THE EULER MEASURE OF FINITE CATEGORIES

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By
Mustafa Akkaya
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We certify that we have read this dissertation and that in our opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of Doctor of Philosophy.

Özgün Ünlü(Advisor)

Laurence John Barker

Cihan Okay

Yıldıray Ozan

Mehmet Fırat Arıkan

Approved for the Graduate School of Engineering and Science:

Orhan Arıkan
Director of the Graduate School

ABSTRACT

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Mustafa Akkaya
Ph.D. in Mathematics
Advisor: Özgün Ünlü
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We associate a rational number $\chi(\mathcal{A})$ to every category $\mathcal{A}$ whose object and morphism sets are finite. The assignment χ is additive under disjoint union and it preserves products. Leinster's Euler characteristic χ_{Lein} and χ agrees whenever χ_{Lein} is defined. Hence χ is different from the series Euler characteristic χ_Σ and χ is preserved under the weak equivalences of canonical model structure when it is restricted to the family of categories for which χ_{Lein} is defined. However, χ is not preserved under the weak equivalences of canonical model structure on its whole domain. For this reason χ is not called the Euler characteristic. When the domain of χ is restricted to the family of categories admitting a weighting, χ satisfies the inclusion exclusion principle. Hence we can call this restriction the Euler measure. By abuse of notation we will denote this restriction by χ again. Since the family of categories admitting both weighting and coweighting is contained by the family of categories admitting weighting, the Euler measure of categories is a proper extension of Leinster's Euler characteristic. We also showed that Leinster's formula for the Grothendieck construction is still valid for diagrams from a poset to the categories in the domain of this Euler measure. The situation for the Thomason model structure is more intricate. We give an example to show that none of χ, χ_{Lein} and χ_Σ is invariant under the weak equivalences of the Thomason model structure and show that such examples can be eliminated by putting extra conditions on weak equivalences of the Thomason model structure.

Keywords: Euler Characteristic, Euler Measure, Homotopy, Category.

ÖZET

SONLU KATEGORİLERİN EULER ÖLÇÜSÜ

Mustafa Akkaya
Matematik, Doktora
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Obje ve morfizm kümeleri sonlu olan her $\mathcal{A}$ kategorisini, $\chi(\mathcal{A})$ rasyonel sayısı ile ilişkilendiriyoruz. χ ilişkilendirmesi ayrık bileşim altında toplamsaldır ve çarpımı korur. Leinster'ın Euler karakteristiği χ_{Lein} tanımlı olduğunda, χ ile χ_{Lein} aynıdır. Böylece χ seri Euler karekteristiği χ_{Σ}'dan farklıdır ve χ_{Lein}'in tanımlı olduğu kategoriler ailesine kısıtlandığında, kanonik model yapısının zayıf denklikleri altında korunur. Fakat, tanımlı olduğu tüm aile üzerinde, χ, kanonik model yapısının zayıf denklikleri altında korunmaz. Bu sebeple χ'ye Euler karakteristiği demeyiz. χ, ağırlıklandırma kabul eden kategoriler ailesine kısıtlandığında, içerme dışlama ilkesine uyar. Böylece bu kısıtlanamaya Euler ölçüsü deriz. Karışıklığa sebep olmaması için bu kısıtlamaya da χ deriz. Hem ağırlıklandırma hem de karşıt-ağırlıklandırma kabul eden kategoriler ailesi, ağırlıklandırma kabul eden kategoriler ailesi tarafından kapsandığından, kategorilerin Euler ölçüsü, Leinster'ın Euler karakteristiğinin bir öz genişlemesidir. Ayrıca Leinster'ın Grothendieck inşası için formülünün, indeks kategorisi bir posetken ve görüntüleri Euler ölçüsünün tanımlı olduğu ailedeyken de geçerli olduğunu gösterdik. Durum Thomason model yapısında daha karmaşıktır. Ne χ'ın, ne χ_{Lein}'ın, ne de χ_{Σ}'nın Thomason model yapısının zayıf denklikler altında korunmadığını göstermek için bir örnek sunuyoruz ve bu tip örneklerin, Thomason model yapısının zayıf denklikleri üzerine bazı ek şartlar koyarak, elenebileceğini gösteriyoruz.

Anahtar sözcükler: Euler Karakteristiği, Euler Ölçüsü, Homotopi, Kategori.

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Contents

Chapter 1

Introduction

The roots of Euler characteristic can be traced back to the paper titled "The Seven Bridges of Königsberg" written by Euler in 1736. He transferred the problem to graph theory. He defined the degree of a vertex and showed that if more than two vertices have odd degree, then it is not possible to walk over edges exactly once. This paper was the first time graph theory took place to study geometric objects. For planar graphs he turned to that direction again and recognized the formula $\chi = V - E + F = 2$ where V denotes the number of vertices, E the number of edges and F the number of faces including the outer unbounded face. Then he proved the formula in 1752, for surfaces of convex polyhedra.

Then it was generalized as the alternating sum of the ranks of the homology modules of each dimension if every homology module is finitely generated and projective, and only finitely many of them are nonzero. For finite complexes, it is alternating sum of the number of cells of each dimension. The Euler characteristic of a topological space satisfies many useful properties on the spaces it is defined. For example, the Euler characteristic of disjoint union of two topological spaces, is the sum of their Euler characteristics, it obeys inclusion exclusion principle, and the Euler characteristic of cartesian product of two topological spaces is the multiplication of their Euler characteristics. It is also invariant under homotopy equivalence.

For every category, we have a simplicial set called the nerve of it whose n-simplices are the n-length of chains of its morphisms. And for every simplicial set, we have a topological space which is called geometric realization of it and obtained by gluing the n-simplicies each other according to face maps. Thus, nerve functor grants us a way of relating some topological spaces with some categories. One may question whether we can define the Euler characteristic of a category as the Euler characteristic of the geometric realization of the nerve of the category. Consider the groupoid category C_2 with only one nontrivial morphism. Then observe that although C_2 is a small category, the corresponding topological space BC_2 has nonzero homology in infinitely many dimensions. Thus, Euler characteristic of C_2 cannot be defined in that way.

The assignment of the Euler characteristic of BC_2 as $\frac{1}{2}$ is due to studies in homotopy theory and group actions on topological spaces. As a reference see [1] and for orbifold Euler characteristic see [2]. The orbifold Euler characteristic is a weighted average of the Euler characteristics of the fixed point sets of a group action on a space. By considering the free C_2 action on the contractible space EC_2, we get $\chi(BC_2) = \frac{1}{2}(\chi(EC_2) + \chi(\emptyset)) = \frac{1}{2}(1+0) = \frac{1}{2}$.

In 2008, Tom Leinster defined the Euler characteristic χ_{Lein} in [3] for the categories whose adjacency matrix of the directed multigraph associated to the category which is also known as the incidence matrix of a category has both weighting and coweighting and showed that this new Euler characteristic agrees with the Euler characteristics of topological spaces obtained by their nerves. Moreover there are some categories whose Euler characteristic is defined and its nerve is not a finite complex. Thus, it also assigns a rational number for some infinite complexes. Hence, it generalizes the Euler characteristics to some larger family of topological spaces. This generalized Euler characteristic also satisfies many nice properties. It obeys a version of the inclusion–exclusion principle. It is preserved under the adjointness of categories and so under the equivalence of categories as well.

Again in 2008 Tom Leinster and Clemens Berger defined the series Euler characteristic χ_Σ in [4]. The idea depends on the formula for finite complexes. It generalizes the formula computing the Euler characteristic as alternating sum of the number of cells of each dimension to some infinite complexes. By writing a variable instead of -1, they get a formal power series. If the obtained formal power series converges around a neighborhood of 0 and analytically continues to -1 with the same value for all analytic continuations, then that value is called the series Euler characteristic of the complex. In [4] there are some examples of categories that admit only one of the series Euler characteristic χ_Σ and Leinster's Euler characteristic χ_{Lein}. There are also many categories χ_Σ agrees with χ_{Lein}. However, there are some examples of categories taking different values under χ_Σ and χ_{Lein}. Thus, in general they do not agree.

For a given category, if the adjacency matrix of the directed multigraph associated to the category is invertible, then the column matrix obtained by sum of the entries of each row and the row matrix obtained by sum of the entries of each column gives us a weighting and a coweighting respectively. But the inverse of a matrix is defined only for non-singular square matrices. The Moore-Penrose inverse generalizes the idea of inverse matrix to all matrices not necessarily square. In Lemma 3.2.1 and Lemma 3.2.2 we show that if a square matrix has a weighting or a coweighting, taking the sum of the entries of each row and column of the Moore-Penrose inverse gives us a weighting and a coweighting respectively. Thus, the Moore-Penrose inverse gives us a unique generalized weighting and coweighting for every finite category. Hence, by taking the sum of all entries in the Moore-Penrose inverse of the incidence matrix of a category $\mathcal{A}$ gives us the rational number $\chi(\mathcal{A})$ of that category. This is how we generalized Leinster's Euler characteristic to all finite categories.

Stephanie Chen and Juan Pablo Vigneaux independently came upon the idea of using the Moore-Penrose inverse to study categorical magnitude in [5] just after few months we did.

In Chapter 2 we give an algorithm to compute the Moore-Penrose inverse of a square matrix with rational entries. There are many different algorithms to compute the Moore-Penrose inverse in the literature. Some of the most popular methods for computing the Moore-Penrose inverse are based on the singular value decomposition (SVD) of matrices as the method used by Penrose in [6]. However, by using this method we cannot prove that the entries of the Moore-Penrose inverse of the incidence matrix of a category are rational. Another common tool for this purpose is the full rank decomposition of matrices (See Theorem 5, page: 48 in [7]). Using this method one can see that the entries of the Moore-Penrose inverse of the incidence matrix of a category are all rational. There are also several algorithms (See [8], [9], [10] and [11]) which compute the Moore-Penrose inverse by generalizing the normal equation method. We also give an algorithm that generalizes normal equation method here. Although this is not a new method, we give it here to stress why the entries of the matrices are all rational numbers that appear during the computations.

In Chapter 3, first we defined χ for all finite categories and showed that it preserves products in Theorem 3.3.13 and it is additive under disjoint union of categories in Theorem 3.3.14. Then we recognized that it is not preserved under the equivalence of categories by the Example 3.4.2. Thus, we decided to not to call it as Euler characteristic anymore. Then, we conclude that the family of categories admitting a weighting obeys the inclusion exclusion principle by Theorem 3.3.10. This explains the reason of calling χ as measure instead of characteristic. On the other hand restricting χ to the family of categories admitting a weighting still makes χ a strict extension of Leinster's Euler characteristic χ_{Lein} since the family of categories admitting both weighting and coweighting is a proper subset of the family of categories having weighting. By abuse of notation, we still call χ as the Euler measure for every finite category. The reader should note that Example 3.4.2 is still valid, since they admit a weighting. Thus, we still cannot call χ as characteristic on the family of finite categories admitting a weighting.

One of the main theorems of this work is Theorem 3.3.10. In Theorem 3.3.9, we showed that the Euler characteristic formula for Grothendieck construction given in Proposition 2.8 in [3] works for χ even if the category $G(F)$ does not admit a weighting. Theorem 3.3.10 shows that if the index category is a finite poset the same formula works for χ without requiring image categories to admit a coweighting. Example 3.4.3 shows that the poset condition cannot be eliminated.

In Chapter 4, we discuss some homotopical properties of the Euler measure. Proposition 4.1.1 shows that the geometric realization of the nerve of a monoid whose only nonidentity morphism is an idempotent morphism is contractible. And then it is shown that the Euler measure χ, the Euler characteristic χ_{Lein} and the series Euler characteristic χ_{Σ} of the monoid above is $1/2$. Thus, we conclude that weak equivalences of the Thomason model category structure preserves none of the Euler characteristics and measure discussed here. We also discuss Euler characteristic $\chi^{(2)}$ as defined in [12] and [13]. We showed that this definition also does not eliminate the example given in Chapter 4. Hence, we give the definition of being strongly contractible and show that the nerve of the monoid mentioned above is not strongly contractible. Thus, one could ask if being strongly contractible is enough to say Euler characteristic is equal to 1.

Some parts of this work will appear in the journal "Homology, Homotopy and Applications".

Chapter 2

The Moore-Penrose Inverse

The main goal of this chapter is to explain why $\chi(\mathcal{A})$ is a rational number for every category $\mathcal{A}$. For this purpose we give an algorithm for computing the Moore-Penrose inverse (also known as pseudo inverse) of a matrix not necessarily square. Then we give some useful properties and use them in Chapter 3.

2.1 Construction of the Moore-Penrose Inverse

The inverse of a matrix is defined only for non-singular square matrices. The Moore-Penrose inverse generalizes the idea of inverse matrix to all matrices with complex entries not necessarily square. However, we consider only rational valued matrices throughout this chapter, unless otherwise stated.

For an $m \times n$ matrix M, its conjugate transpose is denoted by M^*. Because the entries are rational numbers, conjugate transpose is the same as transpose.

Theorem 2.1.1. [7] *Let M be an $m \times n$-matrix with rational entries. If there* *exists an $n \times m$ matrix M^+ with rational entries satisfying the equations*

$$MM^+M = M \tag{2.1}$$

$$M^+MM^+ = M^+ \tag{2.2}$$

$$(M^+M)^* = M^+M \tag{2.3}$$

$$(MM^+)^* = MM^+, \tag{2.4}$$

then it is unique.

Proof. Let X and Y be $n \times m$ matrices with rational entries satisfying the equations 2.1, 2.2, 2.3 and 2.4. Then,

$$\begin{aligned} X &= XAX = X(X^*A^*) = XX^*(A^*Y^*A^*) = X(AX)(AY) = XAY \\ &= (XAX)AY = (A^*X^*)XAY = (A^*Y^*A^*)X^*XAY = Y(AXA)XAY \\ &= Y(AXA)Y = YAY = Y. \end{aligned}$$

□

Now, we can define the Moore-Penrose inverse of a matrix with rational entries.

Definition 2.1.2. Let M be an $m \times n$-matrix with rational entries. The matrix satisfying the equations 2.1, 2.2, 2.3 and 2.4 is called the Moore-Penrose inverse of M and denoted by M^+ if it exists.

We want to show that every matrix with rational entries has the Moore-Penrose inverse. For this purpose, we give an algorithm. First, we need to give some basics and notation.

Let $\mathcal{C}_n$ and $\mathcal{R}_n$ denote the vector space of all $n \times 1$-matrices and $1 \times n$-matrices with rational entries respectively. We call the elements of $\mathcal{C}_n$ and $\mathcal{R}_n$ as **column vectors** and **row vectors** (If it is obvious from the context, just vectors) respectively. Two column vectors c_1, c_2 of $\mathcal{C}_n$ are called **perpendicular** or **orthogonal** and denoted by $c_1 \perp c_2$, if $c_1^* c_2 = 0$. Two subsets of $\mathcal{C}_n$ are called **perpendicular**

sets if each vector from one is perpendicular to every vector of the other. By considering a single vector as a one-element set, we can define the perpendicularity of one element to a set of vectors. Let V be a subspace of $\mathcal{C}_n$. **Orthogonal complement** of V is defined as the subspace consisting of all vectors perpendicular to V and denoted by $V^\perp$. **Projection map** Proj_V onto V is the orthogonal projection from the from the vectors of $\mathcal{C}_n$ onto V where we consider the standard inner product on $\mathcal{C}_n$. Dually, we can define the same concepts for row vectors.

We define the following vector spaces over rational numbers for an $m \times n$ matrix M:

- $\mathrm{im}(M) = \{Mx \in \mathcal{C}_m \,|\, x \in \mathcal{C}_n\}$
- $\mathrm{Null}(M) = \{x \in \mathcal{C}_n \,|\, Mx = 0\}$
- $\mathrm{Row}(M) = \{x^* \in \mathcal{R}_m \,|\, x \in \mathrm{im}(M^*)\}$
- $\mathrm{im}(M)^\perp = \{x \in \mathcal{C}_m \,|\, x \perp \mathrm{im}(M)\}$.

Let $M = [\ c_1\ c_2\ \cdots\ c_n\ ]$ where $c_i \in \mathcal{C}_m$ for $i = 1, 2, \ldots, n$. Observe that

$$\begin{aligned}
\mathrm{im}(M)^\perp &= \{x \in \mathcal{C}_m \,|\, x \perp \mathrm{im}(M)\} \\
&= \{x \in \mathcal{C}_m \,|\, x \perp c_i, \text{ for } i = 1, 2, \ldots, m\} \\
&= \{x \in \mathcal{C}_m \,|\, c_i^* x = 0, \text{ for } i = 1, 2, \ldots, m\} \\
&= \{x \in \mathcal{C}_m \,|\, M^* x = 0\} \\
&= \mathrm{Null}(M^*).
\end{aligned}$$

Then, assuming that $\dim(\mathrm{Row}(M)) = k$ and using the rank nullity theorem, gives us that

$$\begin{aligned}
\dim(\mathrm{im}(M)^\perp) &= \dim(\mathrm{Null}(M^*)) \\
&= m - \dim(\mathrm{im}(M^*)) \\
&= m - \dim(\mathrm{Row}(M)) \\
&= m - k.
\end{aligned}$$

For the rest of the chapter, we fix some notation so that we can construct the Moore-Penrose inverse of a matrix with rational entries. Let $M = [\ c_1\ c_2\ \cdots\ c_n\ ]$ be an $m{\times}n$-matrix where $c_i \in \mathcal{C}_m$ for $i = 1, 2, \ldots, n$. Observe that $\mathrm{Null}(M^*)$ is the solution set of system of equations whose matrix representation is

$$M^*X = \begin{bmatrix} c_1^* \\ c_2^* \\ \vdots \\ c_n^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = 0.$$

Let

$$S = \{s_1, s_2, \ldots s_{m-k}\}$$

be the basis for the solution space of the above system obtained by applying back substitution method to the reduced row echelon form of augmented matrix of the system,

$$\mathrm{rref}(M) = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

the reduced row echelon form of M where $s_1, s_2, \ldots s_{m-k} \in \mathcal{C}_m$, and $r_1, r_2, \ldots, r_k \in \mathcal{R}_n$. Then, we define the $n{\times}m$-matrix A_M and $m{\times}m$-matrix B_M as follows:

$$A_M = [\ r_1^*\ r_2^*\ \cdots\ r_k^*\ \mathbf{0}\ ]$$

and

$$B_M = [\ Mr_1^*\ Mr_2^*\ \cdots\ Mr_k^*\ s_1\ s_2\ \cdots s_{m-k}\ ]^{-1}$$

where $\mathbf{0}$ is the zero matrix of size $n{\times}(m-k)$. Here the question whether $[\ Mr_1^*\ Mr_2^*\ \cdots\ Mr_k^*\ s_1\ s_2\ \cdots s_{m-k}\ ]$ is invertible or not naturally arises. The following lemma shows that it is invertible.

Remark 2.1.3. The following lemma shows that the matrix mentioned above is invertible for every choice of basis for $\mathrm{Null}(M^*)$ and $\mathrm{Row}(M)$.

Lemma 2.1.4. *Let M be an $m \times n$-matrix with rational entries, $\{r_1, r_2, \cdots, r_k\}$ a basis for $\mathrm{Row}(M)$, and $\{s_1, s_2, \cdots, s_{m-k}\}$ a basis for $\mathrm{im}(M)^{\perp}$. Then the $m \times m$-matrix $[Mr_1^* \; Mr_2^* \; \cdots \; Mr_k^* \; s_1 \; s_2 \; \cdots \; s_{m-k}]$ is invertible.*

Proof. For every row vector $r \in \mathrm{Row}(M) = \mathrm{Span}(r_1, r_2, \ldots, r_k)$ and $n \in \mathrm{Null}(M)$ we have $rn = 0$, since $Mn = 0$. Then $r_i^* \perp n$ for every $i = 1, 2, \ldots, k$. And so $\mathrm{Span}(r_1^*, r_2^*, \ldots, r_k^*) \perp \mathrm{Null}(M)$. Thus, we can conclude that

$$\mathrm{Span}(r_1^*, r_2^*, \ldots, r_k^*) \cap \mathrm{Null}(M) = \{0\}.$$

On the other hand we have

$$\dim(\mathcal{C}_n) = n = k + (n-k) = \dim(\mathrm{Span}(r_1^*, r_2^*, \cdots, r_k^*)) + \dim(\mathrm{Null}(M)),$$

hence $\mathcal{C}_n = \mathrm{Span}(r_1^*, r_2^*, \cdots, r_k^*) \bigoplus \mathrm{Null}(M)$. Since multiplying the column vectors of $\mathrm{Null}(M)$ by M gives us just zeros, all nonzero vectors of $\mathrm{im}(M)$ are obtained by multiplying M by $\mathrm{Span}(r_1^*, r_2^*, \ldots, r_k^*)$. Thus, $\mathrm{im}(M)$ is generated by $\{Mr_1^*, Mr_2^*, \cdots, Mr_k^*\}$. Since $\dim(\mathrm{im}(M)) = \dim(\mathrm{Row}(M)) = k$, we say $\{Mr_1^*, Mr_2^*, \cdots, Mr_k^*\}$ is a basis for $\mathrm{im}(M)$. Finally we have two linearly independent set of vectors that are perpendicular which means that $\{Mr_1^*, Mr_2^*, \ldots, Mr_k^*, s_1, s_2, \ldots, s_{m-k}\}$ is linearly independent. And so $[Mr_1^* \; Mr_2^* \; \cdots \; Mr_k^* \; s_1 \; s_2 \; \cdots \; s_{m-k}]$ is invertible.

□

Observation 2.1.5. Let $\{t_1, t_2, \ldots, t_{n-k}\}$ be a basis for $\mathrm{Null}(M)$, and consider that $\mathcal{C}_n = \mathrm{Span}(r_1^*, r_2^*, \cdots, r_k^*) \bigoplus \mathrm{Null}(M)$ in the proof of the Lemma 2.1.4. Since the sets $\{r_1^*, r_2^*, \ldots, r_k^*\}$ and $\{t_1, t_2, \ldots, t_{n-k}\}$ are bases for $\mathrm{Span}(r_1^*, r_2^*, \ldots, r_k^*)$ and $\mathrm{Null}(M)$ respectively. This implies that $\{r_1^*, r_2^*, \ldots, r_k^*, t_1, t_2, \ldots, t_{n-k}\}$ is a basis for $\mathcal{C}_n$.

Observation 2.1.6. Since $\{Mr_1^*, Mr_2^*, \ldots, Mr_k^*, s_1, s_2, \ldots, s_{m-k}\}$ is a linearly independent set of vectors of $\mathcal{C}_m$ and the number of the elements of

$\{Mr_1^*, Mr_2^*, \ldots, Mr_k^*, s_1, s_2, \ldots, s_{m-k}\}$ is the same as the dimension of $\mathcal{C}_m$, $\{Mr_1^*, Mr_2^*, \ldots, Mr_k^*, s_1, s_2, \ldots, s_{m-k}\}$ is a basis of $\mathcal{C}_m$.

Theorem 2.1.7. *Let M be an $m \times n$-matrix with rational entries. Then $M^+ = A_M B_M$ is the Moore-Penrose inverse of M i.e. $A_M B_M$ satisfies the equations 2.1, 2.2, 2.3 and 2.4.*

Proof. Equation 2.1: Let $\beta_1 = \{r_1^*, r_2^*, \ldots, r_k^*, t_1, t_2, \ldots, t_{n-k}\}$ be the basis of $\mathcal{C}_n$ in the Observation 2.1.5 and $\{e_1, e_2, \ldots, e_n\}$ the standard basis of $\mathcal{C}_n$. Please note that we use same notation for the standard basis of $\mathcal{C}_m$ We want to show that $MA_M B_M M$ is equal to M. If we consider these two matrices as linear transformations, showing that they agree on the elements of a basis of the domain space $\mathcal{C}_n$ implies $MA_M B_M M = M$. For this purpose we consider the basis β_1. Then for every $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} MA_M B_M M r_i^* &= MA_M B_M B_M^{-1} e_i \\ &= MA_M e_i \\ &= Mr_i^* \end{aligned}$$

and for every $i = 1, 2, \ldots, n-k$ we have

$$\begin{aligned} MA_M B_M M t_i &= MA_M B_M B_M^{-1} e_{k+i} \\ &= MA_M e_{k+i} \\ &= M0 \\ &= 0 \\ &= Mt_i. \end{aligned}$$

Equation 2.2: Now we want to show that $A_M B_M M A_M B_M = A_M B_M$ by the same way used in the previous equation. Observe that $A_M B_M$ is an $n \times m$ matrix and we can consider is it as a linear transformation from $\mathcal{C}_m$ to $\mathcal{C}_n$. Let $\beta_2 = \{Mr_1^*, Mr_2^*, \ldots, Mr_k^*, s_1, s_2, \ldots, s_{m-k}\}$ be the basis of $\mathcal{C}_m$ in the Observation

2.1.6. Then for every $i = 1, 2, \ldots, k$, we have

$$\begin{aligned}
A_M B_M M A_M B_M M r_i^* &= A_M B_M M A_M B_M B_M^{-1} e_i \\
&= A_M B_M M A_M e_i \\
&= A_M B_M M r_i^*,
\end{aligned}$$

and for every $i = 1, 2, \ldots m - k$ we have

$$\begin{aligned}
A_M B_M M A_M B_M s_i &= A_M B_M M A_M B_M B_M^{-1} e_{k+i} \\
&= A_M B_M M A_M e_{k+i} \\
&= A_M B_M M 0 \\
&= 0 \\
&= A_M e_{k+i} \\
&= A_M B_M B_M^{-1} e_{k+i} \\
&= A_M B_M s_i.
\end{aligned}$$

Equation 2.3: We want to show that $(A_M B_M M)^* = A_M B_M M$. Since $\beta_1 = \{r_1^*, r_2^*, \ldots, r_k^*, t_1, t_2, \ldots, t_{n-k}\}$ is a basis of $\mathcal{C}_n$, the $n \times n$ matrix $\boldsymbol{\beta}_1 = [r_1^* \; r_2^* \; \cdots \; r_k^* \; t_1 \; t_2 \; \cdots \; t_{n-k}]$ is invertible. Thus, showing that

$$\boldsymbol{\beta}_1^* (A_M B_M M)^* \boldsymbol{\beta}_1 = \boldsymbol{\beta}_1^* A_M B_M M \boldsymbol{\beta}_1 \tag{2.5}$$

implies the Equation 2.3. For this purpose, we show that the ij^{th} entry of the matrices in the both sides of the Equation 2.5. Then for every $i \in \{1, 2, \ldots, k\}$

and $j \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M)^* \boldsymbol{\beta}_1 e_i &= r_j (A_M B_M M)^* r_i^* \\
&= (A_M B_M M r_j^*)^* r_i^* \\
&= (A_M B_M B_M^{-1} e_j)^* r_i^* \\
&= (A_M e_j)^* r_i^* \\
&= (r_j^*)^* r_i^* \\
&= r_j r_i^*
\end{aligned}$$

and

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M) \boldsymbol{\beta}_1 e_i &= r_j (A_M B_M M) r_i^* \\
&= r_j A_M B_M B_M^{-1} e_i \\
&= r_j A_M e_i \\
&= r_j r_i^*.
\end{aligned}$$

For every $i \in \{1, 2, \ldots, k\}$ and $j \in \{k+1, k+2, \ldots, n\}$, we have

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M)^* \boldsymbol{\beta}_1 e_i &= t_{j-k}^* (A_M B_M M)^* r_i^* \\
&= (A_M B_M M t_{j-k})^* r_i^* \\
&= (A_M B_M B\, 0)^* r_i^* \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M) \boldsymbol{\beta}_1 e_i &= t_{j-k}^* (A_M B_M M) r_i^* \\
&= t_{j-k}^* A_M B_M B_M^{-1} e_i \\
&= t_{j-k}^* A_M e_i \\
&= t_{j-k}^* r_i^* \\
&= (r_i t_{j-k})^* \\
&= 0.
\end{aligned}$$

For every $i \in \{k+1, k+2, \ldots, n\}$ and $j \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M)^* \boldsymbol{\beta}_1 e_i &= r_j (A_M B_M M)^* t_{i-k} \\
&= (A_M B_M M r_j^*)^* t_{i-k} \\
&= (A_M B_M B_M^{-1} e_j)^* t_{i-k} \\
&= (A_M e_j)^* t_{i-k} \\
&= (r_j^*)^* t_{i-k} \\
&= r_j t_{i-k} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M) \boldsymbol{\beta}_1 e_i &= r_j (A_M B_M M) t_{i-k} \\
&= r_j A_M B_M 0 \\
&= 0.
\end{aligned}$$

For every $i \in \{k+1, k+2, \ldots, n\}$ and $j \in \{k+1, k+2, \ldots, n\}$, we have

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M)^* \boldsymbol{\beta}_1 e_i &= t_{j-k}^* (A_M B_M M)^* t_{i-k} \\
&= (A_M B_M M t_{j-k})^* t_{i-k} \\
&= (A_M B_M B\, 0)^* t_{i-k} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
e_j^* \boldsymbol{\beta}_1^* (A_M B_M M) \boldsymbol{\beta}_1 e_i &= t_{j-k}^* (A_M B_M M) t_{i-k} \\
&= t_{j-k}^* A_M B_M 0 \\
&= 0.
\end{aligned}$$

Equation 2.4: We want to show that $(M A_M B_M)^* = M A_M B_M$ by the same argument as in the previous equation. $\boldsymbol{\beta}_2 = [M r_1^* \, M r_2^* \, \cdots \, M r_k^* \, s_1 \, s_2 \, \cdots \, s_{m-k}]$ is an invertible $m \times m$ matrix, since β_2 is a basis for $\mathcal{C}_m$. Thus, showing that

$$\boldsymbol{\beta}_2^*(MA_MB_M)^*\boldsymbol{\beta}_2 = \boldsymbol{\beta}_2^*(MA_MB_M)\boldsymbol{\beta}_2 \tag{2.6}$$

implies the Equation 2.4. Then for every $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned} e_j^*\boldsymbol{\beta}_2^*(MA_MB_M)^*\boldsymbol{\beta}_2 e_i &= (Mr_j^*)^*(MA_MB_M)^*Mr_i^* \\ &= (MA_MB_MMr_j^*)^*Mr_i^* \\ &= (MA_MB_MB_M^{-1}e_j)^*Mr_i^* \\ &= (MA_Me_j)^*Mr_i^* \\ &= (Mr_j^*)^*Mr_i^* \end{aligned}$$

and

$$\begin{aligned} e_j^*\boldsymbol{\beta}_2^*(MA_MB_M)\boldsymbol{\beta}_2 e_i &= (Mr_j^*)^*(MA_MB_M)Mr_i^* \\ &= (Mr_j^*)^*MA_MB_MB_M^{-1}e_i \\ &= (Mr_j^*)^*MA_Me_i \\ &= (Mr_j^*)^*Mr_i^*. \end{aligned}$$

For every $i \in \{1, 2, \ldots, k\}$ and $j \in \{k+1, k+2, \ldots, m\}$, we have

$$\begin{aligned} e_j^*\boldsymbol{\beta}_2^*(MA_MB_M)^*\boldsymbol{\beta}_2 e_i &= s_{j-k}^*(MA_MB_M)^*Mr_i^* \\ &= (MA_MB_Ms_{j-k})^*Mr_i^* \\ &= (MA_MB_MB_M^{-1}e_j)^*Mr_i^* \\ &= (MA_Me_j)^*Mr_i^* \\ &= (M0)^*Mr_i^* \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
e_j^*\boldsymbol{\beta}_2^*(MA_MB_M)\boldsymbol{\beta}_2e_i &= s_{j-k}^*(MA_MB_M)Mr_i^* \\
&= s_{j-k}^*MA_MB_MB_M^{-1}e_i \\
&= s_{j-k}^*MA_Me_i \\
&= s_{j-k}^*Mr_i^* \\
&= (r_iM^*s_{j-k})^* \\
&= (r_i0)^* \\
&= 0.
\end{aligned}$$

For every $i \in \{k+1, k+2, \ldots, n\}$ and $j \in \{1, 2, \ldots, k\}$, we have

$$\begin{aligned}
e_j^*\boldsymbol{\beta}_2^*(MA_MB_M)^*\boldsymbol{\beta}_2e_i &= (Mr_j^*)^*(MA_MB_M)^*s_{i-k} \\
&= (MA_MB_MMr_j^*)^*s_{i-k} \\
&= (MA_MB_MB_M^{-1}e_j)^*s_{i-k} \\
&= (MA_Me_j)^*s_{i-k} \\
&= (Mr_j^*)^*s_{i-k} \\
&= r_jM^*s_{i-k} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
e_j^*\boldsymbol{\beta}_2^*(MA_MB_M)\boldsymbol{\beta}_2e_i &= (Mr_j^*)^*(MA_MB_M)s_{i-k} \\
&= (Mr_j^*)^*(MA_MB_MB_M^{-1}e_i) \\
&= (Mr_j^*)^*(MA_Me_i) \\
&= (Mr_j^*)^*(M0) \\
&= 0.
\end{aligned}$$

For every $i \in \{k+1, k+2, \ldots, n\}$ and $j \in \{k+1, k+2, \ldots, n\}$, we have

$$\begin{aligned} e_j^* \boldsymbol{\beta}_2^* (MA_M B_M)^* \boldsymbol{\beta}_2 e_i &= s_{j-k}^* (MA_M B_M)^* s_{i-k} \\ &= (MA_M B_M s_{j-k})^* s_{i-k} \\ &= (MA_M B_M B_M^{-1} e_j)^* s_{i-k} \\ &= (MA_M e_j)^* s_{i-k} \\ &= (M0)^* s_{i-k} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} e_j^* \boldsymbol{\beta}_2^* (MA_M B_M) \boldsymbol{\beta}_2 e_i &= s_{j-k}^* (MA_M B_M) s_{i-k} \\ &= s_{j-k}^* (MA_M B_M B_M^{-1} e_i) \\ &= s_{j-k}^* (MA_M e_i) \\ &= s_{j-k}^* (M0) \\ &= 0. \end{aligned}$$

Thus, $M^+ = A_M B_M$.

$\square$

We have defined the Moore-Penrose inverse for an $m \times n$-matrix with rational entries and now we want give some properties of it.

2.2 Properties of the Moore Penrose Inverse

The following lemma shows that the Moore-Penrose inverse of the conjugate transpose of a matrix is the conjugate transpose of its Moore-Penrose inverse.

Lemma 2.2.1. *[6] Let M be an $m \times n$-matrix with rational entries and M^+ the Moore-Penrose inverse of M. Then $(M^*)^+ = (M^+)^*$.*

Proof. By taking their conjugate transpose of $MM^+M = M$ and $M^+MM^+ = M^+$, we have

$$M^*(M^+)^*M^* = M^* \text{ and } (M^+)^*M^*(M^+)^* = (M^+)^*.$$

Similarly by taking conjugate transpose of third and fourth equations for M^+ we get

$$(M^+M)^* = M^+M \text{ and } (MM^+)^* = MM^+.$$

Thus, $(M^+)^*$ satisfies the four conditions for the Moore-Penrose inverse of M^*. Because of the uniqueness of the Moore-Penrose inverse we conclude that

$$(M^*)^+ = (M^+)^*.$$

□

In the following observation, we show how we can consider M^+M and MM^+ as projections onto $\text{im}(M^*)$ and $\text{im}(M)$ respectively.

Observation 2.2.2. Let M be an $m \times n$-matrix with rational entries and M^+ the Moore-Penrose inverse of M. Assume $P = M^+M$. Since $P^2 = M^+MM^+M = M^+M = P$, we say that P is a projection onto its image. On the other hand $P^* = (M^+M)^* = M^+M = P$. Now we see that P is a self-adjoint and idempotent linear transformation. Thus, P is an orthogonal projection onto its image. Since $P = P^* = M^*(M^+)^*$, we have

$$\text{im}(P) \subseteq \text{im}(M^*).$$

Since $M(I - P) = M - MM^+M = M - M = 0$, the image of the complementary projector of P is contained my the null space of M i.e. $\text{im}(I - P) \subseteq \text{Null}(M)$. By taking the orthogonal complement of both sides we have $\text{Null}(M)^\perp \subseteq \text{im}(I - P)^\perp = \text{im}(P)$. Since $\text{Null}(M)^\perp = \text{im}(M^*)$, we write

$$\text{im}(M^*) \subseteq \text{im}(P).$$

Thus,

$$M^+M = \operatorname{Proj}_{\operatorname{im}(M^*)}.$$

Since $MM^+ = (MM^+)^* = (M^+)^*M^* = (M^*)^+M^*$, we can say that MM^+ is orthogonal projection onto $\operatorname{im}((M^*)^*)$ by the same argument. Hence,

$$MM^+ = \operatorname{Proj}_{\operatorname{im}(M)}.$$

Because we deal with only matrices which is the incidence matrix of a category, we consider only square matrices.

Lemma 2.2.3. *Let M be an $m \times m$-matrix, x an $m \times 1$-matrix, and y an $m \times 1$-matrix. Then $M^+x = y$ if and only if $y^* \in \operatorname{Row}(M)$ and $My = \operatorname{Proj}_{\operatorname{im}(M)}(x)$*

Proof. Assume that $M^+x = y$. By multiplying both sides M^+M, we have $M^+My = M^+MM^+x = M^+x = y$. Then $y = \operatorname{Proj}_{\operatorname{im}(M^*)}(y)$ by the Observation 2.2.2. This means that $y \in \operatorname{im}(M^*)$. And so

$$y^* \in \operatorname{Row}(M)$$

By multiplying both sides of $M^+x = y$ by M, we have $MM^+x = My$. Then we can write

$$My = \operatorname{Proj}_{\operatorname{im}(M)}(x)$$

For the reverse direction assume that $y^* \in \operatorname{Row}(M)$ and $My = \operatorname{Proj}_{\operatorname{im}(M)}(x)$. Since $y^* \in \operatorname{Row}(M)$, we have $y \in \operatorname{im}(M^*)$. Then $\operatorname{Proj}_{\operatorname{im}(M^*)}(y) = y$. Since $\operatorname{Proj}_{\operatorname{im}(M^*)} = M^+M$, we have

$$M^+My = y \tag{2.7}$$

$My = \operatorname{Proj}_{\operatorname{im}(M)}(x)$ implies that

$$MM^+(x) = My \tag{2.8}$$

again by the Observation 2.2.2. Multiplying both sides of the equation 2.8 M^+

gives us $M^+(x) = M^+My$. Substituting this in the equation 2.7 gives us

$$M^+x = y$$

$\square$

Let P be an invertible matrix with rational entries. One can easily show that the inverse of P satisfies the required equations to be the Moore-Penrose inverse of P. Because of the uniqueness of the Moore-Penrose inverse, $P^+ = P^{-1}$. If P is orthogonal, $P^+ = P^*$. Moreover for orthogonal matrices we have the following lemma.

Lemma 2.2.4. *Let P be an orthogonal $m \times m$ matrix and M be any $m \times m$ matrix. Then $(MP)^+ = P^+M^+$ and $(PM)^+ = M^+P^+$.*

Proof. Since P is orthogonal, we have $P^+ = P^*$. The matrix P^+M^+ satisfies Equation (2.1), because

$$(MP)(P^+M^+)(MP) = MM^+MP = MP.$$

The matrix P^+M^+ satisfies Equation (2.2), because

$$(P^+M^+)(MP)(P^+M^+) = P^+M^+MM^+ = P^+M^+.$$

The matrix P^+M^+ satisfies Equation (2.3), because

$$((P^+M^+)(MP))^* = P^*M^*(M^+)^*(P^+)^* = P^+(M^+M)^*P = (P^+M^+)(MP).$$

The matrix P^+M^+ satisfies Equation (2.4), because

$$((MP)(P^+M^+))^* = (MM^+)^* = MM^+ = (MP)(P^+M^+).$$

Similarly, $(PM)^+ = M^+P^+$. $\square$

Observation 2.2.5. Let $M \otimes N$ denote the Kronecker product of the square matrices M and N. Then it is obvious to verify that

$$(M \otimes N)^+ = M^+ \otimes N^+$$

because in general we have $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$ in Section 2.6A of [14] when A, B, C, D are square matrices.

Let A_1, A_2, $\ldots$ A_m be square matrices of possibly different sizes. Then the matrix $\mathrm{diag}(A_1, A_2, \ldots, A_n)$ is defined as the square matrix obtained by putting A_1, A_2, $\ldots$ A_m on the diagonal and zero everywhere else. Then we know that

$$\mathrm{diag}(A_1, A_2, \ldots, A_n)^+ = \mathrm{diag}(A_1^+, A_2^+, \ldots, A_n^+).$$

Chapter 3

The Euler Measure of Finite Categories

3.1 Definition of the Euler Measure

In this chapter we define the Euler measure of a finite category $\chi(\mathcal{A})$

Let $\mathbf{1}_m$ denote the $m \times 1$-matrix whose all entries are equal to 1. Given a finite category $\mathcal{A}$, let $[\mathcal{A}]$ denote the adjacency matrix associated with the directed multigraph obtained from the category $\mathcal{A}$ by forgetting its composition data. In other words, if the object set of $\mathcal{A}$ is $\{a_1, a_2, \ldots, a_m\}$ then $[\mathcal{A}]$ is the $m \times m$-matrix whose ij^{th} entry is the number of elements of the morphism set $\mathcal{A}(a_i, a_j)$. Sometimes we call the matrix $[\mathcal{A}]$ as the **incidence matrix** of $\mathcal{A}$. Let $|\mathcal{A}|$ denote the number of objects of a category $\mathcal{A}$.

Theorem 3.1.1. *Let $\mathcal{A}$ be a finite category then the sum of the entries of the matrix $[\mathcal{A}]^+$ is independent of the order chosen on the set of objects of $\mathcal{A}$.*

Proof. Let P be any $|\mathcal{A}| \times |\mathcal{A}|$-permutation matrix. Observe that multiplying a square matrix from right by P just permutes the columns of it and multiplying left by P^{-1} applies the same permutation to the rows of it. Thus, showing that

$\mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|} = \mathbf{1}^*_{|\mathcal{A}|}(P^{-1}[\mathcal{A}]P)^+\mathbf{1}_{|\mathcal{A}|}$ completes the proof. Since P is an orthogonal matrix, $P^+ = P^{-1}$ and by Lemma 2.2.4

$$(P^{-1}[\mathcal{A}]P)^+ = P^+(P^{-1}[\mathcal{A}])^+ = P^{-1}([\mathcal{A}]^+(P^{-1})^+) = P^{-1}[\mathcal{A}]^+P.$$

Therefore,

$$\mathbf{1}^*_{|\mathcal{A}|}(P^{-1}[\mathcal{A}]P)^+\mathbf{1}_{|\mathcal{A}|} = \mathbf{1}^*_{|\mathcal{A}|}P^{-1}[\mathcal{A}]^+P\mathbf{1}_{|\mathcal{A}|} = \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|}.$$

□

We say an $n \times 1$-matrix w is a **weighting** for an $m \times n$-matrix M if

$$Mw = \mathbf{1}_m.$$

Similarly, for an $m \times n$-matrix M, a $1 \times m$-matrix v is called a **coweighting** if

$$vM = \mathbf{1}^*_n.$$

If $[\mathcal{A}]$ has a weighting or a coweighting for a given category $\mathcal{A}$, We say $\mathcal{A}$ **admits a weighting** or a **admits a coweighting** respectively.

Now we can define the Euler measure χ for a finite category.

Definition 3.1.2. The rational number $\chi(\mathcal{A})$ associated with a category $\mathcal{A}$ is defined as

$$\chi(\mathcal{A}) = \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|}$$

where 1×1-matrix on the right hand side is considered as a rational number. Moreover $\chi(\mathcal{A})$ is called the Euler measure of $\mathcal{A}$ if $[\mathcal{A}]$ has a weighting.

Remark 3.1.3. By abuse of notation, we use χ to denote the Euler measure even though it is actually a restriction of χ to the family of categories admitting weighting. The reason for this restriction is explained in Observation 3.3.11.

Observe that $\chi(\mathcal{A})$ is independent of the chosen order of objects of $\mathcal{A}$ by Theorem 3.1.1.

3.2 Comparison with Leinster's Euler Characteristic

Euler characteristic $\chi_{Lein}(\mathcal{A})$ of a finite category $\mathcal{A}$ is defined in [3] as follows: A finite category $\mathcal{A}$ has **Euler characteristic** $\chi_{Lein}(\mathcal{A})$ if $[\mathcal{A}]$ has both a weighting w and a coweighting v and

$$\chi_{Lein}(\mathcal{A}) = \mathbf{1}^*_{|\mathcal{A}|} w = v \mathbf{1}_{|\mathcal{A}|}.$$

We want to compare the Euler measure $\chi(\mathcal{A})$ with the Euler characteristic $\chi_{Lein}(\mathcal{A})$ of $\mathcal{A}$ when $\chi_{Lein}(\mathcal{A})$ exists. For this purpose, we need to fix some notation and give some useful properties.

Lemma 3.2.1. *Let M be an $m \times n$-matrix. Then $M^+\mathbf{1}_m$ is a weighting of M if and only if M has a weighting.*

Proof. Assume that w is a weighting of M. Then

$$\mathbf{1}_m = Mw = MM^+Mw = MM^+\mathbf{1}_m.$$

Hence, $M^+\mathbf{1}_m$ is a weighting for M. The converse is clear. □

Lemma 3.2.2. *Let M be an $m \times n$-matrix. Then $\mathbf{1}^*_n M^+$ is a coweighting of M if and only if M has a coweighting.*

Proof. Assume that v is a coweighting of M. Then

$$\mathbf{1}^*_n = vM = vMM^+M = \mathbf{1}^*_n M^+ M.$$

Hence, $\mathbf{1}^*_n M^+$ is a coweighting for M. The converse is clear. □

Lemma 3.2.3. *Let M be an $m \times n$-matrix. Assume that w is a weighting for M and v is a coweighting for M. Then*

$$\mathbf{1}_n^* w = v\mathbf{1}_m = \mathbf{1}_n^* M^+ \mathbf{1}_m.$$

Proof. Assume that w is a weighting for M and v is a coweighting for M. Then we have

$$\mathbf{1}_n^* w = vMw = v\mathbf{1}_m.$$

To prove the last equality, note that

$$vMw = vMM^+Mw = \mathbf{1}_n^* M^+ \mathbf{1}_m.$$

□

The following theorem shows that χ is an extension of χ_{Lein} to all finite categories.

Theorem 3.2.4. *Let $\mathcal{A}$ be a finite category and assume that $\chi_{Lein}(\mathcal{A})$ is defined. Then*

$$\chi_{Lein}(\mathcal{A}) = \chi(\mathcal{A}).$$

Proof. Let $\mathcal{A}$ be a finite small category. Assume that $\chi_{Lein}(\mathcal{A})$ is defined as in [3]. This means that there exists a coweighting v and a weighting w for $[\mathcal{A}]$. Hence, by Lemma 3.2.3 we have

$$\chi_{Lein}(\mathcal{A}) = \mathbf{1}_{|\mathcal{A}|}^* w = \mathbf{1}_{|\mathcal{A}|}^* [\mathcal{A}]^+ \mathbf{1}_{|\mathcal{A}|} = \chi(\mathcal{A}).$$

□

3.3 Properties of the Euler Measure

For a functor $L : \mathcal{A} \to \mathcal{B}$, let $[L]$ denote the $|\mathcal{B}| \times |\mathcal{A}|$-matrix whose ij^{th} entry is 1 if L sends the j^{th} object of $\mathcal{A}$ to the i^{th} object of $\mathcal{B}$ and 0 otherwise. We give a proof of Proposition 2.4 in [3] by using the terminolgy developed above.

Theorem 3.3.1. *Let $L : \mathcal{A} \rightleftarrows B : R$ be an adjunction. Assume that $[\mathcal{A}]$ has a coweighting and $[\mathcal{B}]$ has a weighting. Then $[\mathcal{B}]$ has a coweighting and $[\mathcal{A}]$ has a weighting and we have $\chi(\mathcal{A}) = \chi(\mathcal{B})$.*

Proof. Assume that $[\mathcal{A}]$ has a coweighting and $[\mathcal{B}]$ has a weighting and $L : \mathcal{A} \rightleftarrows B : R$ is an adjunction. By Lemma 3.2.1 and Lemma 3.2.2, we know that

$$\mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+[\mathcal{A}] = \mathbf{1}^*_{|\mathcal{A}|} \quad \text{and} \quad [\mathcal{B}][\mathcal{B}]^+\mathbf{1}_{|\mathcal{B}|} = \mathbf{1}_{|\mathcal{B}|}.$$

Moreover, we know that

$$\mathbf{1}^*_{|\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}|}[R] \quad \text{and} \quad [L]^*\mathbf{1}_{|\mathcal{B}|} = \mathbf{1}_{|\mathcal{A}|}$$

because each column of the matrices $[R]$ and $[L]$ contains exactly one entry that is equal to 1. Observe that the ij^{th} entry of the matrix $[\mathcal{A}][R]$ is the unique entry of the 1×1-matrix obtained by the multiplication of the row matrix whose entries are the number of morphisms from i^{th} object of $\mathcal{A}$ to the other objects of $\mathcal{A}$ with the column matrix whose only nonzero entry has row index as the order of the image of j^{th} object of $\mathcal{B}$ under R in $\mathcal{A}$. Thus, ij^{th} entry of $[\mathcal{A}][R]$ is the number of morphisms from i^{th} object of $\mathcal{A}$ to the image of j^{th} object of $\mathcal{B}$ under R. Similarly the ij^{th} entry of the matrix $[L]^*[\mathcal{B}]$ is the unique entry of the 1×1-matrix obtained by the multiplication of the row matrix whose only nonzero entry has column index as the order of the image of i^{th} object of $\mathcal{A}$ under L in $\mathcal{B}$ with the column matrix whose entries are the number of morphisms from the objects of $\mathcal{B}$ to j^{th} object of $\mathcal{B}$. Thus, ij^{th} entry of $[L]^*[\mathcal{B}]$ is the number of morphisms from the image of i^{th} object of $\mathcal{A}$ under L to the j^{th} object of $\mathcal{B}$. Then

$$[\mathcal{A}][R] = [L]^*[\mathcal{B}]$$

because R and L are adjoints. Now notice that

$$\mathbf{1}^*_{|\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}|}[R] = \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+[\mathcal{A}][R] = \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+[L]^*[\mathcal{B}].$$

Hence, $\mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+[L]^*$ is a coweighting for $[\mathcal{B}]$. Similarly, $[R][\mathcal{B}]^+\mathbf{1}_{|\mathcal{B}|}$ is weighting for $[\mathcal{A}]$. Finally, we have

$$\mathbf{1}^*_{|\mathcal{B}|}[\mathcal{B}]^+\mathbf{1}_{|\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}|}[R][\mathcal{B}]^+\mathbf{1}_{|\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+[\mathcal{A}][R][\mathcal{B}]^+\mathbf{1}_{|\mathcal{B}|}$$

$$= \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+[L]^*[\mathcal{B}][\mathcal{B}]^+\mathbf{1}_{|\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+[L]^*\mathbf{1}_{|\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}|}[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|}.$$

Hence, $\chi(\mathcal{A}) = \chi(\mathcal{B})$. □

A pseudofunctor is a generalization of a functor between categories that relaxes some of the strict requirements of functors. A pseudofunctor relaxes the strict preservation of composition and identity found in functors, allowing for more flexibility in the relationships between categories while maintaining a coherent structure through isomorphisms.

Let $\mathcal{A}$ be a category and $F : \mathcal{A} \to \mathbf{Cat}$ be a pseudofunctor. Then the **Grothendieck construction** $G(F)$ is the category that has as objects all pairs (a, x) where a is an object in $\mathcal{A}$ and x is an object in $F(a)$, and morphisms from (a, x) to (b, y) are pairs (f, ζ) where $f : a \to b$ is a morphism in $\mathcal{A}$ and $\zeta : F(f)(x) \to y$ is a morphism in $F(b)$.

For the rest of this section, let $\mathcal{A}$ be a finite category with object set

$$\mathrm{Ob}(\mathcal{A}) = \{a_1, a_2, \dots a_m\}.$$

Let $F : \mathcal{A} \to \mathbf{Cat}$ be a pseudofunctor. Assume that for every i in $\{1, 2, \dots m\}$, the category $F(a_i)$ has object set

$$\mathrm{Ob}(F(a_i)) = \{x_{i1}, x_{i2}, \dots x_{in_i}\}$$

for some natural number n_i. Then $G(F)$ has $n_1 + n_2 + \dots + n_m$ many objects and objects of $G(F)$ are ordered as follows: (a_1, x_{11}), (a_1, x_{12}), $\ldots$, (a_1, x_{1n_1}), (a_2, x_{21}), $\ldots$, (a_2, x_{2n_2}), $\ldots$, (a_m, x_{m1}), $\ldots$, (a_m, x_{mn_m}). Let $U : \mathbf{Cat} \to \mathbf{Set}$ denote the functor that sends a category to its set of objects. Let $T : \mathbf{Set} \to \mathbf{Cat}$ denote the discrete category functor. Define $L_1(F) : \mathcal{A} \to \mathbf{Cat}$ as follows:

$$L_1(F) = T \circ U \circ F.$$

Observe that L_1 takes F and just forgets the nonidentity morphisms of categories of **Cat**. Notice that there exists a natural transformation $i : T \circ U \Rightarrow \mathrm{Id}_{\mathbf{Cat}}$ given by inclusion on each component. Define $L_2(F) : T \circ U(\mathcal{A}) \to \mathbf{Cat}$ as follows:

$$L_2(F) = F \circ i_{\mathcal{A}}.$$

Similarly L_2 takes F and just forgets the nonidentity morphisms of $\mathcal{A}$. Then we have the following lemmas:

Lemma 3.3.2. $[G(L_2(F))] = \mathrm{diag}\left([F(a_1)], [F(a_2)], \ldots, [F(a_m)]\right)$.

Proof. Let a, a' be two objects of $\mathcal{A}$, x an object of $F(a)$ and x' an object of $F(a')$. When $a \neq a'$,

$$G(L_2(F))((a, x), (a', x')) = \emptyset$$

since there is no morphism between a and a' in $\mathcal{A}$.

When $a = a'$,

$$G(L_2(F))((a, x), (a', x')) \cong F(a)(x, x')$$

since there is only identity morphism from a to itself. □

Lemma 3.3.3. $[G(F)] = [G(L_1(F))] \, [G(L_2(F))]$.

Proof. Let a, a' be two objects of $\mathcal{A}$, x an object of $F(a)$ and x' an object of $F(a')$. For any object $\widetilde{a}$ in $\mathcal{A}$ and an object $\widetilde{y}$ in $F(\widetilde{a})$, when $\widetilde{a} \neq a'$ we have

$$|G(L_2(F))((\widetilde{a}, \widetilde{y}), (a', x'))| = 0,$$

by Lemma 3.3.2. Then

$$\sum_{\widetilde{a}\in \mathrm{Ob}(\mathcal{A}) \text{ and } \widetilde{y}\in \mathrm{Ob}(F(\widetilde{a}))} |G(L_1(F))((a,x),(\widetilde{a},\widetilde{y}))||G(L_2(F))((\widetilde{a},\widetilde{y}),(a',x'))| \tag{3.1}$$

equals to

$$\sum_{a'\in \mathrm{Ob}(\mathcal{A}) \text{ and } \widetilde{y}\in \mathrm{Ob}(F(a'))} |G(L_1(F))((a,x),(a',\widetilde{y}))||G(L_2(F))((a',\widetilde{y}),(a',x'))|. \tag{3.2}$$

Note that a' fixed. Then by setting $y = a'$, the summation in 3.2 turns to

$$\sum_{y\in \mathrm{Ob}(F(a'))} |G(L_1(F))((a,x),(a',y))||G(L_2(F))((a',y),(a',x'))|. \tag{3.3}$$

Also by Lemma 3.3.2, we have

$$|G(L_2(F))((a',y),(a',x'))| = |F(a')(y,x')|.$$

Hence, the statement in 3.3 equals to

$$\sum_{y\in \mathrm{Ob}(F(a'))} |G(L_1(F))((a,x),(a',y))||F(a')(y,x')| \tag{3.4}$$

Notice that

$$|G(L_1(F))((a,x),(a',y))| = |\{\, f\in \mathcal{A}(a,a') \mid F(f)(x) = y \,\}|.$$

Then the summation in 3.4 equals to

$$\sum_{y\in \mathrm{Ob}(F(a'))} |\{\, f\in \mathcal{A}(a,a') \mid F(f)(x) = y \,\}||F(a')(y,x')| \tag{3.5}$$

On the other hand

$$G(F)((a,x),(a',x')) \cong \coprod_{y\in \mathrm{Ob}(F(a'))} \{\, f\in \mathcal{A}(a,a') \mid F(f)(x) = y \,\} \times F(a')(y,x').$$

Thus

$$\sum_{\widetilde{a}\in \mathrm{Ob}(\mathcal{A}) \text{ and } \widetilde{y}\in \mathrm{Ob}(F(\widetilde{a}))} |G(L_1(F))((a,x),(\widetilde{a},\widetilde{y}))||G(L_2(F))((\widetilde{a},\widetilde{y}),(a',x'))|$$
$$= |G(F)((a,x),(a',x'))|.$$

Hence, we get

$$[G(F)] = [G(L_1(F))]\,[G(L_2(F))]\,.$$

□

Let v_1, v_2, …, v_m be column vectors of possibly different sizes. Then we write $C(v_1, v_2, \ldots, v_m)$ for the following column vector

$$C(v_1, v_2, \ldots, v_m) = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

whose number of rows is equal to the sum of the number rows of v_1, v_2, …, v_m. Note that if v_1, v_2, …, v_m are all rational numbers then $C(v_1, v_2, \ldots, v_m)$ is an $m \times 1$-matrix.

Lemma 3.3.4. *Let $\mu_1, \mu_2, \ldots, \mu_m, \lambda_1, \lambda_2, \ldots, \lambda_m$ be rational numbers. Assume*

$$[\mathcal{A}]C\left(\mu_1, \mu_2, \ldots, \mu_m\right) = C\left(\lambda_1, \lambda_2, \ldots, \lambda_m\right).$$

Then we have

$$[G(L_1(F))]\,C\left(\mu_1\mathbf{1}_{n_1}, \mu_2\mathbf{1}_{n_2}, \ldots, \mu_m\mathbf{1}_{n_m}\right) = C\left(\lambda_1\mathbf{1}_{n_1}, \lambda_2\mathbf{1}_{n_2}, \ldots, \lambda_m\mathbf{1}_{n_m}\right).$$

Proof. Let a be an object in $\mathcal{A}$ and $x \in F(a)$. Then

$$\sum_{i=1}^{m}\sum_{j=1}^{n_i} |G(L_1(F))((a,x),(a_i,x_{ij}))|\mu_i = \sum_{i=1}^{m} |\mathcal{A}(a,a_i)|\mu_i.$$

Hence, the result follows. $\square$

Remark 3.3.5. When $\mathcal{A}$ is a poset, we can choose a total order on objects of $\mathcal{A}$ that extends the partial order on it. Hence, whenever $\mathcal{A}$ is a poset, without loss of generality we can assume that there exists a morphism from a_i to a_j in $\mathcal{A}$ only if $i \leq j$.

Lemma 3.3.6. *If $\mathcal{A}$ is a poset then there exists an invertible matrix M such that*

$$M\,[G(L_2(F))] = [G(L_1(F))][G(L_2(F))].$$

Proof. Assume that $\mathcal{A}$ is a poset. By Remark 3.3.5, we can assume that the chosen total order a_1, a_2, $\ldots$, a_m on the set of objects of $\mathcal{A}$ extends the partial order on it given by the poset structure on $\mathcal{A}$. For i in $\{1, 2\}$, let M_i denote the matrix $[G(L_i(F))]$. Hence, to prove the lemma we need to find an invertible matrix M such that $(M - M_1)M_2 = 0$. Consider these matrices as functions from $\mathrm{Ob}(G(F)) \times \mathrm{Ob}(G(F))$ to $\mathbb{Z}$. Let M be the matrix defined as follows:

$$M((a,x),(a',x')) = \begin{cases} M_1((a,x),(a',x')) & \text{if } a \neq a' \\ 1 & \text{if } (a,x) = (a',x') \\ 0 & \text{otherwise} \end{cases}$$

for objects (a,x), (a',x') in $G(F)$. First notice that M is an upper triangular matrix and all of its diagonal entries are equal to 1. Hence, M is an invertible matrix. Second notice that for each object a in $\mathcal{A}$, there exists a natural isomorphism from $\mathrm{id}_{F(a)}$ to $F(\mathrm{id}_a)$, since F is a pseudofunctor. In particular, this means for each object (a,x) in $G(F)$ there exists an isomorphism $\sigma_{a,x} : x \to F(\mathrm{id}_a)(x)$ in the category $F(a)$. Therefore, we have

$$M_2((a,x),(a',x')) = M_2((a, F(\mathrm{id}_a)(x)),(a',x'))$$

and

$$(M - M_1)((a,x),(a',x')) = \begin{cases} 1 & \text{if } a = a' \text{ and } x = x' \text{ and } x' \neq F(\mathrm{id}_a)(x) \\ -1 & \text{if } a = a' \text{ and } x \neq x' \text{ and } x' = F(\mathrm{id}_a)(x) \\ 0 & \text{otherwise} \end{cases}$$

for every object (a, x), (a', x') in $G(F)$. Thus, we have

$$\begin{aligned} (M - M_1)M_2((a,x),(a',x')) &= \sum_{y \in \{x, F(\mathrm{id}_a)(x)\}} (M - M_1)((a,x),(a,y))M_2((a,y),(a',x')) \\ &= M_2((a,x),(a',x')) - M_2((a,F(\mathrm{id}_a)(x)),(a',x')) \\ &= 0. \end{aligned}$$

Hence, we have proved that $(M - M_1)M_2 = 0$. □

The next lemma is an immediate corollary of the previous one.

Lemma 3.3.7. *If $\mathcal{A}$ is a poset then* $\mathrm{Row}([G(L_1(F))][G(L_2(F))]) = \mathrm{Row}([G(L_2(F))])$.

Proof. By Lemma 3.3.6, we know that there exists an in invertible matrix M such that $M\,[G(L_2(F))] = [G(L_1(F))][G(L_2(F))]$. We can write M as a product of elementary matrices since M is invertible. Moreover multiplication by an elementary matrix from left is same as performing a row operation. Therefore, the matrix $[G(L_2(F))]$ is row equivalent to $M\,[G(L_2(F))]$. Hence, we have

$$\mathrm{Row}([G(L_1(F))][G(L_2(F))]) = \mathrm{Row}([G(L_2(F))]).$$

□

Lemma 3.3.8. *Let $C(\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a weighting for $[\mathcal{A}]$ and v_i a weighting for $[F(a_i)]$ for every i in $\{1, 2, \ldots m\}$. Then $C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m)$ is a weighting for $[G(F)]$.*

Proof. Let $C(\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a weighting for $[\mathcal{A}]$ and v_i a weighting for $[F(a_i)]$

for every i in $\{1, 2, \ldots m\}$. By Lemma 3.3.2 we have

$$[G(L_2(F))] = \operatorname{diag}\left([F(a_1)], [F(a_2)], \ldots, [F(a_m)]\right).$$

Hence,

$$[G(L_2(F))]C\left(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m\right) = C\left(\lambda_1 \mathbf{1}_{n_1}, \lambda_2 \mathbf{1}_{n_2}, \ldots, \lambda_m \mathbf{1}_{n_m}\right).$$

By Lemma 3.3.4, we have

$$[G(L_1(F))]\,C\left(\lambda_1 \mathbf{1}_{n_1}, \lambda_2 \mathbf{1}_{n_2}, \ldots, \lambda_m \mathbf{1}_{n_m}\right) = C\left(\mathbf{1}_{n_1}, \mathbf{1}_{n_2}, \ldots, \mathbf{1}_{n_m}\right) = \mathbf{1}_{|\mathbf{G(F)}|}.$$

By Lemma 3.3.3, we have

$$[G(F)] = [G(L_1(F))][G(L_2(F))].$$

Hence, we have

$$[G(F)]C\left(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m\right) = \mathbf{1}_{|\mathbf{G(F)}|}.$$

Hence, $C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m)$ is a weighting for $[G(F)]$. ☐

Let $\mathcal{A}$ be a finite category with object set $\{a_1, a_2, \ldots a_m\}$. For a pseudofunctor $F : \mathcal{A} \to \mathbf{Cat}$, let $\chi(F)$ denote the $1 \times m$-matrix

$$\chi(F) = \left[\ \chi(F(a_1))\ \chi(F(a_2))\ \cdots\ \chi(F(a_m))\ \right].$$

Theorem 3.3.9. *Let $\mathcal{A}$ be a finite category and $F : \mathcal{A} \to \mathbf{Cat}$ a pseudofunctor such that for every object a in $\mathcal{A}$, the category $F(a)$ is finite. Assume that $[G(F)]$ has a coweighting, $[\mathcal{A}]$ has weighting, and for every object a in $\mathcal{A}$, $[F(a)]$ has a weighting and a coweighting. Then we have*

$$\chi(G(F)) = \chi(F)[\mathcal{A}]^+ \mathbf{1}_{|\mathcal{A}|}.$$

Proof. Since $[\mathcal{A}]$ has a weighting, by Lemma 3.2.1 we know that $[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|}$ is a weighting for $[\mathcal{A}]$. For some rational numbers $\lambda_1,\ \lambda_2, \ldots,\ \lambda_m$, we have

$$[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|} = C(\lambda_1, \lambda_2, \ldots, \lambda_m).$$

Assume that v_i is a weighting for $[F(a_i)]$ for every i in $\{1, 2, \ldots m\}$. Then by Lemma 3.3.8, $C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m)$ is a weighting for $[G(F)]$. Hence, by Lemma 3.2.3, we have

$$\chi(G(F)) = \mathbf{1}^*_{|G(F)|} C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m).$$

Since for every object a in $\mathcal{A}$, $[F(a)]$ has a weighting and a coweighting, we have

$$\chi(F(a_i)) = \mathbf{1}^*_{n_i} v_i$$

by Lemma 3.2.3. Since $\mathbf{1}^*_{|G(F)|} = [\ \mathbf{1}^*_{n_1}\ \mathbf{1}^*_{n_2}\ \ldots\ \mathbf{1}^*_{n_m}\ ]$, we have

$$\chi(G(F)) = \sum_{i=1}^{m} \lambda_i \mathbf{1}^*_{|F(a_i)|} v_i = \sum_{i=1}^{m} \lambda_i \chi(F(a_i)) = \chi(F)[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|}.$$

□

Though the following theorem puts some extra conditions to index category of Grothendieck construction, it loosens the condition of requiring images of the objects of the idex category to admit a coweighting.

Theorem 3.3.10. *Let $\mathcal{A}$ be a finite poset and $F : \mathcal{A} \to$ **Cat** a pseudofunctor such that for every object a in $\mathcal{A}$, the category $F(a)$ is finite. Assume that for every object a in $\mathcal{A}$, $[F(a)]$ has a weighting. Then we have*

$$\chi(G(F)) = \chi(F)[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|}.$$

Proof. Since $\mathcal{A}$ is a finite poset, the matrix $[\mathcal{A}]$ is invertible. Hence, it has a weighting. Therefore, $[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|}$ is a weighting for $[\mathcal{A}]$ by Lemma 3.2.1. For some rational numbers $\lambda_1,\ \lambda_2, \ldots,\ \lambda_m$, we have

$$[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|} = C(\lambda_1, \lambda_2, \ldots, \lambda_m).$$

For every object a in $\mathcal{A}$, we know that $[F(a)]^{+}\mathbf{1}_{|F(a)|}$ is a weighting for $[F(a)]$ by Lemma 3.2.1 since $[F(a)]$ has a weighting. Define $v_i = [F(a_i)]^{+}\mathbf{1}_{|F(a_i)|}$ for i in $\{1, 2, \ldots, m\}$. Then by Lemma 3.3.8, $C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m)$ is a weighting for $[G(F)]$. Notice that $C(0, \ldots, 0, v_i, 0, \ldots 0)^*$ is vector in $\mathrm{Row}(G(L_2(F)))$ due to Lemma 3.3.2 and the fact that v_i^* is in $\mathrm{Row}([F(a_i)])$ for each $i \in \{1, 2, \ldots, m\}$ by Lemma 2.2.3. Now by Lemma 3.3.7 we know that

$$\mathrm{Row}([G(L_1(F))][G(L_2(F))]) = \mathrm{Row}([G(L_2(F))])$$

and by Lemma 3.3.3 we know that

$$[G(F)] = [G(L_1(F))][G(L_2(F))].$$

Hence, $C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m)^*$ is in $\mathrm{Row}(G(F))$. Therefore, by Lemma 2.2.3, we have

$$[G(F)]^{+}\mathbf{1}_{G(F)} = C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m).$$

So as in the pevious proof we again have

$$\chi(G(F)) = \mathbf{1}^*_{|G(F)|} C(\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_m v_m).$$

Thus, we are done by the same argument. □

Observation 3.3.11. Theorem 3.3.10 shows that χ obeys the inclusion exclusion principle on the family of categories admitting weighting. This is why we have the condition of admitting a weighting to call χ as the Euler measure in the Definition 3.1.2. Note that inspite of the extra condition, χ is still more general than χ_{Lein}, since the family of categories admitting both weighting and coweighting is contained by the family of categories admitting weighting.

Example 3.3.12. Notice that the category

$$\mathcal{P} = \{b \leftarrow a \rightarrow c\}$$

is a poset and

$$[\mathcal{P}] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$[\mathcal{P}]^+ = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now consider any pseudofunctor $F : \mathcal{P} \to \mathbf{Cat}$ such that for every object x in $\mathcal{P}$, the category $F(x)$ is finite and $[F(x)]$ has a weighting. Then we have

$$\chi(G(F)) = \chi(F)[\mathcal{P}]^+\mathbf{1}_{|\mathcal{P}|}$$

by Theorem 3.3.10. Moreover, we have

$$\chi(F) = [\ \chi(F(a))\ \chi(F(b))\ \chi(F(c))\ ]$$

and

$$[\mathcal{P}]^+\mathbf{1}_{|\mathcal{P}|} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence

$$\chi(G(F)) = [\ \chi(F(a))\ \chi(F(b))\ \chi(F(c))\ ] \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \chi(F(b)) + \chi(F(c)) - \chi(F(a)).$$

Theorem 3.3.10 doesn't imply that the χ preserves products, so we prove this separately.

Theorem 3.3.13. *Let $\mathcal{A}$, $\mathcal{B}$ be two finite categories. Then we have*

$$\chi(\mathcal{A} \times \mathcal{B}) = \chi(\mathcal{A})\chi(\mathcal{B}).$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$ be two finite categories. Then

$$[\mathcal{A} \times \mathcal{B}] = [\mathcal{A}] \otimes [\mathcal{B}]$$

where $\otimes$ denote Kronecker product. Hence

$$[\mathcal{A} \times \mathcal{B}]^+ = [\mathcal{A}]^+ \otimes [\mathcal{B}]^+.$$

Also notice

$$\mathbf{1}_{|\mathcal{A}\times\mathcal{B}|} = \mathbf{1}_{|\mathcal{A}|} \otimes \mathbf{1}_{|\mathcal{B}|}.$$

As we discussed in Observation 2.2.5, in general we know that $(M \otimes N)(L \otimes K) = (ML) \otimes (NK)$. Then we have

$$\mathbf{1}^*_{|\mathcal{A}\times\mathcal{B}|}[\mathcal{A} \times \mathcal{B}]^+\mathbf{1}_{|\mathcal{A}\times\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}}[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|} \otimes \mathbf{1}^*_{|\mathcal{B}|}[\mathcal{B}]^+\mathbf{1}_{\mathcal{B}|}.$$

Here the operation $\otimes$ on the right-hand side of the equality above is in between two 1×1-matrices. Hence, it just corresponds to the multiplication of rational numbers. ☐

Now we show that χ is additive under disjoint unions.

Theorem 3.3.14. *Let $\mathcal{A}$, $\mathcal{B}$ be two finite categories. Then we have*

$$\chi(\mathcal{A} \sqcup \mathcal{B}) = \chi(\mathcal{A}) + \chi(\mathcal{B}).$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$ be two finite categories. Then

$$[\mathcal{A} \sqcup \mathcal{B}] = \operatorname{diag}([\mathcal{A}], [\mathcal{B}]).$$

Hence

$$[\mathcal{A} \sqcup \mathcal{B}]^+ = \operatorname{diag}([\mathcal{A}]^+, [\mathcal{B}]^+).$$

Also notice

$$\mathbf{1}_{|\mathcal{A}\sqcup\mathcal{B}|} = C(\mathbf{1}_{|\mathcal{A}|}, \mathbf{1}_{|\mathcal{B}|}).$$

So

$$\mathbf{1}^*_{|\mathcal{A}\sqcup\mathcal{B}|}[\mathcal{A}\sqcup\mathcal{B}]^+\mathbf{1}_{|\mathcal{A}\sqcup\mathcal{B}|} = \mathbf{1}^*_{|\mathcal{A}}[\mathcal{A}]^+\mathbf{1}_{|\mathcal{A}|} + \mathbf{1}^*_{|\mathcal{B}|}[\mathcal{B}]^+\mathbf{1}_{\mathcal{B}|}.$$

$\square$

3.4 Examples and Comparison with the Series Euler Characteristic

There are many artificially created examples of categories with incidence matrix having a weighting and no coweighting. For the following examples, note that Corollary 4.2 in[4] states that every square matrix of natural numbers whose diagonal entries are at least 2 is obtained from a category.

Example 3.4.1. Let $\mathcal{C}_1$ be a category with adjacency matrix

$$[\mathcal{C}_1] = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}.$$

Clearly $[\mathcal{C}_1]$ has a weighting and no coweighting. By using the algorithm in Chapter 2, we find its Moore-Penrose inverse as

$$[\mathcal{C}_1]^+ = \begin{bmatrix} 3/26 & 3/26 \\ 1/13 & 1/13 \end{bmatrix}.$$

So its Euler measure is $\chi(\mathcal{C}_1) = 5/13$.

Example 3.4.2. Consider the category $\mathcal{C}_1$ in Example 3.4.1. Let $\mathcal{C}_2$ be the category obtained by adding an isomorphic copy of the second object of $\mathcal{C}_1$ to itself as a third object. Then

$$[\mathcal{C}_2] = \begin{bmatrix} 3 & 2 & 2 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix}.$$

Observe that $[\mathcal{C}_2]$ has a weighting and no coweighting. Then by using the algorithm to find Moore-Penrose inverse, we get

$$[\mathcal{C}_2]^+ = \begin{bmatrix} 1/17 & 1/17 & 1/17 \\ 2/51 & 2/51 & 2/51 \\ 2/51 & 2/51 & 2/51 \end{bmatrix}.$$

So its Euler measure $\chi(\mathcal{C}_2) = 7/17$.

Clearly $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent categories. However, $\chi(\mathcal{C}_1) \neq \chi(\mathcal{C}_2)$. Thus, the Euler measure χ is not invariant under equivalence of categories.

Example 3.4.3. Consider the catories $\mathcal{C}_1$ and $\mathcal{C}_2$ in Example 3.4.2. Let $F : \mathcal{C}_1 \to$ **Cat** be a functor mapping the first object to the terminal category $*$ and second object to the category $\mathcal{B}$ whose adjacency matrix is

$$[\mathcal{B}] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Observe that

$$[G(F)] = \begin{bmatrix} 3 & 2 & 2 \\ 3 & 2 & 2 \\ 3 & 2 & 2 \end{bmatrix}$$

and it is independent of the choice of the image of the morphisms of the category $\mathcal{C}_1$ under F. Thus, $\chi(G(F)) = 7/17$ as in Example 3.4.2. On the other hand

$$\chi(F) = [\ \chi(*)\ \chi(\mathcal{B})\ ] = [\ 1\ 1\ ]$$

and

$$[\mathcal{C}_1]^+ = \begin{bmatrix} 3/26 & 3/26 \\ 1/13 & 1/13 \end{bmatrix}.$$

By applying the formula in Theorem 3.3.10, we get

$$\chi(F)[\mathcal{C}_1]^+\mathbf{1}_{|\mathcal{C}_1|} = [\ 1\ 1\ ] \begin{bmatrix} 3/26 & 3/26 \\ 1/13 & 1/13 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{5}{13} \neq \frac{7}{17} = \chi(G(F)).$$

Hence, Theorem 3.3.10 is not true without the assumption that the index category $\mathcal{A}$ is a poset.

For a finite category $\mathcal{A}$ define

$$g_{\mathcal{A}}(u) = \frac{\mathrm{adj}([\mathcal{A}] - uI)\mathbf{1}_{|\mathcal{A}|}}{\det([\mathcal{A}] - uI)} \in \mathbb{Q}(u)$$

where adj stands for adjugate which takes transpose of cofactor matrix of a given matrix and $\mathbb{Q}(u)$ denotes the rationals of the polynomials of u over rational numbers $\mathbb{Q}$. The series Euler characteristic $\chi_{\Sigma}(\mathcal{A})$ of $\mathcal{A}$ is defined in [4] as follows: $\mathcal{A}$ has series Euler characteristic $\chi_{\Sigma}(\mathcal{A})$ if $g_{\mathcal{A}}(0) \in \mathbb{Q}$ and

$$\chi_{\Sigma}(\mathcal{A}) = g_{\mathcal{A}}(0).$$

χ_{Σ} and χ_{Lein} agree on many categories. For instance, if $[\mathcal{A}]$ is invertible, $\chi_{Lein}(\mathcal{A}) = \chi_{\Sigma}(\mathcal{A})$ by Theorem 3.2 in [4]. But in general they do not agree.

Example 3.4.4. Example 4.5 in [4] shows that for

$$[\mathcal{A}] = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 8 & 5 \end{bmatrix}$$

we have

$$g_{\mathcal{A}}(u) = \frac{3}{9-u}$$

and so

$$\chi_{\Sigma}(\mathcal{A}) = g_{\mathcal{A}}(0) = \frac{1}{3} \neq \frac{1}{2} = \chi_{Lein}(\mathcal{A}).$$

Since χ agrees with χ_{Lein}, we conclude that

$$\chi(\mathcal{A}) \neq \chi_{\Sigma}(\mathcal{A}).$$

The following example shows that even if the Euler characteristic χ_{Lein} is not defined, χ and χ_{Σ} still do not agree.

Example 3.4.5. It is easy to show that

$$[\mathcal{A}] = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

does not have a coweighting, then χ_{Lein} is not defined. Since

$$[\mathcal{A}]^+ = \begin{bmatrix} 4/65 & 6/65 \\ 6/65 & 9/65 \end{bmatrix},$$

we have

$$\chi(\mathcal{A}) = \frac{5}{13}.$$

It is shown in Example 4.8 in [4] that

$$g_{\mathcal{A}}(u) = \frac{2}{5-u}$$

and so

$$\chi_{\Sigma}(\mathcal{A}) = \frac{2}{5}.$$

Thus, they do not agree.

Example 3.4.6. One can show that

$$[\mathcal{A}] = \begin{bmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

does not have a coweighting, then χ_{Lein} is not defined. On the other hand it is shown in Example 4.7 in [4] that

$$g_{\mathcal{A}}(u) = \frac{2+3u}{u(8-u)}$$

and so χ_{Σ} is not defined. Since

$$[\mathcal{A}]^+ = \begin{bmatrix} -7/4 & 1 & 5/4 \\ -1 & 1 & 0 \\ 3/2 & -1 & -1/2 \end{bmatrix},$$

we have

$$\chi(\mathcal{A}) = \frac{1}{2}.$$

Chapter 4

Some Homotopical Properties of the Euler Measure

A model structure on a category is a concept which permits one to do abstract homotopy theory in that category. A model structure on a category grants us just three specified classes of morphisms, called weak equivalences, fibrations and cofibrations satisfying some axioms. For the axioms see Definition 1.1.3 in [15]. Note that on a category there may be more than one model structure. A category with all small limits and colimits together with a model structure on it is called a model category.

The idea of model category is derived from the category of topological spaces **Top**. In general, a continuous map $f : X \to Y$ between two topological spaces X and Y induces group homomorphisms from the homotopy groups of X to the homotopy groups of Y. If this induced maps are group isomorphisms on every level of homotopy groups of X and Y, f is called a weak homotopy equivalence. Consider the weak equivalences as the weak homotopy equivalences of topological spaces, the cofibrations as the retracts of relative cell complexes and the fibrations as the Serre fibrations. It is shown in Section 8 of [16] that these classes of morphisms on the category of topological spaces provide us a structure of model category. This model structure is known as Quillen model structure on **Top**.

Another important model category structure is the canonical model structure on the category of small categories **Cat**. The weak equivalences of this model structure are the equivalences of categories and cofibrations are functors that are injective on objects and fibrations are isofibration in other words functors such that image of mappings on hom sets cover all isomorphisms.

In the Chapter 3, the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ given in Example 3.4.2 have different Euler measures although they are equivalent categories. Thus, the Euler measure χ is not preserved by categorical equivalences. In other words, the Euler measure of a finite category is not invariant under weak equivalences of the canonical model category structure on category of small categories.

Another important model category structure on **Cat** is the Thomason model category structure which is shown in Theorem 4.9 of [17]. For every category $\mathcal{C}$ we have a simplicial set called its nerve and denoted by $\mathcal{N}(\mathcal{C})$ whose n-simplices are the n-length of chains of its morphisms i.e. functors $[\mathbf{n}] \to \mathcal{C}$. Every functor $F : \mathcal{C} \to \mathcal{D}$, induces a simplicial map on their nerves $\mathcal{N}(F) : \mathcal{N}(\mathcal{C}) \to \mathcal{N}(\mathcal{D})$ and their geometric realization gives a continuous map $|\mathcal{N}(F)| : |\mathcal{N}(\mathcal{C})| \to |\mathcal{N}(\mathcal{D})|$ between topological spaces. In the Thomason model category structure on the categories of small categories **Cat**, a functor $F : \mathcal{C} \to \mathcal{D}$ is called a weak equivalence if it induces a weak equivalence $\mathcal{N}(F) : \mathcal{N}(\mathcal{C}) \to \mathcal{N}(\mathcal{D})$ in the classical model structure on simplicial sets i.e. morphisms whose geometric realization is a weak homotopy equivalence of topological spaces.

In this chapter we will discuss the non invariance of the Euler measure defined in Chapter 3 and some others in the literature under weak equivalences of the Thomason model category structure on category of small categories **Cat**.

4.1 Contractible and Strongly Contractible Simplicial Sets

A simplicial map f from a simplicial set X to a simplicial set Y consists of functions $f_n : X_n \to Y_n$ for $n = 0, 1, 2, ...$ which commute with face and degeneracy maps. A simplicial homotopy between simplicial maps $f, g : X \to Y$ is a family of morphisms $h_n^i : X_n \to Y_{n+1}$ for $i = 0, 1, ..., n$ and $n = 0, 1, 2, ...$ such that

- $d_{n+1}^0 h_n^0 = f_n$,

- $d_{n+1}^{n+1} h_n^n = g_n$,

- $d_{n+1}^i h_n^j = \begin{cases} h_{n-1}^{j-1} d_n^i & \text{if } i \langle j \\ d_{n+1}^i h_n^{i-1} & \text{if } i = j \neq 0 \\ h_{n-1}^j d_n^{i-1} & \text{if } i > j+1, \end{cases}$

- $s_{n+1}^i h_n^j = \begin{cases} h_{n+1}^j s_n^{i-1} & \text{if } i > j \\ h_{n+1}^{j+1} s_n^i & \text{if } i \leq j. \end{cases}$

Here lower indices are omitted for simplicity. In [18], a sequence of maps $t_n : X_n \to X_{n+1}$ for $n = 0, 1, 2, ...$ is called a contraction of X if it satisfies the following conditions:

- $d^0 t = \mathrm{id}$,

- $d^i t = t d^{i-1}$, for $i > 0$,

- $s^i t = t s^{i-1}$, for $i > 0$.

A simplicial set X is called contractible if there exists a contraction of X. The conditions to be a contraction are same as the conditions of having an extra degeneracy map that would satisfy a natural continuation of the conditions for degeneracy maps except the one $s^0 t = tt$. If a contraction also satisfies the condition $s^0 t = tt$, it is called a strong contraction and X is called strongly contractible.

Let t be a contraction of X. Set $h^i = (s^0)^i t (d^0)^i : X_n \to X_{n+1}$. Let $r_0 : X_0 \to A_0$ be the coequalizer of $d_1^0, d_1^1 : X_1 \to X_0$. Define $A_n = A_0$ for all n. We define a simplicial map $r : X \to A$ by $r_n = r_0 (d^0)^n : X_n \to A_0 = A_n$. It is shown in [18] that for some simplicial map $i : A \to X$ the composition $r \circ i$ is equal to the identity map on A and h is a homotopy between the identity map on X and $i \circ r$. Notice that A is a constant simplicial set. In particular, if X is connected then the geometric realization of A is just a point and hence the geometric realization of X is contractible.

Let $\mathcal{M}$ denote the category with a unique object whose endomorphism monoid is the monoid $M = \{1, \tau\}$ where 1 is the identity element and $\tau^2 = \tau$.

Proposition 4.1.1. *The geometric realization of the simplicial set $\mathcal{N}(\mathcal{M})$ is contractible.*

Proof. Let a be the unique object of $\mathcal{M}$. Let $\beta : [\mathbf{n}] \to \mathcal{M}$ be an n-simplex of $\mathcal{N}(\mathcal{M})$. Let β^i in $\mathcal{M}$ be the image of the unique morphism α_n^i from $i-1$ to i in $[\mathbf{n}]$. Set $t_n : \mathcal{N}(\mathcal{M})_n \to \mathcal{N}(\mathcal{M})_{n+1}$ to be the function defined as follows:

$$t_n(\beta)(\alpha_{n+1}^i) = \begin{cases} \tau & \text{if } i = 1 \\ \beta^{i-1} & \text{if } i > 1. \end{cases} \tag{4.1}$$

Now to complete the proof, we show that t is a contraction.

$\bullet d^0t = id$:

$$\begin{aligned}
d^0t_n(\beta)(\alpha_n^i) &= \begin{cases} t_n(\beta)(\alpha_{n+1}^2) & \text{if } i = 1 \\ t_n(\beta)(\alpha_{n+1}^{i+1}) & \text{if } i > 1 \end{cases} \\
&= \begin{cases} \beta^1 & \text{if } i = 1 \\ \beta^{i+1-1} & \text{if } i > 1 \end{cases} \\
&= \beta^i.
\end{aligned}$$

Thus, d^0t is just an identity map.

$\bullet d^jt = td^{j-1}$ for $j > 0$:

For $j = 1$,

$$\begin{aligned}
d^1t_n(\beta)(\alpha_n^i) &= \begin{cases} t_n(\beta)(\alpha_{n+1}^2) \circ t_n(\beta)(\alpha_{n+1}^1) & \text{if } i = 1 \\ t_n(\beta)(\alpha_{n+1}^{i+1}) & \text{if } i > 1 \end{cases} \\
&= \begin{cases} t_n(\beta)(\alpha_{n+1}^2) \circ \tau & \text{if } i = 1 \\ \beta^{i+1-1} & \text{if } i > 1 \end{cases} \\
&= \begin{cases} \tau & \text{if } i = 1 \\ d^0(\beta)(\alpha_{n-1}^{i-1}) & \text{if } i > 1 \end{cases} \\
&= t_{n-1}d^0(\beta)(\alpha_n^i).
\end{aligned}$$

For $1 < j < n+1$,

$$\begin{aligned}
d^j t_n(\beta)(\alpha_n^i) &= \begin{cases} t_n(\beta)(\alpha_{n+1}^i) & \text{if } i < j \\ t_n(\beta)(\alpha_{n+1}^{i+1}) \circ t_n(\beta)(\alpha_{n+1}^i) & \text{if } i = j \\ t_n(\beta)(\alpha_{n+1}^{i+1}) & \text{if } i > j \end{cases} \\
&= \begin{cases} \tau & \text{if } i = 1 \\ \beta^{i-1} & \text{if } 1 < i < j \\ \beta^i \circ \beta^{i-1} & \text{if } i = j \\ \beta^i & \text{if } i > j \end{cases} \\
&= \begin{cases} \tau & \text{if } i = 1 \\ \beta^{i-1} & \text{if } 0 < i-1 < j-1 \\ \beta^i \circ \beta^{i-1} & \text{if } i-1 = j-1 \\ \beta^{i-1+1} & \text{if } i-1 > j-1 \end{cases} \\
&= \begin{cases} \tau & \text{if } i = 1 \\ d^{j-1}(\beta)(\alpha_{n-1}^{i-1}) & \text{if } i > 1 \end{cases} \\
&= t_{n-1} d^{j-1}(\beta)(\alpha_n^i).
\end{aligned}$$

For $j = n+1$,

$$\begin{aligned}
d^{n+1} t_n(\beta)(\alpha_n^i) &= t_n(\beta)(\alpha_{n+1}^i) \\
&= \begin{cases} \tau & \text{if } i = 1 \\ \beta^{i-1} & \text{if } i > 1 \end{cases} \\
&= \begin{cases} \tau & \text{if } i = 1 \\ d^n(\beta)(\alpha_{n-1}^{i-1}) & \text{if } i > 1 \end{cases} \\
&= t_{n-1} d^n(\beta)(\alpha_n^i).
\end{aligned}$$

•$s^j t = t s^{j-1}$ for $j > 0$:

$$\begin{aligned}
s^j t_n(\beta)(\alpha^i_{n+2}) &= \begin{cases} t_n(\beta)(\alpha^i_{n+1}) & \text{if } i < j+1 \\ 1_a & \text{if } i = j+1 \\ t_n(\beta)(\alpha^{i-1}_{n+1}) & \text{if } i > j+1 \end{cases} \\
&= \begin{cases} \tau & \text{if } i = 1 \\ \beta^{i-1} & \text{if } 1 < i < j+1 \\ 1_a & \text{if } i = j+1 \\ \beta^{i-1-1} & \text{if } i > j+1 \end{cases} \\
&= \begin{cases} \tau & \text{if } i = 1 \\ s^{j-1}(\beta)(\alpha^{i-1}_{n+1}) & \text{if } i > 1 \end{cases} \\
&= t_{n+1} s^{j-1}(\beta)(\alpha^i_{n+2}).
\end{aligned}$$

□

Observe that the above contraction t does not satisfy the condition $s^0 t = tt$. It is also straightforward to show that there is no other contraction of $\mathcal{N}(\mathcal{M})$. Thus, $\mathcal{N}(\mathcal{M})$ is not strongly contractible.

4.2 Euler characteristics

Let $\mathcal{M}$ denote the category defined in the previous section. Now the Euler characteristic of the terminal category $*$ with one object and one morphism is equal to 1 for all defined Euler characteristics in the literature. As we proved in the previous section the unique functor from $\mathcal{M}$ to $*$ induces a weak equivalence in the Thomason model category structure. However, the Euler characteristics assigned to $\mathcal{M}$ in the literature are not equal to 1. For example we have

$$\chi_{\Sigma}(\mathcal{M}) = \chi_{Lein}(\mathcal{M}) = \chi(\mathcal{M}) = \frac{1}{2} \neq 1$$

where $\chi_{\Sigma}(\mathcal{M})$ denotes the series Euler characteristic as in [4].

Let R be an associative commutative ring with unit and $\chi^{(2)}$, $\underline{\mathrm{R}}$, $\mathcal{N}$ and S be as defined in [12] and [13]. Assume that $I_i(M)$ denotes the $R\mathcal{M}$-module satisfying $I_i(M)(*) = M$ and $I_i(M)(\tau) = i \cdot \mathrm{id}_M$ for $i = 0, 1$ as in Example 2.18 of [13]. Then a projective resolution of $\underline{R}$ is P_* where $P_0 = I_1(R)$ and $P_n = 0$ if $n \neq 0$. Let x be the unique object of $\mathcal{M}$. Then

$$S_x(I_1(R)) = \mathrm{coker}(id_R) = 0.$$

So considering $R = \mathbb{C}$ we have

$$\chi^{(2)}(\mathcal{M}) = \sum_{n \geq 0} (-1)^n \dim_{\mathcal{N}(x)}(H_n(S_x P_* \otimes_{R[x]} \mathcal{N}(x))) = 0 \neq 1.$$

We know that the Euler characteristic of two finite categories is same if they have isomorphic nerves. It is desirable to define an equivalence relation on nerves of categories more like homotopty equivalence and still have the Euler characteristic well-defined up to such an equivalence. Notice that the unique map $\mathcal{M} \to *$ induces a homotopy equivalence from $\mathcal{N}(\mathcal{M})$ to $\mathcal{N}(*)$ but it is not strong in the sense discussed above. Hence, one could try to define strong homotopy equivalences of nerves of categories using the definitions in the previous section. As a first step, we can ask the following question.

Question 4.2.1. Let $\mathcal{C}$ be a finite category with a connected and strongly contractible nerve. Can we prove that $\chi(C) = 1$?

Chapter 5

Code

The following pseudo code computes M^+ for a given square matrix M:

```
Initialize the square matrix $M$ with rational values.
Set variable $d$ as the size of matrix $M$.
Compute the row-reduced echelon form of matrix $M$ and store it in $\mathrm{rref}(M)$.
Compute the transpose of $\mathrm{rref}(M)$ and store it in $A_M$.
Compute the transpose of $A_M$ and store it in $A_M^*$.
Compute the transpose of $M$ and store it in $M^*$.
Compute a basis of null space of $M^*$ and store it in $(\mathrm{im}(M)^\perp)^*$.
Compute the transpose of $(\mathrm{im}(M)^\perp)^*$ and store it in $\mathrm{im}(M)^\perp$.
Initilize $D_M$ as empty list.
For each row $r$ in the $A_M^*$:
    If $r$ is not a zero row:
        Compute the transpose of (the product of $M$ and the transpose of $[r]$)
        Append the first row of the result to $D_M$.
For each row $r$ in $(\mathrm{im}(M)^\perp)^*$:
    Append $r$ to $D_M$.
Compute the inverse of transpose of $D_M$ and store it in $B_M$.
Compute the product of $A_M$ and $B_M$ and store it in $M^+$.
```

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