HAAR SYSTEMS ON LOCALLY COMPACT GROUPOIDS

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Haar systems are generalizations of Haar measures on groups to groupoids. Naturally, important research directions in the field try to generalize the well known existence of a Haar measure on a locally compact group to the existence of Haar systems in different groupoid settings. The groupoid case differs significantly from the group case, evidenced by a result of Deitmar [5], showing that non-existence is possible even for compact groupoids. We first present the classical theory of locally compact groups and Haar Measures on them. We motivate our investigation by constructing full C^*-algebras on locally compact groups, which uses the existence of Haar measures. Then, we cover the theory of locally compact groupoids and present Renault's [9] result that provides a complete characterization of the existence of Haar systems for the r-discrete locally compact groupoid setting, which are precisely the ones where the range map is a local homeomorphism. We present a question from Williams [13] that investigates if the open range map assumption is redundant for second countable, locally compact and transitive groupoids. Finally, we present Buneci's counter-example [1] that answers this question in the negative.

Keywords: Haar systems, locally compact groupoids, r-discrete groupoids.

ÖZET

LOKAL KOMPAKT GRUPOİDLERDE HAAR SİSTEMLERİ

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Matematik, Yüksek Lisans
Tez Danışmanı: Aurelian B.N. Gheondea Eladi
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Haar sistemleri, gruplar üzerine tanımlı Haar ölçülerinin grupoidlere genellemesidir. Doğal bir araştırma alanı ise, Haar ölçülerinin lokal kompakt gruplardaki varlık sonucunun lokal kompakt grupoidlere genellenmesi üzerinedir. Deitmar'ın [5] kompakt grupoidlerde bile Haar sistemi bulunmayabileceğini gösteren sonucunun da işaret ettiği gibi, grupoidler üzerindeki Haar sistemlerinin varlık sorusu, gruplardan oldukça farklıdır. Bu tezde öncelikle lokal kompakt grupların ve üzerine tanımlı Haar ölçülerinin teorisi sunulmuştur. Gruplardaki varlık sorusunun bir motivasyonu olarak, Haar ölçülerinin kullanıldığı tam C^*-cebiri inşa edilmiştir. Lokal kompakt grupoidlerin teorisi sunulmuş ve Renault'un [9] r-ayrık lokal kompakt grupoidler için varlık sorusunun denkliklerine dair sonucu incelenmiştir. Williams'a [13] ait olan, değer fonksiyonunun, geçişli ve diğer özelliklere sahip grupoidlerde açıklığına dair sorusu sunulmuştur. Son olarak, Buneci'nin [1] bu soruyu olumsuz olarak cevaplayan örneği incelenmiştir.

Anahtar sözcükler: Haar sistemleri, lokal kompakt grupoidler, r-ayrık grupoidler.

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Chapter 1

Introduction

A common problem in early 20^{th} century analysis was on the existence of invariant measures on topological groups. Development on this problem for compact groups by Alfred Haar [7] in 1933, the eponym of a left invariant Radon measure called the *Haar measure*, led to the rich theory of analysis on locally compact groups as we know it today.

The central result of the field, proved by André Weil [12], states that on every locally compact group G, there exists a Haar measure μ. A natural question to ask is can this result be preserved for group like objects. The main group like object of our study will be groupoids, and particularly the locally compact ones.

The chapter dedicated to locally compact groupoids will introduce a generalization of the Haar measure to groupoids, which is a family of left invariant measures with additional properties called a *Haar system*. Thus, the generalization of the existence question to groupoids will be, on a locally compact groupoid, does there always exist a Haar system? The answer is negative in general, due to a counter-example by Deitmar [5], see Section 3.3.

The first half of the second reviews the necessary background on the theory of locally compact groups, to prove the existence of Haar measures in the locally compact case. Assuming that a Haar measure μ on a locally compact group G

exists, invariant properties of the Haar integral will be shown. Afterwards, the main effort will be spent on building a positive and left invariant linear functional on $\mathcal{C}_c^+(G)$, the space of all nontrivial positive continuous functions with compact support in G. The main existence theorem will have the Riesz-Markov Theorem as its engine which will extend the mentioned functional to the Haar integral on $\mathcal{C}_c(G)$.

The second half of Chapter 2 will cover the theory of the *full C^*-algebra* of a locally compact group, which requires the existence of a Haar measure. For this, *Universal C^*-algebras* will be presented, whose C^*-norm is obtained by lifting a C^*-seminorm on a $*$-algebra to its natural quotient space. Furthermore, an explicit example of a Universal C^*-algebra with historical importance called the *Cuntz algebra* will be presented, which is generated by a finite family of isometries on a separable Hilbert space.

Chapter 3 first introduces topological groupoids and defines Haar systems on them. However, the theory of invariant system of measures on locally compact groupoids is much more complicated than the group case as evidenced by Deitmar's counter-example [5], since it shows that even for a compact groupoid there might not be a Haar system associated to it. Furthermore, the importance of the existence of a Haar system is further supported by the need for it to construct a full C^*-algebra on a locally compact groupoid.

Groupoids have two natural maps defined on them, called the *range* and *domain* maps denoted by r and d respectively. Groupoids with an open range map, also called *r-discrete* groupoids, have certain convenient properties that allow characterising the existence of a Haar system on the groupoid easier if the groupoid is also locally compact. As developed by Renault [9] in his foundational lecture notes, for a locally compact groupoid, a purely topological characterisation of being r-discrete and having a Haar system is presented. At the end, we present a conjecture on existence with quite general assumptions due to Deitmar [5], which asks if a second countable, locally compact and r-discrete groupoid has a Haar system in general.

Finally, motivated by the importance of having an open range map, a natural research direction is investigating for which types of locally compact groupoids is the r-discreteness assumption redundant. We present two possible cases to investigate, suggested by Williams [13], one for transitive and one for proper and principal groupoids, see Section 3.4.

The case for the proper and principal groupoid is still unanswered but the transitive case has been answered by Buneci [1] who showed that the open range map assumption is not redundant. We finish our discussion by presenting her result where we equip $G = \mathbb{R} \times \mathbb{R}$ with the trivial groupoid structure which makes the groupoid transitive. The Euclidean topology on G is modified to obtain a topology of modified neighbourhoods τ_1 where for a bounded interval $(a,\, b) \subseteq \mathbb{R}$ that doesn't contain 0, we have that the set $\{0\} \times (a,\, b)$ is an open neighbourhood of points of the form $(0,\, x)$, where $x \in (a,\, b)$. Thus $\{0\} \times (a,\, b)$ is open in τ_1 and by the way the product is defined on a trivial groupoid we calculate,

$$r(\{0\} \times (a,\, b)) = \{(0,\, 0)\}$$

which isn't open in the unit space of G which inherits the subspace topology from τ_1 on G. Thus, the groupoid does not have an open range map.

Chapter 2

Haar Measure on a Locally Compact Group

2.1 Existence and Uniqueness of Haar Measure

We present the construction of a Haar measure on a locally compact group as given in [2] which follows [6].

Definition 2.1.

- On (X, τ), a locally compact and Hausdorff space, we denote a Borel σ-algebra generated by τ with $\mathcal{B}$. A *Radon measure* $\mu : \mathcal{B} \to [0, \infty]$ satisfies,

  i) $\mu(K) < \infty$ for any compact $K \subseteq X$,

  ii) for any $B \in \mathcal{B}$ we have

  $$\mu(B) = \sup\{\mu(K) \,|\, K \subseteq B,\ K \text{ compact}\}.$$

- We denote the σ-algebra of all Borel subsets of a locally compact group G with $\mathcal{B}(G)$. For any compact subset K of G, we define

  $$\mathcal{C}_K(G) := \{f : K \to \mathbb{C} \,|\, f \text{ continuous and } \operatorname{supp}(f) \text{ compact}\}.$$

$\mathcal{C}_K(G)$ is a Banach space with the uniform norm,

$$||f||_K := \sup_{a\in K} |f(a)| = \sup_{x\in G} |f(x)|$$

we denote by τ_K the topology associated to this norm. Now we equip

$$\mathcal{C}_c(G) = \bigcup_{\substack{K\subseteq G \\ K \text{ compact}}} C_K(G)$$

with the strictly inductive limit topology associated to the inductive system $\{\mathcal{C}_K(G) \,|\, K$ a compact subset of $G\}$ denoted by τ_{ind}, and called the *uniform topology*. The convex cone of nontrivial positive functions in $\mathcal{C}_c(G)$ is denoted by $\mathcal{C}_c^+(G)$.

- A *left Haar Measure* on G is a Radon measure μ on G that is left invariant, in the sense that, $\mu(xE) = \mu(E)$ for all $E \in \mathcal{B}(G)$ and all $x \in G$.

From now on by a Haar measure on a locally compact group G, we refer to a left Haar measure unless stated otherwise.

Example 2.1. The restriction of the Lebesgue measure to the Borel subsets of $\mathbb{R}$ is a Haar measure on the topological group $(\mathbb{R}, +)$.

For the proof of the existence of a Haar measure on a locally compact group G, we define the shift maps L_y and R_y for $y \in G$ in the following way,

$$L_y\phi(x) := \phi(y^{-1}x) \text{ and } R_y\phi(x) := \phi(xy)$$

for $x \in G$ and $\phi \in \mathcal{C}_c(G)$.

Definition 2.2. *The* general linear group *over* $\mathbb{R}$ *of degree* $n \in \mathbb{N}$ *denoted by* $GL_n(\mathbb{R})$ *is the set of* $n \times n$ *invertible matrices with real entries, with matrix multiplication as the group operation. We denote* $GL_1(\mathbb{R})$ *as* $GL(\mathbb{R})$.

Example 2.2. We can associate $\mathbb{R}\setminus\{0\}$ with the group $GL(\mathbb{R})$. The measure $\frac{dy}{|y|}$ restricted to Borel subsets of $\mathbb{R}\setminus\{0\}$ and where dy denotes the Lebesgue measure is a Haar measure on $GL(\mathbb{R})$. To see its left invariance, fix $x \in GL(\mathbb{R})$, since

$x \neq 0$, without loss of generality assume that $x > 0$ and take $E \in \mathcal{B}(GL(\mathbb{R}))$. We can make the change of variables $z = xy$, and $dy = x^{-1} \cdot dz$ and

$$\frac{dy}{|y|} = \frac{x^{-1} \cdot dz}{|x^{-1}z|} = \frac{x^{-1}dz}{x^{-1}|z|} = \frac{dz}{|z|}.$$

Thus,

$$\int_E \frac{1}{|y|}dy = \int_{xE} \frac{1}{|z|}dz$$

so $\frac{dy}{|y|}$ is left invariant and thus a Haar measure on $GL(\mathbb{R})$.

For briefness purposes, from now on G will refer to a locally compact group or, otherwise is stated. Now we state a classical theorem as presented in [2].

Theorem 2.1. *(Riesz-Markov Theorem) Let (X, τ) be a locally compact and Hausdorff space. Then, for any linear positive functional $\phi : \mathcal{C}_c(X) \to \mathbb{R}$ there exists a unique Radon measure $\mu : \mathcal{B} \to [0, \infty]$ such that $\phi(f) = \int_X f d\mu$ for all $f \in \mathcal{C}_c(X)$.*

Definition 2.3. A *step function* $f : \mathbb{R} \to \mathbb{R}$ is a simple function where the sets of the characteristic functions whose linear combination sum to f are intervals.

Proposition 2.1. *Let μ be a Radon measure on G. The following are equivalent:*

(i) *μ is a Haar measure,*

(ii) *For any $y \in G$ and any step function $f : G \to \mathbb{R}$ we have,*

$$\int_G (L_y f)(x) d\mu(x) = \int_G f(y^{-1}x) d\mu(x) = \int_G f(x) d\mu(x).$$

(iii) *For every $y \in G$ and every $f \in \mathcal{C}_c^+(G)$ the above equality holds.*

(iv) *For every $y \in G$ and every $f \in \mathcal{C}_c(G)$ the above equality holds.*

Proof. (Colojoară & Gheondea, p. 34)

$(i) \implies (ii)$ Let $y \in G$ and $E \in \mathcal{B}(G)$. Then, since $y^{-1}x \in E$ if and only if $x \in yE$, we have,

$$(L_y\chi_E)(x) = \chi_E(y^{-1}x) = \chi_{yE}(x), \text{ for all } x \in G.$$

Thus by the left invariance property of the Haar measure,

$$\int_G (L_y\chi_E)(x)d\mu(x) = \int_G \chi_{yE}(x)d\mu(x) = \mu(yE) = \mu(E) = \int_G \chi_E(x)d\mu(x).$$

Let $f = \sum_{i=1}^n c_i\chi_{E_i}$ be a step function for mutually disjoint sets $E_1, ..., E_n \in \mathcal{B}(G)$ and distinct positive numbers $c_1, ..., c_n$. Then by the linearity of the integral and the previous equality, f satisfies the condition.

$(ii) \implies (iii)$ Let $y \in G$, $E \in \mathcal{B}(G)$ and $f \in \mathcal{C}_c^+(G)$. For any step function ϕ we have $\phi \leq L_y f$ if and only if $L_{y^{-1}}\phi \leq f$. By $\mathcal{S}_G$, we denote the set of all step functions on G. Observe that for any $g \in G$, we have $\mathcal{S}_G = L_g\mathcal{S}_G$. Then, using the assumption in (ii) and the change of variables $\varphi = L_{y^{-1}}\phi$ we have,

$$\begin{aligned}\int_G (L_y f)(x)d\mu(x) &= \sup_{\substack{\phi\leq L_y f,\\ \phi\in\mathcal{S}_G}} \int_G \phi(x)d\mu(x) = \sup_{\substack{L_{y^{-1}}\phi\leq f,\\ \phi\in\mathcal{S}_G}} \int_G (L_{y^{-1}}\phi)(x)d\mu(x)\\ &= \sup_{\substack{\varphi\leq f,\\ \varphi\in\mathcal{S}_G}} \int_G \varphi(x)d\mu(x) = \int_G f(x)d\mu(x).\end{aligned}$$

$(iii) \implies (iv)$ Let $y \in G$ and $f \in \mathcal{C}_c(G)$. Then if f is nontrivial and $f, -f \notin C_c^+(G)$, it has the unique decomposition into positive elements as $f = f_+ - f_-$ with $f_+,\ f_- \in \mathcal{C}_c^+(G)$ since both $f_+,\ f_-$ are nontrivial. Thus, $L_y f = L_y f_+ - L_y f_-$ and by the assumption in (iii), (iv) holds. If f is trivial or if $-f \in \mathcal{C}_c^+(G)$ then the statement follows immediately.

$(iv) \implies (i)$ Let $y \in G$ be fixed and let μ_y be the Radon measure defined by the left shift as in $\mu_y(E) = \mu(yE)$ for all $E \in \mathcal{B}(G)$. By the left invariance assumption of the integral in (iv), the two functionals $f \to \int_G f d\mu$ and $f \to \int_G f d\mu_y$ coincide and by the uniqueness implication of the Riesz-Markov Theorem, we have $\mu = \mu_y$. Thus μ satisfies the left invariance property and is thus a Haar measure.

$\square$

Remark 2.1. In view of Proposition 2.1, we can consider a Haar measure μ as a left invariant positive linear functional $\mu : \mathcal{C}_c(G) \to \mathbb{C}$. Later on, we will prove that $\mu(f)$ coincides with the Haar integral $\int_G f(x)d\mu(x)$ for $f \in \mathcal{C}_c(G)$.

Lemma 2.1. *Let $f, \phi \in \mathcal{C}_c^+(G)$. Then*

i) There exists $x_1, ..., x_n \in G$ and $c_1, ..., c_n > 0$ such that

$$f \leq \sum_{j=1}^{n} c_j L_{x_j} \phi.$$

ii) Letting

$$(f : \phi) := \inf\{\sum_{j=1}^{n} c_j \mid f \leq \sum_{j=1}^{n} c_j L_{x_j} \phi \text{ for some } x_j \in G,\, c_j > 0\},$$

the following properties hold:

(a) $(f : g) := (L_y f : \phi)$ for all $y \in G$.

(b) $(f_1 + f_2 : \phi) \leq (f_1 : \phi) + (f_2 : \phi)$ for all $f_1,\, f_2 \in \mathcal{C}_c^+(G)$.

(c) $(cf : \phi) = c(f : \phi)$ for all $c > 0$.

(d) $(f : \phi) \geq (g : \phi)$ for any $g \in \mathcal{C}_c^+(G)$ such that $f \geq g$ pointwise.

(e) $(f : \phi) \geq ||f||_\infty / ||\phi||_\infty$ for $\phi \neq 0$.

(f) $(f : \phi) \leq (f : \psi) \cdot (\psi : \phi)$ for any $\psi \in \mathcal{C}_c^+(G)$.

Proof.

i) Since $\phi \neq 0$ and $\phi \in \mathcal{C}_c^+(G)$, we have $|\phi| = \phi$ and for any $\epsilon > 0$ by the approximation property of supremum we have $\phi(x) > ||\phi||_\infty - \epsilon$ for some $x \in G$. Let $\epsilon = \frac{||\phi||_\infty}{2}$ and define $U := \{x \in G \mid \phi(x) > \frac{||\phi||_\infty}{2}\} \neq \emptyset$. Since $\text{supp}(f)$ is compact, for the (non-finite) open cover $\{xU \mid x \in G\}$ of $\text{supp}(f)$, there exists $x_1, ..., x_n \in G$ such that $\text{supp}(f) \subseteq \bigcup_{j=1}^{n} x_j U$ where $n \in \mathbb{N}$ such that $1 \leq n||\phi||_\infty$. Then we have $f \leq ||f||_\infty$ and

$$f \leq n \cdot ||\phi||_\infty \cdot ||f||_\infty = \sum_{j=1}^{n} 2||f||_\infty \cdot \frac{1}{2}||\phi||_\infty < \sum_{j=1}^{n} c_j L_{x_j} \phi$$

for $c_j = 2||f||_\infty$ for all $j \in \{1, \dots n\}$.

ii)

(a) For $f \leq \sum_{j=1}^{n} c_j L_{x_j}\phi$, take $y \in G$ and we have $L_y\Big(f - \sum_{j=1}^{n} c_j L_{x_j}\phi\Big) = L_y f - \sum_{j=1}^{n} c_j L_{yx_j}\phi \leq 0$. Since for any $z_j \in G$ there exists $x_j \in G$ such that $z_j = yx_j$, when taking infimum over $\sum_{j=1}^{n} c_j$'s we have $(f : \phi) = (L_y f : \phi)$.

(b) Take f_1, f_2, $\phi \in \mathcal{C}_c^+(G)$. Then $f_1 + f_2 \in \mathcal{C}_c^+(G)$. Then there exists $a_1, ..., a_n, b_1, ..., b_m > 0$ for without loss of generality $m \geq n$ and $x_1, ..., x_n, y_1, ..., y_m \in G$ such that

$$f_1 \leq \sum_{j=1}^{n} a_j L_{x_j}\phi, \;\; f_2 \leq \sum_{j=1}^{m} b_j L_{y_j}\phi.$$

Let $c_j := a_j + b_j$ and choose $z_j \in G$ such that $L_{z_j}\phi \geq L_{x_j}\phi + L_{y_j}\phi$ for $1 \leq j \leq n$ and (if $m \neq n$) let $c_j := b_j$ and choose $z_j \in G$ such that $L_{z_j}\phi \geq L_{y_j}\phi$ for $n+1 \leq j \leq m$. Then,

$$f_1 + f_2 \leq \sum_{j=1}^{m} c_j L_{z_j}\phi, \;\text{ and }\; \sum_{j=1}^{m} c_j = \sum_{j=1}^{n} a_j + \sum_{j=1}^{m} b_j.$$

So when taking infimum over the sums of coefficients a_j and b_j we have $(f_1 : \phi) + (f_2 : \phi) \geq (f_1 + f_2 : \phi)$.

(c) Since $c > 0$, this follows from the properties of infima.

(d) We have $g \leq f \leq \sum_{j=1}^{n} c_j L_{x_j}\phi$, then the infimum for sum of coefficients of g is taken over a larger set than that of f's. Thus $(g : \phi) \leq (f : \phi)$.

(e) Assume that $f \leq \sum_{j=1}^{n} c_j L_{x_j}\phi$ for some $x_j \in G$ and $c_j > 0$. Then,

$$||f||_\infty \leq \sum_{j=1}^{n} c_j ||L_{x_j}\phi||_\infty = ||\phi||_\infty \sum_{j=1}^{n} c_j.$$

So since $\phi \neq 0$, we have $||\phi||_\infty \neq 0$ and taking infimum over $\sum_{j=1}^{n} c_j$'s we get $||f||_\infty / ||\phi||_\infty \leq (f : \phi)$.

(f) Let $\psi \in \mathcal{C}_c^+(G)$ such that $f \leq \sum_{j=1}^{n} c_j L_{x_j}\psi$ for some $c_j > 0$, $x_j \in G$ and $\psi \leq \sum_{k=1}^{m} b_k L_{y_k}\phi$ for some $b_k > 0$, $y_k \in G$. It follows that,

$$f \leq \sum_{j=1}^{n} c_j L_{x_j}\psi \leq \Bigg(\Bigg(\sum_{j=1}^{n} c_j L_{x_j}\Bigg)\Bigg(\sum_{k=1}^{m} b_k L_{y_k}\Bigg)\Bigg)\phi.$$

Taking infima over $\sum_j c_j$ and $\sum_k b_k$'s implies $(f : \phi) \leq (f : \psi) \cdot (\psi : \phi)$.

□

Now using the $(\cdot : \cdot)$ operator on $\mathcal{C}_c^+(G) \times \mathcal{C}_c^+(G)$, we can define a linear functional on $\mathcal{C}_c^+(G)$ whose extension to $\mathcal{C}_c(G)$ will later on be the Haar integral.

Definition 2.4. Let $\phi,\ f_0 \in \mathcal{C}_c^+(G)$ be fixed and define $\mathcal{I}_\phi : \mathcal{C}_c^+(G) \to (0, +\infty)$ by

$$\mathcal{I}_\phi(f) := \frac{(f : \phi)}{(f_0 : \phi)},\ f \in \mathcal{C}_c^+(G).$$

Notice that in the previous definition, the denominator $(f_0 : \phi)$ can never be 0 since otherwise we would have that f_0 is identically 0.

Lemma 2.2. *With notation as in the previous definition and for fixed $\phi,\ f_0 \in \mathcal{C}_c^+(G)$, the functional $\mathcal{I}_\phi$ has the following properties:*

1. *(Left Invariance) $\mathcal{I}_\phi(f) = \mathcal{I}_\phi(L_y f)$ for all $y \in G$ and all $f \in \mathcal{C}_c^+(G)$.*
2. *(Subadditivity) $\mathcal{I}_\phi(f_1 + f_2) \leq \mathcal{I}_\phi(f_1) + \mathcal{I}_\phi(f_2)$ for all $f_1, f_2 \in \mathcal{C}_c^+(G)$.*
3. *(Positive Homogeneity) $\mathcal{I}_\phi(cf) = c\mathcal{I}_\phi(f)$ for all $c > 0$ and $f \in \mathcal{C}_c^+(G)$.*
4. *(Increasing) $\mathcal{I}_\phi(f_1) \leq \mathcal{I}_\phi(f_2)$ whenever $f_1, f_2 \in \mathcal{C}_c^+(G)$ are such that $f_1 \leq f_2$.*
5. *(Boundedness) $(f_0 : f)^{-1} \leq \mathcal{I}_\phi(f) \leq (f : f_0)$ for all $f \in \mathcal{C}_c^+(G)$.*

Proof. (Colojoară & Gheondea, p. 36) Properties $(1-4)$ follow from $ii)\,(a-d)$ of the previous lemma respectively. For property (5), we will use $ii)\,(f)$ of the previous lemma to obtain,

$$(f : \phi) \leq (f : f_0)(f_0 : \phi) \text{ and } (f_0 : \phi) \leq (f_0 : f)(f : \phi)$$

from which we obtain the two needed inequalities. □

Lemma 2.3. *Any function $f \in \mathcal{C}_c(G)$ is right and left uniformly continuous, in the sense that the maps $G \ni a \to L_a f \in \mathcal{C}_c(G)$ and $G \ni a \to R_a f \in \mathcal{C}_c(G)$ are continuous.*

Proof. (Colojoară & Gheondea, p. 32)
We will only prove right uniform continuity since left uniform continuity is similar. Notice that it is sufficient to prove the continuity of the map $G \ni a \to R_a f \in \mathcal{C}_c(G)$ at e. Let $\epsilon > 0$, then for any $x \in \text{supp}(f)$, there exists a neighbourhood $U_x \in \mathcal{V}(x)$ such that

$$|(R_y f)(x) - f(x)| = |f(xy) - f(x)| < \frac{\epsilon}{2}, \;\; y \in U_x.$$

Let $V_x \in \mathcal{V}(e)$ be symmetric so $V_x = V_x^{-1}$ and open such that $V_x^{-1}V_x = V_xV_x \subseteq U_x$. Then since the families of sets $\{xV \,|\, x \in G, \, v \in \mathcal{V}(e)\}$ are bases of the topology of G, we have that $\{V_x\}_{x \in \text{supp}(f)}$ is an open cover of $\text{supp}(f)$. Then, there exists x_1, ..., $x_n \in \text{supp}(f)$ such that

$$K \subseteq \bigcup_{k=1}^{n} x_k V_{x_k}.$$

Let $V := \bigcap_{k=1}^{n} V_{x_k}$ and fix $y \in V$. There are three cases,

- If $x \in \text{supp}(f)$, then there exists x_1, ..., x_n such that $x \in x_k V_{x_k}$ so $x_k^{-1}x \in V_{x_k}$ and hence $xy = x_k(x_k^{-1}x)y \in U_{x_k}$, and thus by applying triangle inequality we get,

$$|(R_y f)(x) - f(x)| \leq |f(xy) - f(x_k)| + |f(x_k) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

- If $xy \in \text{supp}(f)$, then there exists $k \in \{1, ..., n\}$ such that $xy \in x_k V_{x_k} \subseteq x_k U_{x_k}$ hence the inequality holds for this case too.

- If both x, $xy \notin \text{supp}(f)$ then $f(x) = f(xy) = 0$ so the inequality holds for this case as well.

□

Lemma 2.4. *Let $f_1, f_2 \in \mathcal{C}_c^+(G)$ and $\epsilon > 0$ be arbitrary. Then, there exists a neighbourhood V of e in G such that, whenever* $\text{supp}(\phi) \subseteq V$, *we have*

$$\mathcal{I}_\phi(f_1) + \mathcal{I}_\phi(f_2) \leq \mathcal{I}_\phi(f_1 + f_2) + \epsilon.$$

Proof. (Colojoară & Gheondea, p. 36)
Let $f \in \mathcal{C}_c^+(G)$ be such that $g(x) = 1$ for all $x \in \text{supp}(f_1 + f_2)$ and $\delta > 0$. Consider

$$h := f_1 + f_2 + \delta, \;\; h_i := \frac{f_i}{h} \text{ for } i = 1, 2.$$

Then $\text{supp}(h_i) = \text{supp}(f)$ so $h_i \in \mathcal{C}_c^+(G)$ for $i = 1, 2$. So by the previous lemma both h_1 and h_2 are left and right continuous, hence there exists V, a neighbourhood of e such that

$$|h_i(x) - h_i(y)| < \delta \text{ whenever } y^{-1}x \in V, \, i = 1, 2.$$

Let $\phi \in \mathcal{C}_c(G)$ be such that $\text{supp}(\phi) \subseteq V$. If $h \leq \sum_{i=1}^n c_i L_{x_i}\phi$ for some $x_1, ..., x_n \in G$ and $c_1, ..., c_n > 0$. We have,

$$f_i(x) = h(x)h_i(x) \leq \sum_{j=1}^n c_j \phi(x_j^{-1}x)h_i(x), \, x \in G.$$

We will show that

$$f_i(x) \leq \sum_{j=1}^n c_j \phi(x_j^{-1}x)(h_i(x_j) + \delta) \text{ for } i = 1, 2 \quad (*)$$

Fix $j \in \{1, ..., n\}$. We first prove that,

$$\phi(x_j^{-1}x)h_i(x) \leq \phi(x_j^{-1}x)(h_i(x_j) + \delta) \quad (**)$$

If $x_j^{-1}x \in \text{supp}(\phi) \subseteq V$ then $|h_i(x) - h_i(x_j)| < \delta$, $i = 1, 2$ so $(**)$ is satisfied. If $x_j^{-1}x \notin \text{supp}(\phi)$, then both sides of the inequality is 0 so $(**)$ is trivial. Then by multiplying both sides of the inequality with positive scalars c_j and summing over $j \in \{1, ..., n\}$ we get $(*)$. Now since $h_1 + h_2 = \frac{f_1+f_2}{f_1+f_2+\delta g} < 1$, we get

$$(f_1 : \phi) + (f_2 : \phi) \leq \sum_{j=1}^n c_j(h_1(x_j) + \delta) + \sum_{j=1}^n c_j(h_2(x_j) + \delta) + \sum_{j=1}^n c_j(1 + 2\delta)$$

Taking infimum over all finite sums of the form $\sum_{j=1}^n c_j$ such that $h \leq \sum_{j=1}^n c_j L_{x_j}$ we have the following,

$$\mathcal{I}_\phi(f_1) + \mathcal{I}_\phi(f_2) \leq (1 + 2\delta)\mathcal{I}_\phi(h) \leq (1 + 2\delta)(\mathcal{I}_\phi(f_1 + f_2) + \delta\mathcal{I}_\phi(g)).$$

Finally letting $\delta > 0$ be small enough such that

$$2\delta(f_1 + f_2 : f_0) + \delta(1 + 2\delta)(g : f_0) < \epsilon$$

we get the desired result. □

The development of the theory of invariant measures on topological groups had seen rapid development at the first half of the 20^{th} century. In 1933, Alfréd Haar proved that there exists an invariant measure on a Hausdorff compact group [7]. Shortly after in 1933, John von Neumann showed it's uniqueness [11]. André Weil was the first to come up with a proof for the locally compact case [12]. Now we present a proof for this celebrated result.

Theorem 2.2. *On any locally group G there exists a left Haar measure μ.*

Proof. (Colojoară & Gheondea, p. 37)
Let $f_0 \in \mathcal{C}_c^+(G)$ be fixed and for each $f \in \mathcal{C}_c^+(G)$ consider the compact interval $X_f := [\frac{1}{(f_0:f)}, (f : f_0)]$. Note that by the boundedness property of Lemma 2.2, for each $\phi, f \in \mathcal{C}_c^+(G)$ we have $\mathcal{I}_\phi(f) \in X_f$ so the interval is nonempty. Then by Tychonoff's Theorem, the topological space

$$X := \prod_{f \in \mathcal{C}_c^+(G)} X_f$$

is compact and nonempty. As a set we have the identification

$$X = \{F \,|\, F : \mathcal{C}_c^+(G) \to (0, +\infty) \text{ function, } F(f) \in X_f \text{ for all } f \in \mathcal{C}_c^+(G)\}$$

In particular, $\mathcal{I}_\phi \in X$. For each neighbourhood V of $e \in G$, let

$$K(V) := \overline{\{\mathcal{I}_\phi \,|\, \phi \in \mathcal{C}_c^+(G),\ \text{supp}(\phi) \subseteq V\}}^X$$

Since X is compact, it follows that $K(V)$ is compact. $\{K(V)\}_{V \in \mathcal{V}(e)}$ has the finite intersection property. Indeed for $V_1, V_2, ..., V_n \in \mathcal{V}(e)$ we have

$$K\left(\bigcap_{j=1}^{n} V_j\right) \subseteq \bigcap_{j=1}^{n} K(V_j)$$

By compactness, intersection of all $K(V)$'s is nonempty, thus there exists $\mathcal{I} \in X$ that lies in each $K(V)$. Further, for all $V \in \mathcal{V}(e)$, $\epsilon > 0$ and $f_1, ...f_n \in \mathcal{C}_c^+(G)$ there exists $\phi \in \mathcal{C}_c^+(G)$ with $\text{supp}(\phi) \subseteq V$ and $|\mathcal{I}(f_j) - \mathcal{I}_\phi(f_j)| < \epsilon$ for all $j = 1, 2, ..., n$. By Lemma 2.4, $\mathcal{I}$ is additive, positively homogeneous and left invariant. Finally, for $f \in \mathcal{C}_c(G)$, if f is nontrivial and $f, -f \notin C_c^+(G)$, there exist uniquely

$f_\pm \in \mathcal{C}_c^+(G)$ such that $f = f_+ - f_-$ and $f_+ f_- = 0$ so let $\mathcal{I}(f) := \mathcal{I}(f_+) - \mathcal{I}(f_-)$. This extension is a nontrivial linear functional, positive and left invariant. By Riesz-Markov Theorem, there exists uniquely a left Haar measure μ such that $\mathcal{I}(f) = \int\limits_G f(x) d\mu(x)$ for all $f \in \mathcal{C}_c(G)$. □

Theorem 2.3. *If μ and λ are Haar measures on G, then there exists $c \in (0, \infty)$ such that $\mu = c\lambda$.*

Proof. (Colojoară & Gheondea, p. 39)
We denote the inverse map on G with j, so $j(x) = x^{-1}$ for all $x \in G$. Now considering μ as a linear functional, define $\tilde{\lambda} := \lambda(f \circ j)$ for $f \in \mathcal{C}_c(G)$. Then $\tilde{\lambda}$ is a *right* Haar measure. For any $f \in \mathcal{C}_c(G)$ such that $f \in \operatorname{supp}(\mu)$, the function

$$C_f(x) := \frac{1}{\mu(f)} \int_G L_y f(x) d\tilde{\lambda}(y), \ x \in G,$$

is continuous. If $g \in \mathcal{C}_c(G)$ then the function

$$h(x, y) := f(x) g(yx), \ x, y \in G$$

is continuous with support included in $\operatorname{supp}(f) \times \operatorname{supp}(g)$. Then using Fubini's theorem and left invariance of μ and right invariance of $\tilde{\mu}$ we have,

$$\mu(g \cdot (\mu(f) C_f)) = \int_G g(x) \Big(\int_G L_y f(x) d\tilde{\lambda}(y) \Big) d\mu(x)$$
$$\int_G \Big(\int_G L_y f(x) g(x) d\mu(x) \Big) d\tilde{\lambda}(y) = \int_G \int_G h(x, y) d\mu(x) d\tilde{\lambda}(y) =$$
$$\int_G f(x) \Big(\int_G g(yx) d\mu(y) \Big) d\tilde{\lambda}(x) = \int_G f(x) \Big(\int_G g(y) d\mu(y) \Big) d\tilde{\lambda}(x) =$$
$$\Big(\int_G f(x) d\mu(x) \Big) \Big(\int_G g(y) d\tilde{\lambda}(y) \Big) = \mu(f) \tilde{\lambda}(g).$$

Thus $\mu(g C_f) = \tilde{\lambda}(g)$. If $f_1 \in \mathcal{C}_c(G)$ is another function such that $\mu(f) \neq 0$, we also have $\mu(g C_{f_1}) = \tilde{\lambda}(g)$. Then, $\mu(g C_f) = \mu(g C_{f_1})$. Thus, $g C_f = g C_{f_1}$ μ-almost everywhere for every $g \in \mathcal{C}_c(G)$ and then $C_f = C_{f_1}$ μ-almost everywhere. Since $\operatorname{supp}(\mu) = G$ and C_f, C_{f_1} are continuous, the equality holds everywhere on G. Then we can denote $C(x) := C_f(x)$, and for $x = e$, since $L_y f(e) = f(y^{-1}) = (f \circ j)(y)$ we get

$$\tilde{\tilde{\lambda}}(f) = C(e) \mu(f).$$

Also, $\tilde{\tilde{\lambda}}(f) = \tilde{\lambda}(f \circ j) = \lambda(f \circ j \circ j) = \lambda(f)$. Then for $f > 0$, we have $\mu(f), \lambda(f) > 0$ hence it follows that $C(e) = \frac{\lambda(f)}{\mu(f)} > 0$. Since every real valued f can be decomposed into $f = f_+ - f_-$ where $f_+, f_- \geq 0$ and by linearity of the functionals μ, λ, there exists $c := C(e)$ such that $\mu = c\lambda$.

□

Definition 2.5. Let μ be a Haar measure on a locally compact group G. Fixing $x \in G$, $\mu_x(E) := \mu(Ex^{-1})$ for arbitary $E \in \mathcal{B}(G)$. Then μ_x is a left Haar measure and by the uniqueness property, there exists uniquely $\triangle(x) > 0$ such that $\mu_x = \triangle(x)\mu$. $\triangle : G \to (0, +\infty)$ is called the *modular function* of G. A group G is called *unimodular* if $\triangle(x) = 1$ for all $x \in G$.

Example 2.3. An abelian locally compact group G is unimodular. To see this, let μ denote a Haar measure on G, $E \in \mathcal{B}(G)$ and $x \in G$. Then by left invariance of μ, G being abelian and the definition of the modular function we have,

$$\mu(E) = \mu(x^{-1}E) = \mu(Ex^{-1}) = \triangle(x)\mu(E)$$

Thus $\triangle(x) = 1$ for all $x \in G$.

Remark 2.2. For $f \in L^1(G)$, $t \in G$ and μ a Haar measure on G we have,

$$\int_G f(st)d\mu(s) = \triangle(t) \int_G f(s)d\mu(s).$$

Proof. We will prove the statement for simple functions and then the result will hold for all $f \in L^1(G)$. Let $E \in \mathcal{B}(G)$ and $t, s \in G$ we have,

$$\triangle(ts)\mu(E) = \mu(Es^{-1}t^{-1}) = \triangle(t)\triangle(s)\mu(E)$$

thus $\triangle : G \to (0, \infty)$ is a group homomorphism where $(0, \infty)$ is the multiplicative group of positive reals with subspace topology from $\mathbb{R}$ with the standard Euclidean topology.

Now for χ_E where χ denotes the characteristic function and assuming that $\mu(E)$ is finite, we have

$$\int_G \chi_{Et^{-1}} d\mu = \mu(Et^{-1}) = \triangle(t)\mu(E) = \triangle(t) \int_G \chi_E d\mu,$$

which proves the statement for simple functions $f = \sum_{i=1}^{n} s_i \chi_{E_i}$ for scalar s_i's and $E_i \in \mathcal{B}(G)$ where $i \in \{1, ..., n\}$ and $n \in \mathbb{N}$. □

2.2 Universal C^*-algebras

Definition 2.6. A, a Banach algebra over $\mathbb{C}$ with an involution $x \to x^*$, is called a C^*-*algebra* if the following are satisfied,

(i) $(\alpha x + \beta y)^* = \overline{\alpha} x^* + \overline{\beta} y^*$ for $\alpha, \beta \in \mathbb{C}$ and $x, y \in A$,

(ii) $(xy)^* = y^* x^*$ for $x, y \in A$ and

(iii) $||xx^*|| = ||x||^2$ for $x \in A$.

As expected, in the previous definition the C^*-norm property given as in (iii) will usually be the only non-straightforward property to check. A standard example of a finite dimensional C^*-algebra motivated by linear algebra is given below.

Example 2.4. For each $n \in \mathbb{N}$, $M_n(\mathbb{C})$ of $n \times n$ complex matrices with the uniform norm $|| \cdot ||$ is a Banach algebra. The involution is defined by adjoint matrices, so $A^* := \overline{A^T}$ for $A \in M_n(\mathbb{C})$. As the C^*-norm, define the uniform norm as,

$$||A|| := \sup_{||x|| \leq 1} ||Ax||$$

which satisfies the C^*-norm property and thus $M_n(\mathbb{C})$ is a C^*-algebra.

We follow the construction of a universal C^*-algebra as presented in [10].

Definition 2.7. Let A be a $*$-algebra. Let $p : A \to [0, \infty)$ a map. p is a C^*-*seminorm* if

(i) p is a seminorm on A,

(ii) for $x, y \in A$, $p(x^*x) = p(x)^2$ and

(iii) for $x, y \in A$, $p(xy) \leq p(x)p(y)$.

For $x \in A$, define $||x|| := \sup\{p(x) \,|\, p \text{ is a } C^*\text{-seminorm on } A\}$. Suppose that $||x|| < \infty$ for all $x \in A$. Let

$$I := \{x \in A \,|\, ||x|| = 0\}.$$

We will show that I is an ideal. Take $x \in I$ and $a \in A$. Then by the submultiplicativity of the C^*-seminorm, $||ax|| \leq ||a|| \cdot ||x|| = 0$ so $ax \in I$. Thus, I is an ideal in A. Now consider the quotient A/I, where the quotient norm has the definiteness property and thus the seminorm descends to a C^*-norm on A/I. The completion of A/I with respect to this C^*-norm is called the *universal* or the *enveloping C^*-algebra* of A and is denoted by $C^*(A)$.

2.2.1 Cuntz Algebra

An alternative way of constructing universal C^*-algebras is in terms of generators. An important example to illustrate this method is the *Cuntz algebra* that we will denote by $\mathcal{O}_n$ for integer $n \geq 2$. This example is particularly historically important because it was the first known constructed example of a separable infinite simple C^*-algebra [3]. We will start our study of $\mathcal{O}_n$ by defining the notion of simpleness for C^*-algebras.

Definition 2.8. A C^*-algebra A is called *simple* if it contains no non-trivial closed two-sided ideals.

Let $\mathcal{H}$ be a separable Hilbert space and $\{S_i\}_{i=1}^n$ a sequence of isometries on $\mathcal{H}$ so $S_i^*S_i = I$ for $1 \geq i \geq n$. We assume that $\sum_{i=1}^n S_iS_i^* = I$.

Definition 2.9. Given $k \in \mathbb{N}$, let W_k^n be the set of all k-tuples $(i_1, ..., i_k)$ with $i_j \in \{1, ... n\}$. Let $W_o^n := \{0\}$ and $W_\infty := \bigcup_{k=0}^{\infty} W_k^n$. We write $S_0 := I$ and for a multi-index $\alpha = (j_1, ..., j_k) \in W_k^n$ we denote by $S_\alpha := S_{j_1}S_{j_2} \cdots S_{j_k}$. Finally, let $l(\alpha) := k$ be the length of α and $l(0) := 0$.

Lemma 2.5. *a) Let $\mu, \nu \in W_\infty^n$ and $l(\mu) = l(\nu)$. Then $S_\mu^* S_\nu = \delta_{\mu\nu} I$.*

b) Let $\mu, \nu \in W_\infty^n$ and let P, Q be the range projections of S_μ, S_ν respectively. Suppose that $S_\mu^* S_\nu \neq 0$.

(i) If $l(\mu) = l(\nu)$ then $S_\mu = S_\nu$ and $P = Q$.

(ii) If $l(\mu) < l(\nu)$ then $S_\nu = S_\mu S_{\mu'}$ with $\mu' \in W_{l(\nu)-l(\mu)}^n$ and $P < Q$.

(iii) If $l(\mu) > l(\nu)$ then $S_\mu = S_\nu S_{\nu'}$ with $\nu' \in W_{l(\mu)-l(\nu)}^n$ and $P > Q$.

Proof. (Cuntz, p. 175)

a) We have $S_i^* S_j = \delta_{ij} I$ so for $k := l(\mu) = l(\nu)$, $\mu = (i_1, ..., i_k)$ and $\nu = (j_1, ..., j_k)$ we have

$$S_\mu^* S_\nu = S_{i_1}^* \cdots S_{i_k}^* S_{j_1} \cdots S_{j_k}$$

so if $\mu \neq \nu$, $S_\mu^* S_\nu = 0$ since $S_{i_{k-m}}^* S_{j_m} = 0$ for $0 \geq m \geq k-1$ and $S_\mu^* S_\nu = I$ if $\mu = \nu$ since S_μ is a product of isometries.

b) Follows from a).

□

Lemma 2.6. *Let $M \neq 0$ be a word in $\{S_i\} \cup \{S_i^*\}$. Then there are two unique elements μ, $\nu \in W_\infty^n$ such that $M = S_\mu S_\nu^*$.*

Proof. (Cuntz, p. 175)
Let $M = X_1 \cdots X_r$ where $X_j \in \{S_i\} \cup \{S_i^*\}$ for $1 \leq j \leq r$. After cancelling terms of the form $S_i S_i^*$ we get the expression $M = Y_1 \cdots Y_s$ in lowest terms, so $s \leq r$ and $Y_i Y_{i+1} \neq 0$ for all $1 \leq i \leq s-1$. By the previous lemma part a) since $S_i^* S_j = \delta_{ij} I$ and $M \neq 0$, we conclude that if $Y_j = S_i$ for some j such that $2 \leq j \leq s$, then $Y_{j-1} = S_i$. Let j_0 denote the largest integer between 0 and s such that $Y_{j_0} = S_i$, then $Y_j = S_i$ if $1 \leq j \leq j_0$ and $Y_j = S_i^*$ if $j > j_0$. Thus there exists $\mu, \nu \in W_\infty^n$ with $l(\mu) + l(\nu) = s$ such that $M = S_\mu S_\nu^*$. Uniqueness follows from the previous lemma part b). □

Definition 2.10. Let $\mathcal{F}_0^n = \mathbb{C}I$ and let $\mathbb{F}_k^n$ be the C^*-algebra generated by the set $\{S_\mu S_\nu^* \,|\, \mu, \nu \in W_k^n\}$. By $\mathcal{F}^n$, we denote the C^*-algebra generated by the union of all $\mathcal{F}_k^n$'s where $k \geq 0$.

Proposition 2.2. *$\mathcal{F}_k^n$ is $*$-isomorphic to $M_{n^k}(\mathbb{C})$ and $\mathcal{F}_k^n \subseteq \mathcal{F}_{k+1}^n$.*

Proof. (Cuntz, p. 175)
According to Lemma 1.18 part a), for $\mu, \mu', \nu, \nu' \in W_k^n$, we have,

$$(S_\mu S_\nu^*)(S_{\mu'} S_{\nu'}^*) = \delta_{\nu\mu'} S_\mu S_{\nu'}$$

Since also $(S_\mu S_\nu^*)^* = S_\nu S_\mu^* \in \{S_\mu S_\nu^* \,|\, \mu, \nu \in W_k^n\}$, this shows that $\{S_\mu S_\nu^* \,|\, \mu, \nu \in W_k^n\}$ is a self-adjoint system of matrix units. Thus $\mathcal{F}_k^n$ generated by $\{S_\mu S_\nu^* \,|\, \mu, \nu \in W_k^n\}$ is $*$-isomorphic to $M_{n^k}(\mathbb{C})$. Furthermore, since $\sum_{i=1}^n S_i S_i^*$, we have

$$S_\mu S_\nu^* = \sum_{i=1}^{n} S_\mu S_i S_i^* S_\nu^*.$$

Since $l(S_\mu S_i) = l(S_\nu S_i) = k+1$, we have $S_\mu S_\nu^* \in \mathcal{F}_{k+1}^n$. Thus $\mathcal{F}_k^n \subseteq \mathcal{F}_{k+1}^n$. □

Now let $\mathcal{P}$ be the algebra generated by $\{S_i\}_{i=1}^n$ and $\{S_i^*\}_{i=1}^n$ in the sense that all $A \in \mathcal{P}$ is a linear combination of words in $\{S_i\} \cup \{S_i^*\}$ for $1 \leq i \leq n$. Let $V := S_1$ and then $V^{-1} = S_1^*$ since S_1 is an isometry. Let $M = S_\mu S_\nu^*$ be a word in $\{S_i\} \cup \{S_i^*\}$ for $1 \leq i \leq n$. Le $r := l(\mu)$ and $s := l(\nu)$ and $k := r - s$.

(i) If $k > 0$, set $\hat{M} := S_\mu S_\nu^* S_1^{*k}$. Then, since $l(S_\nu^* S_1^{*k}) = s + r - s = r$, we have $\hat{M} \in \mathcal{F}_r^n$ and $M = \hat{M} V^k$,

(ii) If $k < 0$ set $\hat{M} = S_1^{-k} S_\mu S_\nu^*$. Then, since $l(S_1^{-k} S_\mu) = |k| + r = s - r + r = s$, we have $\hat{M} \in \mathcal{F}_s^n$ and $M = V^k M$.

(iii) If $k = 0$ then $M \in \mathcal{F}_r^n = \mathcal{F}_s^n$.

Then for any $A \in \mathcal{P}$, we have an expression of the form,

$$A = \sum_{i=-N}^{-1} V^i A_i + A_0 + \sum_{i=1}^{K} A_i V^i$$

for some non-negative integers N, K and for some words $A_i \in \mathcal{F}^n$ where $i \in \{-N, -N+1, ..., -1, 1, ..., K\}$. Define $F_i(A) := A_i$.

The next technical result we will state without proof.

Proposition 2.3. *(Cuntz, p. 176)*
The elements $A_i = F_i(A)$ are uniquely determined by the construction described above (i.e., they don't depend on the representation of A as a linear combination of words). We also have $||F_i(A)|| \geq ||A||$.

Now we equip $\mathcal{P}$, the algebra generated by the family of isometries $\{S_i\}_{i=1}^n$ with the largest C^*-norm,

$$||X||_0 := \sup\{||\rho(X)|| \mid \rho \text{ is a } *-\text{representation of } \mathcal{P} \text{ on a separable Hilbert space}\}.$$

Let $\mathcal{L}$ be the $||\cdot||_0$-completion of $\mathcal{P}$. Since $||\cdot||_0$ is a C^*-algebra norm which dominates the initial norm on $\mathcal{P}$, the generated C^*-algebra $C^*(S_1, ...S_n)$ is a quotient of $\mathcal{L}$. We will show that $\mathcal{L} \cong C^*(S_1, ..., S_n)$.

Proposition 2.4. *Let $A \in \mathcal{L}$. If $F_i(A) = 0$ for all $i \in \mathbb{Z}$, then $A = 0$.*

Proof. A can be decomposed in the following way,

$$A = \sum_{i=-N}^{-1} V^i F_i(A) + F_0(A) + \sum_{i=1}^{N} F_i(A) V^i$$

where $V = S_1$ and for some $N \in \mathbb{N}$. If each $F_i(A) = 0$ in the decomposition, then $A = 0$. □

Proposition 2.5. *$\mathcal{L}$ is canonically isomorphic to $C^*(S_1, ..., S_n)$.*

Proof. (Cuntz, p. 178)
The identity mapping $\pi : \mathcal{P} \to \mathcal{P}$ extends to a continuous $*$-homomorphism $\tilde{\pi}$ of $\mathcal{L}$ onto $C^*(S_1, ..., S_n)$. To show that $\tilde{\pi}$ is an isomorphism, we will show that it is injective. Since $\mathcal{P}$ contains $\mathcal{F}^n$ for each $n \in \mathbb{N}$, we can identify $\mathcal{F}^n$ with $\tilde{\pi}^{-1}(\mathcal{F}^n)$ and then we have $F_i \circ \tilde{\pi} = \tilde{\pi} \circ F_i$ for each $i \in \mathbb{Z}$. Then if $\tilde{\pi}(A) = 0$ for some

$A \in \mathcal{L}$, we have $F_i(\tilde{\pi}(A)) = 0$ for each $i \in \mathbb{Z}$. Hence since $\tilde{\pi}$ is the extension of the identity mapping and by the previous proposition we have,

$$F_i(A) = \tilde{\pi}(F_i(A)) = F_i(\tilde{\pi}(A)) = 0 \text{ for all } i \in \mathbb{Z} \implies A = 0.$$

Thus $\tilde{\pi}$ is injective. □

Now we will show that the generated C^*-algebra doesn't depend on the choice of family of isometries.

Theorem 2.4. *Let $\{\hat{S}_i\}_{i=1}^n$ be a second family of isometries such that $\sum_{i=1}^n \hat{S}_i \hat{S}_i^* = I$, then*

$$C^*(S_1, ..., S_n) \cong C^*(\hat{S}_1, ...\hat{S}_n).$$

Proof. (Cuntz, p. 177)
$\mathcal{F}^n \cap \mathcal{P} \cong \mathcal{F}^n \cap \hat{\mathcal{P}} \implies \mathcal{F}^n \cong \hat{F}^n \implies \mathcal{P} \cong \hat{P} \implies \mathcal{L} \cong \hat{\mathcal{L}}$. By the last proposition we have $\mathcal{L} \cong C^*(S_1, ..., S_n)$ and $\hat{\mathcal{L}} \cong C^*(\hat{S}_1, ..., \hat{S}_n)$ and thus,

$$C^*(S_1, ..., S_n) \cong \mathcal{L} \cong \hat{\mathcal{L}} \cong C^*(\hat{S}_1, ..., \hat{S}_n).$$

□

We write $\mathcal{O}_n$ for $C^*(S_1, ..., S_n)$ since the isomorphism class of $\mathcal{O}_n$ doesn't depend on the choice of family of isometries. Now to see the simpleness of $\mathcal{O}_n$, take a maximal ideal $\mathcal{J} \subseteq O_n$ and let $\pi : \mathcal{O}_n \to \mathcal{O}_n \backslash \mathcal{J}$ be the canonical projection. Then by the previous theorem, the simple C^* algebra $\mathcal{O}_n \setminus \mathcal{J} = C^*(\pi(S_1), ..., \pi(S_n))$ is isomorphic to $\mathcal{O}_n$ and thus $\mathcal{O}_n$ is simple.

2.3 C^*-algebra Associated to a Locally Compact Group

Now we can define an involutive algebra structure on $\mathcal{C}_c(G)$.

Definition 2.11. For f, $g \in \mathcal{C}_c(G)$, an

- *involution* of f is defined by $f^*(t) := \overline{f(t^{-1})}\triangle(t)^{-1}$ for $t \in G$,
- *convolution* of f and g is defined by $(f * g)(t) := \int_G f(ts)g(s^{-1})d\mu(s)$

Proposition 2.6. *With the* 1*-norm*

$$||f||_1 := \int_G |f(s)|d\mu(s)$$

and the above defined involution and convolution, $\mathcal{C}_c(G)$ becomes an involutive normed algebra.

Proof. (Sundar, p. 36)
To show that $\mathcal{C}_c(G)$ is a $*$-algebra with convolution as multiplication and the defined above involution, we will show the two non-straightforward properties of a $*$-algebra. Mainly, we will show the associativity of the convolution and that $(f * g)^* = g^* * f^*$ for all $f, g \in \mathcal{C}_c(G)$. Take $f, g, h \in \mathcal{C}_c(G)$ and $s \in G$, by applying invariance of μ to $f * g$ and Fubini's theorem we get,

$$((f * g) * h)(t) = \int_G (f * g)(ts)h(s^{-1})d\mu(s) =$$

$$\int_G \left(\int_G f(tsr)g(r^{-1})d\mu(r) \right) h(s^{-1})d\mu(s) = \int_G \left(\int_G f(tr)g(r^{-1}s)d\mu(r) \right) h(s^{-1})d\mu(s) =$$

$$\int_G f(tr) \left(\int_G g(r^{-1}s)h(s^{-1})d\mu(s) \right) d\mu(r) = \int_G f(tr)(g * h)(r^{-1})d\mu(r) = (f * (g * h))(t).$$

Next, since $\triangle$ is a group homomorphism we have,

$$(f * g)^*(t) = \triangle(t)^{-1}\overline{(f * g)(t^{-1})} =$$

$$\triangle(t^{-1}ss^{-1}) \int_G \overline{f(t^{-1}s)g(s^{-1})}d\mu(s) = \int_G \overline{f((s^{-1}t)^{-1})}\triangle((s^{-1}t)^{-1})\overline{g(s^{-1})}\triangle(s^{-1})d\mu(s)$$

$$= \int_G g^*(s)f^*(s^{-1}t)d\mu(s) = (g^* * f^*)(t).$$

□

Definition 2.12. Let A be a $*$-algebra and $\pi : A \to B(\mathcal{H})$ a representation for some Hilbert space $\mathcal{H}$. π is said to be *non-degenerate* if $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$.

Remark 2.3. Given a Banach $*$-algebra A, consider the family $\{\pi_\alpha\}_{\alpha\in I}$ of all non-degenerate $*$-representations for A. The C^*-norm of $C^*(A)$ can be written as

$$||x||_* = \sup_{\alpha\in I}(||\pi_\alpha(x)||)$$

The family of non-degenerate representations that the supremum is taken over is non-empty by the Gelfand-Naimark-Segal construction. Furthermore, the need for the norm $||\cdot||_*$ comes from the fact that $||\cdot||_1$ doesn't generally satisfy the C^*-norm property, in other words for some G and $f \in \mathcal{C}_c(G)$,

$$\begin{aligned} ||f^*f||_1 &= \int_G (f^*f)(t)d\mu(t) = \int_G \triangle(t)^{-1}|f(t^{-1})||f(t)|d\mu(t) \\ &\neq \left(\int_G \triangle(t)^{-1}|f(t^{-1})|d\mu(t)\right)\cdot\left(\int_G |f(t)|d\mu(t)\right) \\ &= ||f^*||_1\cdot||f||_1 \neq ||f||_1^2. \end{aligned}$$

The completion of $\mathcal{C}_c(G)$ in the 1-norm is isomorphic to the space $L^1(G)$ of equivalence classes of functions which are integrable with respect to the Haar measure. Also, $L^1(G)$ is a Banach $*$-algebra with the 1-norm and, the previously defined convolution and involution since it is complete with respect to the 1-norm.

Then $||\cdot||_*$ is a norm on $L^1(G)$ such that

$$||f||_* \leq ||f||_1$$

and

$$||f^* * f||_* = \sup_{\alpha\in I}||\pi_\alpha(f)^*\pi_\alpha(f)|| = \sup_{\alpha\in I}||\pi_\alpha(f)||^2 = ||f||_*^2$$

for $f \in L^1(G)$. Then, we can define the following C^*-algebra that summarizes this development.

Definition 2.13. For a locally compact group G, the *full group C^*-algebra $C^*(G)$* is defined to be the enveloping C^*-algebra of $L^1(G)$.

Chapter 3

Haar System on a Locally Compact Groupoid

3.1 Locally Compact Groupoids

We follow Renault's [9] exposition of groupoids.

Definition 3.1. A *groupoid* is a set G endowed with a product map $G^2 \to G$: $(x, y) \mapsto xy$ where G^2 is a subset of $G \times G$ called the set of composable pairs, and an inverse map $G \to G : x \mapsto x^{-1}$ such that the following relations are satisfied:

(i) $(x^{-1})^{-1} = x$

(ii) $(x, y),\ (y, z) \in G^2 \implies (xy, z),\ (x, yz) \in G^2$ and $(xy)z = x(yz)$

(iii) $(x^{-1}, x) \in G^2$ and if $(x, y) \in G^2$, then $x^{-1}(xy) = y$

(iv) $(x, x^{-1}) \in G^2$ and if $(z, x) \in G^2$ then $(zx)x^{-1} = z$

If $x \in G$, the maps d, $r : G \to G$ defined as $d(x) = x^{-1}x$ and $r(x) = xx^{-1}$ respectively are its *domain* (sometimes called the *source*) and *range* maps respectively. $d(G) = r(G)$ since for any $x \in G$, $r(x) = d(x^{-1})$ and $G^0 = r(G) = d(G)$ is called the *unit space* of G, its elements are units in the sense that $xd(x) = x = r(x)x$.

A common way of referring to the elements of G is as *morphisms*. Thus, a natural question to ask is if groupoids have a category theoretic description. By (ii) of the definition of a groupoid, we know that for composable elements the product map is associative. Also, we know that the class of morphisms, which is the groupoid G itself is a set. In light of this discussion we have the following remark.

Remark 3.1. A groupoid G is a small category where every morphism is invertible and the objects are the elements of the unit space.

A groupoid G is said to be *principal* if the map $G \to G^0 \times G^0$ such that

$$x \mapsto (r(x), d(x))$$

is injective and *transitive* if the map is surjective. For $u, v \in G^0$ we denote the fibers of G with $G^u = r^{-1}(u)$, $G_v = d^{-1}(v)$ and $G^u_v = G^u \cap G_v$. With this notation $G(u) := G^u_u$ becomes a group and is called the *isotropy group* at u. The relation $u \sim v$ if and only if $G^u_v \neq \emptyset$ is an equivalence relation on the unit space G^0. Its equivalence classes are called *orbits* and the orbit of $u \in G^0$ is denoted $[u]$.

Example 3.1. A group G is a groupoid with the usual product and inverse maps. Since all elements of a group are composable we have $G^2 = G \times G$ and $G^0 = \{e\}$.

Example 3.2. Let X be a set and let $G = X \times X$. We define on G the following groupoid structure:

- $G^2 = \{((x, y), (y, z)) \mid x, y, z \in X\}$
- The product map $G^2 \ni ((x, y), (y, z)) \mapsto (x, z) \in G$
- The inverse map $G \ni (x, y) \mapsto (y, x) \in G$

Then we have $r(x, y) = (x, x)$ and $d(x, y) = (y, y)$ so we get $G^{(x,x)} = \{x\} \times X$ and $G_{(x,x)} = X \times \{x\}$ for $x \in X$. G is called the *trivial groupoid.* The trivial groupoid is principal.

Example 3.3. Suppose that the group S acts on the space U on the right. The action of $s \in S$ on $u \in U$ is denoted by $u \cdot s \in S$. We let $G = U \times S$ and define the following *transformation groupoid* structure:

- (u, s) and (v, t) are composable pairs if and only if $v = u \cdot s$ and then $(u, s)(u \cdot s, t) = (u, st)$
- $(u, s)^{-1} = (u \cdot s, s^{-1})$

Then denoting the identity of the group S with e,

$$r(u, s) = (u, s)(u, s)^{-1} = (u, ss^{-1}) = (u, e),$$

$$d(u, s) = (u, s)^{-1}(u, s) = (u \cdot s, s^{-1}s) = (u \cdot s, e)$$

so $G^0 = U \times \{e\}$. Thus we can identify the unit space of G with U.

Definition 3.2. The action of a group S on a non-empty space U is,

- *free* if for any s, $t \in S$, if there exists $u \in U$ such that $u \cdot s = u \cdot t$ then we have $s = t$.
- *transitive* if for any u, $v \in U$, there exists $s \in S$ such that $u \cdot s = v$.

Proposition 3.1. *Let $G = U \times S$ be a transformation groupoid. Then,*

1. *G is principal if and only if S acts freely and,*
2. *G is transitive if and only if S acts transitively.*

Proof. We will denote the identity of the group S with e.

1. [$\Longrightarrow$] Suppose that G is principal so for g, $h \in S$ there exists $x \in U$ such that $x \cdot g = x \cdot h$. Then we have

$$d((x, h)) = (x \cdot h, e) = (x \cdot g, e) = d((x, g)),$$

$$r((x,h)) = (x,e) = r((x,g))$$

since G is principal, we have $(x,h) = (x,g) \implies h = g$. So S acts freely.
[$\Longleftarrow$] Suppose that S acts freely, and for (g,s_g), $(h,s_h) \in G$ we have that $(r((g,s_g)), d((g,s_g))) = (r((h,s_h)), d((h,s_h)))$. Then we have,

$$(g,e) = (h,e) \quad \text{and} \quad (g \cdot s_g, e) = (h \cdot s_h, e)$$

then $g = h$ and $g \cdot s_g = h \cdot s_h \implies g \cdot s_g = g \cdot s_h$ and since S acts freely, $s_g = s_h$. Thus the map $\gamma \to (r(\gamma), d(\gamma))$ is injective and G is principal.

2. [$\Longrightarrow$] Suppose that G is transitive. Take $u, v \in U$ and then (u,e), $(v,e) \in G^0$. Since G is transitive, there exists $(x,s_x) \in G$ such that $r((x,s_x)) = (u,e)$ and $d((x,s_x)) = (v,e)$. Then $(x,e) = (u,e)$ and $(x \cdot s_x, e) = (v,e) \implies x = u$ and $v = x \cdot s_x = u \cdot s_x$. So S acts transitively.
[$\Longleftarrow$] Suppose that S acts transitively on U. Take (u,e), $(v,e) \in G^0$. Since $u, v \in U$ and S acts transitively, there exists $s \in S$ such that $u \cdot s = v \implies r((u,s)) = (u,e)$ and $d((u,s)) = (u \cdot s, e) = (v,e)$. Then the map $\gamma \to (r(\gamma), d(\gamma))$ is surjective and G is transitive.

$\square$

Remark 3.2. A groupoid G is transitive if and only if it has a single orbit.

Proof.

[$\Longrightarrow$] Suppose that G is transitive. So the map $x \to (r(x), d(x))$ is surjective on G. Then take $(u,v) \in G^0 \times G^0$ and there exists $x \in G$ such that $r(x) = u$ and $d(x) = v$. $x \in G^u_v \neq \emptyset$ so $u \sim v$ and $u, v \in [u]$. Since $u, v \in G^0$ were chosen arbitrarily, all the elements in the unit space are equivalent so there exists only one orbit $[u]$.

[$\Longleftarrow$] Suppose that G has a single orbit. Then for any $u, v \in G^0$, $G^u_v \neq \emptyset$. Take $x \in G^u_v$, so $u = r(x)$ and $v = d(x)$. Then for any $(u,v) \in G^0 \times G^0$, there exists $x \in G$ such that $(u,v) = (r(x), d(x))$. Thus $x \to (r(x), d(x))$ is surjective on G.

□

Example 3.4. For a groupoid G, the set of composable elements $F := G^2$ can be turned into another groupoid with the following structure:

- Its set of composable pairs denoted by F^2 is,

$$F^2 := \{((x,\, y),\, (y',\, z)) \in F \times F \,|\, y' = xy\}$$

- The product and inverse maps are defined by $(x, y) \cdot (xy,\, z) := (x,\, yz)$ and $(x,\, y)^{-1} := (xy,\, y^{-1})$

Then if $(x,\, y) \in F$ we have $r(y) = d(x)$ and denoting its range and domain maps with r^2 and d^2 we have,

$$r^2(x,\, y) = (x,\, yy^{-1}) = (x,\, r(y)) = (x,\, d(x)),$$

$$d^2(x,\, y) = (xy,\, y^{-1}y) = (xy,\, d(y))$$

Then the map $x \to (x, d(x)) = r^2(x, y)$ for $(x, y) \in F$ identifies the unit space F^2 of F with G.

Remark 3.3. The groupoid F is principal.

Proof. We will show that the map $(x, y) \to (r^2(x, y), d^2(x, y))$ is injective on F. Suppose that for some (x', y'), $(x'', y'') \in F$ that

$$r^2(x', y') = r^2(x'', y'') \text{ and } d^2(x', y') = d^2(x'', y'').$$

Then from the first equality above, $x' = x''$ and $d(x') = d(x'')$ for $x',\, x'' \in G$. From the second equality, $x'y' = x''y''$ and $d(y') = d(y'')$ for $y',\, y'' \in G$. Also since (x', y'), (x'', y'') are composable pairs in G, $r(y') = d(x')$ and $r(y'') = d(x'')$. Thus by associativity of the product on G and since $r(y)$ is a left unit for all $y \in G$ we get,

$$y' = r(y')y' = d(x')y' = x'^{-1}(x'y') = x''^{-1}(x''y'') = d(x'')y'' = r(y'')y'' = y''$$

Thus F is principal. □

Definition 3.3. A *group bundle* G is a groupoid such that for any $x \in G$, $d(x) = r(x)$. So a group bundle is the union of its isotropy groups $G(u)$ for $u \in G^0$. Here two elements can be composed if and only if they are in the same fiber.

Definition 3.4. A *topological groupoid* consists of a groupoid G and a topology compatible with the groupoid structure such that:

- The map $G \ni x \mapsto x^{-1} \in G$ is continuous
- The map $G^2 \ni (x, y) \mapsto xy \in G$ is continuous where G^2 has the induced topology from $G \times G$.

Remark 3.4. Range and domain maps are continuous on a topological groupoid since they are compositions of the inverse and product maps of the groupoid, which are both continuous.

We will only consider topological groupoids whose topology is Hausdorff and locally compact and we will call such a groupoid G a *locally compact groupoid*. Since singletons are closed in Hausdorff spaces and range and domain maps are continuous on topological groupoids, for $u, v \in G^0$, $G_v = d^{-1}(v)$, $G^u = r^{-1}(u)$ and $G^u_v = G^u \cap G_v$ are closed in G.

Example 3.5. Let G be a topological gropoid that is discrete. Then since G is both locally compact and Hausdorff it is a locally compact groupoid. Particularly, if G is finite, then it is a compact groupoid since compact subsets of a discrete space are the finite ones.

3.2 Haar Systems

Definition 3.5. Let G be a locally compact groupoid. A *left Haar system* for G consists of $\{\lambda^u \,|\, u \in G^0\}$ on G such that,

1. the support of the measure λ^u is G^u,

2. for any $f \in \mathcal{C}_c(G)$, $u \mapsto \lambda(f)(u) := \int\limits_{G^u} f d\lambda^u$ is continuous, and

3. for any $x \in G$ and any $f \in \mathcal{C}_c(G)$,

$$\int\limits_{G^{d(x)}} f(xy) d\lambda^{d(x)}(y) = \int\limits_{G^{r(x)}} f(y) d\lambda^{r(x)}(y)$$

From now on by a Haar system on a locally compact groupoid G, we will refer particularly to a left Haar system unless stated otherwise.

Example 3.6. Let $G = X \times X$ where X is a second countable, locally compact and Hausdorff space be a trivial groupoid, see Example 3.2. Let μ be any positive regular Borel measure on X whose support is G^0. For any $u \in X$, let $\lambda^u := \delta_u \times \mu$ on $G^{(u,u)} = \{u\} \times X$. So for any Borel subset A of X we get $\lambda^u(\{u\} \times A) = \mu(A)$ where $\{u\} \times A \subseteq G^{(u,u)}$. The family $\{\lambda^u\}_{u \in X}$ is called the left Haar system associated to μ. To show continuity of the Haar integral, notice that for $f \in \mathcal{C}_c(G)$ we have

$$\lambda(f)(u) = \int\limits_X f(u,z) d\mu(z)$$

so $X \ni u \mapsto \int\limits_X f(u,z) d\mu(z) \in \mathbb{R}$ is continuous for $f(u, \cdot) \in \mathcal{C}_c(X)$. Invariance follows from the way the product map is defined on G.

For locally compact group bundles we have a similar (to the locally compact group case) uniqueness result.

Proposition 3.2. *Let G be a locally compact group bundle. Then if a Haar system exists, it is unique. In the sense that, any two Haar systems $\{\lambda^u\}_{u \in G^0}$ and $\{\mu^v\}_{v \in G^0}$ on G differ by a continuous positive function h on G^0 such that $\lambda^u = h(u) \cdot \mu^u$ for all $u \in G^0$.*

We provide a proof of the above proposition from Renault [9].

Proof. Let G be a locally compact group bundle with Haar systems $\{\lambda^u\}_{u \in G^0}$ and $\{\mu^v\}_{v \in G^0}$. Since G^u's correspond to $G(u)$'s which are locally compact groups, and thus λ^u and μ^u are Haar measures on the fibers $G(u)$. Then for any $u \in G^0$ there exists $c_u > 0$ such that $\lambda^u = c_u \mu^u$. Define $h : G^0 \to (0, +\infty)$ such that $h(u) := c_u$ so h is strictly positive. Let $f \in \mathcal{C}_C(G)$ be non-negative and its support included in G^u, then

$$\frac{\lambda(f)(u)}{\mu(f)(u)} = \frac{\int_G f d\lambda^u}{\int_G f d\mu^u} = c_u = h(u)$$

is well defined. By continuity of $\lambda(f)(\cdot)$ and $\mu(f)(\cdot)$, h is continuous. □

We can construct a universal C^*-algebra on a locally compact groupoid, similar to the group case using Haar systems. We present this result as stated in [4].

Definition 3.6. Let G be a locally compact groupoid with a Haar system $\lambda = \{\lambda^u\}_{u \in G^0}$ for G. We provide $\mathcal{C}_c(G)$ with the following involutive algebra structure,

- the *involution* by $f^*(x) = \overline{f(x^{-1})}$ for $f \in \mathcal{C}_c(G)$ and $x \in G$,
- the *convolution product* by $(f * g)(x) = \int\limits_{G_{d(x)}} f(xy^{-1})g(y)d\lambda^{d(x)}(y)$,
- The 1-*norm* on $\mathcal{C}_c(G)$ is defined by

$$||f||_1 = \sup_{u \in G^0} \max \left\{ \int\limits_{G_u} |f(x)| d\lambda^u(x), \int\limits_{G_u} |f(x^{-1})| d\lambda^u(x) \right\}$$

The groupoid's *full C^*-algebra* $C^*(G, \lambda)$ is defined to be the enveloping C^*-algebra of the Banach $*$-algebra $\overline{\mathcal{C}_c(G)}^{||\cdot||_1}$.

3.3 Existence of Haar Systems

The following is an example due to Anton Deitmar [5], that shows that let alone locally compact groupoids, even compact groupoids don't necessarily have Haar systems. Notice that similar to a locally compact groupoid, by a *compact groupoid* we mean that the groupoid is both compact and Hausdorff.

Example 3.7. We will construct a trivial groupoid G whose fibers can't support a Radon measure. Let Y be an uncountable set with the discrete topology. Since Y is discrete and not finite it is Hausdorff and locally compact but not compact. Let $X = Y \cup \{\infty\}$ be its Alexandroff extension which is both compact and Hausdorff. We will show that X can't be the support of any Radon measure. To see this, let μ be a Radon measure on X, then $\mu(X) < \infty$, as X is compact. Further $\mu(Y) = \sum_{y \in Y} \mu(\{y\})$, as μ is regular and thus inner regular and the only compact subsets of Y are the finite sets. As $\mu(Y) < \infty$, the set $M := \{y \in Y \mid \mu(\{y\}) > 0\}$ is countable, therefore $M \neq Y$ and μ is supported in $M \cup \{\infty\} \neq X$. Now let $G = X \times X$ with the product topology and make G a groupoid by equipping it with the trivial groupoid structure so the product is defined as $(x, y)(y, z) = (x, z)$ and then $r(x, y) = (x, x)$, $d(x, y) = (y, y)$. Then G is a compact groupoid. The range map is a homeomorphism between $G^{(x,x)}$ and $\{x\} \times X$ since $r(\{x\} \times X) = \{(x, x)\}$, so the fiber $G^{(x,x)}$ can't be the support of any Radon measure, hence no Haar system exists on G.

Proposition 3.3. *Let G be a locally compact groupoid with a Haar system $\{\lambda^u\}_{u \in G^0}$. Then $r : G \to G^0$ is an open map.*

Proof. This follows immediately from the continuity assumption of the Haar system and the continuity and surjectivity of the range map $r : G \to G^0$ since G is a locally compact and hence topological groupoid. □

Definition 3.7. A locally compact groupoid G is *r-discrete* if its unit space is an open subset.

Lemma 3.1. *If G is an r-discrete groupoid, then*

(i) for any $u \in G^0$, the fibers G^u, G_v, G^u_v are discrete spaces,

(ii) the Haar system on G is essentially the counting measures system.

Proof. (Renault, p. 19)

(i) Let $x \in G^u$ and let $v = d(x)$, since $\{v\} = G^0 \cap G^v$, it follows that $\{v\}$ is an open set in G^v, and since $y \mapsto xy$ is a homeomorphism from G^v to G^u, $\{x\}$ is open in G^u. Finally $G^u_v = G_v \cap G^u$ so G^u_v is also open.

(ii) Let $\{\lambda^u\}_{u \in G^0}$ be a Haar system for G. Since the fiber G^u is the support of the measure λ^u and by part (i) is discrete, every element u in G^0 has positive λ^u measure. Let $g(x) := \lambda(\chi_{G^0})(x)$, where χ_{G^0} denotes the characteristic function of G^0. By the continuity condition of the Haar system, g is continuous and positive. Again, since the measure λ^u is supported by the fiber G^u for each unit u, we have for $u \in G^0$,

$$g(u) = \int_G \chi_{G^0} d\lambda^u = \int_{G^0 \cap \operatorname{supp}(\lambda^u)} \chi_{G^0} d\lambda^u = \int_{G^u} \chi_{G^0} d\lambda^u = \lambda^u(G^u) \neq 0$$

replacing λ^u by $\frac{\lambda^u}{g(u)}$ we can assume that $\lambda^u(\{u\}) = 1$ for all $u \in G^0$. Then by invariance, $\lambda^v(\{x\}) = 1$ for any $x \in G^v_u$.

□

Definition 3.8.

- A *G-set* in a groupoid G is a subset $A \subseteq G$ such that the restriction maps r_A and d_A are injections.
- The family of open, Hausdorff subsets A of G such that r_A, d_A are homeomorphisms onto open subsets of G is denoted by G^{op}.
- For A, $B \subseteq G$ we define the inverse and product on sets as,

$$A^{-1} = \{x^{-1} \in G \,|\, x \in A\}$$

$$AB = \{xy \,|\, x \in A,\, y \in B,\, (x, y) \in G^2\}.$$

- An *inverse semi-group* is a set $\mathcal{G}$ endowed with an associative binary operation (multiplication), and inverse such that the below relations hold for all $A \in \mathcal{G}$,

$$AA^{-1}A = A \text{ and } A^{-1}AA^{-1} = A^{-1}.$$

Remark 3.5. $A \subseteq G$ is a G-set if and only if AA^{-1}, $A^{-1}A \subseteq G^0$.

Proof.
[$\Longrightarrow$] Suppose that $A \subseteq G$ is a G-set. Take $ab^{-1} \in AA^{-1}$ so a, $b \in A$ and $d(a) = r(b^{-1}) = d(b)$. Since d is injective on A, $a = b$ and $ab^{-1} = aa^{-1} = r(a) \in G^0$ so $AA^{-1} \subseteq G^0$. The case for $A^{-1}A$ follows similarly from the injectivity of r on A.
[$\Longleftarrow$] Suppose that $AA^{-1} \subseteq G^0$. Take a, $b \in A$ such that

$$d(a) = d(b) \quad \text{and} \quad r(a) = r(b).$$

Then $(a, b^{-1}) \in G^2$ so $ab^{-1} \in AA^{-1} \subseteq G^0$. Then there exists $c \in G$ and thus $r(c) \in G^0$ such that,

$$ab^{-1} = r(c) \quad \text{and also} \quad r(a) = r(ab^{-1}) = r(r(c)) = r(cc^{-1}) = r(c).$$

Then,

$$a = ad(a) = ad(b) = (ab^{-1})b = r(c)b = r(a)b = r(b)b = b.$$

Thus r_A, d_A are injective, and A is a G-set. □

Remark 3.6. The family of G-sets of the groupoid G forms an inverse semi-group.

Proof. This follows from the injectivity characterization of a G-set as given in the previous definition. Let $\mathcal{G}$ be the family of G-sets of the groupoid G.

[$\supseteq$] Take $A \in \mathcal{G}$, for any $a \in A$,

$$a = ad(a) = aa^{-1}a \in AA^{-1}A$$

thus $A \subseteq AA^{-1}A$.

[$\subseteq$] Now take $ab^{-1}c \in AA^{-1}A$ so a, b, $c \in A$ and (a, b^{-1}), $(b^{-1}, c) \in G^2$ so

$$d(a) = r(b^{-1}) = d(b) \text{ and } r(b) = d(b^{-1}) = r(c).$$

Since r and d are injective on A, $a = b$ and $b = c$. Then $ab^{-1}c = aa^{-1}a = ad(a) = a \in A$. Thus $AA^{-1}A \subseteq A$ and then $A = AA^{-1}A$. The case for A^{-1} is similar. Then $\mathcal{G}$ is an inverse semi-group.

□

Remark 3.7. For an r-discrete groupoid G, $A \subseteq G$ belongs to G^{op} if and only if A is an open, Hausdorff G-set.

Proof. (Paterson, p. 45)
[$\Longrightarrow$] Suppose $A \in G^{op}$. Then r_A, d_A are homeomorphisms onto $r(A), d(A)$ respectively. Since r_A, d_A are bijections, they are injections, so A is a G-set.
[$\Longleftarrow$] Since A is a G-set, r_A, d_A's are bijections onto their images and since they are open maps, they have continuous inverse, so they are homeomorphisms. Thus $A \in G^{op}$. □

Our discussion so far allows us to state and prove the next theorem which is the most important result of this section. It characterizes the existence of a Haar system on an r-discrete groupoid in a purely topological way. The previous remark implies that *(ii)* holds if and only if *(iii)* holds in the next theorem so we will only show *(i)* $\Longrightarrow$ *(ii)* $\Longleftrightarrow$ *(iii)* $\Longrightarrow$ *(i)* and *(ii)* $\Longrightarrow$ *(iv)* $\Longrightarrow$ *(iii)*.

Theorem 3.1. *For a locally compact groupoid G, the following properties are equivalent,*

(i) G is r-discrete and admits a Haar system,

(ii) $r : G \to G^0$ is a local homeomorphism,

(iii) G has a base of open G-sets,

(iv) the product map $\cdot : G^2 \to G$ is a local homeomorphism.

Proof. (Renault, p. 19)

(i) $\Longrightarrow$ *(ii)* Since G is r-discrete we can assume that the Haar system $\{\lambda^u\}_{u \in G^0}$ is the system of counting measures on G^u's. Let $x \in G$ and a compact neighbourhood V of x meets G^u in finitely many points $x_1, ..., x_n$. If $x_i \neq x$, there exists a compact neighbourhood V' of x contained in V which doesn't contain x_i. So we can assume that $G^u \cap V = \{x\}$. By continuity of the Haar system, we can assume that $\lambda^u(V) = 1$ for any $u \in r(V)$. So r_V is injective and hence a homeomorphism onto $r(V)$.

(iii) $\implies$ (i) There is an open G-set S such that $r(S) = SS^{-1}$. Take $x \in s$ and $y \in S^{-1}$ and r_S, $d_{S^{-1}}$ are homeomorphisms onto their images. We will show that the product map with domain restricted to $D := SS^{-1}$ is injective. Take (x', y'), $(x'', y'') \in D$ such that $x'y' = x''y''$. Then,

$$r(x') = r(x''), d(y') = d(y'') \implies x' = x'' \text{ and } y' = y''$$

So the product map is a local homeomorphism restricted to D, so $r(S)$ is open in G. Then G^0 is open so G is r-discrete. For the existence of the Haar system, let $\lambda^u = \sum_{x \in G^u} \delta_x$ the counting measure on G^u and $f \in \mathcal{C}_c(G)$. Using a partition of identity, we can write f as a finite sum of functions supported on open G-sets S. So it is enough to consider a function f whose support is contained in an open G-set S. Then since r_S, d_S are local homeomorphisms onto their images,

$$\lambda(f)(u) = \int_{G^u} f d\lambda^u = \sum_{x \in G^u \cap S} f(x) = (f \circ r_S^{-1})(u)$$

So the counting measure system is continuous.

(ii) $\implies$ (iv) We will denote G^2 by F as we previously did in the corresponding example. Take $(x, y) \in F$, since G is locally compact and r is a local homeomorphism, we can choose compact neighbourhoods U and V of x and y respectively such that the restriction maps r_U, d_V are local homeomophisms. Since F inherits the subspace topology from $G \times G$, $(U \times V) \cap F$ is a compact neighbourhood of (x, y) on which the product map is injective. Since the restriction maps r_U, d_V are injective, this can be seen by,

$$\text{Take } (x', y'), (x'', y'') \in F \text{ such that } x'y' = x''y'' \implies$$
$$r(x') = r(x'y') = r(x''y'') = r(x'') \text{ and}$$
$$r(y') = d(x'y') = d(x''y'') = d(y'') \implies$$
$$x' = x'' \text{ and } y' = y'' \implies (x', y') = (x'', y'').$$

(iv) $\implies$ *(iii)* If $x \in G$ and U is a neighbourhood of x, we can find open sets V and W such that $x \in V \subseteq U$, $x^{-1} \in W \subseteq U^{-1}$ and the restriction of the product map to $V \times W$ is injective. So both V, W^{-1} include x, are contained in U and $r_{V \cap W^{-1}}$ is injective by assumption so $V \cap W^{-1}$ is the desired open G-set.

□

Proposition 3.4. *A locally compact groupoid G is r-discrete and admits a Haar system if and only if $F = G^2$ is r^2-discrete and admits a Haar system.*

Now we provide a proof of the above statement from Renault [9].

Proof.

($\implies$) Suppose that G is r-discrete and admits a Haar system. Then by the previous theorem, G has a base of open G-sets. Take $(x, y) \in F$. There exists open G-sets A, $B \subseteq G$ such that $x \in A$ and $y \in B$. Since $A \times B$ is an open set in $G \times G$ (equipped with the product topology), then $(A \times B) \cap F$ is open in F (equipped with the subspace topology inherited from $G \times G$). We will show that r^2 restricted to $(A \times B) \cap F$ is a local homeomorphism of F and for that we will show that it is injective. For some (x', y'), $(x'', y'') \in (A \times B) \cap F$, suppose that

$$r^2(x', y') = r^2(x'', y'')$$

then

$$(x', r(y')) = r^2(x', y') = r^2(x'', y'') = (x'', r(y''))$$

so $x' = x''$ and $r(y') = r(y'')$. Since B is a G-set of G and y', $y'' \in B$, we have $y' = y''$ so $r^2|_{(A \times B) \cap F}$ is injective so it is a homeomorphism onto its image and by the previous theorem F is r-discrete and admits a Haar system.

($\Longleftarrow$) Suppose that F is r-discrete and admits a Haar system. Then by the previous theorem F has a base of open G-sets. We want to show that $\cdot : G^2 \to G$ is a local homeomorphism of G. For convenience by f we will denote the multiplication map $\cdot$ of G and we will show that f is injective. Take $(x, y) \in F$, then there exists V an open G-set in F such that $(x, y) \in V$. Suppose that for some (x', y'), $(x'', y'') \in V \subseteq F$ we have $x'y' = f(x', y') = f(x'', y'') = x''y''$. Then since $d(x'y') = d(y')$ and $d(x''y'') = d(y'')$ we have,

$$\begin{aligned} d^2(x', y') &= (x'y', d(y')) = (x'y', d(x'y')) \\ &= (x''y'', d(x''y'')) = d^2(x''y'', d(y'')) \\ &= d^2(x'', y''). \end{aligned}$$

Since d^2 restricted to V is a homeomorphism, it is injective thus $(x', y') = (x'', y'') \implies x' = x''$ and $y' = y''$. Thus f is injective and by the previous theorem, G is r-discrete and admits a Haar system.

□

We will end our discussion of Haar systems by stating the below important open problem on existence.

Conjecture 3.1 (Deitmar, 2017)**.** *Do every second countable, locally compact groupoid with an open range map admit a Haar system?*

3.4 Groupoids with Open Range Map

Example 3.8. Let $U = S = \mathbb{R}$ and endow both U and S with the standard Euclidean topology generated by the Euclidean metric. Notice that S is also an additive group where its identity denoted by e is 0. Let $G = U \times S$ and let G have the product topology. Let G be the transformation groupoid where S is acting on U from the right by right translations, see Example 3.3. So the group action of s on u denoted by $u \cdot s$ is $u \cdot s := u + s$ for $u \in U$, $s \in S$. So for any $(u, s) \in G$ we have,

$$r(u, s) = (u, e) = (u, 0) \text{ and } d(u, s) = (u \cdot s, e) = (u + s, 0).$$

G is second countable since we can take a basis of $\mathbb{R} \times \mathbb{R}$ consisting of open balls of the type,

$$B_r(q_1,\, q_2), \;\; q_1,\, q_2,\, r \in \mathbb{Q}$$

which forms a countable basis. It is also locally compact and Hausdorff since U and S have the Euclidean topology and G has the product topology. Since U and S are metric spaces, using sequential continuity, showing that the product and the inverse maps of G are continuous is straightforward since addition is continuous on $\mathbb{R}$. Thus G is a locally compact groupoid. For an open subset of G of the form $A \times B$ where $A \subseteq U$ and $B \subseteq S$, since $G^0 = U \times \{0\}$ we have,

$$r(A \times B) = A \times \{0\} = (A \times B) \cap G^0.$$

Notice that the notion of opennes of the range map is in the sense of the subspace topology of G^0, which doesn't necessarily correspond to opennes in G. $r(A \times B) = A \times \{0\}$ is not open in G since $\{0\}$ is a singleton and thus closed in S which is Hausdorff. However, $r(A \times B) = (A \times B) \cap G^0$ is open in G^0, so r is an open map and thus G is r-discrete.

We will show that G doesn't admit a Haar system. To do this we will show that G doesn't have a base of open G-sets and we will use the previous theorem to conclude the desired result. Suppose by contradiction that G has a base of open G-sets. Let A denote an open G-set in this basis. Since products of open intervals in $\mathbb{R}$ forms another basis for G, an open A contains a set of the form $(a,\, b) \times (c,\, d)$ where $a < b$, $c < d$. Let $\epsilon = \min\{\frac{b-a}{4},\, \frac{d-c}{4}\} > 0$. Now let $x_1 = \frac{a+b}{2} + \epsilon$, $y_1 = \frac{c+d}{2} - \epsilon$, $x_2 = \frac{a+b}{2} - \epsilon$ and $y_2 = \frac{c+d}{2} + \epsilon$. Then,

$$d(x_1,\, y_1) = (x_1 + y_1,\, 0) = \left(\frac{a+b+c+d}{2},\, 0\right) = (x_2 + y_2,\, 0) = d(x_2,\, y_2)$$

where $(x_1,\, y_1),\, (x_2,\, y_2) \in (a,\, b) \times (c,\, d)$ and $(x_1,\, y_1) \neq (x_2,\, y_2)$ so d isn't injective on $(a,\, b) \times (c,\, d)$ and thus it isn't injective on A. Then A isn't a G-set, a contradiction with the assumption. Thus, G can't have a basis of open G-sets and thus it doesn't have a Haar system.

Furthermore, we can see that G is trivial and principal

1. **Principal:** We will show that the action of S on U is free and thus that the groupoid is principal. Take $s, t \in S = \mathbb{R}$ and suppose that there exists some $u \in U$ such that $u \cdot s = u \cdot t$. Then $u + t = u + s \implies t = s$. Thus the action is free and G is principal.

2. **Transitive:** We will show that the action of S on U is transitive and thus that the groupoid is transitive. Take $u, v \in U = \mathbb{R}$. Then $s := v - u \in \mathbb{R} = S$. Since $u \cdot s = u + (v - u) = v$, the action is transitive and G is transitive.

Definition 3.9. A groupoid G is called a *proper* groupoid if the map $\rho : Gr(d) \to G^{(0)} \times G^{(0)}$ defined by

$$\rho(y, d(y)) = (r(y), d(y))$$

where $y \in G$, is a proper map.

As seen in the previous example, it is possible to construct r-discrete groupoids in relatively simple settings that are both principal and transitive. The topological characterisation of the existence of a Haar system for r-discrete groupoids is a useful one, especially since we know that if it exists, that it is essentially the counting measures system. Thus, it is natural to ask for which types of groupoids we automatically have r-discreteness. The below two questions from Dana P. Williams (p. 6, 2016) investigate if the r-discreteness assumption is redundant for certain types of groupoids.

Question 3.1. *Must a second countable, locally compact, transitive groupoid have open range and domain maps?*

Question 3.2. *Must a second countable, locally compact, proper principal groupoid have open range and domain maps?*

While the previous example shows that a second countable, locally compact, transitive groupoid (that can even be principle) *can* exist it doesn't show that transitiveness implies an open range map. Indeed, the first question has been answered in the negative by Buneci in 2018 [1] and we finish our discussion by presenting her result below.

Example 3.9. Let $X = \mathbb{R}$ and $G = X \times X$. We will equip G with the trivial groupoid structure, see Example 3.2. To see that G is transitive, we will show that the map $G \ni (x, y) \to (r(x, y),\, d(x, y))$ is surjective. This is straightforward since for any $((x, x),\, (y, y)) \in G^0 \times G^0$ for some $x,\, y \in X$, $r(x, y) = (x, x)$ and $d(x, y) = (y, y)$ where $(x, y) \in G$. We will equip G with a modified Euclidean topology by redefining the neighbourhoods of each $x \in X$. For $x \in X$, let

$$B_x := \begin{cases} \{[\frac{3}{2^{n+2}}, \frac{3}{2^{n+2}} + \epsilon),\, \epsilon > 0\}, & \text{if } x = \frac{3}{2^{n+2}},\ n \in \mathbb{N} \\ \{(\frac{5}{2^{n+2}} - \epsilon, \frac{5}{2^{n+2}}],\, \epsilon > 0\}, & \text{if } x = \frac{5}{2^{n+2}},\ n \in \mathbb{N} \\ \{(x - \epsilon, x + \epsilon),\, \epsilon > 0\}, & \text{otherwise} \end{cases}$$

and using the set of neighbourhoods of each point, we will define the new topology on X as,

$$\mathcal{F}_x = \{V \subseteq X \,|\, \exists U \in \mathcal{B}_x,\, U \subseteq V\}$$

$$\tau_0 = \{\mathcal{O} \subseteq X \,|\, \text{ if } x \in \mathcal{O},\ \text{then } \mathcal{O} \in \mathcal{F}_x\}$$

Now (X, τ_0) is a Hausdorff topological space. Moreover we will show that its topology is locally compact. For all points $x \in X$ such that $x \neq 0$, $x \neq \frac{3}{2^{n+2}}$ and $x \neq \frac{5}{2^{n+2}}$ for all $n \in \mathbb{N}$, we can find $\epsilon > 0$ small enough such that $[x - \epsilon, x + \epsilon]$ contains no points of that form. Then $[x-\epsilon, x+\epsilon]$ is a compact neighbourhood of x. If $x = \frac{3}{2^{n+2}}$ for some $n \in \mathbb{N}$, take $\epsilon > 0$ such that $\epsilon < \frac{2}{2^{2n+2}}$, then $[\frac{3}{2^{2n+2}}, \frac{3}{2^{2n+2}} + \epsilon]$ is a compact neighbourhood. Similarly, for $x = \frac{5}{2^{2n+2}}$ for some $n \in \mathbb{N}$, take $\frac{2}{2^{2n+2}} > \epsilon > 0$ and $[\frac{5}{2^{2n+2}} - \epsilon, \frac{5}{2^{2n+2}}]$ is a compact neighbourhood of x. Thus, since we can find a compact (in τ_0) neighbourhood of each point in X, we have that (X, τ_0) is also a locally compact topological space. Now we will define a new topology on G by describing the neighbourhoods of a point $(x, y) \in G$.

$$B_{(x,y)} := \begin{cases} \{A \times B \,|\, (x, y) \in A \times B,\ A,\, B \in \tau_0\}, & \text{if } x \neq 0,\, y \neq 0, \\ \{\{0\} \times B \,|\, y \in B,\ B \in \tau_0\}, & \text{if } x = 0,\, y \neq 0, \\ \{A \times \{0\} \,|\, x \in A,\ A \in \tau_0\}, & \text{if } x \neq 0,\, y = 0, \\ \{K_n \,|\, n \in \mathbb{N}\}, & \text{if } x = 0,\, y = 0 \end{cases}$$

where

$$K_n := \{(0,0)\} \cup U_n$$

and

$$U_n := \bigcup_{k=n}^{\infty} \left[\frac{3}{2^{k+2}}, \frac{5}{2^{k+2}}\right] \times \left[\frac{3}{2^{k+2}}, \frac{5}{2^{k+2}}\right]$$

for $n \in \mathbb{N}$. Similarly we will define a topology of neighbourhoods on G by,

$$\mathcal{F}_{(x,y)} = \{V \subseteq G \mid \exists U \in B_{(x,y)},\, U \subseteq V\}$$

$$\tau_1 = \{\mathcal{O} \subseteq G \mid \text{ if } (x,y) \in \mathcal{O}, \text{ then } \mathcal{O} \in \mathcal{F}_{(x,y)}\}.$$

We will show that G equipped with the τ_1 topology is a topological groupoid. Since $(x,y)^{-1} = (y,x)$, the preimage of the inverse map maps an open neighbourhood of the form $A \times B$ of (y,x) to $B \times A \in B_{(x,y)}$ and thus to an open set so the inverse map is continuous. The continuity of the product map can be seen by the preimage mapping an open neighbourhood $A \times B$ of (x,z) to $\{((x,y),(y,z)) \mid x \in A,\, y \in X,\, z \in B\} = ((A \times X) \times (X \times B)) \cap G^2$ which is open in G^2 so the product map is also continuous. Thus G is a topological groupoid.

To see that G is second countable, notice that $(0,\,0)$ has countably many neighbourhoods K_n for $n \in \mathbb{N}$. For $(x,\,y) \in G$ and $x \neq 0$ and $y \neq 0$ sets of the type $A \times B$ where

$$A := (q_1 - r_1,\, q_1 + r_1) \;\text{ and }\; B := (q_2 - r_2,\, q_2 + r_2),\; q_1,\, q_2,\, r_1,\, r_2 \in \mathbb{Q}$$

such that $x \in A$ and $y \in B$ form a countable set of neighbourhoods. Without loss of generality, for points of the type $(x,\,0)$, sets of the type $A \times \{0\}$ form a countable set of neighbourhoods.

To see that G is also locally compact, we will show that $(0,0)$ has a compact neighbourhood. Let $((x_n,y_n))_{n\in\mathbb{N}} \in K_m$. Let

$$\mathcal{J} = \{n \in \mathbb{N} \mid (x_n,\, y_n) = (0,0)\}.$$

1. If $\mathcal{J}$ is infinite, then $((x_n,y_n))_{n\in\mathbb{N}}$ has a subsequence converging to $(0,0) \in K_m$. Thus K_m is compact.

2. If $\mathcal{J}$ is finite, then there exists $n_0 \geq m$ such that $(x_n,\, y_n) \in K_n$ for all $n \geq n_0$. Then there exist $u_n,\, v_n \in [3,5]$ and $k_n \in \mathbb{N}$ such that $x_n = \frac{u_n}{2^{k_n+2}}$ and $y_n = \frac{v_n}{2^{k_n+2}}$. If $(k_n)_{n\in\mathbb{N}}$ is not bounded, then $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ have subsequences that converge to 0 in the Euclidean topology of X. For all $n \in \mathbb{N}$,

$$\begin{aligned} &K_m \cap \left(\left(-\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}}\right) \times \left(-\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}}\right)\right) \\ &= \left(U_m \cap \left(\left(-\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}}\right) \times \left(-\frac{5}{2^{n+2}}, \frac{5}{2^{n+2}}\right)\right)\right) \cup \{(0,0)\} \\ &= \begin{cases} U_m \cup \{(0,\,0)\}, & \text{if } m > n, \\ U_{n+1} \cup \{(0,\,0)\}, & \text{if } 0 < m \leq n \end{cases} \end{aligned}$$

In both cases, the set is included in K_n since K_n's are decreasing subsets. It follows that $((x_n,\, y_n))_{n\in\mathbb{N}}$ has a subsequence that converges to $(0,0)$ in τ_1. If $(k_n)_{n\in\mathbb{N}}$ is bounded, then it has a convergent subsequence in the Euclidean topology. Also since u_n and v_n's are bounded, they also have convergent subsequences. Thus $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ have subsequences that converge to $\frac{u}{2^{k+2}}$ and $\frac{v}{2^{k+2}}$ for some $u,\, v \in [3,5]$ and $k \in \mathbb{N}$. Then every neighbourhood of $(\frac{u}{2^{k+2}},\, \frac{v}{2^{k+2}})$ contains another neighbourhood of the form $A \times B$ such that $A,\, B \in \tau_0$. Therefore $((x_n, y_n))_{n\in\mathbb{N}}$ has a subsequence converging to $(\frac{u}{2^{k+2}},\, \frac{v}{2^{k+2}})$ in the topology of τ_1. Thus K_m is compact.

Now we can show that G doesn't have an open range map since for any $B \in \tau_0$,

$$r(\{0\} \times B) = \{(0,0)\}$$

which isn't open in G^0 since it is Hausdorff while $\{0\} \times B$ is a neighbourhood of $(0,\, y)$ for $y \neq 0$ and $y \in B$ and is thus open in τ_1.

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