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A NEW COMPLEX INTEGRAL TRANSFORMATION AND ITS APPLICATIONS

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July 2022

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

A NEW COMPLEX TRANSFORMATION AND ITS APPLICATIONS

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This thesis consists of seven sections. A literature review, including differential equations and integral transforms, has been discussed in the first section. The second section presents the basic definitions and fundamentals required to build a solid background for the work. The third section includes the basic definition of the proposed "Complex (Emad-Faruk-Ghaith) EFG" integral transform. The applicability of the transform to be utilized in solving ordinary and partial differential equations with practical applications is demonstrated in sections four and five, respectively. In section six, the usage of the complex EFG transform in solving Bessel's functions is represented through practical examples. Finally, section seven presents the discussions and conclusions about the concluded work.

2022, 54 pages

Keywords: Complex EFG transform, Ordinary differential equations, Inverse complex EFG transform, Beam, Pharmacokinetics, Bessel’s function.

ÖZET

YENİ BİR KOMPLEKS İNTEGRAL DÖNÜŞÜMÜ VE UYGULAMALARI

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Matematik, Yüksek Lisans

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Bu tez yedi bölümden oluşmaktadır. Birinci bölümde diferansiyel denklemler ve integral dönüşümleri içeren bir literatür taraması ele alınmıştır. İkinci bölüm, çalışma için sağlam bir arka plan oluşturmak için gereken temel tanımları ve esasları sunar. Üçüncü bölüm, önerilen "Karmaşık (Emad-Faruk-Ghaith) EFG" integral dönüşümünün temel tanımını içermektedir. Adi ve kısmi diferansiyel denklemlerin pratik uygulamalarla çözümünde kullanılacak dönüşümün uygulanabilirliği sırasıyla dördüncü ve beşinci bölümlerde gösterilmiştir. Altıncı bölümde, Bessel fonksiyonlarının çözümünde karmaşık EFG dönüşümünün kullanımı pratik örneklerle gösterilmiştir. Son olarak, yedinci bölüm, sonuçlandırılan çalışma hakkındaki tartışmaları ve sonuçları sunmaktadır.

2022, 54 sayfa

Anahtar Kelimeler: Karmaşık EFG dönüşümü, Adi diferansiyel denklemler, Ters karmaşık EFG dönüşümü, Işın, Farmakokinetik, Bessel fonksiyonu.

PREFACE AND ACKNOWLEDGEMENTS

This thesis introduces the "Complex (Emad-Faruk-Ghaith) EFG" transform as a novel general complex integral transformation. The features of the complex EFG transform, its application, and the application of the inverse complex EFG transform to several fundamental functions are reviewed.

I must first thank God Almighty, who enabled me to reach this high scientific stage, and paved the way for me to be among you today to discuss my master’s thesis.

I also extend my highest thanks and gratitude to my thesis advisors, Assoc. Prof. Dr. Faruk KARAASLAN and Asst. Prof. Dr. Emad A. KUFFI, for thier patience, guidance and understanding.

Ghaith Shaeyut Sadkhan ATABI
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CONTENTS

LIST OF SYMBOLS

$\boldsymbol{G^c}\{.\}$	Complex Emad–Faruk–Ghaith integral transform
$\boldsymbol{S^c}(.)$	Complex Sadiq-Emad-Eman integral transform
$\boldsymbol{T^c}\{.\}$	Complex Al-Tememe integral transform
F	Farad
H	Henry
$\boldsymbol{\Omega}$	Ohms
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of Complex numbers
$\boldsymbol{G^{c^{-1}}}\{.\}$	The inverse of Complex Emad–Faruk–Ghaith integral transform
V	Volt

LIST OF ABBREVIATIONS

EFG transform	Emad–Faruk–Ghaith transform
O.D.E	Ordinary Differential Equation.
P.D.E	Partial Differential Equation.
SEE transform	Sadiq-Emad-Eman transform

LIST OF FIGURES

1. INTRODUCTION

It is fundamental in problem-solving to express that problem in a form that can be understood and analyzed (Watanabe 2009). In most scientific fields, this mission can be accomplished through differential equations (Braun and Golubitsky 1983, Hochstadt 2014).

The equation that contains one or multiple variables accompanied by their derivatives is referred to by the term "differential equation." The existence of the variables' derivatives represents the changing rate of these variables at a specific time. The differential equations have two types of variables: the independent variable from which the dependent variables are derived with respect to, and the dependent variables. The general mathematical formula of differential equations is:

$$\frac{dy}{dx} = f(x),$$

where $f(x)$ is the function required to be represented with respect to the variable x; the variable x is the independent and the dependent variable y that is derived with respect to the x variable (Kreyszig 2014, Whitehead 2017).

Differential equations are distinguished by their order or degree, an integer number representing the highest derivative order that appears in the differential equation polynomial. If the differential equation contains variables that are derived with respect to only one variable, then it is called an "Ordinary Differential Equation" (O.D.E), while if the differential equation has multiple variables that are derived partially with respect to multiple variables, then it is called a "Partial Differential Equation" (P.D.E). The P.D.Es are considered a special type of O.D.Es. Furthermore, they can be linear or nonlinear equations, homogeneous or nonhomogeneous equations, as with any equation. Any method that can find a function to satisfy the differential equation is referred to as the solution to that equation. If the solution contains random constants equal to the equation order, then the solution is called a general solution. At the same time, if the solution does not contain any random constants, then it is called an exact solution (Russell 2020).

The importance of differential equations encouraged mathematicians to investigate as many methods to solve them as possible. These investigations produced a decent number of ways to find the general and exact solutions to the differential equations. The integral transforms are the shining star of the methods for solving differential equations (Clark 2017).

Since the introduction of integral transforms by Euler in 1763 (Deakin 1985), a door has been opened for mathematicians to invest their time and effort in studying the existing transforms and suggesting new ones with properties that serve specific applications and can be utilized in other fields (Pundir 2017).

The term "integral transformation" refers to the process of converting a differential equation to an algebraic one. This change should be sufficient to create a more straightforward structure for a mathematical solution (Wolf 1979).

The primary objective of employing integral transforms is to transfer the problem from its current domain, where finding a solution is incredibly challenging, to another domain, where finding a solution is relatively simple.

The general format of integral transform is:

$$G(s) = \int_{a}^{b} k(x,s)g(x)dx \,.$$

The integral boundaries (a and b) are the transformation interval limits, $k(x,s)$ is a function that represents the kernel of the transform and $g(x)$ is the function that is required to transform its domain (integrated with the kernal function) (Saichev *et al.* 2018).

Although the main structure for integral transforms remains the same, the transformation interval limits (a and b) and the transform's kernel function distinguish integral transforms from each other. The main factor that defines the integration domain is represented by the kernel function of the integral transform, which leads to the creation of a new transformation by modifying and changing these factors. The fact that it is possible to get a new integral transform by altering these factors drove the mathematicians to experiment with this aspect. These experiments resulted in the suggestion of many integral transforms that possess different properties (Kuffi *et al.* 2021).

In the entirety of integral transforms, which is an enormous number, some transforms gain popularity among others due to their excessive use in many scientific fields, and sometimes use them as a fundamental pillar to derive new transformations. Some of the most well-known transforms are the Laplace transforms; Laplace introduced them in 1814, representing one of the most celebrated and utilized transforms (Swant 2018, Shahzadi *et al.* 2021). Other transforms that have been introduced recently and which are mostly named after the mathematicians who proposed them are: Smudu transform (Watugala 1993), N-Transform (Khan *et al.* 2008), Elzaki transform (Elzaki 2011), Aboodh transform (Aboodh 2013), ZZ transform (Zafar 2016), Mohand transform (Mohand and Mahgoub 2017), AL-Zughair transform (Mohammed and Abdullah 2018), the Natural Logarithmic transformation (Kuffi and Abbas 2019), Rohit transform (Gupta 2020), Emad-Sara transform (Maktoof *et al.* 2021) and the Generalization of Rangaig transform (Mansour and Kuffi 2022).

The most noticeable property that unites all these transforms are the type of their kernel function, which is a kernel without the complex parameters to which most of the previous years' transforms are inclined. In recent years, however, there has been a strong movement to suggest more transforms with complex parameters at their kernel functions, most of which have the word complex before their names, such as the Complex Al-Tememe transform (Mohammed and Maktoof 2017), the Complex SEE transform (Mansour *et al.* 2021b), the Complex EE transform (Kuffi and Abbas 2022), and the dual-tree complex wavelet transform (Selesnick *et al.* 2005).

This thesis introduces a new complex integral transform called the complex (Emad–Faruk–Ghaith) EFG transform that implements a complex parameter kernel. The basic properties and applications of the EFG transform to the fundamental functions have been studied and proved. The efficiency of the EFG transform in solving ordinary and partial differential equations is demonstrated through examples. The practical applicability of the transform in solving real-life problems in various scientific and engineering fields has been demonstrated via practical applications.

2. GENERAL DEFINITIONS AND BASIC CONCEPTS

This section presents some definitions and basic concepts necessary to fully comprehend the proposed EFG integral transform and its properties.

Definition 2.1. (Goodwine 2010) The boundary condition specifies that the outcome of a solution should always be taken in a particular region of space and time apart. The initial condition specifies a condition that the answer must satisfy in a discrete time interval.

Definition 2.2. (Brychkov and Brychkov 1992) The form that represents an integral transformation as a relationship between two functions $\varphi(t)$ and $\mu(p)$, is

$$\mu(p) = \int_a^b F(p,t)\varphi(t)dt \ ,$$

where:

- The boundaries a and b can be finite or infinite.
- The function $\varphi(t)$ is reffered to as the original function and the domain of the function $\varphi(t)$ is the original space.
- The function $\mu(p)$ is reffered to as the transform and the domain of the $\mu(p)$ function is the image space .
- The function $F(p,t)$ is referred to as the kernel of the transformation, t is a real variable, and $p = \sigma + iw$ is a complex variable.

To shorten the notation, the integral transformation with kernel $F(p,t)$ may be denoted by the symbol T and its called the T transformation as:

$$\mu(p) = T\{\varphi(t)\} \, .$$

Definition 2.3. (Mansour *et al.* 2021a) A new integral transformation, dubbed the SEE (Sadiq-Emad-Eman) integral transform, is stated in set A. Set A is characterized by :

$$A = \{f(t): |f(t)| < M\, e^{\ell_i |t|}, M, \ell_1, \ell_2 > 0,\ \ t \in \mathbb{R}, i = 1,2\},$$

where $t \in X = \begin{cases} (-\infty, 0], & if\ i = 1 \\ [0, \infty), & if\ i = 2 \end{cases}.$

For a given capacity in A's arrangement, the constant M should be a limited number ℓ_1, ℓ_2 might be limited or boundless.

The SEE transformation fundamental change signified by the operator S(.) is defined as follows:

$$S[f(t)] = T(v) = \frac{1}{v^n}\int_0^\infty f(t)e^{-vt}\,dt\,, n \in \mathbb{Z}, t \geq 0, \ell_1 \leq v \leq \ell_2\,.$$

The variable v in this change is utilized to figure out the variable t in the function f. This change has a further association with the Laplace, Aboodh, and Mohanad changes.

Definition 2.4. (Mansour *et al.* 2021b) For set B that is defined by:

$$B = \{\, g(t): |g(t)| < M^{-il_j|t|}, M, \ell_1, \ell_2 > 0, t \in \mathbb{R}, j = 1,2\},$$

where $t \in X = \begin{cases} (-\infty, 0], & if\ j = 1 \\ [0, \infty), & if\ j = 2 \end{cases}.$

where i is a complex number, $i^2 = -1$ for a function $g(t)$ of an exponential nature in the set B, the constant M must be finite number, l_1 and l_2 may be finite or infinite.

The Complex SEE transformation denoted by the operator $S^c(.)$ is defined by :

$$S^c\{g(t)\} = \frac{1}{v^n} \int_{t=0}^{\infty} g(t)e^{-ivt}dt = T(iv), t \geq 0\,, l_1 \leq v \ \leq l_2\ \ , n \in \mathbb{Z}\,.$$

The variable in the complex SEE transformation is used to factor the variable t in the argument of the function $g(t)$.

Definition 2.5. (Mohammed and Kathem 2008) The convergent integral that defines the Al-Tememe transformation for a function $f(x)$; $x > 1$ and for p as a positive constant is as follows :

$$\mathcal{T}[f(x)] = \int_{1}^{\infty} x^{-p}f(x)dx = F(p).$$

Definition 2.6. (Mohammed and Maktoof 2017) The definition of the convergent complex Al-Tememe integral transform in the interval $[1,\infty)$ for the function $g(x); x > 1$ is:

$$T^c\{g(x)\} = \int_{x=1}^{\infty} x^{-ip}g(x)dx = F(ip)\,.$$

Where p is a positive constant and x^{ip} is the kernel of the complex Al-Tememe transformation and $i = \sqrt{-1}$.

Property (2.1) Linearity of Integral Transformations (Pundir 2017)

(i) The following equations can be utilized to express the integral transformations' linearity property:

- $T[a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t)] = a_1 T\big(f_1(t)\big) + a_2 T\big(f_2(t)\big) + \cdots +$ $a_n T\big(f_n(t)\big)$. a_1, a_2 , ..., and a_n are constants, and the functions $f_1(t)$, $f_2(t), \ldots,$ and $f_n(t)$ are predefined.

(ii) The linear property of the inverse integral transformation is characterized as:

$$T^{-1}[a_1\Omega_1(s) + a_2\Omega_2(s) + \cdots + a_n\Omega_n(s)] = a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t) .$$

3. THE COMPLEX EMAD–FARUK–GHAITH (EFG) INTEGRAL TRANSFORM

In this section a new complex integral transform that named after mathematicians how proposed it which is called the complex Emad–Faruk–Ghaith (EFG) integral transfomation. The EFG transformation definition, basic properties and its application on some basic functions are going to be presented and proofed.

3.1. The Complex EFG Integral Transform

For the function of exponential order in set B, which defined as:

$$B = \left\{f(t): |f(t)| < Me^{-iLj|t|} \; M, L_1, L_2 \; > 0, \mathrm{t} \in \mathbb{R}, \mathrm{j} = 1,2\right\} \tag{3.1}$$

where $t \in X = \begin{cases} (-\infty, 0], & if\ j = 1 \\ [0, \infty), & if\ j = 2 \end{cases}$ and i is a complex number and $i^2 = -1$, the M constant must be a finite value for a particular function in the set B, while $L_1\ and\ L_2$ may be finite or infinite.

The complex EFG integral transform denoted by the operator $G^c\{.\}$ is defined as:

$$G^C\{f(t)\} = F(iv) = \lim_{\mathrm{p}\to\infty} \int_{t=0}^{\mathrm{p}} f(t)e^{-iq(v)t}dt\,, \;\; t \geq 0\,, L_1 \leq q(v) \leq L_2 \tag{3.2}$$

$q(v)$ is a function of the parameter v.

The variable iv in the complex EFG transform is used to factor the variable t in the argument of the function $f(t)$.

3.2 The Complex EFG Integral Transformation for Some Functions

Suppose that the complex EFG transformation defined in equation (3.2) exists for any $f(t)$.

The sufficient situation for the existence of the complex EFG integral transform is that, for the function f (t) for $t \geq 0$ being piecewise continuous and of exponential order, the complex EFG integral transform may or may not exist.

1. Let $f(t) = k$, where k is arbitrary constant.

From the definition:

$$G^c\{k\} = F(iv) = \lim_{p\to\infty} \int_{t=0}^{p} e^{-iq(v)t} k\, dt = \frac{-k}{iq(v)} [\, e^{-iq(v)t} \Big|_{0}^{p},$$

$$= \frac{-k}{iq(v)} [0 - 1] = \frac{-ik}{q(v)} .$$

$$G^c\{k\} = \frac{-ik}{q(v)}, \qquad q(v) \neq 0\ ,\ Re\big(q(v)\big) > 0\ .$$

2. Let $f(t) = t$. Then, $G^c\{t\} = \lim_{p\to\infty} \int_{t=0}^{p} e^{-iq(v)t} t\, dt.$

Integration by parts:

$$G^c\{t\} = -\frac{1}{(q(v))^2} \qquad , q(v) \neq 0 \quad , Re\big(q(v)\big) > 0\, .$$

Similarly, performing integration by parts:

(i) $G^c\{t^2\} = \frac{(2!)i}{\left(q(v)\right)^3}$. $\quad q(v) \neq 0, Re\big(q(v)\big) > 0$.

(ii) $G^c\{t^3\} = \frac{(3!)}{\left(q(v)\right)^4}$. $\quad q(v) \neq 0\ , Re\big(q(v)\big) > 0$.

(iii) In general: $G^c\{t^n\} = \frac{(-1)^n (i)^{n-1}\, n!}{(q(v))^{n+1}}$, $q\ (v) \neq 0\ \ Re\big(q(v)\big) > 0$, n is a positive integer number.

3. Let $f(t) = e^{at}$, then $G^c\{e^{at}\} = \lim_{p\to\infty} \int_{t=0}^{p} e^{at}. e^{-iq(v)t} dt$ where a is a constant.

After simple computations:

$$G^c\{e^{at}\} = -\left[\frac{a}{a^2 + \big(q(v)\big)^2} + i\frac{q(v)}{a^2 + \big(q(v)\big)^2}\right],$$

$$a - iq(v) \neq 0\,, \qquad Re\big(a - iq(v)\big) > 0\,.$$

Where a is an arbitrary constant.

This outcome will be useful in determining the complex EFG transform for:

1. $G^c\{\sin(at)\} = \frac{-a}{(q(v))^2 - a^2}$. $q(v) > |a|$, where a is a constant.

2. $G^c\{\cos(at)\} = \frac{-iq(v)}{(q(v))^2 - a^2}$. $q(v) > |a|$, where a is a constant.

3. $G^c\{\sinh(at)\} = \frac{-a}{(q(v))^2 + a^2}$. $Re\big(q(v)\big) > 0,\ q(v) \neq \pm ia$.

4. $G^c\{\cosh(at)\} = \frac{-iq(v)}{(q(v))^2 + a^2}$. $Re\big(q(v)\big) > 0,\ q(v) \neq \pm ia$.

Theorem 3.1. Let $F(iv)$ be the general complex integral transform of $f(t)$

$$F(iv) = G^c\{f(t)\}\,.$$

Then:

(i) $G^c\{\acute{f}(t)\} = -f(0) + iq(v)F(iv)$.

(ii) $G^c\{f''(t)\} = -\acute{f}(t) - iqvf(0) - \big(q(v)\big)^2 F(iv)$

(iii) In general case:

$$G^c\{f^{(n)}(t)\} = -f^{(n-1)}(0) - iqvf^{(n-2)}(0) - \big(iq(v)\big)^2 f^{(n-3)}(0) - \cdots$$
$$- \big(iq(v)\big)^{n-1} f(0) + \big(iq(v)\big)^n F(iv)\,.$$

Where n is a postive integer number

Proof:

(i) From the complex EFG transformation definition:

$$G^c\{\acute{f}(t)\} = \lim_{p\to\infty} \int_0^p e^{-iq(v)t} \acute{f}(t)dt\,.$$

Performing integration by parts:

$$u = e^{-iq(v)t} \quad , \quad dv = \acute{f}(t)dt\,,$$

$$du = -iq(v)e^{-iq(v)t}dt \quad , \quad v = f(t)\,.$$

$$G^c\{\acute{f}(t)\} = f(t)\, e^{-iq(v)t} \Big|_0^p + \lim_{p\to\infty} \int_0^p f(t) \quad iq(v)\, e^{-iq(v)t}dt,$$

$$G^c\{\acute{f}(t)\} = [-f(o)] + iq(v)G^c\{f(t)\}.$$

Thus,

$$G^c\{\acute{f}(t)\} = -f(0) + iq(v)F(iv)\,.$$

(ii) From the complex EFG transformation definition:

$$G^c\{f''(t)\} = \lim_{p\to\infty} \int_0^p e^{-iq(v)t} f''(t)dt\,.$$

Performing integration by parts:

$$u = e^{-iq(v)t} \;,\; dv = f''(t)dt\,.$$

$$du = -iq(v)e^{-iq(v)t} \quad , \quad v = \acute{f}(t).$$

$$G^c\{f''(t)\} = [\acute{f}(t)e^{-iq(v)t} \Big|_0^p + \lim_{p\to\infty} \int_0^p \acute{f}(t) iq(v)\, e^{-iq(v)t} dt \,.$$

$$G^c\{f''(t)\} = -\acute{f}(0) + iqv\,[-f(0) + iqv\, G^c\{f(t)\}] \,.$$

Hence

$$G^c\left\{\acute{\acute{f}}(t)\right\} = -\acute{f}(0) - iqv\, f(0) - \left(q(v)\right)^2 G^c\{f(t)\} \,.$$

(iii) It can be proven by mathematical induction.

Proposition 3.1 (Shifting property) If $\{F(t)\} = F\big(iq(v)\big)$ then, $G^c\{e^{at} f(t)\} = F(iq(v) - a)$, where a is a constant.

Proof: By definition

$$G^C\{f(t)\} = \lim_{p\to\infty} \int_{t=0}^{p} f(t) e^{-iq(v)t} dt \,,$$

$$G^c\{e^{at} f(t)\} = \lim_{p\to\infty} \int_{t=0}^{p} e^{at} f(t)\, e^{-iq(v)t} dt = \lim_{p\to\infty} \int_{t=o}^{p} e^{-[iq(v)-a]t} f(t) dt \,,$$

or

$$G^c\{e^{at} f(t)\} = F(iq(v) - a) \,.$$

$$G^c\{e^{-at} f(t)\} = F(iq(v) + a) \,.$$

Proposition 3.2 (Convolution Property) If $G^c\{f(t)\} = F_1(iv)$ and $G^c\{h(t)\} = F_2(iv)$, then

$$G^c\{f(t) * h(t)\} = G^c\{f(t)\} * G^c\{h(t)\}$$

$$= F_1(iv). F_2(iv).$$

Proposition 3.3 (Linearity Property)

$$G^c[\alpha f(t) + \beta g(t)\} = \alpha\, G^c\{f(t)\} + \beta G^c\{g(t)\}\,,$$

where α and β are arbitrary constants.

Proposition 3.4 (Change of Scale Property) If $G^c\{f(t)\} = F(iv)$ then, $G^c\{f(at)\} = \lim_{p\to\infty} \int_0^p f(at)e^{-iq(v)t}dt$. Subsituting $at = p \Rightarrow adt = dp$ in the above equation:

$$G^c\{f(at)\} = \frac{1}{a}\int_0^p f(p)e^{-iq(v)\frac{p}{a}}\,dp = \frac{1}{a}\,F(iav)\,.$$

Thus, if $G^c\{f(at)\} = F(iv)$ then:

$$G^c\{f(at)\} = \frac{1}{a}F(iav).$$

3.3 The Complex EFG Transform of Derivatives

The complex EFG transform is defined for the function $f(t)$ as:

$$G^C\{f(t)\} = \lim_{p\to\infty}\int_{t=0}^{p} e^{-iq(v)t}f(t)dt = F(iv)\,, t \geq 0\ , L_1 \leq q(v) \leq L_2\,, \qquad (3.3)$$

$q(v)$ is a function of the paramter v. i is an imaginary number , $i = \sqrt{-1}$ and $L_1, L_2 \geq 0$.

To obtain the complex EFG transform of partial differential equations $(\mathrm{P, D, Es})$, integration by parts is used as follows:

$$G^c\left\{\frac{\partial f}{\partial t}\ (x,t)\right\} = \lim_{p\to\infty}\int_{t=0}^{p}\frac{\partial f}{\partial t}e^{-iq(v)t}dt\,.$$

Let

$$u = e^{-iq(v)t}\qquad , \qquad dv^* = \frac{\partial f}{\partial t}\,,$$

$$du = -iq(v)e^{-iq(v)t}dt\qquad , \qquad v^* = f(x,t)\,.$$

$$\int u dv^* = u.v^* - \int v^* du\,.$$

Then

$$\lim_{p\to\infty}\int_{t=0}^{p}\frac{\partial f}{\partial t}e^{-iq(v)t}dt = \lim_{p\to\infty}\left[f(x,t)e^{-iq(v)t}\Big|_{0}^{p} + \int_{t=0}^{p}f(x,t)iq(v)e^{-iq(v)t}dt\,\right].$$

So,

$$G^c\left\{\frac{\partial f}{\partial t}\ (x,t)\right\} = [-f(x,0) + iq(v)F(x,iv)]\,,$$

or

$$G^c\left\{\frac{\partial f}{\partial t}\ (x,t)\right\} = iq(v)F(x,iv) - f(x,0). \qquad (3.4)$$

Assuming $f(x,t)$ is an exponential, piecewise continuous function.

Now,

$$G^c\left\{\frac{\partial f}{\partial x}\right\} = \lim_{p\to\infty}\int_{t=0}^{p} e^{-iq(v)t}\frac{\partial f}{\partial x}.\,dt\,,$$

$$= \frac{\partial}{\partial x}\int_{t=0}^{p} e^{-iq(v)t} f(x,t)dt\,,$$

$$\frac{\partial}{\partial x}[F(x,iv)]\,.$$

Or

$$G^c\left\{\frac{\partial f}{\partial x}\right\} = \frac{d}{dx}[F(x,iv)]\,. \quad (3.5)$$

Also, it is possible to find:

$$G^c\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \frac{d^2}{dx^2}[F(x,iv)]\,. \quad (3.6)$$

To find:

$$G^c\left\{\frac{\partial^2 f}{\partial t^2}\ (x,t)\right\}.$$

Let $\frac{\partial f}{\partial t} = g(x,t)$. Then,

$$G^c\left\{\frac{\partial^2 f}{\partial t^2}\ (x,t)\right\} = G^c\left\{\frac{\partial g}{dt}(x,t)\right\} = iq(v)G^c\{g(x,t)\} - g(x,0)\,,$$

$$G^c\left\{\frac{\partial^2 f}{\partial t^2}\right\} = iq(v)G^c\left\{\frac{\partial f}{\partial t}(x,t)\right\} - \frac{\partial f}{\partial t}(x,0)\,,$$

$$G^c\left\{\frac{\partial^2 f}{\partial t^2}\right\} = iq(v)[ig(v)F(x,iv) - f(x,0)] - \frac{\partial f}{\partial t}(x,0)\ .$$

So,

$$G^c\left\{\frac{\partial^2 f}{\partial t^2}\right\} = -\big(q(v)\big)^2F(x,iv) - iq(v)f(x,0) - \frac{\partial f}{\partial t}(x,0)\ . \qquad (3.7)$$

It is possible to extend this result to the n^{th} partial derivative by "Mathematical Induction".

3.4 The Inverse Complex EFG Integral Transform Definition

If $G^c\{f(t)\} = F(iv)$, then $f(t)$ is said to be the inverse of the complex EFG integral transform of $F(iv)$ and mathematically can be defined by:

$$f(t) = G^{c^{-1}}\{F(iv)\}\ .$$

Where $G^{c^{-1}}\{.\}$ is the invers of the complex EFG integral transform operator.

3.5 The Inverse Complex EFG Transform for Some Elementary Functions

1. $G^{c^{-1}}\{k\} = \frac{-ik}{q(v)}$, k= Constant.
2. $G^{c^{-1}}\{t\} = \frac{-1}{(q(v))^2}$.
3. $G^{c^{-1}}\{t^n\} = \frac{(-1)^n(i)^{n-1}n!}{\big(q(v)\big)^{n+1}}$, n is a positive integer number and i is a complex number .
4. $G^{c^{-1}}\{e^{at}\} = -\left[\frac{a}{a^2+\big(q(v)\big)^2} + i\,\frac{q(v)}{a^2+\big(q(v)\big)^2}\right]$, a is an arbitrary constant.
5. $G^{c^{-1}}\left\{\frac{\sin(at)}{a}\right\} = \frac{-1}{(q(v))^2-a^2}$.
6. $G^{c^{-1}}\{\cos(at)\} = \frac{-iq(v)}{(q(v))^2-a^2}$.
7. $G^{c^{-1}}\left\{\frac{sinh(at)}{a}\right\} = \frac{-1}{\big(q(v)\big)^2+a^2}$.
8. $G^{c^{-1}}\{cosh(at)\} = \frac{-iq(v)}{\big(q(v)\big)^2+a^2}$.

4. APPLYING THE COMPLEX EFG INTEGRAL TRANSFORMATION INTO ORDINARY DIFFERENTIAL EQUATIONS (O.D.E^s)

To prove the competence of the proposed "complex FEG" integral transform, the applicability of the transform in solving the ordinary differential equations that represent different scientific fields has been taken into account and proved via actual examples in this section. The scientific fields that the complex FEG transform has covered include different kinds of differential equations, Bessel's function, and several applications in engineering, including electrical and civil engineering, physics, nuclear physics, Newton's law of cooling, pharmacokinetics, and natural growth and decay.

Application 4.1: Let us consider 1st order differential equation given as follows:

$$\frac{dy}{dx} + y = 0 \text{, with } y(0) = 1. \tag{4.1}$$

Applying complex EFG transform to equation (4.1) gives:

$$G^c\{\acute{y}\} + G^c\{y\} = 0 \,,$$

$$-y(0) + iq(v)G^c\{y\} + G^c\{y\} = 0 \,,$$

$$-1 + iq(v)G^c\{y\} + G^c\{y\} = 0 \,,$$

$$\big(1 + iq(v)\big)G^c\{y\} = 1 \,.$$

$$\Rightarrow G^c\{y\} = \frac{1}{1 + iq(v)} \;.$$

Then

$$G^c\{y\} = \frac{1}{1+iq(v)} \cdot \frac{1-iq(v)}{1-iq(v)} = \frac{1-iq(v)}{1+(q(v)^2} \;.$$

$$G^c\{y\} = -\left[\frac{-1}{1+\big(q(v)\big)^2} + i\,\frac{q(v)}{1+\big(q(v)\big)^2}\right].$$

Using the inverse of complex EFG transform, obtains:

$$y(x) = e^{-x} \,.$$

The obtained result is the solution to equation (4.1).

Application 4.2: Solve the following initial value problem (I.V.P):

$$y' + 2y = x \text{ , with } y(0) = 1 \,. \tag{4.2}$$

Taking the complex EFG transform to equation (4.2):

$$G^c\{y'\} + 2G^c\{y\} = G^c\{x\} \,,$$

$$-y(0) + iq(v)G^c\{y\} + 2G^c\{y\} = \frac{-1}{\big(q(v)\big)^2} \,,$$

$$\big(2 + iq(v)\big)G^c\{y\} = \frac{-1}{\big(q(v)\big)^2} + 1 \,,$$

$$G^c\{y\} = \frac{-1}{\big(q(v)\big)^2 + (2 + iq(v))} + \frac{1}{(2 + iq(v))} \,.$$

Now, taking

$$\frac{-1}{\big(q(v)\big)^2 + (2 + iq(v))} = \frac{A}{\big(q(v)\big)^2} + \frac{B}{q(v)} + \frac{C}{(2 + iq(v))} \,.$$

After simple computations: $A = -\frac{1}{2}$, $B = \frac{i}{4}$ $C = \frac{1}{4}$.

Then,

$$G^c\{y(x)\} = \frac{-1}{2\big(q(v)\big)^2} + \frac{i}{4q(v)} + \frac{1}{4(2 + iq(v))} + \frac{1}{(2 + iq(v))} \,,$$

$$G^c\{y(x)\} = \frac{-1}{2\big(q(v)\big)^2} + \frac{i}{4q(v)} + \frac{5}{4}\left[\frac{1}{\big(2+iq(v)\big)} \cdot \frac{\big(2-iq(v)\big)}{\big(2-iq(v)\big)}\right] ,$$

$$= \frac{-1}{2\big(q(v)\big)^2} + \frac{i}{4q(v)} + \frac{-5}{4}\left[\frac{2}{4+\big(q(v)\big)^2} + \frac{iq(v)}{4+\big(q(v)\big)^2}\right].$$

Applying the inverse of complex EFG transform gives:

$$y(x) = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x} .$$

The obtained result is the solution to equation (4.2).

Application 4.3: Solve the following (I.V.P):

$$\frac{d^2y}{dx^2} + 9y = \cos(2x) \text{, with } y(0) = 1 , y\left(\frac{\pi}{2}\right) = -1 . \quad (4.3)$$

Since $y'(0)$ is unknow, let $y'(0) = b$.

Applying the complex EFG transform to equation (4.3), gives:

$$G^c\{y''\} + 9S^c\{y\} = G^c\{cos(2x)\} ,$$

$$-y'(0) - iq(v)y(0) - \big(q(v)\big)^2 G^c\{y(x)\} + 9G^c\{y(x)\} = \frac{-iq(v)}{\big(q(v)\big)^2 - 4} ,$$

$$-b - iq(v) - \big(q(v)\big)^2 G^c\{y(x)\} + 9G^c\{y(x)\} = \frac{-iq(v)}{\big(q(v)\big)^2 - 4} ,$$

$$\left[9 - \big(q(v)\big)^2\right] G^c\{y(x)\} = b + iq(v) - \frac{-iq(v)}{\big(q(v)\big)^2 - 4} .$$

Then,

$$G^c\{y(x)\} = \frac{b}{9-\left(q(v)\right)^2} + \frac{iq(v)}{9-\left(q(v)\right)^2} - \frac{iq(v)}{\left(q(v)\right)^2 - 4)(9-\left(q(v)\right)^2} \quad .$$

Now, taking

$$\frac{-iq(v)}{[\left(q(v)\right)^2 - 4)]\,[(9-\left(q(v)\right)^2]} = \frac{Aq(v)+B}{\left(q(v)\right)^2 - 4)} + \frac{Cq(v)+D}{9-\left(q(v)\right)^2} \quad .$$

We get $A = C = \frac{-i}{5}$ and $B = D = 0$.

Then,

$$G^c\{y(x)\} = F(iv)$$
$$= \frac{-3b}{3\left[9-\left(q(v)\right)^2\right]} + \frac{-iq(v)}{\left(q(v)\right)^2 - 9} + \frac{-iq(v)}{5\left[\left(q(v)\right)^2 - 4\right]}$$
$$+ \frac{iq(v)}{5\left[\left(q(v)\right)^2 - 9\right]} .$$

Taking the inverse of complex EFG transform:

$$y(x) = \frac{b}{3}\sin(3x) + \frac{4}{5}\cos(3x) + \frac{1}{5}\cos(2x).$$

To determine the value of b, note that: $y\left(\frac{\pi}{2}\right) = -1$ from that it is possible to find: $b = \frac{12}{5}$.

Then, the solution to equation (4.3) is:

$$y(x) = \frac{1}{5}\cos(2x) + \frac{4}{5}\cos(3x) + \frac{4}{5}\sin(3x) \, .$$

Application 4.4: An application of the complex EFG integral transform is give in a simple electric circuit as follows:

Figure 4.1 shows an electric circuit with a switch connected serially with a resistance R, an inductance L, a capacitive condenser C, and a power source of voltage E.

From Kirchhoff's law:

$$L\frac{dI}{dt}+RI+\frac{Q}{C}=E\,. \tag{4.4}$$

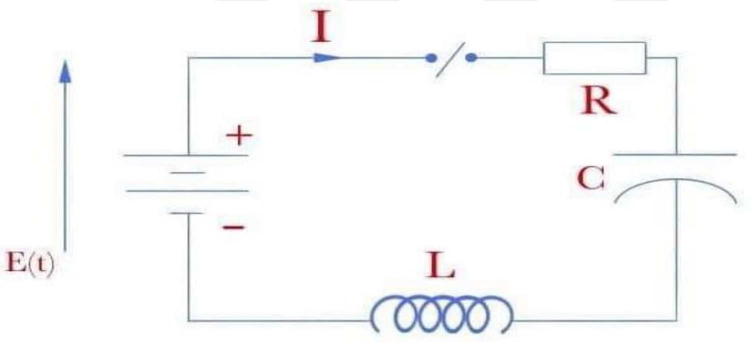

Figure 4.1 Serial electrical circuit.

If the inductance value is 2 H, the resistor value is 16 Ω, and the capacitor value is 0.02 F, which is linked in series with the emf of 300 V, and at time t = 0, the change in the circuit's capacitor and current are both zero, then it is possible to check the charge and current for the circuit at time t > 0.

Let Q and I be the prompt change and the current correspondingly at the time t.

Then by Kirchhoff's law: $L\frac{dI}{dt}+RI+\frac{Q}{C}=E,$

$2Q''(t)+16Q'(t)+50Q(t)=E$ where $I=Q'(t)$.

Then,

$$Q''(t) + 8Q'(t) + 25Q(t) = 150\ .$$

By simplifying: $\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 25y = 150$.

Taking the complex EFG integral transform to the above equation gives:

$$-y'(0) - iq(v)y(0) - \left(q(v)\right)^2 G^c\{y(t)\} + 8(-y(0) + iq(v)G^c\{y(t)\} + 25G^c\{y(t)\}$$
$$= 150G^c\{1\}\ .$$

Using the provided initial conditions:

$$F(iv) = G^c\{y(t)\} = \frac{-150i}{q(v)\left[-\left(q(v)\right)^2 + 8iq(v) + 25\right]},$$

$$G^c\{y(t)\} = 150i\left[\frac{A}{q(v)} + \frac{Bq(v) + D}{\left(q(v)\right)^2 - 8iq(v) - 25}\right].$$

Some simple computations provide: $A = \frac{-1}{25}$, $B = \frac{1}{25}$ and $D = \frac{-8}{25}i$.

So,

$$G^c\{y(t)\} = \frac{-6i}{q(v)} - \frac{6(iq(v) + 4) - 24}{[(iq(v) + 4)^2 + 9]} - \frac{24}{[(iq(v) + 4)^2 + 9]}\quad ,$$

$$F(iv) = G^c\{y(t)\} = \frac{-6i}{q(v)} - \left[\frac{-6i(q(v) - 4i)}{(q(v) - 4i)^2 - 9}\right] - \left[\frac{-(8)(3)}{(q(v) - 4i)^2 - 9}\right]\ .$$

Taking the inverse of complex EFG integral transform and providing the shifting property gives:

$$y(t) = 6 - 6e^{-4t}\cos(3t) - 8e^{-4t}\sin(3t)\ .$$

or

$$Q(t) = 6 - 6e^{-4t}\cos(3t) - 8e^{-4t}\sin(3t) \ .$$

and

$$I = Q' = \frac{dQ}{dt} = 50e^{-4t}\sin(3t) \ .$$

The above eqution is the expresion for the charge and the current at any time, where $t > 0$.

Application 4.5: Applying the complex EFG transform in nuclear physics.

Considering the nuclear physics fundamentals (Kuffi *et al.* 2020).

Taking the following first order linear ordinary differential equation that describes the radioactive decay relation:

$$\frac{dN(t)}{dt} = -\propto N(t) \ . \tag{4.5}$$

Where $N(t)$ during time t denotes the number of undecayed atoms left in a sample of a radioactive isotope, and ($\propto$) is the decay constant.

It is possible to utilize the complex EFG transform $G^c\{.\}$ into equation (4.5) as:

$$G^c\{N'(t)\} + \propto G^c\{N(t)\} = 0.$$

Therefore

$$-N(0) + iq(v)G^c\{N(t)\} + \propto G^c\{N(t)\} = 0 \ ,$$

$$\big(\propto + iq(v)\big)G^c\{N(t)\} = N(0) \ ,$$

$$G^c\{N(t)\} = \frac{N(0)}{\propto + iq(v)} = \frac{N_0}{\propto + iq(v)} \cdot \frac{\propto - iq(v)}{\propto - iq(v)} \quad ,$$

$$G^c\{N(t)\} = -N_0 \left[\frac{-\propto}{\propto^2 + \left(q(v)\right)^2} + \frac{iq(v)}{\propto^2 + \left(q(v)\right)^2} \right] \quad .$$

Applying the inverse of the complex EFG integral transform to the above equation obtains:

$$N(t) = N_0 e^{-\propto t}.$$

The resulted equation is the proper type of radioactive decay.

Application 4.6: Applying the complex EFG to solving pharmacokinetics problem.

$$\acute{C}(t) + \lambda C(t) = \frac{\beta}{volume} \, , t > 0 \, . \tag{4.6}$$

With $C(0) = 0$.

Where:

C(t) at any time t: Is the medication concentration in the blood.

λ: The elimination constant velocity.

β: The infusion proportion (in ${}^{mg}/_{min}$).

Volume: The drug distributed volume.

Applying the complex EFG transform to equation (4.6) gives:

$$-C(0) + iq(v)G^c\{C(t)\} + \lambda G^c\{C(t)\} = \frac{\beta}{volume} G^c\{1\} \, ,$$

$$(iq(v)+\lambda)G^c\{C(t)\} = \frac{\beta}{volume} \cdot \frac{-i}{q(v)},$$

$$G^c\{C(t)\} = F(iv) = \frac{-i\beta}{volume\ q(v)(iq(v)+\lambda)} ,$$

$$F(iv) = \frac{\beta}{volume}\left[\frac{1}{iq(v)(iq(v)+\lambda)}\right] ,$$

$$F(iv) = \frac{\beta}{volume}\left[\frac{A}{iq(v)} + \frac{D}{iq(v)+\lambda}\right] .$$

After simple computations, it is obtained that: $A = \frac{1}{\lambda}\ \ and\ D = \frac{-1}{\lambda}$.

So,

$$G^c\{C(t)\} = F(iv) = \frac{\beta}{volume}\left[\frac{\frac{1}{\lambda}}{iq(v)} + \frac{\frac{-1}{\lambda}}{iq(v)+\lambda}\right].$$

Then,

$$G^c\{C(t)\} = F(iv) = \frac{\beta}{\lambda.volume}\left[\frac{-i}{q(v)} + \frac{-\lambda}{\left(q(v)\right)^2+\lambda^2} + \frac{iq(v)}{\lambda^2+\left(q(v)\right)^2}\right].$$

Taking the inverse of the complex FEG transform gives:

$$C(t) = \frac{\beta}{\lambda.volume}\left[1-e^{-\lambda t}\right].$$

Application 4.7: Applying the complex EFG into solving the beam deflecting problem. A beam that is hinged at its ends, $x = 0$ and $x = L$ and carries a uniform loud w_0 per unit length, as shown in figure 4.2, can be represented via ordinary differential equations and specific initial conditions. Furthermore, as a differential equation, it is possible to solve the beam deflecting problem using the complex EFG integral transformation (Kuffi *et al*. 2019).

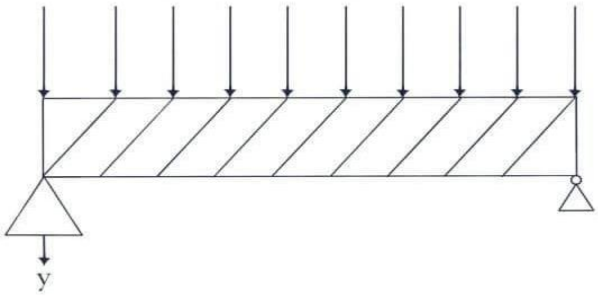

Figure 4.2 A beam under the influanc of uniform load.

The ordinary differential equation and boundary conditions of a deflecting beam are:

$$y^{(4)} = \frac{W_0}{EI} \text{ , } Or \ \frac{d^4y}{dx^4} = \frac{W_0}{EI} \text{ ,} 0 < x < L \text{ ,} \quad (4.7)$$

$$y(0) = y_0 = 0, y''(0) = y''_0 = o, y(L) = 0, y''(L) = y''_L = 0 \text{ .} \quad (4.8)$$

Where:

- E is young's modulus,
- I is the inertia moment for the cross section about an axis normal to the plane of bending,
- EI is the the beam flexural rigidity.

The beam's slope, bending moment, and point's shear are physical quantities associated with the problem. The equations that represent them respectively are:

$$y'(x) \text{ ,} M(x) = EIy''(x) \quad \text{and} \quad S(x) = M'(x) = EIy'''(x) \text{ .}$$

Applying the complex EFG transform to equation (4.7), it is obtained that

$$G^c\left\{\frac{d^4y}{dx^4}\right\} = \frac{W_0}{EI}G^c\{1\},$$

$$-y'''(0) - iqvy''(0) - \left(iq(v)\right)^2 y'(0) - \left(iq(v)\right)^3 y(0) + \left(iq(v)\right)^4 F(iv) = \frac{W_0}{EI}\left(\frac{-i}{q(v)}\right) \tag{4.9}$$

By using the first and second conditions from equation (4.8) in equation (4.9) and the unknown conditions $y'(0) = c_1, y'''(0) = c_2$, it is possible to find:

$$-c_2 - \left(iq(v)\right)^2 c_1 + \left(iq(v)\right)^4 F(iv) = \frac{-W_0 i}{EI\, q(v)} \quad ,$$

$$\left(iq(v)\right)^4 F(iv) = c_2 + \left(iq(v)\right)^2 c_1 - \frac{W_0 i}{EI\, q(v)} \quad ,$$

$$F(iv) = \frac{c_2}{i^4(q(v))^4} + \frac{i^2(q(v))^2 c_1}{i^4(q(v))^4} - \frac{W_0 i}{EIi^4\left(q(v)\right)^5} \quad ,$$

$$F(iv) = \frac{c_2}{(q(v))^4} + \frac{-c_1}{(q(v))^2} + \frac{W_0 i}{EIi^4\left(q(v)\right)^5} \quad ,$$

$$F(iv) = \frac{-c_1}{(q(v))^2} + \frac{c_2}{(q(v))^4} + \frac{W_0 i}{EIi^4\left(q(v)\right)^5} \quad .$$

Applying the invers of the complex EFG transformation to find:

$$y(x) = c_1 x + c_2 \cdot \frac{x^3}{6} + \frac{w_0}{EI} \cdot \frac{x^4}{24}.$$

From the last two conditions in equation (4.9):

$$c_1 = \frac{w_0 L^3}{24EI}, c_2 = \frac{w_0 L}{2EI}.$$

Thus the required deflection of the beam is:

$$y(x) = \frac{w_0}{24EI} x(L - x)(L^2 - Lx - x^2)\,.$$

The concluding equation facilitated the calculation of the bending moment and shear at any point along the beam, most particularly at the ends.

Application 4.8: Utilizing the complex EFG transform into Newton's Law of cooling. The basic statement of Newton's Law is that the rate at which a body's temperature changes is proportional to the temperature differential between the body and its surrounding medium.

Assuming that θ would be the body temperature at any time t and θ_0 would be the surrounding medium temperature. Then from Newton's Law of cooling:

$$\frac{d\theta}{dt} \alpha[\theta(t) - \theta_0(t)] \Rightarrow \frac{d\theta}{dt} = -\mu[\theta(t) - \theta_0(t)]. \qquad (4.10)$$

Where μ is the proportionality constant ($\mu > 0$).

Equation (4.10) is a differential equation that can be solved via the usage of the complex EFG transformation.

For a body temperature that is come down from 80℃ to $60^{\circ}C$ in 20 minutes, when it was placed at a position in which the surrounding air temperature is at 40℃ .Then what will be the body temperature after 40 minutes? When will the temperature reaches 55℃.

Solution: Given that the ambient air temperature is: $\theta_0 = 40$℃ , and the initial temperature i.e. at time $t = 0$ is $\theta(0) = 80$℃ and after time $t = 20$ min. is $\theta(20) = 60$℃.

Newton's Law of cooling in eqation (4.10) is:

$$\frac{d\theta}{dt}\alpha[\theta(t)-\theta_0(t)],$$

$\Rightarrow \frac{d\theta}{dt} = -\mu[\theta(t)-\theta_0(t)]$, where μ is a constant.

$$\Rightarrow \frac{d\theta}{dt} = -\mu\theta(t) + \mu\theta_0(t)\,,$$

$$\frac{d\theta}{dt} + \mu\theta(t) = \mu\theta_0(t)\,.$$

Or

$$\theta'(t) + \mu\theta(t) = 40\mu\,. \tag{4.11}$$

By applying the complex EFG transform to equation (4.11):

$$G^c\left\{\frac{d\theta}{dt} + \mu\theta(t)\right\} = G^c\{\mu\theta_0(t)\}\,.$$

Using the linear property, it is possible to write the above equation as:

$$G^c\left\{\frac{d\theta}{dt}\right\} + \mu G^c\{\theta(t)\} = 40\mu G^c\{1\}\,,$$

$$-\theta(0) + iq(v)\mathrm{G^c}\{\theta(t)\} + \mu\mathrm{G^c}\{\theta(t)\} = \frac{-40i\mu}{q(v)}\ ,$$

$$-80 + (iq(v)+\mu)\mathrm{G^c}\{\theta(t)\} = \frac{-40i\mu}{q(v)}\ ,$$

$$(iq(v)+\mu)\mathrm{G^c}\{\theta(t)\} = \frac{-40i\mu}{q(v)} + 80\,,$$

$$(iq(v)+\mu)\mathrm{G^c}\{\theta(t)\} = \frac{-40i\mu + 80q(v)}{q(v)}\ ,$$

$$G^c\{\theta(t)\} = \frac{-40i\mu + 80q(v)}{q(v)(iq(v)+\mu)} \quad ,$$

$$G^c\{\theta(t)\} = \frac{-40(\mu i - 2q(v))}{q(v)(iq(v)+\mu)} \quad .$$

Now, taking

$$\frac{(\mu i - 2q(v))}{q(v)(iq(v)+\mu)} = \frac{A}{q(v)} + \frac{B}{iq(v)+\mu} \quad ,$$

$$\frac{(\mu i - 2q(v))}{q(v)(iq(v)+\mu)} = \frac{iAq(v) + A\mu + Bq(v)}{q(v)(iq(v)+\mu)} .$$

After simple computations, it is obtained that: $A = i \ , B = -1$.

Then,

$$G^c\{\theta(t)\} = -40\left(\frac{i}{q(v)} + \frac{-1}{iq(v)+\mu}\right),$$

$$G^c\{\theta(t)\} = -40\left(\frac{i}{q(v)} + \frac{-1}{iq(v)+\mu} . \frac{-iq(v)+\mu}{-iq(v)+\mu}\right),$$

$$G^c\{\theta(t)\} = -40\left(\frac{i}{q(v)} + \frac{-\mu}{\left(q(v)\right)^2 + \mu^2} + \frac{iq(v)}{\left(q(v)\right)^2 + \mu^2}\right),$$

$$G^c\{\theta(t)\} = -40\left(\frac{i}{q(v)}\right) - 40\left(\frac{-\mu}{\left(q(v)\right)^2 + \mu^2} + i\frac{q(v)}{\left(q(v)\right)^2 + \mu^2}\right).$$

Taking the inverse of the complex EFG transform, gives :

$$\theta(t) = 40(1) + 40e^{-\mu t},$$

$$\theta(t) = 40 + 40e^{-\mu t} \qquad (4.12)$$

By applying the condition at $t = 20$ min , $\theta(20) = 60$℃ , to equation (4.12):

$$60 = 40 + 40e^{-20\mu} \Longrightarrow 20 = 40e^{-20\mu} \Longrightarrow \frac{1}{2} = e^{-20\mu} \Longrightarrow e^{-\mu} = \left(\frac{1}{2}\right)^{\frac{1}{20}} \qquad (4.13)$$

To find the temperature after time $t = 40$ min, from equation (4.13) as:

$$\theta(t) = 40 + 40e^{-40\mu} \,,$$

$$\theta(t) = 40 + 40(e^{-\mu})^{40} = 40 + 40\left(\frac{1}{2}\right)^{\frac{20}{40}} = 40 + 40\left(\frac{1}{2}\right)^{\frac{1}{2}} = 50℃ \,.$$

Therefore, after 40 minutes of time, the body's temperature would become 50℃.

To find the time required for the body to reach the temperature 55℃, Again, using equation (4.13), we can deduce:

$$55 = 40 + 40e^{-\mu t} \Longrightarrow 15 = 40(e^{-\mu})^{t} \Longrightarrow \frac{15}{40} = (e^{-\mu})^{t} \qquad (4.14)$$

Substituting equation (4.13) into equation (4.14) gives:

$$\frac{15}{40} = \left(\frac{1}{2}\right)^{\frac{1}{20}}.$$

Taking natural logarithms to both sides of this equation, provides:

$$\log_e\left(\frac{15}{40}\right) = \frac{1}{20}\log_e\left(\frac{1}{2}\right) \Longrightarrow t = \frac{20\log_e\left(\frac{15}{40}\right)}{\log_e\left(\frac{1}{2}\right)}\text{min} = 28.3 \text{ minutes} \,.$$

Therefore after 28.3 minutes of time, the body temperature will reach 55℃ .

Application 4.9: Utilizing the complex EFG into the law of natural growth and decay.

Let $x(t)$ denote the substance amount that is being chemically transformed at time 't'. According to the Law of Chemical Conversion, the change rate of quantity $x(t)$ is proportionate to the quantity of the material available at the time 't.' (Mansour *et al*, 2021c).

The Law of Chemical Conversion can be translated into the following mathematical formula:

$$\frac{dx}{dt} \alpha x \Longrightarrow \frac{dx}{dt} = kx \,.$$

Where $k > 0$ is the rate constant.

The Law of Chemical Conversion equation is a first-order first-degree ordinary differential equation representing the natural growth.

Similarly, the differential equation for natural decay is:

$$\frac{dx}{dt} = -kx \,,$$

where $k > 0$ is the rate constant .

Bacteria proliferate at a pace proportionate to their immediate number. If the initial value doubles in two hours, how long will it take to triple?

Let N represents the number of bacteria at any time t. And at the initial time $t = 0, N = x$.

Then, from the natural growth law:

$$\frac{dN(t)}{dt}\alpha N(t) \Rightarrow \frac{dN(t)}{dt} = kN(t) \,,$$

where k is a constant.

$$\frac{dN(t)}{dt} - kN(t) = 0 \tag{4.15}$$

By taking the complex EFG transform:

$$G^c\left\{\frac{dN(t)}{dt} - kN(t)\right\} = G^c\{0\} \,,$$

$$G^c\left\{\frac{dN(t)}{dt}\right\} - kG^c\{N(t)\} = 0 \,,$$

$$-N(0) + iq(v)G^c\{N(t)\} - kG^c\{N(t)\} = 0 \,,$$

$$G^c\{N(t)\}(iq(v) - k) = N(0) \,,$$

$$G^c\{N(t)\} = \frac{N(0)}{(iq(v) - k)} \,,$$

$$G^c\{N(t)\} = \frac{N(0)}{(iq(v) - k)} \cdot \frac{-iq(v) - k}{-iq(v) - k} \,,$$

$$G^c\{N(t)\} = N(0)\left[\frac{-k}{\left(q(v)\right)^2 + k^2} - i\frac{q(v)}{\left(q(v)\right)^2 + k^2}\right] ,$$

$$G^c\{N(t)\} = -N(0)\left[\frac{k}{\left(q(v)\right)^2 + k^2} + i\frac{q(v)}{\left(q(v)\right)^2 + k^2}\right] .$$

Applying the inverse of the complex EFG transform to the above equation obtains:

$N(t) = N_0 e^{kt}$,

$$N(t) = xe^{kt} \tag{4.16}$$

Given that at $t = 2$ hours , $N = 2x$, then equation (4.16) would be:

$$2x = xe^{2k} \Longrightarrow 2 = e^{2k} \Longrightarrow e^{k} = (2)^{\frac{1}{2}} \qquad (4.17)$$

If $N = 3x$, then equation (4.16) gives:

$$3x = xe^{kt} \Longrightarrow 3 = e^{kt} \Longrightarrow 3 = (e^{k})^{t} \Longrightarrow 3 = (2)^{\frac{t}{2}}.$$

Applying the logarithm to both sidesof the above equation gives:

$$\log 3 = \frac{t}{2}\log 2 \Longrightarrow t = \frac{2\log 3}{\log 2} \text{ hours.}$$

A culture of N bacteria grew at a pace proportional to N. N began at 100 and rose to 332 in an hour. After 90 minutes, what was the value of N?

To solve this problem, it must be considered that at $t = 0, N = 100$, and at $t = 0$ minutes, $N = 332$.

It is required to find that at $t = 90$ minutes, what would the value of N be?

By applying the complex EFG to the law of natural growth in equation (4.15):

$$G^{c}\left\{\frac{dN(t)}{dt} - kN(t)\right\} = G^{c}\{0\} \quad ,$$

$$G^{c}\left\{\frac{dN(t)}{dt}\right\} - kG^{c}\{N(t)\} = G^{C}\{0\} \quad ,$$

$$-N(0) + iq(v)G^{c}\{N(t)\} - kG^{c}\{N(t)\} = 0 \quad ,$$

$$G^{c}\{N(t)\}(iq(v) - k) = N(0) \quad ,$$

$$G^c\{N(t)\} = \frac{N(0)}{(iq(v) - k)},$$

$$G^c\{N(t)\} = \frac{N(0)}{(iq(v) - k)} \cdot \frac{-iq(v) - k}{-iq(v) - k},$$

$$G^c\{N(t)\} = N(0)\left[\frac{-k}{\left(q(v)\right)^2 + k^2} - i\frac{q(v)}{\left(q(v)\right)^2 + k^2}\right],$$

$$G^c\{N(t)\} = -N(0)\left[\frac{k}{\left(q(v)\right)^2 + k^2} + i\frac{q(v)}{\left(q(v)\right)^2 + k^2}\right].$$

Applying the inverse of the complex EFG transform to the above equation obtains:

$N(t) = N_0 e^{kt}$,

$$N(t) = 100e^{kt} \tag{4.18}$$

At $t = 0$ min , $N(t) = 332$ and from equation (4.18):

$$332 = 100e^{kt} \Longrightarrow e^k = \left(\frac{332}{100}\right)^{\frac{1}{60}} \tag{4.19}$$

Now to find $N(t)$ at $t = 90$ minutes, form equation (4.19):

$$N(t) = 100e^{90k} = 100(e^k)^{90} \tag{4.20}$$

Substituting equation (4.19) into equation (4.20) gives:

$$N(t) = 100\left(\frac{332}{100}\right)^{\frac{90}{60}} = 604.93 \Longrightarrow N(t) \cong 605 .$$

Therefore, the bacteria number after 90 minutes is 605.

5. APPLYING THE COMPLEX EFG INTEGRAL TRANSFORMATION INTO PARTIAL DIFFERENTIAL EQUATIONS (P.D.Es)

This section demonstrates the efficiency of the complex EFG transformation in solving the other kind of differential equations, the partial differential equations (P.D.E).

Linear first-order and second-order partial differential equations, including Wave, Heat, Laplace, Telegraph, and Klein –Gordan PDEs, will be solved by using the complex EFG transformation to prove its ability to find the exact solution to these differential equations.

Problem 5.1: Consider the first order initial value problem:

$$\left.\begin{array}{l} \frac{\partial y}{\partial x} = 2\frac{\partial y}{\partial t} + y \\ y(x,0) = 6e^{-3x} \end{array}\right\} .\ \text{y is bounded for } x > 0 \text{ and } t > 0 \qquad (5.1)$$

This problem can be solved using the complex EFG transform as:

Let F be the complex EFG transform of y , then taking the complex EFG transform to equation (5.1) gives:

$$\frac{dF(x,iv)}{dx} - 2[iq(v)F(x,iv) - y(x,0)] = F(x,iv),$$

$$\frac{dF(x,iv)}{dx} + (-2iq(v) - 1)F(x,iv) = -2y(x,0),$$

$$\frac{dF(x,iv)}{dx} + (-2iq(v) - 1)F(x,iv) = -12e^{-3x} .$$

The linear ordinary differential equation is:

$$P = e^{\int -(2iq(v)+1)dx} = e^{-(1+2iq(v))x} .$$

Therefore,

$$F(x,iv) = e^{(1+2iq(v))x}.\int e^{-(1+2iq(v))x}.(-12)\,e^{-3x}dx\,,$$

$$F(x,iv) = e^{(1+2iq(v))x}\left[\int e^{-(-4-2iq(v))x}(-12)dx\right],$$

$$F(x,iv) = e^{(1+2iq(v))x}.\left[\frac{-12}{-4-2iq(v)}\,e^{-(-4-2iq(v))x}+c\right],$$

$$F(x,iv) = \frac{6}{2+iq(v)}e^{-3x}\,,$$

$$F(x,iv) = 6\left[\frac{1}{iq(v)+2}.\frac{-iq(v)+2}{-iq(v)+2}\right]e^{-3x}\,,$$

$$F(x,iv) = 6\left[\frac{2}{4+\left(q(v)\right)^2}+\frac{-iq(v)}{4+\left(q(v)\right)^2}\right]e^{-3x}\,,$$

$$F(x,iv) = -6\left[\frac{-2}{4+\left(q(v)\right)^2}+\frac{iq(v)}{4+\left(q(v)\right)^2}\right]e^{-3x}\,.$$

Taking the invers complex EFG transform:

$$y(x,t) = 6e^{-2t}e^{-3x} = 6e^{-(2t+3x)}\,.$$

Problem 5.2: Consider the two – dimensional Laplace's equation:

$$\left.\begin{array}{l} u_{xx}+u_{tt}=0 \\ u(x,0)=0 \quad ,u_t(x,0)=\cos x\ , \qquad x,t>0 \end{array}\right\}. \tag{5.2}$$

Let $F(iv)$ be the complex EFG transform of u . Then taking the complex EFG transform to equation (5.2) gives:

$$\acute{\acute{F}}(x,iv)+\left[-\left(q(v)\right)^2F(x,iv)-iq(v)u(x,0)-u_t(x,0)\right]=0\,,$$

$$F(x,iv)-\frac{1}{\left(q(v)\right)^2}\acute{\acute{F}}(x,iv)=\frac{-\cos(x)}{\left(q(v)\right)^2}.$$

The resulted 2^{nd} order ordinary differential equation, have the particular solution in the form:

$$F(x,iv)=\frac{\frac{-\cos x}{\left(q(v)\right)^2}}{1-\frac{1}{\left(q(v)\right)^2}D},$$

$$F(x,iv)=\frac{-\cos(x)}{\left(q(v)\right)^2+1}.$$

Where $D^2=\frac{d^2}{dx^2}$.

Now taking the inverse of the EFG transform, the exact solution is obtained:

$$u=(x,t)=\sinh(t)\cos(x).$$

Problem 5.3: Consider the wave equation:

$$\left.\begin{array}{l} u_{xx}-4\,u_{tt}=0 \\ u(x,0)=\sin(\pi x),\ and\ \ u_t(x,0)=0\ ,\quad x,t>0 \end{array}\right\} \tag{5.3}$$

Taking the complex EFG transform to equation (5.3) gives:

$$F''(x,iv)-4\left[-\left(q(v)\right)^2F(x,iv)-iq(v)u(x,0)-u_t(x,0)\right]=0\,,$$

$$F''(x,iv)+4\left(q(v)\right)^2F(x,iv)+4\,iq(v)u(x,0)+4u_t(x,0)=0\,,$$

$$\frac{1}{4\left(q(v)\right)^2}F''(x,iv)+F(x,iv)=\frac{-4iq(v)\sin(\pi x)}{4\left(q(v)\right)^2},$$

$$\frac{1}{4\big(q(v)\big)^2}\acute{\acute{F}}(x,iv)+F(x,iv)=\frac{-i}{q(v)}\sin(\pi x)\,,$$

$$F(x,iv)=\frac{\frac{-i}{q(v)}\sin(\pi x)}{\frac{1}{4\left(q(v)\right)^2}D^2+1}=\frac{\frac{-i}{q(v)}\sin(\pi x)}{\frac{1}{4\left(q(v)\right)^2}(-\pi)^2+1}\,,$$

$$F(x,iv)=\frac{-4iq(v)\sin(\pi x)}{4\big(q(v)\big)^2-\pi^2}=\frac{-iq(v)\sin(\pi x)}{\big(q(v)\big)^2-\left(\frac{\pi}{2}\right)^2}\,.$$

Take the inverse of the complex EFG transform:

$$u(x,t)=\cos\left(\frac{\pi}{2}t\right)\cdot\sin(\pi x)\,.$$

Problem 5.4: Consider the homogeneous heat equation in the following normalized one dimension form:

$$\left.\begin{aligned}&u_{xx}-4u_t=0\,, \qquad u(x,0)=\sin\left(\frac{\pi}{2}x\right)\\ &x,t>0\,.\end{aligned}\right\} \tag{5.4}$$

By using the complex EFG transform in equation (5.4):

$$F''(x,iv)-4[iq(v)F(x,iv)-u(x,0)]=0\,,$$

$$F''(x,iv)-4iq(v)F(x,iv)+4\sin\left(\frac{\pi}{2}x\right)=0\,,$$

$$F(x,iv)=\frac{-4\sin\left(\frac{\pi}{2}x\right)}{D^2-4iq(v)}=\frac{-4\sin\left(\frac{\pi}{2}x\right)}{-\left(\frac{\pi}{2}\right)^2-4iq(v)}\,,$$

$$F(x,iv)=\frac{16\sin\left(\frac{\pi}{2}x\right)}{\pi^2+16iq(v)}\,,$$

$$F(x,iv) = \frac{1}{\frac{\pi^2}{16} + iq(v)} \ . \ \sin\left(\frac{\pi}{2}x\right).$$

So,

$$F(x,iv) = -\left[\frac{-\frac{\pi^2}{16}}{\left(\frac{\pi^2}{16}\right)^2 + \left(q(v)\right)^2} + i\frac{q(v)}{\left(\frac{\pi^2}{16}\right)^2 + \left(q(v)\right)^2}\right] \cdot \sin\left(\frac{\pi}{2}x\right).$$

Taking the inverse of the complex EFG transform gives:

$$u(x,t) = e^{-\frac{\pi^2}{16}t} \ \cdot \ \sin\left(\frac{\pi}{2}x\right).$$

Problem 5.5: Consider the linear telegraph equation:

$$u_{xx} = u_{tt} + 2u_t + u \qquad (5.5)$$

Subject to the following initial conditions:

$$u(x,0) = e^x \ , \quad u_t(x,0) = -2e^x \ .$$

By using the complex EFG transform in equation (5.5):

$$F''(x,iv) + \left(q(v)\right)^2 F(x,iv) + iq(v)u(x,0) + \frac{\partial u}{\partial t}(x,0) - 2iq(v)F(x,iv) + 2u(x,0) - F(x,iv) = 0 \ ,$$

$F''(x,iv) + \left(q(v)\right)^2 F(x,iv) + iq(v)e^x - 2e^x - 2iq(v)F(x,iv) + 2e^x - F(x,iv) = 0$,

$$F''(x,iv) + \left(\left(q(v)\right)^2 - 2iq(v) - 1\right) F(x,iv) = -iq(v)e^x \ ,$$

$$F(x,iv) = \frac{\frac{-iq(v)e^x}{\left((q(v))^2-2iq(v)-1\right)}}{\frac{1}{\left((q(v))^2-2iq(v)-1\right)}.D^2+1},$$

$$F(x,iv) = \frac{-iq(v)e^x}{(q(v))^2-2iq(v)} = \frac{-ie^x}{(q(v)-2i)},$$

$$F(x,iv) = -\left[\frac{i(q(v)+2i)}{(q(v)-2i)(q(v)+2i)}\right]e^x,$$

$$F(x,iv) = -\left[\frac{iq(v)-2}{(q(v))^2+4}\right]e^x,$$

$$F(x,iv) = -\left[\frac{-2}{(q(v))^2+4} + i\frac{q(v)}{(q(v))^2+4}\right]e^x.$$

Take the inverse of the complex EFG transform gives the following exact solution:

$$u(x,t) = e^{-2t}.e^x = e^{x-2t}.$$

Problem 5.6: Consider the second order linear homogeneous Klein-Gordan equation:

$$u_{tt} = u_{xx} + u_x + 2u \qquad \left.\begin{array}{c}-\infty < x < \infty \\ t > 0\end{array}\right\} \tag{5.6}$$

Subject to the following initial conditions:

$$u(x,0) = e^x, \qquad u_t(x,0) = 0.$$

By using the complex EFG transform in equation (5.6):

$$-(q(v))^2F(x,iv) - iq(v)u(x,0) - u_t(x,0) - F''(x,iv) - F'(x,iv) - 2F(x,iv) = 0,$$

$$-F''(x,iv) - \acute{F}(x,iv) - \left(\left(q(v)\right)^2 + 2\right)F(x,iv) = iq(v)e^x \,,$$

$$F''(x,iv) + \acute{F}(x,iv) + \left(\left(q(v)\right)^2 + 2\right)F(x,iv) = -iq(v)e^x.$$

Thus,

$$F(x,iv) = \frac{\frac{-iq(v)e^x}{\left(q(v)\right)^2+2}}{\frac{1}{\left(q(v)\right)^2+2}.D^2 + \frac{1}{\left(q(v)\right)^2+2}.D + 1}\,,$$

$$F(x,iv) = \frac{-iq(v)e^x}{\left(q(v)\right)^2 + 4}\,.$$

Take the inverse of the complex EFG transform gives the following exact solution:

$$u(x,t) = \cosh(2t)e^x\,.$$

6. UTILIZING THE COMPLEX EFG TRANSFORM IN SOLVING BESSEL'S FUNCTIONS

Many fields manifest the importance of the Bessel function (Watson 1995, Bowman 2012). Furthermore, due to the importance of such functions, any method that is capable of solving these functions and showing efficiency in doing so proves its worth in the mathematical world. For that reason, in this section, the complex EFG transform is used to solve Bessel's functions to prove the competence of this novel transform.

6.1 Bessel's Function Basic Principles

Some definition that are related to the Bessel's functions are necessary to be mentioned before preceding in solving the functions using the complex EFG transforms.

- Bessel's function of order n, where n is the natural number is given by (Farrell and Ross 2013):

$$J_n(t) = \frac{t^n}{2^n n!}\,[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} - \frac{t^6}{2.4.6(2n+2)(2n+4)(2n+6)} + \cdots] \tag{6.1}$$

For $n = 0$, Bessel's function of order zero that is denoted by $J_0(t)$ would be:

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + \cdots \tag{6.2}$$

For $n = 1$, Bessel's function of order one that is denoted by $J_1(t)$ would be:

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3.2!} + \frac{t^5}{2^5.2!\ .3!} - \frac{t^7}{2^7.2!\,.3!\ .4!} + \cdots \tag{6.3}$$

For $n = 2$, Bessel's function of order two that is denoted by $J_2(t)$ would be:

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2.4.6} + \frac{t^6}{2^2.4^2.6.8} - \frac{t^8}{2^2.4^2.6^2.8.10} + \cdots \qquad (6.4)$$

If $t = 0$, then $J_0(0) = 1$ and $J_1(0) = J_2(0) = 0$.

- Relationship between $\boldsymbol{J_0(t)}$ and $\boldsymbol{J_1(t)}$ (Farrell and Ross 2013)

$$\frac{\mathrm{d}}{dt} J_0(t) = -J_1(t) \qquad (6.5)$$

- Relation between $\boldsymbol{J_0(t)}$ and $\boldsymbol{J_2(t)}$ (Farrell and Ross 2013)

$$J_2(t) = J_0(t) + 2J_0''(t) \qquad (6.6)$$

6.2 Complex EFG Transform of Bessel's Functions

- Complex EFG Transform of J_0(t).

Taking complex (EFG) transform to equation (6.2) gives:

$$G^c[J_0(t)] = G^c[1] - \frac{1}{2^2} G^c[t^2] + \frac{1}{2^2.4^2} G^c[t^4] - \frac{1}{2^2.4^2.6^2} G^c[t^6] + \cdots ,$$

$$= \frac{-i}{q(v)} - \frac{1}{2^2} . \frac{2!\, i}{\left(q(v)\right)^3} + \frac{1}{2^2.4^2} . \frac{(4!)(-i)}{\left(q(v)\right)^5} - \frac{(6!)(i)}{2^2.4^2.6^2\left(q(v)\right)^7} + \cdots,$$

$$= \frac{-i}{q(v)} \left[1 + \frac{1}{2\left(q(v)\right)^2} + \frac{1.3}{2.4} \left(\frac{1}{\left(q(v)\right)^2} \right)^2 + \frac{5.3.1}{2.4.6} \left(\frac{1}{\left(q(v)\right)^2} \right)^3 + \cdots \right] ,$$

$$\frac{-i}{q(v)} \left[1 - \frac{1}{\left(q(v)\right)^2} \right]^{\frac{-1}{2}} = \frac{-i}{q(v)\sqrt{1 - \frac{1}{\left(q(v)\right)^2}}} = \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} .$$

Thus,

$$G^c[J_0(t)] = \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} \,. \tag{6.7}$$

- Complex EFG Transform of $J_1(t)$.

$$G^c[J_1(t)] = -G^c\left[\frac{d}{dt}J_0(t)\right] \tag{6.8}$$

Now, applying the property complex EFG integral transform of derivative on the function on equation (6.8), gives:

$$G^c[J_1(t)] = -\left[-J_0(0) + iq(v)G^c[J_0(t)]\right],$$

$$= -\left[-1 + iq(v)G^c[J_0(t)]\right],$$

$$= 1 - iq(v)\left(\frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}}\right).$$

Thus,

$$G^c[J_1(t)] = \left[1 - \frac{q(v)}{\sqrt{\left(q(v)\right)^2 - 1}}\right] \tag{6.9}$$

- Complex (EFG) Transform of $J_2(t)$

Since

$$G^c[J_2(t)] = G^c[J_0(t)] + 2G^c[\grave{J}_0(t)],$$

$$G^c[J_2(t)] = \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} + 2\left[-\grave{J}_0(0) - iq(v)J_0(0) - \left(q(v)\right)^2 G^c[J_0(t)]\right],$$

$$= \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} + 2\left[-iq(v) + \left(q(v)\right)^2 \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}}\right],$$

$$G^c[J_2(t)] = \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} - 2iq(v) + \frac{2i\left(q(v)\right)^2}{\sqrt{\left(q(v)\right)^2 - 1}} \;,$$

$$G^c[J_2(t)] = \frac{-i - 2iq(v)\sqrt{\left(q(v)\right)^2 - 1} + 2i\left(q(v)\right)^2}{\sqrt{\left(q(v)\right)^2 - 1}} \tag{6.10}$$

6.3 Applications

Practical Bessel's functions applications will be solved to demonstrate the actual capability of the complex EFG transform in solving them.

Application 6.3.1 Calculate the integral:

$$I(t) = \int_0^t J_0(t)\, J_0(t-u)du \tag{6.11}$$

Applying the complex EFG transform to equation (6.11), gives:

$$G^c[I(t)] = G^c\left[\int_{u=0}^{t} J_0(t)\, J_0(t-u)du\right],$$

Using the convolution theorem:

$$G^c[I(t)] = G^c[J_0(t)].\, G^c[J_0(t)]\,,$$

$$= \left[\frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} \cdot \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} \right],$$

$$= \frac{-1}{\left(q(v)\right)^2 - 1}.$$

Take the inverse complex EFG transform:

$$I(t) = \sin(t).$$

The concluded equation the exact solution to equation (6.11).

Application 6.3.2 Evaluate the integral:

$$I(t) = \int_0^t J_0(t)\, J_1(t-u) du \tag{6.12}$$

Applying the complex EFG transform to equation (6.12) gives:

$$G^c[I(t)] = G^c\left[\int_{u=0}^t J_0(t)\, J_1(t-u) du \right].$$

Using the convolution theorem:

$$G^c[I(t)] = G^c[J_0(t)].\, G^c[J_1(t)]\,,$$

$$= \left[\frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} \right] \cdot \left[1 - \frac{q(v)}{\sqrt{\left(q(v)\right)^2 - 1}} \right],$$

$$= \frac{-i}{\sqrt{\left(q(v)\right)^2 - 1}} + \frac{iq(v)}{\left(q(v)\right)^2 - 1} \ .$$

Taking the inverse of the complex EFG transform provieds:

$$I(t) = J_0(t) - \cos(t) \, .$$

Application 6.3.3 Evaluate the integral:

$$I(t) = \int_{u=0}^{t} J_1(t-u)du \tag{6.13}$$

Applying complex EFG transform to equation (6.13) gives:

$$G^c[I(t)] = G^c\left[\int_{u=0}^{t} J_1(t-u)du\right].$$

Using the convolution theorem:

$$G^c[I(t)] = G^c[1].\, G^c[J_1(t)] \, ,$$

$$= \frac{-i}{q(v)} \cdot \left[1 - \frac{q(v)}{\sqrt{\left(q(v)\right)^2 - 1}}\right],$$

$$= \frac{-i}{q(v)} + \frac{i}{\sqrt{\left(q(v)\right)^2 - 1}} \, .$$

Taking the inverse of the complex EFG transform gives:

$$I(t) = 1 - J_0(t) \, .$$

7. CONCLUSIONS AND RECOMMENDATION

This thesis offers a promising new integral transform, denoted by "Complex (Emad-Faruk-Ghaith) EFG." The fundamental properties and applications of the complex EFG and inverse complex EFG transform to various fundamental functions demonstrated the transform's fantastic ability to handle these functions efficiently and with the least amount of computing possible.

The EFG transform has been used to solve versatile applications represented by ordinary differential equations, including nuclear physics, electrical circuits, civil engineering beam deflection, pharmacokinetics, natural growth and decay, and Newton's law of cooling. The transform has also been used to solve partial differential equations. And finally, to highlight the magnificent capability of this transformation in being a valuable tool in the integral transform community, the EFG transform has been utilized in solving the Bessel's function through practical examples.

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