \int_{-\infty}^{\infty} e^{\frac{i\pi(x-x')^2}{\lambda d}} \hat{f}(x') dx' \tag{1.1}$$

where λ is the wavelength and d is the propagation distance. Please note that we will refer to propagating wave field as optical field for simplicity in the rest of the discussion, although our analysis is valid for all the waves.

In some applications, on the other hand, the optical field data is not available on a planar surface because of either the definition of the problem or the configuration of the optical system. For example, the sample locations might be randomly or irregularly selected, as it might be the case for some array of sensors distributed over the space or as in swarm robot applications which uses reconstructed random point measurements of the field [2]. Also, it might not be possible to take all the samples on a planar surface due to the existence of a blocking object such as an optical instrument on the way, which brings the need of obtaining information over different planes and reconstructing the original data. In addition, the measurement device might not have the desired resolution and/or accuracy. As a solution to this problem, multiple measurements can be combined to succeed a higher resolution and/or accuracy. Obtaining a higher resolution diffraction patterns from low resolution ones by using Wigner transform, Canny edge detection and Hough transform is discussed in [3].

Another scenario is where samples are positioned over a curved surface as encountered in computer generated holography and three-dimensional television applications [4–12]. The intuitive solution of this problem is considering each sample on the curved surface as a source and computing the diffraction field arising from those samples and superpose the fields emitted by them. However, finding the diffraction field by direct superposition is not correct which is a often ignored fact [13–15]. This "source model" method does not take mutual interactions into account which can be significant. A comparison between "source model" method and "field method", which is defined as a method taking the mutual interactions into account, is included in [7], and a performance assessment of the "source model" approach can be found in [16]. As a solution to this problem, expressing the field on a curved surface in terms of that on a planar surface and solving the inverse problem where the aim is calculating the scalar diffraction field over the entire space is proposed in [7].

In some studies, the problem is reduced to the diffraction field calculation between tilted planes. In [17] the problem is analyzed as Fraunhofer approximated diffraction problem due to a plane wave passing through a tilted slit. In [18], the relationship between the rotated planes' plane wave spectrum is established. A

method of finding the exact diffraction field by using Rayleigh-Sommerfeld scalar diffraction integral between tilted and translated planes described in [19–21], which is based on the algorithm proposed in [22]. Satisfactory results up to 60° tilt angle are reported. It should be noted the problems mentioned so far are the subset of our problem definition. We do not restrict our samples to be on a specific surface or force the partially distributed information to be on multiple transversal planes. Any distribution of the samples or arbitrary sampling surface can be the subject of this work.

In this work, Fractional Fourier Transform is used to formulate the problem and bring it into the signal processing context. Fractional Fourier Transform (FRT) creates a smooth transition between original signal and its Fourier transform (FT). It has found many applications in optics. It is well known that Fourier transform of a field distribution is observed at the far field. FRT on the other hand can be observed at closer distances. Mathematically, Fresnel integral can be rewritten as a FRT with an additional magnification and chirp multiplication terms. In the light of this analogy between wave propagation and FRT, using partial FRT information to recover the signal has been studied [2, 23–27]. This partial FRT information can be obtained by taking samples of the field distribution on different fractional Fourier domains. These domains can be represented by axes rotated by angle $\alpha = a\pi/2$ from space axis in phase space representation where a is the FRT order [28]. Physically, they correspond to different transversal surfaces along the direction of propagation. Therefore it is possible to set the relationship between the sample locations and FRT orders.

In [2], POCS method is used to recover the signal from partial FRT information. In [23, 24], the problem of recovering the complex signal by using the magnitude information given in different FRT domains is studied. It has been shown that, the choice of the number and the location of FRT domains effect the solution such that higher number of orders ends up with better results. In addition, the adequate separation between the FRT domains leads to an improvement in the results. In [25, 26], the samples taken on two different FRT domains with different distributions are examined and the linear relationships between them are established.

In this thesis, the sampling surfaces are considered as sections of different order FRT domains. By this way, the relationship between the samples on any arbitrary surface can be written by using FRT properties. The main objective of this work is analyzing the linear relationships between high number of fractional Fourier domain samples under Fresnel approximation and investigating the effect of sample locations on the recoverability of the signal.

In Chapter 2, a brief background information on FRT and sampling theorem is provided. The condition number concept which is used as a measure to assess the recoverability of the signals is explained. In Chapter 3, the linear relationship between high number of FRT domains is discussed. The methodology is established and problem is formulated in a general content. Then, different scenarios have been implemented and the influence of the sample distribution on the condition number is analyzed. In Chapter 4 we discussed possible other methodologies. Then, we explain the physical correspondence of the mathematical frame used to express the problem. We conclude this thesis with Chapter 5.

Chapter 2

Background

In this chapter, the subjects that are used in this thesis are summarized in a general content. Firstly, sampling and interpolation equations are given as it is defined in Shannon's work. A more generic form of the interpolation equation is also included. Nonuniform sampling is mentioned as a concept. Then fractional Fourier transform, which is used to formulate our problem, is defined and its basic properties and relation to wave propagation is given. The fractional Fourier domain concept is explained and its correspondence on time/space-frequency space is discussed. It is followed by the formulation of sampling theorem in fractional Fourier domain. Finally, we give the definition of condition number and its characteristics in the case of an ill-posed system.

2.1 Sampling Theorem

Sampling theorem is stated by Shannon as: "If a function contains no frequencies higher than W cycles per second, it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart." as a method of converting an analog signal into a digital one in the process of formulating his rate/distortion theory in his paper titled "Communication in the Presence of Noise" which can be

considered as the basis of the information theory [29]. This theorem can be justified by the following steps. First, we can represent any signal as a superposion of harmonics which is known as inverse FT:

$$f(u) = \int_{-\infty}^{\infty} F(\mu) e^{i2\pi\mu u} d\mu = \int_{-2W}^{2W} F(\mu) e^{i2\pi\mu u} d\mu \tag{2.1}$$

where $F(\mu)$ is zero outside of the bandwidth $2W$. When the continuous function f is evaluated at the sample locations $u = \frac{n}{2W}$ where n is an integer, the equation below is obtained:

$$f\left(\frac{n}{2W}\right) = \int_{-2W}^{2W} F(\mu) e^{i2\pi\mu \frac{n}{2W}} d\mu \tag{2.2}$$

The equation above is nth coefficient of the Fourier series expansion of $F(\mu)$. In other words, the samples obtained by evaluating the function f at locations $u = \frac{n}{2W}$ are equal to the Fourier coefficients of $F(\mu)$. Since the initial assumption is that the signal is limited in the frequency domain such that the highest frequency is W, the samples of the signal determine $F(\mu)$ which is the frequency domain representation of the abstract signal f. Obtaining the time domain representation can be accomplished by a transformation. Therefore, the samples of the function contain all the necessary information to define the signal completely. Although formalization and interpretation of this result is done by Shannon, similar statements are made before by Whittaker and Korel'nikov [30, 31].

The equation to reconstruct the signal from its samples is given as:

$$f(u) = \sum_{n \in Z} f(nT) \mathrm{sinc}(u/T - n) \tag{2.3}$$

where n is an integer and T is the sampling period. If we consider the shifted sinc functions as the basis functions, the interpretation of this equation might be that the samples $f(nT)$ are the representation of the function $f(u)$ in that space. This equation is exact if $f(u)$ is bandlimited such that $\mu_{\max} \leq \frac{1}{2T}$ where the limit of maximum frequency can be recognized as the Nyquist frequency [32].

Sampling theorem is one of the keystones in signal processing and communications. However, in practical uses, Shannon's equation is limited since it is based on the assumption that the signal is bandlimited which is not possible for real signals and images [33]. It is also noted that, using Shannon's reconstruction

equation might be problematic due to the lack of ideal anti-aliasing filters and the slow decay of sinc functions, especially while working with images [34]. These are the basic reasons which leads to the more generic and extended model:

$$V(\varphi) = \left\{ s(u) = \sum_{n \in Z} c(n)\varphi(u-n) : c \in l_2 \right\} \tag{2.4}$$

where sinc function is replaced by *generic function* : φ in space V [35]. This equation means that any continuous function $s(u) \in V(\varphi)$ can be fully represented by the coefficient set $c(n)$, which does not have to be the samples of the signal. Different extensions and variations of sampling theory can be listed as: wavelets, multichannel sampling, finite elements and multi-wavelets, frames and irregular sampling. These topics are beyond the scope of this thesis. A more detailed information about the history and applications of interpolation and sampling can be found at [36,37]. Irregular or nonuniform sampling will be briefly explain next.

Although the base of the analog to digital signal conversion and reconstruction is taking uniformly spaced samples, different sampling schemes might be a necessity depending on the application. For instance, polar sampling grid is used in computerized tomography whereas for fast MRI, spiral sampling grids are more convenient. Irregularly spaced sampling on the other hand has found applications in spectroscopy, geophysics, astronomy and other signal and image processing problems [38].

Fitting a model to the measurements is proposed to get rid of the complications arising from nonuniform sampling. An example problem where the aim is applying Lagrange interpolation to reconstruct the periodically irregularly spaced samples can be found in [39]. Another method is changing the basis functions so that it will be proper to the nonuniform structure of the problem. Although the foundation of our problem is an extended version of nonuniform sampling, where we do not put any restrictions on the sample locations, we will not be directly using its procedures. Therefore, we do not go much more into the details of nonuniform sampling.

2.2 Fractional Fourier Transform

Fractional Fourier transform (FRT), which is a special case of linear canonical transforms, can be considered as a generalized version of the ordinary Fourier transform (FT). It has a variety of application areas in optics and signal processing [40–64].

There are different definitions of FRT which leads to different physical interpretations. The integral based one is given as:

$$\begin{aligned} f_a(u) &= \int_{-\infty}^{\infty} K_a(u,u')f(u')du', \\ K_a(u,u') &= A_\alpha exp\left[i\pi(\cot\alpha u^2 - 2\csc\alpha uu' + \cot\alpha u'^2)\right], \\ A_\alpha &= \sqrt{1-i\cot\alpha}, \alpha \equiv \frac{a\pi}{2} \end{aligned} \tag{2.5}$$

[28], when $a \neq 2k$ for integer k. The FRT kernel $K_a(u,u')$ has the following spectral expansion [65]:

$$K_a(u,u') = \sum_{k=0}^{\infty} \psi_k(u) e^{-i\frac{\pi}{2}ka} \psi_k(u') \tag{2.6}$$

In eq. 2.6, $\psi_k(u)$ denotes the kth Hermite-Gaussian function, and $e^{-i\pi ka/2}$ is the ath power of the eigenvalue of the ordinary Fourier Transform: $\lambda_k = e^{-i\pi k/2}$.
It is straightforward to see that the kernel simplifies to $K_a(u,u') = \delta(u-u')$ when $a = 4k$ and to $\delta(u+u')$ when $a = 4k+2$. It can be shown that the kernel is equal to $e^{-i2\pi uu'}$ when $a = 1$ so that FRT reduces to ordinary FT. Based on these reductions, the following identities can be written for special cases:

$$\begin{aligned} \mathcal{F}^0 &= \mathcal{I}, \\ \mathcal{F}^1 &= \mathcal{F}, \\ \mathcal{F}^2 &= \mathcal{P}, \\ \mathcal{F}^3 &= \mathcal{F}\mathcal{P} = \mathcal{P}\mathcal{F}, \\ \mathcal{F}^4 &= \mathcal{F}^0 = \mathcal{I}, \\ \mathcal{F}^{4k+a} &= \mathcal{F}^{4k'+a} \end{aligned} \tag{2.7}$$

for arbitrary integers k, k' where $\mathcal{I}$ is the identity, $\mathcal{P}$ is the parity, F is the FT and $\mathcal{F}^a$ is the ath order FRT operators. For the other values of a, the FRT operation

can considered as the ath power of ordinary FT operation [66]. In the procedure of deriving FRT by taking the operator power of FT eigenvalue equation, different definitions are obtained based on the choice of the eigenfunctions or the fractional powers of the eigenfunctions [67–84].

In this thesis, FRT will be denoted by $f_a(u)$ or it will be treated as an operator and denoted by $\mathcal{F}^a f$. In some content, to avoid confusion, the ath order FRT domain argument will be represented by a subscript: u_a. For more information about FRT, the reader is referred to [28, 66, 85–87].

2.2.1 FRT and Wave Propagation Relation

The relationship between FRT and wave propagation is well established [88–90]. It is well known that FT of a field distribution is observed at the far field. FRT on the other hand is observed at closer distances such that the free space propagation can be thought as continuous fractional Fourier transformation along the direction of propagation. To be able to show this relationship mathematically, consider an input output optical field pair $\hat{f}(x)$ and $\hat{g}(x)$ which is related to each other through Fresnel integral as in eq. 1.1. Since Fresnel integral can be decomposed into a FRT, followed by magnification, and followed by chirp multiplication, the output field $\hat{g}(x)$ can be expressed as [91, 92]:

$$\hat{g}(x) = e^{i2\pi d\lambda} e^{-ia\pi/4} \sqrt{\frac{1}{sM}}\, e^{\frac{i\pi x^2}{\lambda R}}\, f_a\left(\frac{x}{sM}\right) \tag{2.8}$$

where s is the scaling parameter to convert $\hat{f}(x)$ into dimensionless version $f(u)$ such that $f(x) = \frac{1}{\sqrt{s}} f(x/s) = \frac{1}{\sqrt{s}} f(u)$. The other parameters are defined as:

$$\begin{aligned} a =& \frac{2}{\pi} \arctan \frac{\lambda d}{s^2}, \\ M =& \sqrt{1 + \frac{\lambda^2 d^2}{s^4}} = \sec \frac{a\pi}{2}, \\ R =& \frac{s^4 + \lambda^2 d^2}{\lambda^2 d} = \csc^2 \frac{a\pi}{2}. \end{aligned} \tag{2.9}$$

The relationship between the order a and the propagation distance is given in eq. 2.9 as $a = \frac{2}{\pi} \arctan\left(\frac{\lambda d}{s^2}\right)$. As an alternative formulation, propagation between two

spherical surfaces in terms of fractional Fourier transformation can be found in [88], and the relation between the FRT order and the Gouy phase shift between these two surfaces is given in [89]. In quadratic graded index media on the other hand, the FRT order increases linearly with propagation distance.

This direct relation between wave propagation and FRT creates the link between wave propagation problems where the measurements are partial, spread over several observation planes, have insufficient accuracy and/or resolution etc. and the partial FRT information related recovery problem analysis.

2.2.2 Fractional Fourier Domains

Signals can be considerd as information bearing entities, so as an abstact concept. The representation of these signals in different domains are possible. We are familiar with time, space and frequency domains. However, the signals can also be represented in time/space-frequency domain which is also called phase space. Phase space is represented by a coordinate system where the horizontal axis is for time/space with argument u, and the vertical axis is for frequency with argument μ. Speaking of phase space, the definition of the Wigner distribution, which is a time/space-frequency representation, is included to consolidate the understanding of FRT domains in eq. 2.10.

$$W_f(u,\mu) \equiv \int f(u+u'/2)f^*(u-u'/2)e^{-i2\pi\mu u'}du' \tag{2.10}$$

Wigner distribution gives an intuition about the spread of the signal's energy in phase space under the limits of uncertainty principle. Some important properties of the Wigner distribution are given in 2.11. They suggest that the projection of $W_f(u,\mu)$ onto the time/space axis is equal to $|f(u)|^2$ and the projection onto the frequency axis is given by $|F(\mu)|^2$.

$$\begin{aligned} \int W_f(u,\mu)d\mu =& |f(u)|^2, \\ \int W_f(u,\mu)du =& |F(\mu)|^2, \\ \int\int W_f(u,\mu)dud\mu =& ||f(u)||^2 = \mathrm{En}[f] \end{aligned} \tag{2.11}$$

Figure 2.1: Phase space and the ath order FRT domain

Another important property of the Wigner distribution is its relation to FRT. The ath order FRT corresponds to rotation in the phase space. It can be formulated by the rotation of the Wigner distribution of the signal in the clockwise direction by angle $\alpha = a\pi/2$ as in eq. 2.12 [43,93]. Then, the FRT domain can be represented by the axis making the angle $\alpha = a\pi/2$ with u axis as seen in figure 2.1.

$$W_{f_a}(u,\mu) \equiv W_f(u\cos\alpha - \mu\sin\alpha, u\sin\alpha + \mu\cos\alpha) \tag{2.12}$$

The physical interpretation of the FRT domain concept is based on the FRT and wave propagation relation. It is known that FRT order $a = 0$ corresponds to the reference plane and $a = 1$ to the Fourier plane which occurs at distance infinity. As explained in the previous section, the signal is fractional Fourier transformed as it propagates between these planes. Therefore, the transversal planes between those two planes can be considered as different order FRT domains. Since we worked on one-dimensional wave propagation problems, the transversal planes are represented by vertical lines as seen in eq. 2.2. Although the domains demonstrated in the figure are equally spaced, there is not such an obligation. In fact, the FRT order a is a continuous argument. Therefore, a different domain

Figure 2.2: The u-a coordinate system and the FRT domains

exists passing through each a coordinate.

2.2.3 The Discrete Fractional Fourier Transform

Discrete fractional Fourier transform is defined such that it generalizes the discrete Fourier transform (DFT) in the same sense that the continuous fractional Fourier transform generalizes the continuous ordinary Fourier transform [94–99]. Some definitions of discrete fractional Fourier transform matrix (dFrFT) can be found in [100–104]. These dFrFT matrices map the samples of a reference signal to the same point samples of its FRT.

In this thesis, a unitary and index additive the discrete FRT definition which approximates the continuous FRT and reduces to ordinary DFT when the FRT order is equal to 1 is discussed. This definition is based on the discrete analog of the spectral expansion of the FRT kernel, which is:

$$F^a[m,n] = \sum_{k=0}^{N-1} u_k[m](\lambda_k)^a u_k[n] \tag{2.13}$$

where a is the FRT order, $u_k[n]$ denotes the discrete Hermite-Gaussians and λ_k

is the associated eigenvalue [105].

A different definition for all linear canonical transforms (LCT) is examined in [106]. The definition of LCT is given as:

$$\begin{aligned} \hat{f}_{\mathbf{M}}(u) \equiv & (\mathcal{C}_{\mathbf{M}} f)(u) \equiv \int_{-\infty}^{\infty} C_{\mathbf{M}}(u,u') f(u') du', \\ C_{\mathbf{M}}(u,u') \equiv & \sqrt{\beta} e^{-i\pi/4} e^{i\pi(\alpha u^2 - 2\beta u u' + \gamma u'^2)} \end{aligned} \tag{2.14}$$

where $\mathbf{M}$, α, γ and β are LCT parameters and $\mathcal{C}_{\mathbf{M}}$ is the LCT operator. Then, the relationship between continuous and discrete LCT is given as:

$$\begin{aligned} \hat{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}}) \equiv & \delta u \sum_{k=-N/2}^{N/2-1} \hat{f}(k\delta u) C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) \\ C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) = & \sqrt{\beta} e^{-\frac{i\pi}{4}} e^{\frac{i\pi}{N|\beta|}\left(\alpha \frac{\delta u_{\mathbf{M}}}{\delta u} m^2 - 2\beta k m + \gamma \frac{\delta u}{\delta u_{\mathbf{M}}} k^2\right)} \end{aligned} \tag{2.15}$$

where $\hat{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}})$ is the discrete LCT expression of $\hat{f}(k\delta u)$, δu and $\delta u_{\mathbf{M}}$ are the sampling intervals in reference and LCT plane correspondingly. The sampling intervals in reference plane and LCT domains are related by $\delta u_{\mathbf{M}} = (|\beta| N \delta u)^{-1}$. The number of samples N is equal to $\frac{1}{|\beta|\delta u \delta u_{\mathbf{M}}} = \Delta u \Delta u_{\mathbf{M}} |\beta|$, where Δu and $\Delta u_{\mathbf{M}}$ are the extent of the signal in the reference and the LCT domains. This number of samples is called "bicanonical width product".

A discrete FRT equation and dFrFT matrix can be obtained from 2.15 since FRT is in the family of LCT. For this transition, the LCT matrix $\mathbf{M}$ should be chosen as:

$$\mathbf{M} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \tag{2.16}$$

or equivalently, the parameters α, γ and β should be equal to:

$$\begin{aligned} \alpha = \gamma = & \cot\alpha, \\ \beta = & \csc\alpha. \end{aligned} \tag{2.17}$$

2.2.4 Sampling in Fractional Fourier Transform Domain

So far, sampling theorem has been intuitively thought as measurements in time or space domain. However, it is possible to write the interpolation equation in

frequency domain on the condition that the signal is time limited. If the signal is limited to $[-\Delta u/2, \Delta u/2)$ range in time domain, its Fourier transform can be expressed as in eq. 2.18.

$$F(\mu) = \sum_n F\left(\frac{n}{\Delta u}\right) \operatorname{sinc}(\Delta u \mu - n) \tag{2.18}$$

To be able to write the sampling theorem and the series expansion in fractional Fourier transform domains on the other hand, different theorems have been derived [107–111]. The equation below represents the fractional Fourier domain representation of the signal f in terms of its samples taken on the same u domain [112]. A similar expression is also given in [113].

$$f(u) = \delta u |\beta| \Delta u_a e^{-i\pi\gamma u^2} \sum_{n=-\infty}^{\infty} f(n\delta u) e^{i\pi\gamma(n\delta u)^2} \operatorname{sinc}(\beta \Delta u_a (u - n\delta u)) \tag{2.19}$$

By this contribution, the requirement for limited bandwidth to be able to recover the signal expands to a more general statement: a signal can be represented by its samples taken on a fractional domain if it is limited at any fractional domain. This procedure of sampling obtained by summation of shifted cells multiplied by a linear and constant phase term is referred as chirp-periodization in [114].

The studies on sampling in fractional domains showed that required sampling rate to reconstruct the signal from its samples is not always equal to Nyquist rate for an optimal computation cost [115]. It has been proven that the sampling frequency should satisfy the equation:

$$f_s \geq 2 f_{\max} \csc \alpha \tag{2.20}$$

where it is assumed that the maximum frequency of the signal is $f_{\max}$. When $\csc \alpha = 1$ the inequality gives the ordinary sampling limit.

2.3 Condition Number

Before closing the chapter, condition number concept will be reviewed since it is used as an evaluation criteria in linear inverse problems. The problem we deal

with in this thesis is based on the assessment of the impact of sample distribution on the the solution and we need a measure of how easy recovering the signal is, which depends on the problem's nature and sensitivity to disturbances. In other words, how the solution will be affected in response to a small change in the input data should be tracked and analyzed. For that purpose, we used condition number which is defined as the ratio of the relative changes in the output to the input. Therefore, as the condition number increases, the problem becomes more ill-conditioned or ill-posed one.

For equations in the form of $\mathbf{Ax} = \mathbf{b}$, the matrix $\mathbf{A}$, which defines the linear system, the condition number is defined in [116] as:

$$\text{cond}(\mathbf{A}) = \|\mathbf{A}\| \, \|\mathbf{A^{-1}}\| \tag{2.21}$$

When l_2 norm is used, the condition number expression reduces to the ratio of the maximum singular value to the minimum one:

$$\text{cond}(\mathbf{A}) = \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})} \tag{2.22}$$

Some important properties of condition number are summarized in [26]: Firstly, condition number is always greater than or equal to 1. When it is equal to 1, the matrix $\mathbf{A}$ is unitary. It is ∞ when the matrix is rank deficient such that $\text{rank}(\mathbf{A}) < N$ for M by N matrix $\mathbf{A}$. Thus, it can be concluded that when a matrix is well-conditioned, the associated condition number is 1 and becomes ∞ when the matrix is singular.

Chapter 3

Analysis of The Signal Recovery Problem Under Arbitrarily Distributed Information

In this chapter, we first present the problem formulation. It is based on writing linear equations between samples distributed over the space, and the unknown signal on the reference plane. It is followed by the simulation results section where different scenarios are considered. Condition number and rank concepts are used to assess the impact of the sample locations on the solution.

3.1 Basic Assumptions and Dimensional Normalizations

In this thesis, we assume that most of the energy of the signal is confined to an ellipse in the space-frequency domain. In other words, the signals which have an elliptical phase space representation with semi major and minor axises $\Delta x/2$ and $\Delta\sigma_x/2$ are the subject of this work. In this notation Δx and $\Delta\sigma_x$ denotes the space and frequency extents correspondingly. This assumption does not impair

the generality of the results since for any non-elliptical space-frequency support, the major or minor radius of the ellipse can be chosen larger to contain signal energy. Then, the methodology proposed here can be applied. Also, the small error introduced since it is not possible to have both space and frequency limited signal can be neglected or made sufficiently small by enlarging this ellipse.

The abstract signal f is represented by $f(x)$ and $F(\sigma_x)$ in space and frequency domains correspondingly. To obtain the dimensionless mathematical representations that we will work on, we scale both space and frequency argument by parameter s as in eq. 3.1.

$$f(x) = \frac{1}{\sqrt{s}} f(x/s) = \frac{1}{\sqrt{s}} f(u) \tag{3.1}$$

Therefore, dimensionless time/space extent Δu, and dimensionless frequency extent $\Delta\mu$ of the signal are related to the Δx and $\Delta\sigma_x$ by:

$$\begin{aligned} \Delta u =& \frac{\Delta x}{s}, \\ \Delta\mu =& s\Delta\sigma_x. \end{aligned} \tag{3.2}$$

Then, by choosing the scale parameter $s = \sqrt{\Delta x/\Delta\sigma_x}$, we can convert the phase space representation into an circular one. This way, Δu, $\Delta\mu$ and ath order FRT domain extent Δu_a of the signal are equal to each other.

In this work, the time/space-bandwidth product is assumed to be equal to the number of degrees of freedom such that $\Delta u \Delta\mu = N$. These two concepts are generally used as if they were equivalent. However, number of degrees of freedom is defined as the area of the time/space-frequency support of the signal where the the signal energy is not negligible. Therefore, time/space-bandwidth product is always greater than or equal to the number of degrees of freedom. Under our assumption, the field is sampled with the Nyquist rate such that the sampling period is $1/\Delta\mu$ in the reference domain and $1/\Delta u$ in the frequency domain. The required number of samples is found this way: $\frac{\Delta u}{1/\Delta\mu} = \frac{\Delta\mu}{1/\Delta u} = \Delta u \Delta\mu = N$.

3.2 Methodology and Problem Setup

Before going into the details of the methodology, we first introduce the working frame we use in our formulation. The coordinate system with argument a as the horizontal axis, and the dimensionless space domain argument u as the vertical axis is created for that purpose, as seen in figure 3.1. In this coordinate system, FRT order a is taken in the range $(0, 1]$ where $a = 0$ corresponds to the reference plane and $a = 1$ is the Fourier plane which occurs at distance ∞ in the case of free space propagation. The problem is defined for the one-dimensional case without loss of generality. Therefore, FRT domains along the propagation direction are represented by vertical lines.

In the figure 3.1, we see some samples distributed over the rectangular region, which are demonstrated by dots. For each sample, a space domain location u and a FRT order a is determined according to the given sampling surface or sampling conditions. Each sample is treated as it is on a different FRT domain. If any two samples are on the same FRT domain such that they have the same a coordinate, we do not do any special treatment. We would like to emphasis that there is not any restriction on the sample locations for the method described in this chapter to be applicable. It also leads to the statement that any arbitrary sampling surface or distribution of the samples can be the subject of this work. However in our simulations, we chose the scenarios such that our samples are inside the rectangular region with the exclusion of the reference plane.

In order to set the linear relationship between the distributed samples and the unknown field distribution on the reference plane, the discrete FRT matrix (dFrFT) defined in [105], which satisfies all the requirements of a legitimate definition, is used. If $\mathbf{f} = [\mathrm{f}[-N/2], \cdots, \mathrm{f}[N/2-1]]^T$ and $\mathbf{f_a} = [\mathrm{f}_a[-N/2], \cdots, \mathrm{f}_a[N/2-1]]^T$, denotes the samples of the abstract signal f on the reference and ath order FRT plane correspondingly, the linear relationship between $\mathbf{f}$ and $\mathbf{f_a}$ can be expressed as a matrix vector multiplication via the dFrFT_a

Figure 3.1: The u-a coordinate system and sample locations

matrix:

$$\begin{bmatrix} \mathrm{f}_a[-N/2] \\ \mathrm{f}_a[-N/2+1] \\ \vdots \\ \mathrm{f}_a[N/2-1] \end{bmatrix} = \begin{bmatrix} & & \\ & \mathrm{dFrFT}_a & \\ & & \end{bmatrix} \begin{bmatrix} \mathrm{f}[-N/2] \\ \mathrm{f}[-N/2+1] \\ \vdots \\ \mathrm{f}[N/2-1] \end{bmatrix} \tag{3.3}$$

However, in our problem, samples are on different FRT domains. Therefore, we need an expression which relates the collection of samples taken on different domains to the unknown signal on the reference plane.

This collection of samples are represented by vector:

$$\mathbf{h} = \left[\mathrm{h}_{a_{k_1},u_{p_1}}[1], \cdots, \mathrm{h}_{a_{k_N},u_{p_N}}[N]\right]^T \tag{3.4}$$

In this notation, the integer index represents the sample number. The subscript a_k refers to which FRT domain the sample is on. The subscript u_p stands for the space index of the sample, which varies between $-N/2, \cdots, N/2-1$. It corresponds to a measurement value of $\mathbf{f_{a_k}}$ in the continuous extent of the signal $[-\Delta u/2, \Delta u/2)$. As an example, the fifth sample during a measurement is taken

on coordinate $(a, u) = (0.8, 9)$ is represented by $\mathrm{h}_{a_{0.8},u_9}[5]$, which is equal to $\mathrm{f}_{0.8}[9]$. Therefore, each element of the vector $\mathbf{h}$ is related to the signal on the reference field $\mathbf{f}$, by a row of a the dFrFT matrix with order a_k. Then, the linear relationship between $\mathbf{f}$ and $\mathbf{h}$ can be written via the "constructed matrix":

$$\begin{bmatrix} \mathrm{h}_{a_{k_1},u_{p_1}}[1] \\ \mathrm{h}_{a_{k_2},u_{p_2}}[2] \\ \vdots \\ \mathrm{h}_{a_{k_N},u_{p_N}}[N] \end{bmatrix} = \begin{bmatrix} \\ \textit{constructed} \\ \textit{matrix} \\ \\ \end{bmatrix} \begin{bmatrix} \mathrm{f}[-N/2] \\ \mathrm{f}[-N/2+1] \\ \vdots \\ \mathrm{f}[N/2-1] \end{bmatrix} \tag{3.5}$$

where (a_{k_1}, u_{p_1}) is the coordinate of the first sample which can be anywhere in the defined sampling region, a_{k_2}, u_{p_2} is the coordinate of the second sample and so on.

During the application of the method, for each sample location, a dFrFT matrix is generated with the order of a coordinate of the sample. Then the u index is approximated to the discrete u index the sample location is closer to. In other words, each sample location's projection is taken onto the reference plane and samples are assumed to be given on the closest u grid. This is a mapping between the continuous and discrete space locations such that a point in the interval $[-\Delta u/2, \Delta u/2)$ is mapped to the corresponding point in the interval $[-N/2, N/2-1]$. Although this assumption introduces an error and might worsen the condition number, the maximum shift introduced on the u coordinate of a sample is $0.5/\Delta\mu$ and equal to $1/32$ for $N = 256$. When compared to possible measurement precision, this amount of shift is not considered as vital.

Therefore the "constructed matrix" is obtained by the taking the u_pth row of the a_k order dFrFT matrix for each element of the vector $\mathbf{h}$. By this way, the linear relationships in the form $\mathbf{C}\,\mathbf{f} = \mathbf{h}$ between the unknown signals and the samples on different FRT planes are obtained.

It is known that the number of samples taken on a planar surface should be equal to the number of degrees of freedom of the signal to be able to recover it. However, we do not know the effect of the distribution of the sample locations

though the propagation direction on the information content. Therefore, condition number is used as a measure of recoverability of a signal thanks to its direct relationship to the amount of redundant information in the system.

The value of condition number is critical to say if the signal can be recovered; since if it is equal to 1, the matrix is unitary and in the case of a singular matrix, the condition number is infinity. It is also important in terms of sensitivity analysis. However, it might be giving insufficient information where the problem is highly ill-posed. In other words, comparing two highly ill-posed problems and understanding which one contains more information about the signal is not possible with condition number. Therefore, we also included a "rank" definition to obtain a relative insight into the information capacity of highly ill-posed problems.

Although it is not possible to recover the rank deficient systems by taking the inverse of the system matrix, there are different algorithms and research on the recovery of such systems. The success of these algorithms depends on the amount of information the system contains. Since the scope of this thesis is not recovering signals from partial information but to analyze the effect of the sample distribution on the recoverability of the signals, we should be able to distinguish the change in the amount of information in response to a control variable. This way, we can determine the effect of control variable on the solution. Therefore, having a measure of the linearly independent columns of the constructed matrices is essential. The rank can be defined as the number of non-zero singular values. However, a "tolerance" is needed to define the accuracy of non-zero. In theory, the zero can be determined as the smallest precision that machine can detect. However, this tolerance can be chosen higher to compensate for noise and measurement errors. In that case, the rank is roughly found and possibly lower than the actual number of linearly independent rows of the matrix. Thus, there is a link between the error obtained after recovering the signal and the tolerance. This tolerance can also be used to measure sensitivity. For a full-rank matrix, we can assess the similarity between its rows by increasing the tolerance and observe when it becomes rank deficient.

In the simulations, the reference plane is chosen as the FRT domain with order

$a = 0$. There is not any sample taken on that plane intentionally. The choice of the reference plane does not affect the solution in theory. It is based on the fact that, when the field is known on a transversal plane, it can be propagated into any other one without any loss. However, the numeric studies in [26]showed that taking less samples on the reference plane causes an increase in the condition number. Therefore, the existence of an artificial effect which increase the condition number should be taken into account while evaluating the results.

While discussing the results, we consider the spread of light over space through the propagation axis. As the light propagates, it diffracts. Therefore, the information represented by a sample affects a broader region in the propagation direction. It leads to a mutual interaction and coupling between samples. For example, consider the samples taken on two FRT domains which are close to each other. In that case, each sample on one domain will mostly affect the sample with the same index on the other domain. As the domains move away from each other, each sample on one domain impacts more samples on the other domain. To demonstrate the point, two dFrFT matrices are generated for $a = 0.01$ in table 3.1 and for $a = 1$ in table 3.2 for $N = 4$. These matrices gives the relationship between the signal on the reference plane and the samples taken on the ath order FRT domain. It can be seen that dFrFT for $a = 0.01$ is close to the identity matrix such that the diagonal elements are dominant. It means that, the samples on $a = 0.01$th domain are mostly influenced by only one sample on the reference plane. This is because these two domains are close to each other. In table 3.2 on the other hand, each element of the matrix has the same magnitude. This is because the Fourier domain and the reference plane are far from each other. Therefore, each sample on the $a = 1$th domain is affected by all the samples of the signal on the reference plane equally.

Table 3.1: The dFrFT matrix for $a = 0.01$

0,9998-0,0091i	0,0002+0,011i	-0,0001	0,0002+0,011i
0,0002+0,011i	0,999-0,0235i	0,0004+0,011i	-0,0003-0,007i
-0,0001	0,0004+0,011i	0,998-0,053i	0,0004+0,011i
0,0002+0,011i	-0,0003-0,007i	0,0004+0,01i	0,999-0,023i

Figure 3.2: The cones originating from the unknown samples on reference domain

Table 3.2: The dFrFT matrix for $a = 1$

0,5	0,5	0,5	0,5
0,5	-0,5i	-0,5	0,5i
0,5	-0,5	0,5	-0,5
0,5	0,5i	-0,5	-0,5i

Consider the region that a known sample has considerable effect on. It is suggested that this region can be represented by cones originating from that sample [26]. Similarly, the dependency of a sample to an unknown field can be determined by examining the cones reaching to the known sample from the unknown samples. The logic behind this cone argument is based on the fact that as the light diffracts, it affects the field ahead of it. Therefore, the cones expand with the propagation distance. Similarly, every sample is affected by the field behind it. Since the reference plane is chosen as the FRT domain with order $a = 0$, we will concentrate on the cones originating from the unknown samples.

In the figure 3.2, a symbolic drawing is given. The reference plane is demonstrated as a_0. The unknown sample locations on the reference plane are u_1, u_2, u_3

and u_4. The propagation direction is through the increasing FRT order. Therefore, the cones are originating from the reference plane. It is seen that the cones reaching to a_1 plane, have a small area R1. Only one known sample is inside this region. This is because a_0 and a_1 are close to each other. Therefore, it can be stated that the sample p_1 mostly depends on the unknown sample u_2. Since there is not any other cone reaching to p_1, it has enough information to determine u_2. As another example, there are three cones reaching to the samples on a_4. It means that each sample on a_4 is related to three unknown samples on the reference plane. It should be noted that, the information carried by each sample is equivalent. Therefore, p_5 is not enough to completely define any of the unknown samples. However, it partially represents u_1, u_2 and u_3. If all the samples are taken on the same FRT domain a_4, each sample would be depended to three unknown samples. Therefore, the balance of the intersections of regions coming from different cones would be satisfied naturally. Roughly speaking, each cone can be considered as containing the 1/3 of the information for each unknown sample. Since total of 3 cones cover each sample, they complement each other. However, different samples on different FRT domains, do not create this perfect information integration. The dependency between each sample and the unknown samples due to the cone argument is given in 3.3.

Table 3.3: Known samples versus the unknown samples that affect them

sample	unknown sample
p_1	u_2
p_2	u_2
p_3	u_4
p_4	u_1, u_2
p_5	u_1, u_2, u_3

Now we will analyze different cone interactions that might occur depending on the sample locations. Firstly, two samples on different domains might only be covered by the same cone. It is the case with samples p_1 and p_2 in figure 3.2. These two samples depend solely on u_2. Therefore, they have the same information about the unknown signal. Thus, taking both p_1 and p_2 results in redundancy in the equation set. Therefore, we can conclude that, each sample

should be covered by at least another cone originating from a different unknown sample. This unknown sample should not be fully represented by the samples taken before to avoid redundancy. In other words, any two samples should not be covered by the cones originating from the same unknown samples. This way, each sample has a new information about the reference signal. Therefore, if the set of cones reaching to two samples are intersecting, we cannot directly talk about redundancy. For example, the region R3 is the intersection of the cones originating from u_2 and u_1 and region R4 is intersection of the cones originating from u_1, u_2 and u_3. Therefore, we cannot directly say that taking both p_4 and p_5 results in redundancy. It is because p_5 also depends on u_3. In that case, there are two common unknowns u_1 and u_2. Both p_4 and p_5 are linearly related to u_1 and u_2. However, neither p_4 exactly has the information carried by p_5 nor the opposite. Therefore, the rank and condition number depends on the distribution of the other samples and their mutual interactions. Lastly, the samples p_3 and p_4 are not related to each other because p_3 depends on u_4, p_4 on the other hand, depends on u_2 and u_1. Since we know that p_3 and p_4 do not have a common information about the signal, we can say that taking p_3 and p_4 gives more information than taking p_4 and p_5.

3.3 Simulation Results

In the simulations, a total of N samples are taken inside the rectangular region defined in 3.1. N is assumed to be equal to the number of degrees of freedom of signal f. It is chosen as 256 for all the scenarios. Firstly, we considered the samples taken on a line and observe the effect of varying the slope, length and location of the line on the recoverability of the signal. Then, a line pair with complementary and overlapping sample locations is examined. By altering "depths" of the lines and "frequency" of the samples, different configurations are obtained. It is followed by the samples taken on a circular surface. The control variable is chosen as the radius of the circle. Finally, samples distributed all over the u-a plane are taken into the consideration. Samples are first distributed over the space randomly. Then, they are taken on the rectangular grid uniformly. A

Figure 3.3: Horizontally sheared line sample locations

comparison between these two cases are made.

3.3.1 Lines with Varying Slope

In this part, the effect of the slope of the line on the solution is investigated. The slope is defined as $u_{increment}/a_{increment}$ such that the slope of the vertical line is ∞, the diagonal line is 1 and the horizontal line is 0. Note that we refer to the line with the leftmost sample is on coordinate $(a, u) = (1/N, -N/2)$ and rightmost sample is on coordinate $(a, u) = (1, N/2 - 1)$ as the diagonal line. To be able to make the problem physically meaningful, we examined the effect of the change in the slope of the line in two categories. This distinction is done because horizontal and vertical lines does not have the same length in the physical world. For that reason, we first looked at the horizontal shearing, where the samples are initially on $a = 0.5$th order FRT domain. Then, they are separated from each other in the a direction as seen in figure 3.3.

The condition number versus inverse slope of the line is seen in figure 3.4 and the rank versus inverse slope of the line is given in figure 3.5. We explain the curvature like behavior seen in figure 3.4 by the existence of two effects acting

on the condition number. The first one is the fact that as the lines are tilted, recovering the signal is harder. It can be explained by the cone argument. It is known that when all of the samples are taken on a vertical line, the cones originating from the unknown samples perfectly complement each other for each known sample. When the samples are ahead or behind of each other, on the other hand, they influence or being influenced by each other. Therefore, a worsening effect is expected due to their interactions. These interactions are inversely related to the slope of the line. Thus, condition number increases as the line is tilted and becomes a horizontal one eventually.

The other effect acting on the condition number relies on the fact that taking samples on a larger extent improves the solution. When the inverse slope increases, samples are farther away from each other. Therefore, condition number tends to decrease. In addition, the cone argument suggests that samples should not be too close to each other. As they are separated, they will depend on different unknowns. This effect reduces redundancy and lowers the condition number.

Although there is an ambiguity in the definition of rank due to the choice of the tolerance, we would not expect the constructed matrix to be rank deficient for horizontally sheared lines, since condition number is not infinity. As seen from figure 3.5, the matrix is full rank for all the slopes.

Lastly, we want to share our insight about a possible cause of the sudden jump of the condition number for low values of inverse slope. Please note that, there is not a linear increase, but a jump. To demonstrate this, we repeated this scenario with a higher resolution. Total of 17 sheared lines are evaluated instead of 9. The condition number versus inverse slope is given in 3.6. Although we do not have any proof for that, the reason behind it might be the nonuniform convergence of chirp functions. If this is true, the date we have for very small values of the inverse slope, i.e. inverse slope < 0.1, may not be valid.

Now we turn our attention to vertically sheared line. The sample locations for vertically sheared line, starting from the diagonal line and reaches to the

Figure 3.4: Condition number of horizontally sheared line

Figure 3.5: Rank of horizontally sheared line

Figure 3.6: Condition number of horizontally sheared line with higher resolution

Figure 3.7: Vertically sheared line sample locations

horizontal line are seen in figure 3.7. Vertically sheared line can be though as the continuation of the horizontally sheared line such that they match up with the diagonal line. The condition number versus the slope of the line is seen in figure 3.8 and the rank versus the the slope of the line is given in figure 3.9.

It is seen that just after the diagonal line, the condition number increases rapidly. The matrix is full rank up to the slope of 0.875. Then, it slowly decreases until the sharp transition at the slope of 0.125. The decay in the rank is an expected result, firstly because of the fact that we only use a narrow spatial extent. Therefore, the samples do not carry enough information. The unknown samples on the edges of the signal extent of the reference field, i.e. f$[N/2-1]$ and f$[-N/2]$ are not represented by the known samples. Only a limited number of the cones originating from these unknown samples on the edges reaches to the taken samples. Therefore, there is not enough information of the unknown signal to keep the constructed matrix away from being rank deficient. In addition, the cones of the successive samples becomes the subsets of each others as the slope goes to 0. Therefore the redundancy is introduced which results in infinite condition number and rank deficiency.

In the figure 3.10 we demonstrated the effect of increasing the tolerance to the rank of the matrix. As seen from the figure, the new rank definition identifies the line with slope 0.875 as rank deficient. In addition, the rank decreases linearly with slope. Therefore, it can be considered as a rough approximation of the ideal rank.

3.3.2 Lines with Varying Length

In this part, the effect of the change in the length of the sampling surface on the recoverability of the problem is examined. For horizontal and tilted lines, the extent of the sampling surface is decreased until it is halved. The number of samples are fixed to 256 for all lengths. Therefore, we take samples on a smaller region and observe its effect on the solution. The normalized length is defined as the ratio of the FRT order a coverage of the samples to the full a extent. In

Figure 3.8: Condition number of vertically sheared line

Figure 3.9: Rank of vertically sheared line

Figure 3.10: Vertically sheared line rank with increased tolerance

other words, normalized length is equal to $a_{\max} - a_{\min}$ since the full extent is equal to 1. Therefore, the normalized length of the longest line is 1, and the shortest line is 0.5. All of the sampling surfaces are placed to be symmetric around $(a, u) = (0.5, 0)$ point.

The sample locations for horizontal lines with different lengths is seen in figure 3.11. The normalized lengths are demonstrated by the arrows. The condition number versus the normalized length of the line is seen in figure 3.12 and the rank versus the the normalized length of the line is given in figure 3.13. The condition number is infinity for all the lengths as expected. As explained by the cone argument, some samples to be dependent to the same unknown samples does not directly imply rank deficiency. However, there is a condition for that. Each sample should be covered by a different cone originating from a different unknown sample. However, as the normalized length decreases, samples get closer to each other and align as a horizontal line. Thus, consecutive samples are covered by only the same samples which creates the rank deficiency. From figure 3.13, it is seen that the rank increased by %53 when the length is doubled.

The sample locations for tilted lines passing through the coordinate $(a, u) =$

Figure 3.11: Changing length horizontal line sample locations

Figure 3.12: Condition number of changing length horizontal lines

Figure 3.13: Rank of changing length horizontal lines

$(0.5, 0)$ with the slope of $a_{increment}/u_{increment} = 1$ are seen in figure 3.14 with different lengths. The normalized length of the longest line is 1, and the shortest one is 0.5. The condition number versus the normalized length of the line is seen in figure 3.15 and the rank versus the the normalized length of the line is given in figure 3.16. The condition number is increases rapidly after the normalized length of 1. The constructed matrix is full rank until the normalized length of 0.875. The rank slowly decreases as we shorten the length of the line. From the figure 3.16, it is seen that the rank is increased by %37 when the length is doubled.

Please note that we are not able to observe the effect of varying the length of the vertical line on the solution since our method fails to represent this scenario. Since we round the u coordinate to the closest u grid, when we shorten the length of the vertical line, we take the same sample point at least twice. This leads to choosing the same row of the $a = 0.5$ order dFrFT matrix more than one times to obtain the constructed matrix. Therefore, the rank deficiency is not a result of the nature of the problem but due to the limitation of the method. If we used a method which allows continuous spatial argument u selection, we would not expect the rank and condition number to change dramatically up to a point.

Figure 3.14: Changing length tilted line sample locations

Figure 3.15: Condition number of changing length tilted line

Figure 3.16: Rank of changing length tilted line

3.3.3 Line with Varying Locations

In this section, we investigate the effect of the sampling surface on the location. The impact of shifting a sampling surface up/down or left/right is examined via the simple cases. Firstly, the location of horizontal lines are changed. Then, we alter the location of the vertical lines.

The samples on horizontal lines with different locations are seen in figure 3.17. The normalized locations are defined as the ratio of the distance of the sampling surface spatial coordinate u from $-\Delta u/2$ to the full spatial extent of the signal. Therefore, the normalized location can be expressed by the equation : $\left(\frac{u-(-\Delta u/2)}{\Delta u}\right)$, such that the normalized location of the line at the bottom is 0, and the one top is 1. The condition number versus the normalized location of the line is seen in figure 3.18 and the rank versus the normalized length of the line is given in figure 3.19. The condition number is infinity for all of the locations as expected. However, rank depends on the normalized location. It is seen that when the lines are close to the edges of the signal extent, rank is lower. This result can be explained by the cone argument. The half extent of the cones originating from the unknown samples does not intersect the samples taken on the

Figure 3.17: Changing location horizontal line sample locations

lines close to the edges i.e. the lines with the normalized location around 0 and 1. This argument explains the decrease of rank around the edges. However, one would expect rank to be maximum when it is on the center which is not the case. It is maximum when the normalized length is 0.25 and 0.75 which corresponds to the lines with u coordinate of -4 and 4. At this point, we cannot explain the reason behind the decrease in rank when the line is on the center.

The samples on vertical lines with different locations are seen in figure 3.20. The normalized locations are defined as the order of the FRT domain that the samples are taken on, such that the normalized location of the leftmost line is $1/N$, and rightmost one is 1. The condition number versus the normalized location of the line is seen in figure 3.21 and the rank versus the the normalized length of the line is given in figure 3.22. The condition number is zero for all the normalized locations. Therefore, the constructed matrix is always full rank. It is a numeric demonstration of the statement that samples taken on a transversal plane completely defines the field and there is not any importance of which FRT domain this transversal plane corresponds to.

Figure 3.18: Condition number of changing location horizontal line

Figure 3.19: Rank of changing location horizontal line

Figure 3.20: Changing location vertical line sample locations

Figure 3.21: Condition number of changing location vertical line

Figure 3.22: Rank of changing location vertical line

3.3.4 Samples on Two Planes with Different Distributions

In this part, the different sample distributions on two planar surfaces with FRT orders a_1 and a_2, are analyzed. The separation of these surfaces is defined as "depth". The sequence of the samples is represented by "period" which varies between 2 to 256. Two different distributions are taken into consideration. Firstly, samples are located on two lines such that they would complement each other if they were taken on the same line. Then, they are taken such that they would overlap each other if they were taken on the same line. In that case, the sample locations are identical to each other for u argument. The effect of the change in the depth and period is investigated for these two scenarios.

The samples taken on a pair of lines is studied before in [26]. Total of m_1 sample is taken on the first plane and m_2 sample are taken on the second plane. The line with higher number of known samples is determined as the reference plane. In the simulations, (m_1, m_2) pair is chosen as $(1, 255), (2, 254), (4, 252), (8, 248), (16, 240), (32, 224), (64, 192), (128, 128)$. Uniform-complementary, uniform-overlapping, accumulated-complementary and accumulated-overlapping distributions are considered. It is reported that taking

equal number of samples on both domain gives the worst results. Therefore, our analysis in this section can be considered as the continuation of this work. We take equal number of samples on both domains for all scenarios. Thus, we work on the worst case described in [26].

3.3.4.1 Complementary Sample Locations

An example of the sample locations on two complementary FRT domains with order a_1 and a_2 are seen in figure 3.23. Although N is taken as 256 in the simulations, 3.23 is generated for N=16 for clarity. Period equal to 2 represents the case of taking one sample on a_1 and next on a_2. It corresponds to the physical problem of having multiple low resolution information from two domains and reconstruction high resolution one. Period equal to 256 on the other hand, means taking the first half of the samples on a_1 domain and the other half on a_2 domain. This can be a scenario of inconvenient measurement environment of field due to an optical instrument blocking the way. Depending on the configuration of the surrounding, different periods might have different physically meanings.

Different depths on the other hand stands for the location of FRT domains and represented by the notation "a_1-a_2" such that, depth equals to 0-1 is the case of $a_1 = 0$ and $a_2 = 1$ so that they are farthest away from each other. Then, they get closer and reach to the same plane for $a_1 = a_2 = 0.5$. In the figure 3.24, the depths are demonstrated for $N = 256$ and period $= 64$.

The condition numbers versus depth for different periods are given in figure 3.25 and the rank versus depth for different periods are seen in 3.26. In figure 3.25, the first observation is that for period 256 and 128 condition number is so high unless the planes are close to each other. For intermediate depths, the condition number is in an acceptable level for small periods. When the separation between the planes increases, condition number worsens. When period is 32, on the other hand, condition number is good for all the depths. It might be an indication of a concept like optimum cone expansion. When period is $2, 4, 8, 16$, the constructed matrix is not rank deficient unless depth is 0-1. When period is

(a) Period = 2

(b) Period = 4

(c) Period = 8

(d) Period = 16

Figure 3.23: Complementary sample locations for N=16

Figure 3.24: Different depths

Figure 3.25: Condition number of complementary planes

256 and 128, the matrix is full rank for depths 0.125-0.875 and 0.5-0.5. When period equals to 64 and 32, the matrix is full rank for all the depths.

3.3.4.2 Overlapping Sample Locations

The sample locations for two overlapping FRT domains with order a_1 and a_2 are seen in figure 3.28. Although N is taken as 256 in the simulations, 3.28 is generated for N=16 for clarity. The sample positions on a_1 are taken to be the same with the previous case, the samples on a_2 are aligned with the ones on a_1. This way, the samples are located on two lines have the same u coordinate for each pair of samples. The $a_1 = a_2 = 0.5$ depth is not included to the simulations since the samples would overlap on the same line and only $N/2$ different rows

Figure 3.26: Rank of complementary planes versus depth

Figure 3.27: Rank of complementary planes versus period

would be generated.

The condition number versus depth for different periods are given in figure 3.29 and the rank versus depth for different periods are seen in 3.30. It is seen that for period equals to 256 and 128, condition number is always high. It is an expected result due to the cone argument. When period is 256 for example, samples are taken on the same half extent of the signal. The samples taken on a_1 do not depend on the unknown samples on the other half extent of the signal. Although the samples on a_2 has information of these samples, it is limited and not enough to represent them. It leads to rank deficiency. Note that a_1 is defined as the line close to the reference field and a_2 is the line far from the reference plane. For smaller periods, condition number is acceptable for the intermediate depths. In that region, the condition number decreases as lines get closer for periods $4, 8, 16$. As the lines are separated, the distinction between overlapping and complementary samples eradicates for small periods. When period is equal to $2, 4, 8, 16$ the constructed matrix is full rank except the depth 0-1. Similarly to the complementary case, when period is 32 and 64 the matrix is full rank.

(a) Period = 2

(b) Period = 4

(c) Period = 8

(d) Period = 16

Figure 3.28: Overlapping sample locations for N=16

Figure 3.29: Condition number of overlapping planes

Figure 3.30: Rank of overlapping planes

Figure 3.31: Rank of overlapping planes versus period

Figure 3.32: Circular Surface Sample Locations

3.3.5 Circular Sampling Surface

The sample locations taken on circular surfaces are seen in figure 3.32. These circles corresponds to ellipses in the physical world. The normalized radius is defined as the ratio of the half of the difference between maximum and minimum u coordinates of the samples to the full spatial extent of the signal: $\frac{(u_{\max}-u_{\min})/2}{\Delta u}$. The condition numbers versus the normalized radius are given in figure 3.33 and the rank versus the normalized radius is seen in 3.34. Although the condition number is so high for all radius and does not give a considerable information, the increase in the rank as radius increases suggests that the samples taken on a larger area gives more information, as expected. When the normalized radius equals to 0.5, the constructed matrix is full rank. The affect of increasing the radius improves the results in a higher rate for smaller radius.

Figure 3.33: Condition of circular sampling surface

Figure 3.34: Rank of circular sampling surface

3.3.6 Sample Distributed All Over the u-a Coordinate System

In this section, two different scenarios are considered. Firstly, an example of randomly distributed sample locations are seen in figure 3.35. The condition number versus the number of trials is given in figure 3.36 and the rank versus the the number of trials is seen in 3.37. The constructed matrix is always full rank. The mean of the condition number is 2.9936 and the variance is 0.1155. We observed that randomly distributed samples result in a much better condition number in comparison to the other scenarios. Please also note that reconstructing the signal from its randomly distributed samples is implemented via POCS method in [2]. It strengthens the idea that this condition number is acceptable to be able to recover the signal and supports our definition of rank.

By randomly distributing the samples on the pre-determined FRT domains, we can ensure that the samples will not be too close to each other in the price of damaging the randomness of the problem. The mean of the condition number is 2.87 and the variance is 0.0865 when we only let the samples to be on the FRT domains with orders k/N where k is an integer $\in [1, N]$. Although there is a slight improvement in the condition number, it is still on the order of 3.

Then, we turned our attention to the samples uniformly distributed over the rectangular grid. The sample locations for this scenario are seen in figure 3.38. In this case, the condition number is infinity and the rank is 255. This result might lead to the conclusion that having samples distributed all over the u-a coordinate system is not a way of obtaining a recoverable system and the randomness is the key. But we looked further and realized that there is a mathematical artifact based on the symmetry around the $u = 0$ line and causing the constructed matrix to be rank deficient. Please note that this is not a result of the physics behind the problem.

Now, we will present the results which lead us to the conclusion that the issue is about the strict symmetric problem definition. For that purpose, we tried

Figure 3.35: Randomly distributed sample locations

Figure 3.36: Condition number versus the number of trials

Figure 3.37: Rank versus the number of trials

Figure 3.38: Condition number versus the number of trials

lots of different scenarios which break the symmetry. Firstly, we shifted all the samples up by all possible amounts. Figure 3.39 is an example of these scenarios with shift amount is at its maximum. In that case the samples on top are taken on the line $\Delta u/2 - \delta u$. For all the scenarios of shifting the samples up by different amounts, we observed that the condition number is always in the order of 3 and the constructed matrix is full rank.

When we tried the same procedure for shifting samples down, we realized that the condition number is still around 3 until we reach to the maximum possible shift. One would expect the problem to be symmetrical and obtain the same rank and condition number in the case of maximum shift too. However, if the samples are distributed as it is seen in figure 3.40, the condition number is infinity and the matrix is rank deficient. This result might seem like collapsing our hypothesis on the symmetry of u, since we ruined the symmetry by shifting the samples down and still obtained a not recoverable system. However, please note that the system is still symmetric around u except the samples taken on the line $u = -\Delta u/2$. We know that we could prefer to take samples on $u = \Delta u/2$ instead of $u = -\Delta u/2$ since this choice does not create any difference due to the cyclic periodicity of DFT. Therefore, $u = \Delta u/2$ and $u = -\Delta u/2$ lines are equivalent. In terms of matrix algebra, there is not any difference between having one or the other or the both which implies that the samples in figure 3.40 are symmetric around u.

Next, we break the symmetry around u only by one sample as seen in figure 3.41. When any sample is randomly assigned to another location or shifted to any direction by any amount, the condition number is found to be around 3. This shows that, how dramatically we break the symmetry is not important.

Then, we analyze if there is an effect of the regularity of the grid on the solution or not. For that purpose, we first introduced a systematic irregularity to the system and obtained the sample distribution seen in figure 3.42, where the rank is 256 and condition number is 2.9605. Then we shifted the upper-half of the signal up and lower-half of the signals down for all possible shift amounts as seen in figure 3.43. Please note that, the samples are irregular but symmetric around $u = 0$ in this example. The constructed matrix is found to be rank deficient for

this case. A similar scenario is created by shifting the upper-half of the signal down and lower-half of the signal up for all possible shift amounts as seen in figure 3.44. It is seen that the constructed matrix is still rank deficient.

The results of the configurations in the figures are summarized in table 3.4. In the light of this analysis, we created enough numeric justification for our hypothesis. Please note that this common condition number 3 would change if we worked with a different N instead of 256.

Table 3.4: Samples taken on the rectangular grid with different configurations

Figure no	Description	Condition number	Rank
3.35	Randomly distributed samples, not symmetric around $u = 0$	2.9936	256
3.38	Centered rectangular grid, symmetric around $u = 0$	∞	255
3.39	Rectangular grid is shifted up until reaching $\Delta u/2 - \delta u$ line, not symmetric around $u = 0$	3.0838	256
3.40	Rectangular grid is shifted down until reaching $-\Delta u/2$ line, symmetric around $u = 0$	∞	241
3.41	Rectangular grid with only one arbitrary sample is shifted up, not symmetric	3.22	256
3.42	Rotated grid, not symmetric around $u = 0$	2.9605	256
3.43	Upper-half shifted up & lower-half shifted down, symmetric around $u = 0$	∞	255
3.44	Upper-half shifted down & lower-half shifted up, symmetric around $u = 0$	∞	255

Figure 3.39: Shifted up uniform grid

Figure 3.40: Shifted down uniform grid

Figure 3.41: Uniform grid with one irregular point

Figure 3.42: Systematically irregular samples

Figure 3.43: Irregular grid with upper-half shifted up and lower-half shifted down

Figure 3.44: Irregular grid with upper-half shifted down and lower-half shifted up

Chapter 4

Discussion

In this chapter, we first briefly discuss one of the main consequences of the simulations: the rank deficiency of the samples taken on the horizontal line. Then we examined different methodologies to express our problem. We conclude this chapter by creating the link between the mathematical working space defined in this work and the physical mediums.

4.1 Why the horizontal line is unsolvable?

It is seen that when the samples are taken on a horizontal plane, the condition number is infinity which means that "constructed matrix" is singular. It is the result of the case that at least two rows of the matrix are linearly dependent or in other words, the information obtained by sampling the signal over a horizontal plane is not unique. It is intuitively right to think that the solution gets worse when the sampling plane is becoming more like a horizontal one, basically because the wave equation is constructed between planar surfaces. However, we are not aware of any physical reason why samples on horizontal surface are linearly dependent to each other. Consider the continuous expressions for N samples on the horizontal line passing through the optical axis. Let's say these N points

are on the FRT domains $a_1, a_2, ...a_N$. These samples are represented by different order FRT integrals evaluated on $u = 0$ as seen in eq. 4.1. As $N \to +\infty$, this example represents the complete continuous case.

$$\begin{aligned} f_{a_1}(0) &= \int_{-\infty}^{\infty} K_{a_1}(0, u') f(u') du' = A_{\alpha_1} \int_{-\infty}^{\infty} e^{i\pi \cot \alpha_1 u'^2} f(u') du' \\ f_{a_2}(0) &= \int_{-\infty}^{\infty} K_{a_2}(0, u') f(u') du' = A_{\alpha_2} \int_{-\infty}^{\infty} e^{i\pi \cot \alpha_2 u'^2} f(u') du' \\ &\vdots \\ f_{a_N}(0) &= \int_{-\infty}^{\infty} K_{a_N}(0, u') f(u') du' = A_{\alpha_N} \int_{-\infty}^{\infty} e^{i\pi \cot \alpha_N u'^2} f(u') du' \end{aligned} \tag{4.1}$$

It turns out that, for this set of equations to be linearly dependent, the linear combination of the chirp functions should be equal to another chirp function.

$$e^{i\pi \cot \alpha_k u'^2} = \sum_{\substack{p=1 \\ p \neq k}}^{N} c_p e^{i\pi \cot \alpha_p u'^2} \tag{4.2}$$

If it is indeed true that the condition number of this system is infinity and not a nearly very large number, then that would imply that eq. 4.2 might be an identity.

4.2 Examination of Different Possible Methodologies

In our chapter 3, we used the discrete dFrFT matrix that defined in [105]. As a result of this choice, we discretized the u coordinate of the samples. In this chapter, we explain possible other methodologies that have different foundations and therefore, have different characteristics in terms of the discretization of the u and a arguments. For this purpose, different interpolation equations are taken into consideration. Firstly, we started with the "ordinary" interpolation equation which is written in u domain. This equation is based on the knowledge of the

extent of the signal in frequency domain. As a second method, we established the interpolation equation in FRT domain. Then, we consider the interpolation equation written in u domain with the knowledge of the signal extent in FRT domain. Finally, we briefly discuss using discrete FRT orders as well as discrete u.

4.2.1 A Problem Definition Based on Ordinary Interpolation Equation

This method is based on taking ath order FRT of both sides of the interpolation equation and evaluating it on the continuous sample location u. If the u coordinates are chosen as discrete locations, the equation we obtain by this method should reduce to the discrete FRT expression. This reduction can be considered as a validity check for the method. Therefore, we first showed that it is possible to go from the interpolation equation to DFT equation and later to FRT equation.

$$f(u) = \sum_{n=-\infty}^{\infty} \mathrm{f}[n]\mathrm{sinc}(\Delta\mu u - n) \tag{4.3}$$

If we take FT of both sides of this equation, we obtain the following equation:

$$F(\mu) = \sum_{n=-\infty}^{\infty} \mathrm{f}[n]\mathrm{rect}(\frac{\mu}{\Delta\mu})e^{-j2\pi\mu\frac{n}{\Delta\mu}}\frac{1}{\Delta\mu} \tag{4.4}$$

where the rect function refers to the continuous FT of sinc function. When the terms are rearranged, the equation 4.5 can be obtained:

$$F(\mu) = \mathrm{rect}(\frac{\mu}{\Delta\mu})\sum_{n=-\infty}^{\infty}\frac{1}{\Delta\mu}\mathrm{f}[n]\mathrm{rect}(\frac{\mu}{\Delta\mu})e^{-j2\pi\mu\frac{n}{\Delta\mu}} \tag{4.5}$$

The limits of the summation can be changed such that n goes from -N/2 to N/2-1 since the time/space extent of the signal is assumed to be Δu and the rect function can be removed since there is not any frequency component outside the $\Delta\mu$ band. When this equation is sampled with sampling interval of $1/\Delta u$, we obtained the following equation:

$$F(\frac{k}{\Delta u}) = \frac{1}{\Delta\mu}\sum_{n=-N/2}^{N/2-1}\mathrm{f}[n]e^{-j2\pi\frac{k}{\Delta u}\frac{n}{\Delta\mu}} \tag{4.6}$$

Since $\Delta u \Delta\mu = N$and $\Delta\mu = \sqrt{N}$, the expression simplifies to DFT equation.

$$\mathrm{F}[k] = \frac{1}{\sqrt{N}} \sum_{n=-N/2}^{N/2-1} \mathrm{f}[n] e^{\frac{-j2\pi nk}{N}} \tag{4.7}$$

This result shows that if we can take the continuous FT of the sinc function, it is straight-forward to go from interpolation equation to DFT without any assumption.

When we take the FRT of both sides of the interpolation equation, on the other hand, where ath FRT domain variable is represented by u_a for clarity, and the FRT is represented by the operator F^a, we obtain eq. 4.8:

$$F_a(u_a) = \sum_{n=-\infty}^{\infty} \mathrm{f}[n] F^a \left\{\mathrm{sinc}(\Delta\mu u - n)\right\} \tag{4.8}$$

The FRT of the $\mathrm{sinc}(\Delta\mu u - n)$ can be found in terms of the FRT of the $\mathrm{sinc}(\Delta\mu u)$ by using the shift property [28]:

$$\begin{aligned} \mathcal{F}^a \left\{\mathrm{sinc}(\Delta\mu u - n)\right\} =& e^{i\pi \frac{n^2}{\Delta\mu^2} \sin\alpha\cos\alpha} e^{-i2\pi u_a \frac{n}{\Delta\mu}\sin\alpha} \\ & \times \left(\int_{-\infty}^{\infty} K_a \left(u_a - \frac{n}{\Delta\mu}\cos\alpha, u' \right) \mathrm{sinc}(\Delta\mu u') du' \right) \end{aligned} \tag{4.9}$$

since the amount of shift in u_a is $\xi = n/\Delta\mu$. Then we insert this equation into the eq. 4.8.

$$\begin{aligned} F_a(u_a) =& \sum_{n=-\infty}^{\infty} \mathrm{f}[n] e^{i\pi \frac{n^2}{\Delta\mu^2} \sin\alpha\cos\alpha} e^{-i2\pi u_a \frac{n}{\Delta\mu}\sin\alpha} \\ & \times \left(\int_{-\infty}^{\infty} K_a \left(u_a - \frac{n}{\Delta\mu}\cos\alpha, u' \right) \mathrm{sinc}(\Delta\mu u') du' \right) \end{aligned} \tag{4.10}$$

After expanding $K_a \left(u_a - \frac{n}{\Delta\mu}\cos\alpha, u' \right)$ and re-arrange the terms accordingly, the equation simplifies to:

$$\begin{aligned} F_a(u_a) =& A_\alpha \sum_{n=-\infty}^{\infty} \mathrm{f}[n] e^{i\pi \frac{n^2}{\Delta\mu^2} \cot\alpha} e^{-i2\pi u_a \frac{n}{\Delta\mu}\csc\alpha} e^{i\pi\cot\alpha u_a^2} \\ & \times \left(\int_{-\infty}^{\infty} e^{i2\pi u' \cot\alpha \frac{n}{\Delta\mu}} e^{-i2\pi u' \csc\alpha u_a} e^{i\pi\cot\alpha u'^2} \mathrm{sinc}(\Delta\mu u') du' \right) \end{aligned} \tag{4.11}$$

Since this equation does not have a closed form solution, the integral inside the brackets should be approximated numerically. Riemann sum approximated FRT of f is demonstrated as $\breve{F}_a(u_a)$:

$$\begin{aligned}\breve{F}_a(u_a) =& A_\alpha \sum_{n=-\infty}^{\infty} \mathrm{f}[n] e^{i\pi \frac{n^2}{\Delta\mu^2} \cot\alpha} e^{-i2\pi u_a \frac{n}{\Delta\mu} \csc\alpha} e^{i\pi \cot\alpha u_a^2} \\ &\times \left(\sum_{p=1}^{M} \Delta x e^{i2\pi x_p \cot\alpha \frac{n}{\Delta\mu}} e^{-i2\pi x_p \csc\alpha u_a} e^{i\pi \cot\alpha x_p^2} \operatorname{sinc}(\Delta\mu x_p) \right)\end{aligned} \tag{4.12}$$

where it is assumed that the sinc function is approximately 0 outside an interval Δu_n so that the summation is finite and M is the number of required samples which can approximately be expressed as $M = \Delta u_n^2$. Then $\Delta x = \frac{\Delta u_n}{M}$ and $x_p = -\frac{\Delta u_n}{2} + (p-1)\Delta x$ for $p = 1, ..., M$.

Then this equation is sampled to see its relation of discrete FRT equations.

$$\begin{aligned}\breve{F}_a(\frac{k}{\Delta u}) =& A_\alpha \sum_{n=-\infty}^{\infty} \mathrm{f}[n] e^{i\pi \frac{n^2}{N} \cot\alpha} e^{-2\pi \frac{nk}{N} \csc\alpha} e^{i\pi \cot\alpha \frac{k^2}{N}} \\ &\times \left(\sum_{p=1}^{M} \Delta x e^{i2\pi x_p \cot\alpha \frac{n}{\Delta\mu}} e^{-i2\pi x_p \csc\alpha \frac{k}{\Delta u}} e^{i\pi \cot\alpha x_p^2} \operatorname{sinc}(\Delta\mu x_p) \right)\end{aligned} \tag{4.13}$$

and that is equal to:

$$\breve{F}_a[k] = \sum_{n=-\infty}^{\infty} \mathrm{f}[n] C_{\mathbf{FRT}}[k,n] \left(\sum_{p=1}^{M} \Delta x e^{i2\pi x_p \cot\alpha \frac{n}{\Delta\mu}} e^{-i2\pi x_p \csc\alpha \frac{k}{\Delta u}} e^{i\pi \cot\alpha x_p^2} \operatorname{sinc}(\Delta\mu x_p) \right) \tag{4.14}$$

where $C_{\mathbf{FRT}}[k,n]$ is the discrete FRT matrix given in [106]. The sinc inside the summation can be expanded as:

$$\operatorname{sinc}(\Delta\mu x_p) = \operatorname{sinc}\left(-\Delta\mu \frac{\Delta u_n}{2} + \Delta\mu(p-1)\Delta x \right) = \operatorname{sinc}\left(\frac{\Delta\mu}{\Delta u_n}(p - (M/2+1)) \right) \tag{4.15}$$

It can be seen that, for this equation to be equivalent to discrete FRT equation, the extent of the original signal should be equal to the extent of the sinc function such that $\Delta\mu = \Delta u_n$. The sinc function is zero when $p \neq M/2+1$ and is equal to 1 otherwise. When $p \neq M/2+1$, $x_p = 0$ and the exponentials inside the summation are equal to 1 and $\Delta x = \delta u$. On that condition, eq. 4.14 reduces to

Figure 4.1: Condition number versus the inverse slope of the horizontally sheared line by using method in section 4.2.1

eq. 4.16.

$$F_a[k] = \delta u \sum_{n=-\infty}^{\infty} \mathrm{f}[n] C_{\mathbf{FRT}}[k,n] \tag{4.16}$$

Therefore, we showed that, if we take $M = N = 256$, we can obtain the discrete FRT expression starting from the interpolation equation.

While implementing this method, a different approach can be used instead of approximating the integral by Riemann sum. To approximate the continuous FRT of sinc function, it is possible to take M point FRT of it, where $M \gg N$, by using the dFrFT matrix defined in [105]. Above we showed that when $M = N$, equation 4.14 reduces to a discrete FRT equation. We cannot have an analytical proof when we use the dFrFT matrix in [105] since it does not have a closed form expression. However, we include a numeric result obtained for horizontally sheared line problem by using M point FRT of the sinc function in figure 4.1 where we take $M = N$. This way, we demonstrate the numeric equivalence of the method we used in chapter 3 and the one described here.

4.2.2 Problem Definition Based on Interpolation Equation Written in FRT Domain

In this method, we start by writing the interpolation equation written in FRT domain which enable us to find continuous $f_a(u_a)$ from its samples on the same domain.

$$f_a(u_a) = \sum_{n=-N/2}^{N/2-1} \mathrm{f}_a[n]\mathrm{sinc}(\Delta\mu_a u_a - n) \tag{4.17}$$

Note that $\Delta\mu_a$ is the extent of signal on the Fourier domain of the ath FRT domain which is $(a+1)$th FRT domain. This equation can be written as a product of two finite vectors.

$$f_a(u_a) = \left[\mathrm{f}_a\left[\frac{-N}{2}\right]\cdots\mathrm{f}_a[0]\cdots\mathrm{f}_a\left[\frac{N}{2}-1\right]\right]\begin{bmatrix}\mathrm{sinc}\left(\Delta\mu_a u_a - \left(-\frac{N}{2}\right)\right)\\ \vdots \\ \mathrm{sinc}(\Delta\mu_a u_a)\\ \vdots \\ \mathrm{sinc}\left(\Delta\mu_a u_a - \left(\frac{N}{2}-1\right)\right)\end{bmatrix} \tag{4.18}$$

Now we can realize that the vector containing the samples of $\mathbf{f_a}$ can be represented in terms of the discrete signal on the reference plane: $\mathbf{f}$, by using dFrFT matrix.

$$f_a(u_a) = (\mathrm{dFrFT}\times\mathbf{f})^T\begin{bmatrix}\mathrm{sinc}\left(\Delta\mu_a u_a - \left(-\frac{N}{2}\right)\right)\\ \vdots \\ \mathrm{sinc}(\Delta\mu_a u_a)\\ \vdots \\ \mathrm{sinc}\left(\Delta\mu_a u_a - \left(\frac{N}{2}-1\right)\right)\end{bmatrix} \tag{4.19}$$

If the transpose operation is applied separately to the terms inside the brackets:

$$f_a(u_a) = \mathbf{f}^T\left\{\mathrm{dFrFT}^T\begin{bmatrix}\mathrm{sinc}\left(\Delta\mu_a u_a - \left(-\frac{N}{2}\right)\right)\\ \vdots \\ \mathrm{sinc}(\Delta\mu_a u_a)\\ \vdots \\ \mathrm{sinc}\left(\Delta\mu_a u_a - \left(\frac{N}{2}-1\right)\right)\end{bmatrix}\right\} \tag{4.20}$$

Since dFrFT matrix is symmetric, its transpose is equal to itself. Due to the assumption of the circular phase space representation, μ_a is equal

Figure 4.2: Condition number versus the inverse slope of the horizontally sheared line by using method in section 4.2.2

to $\Delta\mu$. Therefore, the equation to set the linear relationships between samples and $\mathbf{f}$ can be found by taking the FRT of the vector $[\mathrm{sinc}\,(\Delta\mu u_a - (-N/2))\,, \cdots, \mathrm{sinc}\,(\Delta\mu u_a - (N/2-1))]$.

We include a numeric result obtained for horizontally sheared line problem defined in Chapter 3 by using the method described in this section. The same numeric result is obtained with the method we used in our simulations.

4.2.3 Problem Definition Based on LCT Interpolation Equation

In this part, a different methodology to set the linear relationships is presented which is based on the interpolation formula for the general LCT given in [112]. It is repeated here for convenience:

$$f(u) = \delta u|\beta|\Delta u_{\mathbf{M}} e^{-i\pi\gamma u^2} \sum_{n=-\infty}^{\infty} f(n\delta u) e^{i\pi\gamma(n\delta u)^2} \mathrm{sinc}(\beta\Delta u_{\mathbf{M}}(u - n\delta u)) \qquad (4.21)$$

Since FRT is a special case of LCT, eq. 4.21 can be rewritten for FRT by substituting $\beta = \csc\,(a\pi/2)$ and $\gamma = \cot\,(a\pi/2)$. This interpolation equation is

based on the knowledge of limited extent in FRT domain, instead of the frequency domain.

$$f(u) = e^{-i\pi \cot \alpha u^2} \sum_{n=-\infty}^{\infty} f[n] e^{i\pi \cot \alpha (n\delta u)^2} \mathrm{sinc}(\csc \alpha \Delta u_a u - n) \tag{4.22}$$

Note that we used $\delta u = \frac{1}{|\beta|\Delta u_a}$ to simplify the equation. In order to avoid complication, it is assumed that $|\beta| \geq 0$, which covers the range of $a \in [0, 2]$. This assumption is valid since all the information in any domain can be reduced to a domain $a \in [0, 2]$ as seen in eq. 4.23 and this thesis deals with the range of $a \in (0, 1]$.

$$\begin{aligned} K_{2+a}(u, u') =& \sqrt{1 + i \cot \alpha}\ e^{[i\pi(\cot \alpha u^2 - 2 \csc \alpha (-u) u' + \cot \alpha u'^2)]} \\ K_a(u, u') =& K_{2+a}(-u, u') \end{aligned} \tag{4.23}$$

The method described in this section is based on taking ath order FRT of both sides of the interpolation equation 4.22. The discrete FRT expression we try to obtain by manipulating the interpolation equation is defined in [106] for general LCT as:

$$\hat{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}}) \equiv \delta u \sum_{k=-N_a/2}^{N_a/2-1} \hat{f}(k\delta u) C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) \tag{4.24}$$

for $m = N_a/2, ..., N_a/2 - 1$ where
$C_{\mathbf{M}}(m\delta u_{\mathbf{M}}, k\delta u) = \sqrt{\beta} e^{-\frac{i\pi}{4}} e^{\frac{i\pi}{N_a|\beta|}\left(\alpha \frac{\delta u_{\mathbf{M}}}{\delta u} m^2 - 2\beta k m + \gamma \frac{\delta u}{\delta u_{\mathbf{M}}} k^2\right)}$ and $\hat{f}_{\mathbf{M}}(m\delta u_{\mathbf{M}})$ is defined as discrete LCT of $\hat{f}(k\delta u)$ [117, 118].

Therefore, it is possible to write the dFrFT expression between the reference domain u and FRT domain u_a as:

$$\hat{f}_a(m\delta u_a) \equiv \delta u \sum_{k=-N_a/2}^{N_a/2-1} \hat{f}(k\delta u) C_a(m\delta u_a, k\delta u) \tag{4.25}$$

where $N_a = \Delta u \Delta u_a |\beta|$, $\delta u = \frac{1}{\Delta u_a |\beta|}$ and $\delta u_a = \frac{1}{\Delta u |\beta|}$.

When we truncate the summation in eq. 4.22 since we assumed time/space limited signal, and take the FRT of both sides:

$$f_a(u_a) = \sum_{n=-N_a/2}^{N_a/2-1} f[n] e^{i\pi \cot \alpha (n\delta u)^2} \int_{-\infty}^{\infty} K_a(u_a, u') e^{-i\pi \cot \alpha u'^2} \mathrm{sinc}(\csc \alpha \Delta u_a u' - n) du' \tag{4.26}$$

After expanding the FRT kernel, we obtain:

$$f_a(u_a) = A_\alpha \sum_{n=-N_a/2}^{N_a/2-1} f[n] e^{i\pi \cot\alpha (n\delta u)^2} e^{i\pi \cot\alpha u_a^2} \int_{-\infty}^{\infty} e^{-i\pi 2 \csc\alpha u_a u'} \mathrm{sinc}(\csc\alpha \Delta u_a u' - n) du' \tag{4.27}$$

The integral in eq. 4.27 is a FT expression and is equal to eq. 4.28 when the integral variable is changed to $w = \csc\alpha u'$, and multiplication and shift properties of the FT is applied.

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-i\pi 2 \csc\alpha u_a u'} \mathrm{sinc}(\csc\alpha \Delta u_a u' - n) du' \\ &= \frac{1}{\csc\alpha} \int_{-\infty}^{\infty} e^{-i\pi 2 u_a w} \mathrm{sinc}(\Delta u_a w - n) dw = \frac{1}{\csc\alpha \Delta u_a} \mathrm{rect}\left(\frac{u_a}{\Delta u_a}\right) e^{-i2\pi u_a \frac{n}{\Delta u_a}} \end{aligned} \tag{4.28}$$

When eq. 4.28 is inserted into eq. 4.27 and the rect $\left(\frac{u_a}{\Delta u_a}\right)$ is removed since it does not have any effect due to the limited phase space representation assumption:

$$f_a(u_a) = A_\alpha \delta u \sum_{n=-N_a/2}^{N_a/2-1} f[n] e^{i\pi \cot\alpha (n\delta u)^2} e^{i\pi \cot\alpha u_a^2} e^{-i2\pi u_a \frac{n}{\Delta u_a}} \tag{4.29}$$

This equation is sampled with sampling interval $\delta u_a = \frac{1}{\beta \Delta u}$ in eq. 4.30.

$$f_a(\frac{k}{\beta \Delta u}) = A_\alpha \delta u \sum_{n=-N_a/2}^{N_a/2-1} f[n] e^{i\pi \cot\alpha \frac{n^2}{\beta^2 \Delta u_a^2}} e^{-i\pi \cot\alpha \frac{k^2}{\beta^2 \Delta u^2}} e^{-i2\pi \frac{k}{\beta \Delta u} \frac{n}{\Delta u_a}} \tag{4.30}$$

The final equation is established by noting that $\beta \Delta u \Delta u_a = N_a$ and $\Delta u = \Delta u_a$:

$$f_a[k] = A_\alpha \delta u \sum_{n=-N_a/2}^{N_a/2-1} f[n] e^{\frac{i\pi}{\csc\alpha N_a}\left(\cot\alpha n^2 - 2\csc\alpha k n + \cot\alpha k^2\right)} \tag{4.31}$$

Since eq. 4.31 is equal to eq. 4.25, we conclude that we can use the interpolation formula given in 4.21 for continuous space extent. And the continuous expression that is used to set the linear relationships is eq. 4.29. The N_a by N_a "constructed matrix", $\mathbf{C}$ can be written as:

$$\mathbf{C} = \delta u \begin{bmatrix} C_{11} & \cdots & C_{N_a 1} \\ \vdots & \ddots & \vdots \\ C_{N_a 1} & \cdots & C_{N_a N_a} \end{bmatrix} \tag{4.32}$$

where;

$$
\begin{aligned}
C_{11} =& A_{\alpha_{k_1}} e^{i\pi\left(\cot\alpha_{k_1}(-\frac{N_a}{2})^2\delta u^2+\cot\alpha_{k_1}u_{p_1}^2-2u_{p_1}\frac{-N_a/2}{\Delta u_a}\right)} \\
C_{1N_a} =& A_{\alpha_{k_1}} e^{i\pi\left(\cot\alpha_{k_1}(\frac{N_a}{2}-1)^2\delta u^2+\cot\alpha_{k_1}u_{p_1}^2-2u_{p_1}\frac{N_a/2-1}{\Delta u_a}\right)} \\
C_{N_a1} =& A_{\alpha_{k_n}} e^{i\pi\left(\cot\alpha_{k_n}(-\frac{N_a}{2})^2\delta u^2+\cot\alpha_{k_n}u_{pn}^2-2u_{pn}\frac{-N_a/2}{\Delta u_a}\right)} \\
C_{N_aN_a} =& A_{\alpha_{k_n}} e^{i\pi\left(\cot\alpha_{k_n}(\frac{N_a}{2}-1)^2\delta u^2+\cot\alpha_{k_n}u_{pn}^2-2u_{pn}\frac{N_a/2-1}{\Delta u_a}\right)}
\end{aligned}
\tag{4.33}
$$

However, there is an issue about the choice of the space domain extent of the signal Δu. "Bicanonical width product" for general LCT which gives the minimum number of samples needed to represent a signal is defined as $N_a = \Delta u \Delta u_T |\beta|$ [26]. We assumed that $\Delta u = \Delta u_T$ but $|\beta|$ changes with FRT order a which results in different N_a for each FRT domain. This is not applicable for us since taking different order FRT's of a signal would result in change in the characteristic of it. Therefore, making any comparison between different scenarios would not be logical. A solution to this problem can be choosing N_a and δu such that they are valid for each FRT domain. It is the case if they satisfy eq. 4.34 and 4.35 for each FRT domain u_a.

$$N_a \geq \Delta u \Delta u_a |\beta| \tag{4.34}$$

$$\delta u \leq \frac{1}{\Delta u_a |\beta|} \tag{4.35}$$

To make sure that these conditions are satisfied so that the interpolation equation is applicable, we determined the biggest β to cover all the cases such that:

$$\beta = \csc(a_{\min}\pi/2) = \csc\left(\frac{\pi}{2N_a}\right) \tag{4.36}$$

In our simulations, the smallest FRT order that we take samples on is $a = 1/N_a$. Then Δu should be set to $\sqrt{N_a/\beta}$ and $\delta u = \frac{1}{\beta \Delta u_a}$. Although this method is mathematically correct, Δu is decreased by the ratio of $\sqrt{1/\beta}$ which harms the physical interpretation behind the problem.

A different solution to this issue is described here. Firstly, the unknown signal on the reference plane f is sampled with the Nyquist rate, so that we have the

minimum number of samples, which is enough to recover the signal : N. Since we assumed circular phase space representation, $\Delta u = \Delta\mu = \Delta u_a = \sqrt{N}$ and the sampling interval on the reference plane is $\delta u' = 1/\Delta\mu = 1/\Delta u$ where N is the number of degrees of freedom. However, eq. 4.29 gives a relationship between the sample location and a different discrete representation of the signal on the reference plane. This representation has N_a samples with interval $\delta u = \frac{1}{\beta \Delta u_a}$ on the signal extent Δu.

These two discrete representations of f are related to each other though an interpolation equation. For this purpose, we first write the interpolation equation which enables us to recover the signal by using N samples as seen in eq. 4.37. Then we can evaluate the interpolated signal on the other discrete representation's sample locations, which is equal to $n\delta u$ for nth sample, as in eq. 4.38.

$$f(u) = \sum_{p=-N/2}^{N/2-1} f[p] \operatorname{sinc}(\Delta\mu u - p) \tag{4.37}$$

$$f(n\delta u) = \sum_{p=-N/2}^{N/2-1} f[p] \operatorname{sinc}(\Delta\mu \delta u n - p) = f[n] \tag{4.38}$$

If we substitute $f[n]$ into eq. 4.29:

$$\begin{aligned} f_a(u_a) =& A_\alpha \delta u \sum_{n=-N_a/2}^{N_a/2-1} e^{i\pi \cot\alpha (n\delta u)^2} e^{i\pi \cot\alpha u_a^2} e^{-i2\pi u_a \frac{n}{\Delta u_a}} \\ & \times \left(\sum_{p=-N/2}^{N/2-1} f[p] \operatorname{sinc}(\Delta\mu \delta u n - p) \right) \end{aligned} \tag{4.39}$$

When we change the order of summations, we obtained eq. 4.40.

$$\begin{aligned} f_a(u_a) =& A_\alpha \delta u \sum_{p=-N/2}^{N/2-1} f[p] e^{i\pi \cot\alpha u_a^2} \\ & \times \left(\sum_{n=-N_a/2}^{N_a/2-1} \operatorname{sinc}(\Delta\mu \delta u n - p) e^{-i2\pi u_a \frac{n}{\Delta u_a}} e^{i\pi \cot\alpha (n\delta u)^2} \right) \end{aligned} \tag{4.40}$$

When a comparison between this method and the one we used in our simulations in terms of performance and cost, we have decided on a possible improvement due to the exactness of eq. 4.40 might not be very crucial and it is computationally expensive.

4.2.4 Using Discrete FRT Orders

In the literature, the FRT order a is treated as a continuous argument traditionally. However, in [91, 92] it is suggested that the discretization in u naturally implies a discretization in a. Therefore, it is possible to determine a prior resolution of the FRT domains depending on N. This way, dFrFT matrices can be generated for all possible FRT domains and stored. During simulations, a coordinate of the samples should be rounded to the closest a which is one of the pre-determined domains' FRT order.

4.3 Physical Correspondence of the Problem

In this section, our aim is to set the link between the sample locations described here and the ones on the physical world. Therefore, we need a mapping between the dimensionless u-a frame and the x[m]-z[m] space. We will set this link for graded index medium, free space propagation and quadratic phase systems.

4.3.1 Graded Index Medium

It is known that an optical field is fractional Fourier transformed with order $a = d/d_0$ as it propagates through a graded index media where d is the propagation distance if the scale parameter is chosen as $s = \sqrt{\lambda\chi}$ where χ is the graded index medium parameter. This choice of scale parameter gets rid of the magnification and the quadratic phase term due the the curvature R. Then the physical extent of the signal in space and frequency domains, Δx and $\Delta\sigma_x$ are related to the normalized versions as:

$$\begin{aligned} \Delta x =& \Delta u s = \sqrt{\lambda\chi N}, \\ \Delta\sigma_x =& \frac{\Delta\mu}{s} = \sqrt{\frac{N}{\lambda\chi}} \end{aligned} \tag{4.41}$$

The framework of the simulations performed can be considered as a rectangle

where the horizontal edge goes from $a = 0$ to $a = 1$ and the vertical edge goes from $u = -\Delta u/2$ to $u = \Delta u/2$. The ratio of the edges of this rectangle is :

$$\frac{\Delta x}{d_0} = \frac{4\lambda N}{\chi \pi^2} \tag{4.42}$$

This ratio gives the relationship between our examples and the physical world such that the rectangle are scaled with this ratio. The closest physical system to our working frame is graded index medium.

4.3.2 Free Space Propagation

The input and output planes of an optical system with field distributions $\hat{f}(x)$ and $\hat{g}(x)$, are related to each other by the Fresnel integral:

$$\hat{g}(x) = e^{i2\pi d/\lambda} e^{-i\pi/4} \sqrt{\frac{1}{\lambda d}} \int_{-\infty}^{\infty} e^{\frac{i\pi(x-x')^2}{\lambda d}} \hat{f}(x') dx' \tag{4.43}$$

where λ is the wavelength and d is the propagation distance.

To find the mapping between u-a coordinate system and the free space propagation, the Fresnel integral is decomposed into a FRT, followed by magnification, and followed by chirp multiplication. Then the output field $\hat{g}(x)$ can be expressed as [91, 92]:

$$\hat{g}(x) = e^{i2\pi d\lambda} e^{-ia\pi/4} \sqrt{\frac{1}{sM}}\, e^{\frac{i\pi x^2}{\lambda R}}\, f_a\left(\frac{x}{sM}\right) \tag{4.44}$$

where s is the scaling parameter to convert $\hat{f}(x)$ into dimensionless version $f(u)$ so that FRT can be applied.

$$f(x) = \frac{1}{\sqrt{s}} f(x/s) = \frac{1}{\sqrt{s}} f(u) \tag{4.45}$$

The other parameters are defined as:

$$\begin{aligned} a =& \frac{2}{\pi} \arctan \frac{\lambda d}{s^2}, \\ M =& \sqrt{1 + \frac{\lambda^2 d^2}{s^4}} = \sec \frac{a\pi}{2}, \\ R =& \frac{s^4 + \lambda^2 d^2}{\lambda^2 d} = \csc^2 \frac{a\pi}{2}. \end{aligned} \tag{4.46}$$

The relationship between the order a and the propagation distance is given in eq. 4.46 as $a = \frac{2}{\pi}\arctan\left(\frac{\lambda d}{s^2}\right)$. The scale in the space argument is governed by M and s such that $u = \frac{x}{sM}$.

Please note that, the $a = a_0$ lines in our u-a coordinate system corresponds to spherical surfaces in the real world. Working on the spherical surfaces also enable us to get rid of the chirp factor in the eq. 4.44.

4.3.3 Quadratic Phase Systems

Quadratic phase systems define the optical system with sections of free space, thin lenses ans sections of graded index media and any arbitrary combination of them. The mathematical formulation of quadratic phase systems is the same with LCT such that the relation of the output field is given in terms of the input field by:

$$\hat{g}(x) = \int \hat{h}(x, x')\hat{f}(x')dx' \tag{4.47}$$

where the kernel is defined as:

$$\begin{aligned} h(x, x') =& Ke^{i\pi x^2/\lambda R}\sqrt{\frac{1}{s^2 M}}A_{a\pi/2} \\ &\times exp\left[\frac{i\pi}{s^2}\left(\frac{x^2}{M^2}\cot(a\pi/2) - 2\frac{xx'}{M}\csc(a\pi/2) + x'^2\cot(a\pi/2)\right)\right] \end{aligned} \tag{4.48}$$

This quadratic phase system maps a function $\hat{f}(x) = \frac{1}{s^2}f(x/s) = \frac{1}{s^2}f(u)$ into:

$$\hat{g}(x) = Ke^{\frac{i\pi x^2}{\lambda R}}\sqrt{\frac{1}{sM}}f_a\left(\frac{x}{sM}\right) \tag{4.49}$$

This sets the relationship between quadratic phase systems and FRT. The relation to the matrix parameters is as follows:

$$\begin{aligned} \hat{A} =& M\cos(a\pi/2), \\ \hat{B} =& s^2 M\sin(a\pi/2), \\ \hat{C} =& -\frac{\sin(a\pi/2)}{s^2 M} + \frac{M\cos(a\pi/2)}{\lambda R}, \\ \hat{D} =& \frac{\cos(a\pi/2)}{M} + \frac{s^2 M\sin(a\pi/2)}{\lambda R}. \end{aligned} \tag{4.50}$$

where R is the radius of the spherical surface on which the FRT is observed [28].

It is known that free space, thin lenses and sections of graded index media can be expressed by LCT matrices. Any combination of them can be characterized by the multiplication of those matrices. Therefore, any point in a quadratic phase system can be expressed by the multiplication of those LCT matrices which is another LCT matrix with parameters $\hat{A}, \hat{B}, \hat{C}, \hat{D}$. To set the link between quadratic phase system and our mathematical working frame, these LCT matrix coefficients should be determined first. Then, the FRT order can be found by solving the equation set given in 4.50. The scale in the space argument is governed by M and s such that $u = \frac{x}{sM}$.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this thesis, we worked on a wave propagation based interpolation problem. In the literature, similar problems are discussed in different contents. The main focus of these works are solving a particular problem defined over a specific sampling surface. In this work, however, we did not put any restriction on the sample locations and let them be arbitrarily distributed over the space. Therefore, any of the mentioned problems can be considered as a special case of the problem we deal with.

We formulated the problem by using the analogy between wave propagation and fractional Fourier transformation. We obtained a mathematical working frame: the u-a coordinate system. In this system a is the FRT order. Therefore, it increases along the propagation direction. The dimensionless u is the spatial argument. The linear relationships between the samples taken over the space and the unknown field distribution on the reference plane $\mathbf{f}$, are established in the form of $\mathbf{h} = \mathbf{C}\,\mathbf{f}$ where $\mathbf{h}$ is the vector containing collection of samples. To be able to construct the $\mathbf{C}$ matrix, we first generated dFrFT matrices for each a coordinate. Then, we determined which row of each dFrFT matrix sets the linear relationship

between the sample and unknown field distribution $\mathbf{f}$. For that purpose, the u coordinate of each sample is mapped to the corresponding integer in [-N/2,N/2) range. This way, we found the index of the row of the each dFrFT matrix that we are interested in. Then, we combined these rows and obtain the $\mathbf{C}$ matrix. This constructed $\mathbf{C}$ matrix defines our system. Therefore, to examine the system's performance, we looked for the rank and condition number of this matrix. These concepts give an information about how ill-posed and sensitive the system is. Different distributions of samples are taken into consideration and the effect of samples' location on recoverability of the signal is examined.

It has been seen that when the samples are taken on a vertical line the perfect recovery is possible. However, as the inverse slope of the lines increases and samples starts to be side by side, we observed a increase in the condition number. When the slope is less than 0.875, the system becomes rank deficient. It leads to one of the main results of this work: the samples should not be horizontally spread for a good condition number.

Then we analyzed the closeness of the samples to each other. When we shorten the length of the lines, the rank deficiency is unavoidable. Therefore, taking samples on a larger extent results in better condition numbers. Since the distance between the samples can be expanded by horizontally shearing them, it can improve the condition number up to a point where the side by side'ness effect of the samples does not dominate the condition number. It is also seen that the locations of the samples are not that important. Therefore, shifting a sampling surface in any direction does not have a considerable affect on the solution.

Then the samples are taken on a pair of line with different depths, and periods. Two different scenario are realized: samples are chosen such that the projection of them onto a transversal plane complement or overlap each other. In the case of complementary sample locations, the best results are obtained when the lines are close to each other. When the high period is high, the system becomes rank deficient slightly after the lines are moving away from each other. In the case of lower periods, a stationary behavior is observed where the condition number does not change considerably. When the lines are far from each other, the condition

number increases dramatically. When the samples are overlapping each other, the condition number is not in an acceptable level for high periods. Therefore it can be said that lower periods are better that the higher ones. High period can be considered as sparsely distributed samples while lower ones corresponds to a more nested structure. As the separation between the planes increases, the overlapping and complementary distributions becomes similar for small periods.

Later we examined the samples taken on a circular surface. We observed that the condition number is high for all the radiuses. Then we considered the samples taken all over the u-a coordinate system. We realized that both randomly or uniformly distributed samples, which cover all the area, result in a condition number in the order of 3. However, if the system is strictly symmetric around $u = 0$ axis, an artifact is introduced to the system which results in infinite condition number. The circular surface can also be considered as a special case of having samples all over the plane. In that case, the samples are taken on the boundary only, which turns out not to be enough to recover the signal.

To sum up, it can be stated that there are lots of effects acting on the solution of the recovery problem with distributed samples. In wave propagation problems, signals are fully represented by the samples taken on the transversal planes. If it is not possible to taken all the samples on the transversal planes, they are ahead or behind each other. Thus, the samples influence or being influence by the other samples due to the diffraction of light field. It leads to rank deficient and ill-posed systems. In our analysis, we observed that locating samples side by side increases the condition number. In addition, the samples should not be close to each other. Furthermore, covering all the working frame with samples, which can be considered as the combination of the first two outcomes, results in recoverable systems.

Then, in the discussion chapter, we looked for other possible methodologies. In the process, we analyzed and derived the relationship between interpolation equation and fractional Fourier transformation. We set the link between the methodology we used in our simulations and the ones derived in chapter 4.

Figure 5.1: The u-a coordinate system and sample locations for 2D case

5.2 Future Work

In this section, the future works are listed and briefly explained.

- In this work, the wave propagation problem is solved for the one-dimensional case. However, in the physical systems, the waves propagate through two dimensional transversal planes as seen from figure 5.1. Generalizing our methodology and simulations to the two-dimensional working frame is defined as a future work.

- In this thesis, we worked on a purely mathematical framework where the samples are distributed over different FRT domains. There is a direct connection between this working frame and the physical world. Therefore, we can take samples on the physical world and find the corresponding sample locations in our system. Applying our methodology to that samples is left as a future work. This would increase the understanding of the sampling scheme. Thus, it might result in some conclusions that may not be very deducible in the present frame.

- The problems we deal with are mostly ill-posed and rank deficient. Taking samples more than the number of degrees of freedom of the signal and

analyzing the impact of it on the condition number is another future work.

- We want to explore the other methods and their possible improvement on the solution. A comparison between the methodologies which use discrete u and a, discrete u, continuous a and continuous u and a can be made.

- Different scenarios will be realized. The problems solved for a specific sampling surface can be implemented by our method for comparison.

Bibliography

[1] J. W. Goodman, *Introduction to Fourier optics*. McGraw-Hill, 2nd ed., 1996.

[2] H. Guven, H. M. Ozaktas, A. E. Cetin, and B. Barshan, "Signal recovery from partial fractional Fourier domain information and its applications," *IET Signal Processing*, vol. 2, no. 1, pp. 15–25, 2008.

[3] V. Yücesoy, "Improving the resolution of diffraction patterns from many low resolution rcordings," Master's thesis, Bilkent University, Ankara, Turkey, 9 2010.

[4] L. Onural, A. Gotchev, H. M. Ozaktas, and E. Stoykova, "A survey of signal processing problems and tools in holographic three-dimensional television," *IEEE Transactions on Circuits and Systems for Video Technology*, vol. 17, no. 11, pp. 1631–1646, 2007.

[5] V. Uzunov, G. B. Esmer, A. Gotchev, L. Onural, and H. M. Ozaktas, "Bessel functions-based reconstruction of non-uniformly sampled diffraction fields," in *3DTV Conference 2007*, pp. 1–4, IEEE, 2007.

[6] E. Şahin and L. Onural, "Calculation of the scalar diffraction field from curved surfaces by decomposing the three-dimensional field into a sum of Gaussian beams," *JOSA A*, vol. 30, no. 3, pp. 527–536, 2013.

[7] G. B. Esmer, L. Onural, and H. M. Ozaktas, "Exact diffraction calculation from fields specified over arbitrary curved surfaces," *Optics Communications*, vol. 284, no. 24, pp. 5537–5548, 2011.

[8] G. B. Esmer, V. Uzunov, L. Onural, A. Gotchev, and H. M. Ozaktas, "Reconstruction of scalar diffraction field from distributed data points over 3d space," in *3DTV Conference 2007*, pp. 1–4, IEEE, 2007.

[9] E. Şahin and L. Onural, "Scalar diffraction field calculation from curved surfaces via Gaussian beam decomposition," *JOSA A*, vol. 29, no. 7, pp. 1459–1469, 2012.

[10] L. Onural, "Signal processing and 3DTV," *IEEE Signal Processing Magazine*, vol. 27, no. 5, pp. 144–142, 2010.

[11] E. Ulusoy, G. B. Esmer, H. M. Ozaktas, L. Onural, A. Gotchev, and V. Uzunov, "Signal processing problems and algorithms in display side of 3DTV," in *IEEE International Conference on Image Processing 2006*, pp. 2985–2988, IEEE, 2006.

[12] G. B. Esmer, V. Uzunov, L. Onural, H. M. Ozaktas, and A. Gotchev, "Diffraction field computation from arbitrarily distributed data points in space," *Signal Processing: Image Communication*, vol. 22, no. 2, pp. 178–187, 2007.

[13] M. Huebschman, B. Munjuluri, and H. Garner, "Dynamic holographic 3-d image projection," *Optics Express*, vol. 11, no. 5, pp. 437–445, 2003.

[14] M. Lucente, *Diffraction-specific fringe computation for electro-holography.* PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, USA, 1994.

[15] K. Matsushima, "Computer-generated holograms for three-dimensional surface objects with shade and texture," *Applied Optics*, vol. 44, no. 22, pp. 4607–4614, 2005.

[16] G. B. Esmer, L. Onural, H. M. Ozaktas, V. Uzunov, and A. Gotchev, "Performance assessment of a diffraction field computation method based on source model," in *3DTV Conference: The True Vision-Capture, Transmission and Display of 3D Video 2008*, pp. 257–260, IEEE, 2008.

[17] S. Ganci, "Fourier diffraction through a tilted slit," *European Journal of Physics*, vol. 2, no. 3, pp. 158–150, 1981.

[18] T. Tommasi and B. Bianco, "Frequency analysis of light diffraction between rotated planes," *Optics Letters*, vol. 17, no. 8, pp. 556–558, 1992.

[19] G. B. Esmer and L. Onural, "Computation of holographic patterns between tilted planes," in *Holography 2005: International Conference on Holography, Optical Recording, and Processing of Information*, pp. 62521K–1–62521K–6, International Society for Optics and Photonics, 2006.

[20] L. Onural, "Exact solution for scalar diffraction between tilted and translated planes using impulse functions over a surface," *JOSA A*, vol. 28, no. 3, pp. 290–295, 2011.

[21] G. B. Esmer and L. Onural, "Simulation of scalar optical diffraction between arbitrarily oriented planes," in *First International Symposium on Control, Communications and Signal Processing 2004*, pp. 225–228, IEEE, 2004.

[22] N. Delen and B. Hooker, "Free-space beam propagation between arbitrarily oriented planes based on full diffraction theory: a fast Fourier transform approach," *JOSA A*, vol. 15, no. 4, pp. 857–867, 1998.

[23] M. G. Ertosun, H. Atli, H. M. Ozaktas, and B. Barshan, "Complex signal recovery from multiple fractional Fourier-transform intensities," *Applied Optics*, vol. 44, no. 23, pp. 4902–4908, 2005.

[24] M. G. Ertosun, H. Atlı, H. M. Ozaktas, and B. Barshan, "Complex signal recovery from two fractional Fourier transform intensities: order and noise dependence," *Optics Communications*, vol. 244, no. 1, pp. 61–70, 2005.

[25] F. S. Öktem and H. M. Özaktaş, "Linear algebraic analysis of fractional Fourier domain interpolation," in *IEEE 17th Signal Processing and Communications Applications Conference, 2009*, pp. 873–875, 2009.

[26] S. F. Öktem, "Signal representation and recovery under partial information, redundancy, and generalized finite extent constraints," Master's thesis, Bilkent University, Ankara, Turkey, 7 2009.

[27] H. E. Guven, "Signal recovery from partial fractional Fourier domain information and pulse shape design using iterative projections," Master's thesis, Bilkent University, Ankara, Turkey, 7 2005.

[28] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform with Applications in Optics and Signal Processing.* New York: Wiley, 2001.

[29] C. E. Shannon, "Communication in the presence of noise," *Proceedings of the IRE*, vol. 37, no. 1, pp. 10–21, 1949.

[30] J. M. Whittaker, "The Fourier theory of the cardinal function," *Proceedings of the Edinburgh Mathematical Society (Series 2)*, vol. 1, no. 03, pp. 169–176, 1928.

[31] V. A. Kotel'nikov, "On the transmission capacity of'ether'and wire in electric communications," *Physics-Uspekhi*, vol. 49, no. 7, pp. 736–744, 2006.

[32] H. Nyquist, "Certain topics in telegraph transmission theory," *Transactions of the American Institute of Electrical Engineers*, vol. 47, no. 2, pp. 617–644, 1928.

[33] D. Slepian, "On bandwidth," *Proceedings of the IEEE*, vol. 64, no. 3, pp. 292–300, 1976.

[34] J. A. Parker, R. V. Kenyon, and D. E. Troxel, "Comparison of interpolating methods for image resampling," *IEEE Transactions on Medical Imaging*, vol. 2, no. 1, pp. 31–39, 1983.

[35] M. Unser, "Sampling-50 years after Shannon," *Proceedings of the IEEE*, vol. 88, no. 4, pp. 569–587, 2000.

[36] E. Meijering, "A chronology of interpolation: from ancient astronomy to modern signal and image processing," *Proceedings of the IEEE*, vol. 90, no. 3, pp. 319–342, 2002.

[37] A. G. García, "A brief walk through sampling theory," *Advances in Imaging and Electron Physics*, vol. 124, pp. 64–139, 2002.

[38] A. Aldroubi and K. Gröchenig, "Nonuniform sampling and reconstruction in shift-invariant spaces," *SIAM Review*, vol. 43, no. 4, pp. 585–620, 2001.

[39] A. Shahmansoori, "Reconstruction of periodic nonuniform samples using Lagrange interpolation in time-interleaved ADCs," *Signal, Image and Video Processing*, vol. 9, no. 2, pp. 471–479, 2015.

[40] L. B. Almeida, "The fractional Fourier transform and time-frequency representations," *IEEE Transactions on Signal Processing*, vol. 42, no. 11, pp. 3084–3091, 1994.

[41] A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," *JOSA A*, vol. 10, no. 10, pp. 2181–2186, 1993.

[42] D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transforms and their optical implementation: I," *JOSA A*, vol. 10, no. 9, pp. 1875–1881, 1993.

[43] H. M. Ozaktas, D. Mendlovic, L. Onural, and B. Barshan, "Convolution, filtering, and multiplexing in fractional Fourier domains and their relation to chirp and wavelet transforms," *JOSA A*, vol. 11, no. 2, pp. 547–559, 1994.

[44] H. M. Ozaktas and D. Mendlovic, "Fourier transforms of fractional order and their optical interpretation," *Optics Communications*, vol. 101, no. 3, pp. 163–169, 1993.

[45] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transforms and their optical implementation. ii," *JOSA A*, vol. 10, no. 12, pp. 2522–2531, 1993.

[46] M. A. Kutay, H. M. Ozaktas, O. Arıkan, and L. Onural, "Optimal filtering in fractional Fourier domains," *IEEE Transactions on Signal Processing*, vol. 45, no. 5, pp. 1129–1143, 1997.

[47] G. Cariolaro, T. Erseghe, P. Kraniauskas, and N. Laurenti, "A unified framework for the fractional Fourier transform," *IEEE Transactions on Signal Processing*, vol. 46, no. 12, pp. 3206–3219, 1998.

[48] D. Mendlovic, Z. Zalevsky, and H. M. Ozaktas, "Applications of the fractional Fourier transform to optical pattern recognition," *Optical Pattern Recognition*, pp. 89–125, 1998.

[49] M. F. Erden, M. A. Kutay, and H. M. Ozaktas, "Repeated filtering in consecutive fractional Fourier domains and its application to signal restoration," *IEEE Transactions on Signal Processing*, vol. 47, no. 5, pp. 1458–1462, 1999.

[50] G. Cariolaro, T. Erseghe, P. Kraniauskas, and N. Laurenti, "Multiplicity of fractional Fourier transforms and their relationships," *IEEE Transactions on Signal Processing*, vol. 48, no. 1, pp. 227–241, 2000.

[51] M. A. Kutay, H. Özaktaş, H. M. Ozaktas, and O. Arıkan, "The fractional Fourier domain decomposition," *Signal Processing*, vol. 77, no. 1, pp. 105–109, 1999.

[52] M. A. Kutay and H. M. Ozaktas, "Optimal image restoration with the fractional Fourier transform," *JOSA A*, vol. 15, no. 4, pp. 825–833, 1998.

[53] B. Barshan and B. Ayrulu, "Fractional Fourier transform pre-processing for neural networks and its application to object recognition," *Neural Networks*, vol. 15, no. 1, pp. 131–140, 2002.

[54] S.-C. Pei, M.-H. Yeh, and C.-C. Tseng, "Discrete fractional Fourier transform based on orthogonal projections," *IEEE Transactions on Signal Processing*, vol. 47, no. 5, pp. 1335–1348, 1999.

[55] M. J. Bastiaans and A. J. Van Leest, "From the rectangular to the quincunx Gabor lattice via fractional Fourier transformation," *IEEE Signal Processing Letters*, vol. 5, no. 8, pp. 203–205, 1998.

[56] A. Zayed *et al.*, "A convolution and product theorem for the fractional Fourier transform," *IEEE Signal Processing Letters*, vol. 5, no. 4, pp. 101–103, 1998.

[57] O. Akay and G. F. Boudreaux-Bartels, "Fractional convolution and correlation via operator methods and an application to detection of linear fm

signals," *IEEE Transactions on Signal Processing*, vol. 49, no. 5, pp. 979–993, 2001.

[58] T. Alieva and M. J. Bastiaans, "On fractional Fourier transform moments," *IEEE Signal Processing Letters*, vol. 7, no. 11, pp. 320–323, 2000.

[59] S.-C. Pei and M.-H. Yeh, "The discrete fractional cosine and sine transforms," *IEEE Transactions on Signal Processing*, vol. 49, no. 6, pp. 1198–1207, 2001.

[60] İ. Ş. Yetik, H. M. Ozaktas, B. Barshan, and L. Onural, "Perspective projections in the space-frequency plane and fractional Fourier transforms," *JOSA A*, vol. 17, no. 12, pp. 2382–2390, 2000.

[61] S. Liu, L. Yu, and B. Zhu, "Optical image encryption by cascaded fractional Fourier transforms with random phase filtering," *Optics Communications*, vol. 187, no. 1, pp. 57–63, 2001.

[62] J. T. Sheridan and R. Patten, "Holographic interferometry and the fractional Fourier transformation," *Optics Letters*, vol. 25, no. 7, pp. 448–450, 2000.

[63] H. M. Ozaktas and O. Aytür, "Fractional Fourier domains," *Signal Processing*, vol. 46, no. 1, pp. 119–124, 1995.

[64] L. Barker, C. Candan, T. Hakioğlu, M. A. Kutay, and H. M. Ozaktas, "The discrete harmonic oscillator, Harper's equation, and the discrete fractional Fourier transform," *Journal of Physics A: Mathematical and General*, vol. 33, no. 11, pp. 2209–2222, 2000.

[65] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *IMA Journal of Applied Mathematics*, vol. 25, no. 3, pp. 241–265, 1980.

[66] H. M. Ozaktas, M. A. Kutay, and C. Candan, "Fractional Fourier transform," in *Transforms and Applications Handbook* (A. D. Poularikas, ed.), ch. 14, Boca Raton: CRC Press, 2010.

[67] A. Cetin, H. Özaktaş, and H. Ozaktas, "Resolution enhancement of low resolution wavefields with POCS algorithm," *Electronics Letters*, vol. 39, no. 25, pp. 1808–1810, 2003.

[68] L. Chen and D. Zhao, "Optical color image encryption by wavelength multiplexing and lensless Fresnel transform holograms," *Optics Express*, vol. 14, no. 19, pp. 8552–8560, 2006.

[69] L. Chen and D. Zhao, "Color information processing (coding and synthesis) with fractional Fourier transforms and digital holography," *Optics Express*, vol. 15, no. 24, pp. 16080–16089, 2007.

[70] S. Chiu, "Application of fractional Fourier transform to moving target indication via along-track interferometry," *EURASIP Journal on Applied Signal Processing*, vol. 2005, pp. 3293–3303, 2005.

[71] S. Coëtmellec, D. Lebrun, and C. Özkul, "Application of the two-dimensional fractional-order Fourier transformation to particle field digital holography," *JOSA A*, vol. 19, no. 8, pp. 1537–1546, 2002.

[72] L. Cohen, *Time-frequency analysis*, vol. 1. Englewood Cliffs,NJ: Prentice Hall, 1995.

[73] W.-X. Cong, N.-X. Chen, and B.-Y. Gu, "Beam shaping and its solution with the use of an optimization method," *Applied Optics*, vol. 37, no. 20, pp. 4500–4503, 1998.

[74] I. Djurovic, S. Stankovic, and I. Pitas, "Digital watermarking in the fractional Fourier transformation domain," *Journal of Network and Computer Applications*, vol. 24, no. 2, pp. 167–173, 2001.

[75] B.-Z. Dong, Y. Zhang, B.-Y. Gu, and G.-Z. Yang, "Numerical investigation of phase retrieval in a fractional Fourier transform," *JOSA A*, vol. 14, no. 10, pp. 2709–2714, 1997.

[76] R. G. Dorsch and A. W. Lohmann, "Fractional Fourier transform used for a lens-design problem," *Applied Optics*, vol. 34, no. 20, pp. 4111–4112, 1995.

[77] R. G. Dorsch, A. W. Lohmann, Y. Bitran, D. Mendlovic, and H. M. Ozaktas, "Chirp filtering in the fractional Fourier domain," *Applied Optics*, vol. 33, no. 32, pp. 7599–7602, 1994.

[78] D. Dragoman and M. Dragoman, "Near and far field optical beam characterization using the fractional Fourier transform," *Optics Communications*, vol. 141, no. 1, pp. 5–9, 1997.

[79] D. Dragoman, M. Dragoman, and K.-H. Brenner, "Variant fractional Fourier transformer for optical pulses," *Optics Letters*, vol. 24, no. 14, pp. 933–935, 1999.

[80] L. Durak and O. Arikan, "Short-time Fourier transform: two fundamental properties and an optimal implementation," *IEEE Transactions on Signal Processing*, vol. 51, no. 5, pp. 1231–1242, 2003.

[81] S.-i. Jin, Y.-S. Bae, and S.-Y. Lee, "Generalized Vander Lugt correlator as an optical pattern classifier and its optimal learning rate," *Optics Communications*, vol. 206, no. 1, pp. 19–25, 2002.

[82] F. Di, Y. Yingbai, L. Si, K. Tian, and J. Guofan, "Designing diffractive phase plates for beam smoothing in the fractional Fourier domain," *Journal of Modern Optics*, vol. 49, no. 7, pp. 1125–1133, 2002.

[83] R. Khanna, K. Singh, and R. Saxena, "Fractional Fourier transform based beamforming for next generation wireless communication systems," *IETE Technical Review*, vol. 21, no. 5, pp. 357–366, 2004.

[84] A. Koc, H. M. Ozaktas, C. Candan, and M. A. Kutay, "Digital computation of linear canonical transforms," *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2383–2394, 2008.

[85] A. Bultheel and H. Martinez-Sulbaran, "Recent developments in the theory of the fractional Fourier and linear canonical transforms," *Bulletin of the Belgian Mathematicak Society-Simon Stevin*, vol. 13, pp. 971–1005, 2006.

[86] H. M. Ozaktas and M. A. Kutay, "The fractional Fourier transform with applications in optics and signal processing-supplementary bibliography," 2008.

[87] H. M. Ozaktas, M. A. Kutay, and D. Mendlovic, "Introduction to the fractional Fourier transform and its applications," in *Advances in Imaging and Electrol Physics 106*, pp. 239–291, San Diego: Academic Press, 1999.

[88] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," *JOSA A*, vol. 12, no. 4, pp. 743–751, 1995.

[89] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transform as a tool for analyzing beam propagation and spherical mirror resonators," *Optics Letters*, vol. 19, no. 21, pp. 1678–1680, 1994.

[90] M. F. Erden, H. M. Ozaktas, and D. Mendlovic, "Synthesis of mutual intensity distributions using the fractional Fourier transform," *Optics Communications*, vol. 125, no. 4, pp. 288–301, 1996.

[91] H. M. Ozaktas, S. Ö. Arık, and T. Coşkun, "Fundamental structure of Fresnel diffraction: natural sampling grid and the fractional Fourier transform," *Optics Letters*, vol. 36, no. 13, pp. 2524–2526, 2011.

[92] H. M. Ozaktas, S. Ö. Arık, and T. Coşkun, "Fundamental structure of Fresnel diffraction: longitudinal uniformity with respect to fractional Fourier order," *Optics Letters*, vol. 37, no. 1, pp. 103–105, 2012.

[93] D. Mustard, "The fractional Fourier transform and the Wigner distribution," *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, vol. 38, no. 02, pp. 209–219, 1996.

[94] S.-C. Pei, C.-C. Tseng, M.-H. Yeh, and J.-J. Shyu, "Discrete fractional Hartley and Fourier transforms," *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing*, vol. 45, no. 6, pp. 665–675, 1998.

[95] H. M. Ozaktas, O. Arikan, M. A. Kutay, and G. Bozdağı, "Digital computation of the fractional Fourier transform," *IEEE Transactions on Signal Processing*, vol. 44, no. 9, pp. 2141–2150, 1996.

[96] O. Arikan, M. A. Kutay, H. M. Ozaktas, and O. K. Akdemir, "The discrete fractional Fourier transformation," in *Proc. Int. Symp. Time-Frequency, Time-Scale Analysis*, pp. 205–207, 1996.

[97] N. M. Atakishiyev and K. B. Wolf, "Fractional Fourier-Kravchuk transform," *JOSA A*, vol. 14, no. 7, pp. 1467–1477, 1997.

[98] B. W. Dickinson and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 30, no. 1, pp. 25–31, 1982.

[99] B. Santhanam and J. H. McClellan, "The discrete rotational Fourier transform," *IEEE Transactions on Signal Processing*, vol. 44, no. 4, pp. 994–998, 1996.

[100] S.-C. Pei and M.-H. Yeh, "Improved discrete fractional Fourier transform," *Optics Letters*, vol. 22, no. 14, pp. 1047–1049, 1997.

[101] S.-C. Pei, W.-L. Hsue, and J.-J. Ding, "Discrete fractional Fourier transform based on new nearly tridiagonal commuting matrices," *IEEE Transactions on Signal Processing*, vol. 54, no. 10, pp. 3815–3828, 2006.

[102] S.-C. Pei, M.-H. Yeh, and C.-C. Tseng, "Discrete fractional Fourier transform based on orthogonal projections," *IEEE Transactions on Signal Processing*, vol. 47, no. 5, pp. 1335–1348, 1999.

[103] J. G. Vargas-Rubio and B. Santhanam, "On the multiangle centered discrete fractional Fourier transform," *IEEE Signal Processing Letters*, vol. 12, no. 4, pp. 273–276, 2005.

[104] B. Santhanam and J. G. Vargas-Rubio, "On the Grunbaum commuter based discrete fractional Fourier transform," in *IEEE International Conference on Acoustics, Speech, and Signal Processing, 2004. Proceedings.(ICASSP'04)*, vol. 2, pp. 641–641, IEEE, 2004.

[105] C. Candan, M. A. Kutay, and H. M. Ozaktas, "The discrete fractional Fourier transform," *IEEE Transactions on Signal Processing*, vol. 48, no. 5, pp. 1329–1337, 2000.

[106] F. S. Oktem and H. M. Ozaktas, "Exact relation between continuous and discrete linear canonical transforms," *IEEE Signal Processing Letters*, vol. 16, no. 8, pp. 727–730, 2009.

[107] S.-C. Pei, M.-H. Yeh, and T.-L. Luo, "Fractional Fourier series expansion for finite signals and dual extension to discrete-time fractional Fourier transform," *IEEE Transactions on Signal Processing*, vol. 47, no. 10, pp. 2883–2888, 1999.

[108] H. Sundaram, S. D. Joshi, and R. Bhatt, "Scale periodicity and its sampling theorem," *IEEE Transactions on Signal Processing*, vol. 45, no. 7, pp. 1862–1865, 1997.

[109] X.-G. Xia, "On bandlimited signals with fractional Fourier transform," *IEEE Signal Processing Letters*, vol. 3, no. 3, pp. 72–74, 1996.

[110] A. I. Zayed, "On the relationship between the Fourier and fractional Fourier transforms," *IEEE Signal Processing Letters*, vol. 3, no. 12, pp. 310–311, 1996.

[111] A. I. Zayed and A. G. Garcıa, "New sampling formulae for the fractional Fourier transform," *Signal Processing*, vol. 77, no. 1, pp. 111–114, 1999.

[112] H. M. Özaktaş and F. S. Öktem, "Linear canonical transforms, degrees of freedom, and sampling in optical signals and systems," in *Signal Processing and Communications Applications Conference (SIU), 2014 22nd*, pp. 429–432, IEEE, 2014.

[113] Ç. Candan and H. M. Ozaktas, "Sampling and series expansion theorems for fractional Fourier and other transforms," *Signal Processing*, vol. 83, no. 11, pp. 2455–2457, 2003.

[114] T. Erseghe, P. Kraniauskas, and G. Cariolaro, "Unified fractional Fourier transform and sampling theorem," *IEEE Transactions on Signal Processing*, vol. 47, no. 12, pp. 3419–3423, 1999.

[115] R. Tao, B. Deng, W.-Q. Zhang, and Y. Wang, "Sampling and sampling rate conversion of band limited signals in the fractional Fourier transform domain," *IEEE Transactions on Signal Processing*, vol. 56, no. 1, pp. 158–171, 2008.

[116] M. T. Heath, *Scientific Computing: An Introductory Survey.* New York: Mc Graw Hill, 2002.

[117] S.-C. Pei and J.-J. Ding, "Closed-form discrete fractional and affine Fourier transforms," *IEEE Transactions on Signal Processing*, vol. 48, no. 5, pp. 1338–1353, 2000.

[118] B. M. Hennelly and J. T. Sheridan, "Fast numerical algorithm for the linear canonical transform," *JOSA A*, vol. 22, no. 5, pp. 928–937, 2005.