ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

HESSENBERG MATRICES WITH SECOND ORDER RECURRENCE RELATION ENTRIES

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

KÜBRA KARATAŞ SELAM

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“With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by academic rules and ethical principles all works that are not result of this study are cited properly.”

Kübra KARATAŞ SELAM

ABSTRACT

Master of Science Thesis

HESSENBERG MATRICES WITH SECOND ORDER RECURRENCE RELATION ENTRIES

Kübra KARATAŞ SELAM

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Assist. Prof. Dr. Can Murat DİKMEN
August 2024, 43 pages

In this thesis, studies on the properties of Jacobsthal sequences and Fibonacci-like sequences were examined, resulting in findings about the preliminary characteristics and properties of the Jacobsthal-like sequences.

Additionally, in this study, $n \times n$ Hessenberg matrices were defined by considering research on Fibonacci-like sequences, and the relationships between generalized Jacobsthal-like sequences were explored by analyzing the determinants and permanents of these matrices.

This thesis consists of five chapters.

In Chapter 1, the history of Fibonacci and Lucas sequences, which are considered the foundation of integer sequences, and the relationships between these numbers and matrices have been presented. The definitions and fundamental properties of certain integer sequences

ABSTRACT (continued)

that hold an important place in the literature have been provided. Examples from studies related to the determinants and permanents of Fibonacci-type integer sequences and Hessenberg matrices have also been presented.

In Chapter 2, generalized Jacobsthal-like sequences have been defined using Jacobsthal and Jacobsthal-Lucas sequences, and the algebraic properties of these sequences, such as the Binet's formula, generating functions, Simson formula, and sum formula, have been discussed. Additionally, other summation formulas, including the sum of even and odd indices and the alternating sum of generalized Jacobsthal-like sequences, have been proven.

In Chapter 3, the definitions of Hessenberg matrices and tridiagonal matrices have been provided. Then, the basic properties of permanent and determinant functions have been presented. Finally, a method for calculating the permanents of matrices has been explained.

In Chapter 4, the results from our study on the relationship between the determinants and permanents of Hessenberg matrices defined by generalized Jacobsthal-like sequences have been presented.

In Chapter 5, comments on the thesis and suggestions for future studies have been provided.

Keywords: Jacobsthal sequences, Jacobsthal- Lucas sequences, generalized Jacobsthal-like sequences, Hessenberg matrix, determinant, permanent.

Science Code: 403.01.01

ÖZET

Yüksek Lisans Tezi

İKİNCİ DERECEDEN TEKRARLAMA BAĞINTILI GİRİŞLİ HESSENBERG MATRİSLERİ

Kübra KARATAŞ SELAM

Zonguldak Bülent Ecevit Üniversitesi
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Bu tezde Jacobsthal dizileri ve Fibonacci benzeri dizilerin özellikleri üzerine yapılan çalışmalar incelenmiş ve Jacobsthal benzeri dizilerin özellikleri hakkında bulgular elde edilmiştir.

Ayrıca bu çalışmada Fibonacci benzeri diziler üzerinde yapılan araştırmalar dikkate alınarak $n \times n$ Hessenberg matrisleri tanımlanmış ve bu matrislerin determinantları ve permanentları analiz edilerek genelleştirilmiş Jacobsthal benzeri diziler arasındaki ilişkiler araştırılmıştır.

Bu tez beş bölümden oluşmaktadır.

Bölüm 1'de, tam sayı dizilerinin temeli olarak düşünülen Fibonacci ve Lucas dizilerinin tarihçesi ve bu sayılarla matrisler arasındaki ilişkiler sunulmuştur. Literatürde önemli bir yere sahip olan bazı tam sayı dizilerinin tanımları ve temel özellikleri verilmiştir. Fibonacci tipi tam sayı dizileri ve Hessenberg matrislerinin determinantları ve permanentleri ile ilgili yapılan çalışmalardan örnekler sunulmuştur.

ÖZET (devam ediyor)

Bölüm 2'de, Jacobsthal ve Jacobsthal –Lucas dizileri kullanılarak genelleştirilmiş Jacobsthal-Benzeri diziler tanımlanmış ve bu dizilerin Binet formülü, üreten fonksiyonları, Simson formülü ve toplam formülü gibi cebirsel özellikleri sunulmuştur. Ayrıca çift ve tek indekslerin toplamı ve genelleştirilmiş Jacobsthal-benzeri dizilerin alterne toplamı gibi diğer toplama formülleri de kanıtlanmıştır.

Bölüm 3'te, Hessenberg matrisleri ve üç köşegen matrislerin tanımları verilmiştir. Daha sonra permanent ve determinant fonksiyonlarının temel özellikleri sunulmuştur. Son olarak matrislerin permanentlerini hesaplamak için bir yöntem anlatılmıştır.

Bölüm 4'te, genelleştirilmiş Jacobsthal benzeri dizilerle tanımlanan Hessenberg matrislerin determinantları ve permenantler arasındaki ilişkiye yönelik çalışmamızdan elde edilen sonuçlar sunulmuştur.

Bölüm 5'te, tezle ve gelecek çalışmalarla ilgili yorumlar verilmiştir.

Anahtar Kelimeler: Jacobsthal dizileri, Jacobsthal-Lucas dizileri, genelleştirilmiş Jacobsthal-benzeri diziler, Hessenberg matrisi, determinant, permanent.

Bilim Kodu: 403.01.01

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

$\{F_n\}$: Fibonacci sequence

F_n : n-th Fibonacci number

$\{L_n\}$: Lucas sequence

L_n : n-th Lucas number

$\{P_n\}$: Pell sequence

P_n : n-th Pell number

$\{Q_n\}$: Pell-Lucas sequence

Q_n : n-th Pell number

$\{J_n\}$: Jacobsthal sequence

J_n : n-th Jacobsthal number

$\{j_n\}$: Jacobsthal-Lucas sequence

j_n : n-th Jacobsthal-Lucas number

$\{D_n\}$: Generalized Fibonacci-like sequence

D_n : n-th Generalized Fibonacci-like number

$\{V_n\}$, $\{B_n\}$, $\{R_n\}$, $\{K_n\}$: Generalized Jacobsthal-like sequences

V_n, B_n, R_n, K_n : n-th Generalized Jacobsthal-like numbers

A_n : n-square matrix

$T_n(m)$: n-square Hessenberg Matrix for B_n and R_n, for determinant

$U_n(s)$: n-square Hessenberg Matrix for K_n, for determinant

$C_n(m)$: n-square Hessenberg Matrix for V_n, for determinant

$P_n(m)$: n-square Hessenberg Matrix for B_n and R_n, for permanent

$P_n(s)$: n-square Hessenberg Matrix for K_n, for permanent

$S_n(m)$: n-square Hessenberg Matrix for V_n, for permanent

LIST OF SYMBOLS AND ABBREVIATIONS (continued)

ABBREVIATIONS

$\mathbf{det}\,\boldsymbol{A_n}$: Determinant of matrix A_n

$\mathbf{det}\,\boldsymbol{T_n(m)}$: Determinant of matrix $T_n(m)$

$\mathbf{det}\,\boldsymbol{U_n(s)}$: Determinant of matrix $U_n(s)$

$\mathbf{det}\,\boldsymbol{C_n(m)}$: Determinant of matrix $C_n(m)$

$\mathbf{per}\,\boldsymbol{A_n}$: Permanent of matrix A_n

$\mathbf{per}\,\boldsymbol{P_n(m)}$: Permanent of matrix $P_n(m)$

$\mathbf{per}\,\boldsymbol{P_n(s)}$: Permanent of matrix $P_n(s)$

$\mathbf{per}\,\boldsymbol{S_n(m)}$: Permanent of matrix $S_n(m)$

CHAPTER 1

INTRODUCTION

Among the greatest mathematicians of the Middle Ages was Leonardo Fibonacci, whose life remains somewhat shrouded in mystery. It is believed that he was born in the city of Pisa in Italy in the 1170s. Due to his father's profession, he was assigned to the North African port of Bugia, where he received mathematics lessons from an Arab teacher. It was here that he learned about Hindu-Arabic numerals (1, 2, 3, ...) from the Arab culture. Fibonacci is recognized as the person who introduced mathematics from the Arab world to Europe. Information about Fibonacci is primarily obtained through the books he wrote. His most famous work, "Liber Abaci" (The Book of Calculation), written in 1202, is considered the first book he authored. In addition to this, he wrote other mathematical books such as "Practice Geometria" (The Practice of Geometry) (1220), "Liber Quadratorum" (The Book of Square Numbers) (1225), and "Flos" (The Flower) (1225). Among his works, "Liber Abaci" stands out, as it explains the Fibonacci numbers, which are obtained as ratios of successive terms, known as the Golden Ratio. In this book, Fibonacci presents a problem related to his friend's rabbit breeding. Initially, there is one male and one female rabbit in the breeding enclosure. One month later, these rabbits mature and produce a litter. Each month, the mature rabbits give birth to another litter of rabbits. Assuming the rabbits do not die, Fibonacci poses the question of how many rabbits will be in the breeding enclosure in the nth month if they continue to reproduce in this manner. This question leads to the derivation of the Fibonacci sequence. Let's consider that in the first month, there is one newborn rabbit. Since they have not yet reproduced, there is still one rabbit in the enclosure in the second month. In the third month, the two mature rabbits will produce a litter, resulting in three rabbits in the enclosure. If the number of rabbit pairs in the farm at the nth month is denoted by F_n, where $F_1 = 1$ and $F_2 = 1$, then for $n \geq 1$, is determined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$. In other words, each term after the first two is found by adding the two preceding terms (Koshy, 2001). Hence, Fibonacci numbers are represented as

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987...

and so on.

When one of the consecutive terms of the Fibonacci numbers is divided on one side, the resulting ratios gradually converge to the Golden Ratio. The Golden Ratio, which is the number 1,618…, is one of the most harmonious ratios in the structure and shapes of living and non-living entities in nature. For example, it can be seen in the rates in the shaped structures of the sunflower, in the pinecone, in the shell of the snail, and in the human body. In addition, it is frequently used in different branches of science such as finance, computer science, cryptography, architecture, and art and operates in many fields. Many studies have been conducted on Fibonacci sequences, uncovering numerous identities. For instance, the general method of obtaining any term in the Fibonacci sequence without calculating the preceding terms was discovered by Jacques Philippe Marie Binet. Over time, similar formulas have been introduced into the literature for other number sequences and polynomials. These formulas are commonly referred to as Binet's formula or Binet-like formulas and have been utilized to represent the general term. Consequently, through Binet’s formulas, many new identities have been derived for Fibonacci sequences and other mathematical structures (Koshy, 2001).

Fibonacci numbers are used in almost every branch of mathematics (number theory, differential equations, probability, statistics, numerical analysis, linear algebra). In addition, Fibonacci numbers have a wide range of applications in biology, chemistry, cryptology and electrical engineering (Philippou et al., 2001). In recent years, especially in physics, Fibonacci and Lucas number sequences have a wide range of applications (Kiliç and Stakhov, 2009).

Considering that many generalizations of Fibonacci type sequences have been found, the necessity of a study that will encompass as many of them as possible is revealed. In recent years, some techniques have been developed in order to calculate the desired terms of these number sequences, and some new methods have been proposed. One of these methods involves defining various Hessenberg matrices and calculating the terms of Fibonacci and Lucas-like numbers and some of their generalizations using the determinants and permanents of these matrices. When considering that computer algorithms can also be used in these calculations, it is important for these matrices to be mathematically expressed easily. While calculating maybe the first 100 terms of Fibonacci type numbers using recursion can be easily done with the help of a computer, for larger numbers, an algorithm other than recursive relationship is needed (Şahin, 2013).

Over the past 50 years, various methods have been developed using Hessenberg matrices a significant tool in matrix theory to calculate terms in Fibonacci and similar sequences. Notably, studies on properties such as the determinant and permanent of these matrices have enabled efficient computation of not only Fibonacci and Lucas sequences but also their generalized derivatives. Minc (1964) defined an $(n \times n)$ matrix composed solely of 0s and 1s. He demonstrated that the permanent of this matrix is equal to the generalized k-step Fibonacci numbers defined by Miles (1960).

Under the heading of general definitions, some integer sequences and their basic properties have been provided.

1.1 GENERAL DEFINITIONS

Recurrence (iteration, reduction) related number sequences and polynomials have attracted the interest of many researchers, and a lot of research has been conducted on them. Various number sequences such as Fibonacci, Lucas, Pell, Perrin, Jacobsthal, Pell-Lucas, and Jacobsthal-Lucas have been studied. These types of sequences are generally referred to as Fibonacci-type sequences. Many definitions have been made on Fibonacci type sequences and their generalizations. Some of these are given below.

Definition 1.1.1 For each natural number $n \geq 2$, numbers defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2} \tag{1.1}$$

where $F_0 = 0$, $F_1 = 1$, are called Fibonacci numbers. Here, F_n; represents the n-th Fibonacci number. The sequence of Fibonacci numbers is called the Fibonacci sequence (A000045 OEIS sequence) and is represented by $\{F_n\}$ (Vajda, 1989). This is one of the most famous sequences defined so far.

Now we define another famous sequence below.

Definition 1.1.2 For each natural number $n \geq 2$, numbers defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2} \tag{1.2}$$

where $L_0 = 2$, $L_1 = 1$, are called Lucas numbers. The sequence of Lucas numbers is represented by $\{L_n\}$ and called by the Lucas sequence (A000032 OEIS sequence) (Vajda, 1989).

Fibonacci and Lucas sequences have the same characteristic equation, which is obtained by the general terms of the sequences. The characteristic equation of these sequences is $x^2 - x - 1 = 0$ and its roots are $\theta = \frac{1+\sqrt{5}}{2}, \sigma = \frac{1-\sqrt{5}}{2}$. With these roots, we can get the Binet's formulas of the Fibonacci and Lucas sequences as follows:

$$F_n = \frac{\theta^n - \sigma^n}{\theta - \sigma}\ ,\ L_n = \theta^n - \sigma^n. \tag{1.3}$$

Definition 1.1.3 For each natural number $n \geq 2$, numbers defined by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2} \tag{1.4}$$

where $P_0 = 0$, $P_1 = 1$, are called Pell numbers. The sequence of Pell numbers is represented by $\{P_n\}$ and called by the Pell sequence (A000129 OEIS sequence) (Koshy, 2001).

By chancing the initial values of the above definition, we have the definition of the Pell-Lucas sequence.

Definition 1.1.4 For each natural number $n \geq 2$, numbers defined by the recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2} \tag{1.5}$$

where $Q_0 = 2$, $Q_1 = 2$, are called Pell-Lucas numbers. The sequence of Pell-Lucas numbers is represented by $\{Q_n\}$ and called by the Pell-Lucas sequence (A122075 OEIS sequence) (Koshy, 2001).

Like Fibonacci and Fibonacci-Lucas sequences, Pell and Pell-Lucas sequences have the same characteristic equation. The characteristic equation of these sequences is $x^2 - 2x - 1 = 0$ and its roots are $\delta = 1 + \sqrt{2}, \gamma = 1 - \sqrt{2}.$ Using these two roots, we have the Binet's formulas of the Pell and Pell-Lucas sequences as belove:

$$P_n = \frac{\delta^n - \gamma^n}{2\sqrt{2}} \text{ , } Q_n = \delta^n - \gamma^n. \tag{1.6}$$

Definition 1.1.5 For each natural number $n \geq 2$, numbers defined by the recurrence relation

$$J_n = J_{n-1} + 2J_{n-2} \tag{1.7}$$

where $J_0 = 0, J_1 = 1$, are called Jacobsthal numbers.

The sequence of Jacobsthal numbers is represented by $\{J_n\}$ and called by the Jacobsthal sequence (A001045 OEIS sequence) (Horadam 1996).

The first few terms of the Jacobsthal sequence J_n are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, ….

Definition 1.1.6 For each natural number $n \geq 2$, numbers defined by the recurrence relation

$$j_n = j_{n-1} + 2j_{n-2} \tag{1.8}$$

where $j_0 = 2, j_1 = 1$, are called Jacobsthal-Lucas numbers. The sequence of Jacobsthal-Lucas numbers is represented by $\{j_n\}$ and called by the Jacobsthal-Lucas sequence (A014551 OEIS sequence) (Horadam 1996).

The first few terms of the Jacobsthal-Lucas sequence j_n are 2,1,5,7,17,31,65,127,257,511, 1025, …

Jacobsthal numbers, similar to Fibonacci numbers, were introduced by the German mathematician Ernst Jacobsthal (1882-1965).

The characteristic equation of Jacobsthal and Jacobsthal-Lucas sequences is the same, and this equation is obtained through the general terms of the sequences. The characteristic equation of this sequences is $x^2 - x - 2 = 0$ is and its roots are $\alpha = 2, \beta = -1$. Therefore, the Binet's formulas for these two series are respectively,

$$J_n = \frac{2^n - (-1)^n}{3}\ , \ j_n = 2^n + (-1)^n. \tag{1.9}$$

The negative terms of the famous sequences defined above can also be given. Many studies include these terms in their results. However, we are not interested in these terms in this study.

In the next chapter, the characteristics of Generalized Fibonacci-like sequences obtained by Sanjay, Bijendra and Shubhraj (2014) will be given. The results for the Generalized Jacobsthal-like sequence that we defined in a similar manner will be shown.

CHAPTER 2

EXAMINATION OF GENERALIZED JACOBSTHAL-LIKE SEQUENCES

In this section, generalized Jacobsthal-like sequences and their properties will be stated.

2.1 GENERALIZED JACOBSTHAL-LIKE SEQUENCES

Generalized Fibonacci-like sequence was obtained by Sanjay, Bijendra and Shubhraj (2014). (Sanjay at al., 2014). The new number sequence, which has a form and pattern similar to the recursive formula of the Fibonacci sequences and is obtained by using the initial conditions of the Fibonacci and Lucas sequences, is called a Fibonacci-like sequence.

Extensive work has been done on generalized Fibonacci-like sequences for many years, investigating both their properties and preliminary results. (Sanjay et al., 2014, Singh et al., 2010, Gupta et al., 2014). Using these studies on generalized Fibonacci-like sequences, we similarly extended these studies to generalized Jacobsthal-like sequences (Dikmen and Karatas Selam, 2024).

Definition 2.1.1 Generalized Jacobsthal-like sequence $\{V_n\}$ defined by

$$V_n = V_{n-1} + 2V_{n-2}, \qquad n \geq 2 \tag{2.1}$$

with $V_0 = 2$ and $V_1 = 1 + m$, m being a fixed positive integer.

Here the initial conditions V_0 and V_1 are the sum of m times the initial conditions of Jacobsthal sequence and the initial conditions of Jacobsthal-Lucas sequence respectively.

The relation between Jacobsthal sequence and generalized Jacobsthal-like sequence can be written as

$$V_n = mJ_n + j_n, \qquad n \geq 0 \tag{2.2}$$

Then, the terms of the sequence $\{V_n\}$ are given by

$$\{V_n\} = \{2, 1+m, 5+m, 7+3m, 17+5m, 31+11m, \ldots\}.$$

The first, we introduce some basic results of generalized Jacobsthal-like sequence and Jacobsthal sequence.

The corresponding characteristic equation of relation (1.7) is

$x^2 - x - 2 = 0$ and its roots are $\alpha = 2$ and $\beta = -1$ using these two roots, we obtain Binet's formula of recurrence relation (2.1)

$$V_n = \frac{m}{3}(2^n - (-1)^n) + (2^n + (-1)^n). \tag{2.3}$$

Generating function of $\{V_n\}$ is defined as

$$\sum_{k=0}^{\infty} V_k x^k = \frac{2+(m-1)x}{1-x-2x^2}. \tag{2.4}$$

Sums of generalized Jacobsthal-like terms can be given in the following theorems.

Theorem 2.1.2 Sum of first n terms of the generalized Jacobsthal-like sequence $\{V_n\}$ is

$$V_1 + V_2 + V_3 + \cdots + V_n = \sum_{k=1}^{n} V_k = \frac{V_{n+2} - V_2}{2}. \tag{2.5}$$

This identity becomes

$$V_1 + V_2 + V_3 + \cdots + V_{2n} = \sum_{k=1}^{2n} V_k = \frac{V_{2n+2} - V_2}{2}. \tag{2.6}$$

Proof. We know that the following relations hold:

$2V_1 = V_3 - V_2,$

$2V_2 = V_4 - V_3,$

$2V_3 = V_5 - V_4,$

$\vdots$

$2V_{n-1} = V_{n+1} - V_n,$

$2V_n = V_{n+2} - V_{n+1}.$

Term wise addition of all above equations, we obtain

$2(V_1 + V_2 + V_3 + \cdots + V_n) = V_{n+2} - V_2,$

$V_1 + V_2 + V_3 + \cdots + V_n = \dfrac{V_{n+2} - V_2}{2}.$

Theorem 2.1.3 Sum of first $2n$ terms of the generalized Jacobsthal-like sequence $\{V_n\}$ is

$$V_1 + V_2 + V_3 + \cdots + V_{2n} = V_{2n+1} - V_1. \tag{2.7}$$

Proof. We know that the following relations hold:

$V_2 = V_3 - 2V_1,$

$V_4 = V_5 - 2V_3,$

$V_6 = V_7 - 2V_5,$

$\vdots$

$V_{2n-2} = V_{2n-1} - 2V_{2n-3},$

$V_{2n} = V_{2n+1} - 2V_{2n-1}$.

Term wise addition of all above equations, we obtain

$$V_2 + V_4 + V_6 + \cdots + V_{2n} = -(V_1 + V_3 + \cdots + V_{2n-1}) + V_{2n+1} - V_1.$$

Adding odd indices to the both sides of the equation, we have

$$V_1 + V_2 + V_3 + \cdots + V_{2n} = V_{2n+1} - V_1.$$

Theorem 2.1.4 Sum of first $2n - 1$ terms of the generalized Jacobsthal-like sequence $\{V_n\}$ is

$$V_0 + V_1 + V_2 + V_3 + \cdots + V_{2n-1} = V_{2n} - V_0. \quad (2.8)$$

Proof.

$V_1 = V_2 - 2V_0$,

$V_3 = V_4 - 2V_2$,

$V_5 = V_6 - 2V_4$,

$\vdots$

$V_{2n-3} = V_{2n-2} - 2V_{2n-4}$,

$V_{2n-1} = V_{2n} - 2V_{2n-2}$.

Term wise addition of all above equations, we obtain

$$V_1 + V_3 + V_5 + \cdots + V_{2n-1} = -(V_0 + V_2 + \cdots + V_{2n-2}) + V_{2n} - V_0,$$

$$V_0 + V_1 + V_2 + V_3 + \cdots + V_{2n-1} = V_{2n} - V_0.$$

We state and prove the following identity for the generalized Jacobsthal-like sequence $\{V_n\}$.

Lemma 2.1.5 For every positive integer n, we have

$$2V_{2n} - V_{2n+1} = 3 - m. \tag{2.9}$$

Proof. Combining (2.6) and (2.7) and putting $V_1 = 1 + m, V_2 = 5 + m$, we obtain

$$V_1 + V_2 + V_3 + \cdots + V_{2n} = \sum_{k=1}^{2n} V_k = \frac{V_{2n+2} - (5+m)}{2} = V_{2n+1} - (1+m),$$

$$V_{2n+2} - (5+m) = 2V_{2n+1} - 2(1+m),$$

$$V_{2n+2} - 2V_{2n+1} = 3 - m,$$

$$V_{2n+1} + 2V_{2n} - 2V_{2n+1} = 3 - m,$$

$$2V_{2n} - V_{2n+1} = 3 - m.$$

Theorem 2.1.6 Sum of the first $(n+1)$ terms of the generalized Jacobsthal-like sequence $\{V_n\}$ with odd and even indices are

$$V_1 + V_3 + V_5 + \cdots + V_{2n+1} = \frac{2V_{2n+2} - (n+1)(3-m) - 4}{3}, \tag{2.10}$$

and

$$V_0 + V_2 + V_4 + \cdots + V_{2n} = \frac{V_{2n+2} + (n+1)(3-m) - 2}{3} \tag{2.11}$$

respectively.

Proof.

Using (2.8),

$$V_0 + V_1 + V_2 + \cdots + V_{2n} + V_{2n+1} = V_{2n+2} - 2.$$

For, $V_0 + V_2 + V_4 + \cdots + V_{2n-2} + V_{2n} = X$, $V_1 + V_3 + V_5 + \cdots + V_{2n-1} + V_{2n+1} = Y$

$$X + Y = V_{2n+2} - 2. \tag{2.12}$$

Using (2.9),

$$\sum_{k=0}^{n}(2V_{2k} - V_{2k+1}) = \sum_{k=0}^{n}(3 - m),$$

$$2\sum_{k=0}^{n} V_{2k} - \sum_{k=0}^{n} V_{2k+1} = (n+1)(3-m),$$

$$2X - Y = (n+1)(3-m). \tag{2.13}$$

Using (2.12) and (2.13) we get

$$V_0 + V_2 + V_4 + \cdots + V_{2n} = \frac{V_{2n+2} + (n+1)(3-m) - 2}{3},$$

$$V_1 + V_3 + V_5 + \cdots + V_{2n+1} = \frac{2V_{2n+2} - (n+1)(3-m) - 4}{3}.$$

Corollary 2.1.7 The alternating sum of the first n numbers of the generalized Jacobsthal-like sequence $\{V_n\}$ is given by

$$V_0 - V_1 + V_2 - V_3 + V_4 - V_5 + \cdots + (-1)^n V_n = \frac{(-1)^n V_{n+1} + (n+1)(3-m) + 2}{3}. \tag{2.14}$$

Proof. If we subtract equation (2.10) term wise from equation (2.11), we get alternating sum of first $2n+1$ numbers:

$$\begin{aligned} V_0 - V_1 + V_2 - V_3 + V_4 - V_5 + \cdots + V_{2n} - V_{2n+1} \\ = \frac{V_{2n+2} + (n+1)(3-m) - 2}{3} - \frac{2V_{2n+2} - (n+1)(3-m) - 4}{3} \\ = \frac{-V_{2n+2} + 2(n+1)(3-m) + 2}{3}. \end{aligned}$$

If we want to calculate the alternating sum of first n numbers from the above equation, substituting $2n+1$ by n we get the following result

$$V_0 - V_1 + V_2 - V_3 + V_4 - V_5 + \cdots + (-1)^n V_n = \frac{(-1)^n V_{n+1} + (n+1)(3-m) + 2}{3}.$$

Now, some identities for the generalized Jacobsthal-like sequence $\{V_n\}$ are stated and proven below.

Theorem 2.1.8 For every integer $n \geq 0,$ for each real coefficient m,

$$mV_{n+2} - mV_{n+1} = 2mV_n. \tag{2.15}$$

Proof.

$$m(V_{n+2} - V_{n+1}) = m(2V_n) = 2mV_n.$$

Theorem 2.1.9 For every integer $n \geq 1$, we have

$$V_n^2 = V_n V_{n+1} - 2V_{n-1}V_n\,. \tag{2.16}$$

Proof.

$$V_n V_{n+1} - 2V_{n-1}V_n = V_n(V_{n+1} - 2V_{n-1}) = V_n^2\,.$$

Theorem 2.1.10 (Simson formula) For every integer $n \geq 1$, we have

$$V_{n+1}V_{n-1} - V_n^2 = (-1)^{n+1}2^{n-1}(9 - m^2)\,. \tag{2.17}$$

Proof.

We shall use mathematical induction over n.

It is easy to see that for $n = 1$,

$V_2V_0 - V_1^2 = (-1)^2 2^0 (9 - m^2)$

$2\,(5 + m) - (\,1 + m)^2 = (9 - m^2)$, which is true.

Assume that the result is true for $n = k$. Then

$$V_{k+1}V_{k-1} - V_k^2 = (-1)^{k+1}2^{k-1}(9 - m^2). \tag{2.18}$$

Multiplying by 2 and adding V_kV_{k+1} to each side of equation (2.18), we get

$2V_{k+1}V_{k-1} - 2V_k^2 + V_kV_{k+1} = (-1)^{k+1}2^k(9 - m^2) + V_kV_{k+1}$,

$V_{k+1}(2V_{k-1} + V_k) - V_k(2V_k + V_{k+1}) = (-1)^{k+1}2^k(9 - m^2)$,

$V_{k+1}^2 - V_kV_{k+2} = (-1)^{k+1}2^k(9 - m^2)$,

$-(V_kV_{k+2} - V_{k+1}^2) = (-1)^{k+1}2^k(9 - m^2)$,

$V_{k+2}V_k - V_{k+1}^2 = (-1)^{k+2}2^k(9 - m^2)$.

Therefore, the result is true for $n = k + 1$.

Hence, $V_{n+1}V_{n-1} - V_n^2 = (-1)^{n+1}2^{n-1}(9 - m^2)$, for every $n \geq 1$.

Theorem 2.1.11 For every positive integer n,

$$V_3 + V_6 + V_9 + \cdots + V_{3n} = \begin{cases} \frac{1}{7}(V_{3n+3} - 16), & if\ n\ is\ odd \\ \frac{1}{7}(V_{3n+3} - V_3), & if\ n\ is\ even\ . \end{cases} \tag{2.19}$$

Proof. We use the Binet's formula of generalized Jacobsthal-like,

$$V_3 + V_6 + V_9 + \cdots + V_{3n}$$
$$= \frac{m}{3}(\alpha^3 - \beta^3) + (\alpha^3 + \beta^3) + \frac{m}{3}(\alpha^6 - \beta^6) + (\alpha^6 + \beta^6) + \frac{m}{3}(\alpha^9 - \beta^9)$$
$$+ (\alpha^9 + \beta^9) + \cdots + \frac{m}{3}(\alpha^{3n} - \beta^{3n}) + (\alpha^{3n} + \beta^{3n})$$

$$=\frac{m}{3}[(\alpha^3 + \alpha^6 + \alpha^9 + \cdots + \alpha^{3n}) - (\beta^3 + \beta^6 + \beta^9 + \cdots + \beta^{3n})] + [(\alpha^3 + \alpha^6 + \cdots + \alpha^{3n}) - (\beta^3 + \beta^6 + \beta^9 + \cdots + \beta^{3n})],$$

$$=\frac{m}{3}\left[\left(\frac{\alpha^{3n+3}-\alpha^3}{\alpha^3-1}\right) - \left(\frac{\beta^{3n+3}-\beta^3}{\beta^3-1}\right)\right] + \left[\left(\frac{\alpha^{3n+3}-\alpha^3}{\alpha^3-1}\right) + \left(\frac{\beta^{3n+3}-\beta^3}{\beta^3-1}\right)\right]$$

$$=\frac{m}{3}\left[\left(\frac{2^{3n+3}-8}{7}\right) - \left(\frac{(-1)^{3n+3}+1}{-2}\right)\right] + \left[\left(\frac{2^{3n+3}-8}{7}\right) + \left(\frac{(-1)^{3n+3}+1}{-2}\right)\right]$$

$$=\begin{cases} \frac{m}{3}\left[\left(\frac{2^{3n+3}-8}{7}\right) + 1\right] + \left[\left(\frac{2^{3n+3}-8}{7}\right) - 1\right], & if\ n\ is\ odd \\ \frac{m}{3}\left(\frac{2^{3n+3}-8}{7}\right) + \frac{2^{3n+3}-8}{7} & , \quad if\ n\ is\ even \end{cases}$$

$$=\begin{cases} \frac{m}{3}\left(\frac{2^{3n+3}-1}{7}\right) + \frac{2^{3n+3}-15}{7} & , \quad if\ n\ is\ odd \\ \frac{1}{7}\left[\frac{m}{3}(2^{3n+3} - (-1)^{3n+3} - 9) + (2^{3n+3} + (-1)^{3n+3} - 7)\right], & if\ n\ is\ even \end{cases}$$

$$=\begin{cases} \frac{m}{3}\left(\frac{2^{3n+3}-(-1)^{3n+3}}{7}\right) + \frac{2^{3n+3}+(-1)^{3n+3}-16}{7}, & if\ n\ is\ odd \\ \frac{1}{7}[V_{3n+3} - (3m + 7)\ ] & , \quad if\ n\ is\ even \end{cases}$$

$$=\begin{cases} \frac{1}{7}(V_{3n+3} - 16), & if\ n\ is\ odd \\ \frac{1}{7}(V_{3n+3} - V_3), & if\ n\ is\ even \end{cases}$$

where $V_{3n+3} = \frac{m}{3}(2^{3n+3} - (-1)^{3n+3}) + (2^{3n+3} + (-1)^{3n+3})$ is used.

In this chapter, we defined generalized Jacobsthal-like sequences and examined algebraic properties such as the Binet's formula, generator functions, Simson formula, and addition formula. Some other addition formulas were presented, such as the sum of even and odd indices and the alternative sum of generalized Jacobsthal-like sequences. By considering Jacobsthal-like sequences, we also obtained some other important identities.

We believe that the generalized Jacobsthal-like sequences considered in this study can be extended to generalize other sequences such as Pell and Narayana, and the results given in this chapter may be useful for further research on this topic.

CHAPTER 3

DEFINITIONS OF HESSENBERG MATRICES, DETERMİNANT AND PERMANENT FUNCTIONS

In this chapter, we will first provide definitions for Hessenberg matrices and tridiagonal matrices. Next, we will present fundamental properties of permanent and determinant functions. Finally, we will explain a method for computing the permanents of matrices.

3.1 HESSENBERG MATRICES

Hessenberg matrices were first investigated by Karl Hessenberg (1904-1959), a German engineer (Press et al., 1992). Hessenberg matrices have various applications in numerical calculations and mathematical analysis.

A matrix is said to be lower Hessenberg (Esmaeili, 2006) if all entries above the superdiagonal are zero and transposition of a lower Hessenberg matrix is called as upper Hessenberg matrix. A $n \times n$ matrix $A_n = [a_{ij}]$ is called lower Hessenberg matrix if

$a_{ij} = 0$ when $j - i > 1$, i.e.,

$$A_n = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & \cdots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix}. \tag{3.1}$$

3.2 TRIDIAGONAL MATRICES

An n −square matrix that is both a lower and upper Hessenberg matrix, that is, all elements that are not on the principal diagonal and the lines parallel to it from the bottom and top are 0, is called a Tridiagonal matrix (Başar, 2012). A $n \times n$ matrix $A_n = [a_{ij}]$ is called Tridiagonal matrix if $a_{ij} = 0$ when $|i-j| > 1$, i.e.,

$$A_n = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & 0 & 0 & 0 \\ 0 & a_{3,2} & a_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2,n-2} & a_{n-2,n-1} & 0 \\ 0 & 0 & 0 & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & a_{n,n-1} & a_{n,n} \end{pmatrix}. \quad (3.2)$$

3.3 DETERMINANT AND PERMANENT FUNCTIONS

The concept of permanent was first introduced by Binet and Cauchy in 1812 during the development of determinant theory. Although the properties of determinant and permanent functions were used together in the first studies, over time the subject of permanent was separated from determinant and started to be called with this name by Muir (Minc and Marcus, 1984).

3.3.1 Determinant

Determinant is one of the most important topics in mathematics. Many problems can be easily solved using determinants. Determinant theory was introduced by Leibnitz in 1696.

It was later developed further by mathematicians such as Bezout, Vandermonde, Crammer, Lagrange and Laplace. In the 19th century, these mathematicians were joined by Cauchy, Jacobi and Sylvester. Today, determinant is used in common in many branches of science such as physics, finance and statistics. (Bozkurt and Türen, 2003).

Let $A_n = [a_{ij}]$ be an $n \times n$ matrix and S_n be a symetric group of permutations over the set $\{1,2,\dots,n\}$. The determinant of matrix A_n is defined by

$$\det A_n = \sum_{\alpha \in S_n} sgn(\alpha) \prod_{i=1}^{n} \alpha_{i\alpha(i)} \quad (3.3)$$

where the sum ranges over all the permutations of the integers 1, 2, …, n. (Serre, 2002).

Theorem 3.3.1.1 (Cahill et al. 2002.)

Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and let $det(A_0) = 1$. Then,

$$\det(A_1) = a_{11}$$

and for $n \geq 2$.

$$\det(A_n) = a_{n,n} \det(A_{n-1}) + \sum_{r=1}^{n-1} \left[(-1)^{n-r} \alpha_{n,r} \prod_{j=r}^{n-1} \alpha_{j,j+1} \det(A_{r-1}) \right]. \tag{3.4}$$

3.3.2 Permanent

The permanent of a matrix is similar to the determinant but all the signs used in the Laplace expansion of minors are positive. The permanent of an n-square matrix is defined by

$$perA_n = \sum_{\alpha \in S_n} \prod_{i=1}^{n} \alpha_{i\alpha(i)} \tag{3.5}$$

where the summation extends over all permutations α of the symmetric group S_n (Minc,1978).

$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ for, $perA = per \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}.a_{22} + a_{12}.a_{21}$.

Theorem 3.3.2.1 (Ocal, et al.2005)

Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and let $per(A_0) = 1$. Then

$$\mathrm{per}(A_1) = a_{11}$$

and for $n \geq 2$

$$\text{per}(A_n) = a_{n,n}\,\text{per}(A_{n-1}) + \sum_{r=1}^{n-1}\left[\alpha_{n,r}\prod_{j=r}^{n-1}\alpha_{j,j+1}\text{per}(A_{r-1})\right]. \quad (3.6)$$

Since the definitions of permanent and determinant functions are similar, many properties of permanents are similar to the properties of determinants. However, although the definitions are similar, permanents do not have the two basic features of determinants. These are the "Multiplicativeness" property and the "Invariance" property under some elementary operations on matrices. (That is, if c times of a row are added to another row, the permanent value changes.)

Although the first studies on permanent function were carried out by Binet and Cauchy, Borchardt, Cayley and Muir published many articles on the subject in the ongoing process. In all of these studies, results including permanent and determinant were obtained (Minc and Marcus, 1984).

Theorem 3.3.2.2 (Minc and Marcus, 1984)

If A is an $m \times n$ matrix, $m \leq n$, P and Q are permutation matrices of orders m and n, respectively, then

$$per(PAQ) = per(A). \quad (3.7)$$

In other words, shifting the rows (or columns) of the matrix does not change its permanent.

Theorem 3.3.2.3 (Minc and Marcus, 1984)

If A is an n-square matrix, then

$$per(A^T) = per(A). \quad (3.8)$$

3.4 CONTRACTION METHOD

In matrix theory, various methods have been developed to calculate the permanent and determinants of square matrices. Calculating the permanent of a matrix often requires a lot of processing. Especially in high-order matrices, calculating the determinant or permanent by analytical means causes a lot of time loss. Now let's explain the method known as the "Contraction" method in the literature, which has an important place in the calculation of permanents of matrices, and give the theorem related to the method.

Let $A_n = [a_{ij}]$ be an $m \times n$ matrix with row vectors $r_1, r_2, \dots r_m$. We call A_n is contractible on column k if column k contains exactly two nonzero elements. Suppose that A_n is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A_n replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A_n on column k relative to rows i and j.

If A_n is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A_n on row k relative to columns i and j (Brauldi and Gibson, 1977).

Theorem 3.4.1 (Brauldi and Gibson, 1977)

Let A be a nonnegative integral matrix of order n for $n > 1$ and let B be a contraction of A.

Then,

$$perA = perB. \tag{3.9}$$

In the next section, we will talk about the relationship between $n \times n$ dimensional Hessenberg matrices and Jacobsthal number arrays, on which many studies have been done. Then, we will talk about the content of the articles we plan to publish titled "Determinants and Permanents of Hessenberg Matrices with Jacobsthal-like Sequences" and "On Generalized Jacobsthal-like Sequences by Hessenberg Matrices". In these two articles, we defined a new Hessenberg matrix and focused on the relationship between the determinants and permanents of these matrices and the terms Jacobsthal-like sequences.

CHAPTER 4

CALCULATION OF PERMANENTS AND DETERMINANTS OF SOME HESSENBERG MATRICES WITH JACOBSTHAL-LIKE SEQUENCE ENTRIES

In the last 50 years, Hessenberg matrices have been used to calculate the terms of Fibonacci type sequences. Some of those; In his book published in Strank (1998), he gave some Hessenberg matrices and showed that their determinants are equal to Fibonacci numbers. Öcal et al. (2005) used the determinant and permanent of various Hessenberg matrices to calculate generalized k-digit Fibonacci and Lucas numbers. Tasyurdu and Işık (2019) defined Hessenberg matrices and showed that the determinants and permanents of these Hessenberg matrices are equal to the nth term of Fibonacci-like sequences. Additionally, Tasyurdu (2018) defined some Hessenberg matrices and showed that the determinants and permanents of these Hessenberg matrices are the terms of generalized Fibonacci-like sequences.

Inspired by Tasyurdu and Işık's (2019) study on Fibonacci-like sequences and Hessenberg Matrices, we examined the relationship of determinants and permanents of Hessenberg matrices with Jacobsthal-like sequences and obtained some new results. In this section, we present our results. It will be shown that the determinants and permanents of the $n \times n$ Hessenberg matrices we define are equal to the nth term of the defined Jacobsthal-like sequences.

4.1 GENERALIZED JACOBSTHAL-LIKE SEQUENCES

Generalized Jacobsthal-like sequences can be defined in relation to Jacobsthal and Jacobsthal-Lucas sequences. Now we will give the definitions, recurrence relations, initial conditions, Binet's formulas and terms of the series of the generalized Jacobsthal-like sequences we have defined. In the next section, we will show that we obtain the terms of these sequences using Hessenberg matrices.

Definition 4.1.1 Generalized Jacobsthal-like sequence $\{B_n\}$ is defined by recurrence relation

$$B_n = B_{n-1} + 2B_{n-2}, \qquad n \geq 2 \tag{4.1}$$

with initial conditions $B_0 = 0$ and $B_1 = m$, where m is a fixed positive integer.

The relation between Jacobsthal sequences and generalized Jacobsthal-like sequences can be written as

$$B_n = mJ_n$$

where m is a positive integer. A few terms of this sequence are

$$\{B_n\} = \{0, m, m, 3m, 5m, 11m, 21m, \ldots\}.$$

Definition 4.1.2 Generalized Jacobsthal-like sequence $\{R_n\}$ is defined by recurrence relation

$$R_n = R_{n-1} + 2R_{n-2}, \qquad n \geq 2 \tag{4.2}$$

with initial conditions $R_0 = 2$ and $R_1 = 2$.

Here initial conditions R_0 and $R_1$are the sum initial conditions of Jacobsthal and Jacobsthal-Lucas sequences respectively, i.e. $R_0 = J_0 + j_0$, $R_1 = J_1 + j_1$.

The relation between Jacobsthal sequences and generalized Jacobsthal-like sequences can be written as

$$R_n = 2J_{n+1}$$

A few terms of this sequence are

$$\{R_n\} = \{2, 2, 6, 10, 22, 42, \ldots\}.$$

Definition 4.1.3 Generalized Jacobsthal-like sequence $\{K_n\}$ associated with Jacobsthal and Jacobsthal-Lucas sequences is defined by the second order recurrence relation.

$$K_n = K_{n-1} + 2K_{n-2}, \qquad n \geq 2 \tag{4.3}$$

with initial conditions $K_0 = 2s$ and $K_1 = s + 1$, where s is a fixed positive integer.

The relation between Jacobsthal and Jacobsthal-Lucas sequences and generalized Jacobsthal-like sequences can be written as

$$K_n = J_n + sj_n$$

where s is a fixed positive integer. A few terms of this sequence are

$$\{K_n\} = \{2s, 1 + s, 1 + 5s, 3 + 7s, 5 + 17s, \dots\}.$$

Now, we will obtain the terms of the generalized Jacobsthal-like sequence $\{V_n\}$ defined in the second section (2.1) using the determinant and permanents of the Hessenberg matrix. Therefore, let us recall the recurrence relation, Binet's formula, and terms of the sequence.

Generalized Jacobsthal-like sequence $\{V_n\}$ associated with Jacobsthal and Jacobsthal-Lucas sequences is defined by the second order recurrence relation.

$$V_n = V_{n-1} + 2V_{n-2}, \qquad n \geq 2 \tag{4.4}$$

with initial conditions $V_0 = 2$ and $V_1 = m + 1$, m being a fixed positive integer.

The relation between Jacobsthal and Jacobsthal-Lucas sequences and generalized Jacobsthal-like sequences can be written as

$$V_n = mJ_n + j_n$$

where m is a fixed positive integer. A few terms of this sequence are

$$\{V_n\} = \{2,1 + m, 5 + m, 7 + 3m, 17 + 5m, \dots\}.$$

The characteristic equation for equations (4.1), (4.2), (4.3) and (4.4) can be expressed as follows:

$$x^2 - x - 2 = 0$$

and its roots are $\alpha = 2$ and $\beta = -1$.

Moreover, we can find the Binet's formulas for the generalized Jacobsthal-like sequences $\{B_n\}$, $\{R_n\}, \{K_n\}$ and $\{V_n\}$ respectively, as follows:

$$B_n = m\left(\frac{2^n - (-1)^n}{3}\right),$$

$$R_n = 2\left(\frac{2^{(n+1)} - (-1)^{(n+1)}}{3}\right),$$

$$K_n = \left(\frac{2^n - (-1)^n}{3}\right) + s(2^n + (-1)^n),$$

$$V_n = m\left(\frac{2^n - (-1)^n}{3}\right) + (2^n + (-1)^n).$$

4.2 THE DETERMINANTAL REPRESENTATIONS OF GENERALIZED JACOBSTHAL-LIKE SEQUENCES

Many studies have been carried out to date to examine the relationships between matrix theory and number theory. As a result of these studies, some special number sequences have been obtained by various methods using some special types of matrices. On the other hand, new $n \times n$ dimensional Hessenberg and Tridiagonal type matrices were defined and number sequences such as Padovan, Perrin, Tribonacci, Jacobsthal, Jacobsthal-Lucas and Pell-Lucas were obtained with these matrices (Aktaş, 2013). Aktaş and Köse (2015) showed the relationship between the permanents of Hessenberg matrices and Jacobsthal numbers. We defined Hessenberg matrices and then showed that the permanents and permanents of these matrices are the terms of Jacobsthal-like sequences.

Now, we define three types of $n \times n$ upper Hessenberg matrices.

Definition 4.2.1 The $n \times n$ Hessenberg matrix $T_n(m) = (t_{i,j})$ is defined by

$$T_n(m) = \begin{pmatrix} m & -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ m & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\ 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \tag{4.5}$$

with $t_{i,i} = 1$, $t_{i,i+1} = -2, t_{i+1,i} = 1$, for $1 \leq i \leq n$, where $t_{11} = t_{21} = m$ and 0 otherwise.

Theorem 4.2.2 Let the matrix $T_n(m)$ be as in equation (4.5). Then for $n \geq 1$,

$$detT_n(m) = B_{n+1}$$

where B_n is the nth term of generalized Jacobsthal-like sequence $\{B_n\}$.

Proof. We can use the mathematical induction on n to prove $detT_n\,(m) = B_{n+1}$. Using the equation (3.4) we have

$n = 1,\ detT_1(m) = t_{1,1} = m = B_2$

$$n = 2, detT_2(m) = t_{2,2}\det T_1(m) + \sum_{r=1}^{1}\left[(-1)^{2-r}t_{2,r}\prod_{j=r}^{1}t_{j,j+1}\mathrm{det}T_{r-1}(m)\right]$$

$$= (1)(m) + (-1)t_{2,1}t_{1,2}detT_0(m)$$

$$= (1)(m) + (-1)(m)(-2)(1)$$

$$= 3m$$

$$= B_3$$

where $detT_0(m) = 1$. We assume that it is true for $n \epsilon \mathbb{Z}^+$, namely.

$$detT_n(m) = B_{n+1}, detT_{n-1}(m) = B_n, \ldots$$

and we show that it is true for $n + 1$. Using induction's hypothesis, we obtain

$$\det T_{n+1}(m) = t_{n+1,n+1}\det T_n(m) + \sum_{i=1}^{n}\left[(-1)^{n+1-r}t_{n+1,r}\prod_{j=r}^{n}t_{j,j+1}\mathrm{det}T_{r-1}(m)\right]$$

$$= (1)\mathrm{det}T_n(m)$$
$$+\sum_{r=1}^{n-1}\left[(-1)^{n+1-r}t_{n+1,r}\prod_{j=r}^{n}t_{j,j+1}\mathrm{det}T_{r-1}(m)\right] + (-1)t_{n+1,n}t_{n,n+1}detT_{n-1}(m)$$

$$= \det T_n(m) + [(-1)(1)(-2)\ detT_{n-1}(m)]$$

$$= detT_n(m) + 2\ detT_{n-1}(m)$$

$= B_{n+1} + 2B_n$

$= B_{n+2}$.

So, the proof is completed.

Note that for $m = 2$ in Definition 4.2.1 , we have the Hessenberg matrices of Jacobsthal-like sequences $\{R_n\}$. So we can write following corollary.

Corollary 4.2.3 Let the matrix $T_n(2) = (t_{i,j})$ be as in equation (4.5). Then for $n \geq 1$,

$$detT_n(2) = R_n$$

where R_n is the nth term of generalized Jacobsthal-like sequence $\{R_n\}$.

Definition 4.2.4 The *n* square Hessenberg matrix $U_n(s) = (u_{i,j})$ is defined by

$$U_n(s) = \begin{pmatrix} 1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2s & 1+s & 2(1+s) & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \tag{4.6}$$

with $u_{i,i} = 1, u_{i,i+1} = 2, u_{i+1,i} = -1$, for $3 \leq i \leq n, u_{21} = -2s, u_{22} = 1 + s, u_{23} = 2(1+s)$, $u_{32} = -1$ and 0 otherwise.

Theorem 4.2.5 Let the matrix $U_n(s)$ be as in equation (4.6). Then for $n \geq 2$,

$$detU_n(s) = K_n$$

where K_n is the nth term of generalized Jacobsthal-like sequence $\{K_n\}$.

Proof. Mathematical induction can be employed to establish the proof for n. We first show that it is true for $n = 2, 3$ and 4.

$$n = 2, \qquad detU_2(s) = \begin{vmatrix} 1 & 2 \\ -2s & 1+s \end{vmatrix} = 1 + 5s = K_2,$$

$$n = 3, \qquad detU_3(s) = \begin{vmatrix} 1 & 2 & 0 \\ -2s & 1+s & 2(1+s) \\ 0 & -1 & 1 \end{vmatrix} = 3 + 7s = K_3,$$

$$n = 4, \qquad detU_4(s) = \begin{vmatrix} 1 & 2 & 0 & 0 \\ -2s & 1+s & 2(1+s) & 0 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{vmatrix} = 5+17s = K_4.$$

We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$detU_n(s) = K_n, detU_{n-1}(s) = K_{n-1}, \dots .$$

Then, we demonstrate the validity of this statement for $n+1$. By our assumption and using equations (4.3) and (3.4), we have

$$\det U_{n+1}(s) = u_{n+1,n+1} \det U_n(s) + \sum_{i=1}^{n} \left[(-1)^{n+1-r} u_{n+1,r} \prod_{j=r}^{n} u_{j,j+1} \mathrm{det} U_{r-1}(s) \right]$$

$$= (1) \det U_n(s)$$

$$+ \sum_{i=1}^{n-1} \left[(-1)^{n+1-r} u_{n+1,r} \prod_{j=r}^{n} u_{j,j+1} \mathrm{det} U_{r-1}(s) \right] + (-1) u_{n+1,n} u_{n,n+1} detU_{n-1}(s)$$

$$= \det U_n(s) + [(-1)(-1)(2)\, detU_{n-1}(s)]$$

$$= detU_n(s) + 2\, detU_{n-1}(s)$$

$$= K_n + 2K_{n-1}$$

$$= K_{n+1}$$

Hence, we have the proof.

Definition 4.2.6 The n-square Hessenberg matrix $C_n(m) = (c_{i,j})$ is defined by

$$C_n(m) = \begin{pmatrix} 1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1+m & 2(1+m) & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \tag{4.7}$$

with $c_{i,i} = 1$, $c_{i,i+1} = 2, c_{i+1,i} = -1$, for $3 \leq i \leq n$, $c_{21} = -2$, $c_{22} = 1 + m$, $c_{23} = 2(1 + m)$, $c_{32} = -1$ and 0 otherwise.

Theorem 4.2.7 Let the matrix $C_n(m)$ be as in equation (2.2). Then for $n \geq 2$,

$$detC_n(m) = V_n$$

where V_n is the nth term of generalized Jacobsthal-like sequence $\{V_n\}$.

Proof. Mathematical induction can be employed to establish the proof for n. We first show that it is true for $n = 2, 3$ and 4.

$n = 2,$ $\qquad detC_2(m) = \begin{vmatrix} 1 & 2 \\ -2 & 1+m \end{vmatrix} = 5 + m = V_2,$

$n = 3,$ $\qquad detC_3(m) = \begin{vmatrix} 1 & 2 & 0 \\ -2 & 1+m & 2(1+m) \\ 0 & -1 & 1 \end{vmatrix} = 7 + 3m = V_3,$

$n = 4,$ $\qquad detC_4(m) = \begin{vmatrix} 1 & 2 & 0 & 0 \\ -2 & 1+m & 2(1+m) & 0 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{vmatrix} = 17+5m = V_4.$

We assume that it is true for $n \in \mathbb{Z}^+$, namely

$$detC_n(m) = V_n, detC_{n-1}(m) = V_{n-1}, \dots .$$

Then, we demonstrate the validity of this statement for $n + 1$.

By our assumption and using equation (4.4) and (3.4), we have

$$\det C_{n+1}(m) = c_{n+1,n+1} \det C_n(m) + \sum_{i=1}^{n}\left[(-1)^{n+1-r} c_{n+1,r} \prod_{j=r}^{n} c_{j,j+1} \det C_{r-1}(m)\right]$$

$$= (1)\det C_n(m)$$

$$+ \sum_{i=1}^{n-1}\left[(-1)^{n+1-r} c_{n+1,r} \prod_{j=r}^{n} c_{j,j+1} \det C_{r-1}(m)\right] + (-1) c_{n+1,n} c_{n,n+1} det C_{n-1}(m)$$

$$= \det C_n(m) + [(-1)(-1)(2)\, det C_{n-1}(m)]$$

$$= \, det C_n(m) + 2\, det C_{n-1}(m)$$

$$= V_n + 2V_{n-1}$$

$$= V_{n+1}.$$

Hence, we have the proof.

4.3 THE PERMANENTAL PEPRESENTATIONS OF GENERALIZED JACOBSTHAL-LIKE SEQUENCES

Now, we define three types of $n \times n$ upper Hessenberg matrices.

Definition 4.3.1 The $n \times n$ Hessenberg matrix $P_n(m) = (p_{i,j})$ is defined by

$$P_n(m) = \begin{pmatrix} m & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ m & 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix} \qquad (4.8)$$

with $p_{i,i} = 1$, $p_{i,i+1} = 2, p_{i+1,i} = 1$, for $1 \leq i \leq n$, where $p_{11} = \; p_{21} = m$ and 0 otherwise.

Theorem 4.3.2 Let the matrix $P_n(m)$ be as in equation (4.8). Then for $n \geq 1$,

$$perP_n(m) = perP_n^{(n-2)}(m) = B_{n+1}$$

where B_n is the nth term of generalized Jacobsthal-like sequence $\{B_n\}$.

Proof. With the Definition 4.3.1, it can be contracted on first column. Let $P_n^r(m)$ be rth contraction of $P_n(m)$, $1 \leq r \leq n-2$. The matrix $P_n(m)$ can be contracted on first column 1, so that

$$P_n^1(m) = \begin{pmatrix} 3m & 2m & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} B_3 & 2B_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

where $B_3 = 3m$ and $B_2 = m$. Since $P_n^1(m)$ also can be contracted on first column,

$$P_n^2(r) = \begin{pmatrix} 5m & 2.3m & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} B_4 & 2B_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

where $B_4 = 5m$ and $B_3 = 3m$. Continuing with this process, we have the rth contraction of the matrix $P_n(m)$ as follows

$$P_n^r(m) = \begin{pmatrix} B_{r+2} & 2B_{r+1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

for $3 \leq r \leq n-4$. Hence

$$P_n^{n-3}(m) = \begin{pmatrix} B_{n-1} & 2B_{n-2} & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}.$$

which by contraction of $P_n^{n-3}(m)$ on column 1, we obtain

$$P_n^{n-2}(m) = \begin{pmatrix} B_{n-1} + 2B_{n-2} & 2B_{n-1} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} B_n & 2B_{n-1} \\ 1 & 1 \end{pmatrix}$$

by using equation (3.9). From the equation (4.1), we have

$$perP_n(m) = perP_n^{(n-2)}(m) = B_n + 2B_{n-1} = B_{n+1}.$$

So, the proof is completed.

Note that for $m = 2$ in Definition 4.3.1 , we have the Hessenberg matrices of Jacobsthal-like sequences $\{R_n\}$. So we can write following corollary.

Corollary 4.3.2 Let the matrix $P_n(2) = (p_{i,j})$ be as in equation (4.8). Then for $n \geq 1$,

$$perP_n(2) = perP_n^{(n-2)}(2) = R_n$$

where R_n is the nth term of generalized Jacobsthal-like sequence $\{R_n\}$.

Definition 4.3.3 The n-square Hessenberg matrix $P_n(s) = p_{ij}$ is defined by

$$P_n(s) = \begin{pmatrix} 1 & -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2s & 1+s & -2(1+s) & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \tag{4.9}$$

with $p_{i,i} = 1$, $p_{i,i+1} = -2,\ p_{i+1,i} = -1$, for $3 \leq i \leq n$, $p_{21} = -2s$, $p_{22} = 1+s$, $p_{23} = -2(1+s)$, $p_{32} = -1$ and 0 otherwise.

Theorem 4.3.4 Let matrix $P_n(s)$ be as in equation (4.9). Then for $n \geq 2$,

$$perP_n(s) = perP_n^{(n-2)}(s) = K_n$$

where K_n is the nth term of generalized Jacobsthal-like sequence $\{ K_n\}$.

Proof. With the Definition 4.3.3, the matrix can be contracted on first column. Let $P_n^r(s)$ be rth contraction of $P_n(s)$, $1 \leq r \leq n-2$. The matrix $P_n(s)$ can be contracted on first column, so that we get

$$P_n^1(s) = \begin{pmatrix} 1+5s & -2(1+s) & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} K_2 & -2K_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

where $K_2 = 1 + 5s$ and $K_1 = 1 + s$. Since $P_n^1(s)$ also can be contracted on first column,

$$P_n^2(s) = \begin{pmatrix} 3+7s & -2(1+5s) & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} K_3 & -2K_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

where $K_3 = 3 + 7s$ and $K_2 = 1 + 5s$. Continuing current process, we have the rth contraction of the matrix $P_n(s)$ as

$$P_n^r(s) = \begin{pmatrix} K_{r+1} & -2K_r & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

for $3 \le r \le n-4$. Hence

$$P_n^{n-3}(s) = \begin{pmatrix} K_{n-2} & -2K_{n-3} & 0 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

which by contraction of $P_n^{n-3}(s)$ on first column, we obtain

$$P_n^{n-2}(s) = \begin{pmatrix} K_{n-2} + 2K_{n-3} & -2K_{n-2} \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} K_{n-1} & -2K_{n-2} \\ -1 & 1 \end{pmatrix}$$

by using equation (3.9). From the equation (4.3), we have

$$perP_n(s) = perP_n^{(n-2)}(s) = K_{n-1} + 2K_{n-2} = K_n.$$

So the proof is finished.

Definition 4.3.5 The n-square Hessenberg matrix $S_n(m) = (s_{ij})$ is defined by

$$S_n(m) = \begin{pmatrix} 1 & -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 1+m & -2(1+m) & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \tag{4.10}$$

with $s_{i,i} = 1$, $s_{i,i+1} = -2, s_{i+1,i} = -1$, for $3 \leq i \leq n$, $s_{21} = -2$, $s_{22} = 1 + m$, $s_{23} = -2(1 + m)$, $s_{32} = -1$ and 0 otherwise.

Theorem 4.3.6 Let the matrix $S_n(m)$ be as in equation (4.10). Then for $n \geq 2$,

$$perS_n(m) = perS_n^{(n-2)}(m) = V_n$$

where V_n is the nth term of Jacobsthal-like sequence $\{V_n\}$.

Proof. Because of Definition 4.3.5, the matrix can be contracted on first column. Let $S_n^r(m)$ be rth contraction of $S_n(m)$, $1 \leq r \leq n-2$. The matrix $S_n(m)$ can be contracted on first column, so that we get

$$S_n^1(m) = \begin{pmatrix} 5+m & -2(1+m) & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} V_2 & -2V_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

where $V_2 = 5 + m$ and $V_1 = 1 + m$. Since $S_n^1(m)$ also can be contracted on first column,

$$S_n^2(m) = \begin{pmatrix} 7+3m & -2(5+m) & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} V_3 & -2V_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

where $V_3 = 7 + 3m$ and $V_2 = 5 + m$. Continuing by this way, we have the rth contraction of the matrix $S_n(m)$ as

$$S_n^r(m) = \begin{pmatrix} V_{r+1} & -2V_r & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$$

for $3 \leq r \leq n-4$. Hence

$$S_n^{n-3}(m) = \begin{pmatrix} V_{n-2} & -2V_{n-3} & 0 \\ -1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$$

then by contraction of $S_n^{n-3}(m)$ on first column, we obtain

$$S_n^{n-2}(m) = \begin{pmatrix} V_{n-2} + V_{n-3} & -2V_{n-2} \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} V_{n-1} & -2V_{n-2} \\ -1 & 1 \end{pmatrix}$$

by using equation (3.9). From the equation (4.4), we have

$$perS_n(m) = perS_n^{(n-2)}(m) = V_{n-1} + 2V_{n-2} = V_n.$$

Hence, we get the proof.

Moreover, Theorem 4.3.4 and Theorem 4.3.6 can be proved with equation (3.6) as the other way.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

In this study, Jacobsthal and Jacobsthal-Lucas sequences are well-known second-order recursive sequences consisting of integers in mathematics. This research article includes the examination of generalized Jacobsthal-like sequences related to these sequences. Generalized Jacobsthal-like sequences are defined in the study, and algebraic properties such as Binet's formula, generating functions, Simpson formula, and sum formula are analyzed. Other sum formulas such as the sum of even and odd indices and alternative sum of generalized Jacobsthal-like sequences are also presented. Various forms of Hessenberg matrices are defined, and the determinants and permanent representations of generalized Jacobsthal-like sequences are investigated. It is shown that the determinants and permanents of the defined Hessenberg matrices are equal to the *n*-th term of the generalized Jacobsthal-like sequence. Thus, it is demonstrated that different Hessenberg matrices can be obtained depending on positive integers m and Jacobsthal-like sequences can be obtained utilizing the determinants and permanents of these matrices.

We believe that this study can be generalized for Catalan, Mersenne, Fermat, Pell, and Narayana number sequences, and it will be useful in exploring the relationships between Hessenberg matrices and these number sequences in terms of determinants and permanents.

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CURRICULUM VITAE

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