ZONGULDAK BÜLENT ECEVİT UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

TOEPLITZ MATRICES WITH SOME WELL-KNOWN NUMBERS

DEPARTMENT OF MATHEMATICS

MASTER OF SCIENCE THESIS

SEVDA AKTAŞ

JUNE 2023

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APPROVAL OF THE THESIS:

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"With this thesis it is declared that all the information in this thesis is obtained and presented according to academic rules and ethical principles. Also as required by the academic rules and ethical principles all works that are not result of this study are cited properly."

Sevda AKTAŞ

ABSTRACT

Master of Science Thesis

TOEPLITZ MATRICES WITH SOME WELL-KNOWN NUMBERS

Sevda AKTAŞ

Zonguldak Bülent Ecevit University
Graduate School of Natural and Applied Sciences
Department of Mathematics

Thesis Advisor: Prof. Dr. Yüksel SOYKAN
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In this thesis, we consider special norms of the Toeplitz matrices with some well-known number sequences. We summarize the chapters as follows.

In Chapter 1, we present some basic important definitions, recurrence relations, Binet's formulas and sum formulas of second order generalized number sequences used throughout the thesis.

In Chapter 2, we present some basic important definitions and some results used throughout the thesis. Moreover, we review the literature on Toeplitz matrices of special second order recurrence relations by scanning several research papers.

ABSTRACT (continued)

In Chapter 3, the Toeplitz matrices whose elements are the Oresme numbers are created and then the Frobenius (Euclidian), row and column norms of these matrices found. Furthermore, we obtain lower and upper bounds for the spectral norms of these matrices. In addition, we calculate the upper bounds for the Frobenius and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the Oresme numbers. This chapter contains our original work.

In Chapter 4, we obtain and investigate the Toeplitz matrices whose elements are the Mersenne numbers. First the matrices are created and then the Frobenius (Euclidian), row and column norms of these matrices are found. Furthermore, we obtain lower and upper bounds for the spectral norms of these matrices. In addition, we calculate the upper bounds for the Frobenius and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the Mersenne numbers. The work presented in this chapter is original.

In Chapter 5, we define and investigate the Toeplitz matrices whose elements are the Balancing numbers and then the Frobenius (Euclidian), row and column norms of these matrices are found. Furthermore, we obtain lower and upper bounds for the spectral norms of these matrices. In addition, we calculate the upper bounds for the Frobenius and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the balancing numbers. This chapter contains our original work.

Keywords: Oresme numbers, Mersenne numbers, balancing numbers, norm, Toeplitz matrix, Kronecker product and Hadamard product.

Science Code: 403.01.01

ÖZET

Yüksek Lisans Tezi

BAZI İYİ BİLİNEN SAYILARLA TOEPLİTZ MATRİSLERİ

Sevda AKTAŞ

Zonguldak Bülent Ecevit Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı

Tez Danışmanı: Prof Dr. Yüksel SOYKAN
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Bu tez de, bazı iyi bilinen sayı dizileriyle Toeplitz matrislerinin özel normlarını ele alıyoruz. Bölümleri aşağıdaki gibi özetliyoruz.

Bölüm 1'de, tez boyunca kullanılacak olan ikinci mertebeden genelleştirilmiş sayı dizilerinin bazı temel önemli tanımlarını, rekürans bağıntılarını, Binet's formüllerini ve toplam formüllerini sunuyoruz.

Bölüm 2'de, tez boyunca kullanılan bazı temel önemli tanımları ve bazı sonuçları sunuyoruz. Ayrıca birkaç araştırma makalesini tarayarak özel ikinci mertebe rekürans bağıntılarının Toeplitz matrisleri hakkındaki literatürü gözden geçiriyoruz .

Bölüm 3'de, elemanları Oresme sayıları olan Toeplitz matrisleri oluşturulmuş ve bu matrislerin Frobenius (Öklidyen), satır normunu, sütun normları bulunmuştur. Ayrıca bu matrislerin spectral norm için alt ve üst sınırlar elde ediyoruz. Ek olarak Oresme sayılarla

ÖZET (devam ediyor)

Toeplitz matrislerin Kronecker ve Hadamard çarpım matrislerinin Frobenius ve spectral normları için üst sınırlar hesaplıyoruz. Bu bölüm orjinal çalışmamızı içermektedir.

Bölüm 4'de, elemanları Mersenne sayıları olan Toeplitz matrislerini elde edip inceledik. Önce matrisler oluşturulur ve daha sonra bu matrislerin Frobenius (Öklidyen), satır ve sütun normunları bulunur. Ayrıca bu matrislerin spectral normları için alt ve üst sınırlar elde ediyoruz. Ek olarak Mersenne sayılarla Toeplitz matrislerin Kronecker ve Hadamard çarpım matrislerinin Frobenius ve spectral normları için üst sınırlar hesaplıyoruz. Bu bölümde sunulan çalışma orjinaldir.

Bölüm 5'de, elemanları balancing sayıları olan Toeplitz matrislerini tanımlayıp inceliyoruz ve ardından bu matrislerin Frobenius (Öklid), satır ve sütun normunu buluyoruz. Ayrıca spectral norm için alt ve üst sınırlar elde ediyoruz. Ek olarak balancing sayılarla Toeplitz matrislerin Kronecker ve Hadamard çarpım matrislerinin Frobenius ve spectral normları için üst sınırlar hesaplanmıştır. Bu bölüm orjinal çalışmamızı içermektedir.

Anahtar Kelimeler: Oresme sayıları, Mersenne sayıları, balancing sayıları, Toeplitz matris, norm, Hadamard çarpım, Kronecker çarpım.

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I dedicate this thesis my dear sons Yiğitalp AKTAŞ and Mehmet Fatih AKTAŞ.

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LIST OF SYMBOLS AND ABBREVIATIONS

SYMBOLS

W_n : n-th generalized (Oresme, Mersenne, balancing) number

G_n : n-th modified Oresme number

H_n : n-th Oresme-Lucas, Mersenne-Lucas, modified Lucas-balancing number

O_n : n-th Oresme number

M_n : n-th Mersenne number

B_n : n-th balancing number

C_n : n-th Lucas-balancing number

CHAPTER 1

LINEAR SECOND ORDER RECURRENCE RELATIONS

In this chapter, we present some basic important definitions, recurrence relations, Binet's formulas and sum formulas of special second order generalized number sequences used throughout the thesis.

1.1 GENERALIZED ORESME NUMBERS

In this section, we present some definition, recurrence relations, Binet's formulas and sum formulas of generalized Oresme numbers. We use Soykan [6] and [7].

A generalized Oresme sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second -order recurrence relation

$$W_n = W_{n-1} - \frac{1}{4}W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 4W_{-(n-1)} - 4W_{-(n-2)}$$

for $n = 1, 2, 3, \cdots$. Therefore, recurrence equation (1.1) holds for all integer n.

Characteristic equation of generalized Oresme sequence $\{W_n\}_{n\geq 0}$ is given as the quadratic equation

$$x^2 - x + \frac{1}{4} = 0,$$

whose roots are α, β and

$$\alpha = \beta = \frac{1}{2}.$$

Binet's formula of generalized Oresme sequence is given as

$$W_n = \left(nW_1 - \frac{1}{2}(n-1)W_0\right)\left(\frac{1}{2}\right)^{n-1}.$$

The first few generalized Oresme numbers with positive subscript and negative subscript are given in the following Table 1.1.

Table 1.1 A few generalized Oresme numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$4W_0 - 4W_1$
2	$W_1 - \frac{1}{4}W_0$	$12W_0 - 16W_1$
3	$\frac{3}{4}W_1 - \frac{1}{4}W_0$	$32W_0 - 48W_1$
4	$\frac{1}{2}W_1 - \frac{3}{16}W_0$	$80W_0 - 128W_1$
5	$\frac{5}{16}W_1 - \frac{1}{8}W_0$	$192W_0 - 320W_1$
6	$\frac{3}{16}W_1 - \frac{5}{64}W_0$	$448W_0 - 768W_1$
7	$\frac{7}{64}W_1 - \frac{3}{64}W_0$	$1024W_0 - 1792W_1$
8	$\frac{1}{16}W_1 - \frac{7}{256}W_0$	$2304W_0 - 4096W_1$
9	$\frac{9}{256}W_1 - \frac{1}{64}W_0$	$5120W_0 - 9216W_1$
10	$\frac{5}{256}W_1 - \frac{9}{1024}W_0$	$11264W_0 - 20480W_1$

For more information on generalized Oresme numbers, see for example, Soykan [6].

Modified Oresme sequence $\{G_n\}_{n\geq 0}$, Oresme-Lucas sequence $\{H_n\}_{n\geq 0}$ and Oresme sequence $\{O_n\}_{n\geq 0}$ are defined respectively, by the second order recurrence relations;

$$G_{n+2} = G_{n+1} - \frac{1}{4}G_n, \quad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n, \quad H_0 = 2, H_1 = 1, \tag{1.3}$$

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n \quad O_0 = 0, O_1 = \frac{1}{2}. \tag{1.4}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{O_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= 4G_{-(n-1)} - 4G_{-(n-2)}, \\ H_{-n} &= 4H_{-(n-1)} - 4H_{-(n-2),} \\ O_{-n} &= 4O_{-(n-1)} - 4O_{-(n-2)}. \end{aligned}$$

for $n = 1, 2, 3, \cdots$ respectively.

Therefore recurrence Equ. (1.2), Equ. (1.3) and Equ. (1.4) hold for all integer n.

Next, we present the first few values of the modified Oresme, Oresme-Lucas and Oresme numbers with positive and negative subscripts:

Table 1.2 The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9
G_n	0	1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{3}{16}$	$\frac{7}{64}$	$\frac{1}{16}$	$\frac{9}{256}$
G_{-n}	....	-4	-16	-48	-128	-320	-768	-1792	-4096	-9216
H_n	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
H_{-n}	....	4	8	16	32	64	128	256	512	1024
O_n	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{5}{32}$	$\frac{3}{32}$	$\frac{7}{128}$	$\frac{1}{32}$	$\frac{9}{512}$
O_{-n}	....	-2	-8	-24	-64	-160	-384	-896	-2048	-4608

Binet's formulas of modified Oresme, Oresme-Lucas and Oresme numbers are

$$\begin{aligned} G_n &= n\alpha^{n-1} = \frac{n}{2^{n-1}}, \\ H_n &= 2\alpha^n = \frac{1}{2^{n-1}}, \\ O_n &= n\alpha^n = \frac{n}{2^n}, \end{aligned}$$

and Binet's formulas of modified Oresme, Oresme-Lucas and Oresme numbers at the negative index are

$$\begin{aligned} G_{-n} &= -4^n G_n = -n \times 2^{n+1}, \\ H_{-n} &= 4^n H_n = 2^{n+1}, \\ O_{-n} &= -4^n O_n = -n \times 2^n. \end{aligned}$$

In the following theorem, we present some sum formulas of generalized Oresme numbers.

Theorem 1.1 *For generalized Oresme numbers, we have following sum formulas:*

(a) *[6, Proposition 26. a] If* $\frac{1}{4}(x-2)^2 \neq 0, i.e., x \neq 2$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(x-4)\, x^{n+1} W_n + x^{n+1} W_{n-1} + 4W_0 + 4(W_1 - W_0)\, x}{(x-2)^2}. \tag{1.5}$$

(b) *[6, Proposition 26. d] If* $(2x-1)^2 \neq 0, i.e.,\ x \neq \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{4x^{n+1} W_{-n+1} + 4(x-1)x^{n+1} W_{-n} + W_0 - 4xW_1}{(2x-1)^2}.$$

(c) *[7, Proposition 2.1. a] If* $\frac{1}{64}(x-4)^3 \neq 0, i.e., x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta}{(x-4)^3}$$

where

$$\Delta = (x-4)(x-8)x^{n+1}W_n^2 + (x-4)x^{n+1}W_{n-1}^2 + 16(x-4)W_0^2 - 16x(x-4)(W_0 - W_1)^2 - 2^{-2n+5}(W_0 - 2W_1)^2(2^{2n} - x^n)x.$$

(d) *[7, Proposition 2.1. d] If* $(4x-1)^3 \neq 0, i.e., x \neq \frac{1}{4}$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{(4x-1)^3}$$

where

$$\Delta = 16(4x-1)x^{n+1}W_{-n+1}^2 + 8(2x-1)(4x-1)x^{n+1}W_{-n}^2 + (4x-1)W_0^2 - 16(4x-1)xW_1^2 + 8(W_0 - 2W_1)^2(2^{2n}x^n - 1)x.$$

If we set $x = 1$ in the last Theorem, we have the following corollary.

Corollary 1.1 *For generalized Oresme numbers, we have following sum formulas:*

(a)

$$\sum_{k=0}^{n} W_k = -3W_n + W_{n-1} + 4W_1. \tag{1.6}$$

(b)

$$\sum_{k=0}^{n} W_{-k} = 4W_{-n+1} + W_0 - 4W_1. \tag{1.7}$$

(c)

$$\sum_{k=0}^{n} W_k^2 = -\frac{1}{27}(21W_n^2 - 3W_{n-1}^2 - 48W_1(2W_0 - W_1) - 2^{-2n+5}(W_0 - 2W_1)^2(2^{2n} - 1)). \tag{1.8}$$

(d)

$$\sum_{k=0}^{n} W_{-k}^2 = \frac{1}{27}(48W_{-n+1}^2 + 24W_{-n}^2 + 3W_0^2 - 48W_1^2 + 8(W_0 - 2W_1)^2(2^{2n} - 1)). \tag{1.9}$$

1.2 GENERALIZED MERSENNE NUMBERS

In this section, we present some definition, recurrence relations, Binet's formulas and sum formulas of generalized Mersenne numbers. We use Soykan [8] and [9].

A generalized Mersenne sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second -order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \tag{1.10}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{3}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \cdots$. Therefore, recurrence equation (1.10) holds for all integer n.

Mersenne numbers and Mersenne primes are fascinating topics in number theory and have been the subject of intense research throughout history. When M_n is a prime numbers, it is called a Mersenne prime.

However, it is important to note that not all Mersenne numbers M_n are prime. As n grows larger, it becomes increasingly difficult to find Mersenne primes. Currently, the largest known prime is a Mersenne prime, specifially M82589933, which is equal to $2^{82589933} - 1$.The search for Mersenne primes has been a significant area of investigation in the study of prime numbers. Mersenne primes have also been employed in various areas of mathematics and computer science, particularly in the study of perfect numbers and for certain fast algorithms in number theory.

In the search for Mersenne primes, various tests are employed to determine their primality. The Lucas- Lehemer test and Pepin's test are two such as tests.

Pepin's test, on the other hand is a primality test designed specifially for Mersenne numbers that can provide insight info their primality.

Research continue to explore these and other test, along with searching for new Mersenne primes, to deepen our understanding of prime numbers and their properties.

The first few generalized Mersenne numbers with positive subscript and negative subscript are given in the following Table 2.1.

Table 1.3 A few generalized Mersenne numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{3}{2}W_0-\frac{1}{2}W_1$
2	$3W_1-2W_0$	$\frac{7}{4}W_0-\frac{3}{4}W_1$
3	$7W_1-6W_0$	$\frac{15}{8}W_0-\frac{7}{8}W_1$
4	$15W_1-14W_0$	$\frac{31}{16}W_0-\frac{15}{16}W_1$
5	$31W_1-30W_0$	$\frac{63}{32}W_0-\frac{31}{32}W_1$
6	$63W_1-62W_0$	$\frac{127}{64}W_0-\frac{63}{64}W_1$
7	$127W_1-126W_0$	$\frac{255}{128}W_0-\frac{127}{128}W_1$
8	$255W_1-254W_0$	$\frac{511}{256}W_0-\frac{255}{256}W_1$
9	$511W_1-510W_0$	$\frac{1023}{512}W_0-\frac{511}{512}W_1$
10	$1023W_1-1022W_0$	$\frac{2047}{1024}W_0-\frac{1023}{1024}W_1$
11	$2047W_1-2046W_0$	$\frac{4095}{2048}W_0-\frac{2047}{2048}W_1$
12	$4095W_1-4094W_0$	$\frac{8191}{4096}W_0-\frac{4095}{4096}W_1$

For more information on generalized Mersenne numbers, see for example, Soykan [8].

Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined respectively, by the second order recurrence relations;

$$M_n = 3M_{n-1} - 2M_{n-2} \quad M_0 = 0, M_1 = 1, \tag{1.11}$$

$$H_n = 3H_{n-1} - 2H_{n-2}, \quad H_0 = 2, H_1 = 3. \tag{1.12}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)}, \\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}. \end{aligned}$$

for $n = 1, 2, 3, \cdots$ respectively.

Therefore recurrence equation(1.11), equation (1.12) hold for all integer n.

Next, we present the first few values of the Mersenne and Mersenne-Lucas numbers with positive and negative subscripts:

Table 1.4 The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10
M_n	0	1	3	7	15	31	63	127	255	511	1023
M_{-n}	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$
H_n	2	3	5	9	17	33	65	129	257	513	1025
H_{-n}	2	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{17}{16}$	$\frac{33}{32}$	$\frac{65}{64}$	$\frac{129}{128}$	$\frac{257}{256}$	$\frac{513}{512}$	$\frac{1025}{1024}$

Characteristic equation of generalized Mersenne sequence $\{W_n\}_{n\geq 0}$ is given as the quadratic equation

$$x^2 - 3x + 2 = 0,$$

whose roots are α, β and

$$\begin{aligned} \alpha &= 2 \\ \beta &= 1. \end{aligned}$$

Binet's formula of generalized Mersenne sequence is given as

$$\begin{aligned} W_n &= \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \\ &= (W_1 - W_0)2^n - (W_1 - 2W_0). \end{aligned}$$

Binet's formulas of Mersenne and Mersenne-Lucas are

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha - \beta)} + \frac{\beta^n}{(\beta - \alpha)} = 2^n - 1, \\ H_n &= \alpha^n + \beta^n = 2^n + 1 \end{aligned}$$

and Binet's formulas of Mersenne numbers and Mersenne-Lucas at the negative index are

$$\begin{aligned} M_{-n} &= \frac{1}{\alpha^n} - \frac{1}{\beta^n} = \frac{-2^n + 1}{2^n}, \\ H_{-n} &= \frac{1}{\alpha^n} + \frac{1}{\beta^n} = \frac{2^n + 1}{2^n}. \end{aligned}$$

In the following theorem, we obtain some formulas of generalized Mersenne of numbers.

Theorem 1.2 *For generalized Mersenne numbers, we have following sum formulas:*

(a) *[8, Proposition 22. a] If* $2x^2-3x+1=0$, *i.e.*, $x=1$ *or* $x=\frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(2(n+2)x-3(n+1))x^n W_n + 2(n+1)x^n W_{n-1} + (W_1-3W_0)}{4x-3}.$$

(b) *[9, Proposition 2.1. a] If* $(2x-1)(4x-1)(x-1)=0$, *i.e.*, $x=\frac{1}{4}$ *or* $x=\frac{1}{2}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{(24x^2-28x+7)}$$

where

$$\Psi = (n(2x-1)(4x-5)+24x^2-28x+5)x^n W_n^2 + 4((2x-1)n+4x-1)x^n W_{n-1}^2 + 2W_0^2 - (4x-1)(3W_0-W_1)^2 + 2(W_1^2+2W_0^2-3W_1W_0)(2^n(n+1)x^n-1).$$

(c) *[9, Proposition 2.1. d] If* $(x-2)(x-1)(x-4)=0$, *i.e.*, $x=1$ *or* $x=2$ *or* $x=4$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(3x^2-14x+14)}$$

where

$$\Psi = (n(x-2)+2(x-1))x^n W_{-n+1}^2 + (n(x-2)(x-5)+3x^2-14x+10)x^n W_{-n}^2 + 4W_0^2 - 2(x-1)W_1^2 + 4(W_1^2+2W_0^2-3W_1W_0)(2^{-n}(n+1)x^n-1).$$

If we set $x=1$ in the last Theorem, we have the following corollary.

Corollary 1.2 *For generalized Mersenne numbers, we have following sum formulas:*

(a)

$$\sum_{k=0}^{n} W_k = (1-n)W_n + (2n+2)W_{n-1} + (W_1-3W_0). \tag{1.13}$$

(b)

$$\begin{aligned}\sum_{k=0}^{n} W_k^2 &= \frac{1}{3}((1-n)W_n^2 + 4(n+3)W_{n-1}^2 - 25W_0^2 + 18W_0W_1 - 3W_1^2 \\ &\quad +2(W_1-2W_0)(W_1-W_0)(2^n(n+1)-1)).\end{aligned} \tag{1.14}$$

(c)

$$\sum_{k=0}^{n} W_{-k}^2 = \frac{1}{3}(-nW_{-n+1}^2 + (4n-1)W_{-n}^2 + 4W_0^2 + 4(W_1^2+2W_0^2-3W_1W_0)(2^{-n}(n+1)-1)). \tag{1.15}$$

1.3 GENERALIZED BALANCING NUMBERS

In this section, we present some definition, recurrence relations, binet formulas and sum formulas of generalized balancing numbers. We use Soykan [10] and [11].

A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second -order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2} \tag{1.16}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)}$$

for $n = 1, 2, 3, \cdots$. Therefore, recurrence equation (1.16) holds for all integer n.

The first few generalized balancing numbers with positive subscript and negative subscript are given in the following Table 3.1.

Table 1.5 A few generalized balancing numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$6W_0 - W_1$
2	$6W_1 - W_0$	$35W_0 - 6W_1$
3	$35W_1 - 6W_0$	$204W_0 - 35W_1$
4	$204W_1 - 35W_0$	$1189W_0 - 204W_1$
5	$1189W_1 - 204W_0$	$6930W_0 - 1189W_1$
6	$6930W_1 - 1189W_0$	$40391W_0 - 6930W_1$
7	$40391W_1 - 6930W_0$	$235416W_0 - 40391W_1$
8	$235416W_1 - 40391W_0$	$1372105W_0 - 235416W_1$
9	$1372105W_1 - 235416W_0$	$7997214W_0 - 1372105W_1$
10	$7997214W_1 - 1372105W_0$	$46611179W_0 - 7997214W_1$

For more information on generalized balancing numbers, see for example, Soykan [10].

Balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined respectively, by the second order recurrence re-

lations:

$$B_n = 6B_{n-1} - B_{n-2}, \quad B_0 = 0, B_1 = 1, \tag{1.17}$$

$$H_n = 6H_{n-1} - H_{n-2,} \quad H_0 = 2, H_1 = 6, \tag{1.18}$$

$$C_n = 6C_{n-1} - C_{n-2}, \quad C_0 = 1, C_1 = 3. \tag{1.19}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)} - B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)} - H_{-(n-2),} \\ C_{-n} &= 6C_{-(n-1)} - C_{-(n-2)}. \end{aligned}$$

for $n = 1, 2, 3, \cdots$ respectively.

Therefore recurrence equation (1.17), equation (1.18) and equation (1.19) hold for all integer n.

Next, we present the first few values of the balancing, modified Lucas-balancing and Lucas-balancing numbers with positive and negative subscripts:

Table 1.6 The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8
B_n	0	1	6	35	204	1189	6930	40391	235416
B_{-n}		-1	-6	-35	-204	-1189	-6930	-40391	-235416
H_n	2	6	34	198	1154	6726	39202	228486	1331714
H_{-n}		6	34	198	1154	6726	39202	228486	1331714
C_n	1	3	17	99	577	3363	19601	114243	665857
C_{-n}		3	17	99	577	3363	19 601	114243	665857

Binet's formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n$$

where α and β are the roots of the quadratic equation

$$x^2 - 6x + 1 = 0.$$

Moreover,

$$\alpha = 3+2\sqrt[2]{2},$$
$$\beta = 3-2\sqrt[2]{2}.$$

Binet's formulas of balancing, modified Lucas-balancing and Lucas-balancing numbers are

$$B_n = \frac{\alpha^n}{(\alpha-\beta)}+\frac{\beta^n}{(\alpha-\beta)},$$
$$H_n = \alpha^n+\beta^n,$$
$$C_n = \frac{\alpha^n+\beta^n}{2},$$

respectively.

In the following theorem, we present some sum formulas of generalized balancing numbers.

Theorem 1.3 *For generalized balancing numbers, we have following sum formulas:*

(a) *[10, Proposition 6.2. (a)] If* $x^2-6x+1\neq 0, i.e., x\neq 3+2\sqrt[2]{2}, x\neq 3-2\sqrt[2]{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(x-6)x^{n+1}W_n + x^{n+1}W_{n-1} + (W_1-6W_0)x + W_0}{x^2-6x+1}. \tag{1.20}$$

(b) *[10, Proposition 6.2 (d)] If* $x^2-6x+1\neq 0, i.e.,$ $x\neq 3+2\sqrt[2]{2}, x\neq 3-2\sqrt[2]{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{x^{n+1}W_{-n+1} + (x-6)x^{n+1}W_{-n} - W_1x + W_0}{x^2-6x+1}.$$

(c) *[11, Proposition 2.2.(a)] If* $(x-1)(x^2-34x+1)\neq 0, i.e., x=1$ *or* $x=17-12\sqrt[2]{2}$ *or* $x=17+12\sqrt[2]{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{3x^2-70x+35},$$

where

$$\Psi = ((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^nW_n^2+((n+2)x-(n+1))x^nW_{n-1}^2$$
$$+W_0^2-(2x-1)(W_1-6W_0)^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1).$$

(d) *[11, Proposition 2.2. (d)] If* $(x-1)(x^2-34x+1)\neq 0, i.e., x=1$ *or* $x=17-12\sqrt[2]{2}$ *or* $x=17+12\sqrt[2]{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(3x^2-70x+35)},$$

where

$\Psi = ((n+2)x-(n+1))x^nW_{-n+1}^2+((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)W_{-n}^2$
$+W_0^2-(2x-1)W_1^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1).$

If we set $x=1$ in the last Theorem, we obtain the following corollary.

Corollary 1.3 *For generalized balancing numbers, we have following sum formulas:*

(a)

$$\sum_{k=0}^{n} W_k = \frac{5W_n - W_{n-1} + 5W_0 - W_1}{4}. \tag{1.21}$$

(b)

$$\sum_{k=0}^{n} W_{-k} = \frac{-W_{-n+1} + 5W_{-n} + W_1 - W_0}{4}. \tag{1.22}$$

(c)

$$\sum_{k=0}^{n} W_k^2 = \frac{1}{32}(33W_n^2 - W_{n-1}^2 - W_0^2 + (W_1 - 6W_0)^2 - 2n(W_1^2 + W_0^2 - 6W_1W_0)). \tag{1.23}$$

(d)

$$\sum_{k=0}^{n} W_{-k}^2 = -\frac{1}{32}(W_{-n+1}^2 - 33W_{-n}^2 + W_0^2 - W_1^2 + 2n(W_1^2 + W_0^2 - 6W_1W_0)). \tag{1.24}$$

CHAPTER 2

TOEPLITZ MATRICES AND PROPERTIES

In this chapter, we present some basic important definitions and some results use throughout the thesis.

2.1 DEFINITIONS AND PROPERTIES

A matrix $T = [t_{ij}] \in M_n(\mathbb{C})$ is called a Toeplitz matrix if it is of the form $t_{ij} = t_{i-j}$ for

$$T_n = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{pmatrix}.$$

Now, we give some preliminaries related to our study. Let $A = (a_{ij})$ be an $m \times n$ matrix. The ℓ_p norm of the matrix A is defined by

$$\|A\|_p = (\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p)^{\frac{1}{p}} \qquad (1 \leq p < \infty).$$

If $p = \infty$, then $\|A\|_\infty = \lim_{p\to\infty} \|A\|_p = \max_{i,j} |a_{ij}|$.

The well-known Frobenius (Euclidean) and spectral norms of the matrix A are defined respectively by

$$\|A\|_F = (\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$$

and

$$\|A\|_2 = \sqrt{\max_{1\leq i\leq n} |\lambda_i|} \tag{2.1}$$

where the numbers λ_i are the eigenvalues of matrix $A^H A$ and the matrix A^H is the conjugate transpose of the matrix A. The following inequality between the Frobenius and spectral norms of A holds.

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F . \tag{2.2}$$

It follows that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n} \|A\|_2 .$$

In literature, there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \tag{2.3}$$

and the maximum row sum matrix norm is

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| . \tag{2.4}$$

The maximum column lenght norm $c_1(.)$ and maximum row lenght norm $r_1(.)$ of on matrix of order $m \times n$ are defined as follows

$$c_1(A) \equiv \max_{1 \leq j \leq n} \left(\sum_{i=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}} = \max_{1 \leq j \leq n} \left\| [a_{ij}]_{i=1}^{m} \right\|_F \tag{2.5}$$

and

$$r_1(A) \equiv \max_{1 \leq i \leq m} \left(\sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} = \max_{1 \leq i \leq m} \left\| [a_{ij}]_{j=1}^{n} \right\|_F \tag{2.6}$$

respectively.

For any $A, B \in M_{mn}(\mathbb{C})$, the Hadamard product of $A = (a_{ij})$ and $B = (b_{ij})$ is entrywise product and defined by $A \circ B = (a_{ij} b_{ij})$ and have the following properties

$$\|A \circ B\|_2 \leq r_1(A)\, c_1(B) , \tag{2.7}$$

and

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2 . \tag{2.8}$$

In addition,

$$\|A \circ B\|_F \leq \|A\|_F \|B\|_F . \tag{2.9}$$

Let $A \in M_{mn}(\mathbb{C})$, and $B \in M_{mn}(\mathbb{C})$ be given, then the Kronecker product of A, B is defined by

$$\|A \otimes B\| = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

and have the following properties

$$\begin{aligned} \|A \otimes B\|_2 &= \|A\|_2 \|B\|_2 , \\ \|A \otimes B\|_F &= \|A\|_F \|B\|_F . \end{aligned} \tag{2.10}$$

2.2 LITERATURE REVIEW

In this section, we review the literature on Toeplitz matrices of special second-order recurrence relations by reviewing several research papers.
For Toeplitz matrices with the entry of special second order recurrence relations, we give the largest absolute column sum, the largest absolute row sum, the Frobenius norm and the spectral norm. Moreover, some properties related to the Hadamard and Kronecker products of these Toeplitz matrices are presented.

2.2.1 On The Toeplitz Matrices with Fibonacci and Lucas Numbers:

In this section, we use Solak and Bahşi [5]. We list some properties related to the spectral norms of Toeplitz matrices with Fibonacci and Lucas numbers given in Solak and Bahşi.

- Let the matrix $A = [F_{j-i}]_{i,j=0}^{n-1}$ be Toeplitz matrix, then the singular values of A are

  $$\sigma_{1,2} = \begin{cases} F_n & , \quad \text{if } n \text{ is even} \\ \sqrt{F_n^2 - 1} & , \quad \text{if } n \text{ is odd} \end{cases}$$

  and

  $\sigma_m = 0,$ where $m = 3, 4, \cdots, n.$

- Let $A = [F_{j-i}]_{i,j=0}^{n-1}$ be Toeplitz matrix, then

$$\|A\|_2 = \begin{cases} F_n & , \quad \text{if } n \text{ is even} \\ \sqrt{F_n^2 - 1} & , \quad \text{if } n \text{ is odd} \end{cases}$$

- Let the matrix $B = [L_{j-i}]_{i,j=0}^{n-1}$ be Toeplitz matrix, then the singular values of B are

$$\sigma_{1,2} = \begin{cases} L_n \pm 1 & , \quad \text{if } n \text{ is odd} \\ \sqrt{F_n^2 - 1} & , \quad \text{if } n \text{ is even} \end{cases}$$

and

$\sigma_m = 0$, where $m = 3, 4, \cdots, n$.

- Let the matrix $B = [L_{j-i}]_{i,j=0}^{n-1}$ be Toeplitz matrix, then

$$\|B\|_2 = \begin{cases} L_n + 1 & , \quad \text{if } n \text{ is odd} \\ \sqrt{L_n^2 - 4} & , \quad \text{if } n \text{ is even} \end{cases}$$

2.2.2 On The Toeplitz Matrices with Generalized Fibonacci and Lucas Numbers:

In this section, we use Akbulak, Bozkurt [1]. We list some properties (lower and upper bounds for spectral norms) of Toeplitz matrices with generalized Fibonacci and Lucas numbers.

- Let the $n \times n$ matrix $A = [a_{ij}]$ satisfy $a_{ij} \equiv F_{i-j}$ (as in [1], (3) for $k = 2$). Then,

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{2}{n} F_n^2} & , \quad n \text{ even} \\ \sqrt{\frac{2}{n}(F_n^2 - 1)} & , \quad n \text{ odd} \end{cases}$$

and

$$\|A\|_2 \leq \sqrt{(1 + F_n F_{n-1})(F_n F_{n-1})},$$

where $\|.\|_2$ is the spectral norm, and F_n denotes the n th Fibonacci number.

- Let the $n \times n$ matrix $B = [b_{ij}]$ satisfy $b_{ij} \equiv L_{i-j.}$ Then,

$$\|B\|_2 \geq \begin{cases} \sqrt{\frac{2}{n}(L_n^2 - 4)} & , \quad n \text{ even}, \\ \sqrt{\frac{2}{n}(L_n^2 + 1)} & , \quad n \text{ odd}, \end{cases}$$

and

$$\|B\|_2 \leq \sqrt{(L_n L_{n-1} - 1)(L_n L_{n-1} + 2)},$$

where $\|.\|_2$ is the spectral norm, and L_n denotes the $n\ th$ Lucas number.

2.2.3 On The Toeplitz Matrices with k-Fibonacci and k-Lucas Numbers:

We use Shen [4] in this section. We list some properties (Lower and upper bounds for the spectral norms) of Toeplitz matrices with k-Fibonacci and k-Lucas numbers and some properties for the norms of Kronecker and Hadamard product matrices of these matrices.

- Let $A = [F_{k,i-j}]_{i,j=1}^{n}$ be Toeplitz matrix, then we have

$$\|A\|_2 \geq \begin{cases} \frac{1}{k}\sqrt{\frac{2(F_{k,n}^2 - 1)}{n}} & , \quad n \text{ odd} \\ \frac{1}{k}\sqrt{\frac{2(F_{k,n}^2 - 1)}{n}} & , \quad n \text{ even} \end{cases}$$

and

$$\|A\|_2 \leq \frac{2(F_{k,n} + F_{k,n-1} - 1)}{k}.$$

- Let $B = [L_{k,i-j}]_{i,j=1}^{n}$ be Toeplitz matrix, then we have

$$\|B\|_2 \geq \begin{cases} \frac{1}{k}\sqrt{\frac{2(L_{k,n}^2 + k^2)}{n}} & , \quad n \text{ odd}, \\ \frac{1}{k}\sqrt{\frac{2(L_{k,n}^2 - 4)}{n}} & , \quad n \text{ even} \end{cases}$$

and

$$\|B\|_2 \leq \frac{2(L_{k,n} + L_{k,n-1} - 2)}{k}.$$

- Let $A = [F_{k,i-j}]_{i,j=1}^{n}$ and $B = [L_{k,i-j}]_{i,j=1}^{n}$ be Toeplitz matrices, then we have

$$\|A \circ B\|_2 \leq \frac{4(F_{k,n} + F_{k,n-1} - 1)(L_{k,n} + L_{k,n-1} - 2)}{k^2}.$$

- Let $A = [F_{k,i-j}]_{i,j=1}^{n}$and $B = [L_{k,i-j}]_{i,j=1}^{n}$ be Toeplitz matrices, then we have

$$\|A \otimes B\|_2 \leq \frac{4(F_{k,n} + F_{k,n-1} - 1)(L_{k,n} + L_{k,n-1} - 2)}{k^2}$$

and

$$\|A \otimes B\|_2 \geq \begin{cases} \frac{2}{nk^2}\sqrt{(F_{k,n}^2 - 1)(L_{k,n}^2 + k^2)} & , \quad n \text{ odd}, \\ \frac{2F_{k,n}}{nk^2}\sqrt{L_{k,n}^2 - 4} & , \quad n \text{ even}. \end{cases}$$

2.2.4 On The Toeplitz Matrices with Pell Numbers:

In this section, we use Karpuz at al. [3]. We list some properties of Toeplitz matrices containing Pell numbers.

- Let the $n \times n$ matrix $A = [a_{ij}]_{i,j=0}^{n-1}$ satisfy $a_{ij} = P_{i-j}$. Then

$$\|A\|_2 \geq \begin{cases} \sqrt{\frac{1}{2n}P_n^2} & , \quad n \text{ even}, \\ \sqrt{\frac{1}{2n}(P_n^2 - 1)} & , \quad n \text{ odd} \end{cases}$$

and

$$\|A\|_2 \leq \frac{1}{2}\sqrt{(2 + P_n P_{n-1})(P_n P_{n-1})}.$$

2.2.5 On The Toeplitz Matrices with Pell, Pell-Lucas and Modified Pell Numbers

In this section, we use Daşdemir [2]. We list some properties of Toeplitz matrices with modified Pell, Pell, Pell-Lucas numbers. We give the Frobenius norm, the largest absolute column sum (1-norm), the largest absolute row sum (∞-norm); the lower and upper bounds for the spectral norm.

The first of all, we defined that the entry of the matrices comprise of the Pell, Pell-Lucas and modified Pell numbers in [2].

$$\Gamma_n = [\varphi_{ij} = P_{i+j-1} : i, j = 0, 1, \cdots, n-1]_{n\times n},$$

$$\Lambda_n = [\kappa_{ij} = Q_{i+j-1} : i, j = 0, 1, \cdots, n-1]_{n\times n},$$

$\gamma_n = [\upsilon_{ij} = q_{i+j-1} : 0, 1, \cdots, n-1]_{n\times n}$.

- Let $\Gamma_n = \varphi_{ij} = P_{i+j-1} : i, j = 0, 1, \cdots, n-1$ be an $n \times n$ matrix as in [2, Theorem2.1] .Then, we have following results:

(a)

$$\|\Gamma_n\|_F = \frac{1}{2}\sqrt{P_{n+1}^2 + P_{n+2}P_n - 1} = \frac{1}{4}\sqrt{Q_{n+1}^2 - 4}.$$

(b)

$$\|\Gamma_n\|_1 = \|\Gamma_n\|_\infty = \frac{P_{n+1} + P_n - 1}{2}.$$

(c)

$$\frac{1}{2}\sqrt{\frac{P_{n+1}^2 + P_{n+2}P_n - 1}{n}} \leq \|\Gamma_n\|_2 \leq \frac{1}{2}\sqrt{P_nP_{n-1}(2 + P_{n-1}P_n)}.$$

- Let $\Lambda_n = \kappa_{ij} = Q_{i+j-1} : i, j = 0, 1, \cdots, n-1$ be an $n \times n$ matrix as in [2, Theorem 2.3]. Then, we have following properties:

(a)

$$\|\Lambda_n\|_F = \sqrt{\frac{Q_{2n}^2 - \xi}{2}}.$$

(b)

$$\|\Lambda_n\|_1 = \|\Lambda_n\|_\infty = \frac{1}{2}(Q_n + Q_{n-1}) = 2P_n.$$

(c)

$$\sqrt{\frac{Q_{2n} - \xi}{2n}} \leq \|\Lambda_n\|_2 \leq \frac{1}{4}\sqrt{(Q_{2n-1} - \xi)(Q_{2n-1} - \xi + 4)}.$$

- Let $\gamma_n = \upsilon_{ij} = q_{i+j-1} : 0, 1, \cdots, n-1$ be a $n \times n$ matrix as in [2, Theorem 2.5]. Then, we have

(a)

$$\|\gamma_n\|_F = \frac{1}{2\sqrt{2}}\sqrt{q_{2n}^2 - \frac{\varepsilon}{2}}.$$

(b)

$$\|\gamma_n\|_1 = \|\gamma_n\|_\infty = q_n + q_{n-1} = P_n.$$

(c)

$$\sqrt{\frac{1}{n}(q_{2n} - \varepsilon)} \leq \|\gamma_n\|_2 \leq \frac{1}{2}\sqrt{(q_{2n-1} - \varepsilon)(q_{2n-1} - \varepsilon + 2)}$$

where $\varepsilon = \xi/2$.

2.2.6 On The Toeplitz Matrices with Jacobsthal and Jacobsthal-Lucas Numbers:

We use Uygun [12] in this section. For the special norms of Toeplitz matrices whose elements are Jacobsthal and Jacobsthal-Lucas numbers respectively, we list some properties such as the Frobenius norms and spectral norms of Hadamard and Kronecker product matrices of these matrices.

- Let $A = T(j_0, j_1, \cdots, j_{n-1})$ be the Toeplitz matrix with Jacobsthal numbers entry, then

  - The maximum absolute column sum norm $(1 - norm)$ and the maximum absolute row sum norm $(\infty - norm)$ of matrix A are

    $$\|A\|_1 = \|A\|_\infty = \frac{j_n + 2j_{n-1} - 1}{2}.$$

  - The Frobenious or Euclidean norm of matrix A is

    $$\|A\|_E = \frac{1}{3}\sqrt{\frac{16j_{2n-2} + 8(-1)^n j_{n-1}}{3} - \frac{j_{2n-2}}{32^{2n-2}} + \frac{j_{n-1}}{32^{n-2}} + n^2 - 1}.$$

– The lower and upper bounds for the spectral norm of A are obtained as

$$\frac{1}{3}\sqrt{\frac{16j_{2n-2}+8(-1)^n j_{n-1}}{3n}-\frac{j_{2n-2}}{3n2^{2n-2}}+\frac{j_{n-1}}{3n2^{n-2}}+\frac{n^2-1}{n}} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \frac{1}{9}\sqrt{(j_{2n-2}+2(-1)^{n-1}j_{n-1}+n)(j_{2n}+2(-1)^n j_n+n)}.$$

- Let the elements of the Toeplitz matrix $A = T(c_0, c_1, \cdots, c_{n-1})$ with Jacobsthal Lucas numbers,

  – The $1-norm$, $\infty-norm$ of A are

  $$\|A\|_1 = \|A\|_\infty = \frac{c_{n+2}-1}{2}.$$

  – The Frobenius or Euclidean norm of matrix A is

  $$\|A\|_E = \sqrt{\frac{1}{3}(16j_{2n-2}+8(-1)^n j_{n-1}-\frac{j_{2n-2}}{2^{2n-2}}-\frac{j_{n-1}}{2^{n-2}}+13-7n+3n^2)}.$$

  – The lower and upper bounds for the spectral norm of A are obtained as

  $$\sqrt{\frac{1}{3n}(16j_{2n-2}+8(-1)^n j_{n-1}-\frac{j_{2n-2}}{2^{2n-2}}-\frac{j_{n-1}}{2^{n-2}}+13-7n+3n^2)} \leq \|A\|_2$$

  and

  $$\|A\|_2 \leq \sqrt{(j_{2n-2}+4(-1)^{n-1}j_{n-1}+n)(j_{2n}-2(-1)^n j_n+n)}.$$

- Let $A = C(j_0, j_1, \cdots, j_{n-1})$ and $B = C(c_0, c_1, \cdots, c_{n-1})$ be the Toeplitz matrices with Jacobsthal and the Jacobsthal Lucas numbers, then

  – The Euclidean norm of Kronecker product of these matrices is

  $$\begin{aligned}\|A\otimes B\|_E &= \frac{1}{3}\sqrt{\frac{16j_{2n-2}+8(-1)^n j_{n-1}}{3}-\frac{j_{2n-2}}{3.2^{2n-2}}+\frac{j_{n-1}}{3.2^{n-2}}+n^2-1} \\ &\quad\times\sqrt{\frac{1}{3}(16j_{2n-2}+8(-1)^n j_{n-1}-\frac{j_{2n-2}}{2^{2n-2}}-\frac{j_{n-1}}{2^{n-2}}+13-7n+3n^2)}.\end{aligned}$$

  – The upper bound for any norm of Hadamard product of these matrices is

  $$\begin{aligned}\|A\circ B\|_E &\leq \frac{1}{3}\sqrt{\frac{16j_{2n-2}+8(-1)^n j_{n-1}}{3}-\frac{j_{2n-2}}{3.2^{2n-2}}+\frac{j_{n-1}}{3.2^{n-2}}+n^2-1} \\ &\quad\times\sqrt{\frac{1}{3}(16j_{2n-2}+8(-1)^n j_{n-1}-\frac{j_{2n-2}}{2^{2n-2}}-\frac{j_{n-1}}{2^{n-2}}+13-7n+3n^2)}.\end{aligned}$$

2.2.7 On The Toeplitz Matrices with k-Jacobsthal and k-Jacobsthal Numbers:

We use Uygun [13] in this section. After giving the special norms of Toeplitz matrices with k-Jacobsthal and k-Jacobsthal-Lucas members, we present some properties of these matrices for spectral and Frobenius norms of Hadamard and Kronecker product matrices.

- Let $A = T\left(j_{k,0}, j_{k,1}, \cdots, j_{k,n-1}\right)$ be the Toeplitz matrix with k-Jacobsthal numbers,

  - The largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are

    $$\|A\|_1 = \|A\|_\infty = \frac{(k+2)j_{k,n} + 2j_{k,n-1} - 1}{k+1}.$$

  - The Frobenious (or Euclidean) norm of matrix A is.

$$\|A\|_E = \sqrt{\begin{array}{c} \frac{1}{(k^2+8)(1-k^2)^2}\left(c_{k,2n+2} + \frac{c_{k,2n+2}}{2^{2n-2}} - 8c_{k,2n} - \frac{2c_{k,2n}}{2^{2n-2}} + 16c_{k,2n-2} + \frac{c_{k,2n-2}}{2^{2n-2}} - 18\right) + \\ \frac{(n-1)\left(2k^2-2\right)-4-k^2}{(k^2+8)(k^2-1)} + \frac{1}{3(k^2+8)}\left(8\left(-1\right)^n j_{n-1} - \frac{4j_n}{2^n} - 6n + 8\right) \end{array}}.$$

  - The lower and upper bounds for the spectral norm of A are obtained as

$$\sqrt{\begin{array}{c} \frac{1}{n(k^2+8)(1-k^2)^2}\left(c_{k,2n+2} + \frac{c_{k,2n+2}}{2^{2n-2}} - 8c_{k,2n} - \frac{2c_{k,2n}}{2^{2n-2}} + 16c_{k,2n-2} + \frac{c_{k,2n-2}}{2^{2n-2}} - 10\right) \\ + \frac{(n-1)(2k^2-2)-4-k^2}{n(k^2+8)(k^2-1)} + \frac{1}{3n(k^2+8)}\left(8(-1)^n j_{n-1} + \frac{j_{n-1}}{2^{n-2}} - 6n + 6\right) \end{array}} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \sqrt{\begin{array}{c} \left(\frac{1}{k^2+8}\left(\frac{4c_{k,2n-4} - c_{k,2n-2} - k^2 - 2}{1-k^2} + 2(-1)^{n-1} j_{n-1}\right) + 1\right) \\ \left(\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2} - c_{k,2n} - k^2 - 2}{1-k^2} + 2(-1)^n j_n\right)\right) \end{array}}.$$

- Let the elements of the Toeplitz matrix be k-Jacobsthal Lucas numbers $A = T\left(c_{k,0}, c_{k,1}, \cdots, c_{k,n-1}\right)$.

  - The 1-norm and $\infty-$norm of A are

    $$\|A\|_1 = \|A\|_\infty = \frac{(k+2)j_{k,n} + 2j_{k,n-1} - 1}{k+1}.$$

  - The Frobenious (or Euclidean) norm of matrix A is

$$\|A\|_E = \sqrt{\begin{array}{c} \frac{16c_{k,2n-2} - 8c_{k,2n} + c_{k,2n+2} - k^2 - 20}{(1-k^2)^2} + \frac{c_{k,2n-2} - 2c_{k,2n} + c_{k,2n+2}}{2^{2n-2}(1-k^2)^2} \\ + \frac{(k^2+4) + (n-1)(6k^2-6)}{1-k^2} + \frac{-8(-1)^n j_{n-1} + 6n - 8}{3} + \frac{j_n}{3.2^{n-2}}. \end{array}}$$

– The lower and upper bounds for the spectral norm of A are obtained as

$$\sqrt{\begin{array}{c}\frac{16c_{k,2n-2}-8c_{k,2n}+c_{k,2n+2}-(k^2-4)^2}{n(1-k^2)^2}+\frac{c_{k,2n-2}-2c_{k,2n}+c_{k,2n+2}}{2^{2n-2}n(1-k^2)^2}\\ +\frac{(-4k^2-8)+(n-1)(2k^2-4)}{(1-k^2)n}+\frac{-8(-1)^n j_{n-1}+6n-6}{3n}-\frac{j_{n-1}}{3n.2^{n-2}}\end{array}} \leq \|A\|_2$$

and

$$\|A\|_2 \leq \sqrt{\begin{array}{c}\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2}-2(-1)^{n-1}j_{n-1}+5\\ \frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2}-2(-1)^n j_n+4\end{array}}$$

- Let $A = T(j_{k,0}, j_{k,1}, \cdots, j_{k,n-1})$ and $B = T(c_{k,0}, c_{k,1}, \cdots, c_{k,n-1})$ be the Toeplitz matrix with k-Jacobsthal and the k-Jacobsthal Lucas numbers,

  – The Euclidean norm of Kronecker product of these matrices is given as:

  $$\|A \otimes B\|_E = u_1 \times v_1$$

  where

  $$u_1 = \sqrt{\begin{array}{c}\frac{1}{(k^2+8)(1-k^2)^2}(c_{k,2n+2}+\frac{c_{k,2n+2}}{2^{2n-2}}-8c_{k,2n}-\frac{2c_{k,2n}}{2^{2n-2}}+16c_{k,2n-2}+\frac{c_{k,2n-2}}{2^{2n-2}}-18)\\ +\frac{(n-1)(2k^2-2)-4-k^2}{(k^2+8)(k^2-1)}+\frac{1}{3(k^2+8)}(8(-1)^n j_{n-1}-\frac{4j_n}{2^n}-6n+8)\end{array}}$$

  and

  $$v_1 = \sqrt{\begin{array}{c}\frac{16c_{k,2n-2}-8c_{k,2n}+c_{k,2n+2}-k^2-20}{(1-k^2)^2}+\frac{c_{k,2n-2}-2c_{k,2n}+c_{k,2n+2}}{2^{2n-2}(1-k^2)^2}\\ \frac{(k^2+4)+(n-1)(6k^2-6)}{1-k^2}+\frac{-8(-1)^n j_{n-1}+6n-8}{3}+\frac{j_n}{3.2^{n-2}}\end{array}}.$$

  – The upper bound for the spectral norm of Kronecker product of these matrices is given as:

  $$\|A \otimes B\|_2 \leq u_2 \times v_2$$

  where

  $$u_2 = \sqrt{\begin{array}{c}\left[\frac{1}{k^2+8}(\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2}+2(-1)^{n-1}j_{n-1})+1\right]\\ \left[\frac{1}{k^2+8}(\frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2}+2(-1)^n j_n)\right]\end{array}}$$

  and

  $$v_2 = \sqrt{\begin{array}{c}\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2}-2(-1)^{n-1}j_{n-1}+5\\ \frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2}-2(-1)^n j_n+4\end{array}}.$$

– The upper bound for the spectral norm of Hadamard product of the matrices is

$$\|A \circ B\|_2 \leq u_3 \times v_3$$

where

$$u_3 = \sqrt{\begin{matrix}\left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2}+2(-1)^{n-1}j_{n-1}\right)+1\right] \\ \left[\frac{1}{k^2+8}\left(\frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2}+2(-1)^{n}j_{n}\right)\right]\end{matrix}}$$

and

$$v_3 = \sqrt{\begin{matrix}\left(\frac{4c_{k,2n-4}-c_{k,2n-2}-k^2-2}{1-k^2}-2(-1)^{n-1}j_{n-1}+5\right) \\ \left(\frac{4c_{k,2n-2}-c_{k,2n}-k^2-2}{1-k^2}-2(-1)^{n}j_{n}+4\right)\end{matrix}}.$$

– The upper bound for the Euclid norm of Hadamard product of the matrices is

$$\|A \circ B\|_E \leq u_4 \times v_4$$

where

$$u_4 = \sqrt{\begin{matrix}\frac{1}{(k^2+8)(1-k^2)^2}\left(c_{k,2n+2}+\frac{c_{k,2n+2}}{2^{2n-2}}-8c_{k,2n}-\frac{2c_{k,2n}}{2^{2n-2}}+16c_{k,2n-2}+\frac{c_{k,2n-2}}{2^{2n-2}}-18\right) \\ +\frac{(n-1)(2k^2-2)-4-k^2}{(k^2+8)(k^2-1)}+\frac{1}{3(k^2+8)}\left(8(-1)^{n}j_{n-1}-\frac{4j_n}{2^n}-6n+8\right)\end{matrix}}$$

and

$$v_4 = \sqrt{\begin{matrix}\frac{16c_{k,2n-2}-8c_{k,2n}+c_{k,2n+2}-k^2-20}{(1-k^2)^2}+\frac{c_{k,2n-2}-2c_{k,2n}+c_{k,2n+2}}{2^{2n-2}(1-k^2)^2} \\ +\frac{(k^2+4)+(n-1)(6k^2-6)}{1-k^2}+\frac{8(-1)^{n+1}j_{n-1}+6n-8}{3}+\frac{j_n}{3.2^{n-2}}\end{matrix}}.$$

CHAPTER 3

TOEPLITZ MATRICES WITH GENERALIZED ORESME NUMBERS

This chapter is about Toeplitz matrices with Oresme numbers components. First, the Toeplitz matrices which elements are the Oresme numbers and then the Frobenius (Euclidian), row and column norms of these matrices are found. Furthermore, lower and upper bounds are obtained for the spectral norms of these matrices. In addition, the upper bounds for the Frobenius and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the Oresme numbers are calculated. We use Soykan [5, 6].

3.1 THE TOEPLITZ MATRICES WITH GENERALIZED ORESME NUMBERS COMPONENTS

In this section, we define the Toeplitz matrices which elements are modified Oresme, Oresme-Lucas, Oresme numbers respectively and we use the notation $A = T(W_0, W_1, \cdots, W_{n-1})$ for the Toeplitz matrix with generalized Oresme numbers, i.e.,

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}. \tag{3.1}$$

For special cases, we get

$$A = \begin{pmatrix} G_0 & G_{-1} & G_{-2} & \cdots & G_{1-n} \\ G_1 & G_0 & G_{-1} & \cdots & G_{2-n} \\ G_2 & G_1 & G_0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & G_{n-2} & G_{n-3} & \cdots & G_0 \end{pmatrix} = \begin{pmatrix} 0 & -4 & -16 & \cdots & G_{1-n} \\ 1 & 0 & -4 & \cdots & G_{2-n} \\ 1 & 1 & 0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & G_{n-2} & G_{n-3} & \cdots & 0 \end{pmatrix} \tag{3.2}$$

for the Toeplitz matrix $A = T(G_0, G_1, \cdots, G_{n-1})$ with modified Oresme numbers and

$$A = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ H_1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ H_2 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 8 & \cdots & H_{1-n} \\ 1 & 2 & 4 & \cdots & H_{2-n} \\ \frac{1}{2} & 1 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix} \tag{3.3}$$

for the Toeplitz matrix $A = T(H_0, H_1, \cdots, H_{n-1})$ with Oresme-Lucas numbers and

$$A = \begin{pmatrix} O_0 & O_{-1} & O_{-2} & \cdots & O_{1-n} \\ O_1 & O_0 & O_{-1} & \cdots & O_{2-n} \\ O_2 & O_1 & O_0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & O_{n-2} & O_{n-3} & \cdots & O_0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -8 & \cdots & O_{1-n} \\ \frac{1}{2} & 0 & -2 & \cdots & O_{2-n} \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & O_{n-2} & O_{n-3} & \cdots & 0 \end{pmatrix} \tag{3.4}$$

for the Toeplitz matrix $A = T(O_0, O_1, \cdots, O_{n-1})$ with Oresme numbers.

3.2 THE LARGEST ABSOLUTE COLUMN SUM (1-norm) AND THE LARGEST ABSOLUTE ROW SUM (∞-norm) OF THE TOEPLITZ MATRICES WITH GENERALIZED ORESME NUMBERS

In this section, we calculate the largest absolute column sum and the largest absolute row sum of the Toeplitz matrices with generalized Oresme numbers and obtain their properties.

In the following theorem, we present the norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A.

Theorem 3.1 *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix with generalized Oresme numbers then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are*

$$\|A_1\| = \|A\|_\infty = \begin{cases} -4W_{-n+1} - W_0 + 4W_1 + W_{-n} & , \quad if \ |W_{-k}| \geq |W_k| \quad and \ W_{-k} \leq 0 \\ 4W_{-n+1} + W_0 - 4W_1 - W_{-n} & , \quad if \ |W_{-k}| \geq |W_k| \quad and \ W_{-k} \geq 0 \end{cases}$$

where $k = i - j : i, j = 0, 1, \cdots, n-1;\ (k \in N, -k \in N^{-})$.

Proof. Consider $A = T(W_0, W_1, \cdots, W_{n-1})$ which is given as in (3.1). By the definitions of $1 - norm$ and $\infty - norm$ and Equ. (2.3) and Equ. (2.4) and Equ. (1.7), we conclude that

(i) If $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \leq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{in}| \\
&= |a_{1n}| + |a_{2n}| + |a_{3n}| + \cdots + |a_{nn}| \quad = \sum_{k=0}^{n-1} |W_{-k}| \\
&= -\sum_{k=0}^{n-1} W_{-k} = -(\sum_{k=0}^{n} W_{-k} - W_{-n}) \\
&= -4W_{-n+1} - W_0 + 4W_1 + W_{-n}
\end{aligned}$$

and if $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \geq 0, k \in N$, then we obtain

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{in}| \\
&= |a_{1n}| + |a_{2n}| + |a_{3n}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1} |W_{-k}| \\
&= \sum_{k=0}^{n-1} W_{-k} = \sum_{k=0}^{n} W_{-k} - W_{-n} \\
&= 4W_{-n+1} + W_0 - 4W_1 - W_{-n}.
\end{aligned}$$

(ii) If $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \leq 0, k \in N$, then it follows that

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{1j}| \\
&= |a_{11}| + |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\
&= -\sum_{k=0}^{n-1} W_{-k} = -(\sum_{k=0}^{n} W_{-k} - W_{-n}) \\
&= -4W_{-n+1} - W_0 + 4W_1 + W_{-n}
\end{aligned}$$

and if $|W_{-k}| \geq |W_k|, k \in N$ and $W_{-k} \geq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{1j}| \\
&= |a_{11}| + |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\
&= \sum_{k=0}^{n-1} W_{-k} = \sum_{k=0}^{n} W_{-k} - W_{-n} \\
&= 4W_{-n+1} + W_0 - 4W_1 - W_{-n}.
\end{aligned}$$

Thus, the proof is completed. □

Remark 3.1 *In the statement of the Theorem 3.1, the condion on W_n, W_{-n}, $n \in N$ is given to calculade $\|.\|_1$ and $\|.\|_\infty$ norms of modified Oresme, Oresme-Lucas, Oresme numbers. The other cases can be handled similarly.*

From the last Theorem 3.1, we have the following corollary which present norm values of $\|A\|_1$ and $\|A\|_\infty$ of A with modified Oresme numbers, Oresme-Lucas numbers and Oresme numbers, respectively, (set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Corollary 3.1

(a) For $A = T(G_0, G_1, \cdots, G_{n-1})$, the values of norms of Toeplitz matrices with modified Oresme numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = -4G_{-n+1} + G_{-n} + 4.$$

(b) For $A = T(H_0, H_1, \cdots, H_{n-1})$, the values of norms of Toeplitz matrices with Oresme-Lucas numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = 4H_{-n+1} - H_{-n} - 2.$$

(c) For $A = T(O_0, O_1, \cdots, O_{n-1})$, the values of norms of Toeplitz matrices with Oresme numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = -4O_{-n+1} + O_{-n} + 2.$$

3.3 THE FROBENIOUS NORM OF THE TOEPLITZ MATRICES WITH GENERALIZED ORESME NUMBERS

In this section, we calculate the Frobenious norm of the Toeplitz matrices with generalized Oresme numbers and obtain their results.

Next, theorem presents the Frobenious (Euclidian) norm of a Toeplitz matrix A.

Theorem 3.2 *Let* $A = T(W_0, W_1, \cdots, W_{n-1})$ *be a Toeplitz matrix with generalized Oresme numbers components, then the Frobenious (Euclidian) norm of matrix* A *is*

$$\|A\|_F = \sqrt{\Omega_1}$$

where

$\Omega_1 = \frac{1}{81}(96W_n^2 - 24W_{-n}^2 - 15W_{n-1}^2 + 240W_{-n+1}^2 + (72(2^{2n}) + 192(2^{-2n}) - 96)W_0^2 + (576 - 288(2^{2n}) - 768(2^{-2n}))W_0W_1 + (288(2^{2n}) + 768(2^{-2n}) - 1056)W_1^2)$.

Proof. The matrix A is of the form

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad + (n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \end{aligned}$$

and so

$$\begin{aligned} \|A\|_F^2 &= (2-n)W_0^2 - \frac{7}{9}\sum_{k=1}^{n-1} W_k^2 + \frac{1}{9}\sum_{k=1}^{n-1} W_{k-1}^2 + \frac{16}{9}\sum_{k=1}^{n-1} W_{-k+1}^2 \\ &\quad + \frac{8}{9}\sum_{k=1}^{n-1} W_{-k}^2 + \frac{16}{9}\sum_{k=1}^{n-1} W_1(2W_0 - W_1) \\ &\quad + \frac{1}{27}\sum_{k=1}^{n-1} 2^{-2k+5}(W_0 - 2W_1)^2(2^{2k} - 1) \\ &\quad + \frac{1}{9}\sum_{k=1}^{n-1} W_0^2 - \frac{16}{9}\sum_{k=1}^{n-1} W_1^2 + \frac{8}{27}\sum_{k=1}^{n-1}(W_0 - 2W_1)^2(2^{2k} - 1). \end{aligned}$$

By using the equalities

$$\begin{aligned}\frac{16}{9}\sum_{k=1}^{n-1} W_1(2W_0 - W_1) &= \frac{16}{9}(n-1)W_1(2W_0 - W_1),\\ \frac{1}{9}\sum_{k=1}^{n-1} W_0^2 &= \frac{1}{9}(n-1)W_0^2,\\ -\frac{16}{9}\sum_{k=1}^{n-1} W_1^2 &= -\frac{16}{9}(n-1)W_1^2,\end{aligned}$$

$$\begin{aligned}\frac{1}{27}\sum_{k=1}^{n-1} 2^{-2k+5}(W_0 - 2W_1)^2(2^{2k}-1) &= \frac{(W_0-2W_1)^2}{81}(3n2^5 + 2^7(2^{-2n}-1)),\\ \frac{8}{27}\sum_{k=1}^{n-1}(W_0 - 2W_1)^2(2^{2k}-1) &= \frac{8\,(W_0-2W_1)^2}{81}(2^{2n} - 3n - 1),\end{aligned}$$

we obtain

$$\begin{aligned}P &= \frac{16}{9}\sum_{k=1}^{n-1} W_1(2W_0 - W_1) + \frac{1}{9}\sum_{k=1}^{n-1} W_0^2 - \frac{16}{9}\sum_{k=1}^{n-1} W_1^2\\ &\quad + \frac{1}{27}\sum_{k=1}^{n-1} 2^{-2k+5}(W_0 - 2W_1)^2(2^{2k}-1) + \frac{8}{27}\sum_{k=1}^{n-1}(W_0 - 2W_1)^2(2^{2k}-1),\end{aligned}$$

and it follows that

$$\begin{aligned}P &= \frac{16}{9}\,(n-1)\,W_1(2W_0 - W_1) + \frac{1}{9}(n-1)W_0^2 - \frac{16}{9}(n-1)W_1^2\\ &\quad + \frac{(W_0-2W_1)^2}{81}(3n2^5 + 2^7(2^{-2n}-1)) + \frac{8(W_0-2W_1)^2}{81}(2^{2n} - 3n - 1).\end{aligned}$$

Moreover, we use equation 1.8 and equation 1.9 in Corollary 1.1.

Therefore, we get

$\|A\|_F^2 = (2-n)\,W_0^2 - \frac{7}{9}\sum_{k=1}^{n-1} W_k^2 + \frac{1}{9}\sum_{k=1}^{n-1} W_{k-1}^2 + \frac{16}{9}\sum_{k=1}^{n-1} W_{-k+1}^2 + \frac{8}{9}\sum_{k=1}^{n-1} W_{-k}^2 + P$
$= \frac{1}{81}(96W_n^2 - 24W_{-n}^2 - 15W_{n-1}^2 + 240W_{-n+1}^2 + (72(2^{2n}) + 192(2^{-2n}) - 96)W_0^2 + (576 - 288(2^{2n}) - 768(2^{-2n}))W_0W_1 + (288(2^{2n}) + 768(2^{-2n}) - 1056)W_1^2).$

This complates the proof. □

From the last Theorem 3.2, we have the following corollary which gives Frobenius norm formulas of modified Oreme numbers, Oresme-Lucas numbers and Oresme numbers, respectively, (take

$W_n = G_n$, with $G_0 = 0, G_1 = 1$ and $W_n = H_n$, with $H_0 = 2, H_1 = 1$ and $W_n = O_n$, with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Corollary 3.2 *For $n \geq 0$, Toeplitz matrices with the modified Oresme, Oresme-Lucas and Oresme numbers, respectively have the following properties:*

(a) $\|A\|_F = \sqrt{\Omega_2}$

where A is given as in 3.2

$$\Omega_2 = \frac{1}{81}(96G_n^2 - 24G_{-n}^2 - 15G_{n-1}^2 + 240G_{-n+1}^2 + 288(2^{2n}) + 768(2^{-2n}) - 1056).$$

(b) $\|A\|_F = \sqrt{\Omega_3}$

where A is given as in 3.3

$$\Omega_3 = \frac{1}{81}(96H_n^2 - 24H_{-n}^2 - 15H_{n-1}^2 + 240H_{-n+1}^2 - 288).$$

(c) $\|A\|_F = \sqrt{\Omega_4}$

where A is given as in 3.4

$$\Omega_4 = \frac{1}{81}(96O_n^2 - 24O_{-n}^2 - 15O_{n-1}^2 + 240O_{-n+1}^2 + 72(2^{2n}) + 192(2^{-2n}) - 264).$$

3.4 THE LOWER AND UPPER BOUNDS OF THE SPECTRAL NORMS OF THE TOEPLITZ MATRICES WITH GENERALIZED ORESME NUMBERS

In this section, we investigate the lower and upper bounds of the spectral norms of the Toeplitz matrices with generalized Oresme numbers

In the following theorem, we present the lower and upper bounds of the spectral norms of the Toeplitz matrices with the modified Oresme numbers, Oresme-Lucas numbers and Oresme numbers, respectively, (set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Theorem 3.3

(a) *Consider $A = T(G_0, G_1, \cdots, G_{n-1})$ which is given as in (3.2). Let*

$$C = \begin{pmatrix} 1 & G_{-1} & G_{-2} & \cdots & G_{1-n} \\ 1 & G_0 & G_{-1} & \cdots & G_{2-n} \\ 1 & G_1 & G_0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & G_{n-2} & G_{n-3} & \cdots & G_0 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -16 & \cdots & G_{1-n} \\ 1 & 0 & -4 & \cdots & G_{2-n} \\ 1 & 1 & 0 & \cdots & G_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & G_{n-2} & G_{n-3} & \cdots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} G_0 & 1 & 1 & \cdots & 1 \\ G_1 & 1 & 1 & \cdots & 1 \\ G_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Omega_2}$$

where Ω_2 is as in Corollary 3.2.

(ii)

$$\|A\|_2 \leq \Omega_5$$

where

$$\Omega_5 = (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 + 32(2^{2n} - 1) - 21))^{\frac{1}{2}} \times \sqrt{n}.$$

(b) *Consider $A = T(H_0, H_1, \cdots, H_{n-1})$ which is given as in (3.3). Let*

$$C = \begin{pmatrix} 1 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ 1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ 1 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 8 & \cdots & H_{1-n} \\ 1 & 2 & 4 & \cdots & H_{2-n} \\ 1 & 1 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix}$$

and

$$D = \begin{pmatrix} H_0 & 1 & 1 & \cdots & 1 \\ H_1 & 1 & 1 & \cdots & 1 \\ H_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Omega_3}$$

where Ω_3 is as in Corollary 3.2.

(ii)

$$\|A\|_2 \leq \Omega_6$$

where

$$\Omega_6 = \begin{cases} (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} \times \sqrt{n} & , \quad n \geq 6 \\ (-\frac{1}{27}(48H_n^2 - 3H_{n-1}^2 - 144))^{\frac{1}{2}} \times (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} & , \quad 1 \leq n < 6 \end{cases}.$$

(c) *Consider $A = T(O_0, O_1, \cdots, O_{n-1})$ which is given as in (3.4). Let*

$$C = \begin{pmatrix} 1 & O_{-1} & O_{-2} & \cdots & O_{1-n} \\ 1 & O_0 & O_{-1} & \cdots & O_{2-n} \\ 1 & O_1 & O_0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & O_{n-2} & O_{n-3} & \cdots & O_0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -8 & \cdots & O_{1-n} \\ 1 & 0 & -2 & \cdots & O_{2-n} \\ 1 & \frac{1}{2} & 0 & \cdots & O_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & O_{n-2} & O_{n-3} & \cdots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} O_0 & 1 & 1 & \cdots & 1 \\ O_1 & 1 & 1 & \cdots & 1 \\ O_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & 1 & 1 & \cdots & 1 \\ \frac{1}{2} & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart Product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Omega_4}$$

where Ω_4 is as in Corollary 3.2.

(ii)

$$\|A\|_2 \leq \Omega_7$$

where

$$\Omega_7 = \ (\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 + 15 + 8(2^{2n} - 1)))^{\frac{1}{2}} \times \sqrt{n}.$$

Proof.

(a)

(i) We use equation (2.2).

(ii) By definition, we get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{1j}|^2)^{\frac{1}{2}} = (1 + \sum_{k=1}^{n-1} W_{-k}^2)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 - 24G_0^2 - 48G_1^2 \\ &\quad +8(G_0 - 2G_1)^2(2^{2n} - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 + 32(2^{2n} - 1) - 21))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} \\ &= \sqrt{n} \ (0 \leq G_i \leq 1, \quad \text{for} \quad (i \geq 0 \text{ and } \quad n \geq 1)). \end{aligned}$$

So, from inequality (2.7),

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Omega_5 \\ &= (\frac{1}{27}(48G_{-n+1}^2 - 3G_{-n}^2 + 32(2^{2n} - 1) - 21))^{\frac{1}{2}} \times \sqrt{n} \end{aligned}$$

(b)

(i) We use equation (2.2).

(ii) We get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{1j}|^2)^{\frac{1}{2}} = (1 + \sum_{k=1}^{n-1} H_{-k}^2)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 24H_0^2 - 48H_1^2 \\ &\quad +8(H_0 - 2H_1)^2(2^{2n} - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} \end{aligned}$$

and

$$c_1(D) = \begin{cases} \sum_{k=0}^{n-1} H_k^2 = \sqrt{n} & , \quad n \geq 6 \\ \sum_{k=0}^{n-1} H_k^2 = -\frac{1}{27}(21H_n^2 - 3H_{n-1}^2 - 48H_1(2H_0 - H_1) & \\ \quad -2^{-2n+5}(H_0 - 2H_1)^2(2^{2n} - 1)) - H_n^2 & , \quad 1 \leq n < 6 \\ \quad = (-\frac{1}{27}(48H_n^2 - 3H_{n-1}^2 - 144))^{\frac{1}{2}} & \end{cases}$$

so by defination of Hadamard product and from inequality (2.7)

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Omega_6 \\ &= \begin{cases} (\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} \times \sqrt{n} & , \quad n \geq 6 \\ (-\frac{1}{27}(48H_n^2 - 3H_{n-1}^2 - 144))^{\frac{1}{2}} & \\ \times(\frac{1}{27}(48H_{-n+1}^2 - 3H_{-n}^2 - 117))^{\frac{1}{2}} & , \quad 1 \leq n < 6 \end{cases}. \end{aligned}$$

(c)

(i) We use equation (2.2).

(ii) We get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{1j}|^2)^{\frac{1}{2}} = (1 + \sum_{k=1}^{n-1} O_{-k}^2)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 - 24O_0^2 - 48O_1^2 \\ &\quad +8(O_0 - 2O_1)^2(2^{2n} - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 + 15 + 8(2^{2n} - 1)))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} \\ &= \sqrt{n}\ (0 \leq O_i \leq 1, \quad \text{for} \quad (i \geq 0 \quad \text{and} \quad n \geq 1)). \end{aligned}$$

so, from inequality (2.7)

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Omega_7 \\ &= (\frac{1}{27}(48O_{-n+1}^2 - 3O_{-n}^2 + 15 + 8(2^{2n} - 1)))^{\frac{1}{2}} \times \sqrt{n}. \end{aligned}$$

This complates the proof. □

3.5 THE KRONECKER AND HADAMARD PRODUCT OF THE TOEPLITZ MATRICES WITH GENERALIZED ORESME NUMBERS

In this section, we obtain some results for the Kronecker and Hadamard product of the Toeplitz matrices with generalized Oresme numbers.
From the equation (2.10) and Corollary 3.2, we have the following corollary which gives the Frobenius norms of the Kronecker products of the Toeplitz matrices with special cases of generalized Oresme numbers.

Corollary 3.3

(a) Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have the following property.

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Omega_2}\sqrt{\Omega_3} \end{aligned}$$

where Ω_2 and Ω_3 are as in Corollary 3.2 (a) and (b),

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b) Suppose that $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers respectively, then we obtain the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Omega_2}\sqrt{\Omega_4} \end{aligned}$$

where Ω_2 and Ω_4 are as in Corollary 3.2 (a) and (c),
(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c) Given $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers respectively, then we get the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Omega_3}\sqrt{\Omega_4} \end{aligned}$$

where Ω_3 and Ω_4 are as in Corollary 3.2 (b) and (c),
(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively.)
Proof. (a), (b) and (c) follows from equation (2.10) and Theorem 3.2 and Corollary 3.2. □

From the above inequality (2.9) and Theorem 3.2 and Corollary 3.2, we have the following result, which gives an upper bound for the Frobenius norm of Hadamard products of Toeplitz matrices by exclusive cases of generalized Oresme numbers.

Corollary 3.4

(a) Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Omega_2}\sqrt{\Omega_3} \end{aligned}$$

where Ω_2 and Ω_3 are as in Corollary 3.2 (a) and (b),

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b) Suppose that $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers, respectively, then we obtain the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Omega_2}\sqrt{\Omega_4} \end{aligned}$$

where Ω_2 and Ω_4 are as in Corollary 3.2 (a) and (c),

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c) Assume that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Omega_3}\sqrt{\Omega_4} \end{aligned}$$

where Ω_3 and Ω_4 are as in Corollary 3.2 (b) and (c),

(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).
In the last inequality (2.8) and Theorem 3.3, we have the following Corollary, which gives an upper bound for the spectral norm of Hadamard products of Toeplitz matrices with special cases of generalized Oresme numbers.

Corollary 3.5

(a) Given $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have following property:

$$\|A \circ B\|_2 \leq \Omega_5 \times \Omega_6$$

where Ω_5 and Ω_6 are as in Theorem 3.3,

(take $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b) Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers respectively, then we have the following property:

$$\|A \circ B\|_2 \leq \Omega_5 \times \Omega_7$$

where Ω_5 and Ω_7 are as in Theorem 3.3,

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c) Suppose that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers respectively, then we get the following property:

$$\|A \circ B\|_2 \leq \Omega_6 \times \Omega_7$$

where Ω_6 and Ω_7 are as in Theorem 3.3,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Proof. For (a), (b) and (c) see inequality (2.8) and Theorem 3.3. □

From the related equation (2.10) and Theorem 3.3, we have the following Corollary which gives an upper bound for the spectral norm of Kronocker products of Toeplitz matrices with special cases of generalized Oresme numbers.

Corollary 3.6

(a) Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = (H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme-Lucas numbers, respectively, then we have the following property:

$$\|A \otimes B\|_2 \leq \Omega_5 \times \Omega_6$$

where Ω_5 and Ω_6 are as in Theorem 3.3,

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 1$, respectively).

(b) Let $A = T(G_0, G_1, \cdots, G_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with modified Oresme numbers and Oresme numbers respectively, then we get the following property:

$$\|A \otimes B\|_2 \leq \Omega_5 \times \Omega_7$$

where Ω_5 and Ω_7 are as in Theorem 3.3,

(set $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

(c) Let $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(O_0, O_1, \cdots, O_{n-1})$ be Toeplitz matrices with Oresme-Lucas numbers and Oresme numbers respectively, then we obtain the following property:

$$\|A \otimes B\|_2 \leq \Omega_6 \times \Omega_7$$

where Ω_6 and Ω_7 are as in Theorem 3.3,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 1$ and $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{2}$, respectively).

Proof. For (a), (b) and (c) see equation (2.10) and Theorem 3.3. $\square$

CHAPTER 4

TOEPLITZ MATRICES WITH GENERALIZED MERSENNE NUMBERS

In this chapter, we investigate Toeplitz matrices with Mersenne numbers components. First, the Toeplitz matrices whose elements are the Mersenne numbers are created and then the Frobenius, row and column norms of these matrices are found. Furthermore lower and upper bounds are obtained for the spectral norms of these matrices. In addition, the upper bounds for the Frobenius (Euclidian) and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the Mersenne numbers are calculated. We use Soykan [8, 9].

4.1 THE TOEPLITZ MATRICES WITH GENERALIZED MERSENNE NUMBERS COMPONENTS

In this section, we define the Toeplitz matrices whose elementry Mersenne and Mersenne-Lucas, respectively and we use the notation $A = T(W_0, W_1, \cdots, W_{n-1})$ for the Toeplitz matrix with generalized Mersenne numbers, i.e.,

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}. \tag{4.1}$$

For exclusive cases, we get

$$A = \begin{pmatrix} M_0 & M_{-1} & M_{-2} & \cdots & M_{1-n} \\ M_1 & M_0 & M_{-1} & \cdots & M_{2-n} \\ M_2 & M_1 & M_0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-2} & M_{n-3} & \cdots & M_0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{3}{4} & \cdots & M_{1-n} \\ 1 & 0 & -\frac{1}{2} & \cdots & M_{2-n} \\ 3 & 1 & 0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-2} & M_{n-3} & \cdots & 0 \end{pmatrix}$$

(4.2)

for the Toeplitz matrix $A = T(M_0, M_1, \cdots, M_{n-1})$ with Mersenne numbers and

$$A = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ H_1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ H_2 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{4} & \cdots & H_{1-n} \\ 3 & 2 & \frac{3}{2} & \cdots & H_{2-n} \\ 5 & 3 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix} \tag{4.3}$$

for the Toeplitz matrix $A = T(H_0, H_1, \cdots, H_{n-1})$ with Mersenne- Lucas numbers.

4.2 THE LARGEST ABSOLUTE COLUMN SUM (1-norm) AND THE LARGEST ABSOLUTE ROW SUM (∞-norm) OF THE TOEPLITZ MATRICES WITH GENERALIZED MERSENNE NUMBERS

In this section, we calculate the largest absolute column sum and the largest absolute row sum of the Toeplitz matrices with generalized Mersenne numbers and obtain their properties.

In the following theorem, we present the norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A.

Theorem 4.1 *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix with generalized Mersenne numbers then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are*

$$\|A\|_1 = \|A\|_\infty = \begin{cases} nW_n - (2n+2)W_{n-1} + 3W_0 - W_1 & , \text{ if } |W_k| \geq |W_{-k}| \text{ and } W_k \leq 0 \\ -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0) & , \text{ if } |W_k| \geq |W_{-k}| \text{ and } W_k \geq 0 \end{cases}$$

where $k = i - j : i,\ j = 0, 1, \cdots, n-1;\ (k \in N, -k \in N^-)$.

Proof. Acknowledge $A = T(W_0, W_1, \cdots, W_{n-1})$ which is given as in (4.1). By the definitions of $1-norm$ and $\infty - norm$, and equation (2.3), equation (2.4) and equation(1.13), we conclude that

(i) If $|W_k| \geq |W_{-k}|\,, k \in N$ and $W_k \leq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n}|a_{i1}| \\
&= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| = \sum_{k=0}^{n-1}|W_k| \\
&= -(\sum_{k=0}^{n-1}W_k) = -(\sum_{k=0}^{n}W_k - W_n) = -\sum_{k=0}^{n}W_k + W_n \\
&= -((1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0)) + W_n \\
&= (n-1)W_n - (2n+2)W_{n-1} + (3W_0 - W_1) + W_n \\
&= nW_n - (2n+2)W_{n-1} + 3W_0 - W_1
\end{aligned}$$

and if $|W_k| \geq |W_{-k}|\,, k \in N$ and $W_k \geq 0, k \in N$, then we obtain

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n}|a_{i1}| \\
&= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| = \sum_{k=0}^{n-1}|W_k| \\
&= \sum_{k=0}^{n-1}W_k = \sum_{k=0}^{n}W_k - W_n \\
&= (1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0) - W_n \\
&= -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0).
\end{aligned}$$

(ii) If $|W_k| \geq |W_{-k}|\,, k \in N$ and $W_k \leq 0, k \in N$, then it follows that

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n}|a_{nj}| \\
&= |a_{n1}| + |a_{n2}| + |a_{n3}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1}|W_k| \\
&= -(\sum_{k=0}^{n}W_k - W_n) = -\sum_{k=0}^{n}W_k + W_n \\
&= -((1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0)) + W_n \\
&= nW_n - (2n+2)W_{n-1} + 3W_0 - W_1
\end{aligned}$$

and if $|W_{-k}| \geq |W_k|, k \in N$ and $W_k \geq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{nj}| \\
&= |a_{n1}| + |a_{n2}| + |a_{n3}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1} |W_k| = \sum_{k=0}^{n} W_k - W_n \\
&= (1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0) - W_n \\
&= -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0).
\end{aligned}$$

Thus, the proof is completed. □

Remark 4.1 *In the statement of the Theorem 4.1 the condion on W_n, W_{-n}, $n \in N$ is given to calculade $\|A\|_1$ and $\|A\|_\infty$ norms of Mersenne, Mersenne-Lucas numbers. The other cases can be handled similarly.*

From the last Theorem 4.1, we have the following corollary which gives norm value of $\|A\|_1$ and $\|A\|_\infty$ of A with Mersenne numbers and Mersenne-Lucas numbers, respectively, (set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Corollary 4.1

(a) For $A = T(M_0, M_1, \cdots, M_{n-1})$, the vaules of norms of Toeplitz matrices with Mersenne numbers hold the following property:

$$\|A\|_1 = \|A\|_\infty = -nM_n + (2n+2)M_{n-1} + 1.$$

(b) For $A = T(H_0, H_1, \cdots, H_{n-1})$, the values of norms of Toeplitz matrices with Mersenne-Lucas numbers hold the following property:

$$\|A\|_1 = \|A\|_\infty = -nH_n + (2n+2)H_{n-1} - 3.$$

4.3 THE FROBENIOUS NORM OF THE TOEPLITZ MATRICES WITH GENERALIZED MERSENNE NUMBERS

In this section, we calculate the Frobenious norm of the toeplitz matrices with generalized Mersenne numbers and obtain their results.

Next theorem, present the Frobenious (Euclidian) norm of a Toeplitz matrix A.

Theorem 4.2 *Consider* $A = T(W_0, W_1, \cdots, W_{n-1})$ *which is given in (4.1), then the Frobenious (Euclidian) norm of matrix* A *is*

$$\|A\|_F = \sqrt{\Lambda_1}$$

where

$$\Lambda_1 = (\tfrac{-3n^2+5n}{18})W_{-n+1}^2 + (\tfrac{6n^2-10n+4}{9})W_{-n}^2 - (\tfrac{144n+271}{18})W_0^2 + (6n+11)W_0W_1 - \tfrac{(6n+11)}{6}W_1^2 - (\tfrac{3n^2+23n+34}{18})W_n^2 + \tfrac{6n^2+46n+84}{9}W_{n-1}^2 - 2W_{-1}^2$$
$$+\tfrac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) - 18n + 14)(W_1 - 2W_0)(W_1 - W_0).$$

Proof. The matrix A is of the form

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad +(n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \end{aligned}$$

and so

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_i^2) + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_{-i}^2) - 2(n-1)W_0^2 \\ &= (n-2n+2)W_0^2 + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_i^2) + \sum_{k=1}^{n-1}(\sum_{i=0}^{k} W_{-i}^2) \\ &= (\frac{-3n^2+5n}{18})W_{-n+1}^2 + (\frac{6n^2-10n+4}{9})W_{-n}^2 - (\frac{144n+271}{18})W_0^2 \\ &\quad +(6n+11)W_0W_1 - \frac{(6n+11)}{6}W_1^2 - (\frac{3n^2+23n+34}{18})W_n^2 + \frac{6n^2+46n+84}{9}W_{n-1}^2 \\ &\quad -2W_{-1}^2 + \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) \\ &\quad +14(2^n) - 18n + 14)(W_1 - 2W_0)(W_1 - W_0) \end{aligned}$$

Moreover, we use equation 1.14 and equation 1.15 in Corollary 1.2.

Then, we obtain

$$(\tfrac{-3n^2+5n}{18})W_{-n+1}^2 + (\tfrac{6n^2-10n+4}{9})W_{-n}^2 - \tfrac{144n+271}{18}W_0^2 + (6n+11)W_0W_1 - \tfrac{(6n+11)}{6}W_1^2 - (\tfrac{3n^2+23n+34}{18})W_n^2 + \tfrac{6n^2+46n+84}{9}W_{n-1}^2$$

$-2W_{-1}^2 + \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) - 18n + 14)(W_1 - 2W_0)(W_1 - W_0)$.

Therefore, we get

$\|A\|_F^2 = (\frac{-3n^2+5n}{18})W_{-n+1}^2 + (\frac{6n^2-10n+4}{9})W_{-n}^2 - (\frac{144n+271}{18})W_0^2 + (6n+11)W_0W_1 - \frac{(6n+11)}{6}W_1^2 - (\frac{3n^2+23n+34}{18})W_n^2 + \frac{6n^2+46n+84}{9}W_{n-1}^2$

$-2W_{-1}^2 + \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) - 18n + 14)(W_1 - 2W_0)(W_1 - W_0)$.

This complates the proof. □

From the last Theorem 4.2, we have the following corollary which gives Frobenius norm formulas of Mersenne numbers and Mersenne-Lucas numbers, respectively, (take $W_n = M_n$ with $M_0 = 0, M_1 = 1, M_{-1} = -\frac{1}{2}$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3, H_{-1} = \frac{3}{2}$, respectively).

Corollary 4.2 *For $n \geq 0$, Toeplitz matrices with the Mersenne and Mersenne-Lucas numbers, respectively have the following properties:*

(a) $\|A\|_F = \sqrt{\Lambda_2}$

where A is given as in (4.2)

$\Lambda_2 = (\frac{-3n^2+5n}{18})M_{-n+1}^2 + (\frac{6n^2-10n+4}{9})M_{-n}^2 - (\frac{3n^2+23n+34}{18})M_n^2 + \frac{6n^2+46n+84}{9}M_{n-1}^2 + \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) - 27n - 7)$.

(b) $\|A\|_F = \sqrt{\Lambda_3}$

where A is given as in (4.3)

$\Lambda_3 = (\frac{-3n^2+5n}{18})H_{-n+1}^2 + (\frac{6n^2-10n+4}{9})H_{-n}^2 - (\frac{3n^2+23n+34}{18})H_n^2 + (\frac{6n^2+46n+84}{9})H_{n-1}^2 - \frac{1}{9}(6n^2 2^{-n} + 3n^2 2^n - 10n2^{-n} + 23n2^n - 28(2^{-n}) + 14(2^n) + 27n + 151)$.

4.4 THE LOWER AND UPPER BOUNDS OF THE SPECTRAL NORMS OF THE TOEPLITZ MATRICES WITH GENERALIZED MERSENNE NUMBERS

In this section, we investigate the lower and upper bounds of the spectral norms of the Toeplitz matrices with generalized Mersenne numbers

In the following theorem, we find the lower and upper bounds for the spectral norm of the Toeplitz matrices with the Mersenne numbers, Mersenne-Lucas numbers, respectively, (take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Theorem 4.3

(a) Consider $A = T(M_0, M_1, \cdots, M_{n-1})$ which is given as in (4.2). Let

$$C = \begin{pmatrix} 1 & M_{-1} & M_{-2} & \cdots & M_{1-n} \\ 1 & M_0 & M_{-1} & \cdots & M_{2-n} \\ 1 & M_1 & M_0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & M_{n-2} & M_{n-3} & \cdots & M_0 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{4} & \cdots & M_{1-n} \\ 1 & 0 & -\frac{1}{2} & \cdots & M_{2-n} \\ 1 & 1 & 0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & M_{n-2} & M_{n-3} & \cdots & 0 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} M_0 & 1 & 1 & \cdots & 1 \\ M_1 & 1 & 1 & \cdots & 1 \\ M_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamard Product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Lambda_2}$$

where Λ_2 is as in Corollary 4.2.

(ii)

$$\|A\|_2 \leq \Lambda_4$$

where

$$\begin{aligned} \Lambda_4 &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 + 2^{n+1}(n+1) - 2))^{\frac{1}{2}} \\ &\quad \times (\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 + 2^{n+1}(n+1) - 5))^{\frac{1}{2}}. \end{aligned}$$

(b) Consider $A = T(H_0, H_1, \cdots, H_{n-1})$ which is given as in (4.3). Let

$$C = \begin{pmatrix} 1 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ 1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ 1 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{4} & \cdots & H_{1-n} \\ 1 & 2 & \frac{3}{2} & \cdots & H_{2-n} \\ 1 & 3 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} H_0 & 1 & 1 & \cdots & 1 \\ H_1 & 1 & 1 & \cdots & 1 \\ H_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ 5 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

such that $A = C \circ D$ (Hadamard Product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Lambda_3}$$

where Λ_3 is as in Corollary 4.2

(ii)

$$\|A\|_2 \leq \Lambda_5$$

where

$$\begin{aligned} \Lambda_5 &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \\ &\quad \times (\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}}. \end{aligned}$$

Proof.

(a) (i) We use equation (2.2).

(ii) We get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-2} M_k^2 + 1)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n} M_k^2 - M_n^2 - M_{n-1}^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 - 25M_0^2 + 18M_0M_1 \\ &\quad -3M_1^2 + 2(M_1 - 2M_0)(M_1 - M_0)(2^n(n+1) - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 + 2^{n+1}(n+1) - 2))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-1} M_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} M_k^2 - M_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 - 25M_0^2 + 18M_0M_1 \\ &\quad -3M_1^2 + 2(M_1 - 2M_0)(M_1 - M_0)(2^n(n+1) - 1)))^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 + 2^{n+1}(n+1) - 5))^{\frac{1}{2}} \end{aligned}$$

so, from inequality (2.7),

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Lambda_4 \\ &= (\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 \\ &\quad +2^{n+1}(n+1) - 2))^{\frac{1}{2}} \times (\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 \\ &\quad +2^{n+1}(n+1) - 5))^{\frac{1}{2}}. \end{aligned}$$

(b) (i) We use equation (2.2).

(ii) By definition, we get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-2} H_k^2 + 1)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n} H_k^2 - H_n^2 - H_{n-1}^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 25H_0^2 + 18H_0H_1 - 3H_1^2 \\ &\quad +2(H_1 - 2H_0)(H_1 - H_0)(2^n(n+1) - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} = (\sum_{k=0}^{n-1} H_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} H_k^2 - H_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 25H_0^2 + 18H_0H_1 - 3H_1^2 \\ &\quad +2(H_1-2H_0)(H_1-H_0)(2^n(n+1)-1)))^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}} \end{aligned}$$

so, from inequality (2.7)

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Lambda_5 \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \\ &\quad \times(\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}}. \end{aligned}$$

This complates the proof. □

4.5 THE KRONECKER AND HADAMARD PRODUCT OF THE TOEPLITZ MATRICES WITH GENERALIZED MERSENNE NUMBERS

In this section, we obtain some results for the Kronecker and Hadamard product of the Toeplitz matrices with generalized Mersenne numbers.

From the equation (2.10) and Corollary 4.2, we have the following corollary which gives the Frobenius norms of the Kronecker products of the Toeplitz matrices with special cases of generalized Mersenne numbers.

Corollary 4.3 *Suppose that* $A = T(M_0, M_1, \cdots, M_{n-1})$ *and* $B = T(H_0, H_1, \cdots, H_{n-1})$ *be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers respectively, then we have the following property:.*

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Lambda_2}\sqrt{\Lambda_3} \end{aligned}$$

where Λ_2 *and* Λ_3 *are as in Corollary 4.2 (a) and (b),*

(set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Proof. It can be easily seen from equation (2.10) and Theorem 4.2 and Corollary 4.2. □

From the above inequality (2.9) and Theorem 4.2 and Corollary 4.2, we have the following result, which gives an upper bound for the Frobenius norm of Hadamard products of Toeplitz matrices with different Mersenne sequences.

Corollary 4.4 *Assume that* $A = T(M_0, M_1, \cdots, M_{n-1})$ *and* $B = T(H_0, H_1, \cdots, H_{n-1})$ *be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers, respectively, then we have the following property:*

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Lambda_2}\sqrt{\Lambda_3} \end{aligned}$$

where Λ_2 *and* Λ_3 *are as in Corollary 4.2 (a) and (b),*

(set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).
Proof. For the proof see inequality (2.9) and Theorem 4.2. □
In the last inequality (2.8) and Theorem 4.3, we have the following corollary, which gives an upper bound for the spectral norm of Hadamard products of Toeplitz matrices with different Mersenne sequences.

Corollary 4.5 *Given* $A = T(M_0, M_1, \cdots, M_{n-1})$ *and* $B = T(H_0, H_1, \cdots, H_{n-1})$ *be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers respectively, then we have the following property:*

$$\|A \circ B\|_2 \leq \Lambda_4 \times \Lambda_5$$

where Λ_4 *and* Λ_5 *are as in Theorem 4.3,*

(take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).
Proof. See inequality (2.8) and Theorem 4.3. □
From the related equation (2.10) and Theorem 4.3, we have the following corollary which gives an upper bound for the spectral norm of Kronocker products of Toeplitz matrices with different Mersenne sequences.

Corollary 4.6 *Let* $A = T(M_0, M_1, \cdots, M_{n-1})$ *and* $B = T(H_0, H_1, \cdots, H_{n-1})$ *be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers, respectively, then we have the following property:*

$$\|A \otimes B\|_2 \leq \Lambda_4 \times \Lambda_5$$

where Λ_4 *and* Λ_5 *are as in Theorem 4.3,*

(set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Proof. See equation (2.10) and Theorem 4.3. $\square$

CHAPTER 5

TOEPLITZ MATRICES WITH GENERALIZED BALANCING NUMBERS

In this chapter we investigate, Toeplitz matrices with balancing numbers and their properties. First, the Toeplitz matrices whose elements are the balancing numbers are created and then the euclidian, row and column norms of these matrices are found. Furthermore lower and upper bounds are obtained for the spectral norms of these matrices. In addition, the upper bounds for the Frobenius (Euclidian) and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with the balancing numbers are calculated. We use Soykan [10, 11].

5.1 THE TOEPLITZ MATRICES WITH GENERALIZED BALANCING NUMBERS COMPONENTS

In this section, we define the Toeplitz matrices which elements are balancing, modified Lucas-balancing, Lucas-balancing numbers, respectively and we use the notation $A = T(W_0, W_1, \cdots, W_{n-1})$ for the Toeplitz matrix with generalized balancing numbers, i.e.,

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}. \tag{5.1}$$

For special cases, we get

$$A = \begin{pmatrix} B_0 & B_{-1} & B_{-2} & \cdots & B_{1-n} \\ B_1 & B_0 & B_{-1} & \cdots & B_{2-n} \\ B_2 & B_1 & B_0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & B_{n-2} & B_{n-3} & \cdots & B_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -6 & \cdots & B_{1-n} \\ 1 & 0 & -1 & \cdots & B_{2-n} \\ 6 & 1 & 0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & B_{n-2} & B_{n-3} & \cdots & 0 \end{pmatrix} \tag{5.2}$$

for the Toeplitz matrix $A = T(B_0, B_1, \cdots, B_{n-1})$ with balancing numbers and

$$A = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ H_1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ H_2 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 34 & \cdots & H_{1-n} \\ 6 & 2 & 6 & \cdots & H_{2-n} \\ 34 & 6 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix} \tag{5.3}$$

for the Toeplitz matrix $A = T(H_0, H_1, \cdots, H_{n-1})$ with modified Lucas-balancing numbers and

$$A = \begin{pmatrix} C_0 & C_{-1} & C_{-2} & \cdots & C_{1-n} \\ C_1 & C_0 & C_{-1} & \cdots & C_{2-n} \\ C_2 & C_1 & C_0 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & C_{n-3} & \cdots & C_0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 17 & \cdots & C_{1-n} \\ 3 & 1 & 3 & \cdots & C_{2-n} \\ 17 & 3 & 1 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & C_{n-2} & C_{n-3} & \cdots & 1 \end{pmatrix} \tag{5.4}$$

for the Toeplitz matrix $A = T(C_0, C_1, \cdots, C_{n-1})$ with Lucas-balancing numbers.

5.2 THE LARGEST ABSOLUTE COLUMN SUM (1-norm) AND THE LARGEST ABSOLUTE ROW SUM (∞-norm) OF THE TOEPLITZ MATRICES WITH GENERALIZED BALANCING NUMBERS

In this section, we calculate the largest absolute column sum and the largest absolute row sum of the Toeplitz matrices with generalized balancing numbers and obtain their properties.

In the following theorem, we present the norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A.

Theorem 5.1 *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix with generalized balancing numbers then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are*

$$\|A\|_1 = \|A\|_\infty = \begin{cases} \frac{1}{4}(-W_n + W_{n-1} - 5W_0 + W_1) & , \text{ if } |W_k| = |W_{-k}| \text{ and } W_k \leq 0 \\ \frac{1}{4}(W_n - W_{n-1} + 5W_0 - W_1) & , \text{ if } |W_k| = |W_{-k}| \text{ and } W_k \geq 0 \end{cases}$$

where $k = i - j : i, j = 0, 1, \cdots, n-1$; $(k \in N, -k \in N^-)$.

Proof. Consider $A = T(W_0, W_1, \cdots, W_{n-1})$ which is given as in (5.1). By the definitions of $1-norm$ and $\infty-norm$, and equation (2.3) and equation (2.4) and equation (1.21), we conclude that

(i) If $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \leq 0$, $k \in N$, then we get

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} \\
&= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| \quad = \sum_{i=1}^{n} |a_{i1}| \\
&= \sum_{k=0}^{n-1} |W_k| = -\sum_{k=0}^{n-1} W_k = -\sum_{k=0}^{n} W_k + W_n \\
&= \frac{-W_n + W_{n-1} - 5W_0 + W_1}{4}
\end{aligned}$$

and if $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \geq 0$, $k \in N$, then we obtain

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} \\
&= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| \quad = \sum_{i=1}^{n} |a_{i1}| \\
&= \sum_{k=0}^{n-1} |W_k| = \sum_{k=0}^{n-1} W_k = \sum_{k=0}^{n} W_k - W_n \\
&= \frac{W_n - W_{n-1} + 5W_0 - W_1}{4}.
\end{aligned}$$

(ii) If $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \leq 0$, $k \in N$, then we get

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} \\
&= |a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| = \sum_{j=1}^{n} |a_{nj}| \\
&= \sum_{k=0}^{n-1} |W_k| = -\sum_{k=0}^{n-1} W_k = -\sum_{k=0}^{n} W_k + W_n \\
&= \frac{-W_n + W_{n-1} - 5W_0 + W_1}{4}
\end{aligned}$$

and if $|W_k| = |W_{-k}|$, $k \in N$ and $W_k \geq 0$, $k \in N$, then we obtain

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} \\
&= |a_{n1}| + |a_{n2}| + \cdots + |a_{nn}| = \sum_{j=1}^{n} |a_{nj}| \\
&= \sum_{k=0}^{n-1} |W_k| = \sum_{k=0}^{n-1} W_k = \sum_{k=0}^{n} W_k - W_n \\
&= \frac{W_n - W_{n-1} + 5W_0 - W_1}{4}.
\end{aligned}$$

Thus, the proof is completed. □

Remark 5.1 *In the statement of the Theorem 5.1 the condion on W_n, W_{-n}, $n \in N$ is given to calculade $\|.\|_1$ and $\|.\|_\infty$ norms of balancing, modified Lucas-balancing, Lucas-balancing numbers. The other cases can be handled similarly.*

From the last Theorem 5.1, we have the following corollary which present norm values of $\|A\|_1, \|A\|_\infty$ of A with balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, (set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Corollary 5.1

(a) For $A = T(B_0, B_1, \cdots, B_{n-1})$, the vaules of norms of Toeplitz matrices with balancing numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = \frac{B_n - B_{n-1} - 1}{4}.$$

(b) For $A = T(H_0, H_1, \cdots, H_{n-1})$, the values of norms of Toeplitz matrices with modified Lucas-balancing numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = \frac{H_n - H_{n-1} + 4}{4}.$$

(c) For $A = T(C_0, C_1, \cdots, C_{n-1})$, the values of norms of Toeplitz matrices with Lucas-balancing numbers have the following property:

$$\|A\|_1 = \|A\|_\infty = \frac{C_n - C_{n-1} + 2}{4}.$$

5.3 THE FROBENIOUS NORM OF THE TOEPLITZ MATRICES WITH GENERALIZED BALANCING NUMBERS

In this section, we calculate the Frobenious norm of the Toeplitz matrices with generalized balancing numbers and obtain the following results.

Next theorem present the Frobenious (Euclidian) norm of a Toeplitz matrix A.

Theorem 5.2 *Let* $A = T(W_0, W_1, \cdots, W_{n-1})$ *be a Toeplitz matrix with generalized balancing numbers components, then the Frobenious (Euclidian) norm of matrix* A *is*

$$\|A\|_F = \sqrt{\Upsilon_1}$$

where $\Upsilon_1 = \frac{1}{32}W_n^2 + \frac{1}{32}W_{-n}^2 - \frac{(34n+2)}{32}W_0^2 + \frac{n}{32}W_1^2 + \frac{n}{32}(W_1 - 6W_0)^2 - \frac{2n(n+1)}{32}(W_1^2 + W_0^2 - 6W_1W_0)$.

Proof. The matrix A is of the form

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad + (n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \end{aligned}$$

we obtain it follows that

$$\begin{aligned}
\|A\|_F^2 &= nW_0^2+(n-1)W_{-1}^2+(n-2)W_{-2}^2+(n-3)W_{-3}^2+\cdots+W_{1-n}^2 \\
&\quad +(n-1)W_1^2+(n-2)W_2^2+(n-3)W_3^2+\cdots+W_{n-1}^2 \\
&= nW_0^2+\sum_{k=1}^{n-1}\sum_{i=1}^{k}W_i^2+\sum_{k=1}^{n-1}\sum_{i=1}^{k}W_i^2 \\
&= (2-n)W_0^2+\sum_{k=1}^{n-1}(\frac{1}{32}(33W_n^2-W_{n-1}^2-W_0^2+(W_1-6W_0)^2 \\
&\quad -2n(W_1^2+W_0^2-6W_1W_0))) \\
&\quad +\sum_{k=1}^{n-1}(-\frac{1}{32}(W_{-n+1}^2-33W_{-n}^2+W_0^2-W_1^2+2n(W_1^2+W_0^2-6W_1W_0))) \\
&= (2-n)W_0^2+\frac{1}{32}(W_n^2-(n+33)W_0^2+n(W_1-6W_0)^2 \\
&\quad -n(n+1)(W_1^2+W_0^2-6W_1W_0))+\frac{1}{32}W_{-n}^2-\frac{(n+33)}{32}W_0^2+\frac{n}{32}W_1^2 \\
&\quad -\frac{n(n+1)}{32}(W_1^2+W_0^2-6W_1W_0) \\
&= \frac{1}{32}W_n^2+\frac{1}{32}W_{-n}^2-\frac{(34n+2)}{32}W_0^2+\frac{n}{32}W_1^2+\frac{n}{32}(W_1-6W_0)^2 \\
&\quad -\frac{2n(n+1)}{32}(W_1^2+W_0^2-6W_1W_0)
\end{aligned}$$

Moreover, we use equation (1.23) and equation (1.24) in Corollary 1.3.

Therefore, we get

$\|A\|_F^2=\frac{1}{32}W_n^2+\frac{1}{32}W_{-n}^2-\frac{(34n+2)}{32}W_0^2+\frac{n}{32}W_1^2+\frac{n}{32}(W_1-6W_0)^2-\frac{2n(n+1)}{32}(W_1^2+W_0^2-6W_1W_0)$.

This complates the proof. □

From the last Theorem 5.2, we have the following corollary which gives Frobenius norm formulas of balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, (take $W_n=B_n$ with $B_0=0, B_1=1$ and $W_n=H_n$ with $H_0=2, H_1=6$ and $W_n=C_n$ with $C_0=1, C_1=3$, respectively).

Corollary 5.2 *For $n\geq 0$, Toeplitz matrices with the balancing, modified Lucas-balancing and Lucas-balancing numbers, recpectively, have the following properties:*

(a) $\|A\|_F=\sqrt{\Upsilon_2}$

where A is given as in (5.2)

$\Upsilon_2=\frac{1}{32}(B_n^2+B_{-n}^2-2n^2)$.

(b) $\|A\|_F = \sqrt{\Upsilon_3}$

where A is given as in (5.3)

$$\Upsilon_3 = \frac{1}{32}(H_n^2 + H_{-n}^2 + (64n^2 - 8)).$$

(c) $\|A\|_F = \sqrt{\Upsilon_4}$

where A is given as in (5.4)

$$\Upsilon_4 = \frac{1}{32}(C_n^2 + C_{-n}^2 + 16n^2 - 2).$$

5.4 THE LOWER, UPPER BOUNDS OF THE SPECTRAL NORMS OF THE TOEPLITZ MATRICES WITH GENERALIZED BALANCING NUMBERS

In this section, we investigate the lower and upper bounds of the spectral norms of the Toeplitz matrices with generalized balancing numbers

In the following theorem, we find the lower and upper bounds for the spectral norms of the matrices with the balancing numbers, modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, (take $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Theorem 5.3

(a) Consider $A = T(B_0, B_1, \cdots, B_{n-1})$ which is given as in (5.2). Let

$$C = \begin{pmatrix} 1 & B_{-1} & B_{-2} & \cdots & B_{1-n} \\ 1 & B_0 & B_{-1} & \cdots & B_{2-n} \\ 1 & B_1 & B_0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & B_{n-2} & B_{n-3} & \cdots & B_0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -6 & \cdots & B_{1-n} \\ 1 & 0 & -1 & \cdots & B_{2-n} \\ 1 & 1 & 0 & \cdots & B_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & B_{n-2} & B_{n-3} & \cdots & 0 \end{pmatrix}$$

and

$$D = \begin{pmatrix} B_0 & 1 & 1 & \cdots & 1 \\ B_1 & 1 & 1 & \cdots & 1 \\ B_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 6 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Upsilon_2}$$

where Υ_2 is as in Corollary 5.2.

(ii)

$$\|A\|_2 \leq \Upsilon_5$$

where

$$\Upsilon_5 = (\frac{1}{32}(B_n^2 - 33B_{n-1}^2 - 2n + 33))^{\frac{1}{2}} \times (\frac{1}{32}(B_n^2 - B_{n-1}^2 + 1 - 2n))^{\frac{1}{2}}.$$

(b) Consider $A = T(H_0, H_1, \cdots, H_{n-1})$ which is given as in (5.3). Let

$$C = \begin{pmatrix} 1 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ 1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ 1 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 34 & \cdots & H_{1-n} \\ 1 & 2 & 6 & \cdots & H_{2-n} \\ 1 & 6 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix}$$

and

$$D = \begin{pmatrix} H_0 & 1 & 1 & \cdots & 1 \\ H_1 & 1 & 1 & \cdots & 1 \\ H_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 6 & 1 & 1 & \cdots & 1 \\ 34 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Upsilon_3}$$

where Υ_3 is as in Corollary 5.2.

(ii)

$$\|A\|_2 \leq \Upsilon_6$$

where

$$\Upsilon_6 = (2 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 + 64n))^{\frac{1}{2}} \times (\frac{1}{32}(H_n^2 - H_{n-1}^2 + 32 + 64n))^{\frac{1}{2}}.$$

(c) Consider $A = T(C_0, C_1, \cdots, C_{n-1})$ which is given as in (5.4). Let

$$C = \begin{pmatrix} 1 & C_{-1} & C_{-2} & \cdots & C_{1-n} \\ 1 & C_0 & C_{-1} & \cdots & C_{2-n} \\ 1 & C_1 & C_0 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & C_{n-2} & C_{n-3} & \cdots & C_0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 17 & \cdots & C_{1-n} \\ 1 & 1 & 3 & \cdots & C_{2-n} \\ 1 & 3 & 1 & \cdots & C_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & C_{n-2} & C_{n-3} & \cdots & 1 \end{pmatrix}$$

and

$$D = \begin{pmatrix} C_0 & 1 & 1 & \cdots & 1 \\ C_1 & 1 & 1 & \cdots & 1 \\ C_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ 17 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamart product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Upsilon_4}$$

where Υ_4 is as in Corollary 5.2.

(ii)

$$\|A\|_2 \leq \Upsilon_7$$

where

$$\Upsilon_7 = (\frac{1}{32}(C_n^2 - 33C_{n-1}^2 + 40 + 16n))^{\frac{1}{2}} \times (\frac{1}{32}(C_n^2 - C_{n-1}^2 + 8 + 16n))^{\frac{1}{2}}.$$

Proof.

(a) (i) We use equation (2.2).

(ii) We get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n-2} B_k^2)^{\frac{1}{2}} \\ &= (1 + \frac{1}{32}(B_n^2 - 33B_{n-1}^2 - B_0^2 + (B_1 - 6B_0)^2 \\ &\quad -2n(B_1^2 + B_0^2 - 6B_1B_0)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(B_n^2 - 33B_{n-1}^2 - 2n + 33))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-1} B_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} B_k^2 - B_n^2)^{\frac{1}{2}} \\ &= ((\frac{1}{32}(33B_n^2 - B_{n-1}^2 - B_0^2 + (B_1 - 6B_0)^2 \\ &\quad -2n(B_1^2 + B_0^2 - 6B_1B_0)) - B_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{32}(B_n^2 - B_{n-1}^2 - B_0^2 + (B_1 - 6B_0)^2 \\ &\quad -2n(B_1^2 + B_0^2 - 6B_1B_0)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(B_n^2 - B_{n-1}^2 + 1 - 2n))^{\frac{1}{2}} \end{aligned}$$

so, from inequality (2.7),

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Upsilon_5 \\ &= (\frac{1}{32}(B_n^2 - 33B_{n-1}^2 - 2n + 33))^{\frac{1}{2}} \times (\frac{1}{32}(B_n^2 - B_{n-1}^2 + 1 - 2n))^{\frac{1}{2}}. \end{aligned}$$

(b) **(i)** We use equation (2.2).

(ii) By definition, we get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n-2} H_k^2)^{\frac{1}{2}} \\ &= (1 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 - H_0^2 + (H_1 - 6H_0)^2 \\ &\quad -2n(H_1^2 + H_0^2 - 6H_1H_0)))^{\frac{1}{2}} \\ &= (2 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 + 64n))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} & c_1(D) \\ &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} = (\sum_{k=0}^{n-1} H_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} H_k^2 - H_n^2)^{\frac{1}{2}} \\ &= ((\frac{1}{32}(33H_n^2 - H_{n-1}^2 - H_0^2 + (H_1 - 6H_0)^2 \\ &\quad -2n(H_1^2 + H_0^2 - 6H_1H_0)) - H_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{32}(H_n^2 - H_{n-1}^2 - 4 + 36 - 2n(36 + 4 - 72)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(H_n^2 - H_{n-1}^2 + 32 + 64n))^{\frac{1}{2}}. \end{aligned}$$

so, from inequality (2.7),

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Upsilon_6 \\ &= (2 + \frac{1}{32}(H_n^2 - 33H_{n-1}^2 + 64n))^{\frac{1}{2}} \times (\frac{1}{32}(H_n^2 - H_{n-1}^2 + 32 + 64n))^{\frac{1}{2}}. \end{aligned}$$

(c) **(i)** We use equation (2.2).

(ii) By definition, we get

$$\begin{aligned} r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n-2} C_k^2)^{\frac{1}{2}} \\ &= (1 + \frac{1}{32}(C_n^2 - 33C_{n-1}^2 - C_0^2 + (C_1 - 6C_0)^2 \\ &\quad -2n(C_1^2 + C_0^2 - 6C_1C_0)))^{\frac{1}{2}} \\ &= (\frac{1}{32}(C_n^2 - 33C_{n-1}^2 + 40 + 16n))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} & c_1(D) \\ = & \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} = (\sum_{k=0}^{n-1} C_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} C_k^2 - C_n^2)^{\frac{1}{2}} \\ = & ((\frac{1}{32}(33C_n^2 - C_{n-1}^2 - C_0^2 + (C_1 - 6C_0)^2 \\ & -2n(C_1^2 + C_0^2 - 6C_1C_0)) - C_n^2)^{\frac{1}{2}} \\ = & (\frac{1}{32}(C_n^2 - C_{n-1}^2 - 1 + (3-6)^2 - 2n(9+1-18)))^{\frac{1}{2}} \\ = & (\frac{1}{32}(C_n^2 - C_{n-1}^2 + 8 + 16n))^{\frac{1}{2}}. \end{aligned}$$

so, by definition of Hadamard product and from inequality (2.7),

$$\begin{aligned} \|A\|_2 & \leq r_1(C)c_1(D) = \Upsilon_7 \\ & = (\frac{1}{32}(C_n^2 - 33C_{n-1}^2 + 40 + 16n))^{\frac{1}{2}} \times (\frac{1}{32}(C_n^2 - C_{n-1}^2 + 8 + 16n))^{\frac{1}{2}}. \end{aligned}$$

This complates the proof. □

5.5 THE KRONECKER AND HADAMARD PRODUCT OF THE TOEPLITZ MATRICES WITH GENERALIZED BALANCING NUMBERS

In this section, we obtain some results for the Kronecker and Hadamard product of the Toeplitz matrices with generalized balancing numbers.
From the equation (2.10) and Corollary 5.2, we have the following corollary which gives the Frobenius norms of the Kronecker products of the Toeplitz matrices with special cases of generalized balancing numbers.

Corollary 5.3

(a) Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers respectively, then we have the following property:

$$\begin{aligned} \|A \otimes B\|_F & = \|A\|_F \|B\|_F \\ & = \sqrt{\Upsilon_2}\sqrt{\Upsilon_3} \end{aligned}$$

where Υ_2 and Υ_3 are as in Corollary 5.2, (a) and (b),

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b) Suppose that $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers, respectively, then we obtain the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Upsilon_2}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_2 and Υ_4 are as in Corollary 5.2, (a) and (c),
(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c) Given $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, then we get the following property:

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Upsilon_3}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_3 and Υ_4 are as in Corollary 5.2, (b) and (c),
(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).
Proof. (a), (b) and (c) follows from equation (2.10) and Theorem 5.2 and Corollary 5.2. □

From the above inequality (2.9) and Theorem 5.2 and Corollary 5.2, we have the following result, which gives an upper bound for the Frobenius norm of Hadamard products of Toeplitz matrices by exclusive cases of generalized balancing numbers.

Corollary 5.4

(a) Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers, respectively, then

we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Upsilon_2}\sqrt{\Upsilon_3} \end{aligned}$$

where Υ_2 and Υ_3 are as in Corollary 5.2, (a) and (b),

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b) Suppose that $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and Lucas-balancing numbers, respectively, then we obtain the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Upsilon_2}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_2 and Υ_4 are as in Corollary 5.2, (a) and (c),

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c) Assume that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, then we have the following property:

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Upsilon_3}\sqrt{\Upsilon_4} \end{aligned}$$

where Υ_3 and Υ_4 are as in Corollary 5.2, (a) and (c),

(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).
In the last inequality (2.8) and Theorem 5.3, we have the following corollary, which gives an upper bound for the spectral norm of Hadamard products of Toeplitz matrices with special cases of generalized balancing numbers.

Corollary 5.5

(a) Given $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers respectively, then we have the following property:

$$\|A \circ B\|_2 \leq \Upsilon_5 \times \Upsilon_6$$

where Υ_5 and Υ_6 are as in Theorem 5.3,

(take $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b) Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and Lucas-balancing numbers, respectively, then we have the following property:

$$\|A \circ B\|_2 \leq \Upsilon_5 \times \Upsilon_7$$

where Υ_5 and Υ_7 are as in Theorem 5.3,

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c) Suppose that $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers respectively, then we get the following property:

$$\|A \circ B\|_2 \leq \Upsilon_6 \times \Upsilon_7$$

where Υ_6 and Υ_7 are as in Theorem 5.3,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Proof. For (a), (b) and (c) see equation (2.8) and Theorem 5.3. □

From the related equation (2.10) and Theorem 5.3, we have the following corollary which gives an upper bound for the spectral norm of Kronocker products of Toeplitz matrices with special cases of generalized balancing numbers.

(a) Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with balancing numbers and modified Lucas-balancing numbers, respectively, then we have the following property:

$\|A \otimes B\|_2 \leq \Upsilon_5 \times \Upsilon_6$

where Υ_5 and Υ_6 are as in Theorem 5.3,

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 6$, respectively).

(b) Let $A = T(B_0, B_1, \cdots, B_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with balancing numbers and Lucas-balancing numbers, respectively, then we get the following property:

$\|A \otimes B\|_2 \leq \Upsilon_5 \times \Upsilon_7$

where Υ_5 and Υ_7 are as in Theorem 5.3,

(set $W_n = B_n$ with $B_0 = 0, B_1 = 1$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

(c) Let $A = T(H_0, H_1, \cdots, H_{n-1})$ and $B = T(C_0, C_1, \cdots, C_{n-1})$ be Toeplitz matrices with modified Lucas-balancing numbers and Lucas-balancing numbers, respectively, then we obtain the following property:

$\|A \otimes B\|_2 \leq \Upsilon_6 \times \Upsilon_7$

where Υ_6 and Υ_7 are as in Theorem 5.3,

(set $W_n = H_n$ with $H_0 = 2, H_1 = 6$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3$, respectively).

Proof. For (a), (b) and (c) see equation (2.10) and Theorem 5.3. □

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