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NEW GENERALIZATIONS OF P-EXTENDING MODULES



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NEW GENERALIZATIONS OF P-EXTENDING MODULES

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July 2022

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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ABSTRACT

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Master of Science in Mathematics

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In this study, E is unitary right module over associative ring S with unit element. A sub module K of module E is called rational in E and denoted. by $(K \leq_r E)$, if for each, $y \in E$ and $x \in E \setminus \{0\}$, $\exists r \in S \ni rx \neq 0$ and $ry \in K$ [K.R.Goodearl 1976]. In this work, we used the rational sub module to introduce and study a new generalization of the class of P-extending modules namely rationally principally extending module (for shortly, RP-extending) where *an S -module E is said to be RP-extending ; if every cyclic sub module of E is rational in direct summand of E .* On the other hand, we present a new class of modules namely Rc-sub modules and Rc-modules. Several properties and characterizations of these notions are given. Also we study the relation with our notions and some related well known concepts. In the last part of this study, a new generalization of injectivity are introduced and studied by using Rc-sub modules. The notions of (quasi-) injective modules is generalized to that of *Rcc- (quasi-)injective modules: Let W and F be S -modules. Then F is called Rcc – W -injective if every homomorphism $\varphi: L \rightarrow F$, where L is Rc-closed submodule of W , can be extended to an R -homomorphism $\theta: W \rightarrow F$. An R -module F is called Rcc–quasi-injective or Rcc-self-injective, if F is Rcc – F -injective.* Numerous properties and characterizations of these generalizations are given. Moreover, the relation between these notions is studied.

2022, 64 pages

Keywords: Rational sub modules, Cyclic sub modules, P-extending, Rationally extending, Rcc-injective, Rcc-quasi- injective modules

ÖZET

P-GENİŞLETME MODÜLLERİNİN YENİ GENELLEŞTİRİLMESİ

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Bu çalışmada E , birim elemanlı S ilişkisel halka üzerindeki birimsel sağ modüldür. Modül E 'nin bir alt modülü K , E 'de rasyonel olarak adlandırılır ve gösterilir. ($K \leq_r E$) ile, her biri için ise, $y \in E$ ve $x \in E \setminus \{0\}$, $\exists r \in S \exists rx \neq 0$ ve $ry \in K$ [K.R.Goodearl 1976]. Bu çalışmada, P -genişleyen modüller sınıfının yeni bir genellemesini tanıtmak ve incelemek için rasyonel alt modülü kullandık, yani bir S -modül E 'nin RP -olduğu söylenen rasyonel olarak prensipte genişleyen modül (kısaca, RP -genişleyen) uzanan; E 'nin her çevrimsel alt modülü, E 'nin doğrudan toplamında rasyonel ise. Öte yandan, yeni bir modül sınıfı, yani R_c -alt modüller ve R_c -modülleri sunuyoruz. Bu kavramların çeşitli özellikleri ve karakterizasyonları verilmiştir. Ayrıca kavramlarımızla ve ilgili bazı iyi bilinen kavramlarla olan ilişkiyi de inceliyoruz. Bu çalışmanın son bölümünde, R_c -alt modülleri kullanılarak yeni bir enjektivite genellemesi tanıtılmış ve incelenmiştir. (Yarı-) enjekteli modüllerin ulusları, R_{cc} - (yarı-)enjektif modüllerinkiyle genelleştirilir: W ve F , S -modülleri olsun. O zaman, L 'nin W 'nin R_c -kapalı alt modülü olduğu her homomorfizm $\varphi: L \rightarrow F$, bir R -homomorfizma $\theta: W \rightarrow F'$ 'ye genişletilebilirse, F' 'ye R_{cc} - W -enjektif denir. Bir R -modülü F , R_{cc} -yarı-enjektif veya R_{cc} -kendi kendine enjeksiyonlu olarak adlandırılır, eğer F R_{cc} - F -enjektif ise. Bu genellemelerin çok sayıda özelliği ve karakterizasyonu verilmiştir. Ayrıca bu kavramlar arasındaki ilişki de incelenmiştir.

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Anahtar Kelimeler: Rasyonel alt modüller, Döngüsel alt modüller, P -genişleyen, Rationally genişleyen, R_{cc} -enjektif, R_{cc} -yarı-enjektif modüller

PREFACE AND ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

$\prod_{i \in I} E_i$	The direct product of the modules E_i
$\bigoplus_{i \in I} E_i$	The direct sum of the modules E_i
E	Right module over a ring S
$H \leq_{\oplus} E$	H is a direct summand of E
$H \leq_{\oplus} E$	H is a direct summand of E
$H \leq_{cy} E$	H is cyclic sub module of E
$H \leq E$	H is sub module of E
$H \leq_{Rc} E$	H is Rc-closed submodule of E
$H \leq_c E$	H is closed submodule of E
$H \leq_e E$	H is essential submodule of E
$H \leq_r E$	H is rational submodule of E
$H \leq_{Rc} E$	H is Rc-submodule of E
$H \leq_{rc} E$	H is rationally closed submodule of E
$H \not\cong E$	H is proper sub module of E
$H \not\leq E$	H is not sub module of E
Q	The group of rational numbers
S	An associative ring with identity
Z	The ring of integer numbers
Z_n	The group of all integers module n

LIST OF ABBREVIATIONS

$\text{Hom}_S(E, F)$	All homomorphisms of S-module from E to F
$\text{ann}_S(H)$	Annihilator of a subset H of an S-module E
CS	Extending module
$E \cong F$	E is isomorphic to F
$E(F)$	Injective envelope of a S-module F
$\text{End}(E)$	All homomorphisms of S-module from E to E
$\text{Im}(f)$	Image of a homomorphism f
$\text{Ker}(f)$	Kernel of a homomorphism f
RCC	Module has every submodules are Rc-modules
RCS	Rationally extending module
$Z(E)$	Singular submodule of a S-module E

1. INTRODUCTION

Throughout this work, S is an associative ring with identity and all S -modules are unitary right modules.

Baer (1940), introduced the concept of injective modules. The well known basic properties and equivalent definitions of injective modules can be found in (Faith 1967, Sharpe and Vamos 1972, Kasch 1982). An S -module E is said to be injective, if it satisfies one of the following equivalent conditions:

- (i) For every S -monomorphism from A to B (where A and B are any S -modules) and every S -homomorphism from A to E can be extended to an S -homomorphism from B to E ;
- (ii) (Baer's Criterion) every S -homomorphism from a right ideal I of S into E can be extended to an S -homomorphism from S into E .

Let F be S -module. An S -module E is said to be injective relative to F (or F -injective), if for every S -homomorphism from a submodule of F into E can be extended to an S -homomorphism from F into E see (Mohamed and Muller 1990).

Injective modules have been studied extensively, and many generalizations for the concept of injective modules were given by many authors, we will mention here those which are relevant to our work. In (Johnson and Wong 1961) was introduced the concept of quasi-injective as a proper generalization of injectivity. An S -module E is called quasi-injective, if any S -homomorphism from a submodule of E into E can be extended to an S -homomorphism of E . In, (Tiwary *et al.* 1979) was introduced the notion of C -quasi-injective as a proper generalization of quasi-injective module. An S -module E is called C -quasi-injective, if any S -homomorphism from a closed submodule of E into E can be extended to an S -homomorphism of E . Also, in (Clara 1998) was introduced another generalization of injectivity, namely c -injectivity. Let G_1 and G_2 be

S-modules. The module G_1 is G_2 -c-injective, if every S-homomorphism from a closed submodule of G_2 into G_1 , can be extended to an S-homomorphism from G_2 into G_1 . An S-module G is called quasi or self-c-injective when it is G -c-injective.

Extending modules (or CS modules) have been extensively studied in a monograph by (Dung *et al.* 2019), as well as in earlier book by (Mohamed and Müller 1990). The class of extending modules is a generalization of quasi-injective modules. Since 1980s, the study of the CS-module theory has been a major area of research interest in the ring and module theory. (Kamal and Elmnophy 2005) was introduced the concept namely principally extending modules (for short P-extending modules), an S-module E is called P-extending, if each cyclic submodule of E is essential in summand. Equivalently, E is P-extending iff each EC-closed submodule of E is summand. On other hand a submodule B of a module E , said to be EC-(closed) submodule, if B is (closed) submodule which contains essentially a cyclic submodule; i.e $\exists x \in B$ such that $Sx \leq_e B$ (Kamal and Elmnophy 2005) For more details about the generalizations of extending modules see (Mohamed and Müller 1990, Abd Al-Majeed and Nayef 2021, Adnan and Tercan 2016).

The main aim of this study is to introduce and study a new classes of modules which are a proper generalizations of P-extending modules and weaker than rationally extending module and in the last part of this work we introduce a proper generalization of injective and quasi-injective modules in terms of rational extensions.

This thesis consists of five chapters. The first chapter is devoted to reviewing the introduction to this study.

In chapter two which is subdivided into six sections, we give a review of some basic concepts and results which are required for our work in this study. The first section is devoted for basic definitions and some results of modules and submodules. In second section, In second section we recall the definition of large (essential), closed submodules and uniform modules with state prove for some their basic results which are

relevant to our work. In section three we discuss the properties of the concepts direct summand, semi simple and indecomposable module. Section four is devoted to establish the basic properties of homomorphism of modules. While section five is provides the preliminaries and some background results of singular and non singular modules. In last section of this chapter we give a review of some definitions of rational sub modules and monofrom modules and provides notations and some results which are required in our work.

In chapter three, exactly in first three sections, follows (Dung *et al.* 2019, Mohamed and Muller 1990) we recall the definition of extending modules and some of basic properties of extending module. Also, we present some of their generalizations such as P-extending follows (Kamal and Elmnophy 2005) and rationally extending follows (Abbas and Ahmed 2011), which are required for our study in this work.

In last section of chapter three , we list some definitions and well-known fundamental properties of injective modules and some of their generalizations which are required for our study in this work. Also, in part of this section we introduce some results which will be needed in our work.

The fourth chapter included the main results of this study, which consisted to three sections. In section one we introduce a new class of modules namely principally extending modules(for short, RP-extending) which is stronger properly than P-extending modules and weak than rationally extending modules. Also we study some properties of those concepts and discuss some connections between those concepts and some concepts of well-known modules. In the second section, we will introduce the definition of RCC-modules and discuss when the direct sum of RP-Extending is also RP-Extending. In section three, we introduce and study the concepts Rcc – E -injective and rcc-quasi-injective (or self-Rcc-injective) modules which are proper generalizations of relative injective and quasi-injective, respectively. Let W and F are S -modules. Then F is said to be Rcc – W -injective if every homomorphism $h: L \rightarrow F$, (where $L \leq_{\text{RCC}} W$) ,can be extended to a homomorphism $t: W \rightarrow F$. We gave basic properties and som characterizations of these concepts (see proposition (4.3.3), proposition (4.3.7),

proposition (4.3.19). Also, we show that the class of Rcc-injectivity is weaker than the class of c-injectivity. Moreover, the inherited property for Rcc-quasi-injective modules is studied.

Finally, the conclusions of this study are present in chapter five.



2. PRELIMINARIES

2.1 Modules and Some Types of Submodules

In this section we recall the definition of modules and sub modules and stat some basic results.

Now, we start with definition of module.

Definition2. 1: (Kasch 1982)

Let $(S, +, \cdot)$ is a ring and $(E, +)$ is abelian group . Then E is said to be a right S - module if the mapping $E \times S \rightarrow E$ define by, $(e, s) \mapsto es \in E, \forall e \in E$ and $s \in S$) satisfies the following conditions :

(i) $(a + b) r = er + br$

(ii) $e(r + s) = er + e s$

(iii) $e(rs) = (er) s, \forall r, s \in S,$ and $e, b \in E.$

Similarly, we can define left S – module by operating to the left side of $E.$

Remarks and Examples 2.2 :

(1) If S has identity then $e.1=e, \forall e \in E$, in this case E is called unitary right S -module.

(2) If E is both a left and a right S – module, then E is called an S – module.

(3) Every ring S is an S – module over itself.

(4) Every additive group is a module over the ring \mathbb{Z} .

(5) Let N be a right ideal of a ring S . Then N is a right S -module.

(6) Every vector space over a field F is F -module.

(7) By (3) we get \mathbb{Z} is \mathbb{Z} -module and \mathbb{Z}_n is \mathbb{Z}_n -module.

(8) By (4) \mathbb{Q} is \mathbb{Z} -module, but not hard to show that \mathbb{Z} is not \mathbb{Q} -module.

Definition 2.3: (Kasch 1982)

Let E is S -module. Then H is called an S -sub module of E and denoted by ($H \leq E$) if,

1- $\emptyset \neq H \subseteq E$,

2- H is S -module.

Proposition 2.4:

Let H is a non-empty sub set of an S -module E , then $H \leq E$ if,

i) $x + y \in H$, $\forall x, y \in H$

ii) $x \cdot s \in H$, $\forall s \in S$ and $\forall x \in H$.

Proof: see (Kasch 1982).

Remarks and Examples 2.5:

- 1) Every S -module E has at least two sub modules (0) and E which called trivial sub modules.
- 2) The sub module which is not trivial is called proper sub module of E .
- 3) Every sub group of Z is sub module of Z as Z -module.
- 4) Every ideal is sub module of S as S -module.
- 5) Let $E = Z_9$ as Z -module then all sub modules of Z_9 , are : $\langle 0 \rangle$, Z_9 (trivial) and $\langle 3 \rangle$ (proper).

Proposition 2.6:

- 1- The intersection of sub modules also is sub modules.
- 2- The sum of family of sub modules is sub modules.

Note: *The union of sub module is not sub module in general.*

Definition 2.7 (Mahima and Avishek 2014)

An S - module E is said to be a cyclic module, if $\exists x \in E$ such that $\forall m \in E$ can be write as $m = x \cdot s$ for some $s \in S$. The element a is called the generator of E and we write $E = \langle a \rangle = \{ a \cdot s : s \in S \}$.

If $H \leq E$ such that $H = \langle x \rangle = \{ x \cdot s : x \in E, s \in S \}$ then H is called cyclic sub module generated by x .

Example 2.8: Z as Z -module is cyclic module which is generated by 1 and -1 .

Also $H = \langle 3 \rangle = \{0, 3, 6, 9\}$ is cyclic sub module of \mathbb{Z} -module \mathbb{Z}_{12} .

Definition 2.9: (Mahima and Avishek 2014)

An S -module E is simple module, if $E \neq 0$ and has only trivial sub module. In other word (0) and E are only sub module of E .

Theorem 2.10: (Mahima and Avishek 2014)

Let E be a left unitary S -module. Then E is simple iff for each non-zero

element $x \in E$, implies $E = \langle x \rangle$.

Proof: Let E is a simple unitary S -module. Then for $0 \neq x \in E$, $\Rightarrow x = 1 \cdot x \in$

$Sx \Rightarrow Sx = \emptyset$.

Consequently, Sx is a sub module of E . Since for $0 \neq x \in E \Rightarrow Sx \neq 0$. But by hypothesis, E is a simple S -module, so $Sx = E$.

Conversely, let $Sx = E$ for every $0 \neq x \in E$.

Suppose that $0 \neq L \leq E$. Then $\exists 0 \neq x \in L$ such that $Sx \leq L$ i.e., $E \leq L$.

But since $L \subseteq E$, it follows that $L = E$. Therefore, E is a simple R -module.

Remark 2.11 :

Every simple module is cyclic but the converse is not hold in general.

For example: Z as Z -module is cyclic module but not simple since has a proper sub module in fact every element x in Z is generated a proper sub module of Z such that $0 \neq x$ and $x \neq \pm 1$.

2.2 Large Submodules, Closed sub Modules and Uniform Modules

In this section we recall the definitions and prove some basic properties of the concepts (large or (essential) , closed sub modules and uniform modules) which are relevant to our work.

Definition 2.12: (Goodearl 1976)

Let E be any non-zero S -module. A non-zero submodule H of E (briefly $H \leq E$) is called large or essential in E (briefly $H \leq_e E$), if $H \cap K \neq 0$, for each non-zero submodule K of E (this means, if $H \cap K = 0$ then must be $K = 0$).

Note: If $H \leq_e E$ then we say that E is essential extension of H .

Proposition 2.13: (Kasch 1982)

H is essential in E if and only if, for each $0 \neq m \in E$ there exists $0 \neq s \in S$ such that $0 \neq ms \in H$.

Proof: Suppose that $H \leq_e E$ and take $0 \neq m \in E$.

Then $mS \neq 0$ and so $H \cap mS \neq 0$. Then there is $0 \neq y \in H \cap mS$. So that $0 \neq y \in H$ and there exists $s \in S$ such that $y = ms$.

Therefore $0 \neq y = msr \in H$.

Conversely, let $0 \neq K \leq E$. Take $0 \neq m \in K$.

But by our hypothesis, there is $s \in S$ with $ms \neq 0$ and $ms \in H$. Now, since $ms \in K$, thus we have

$0 \neq ms \in H \cap K$. This means $H \leq_e G$.

Definition 2.14 (Goodearl 1976)

Any R -monomorphism $f: H \rightarrow M$ is said to be essential monomorphism if $mf \leq_e M$.

Notes and Examples 2.15 (1) Every module E is essential in it self.

(2) Zero sub module is large iff $E=0$

(3) Let $E = Z_8$ as Z -module then every non zero sub module of E is large.

4) Let $E = Z_6$ as Z -module, then $Z_6 \leq_e Z_6$, but $\langle 2 \rangle \not\leq_e Z_6$ since $\langle 2 \rangle \cap \langle 3 \rangle = 0$ but $\langle 3 \rangle = \{0,3\} \neq 0$

In the following proposition we state some basic properties of essential submodules.

Proposition 2.16 (Kasch 1982, Anderson and Fuller 1992)

Let E be any S -module with submodules N and H . Then

(i) If $H \leq N$, then $H \leq_e E$ if and only if $H \leq_e N$ and $N \leq_e E$.

(ii) If $H \leq_e E$, $N \leq_e E$ then $H \cap N \leq_e E$. In general if $H_i \leq_e E$, then $\bigcap_{i=1}^n H_i \leq_e E$.

(iii) if $H \leq_e E$, $N \leq_e E$ then $H \oplus N \leq_e E$.

(iv) If $E = \bigoplus_{i \in I} E_i$ and $N_i \leq_e E_i$ for every $i \in I$, then $\bigoplus_{i \in I} N_i \leq_e E$.

(v) If $H \leq_e B$ (where B any R -module) and $f: E \rightarrow B$ be S -homomorphism, then $f^{-1}(H) \leq_e E$.

Proof:

(i) Let $Y \leq N$ be such that $Y \neq 0$. Then $Y \leq E$.

Now, since $H \leq_e E$, $Y \cap H \neq 0$.

Hence, $H \leq_e N$.

Now, must prove that $N \leq_e E$. Let $P \leq E$ such that $P \neq 0$. Then $H \cap P \subseteq N \cap P$. But $H \leq_e E$ thus, $H \cap P \neq 0$ implies $N \cap P \neq 0$. Therefore, $N \leq_e E$.

In the same way we can prove the conversely. \square

Proof :

ii) Suppose that, $H \leq_e E$, $N \leq_e E$ and let $X \leq E \ni X \neq 0$. Then $X \cap (H \cap N) = (X \cap H) \cap N \neq 0$.

Hence $H \cap N \leq_e E$. Similarly we can prove general case by using induction. \square

iv) Assume $H \leq_e E$, $N \leq_e E$ Since we have $H \leq H \oplus N \leq E$ and $H \leq_e E$ then by part

(i) $H \oplus N \leq_e E$. \square

For proof the parts (iv) and (v) see (Kasch 1982). \square

Definition 2.17 (Goodearl 1976)

An S -module E is called uniform if $E \neq (0)$ and any two non-zero submodules of E have a non-zero intersection. Equivalently, M is uniform if and only if $E \neq (0)$ and every non-zero submodule of E is essential in E .

For example: Let $E = \mathbb{Z}_8$ and $M = \mathbb{Z}_6$ are two \mathbb{Z} -module. Then \mathbb{Z}_8 is uniform module see example (2.15)(2). While, \mathbb{Z}_6 is not uniform.

It is clear that, any non-zero submodule of a uniform module is uniform and any essential extension of a uniform module is uniform.

Definition 2.18 : (Goodearl 1976) *Let E be any S -module. A submodule H of E is called closed in E , if it has no proper essential extension in M . If H is closed in E , we write $H \leq_c E$.*

Definition 2.19 (Dung et al. 2019) *Let H be a submodule of E , a relative complement of H in E is any submodule A of E which is maximal with respect to the property that $H \cap A = 0$.*

Notes and Examples 2.20 : 1) (0) and E is always closed sub modules of E (Goodearl 1976).

2) If $A \not\leq_e E$ then $A \leq_c E$

3) Let $E = \mathbb{Z}_{12}$ as \mathbb{Z} -module then $\langle 3 \rangle = \{0,3,6,9\}$ is closed since has no proper essential extension, but $\langle 2 \rangle = \{0,2,4,6,8,10\}$ is not closed since it is essential.

(4) Following (Goodearl 1976) : If U is a closed submodule of an R -module E and $U \leq N \leq E$, then U is closed in N .

Proof: Assume $U \leq_e T$ for some submodule T of N . It is clear that T is a submodule of M . Hence $U = T$, since U is closed in M . Thus U is closed in N .

Lemma 2.21 (Dung *et al.* 2019)

Let A and F be submodules of an S -module E with $A \cap F = 0$. Then

(i) There exists a complement H of A such that $F \leq H$.

(ii) $H \oplus A \leq_e E$.

(iii) $H \leq_c E \not\Rightarrow$ □

Proposition 2.22

(Dung *et al.* 2019) Let H be a sub modules of an S -module E and let L be a complement of H . Then H is closed in E if and only if H is a complement of L in E .

We now list some basic properties of closed sub modules.

Proposition 2.23 (Dung *et al.* 2019, Goodearl 1976)

Let H and L be submodules of an S -module E , with $H \leq L$.

(i) For every $A \leq E$, there exists $H \leq_c E$ such that $A \leq_e H$.

(ii) If $H \leq_c L$ and $L \leq_c E$, then $H \leq_c E$.

(iii) If $L \leq_c E$, then $L/H \leq_c E/H$.

Now, we will prove (iii)

Proof: Assume $(L/H) \leq_e (C/H)$, where $(C/H) \leq (E/H)$.

Let $\pi: E \longrightarrow (E/H)$ be the natural projection. Then $L = \pi^{-1}(L/H)$ and so by Proposition (2.16(V)), $L \leq_e C$. But L is closed in E . Hence $L = C$.

It follows that $(L/H) = (C/H)$ and (L/H) is closed in (E/H) . \square

Remark 2.24: The intersection of two closed submodules of an R -module need not be closed as the following example shows:

Example 2.25 : (Goodearl 1976)

Let $E = \mathbb{Z} \oplus \mathbb{Z}_2$ be the \mathbb{Z} -module, . Let $V = \langle (\bar{1}, \bar{0}) \rangle$ and $U = \langle (\bar{1}, \bar{1}) \rangle$. V, U are direct summands of E , so they are closed in E . Notice that $V \cap U = \langle (\bar{2}, \bar{0}) \rangle$ and $(V \cap U) \leq_e U$; that is $V \cap U$ is not closed in E . \square

2.3 Indecomposable and Semisimple Modules

In this part we discuss the properties of the concepts direct summand, semi simple and indecomposable module.

Definition 2.26 (Wisbauer 1991) Let N_1 and N_2 be submodules of an S -module N . If $N = N_1 + N_2$ and $N_1 \cap N_2 = 0$, then N is called the (internal) direct sum of N_1 and N_2 . This written as $N = N_1 \oplus N_2$ and is called a direct decomposition of N . Also, N_1

and N_2 are called direct summand of N . When a submodule H of an S -module N is a direct summand of N , we write $H \leq_{\oplus} N$.

Moreover, if H is a direct summand of N , then in general there are various submodules K with $N = H \oplus K$.

The following results give some properties of direct summand.

The following result appeared in (Goodearl 1976). However we give the proof for completeness.

Proposition 2.27: *Every direct summand of an S -module is closed.*

Proof: Let V be a direct summand of E . Then $V \oplus B = E$ for some $B \leq E$.

Assume $V \leq_e K \leq E$. To prove $V = K$.

Let $x \in K$, then $x = a + b$ where $a \in V$ and $b \in B$. Suppose $b \neq 0$.

Now, $b = x - a \in K$.

So there is a non-zero element $r \in R$ such that $rb \neq 0$ and $rb \in V$.

Thus $0 \neq rb = r(x - a) \in V$.

So that $rb \in V \cap B = 0$.

Thus $rb = 0$ which is a contradiction.

Hence $b = 0$ and $x = a \in V$.

Therefore $V = K$. □

But, the converse of proposition(2..14) is not true in general .For example, let $E = Z_8 \oplus Z_2$ as Z -module and let $H = Z_8 \oplus (0)$, $K = (0) \oplus Z_2$, and $T = \{(0,0), (2,1), (4,0), (6,1)\}$ be sub modules of E . Then One can easily check that H and K are summand of E , while T is a closed in N but not direct summand of E .

Lemma 2.28 : (Rowen 1991) *Let E be an S -module with submodules $H \leq N \leq E$. If $H \leq^\oplus E$ then $H \leq^\oplus N$. □*

Lemma 2.29: (Anderson and Fuller 1992) (i) *A submodule H is essential and direct summand of a module E if and only if $H = E$.*

(ii) *Let E be an S -module with sub module $H \leq N \leq E$. If H is direct summand of N and N is direct summand of E then H is direct summand of E . □*

Definition 2.30: (Goodearl 1976) *An S -module E is called indecomposable, if $E \neq (0)$, and the only direct summands of E are (0) and E .*

Definition 2.31 : (Kasch 1982) *An S -module E is called semisimple, if it is a sum of simple submodules. Equivalently, E is semisimple if and only if every submodule of E is a direct summand.*

Lemma 2.32 : (Kasch 1982) *Every submodule of a semi simple module is semi simple. □*

Now we will end this section by gave some examples relations between above concepts with related concepts:

Remarks and Examples 2.33 :

(1) Every simple module is indecomposable but the converse is not hold. In fact, the \mathbb{Z} -module $E = \mathbb{Z}_4$ is indecomposable but not simple.

(2) Every uniform module is indecomposable but the opposite is not true.

(3) The \mathbb{Z} -module \mathbb{Z}_6 is semi simple but not indecomposable.

(4) Every simple module is semi simple but the converse is not hold.

As in the \mathbb{Z} -module $E = \mathbb{Z}_{10} = \langle 2 \rangle \oplus \langle 5 \rangle$ and $E = \mathbb{Z}_{10} = \langle 0 \rangle \oplus \langle 1 \rangle$ that is all sub modules of E is summand then E is semi simple but not simple module.

Lemma 2.34: (Wisbauer 1991) (*Modular law*) : If H, K, L are submodules of S -module E and $K \leq L$ then $(H + K) \cap L = (H \cap L) + K$. \square

2.4 Homomorphism of Modules

Definition 2.35 (Kasch 1982, Mahima and Avishek 2014)

Let E and F be two S -modules. Then a mapping $\alpha : E \rightarrow F$ is called a homomorphism (S -homomorphism or module homomorphism) if

(i) $\alpha(e_1 + e_2) = \alpha(e_1) + \alpha(e_2)$, for any $e_1, e_2 \in E$

(ii) $\alpha(r e) = r \alpha(e)$, $\forall r \in S, e \in E$

Remark 2.36 : If $\alpha : E \rightarrow F$ is a homomorphism, then

(i) $\alpha(0) = 0$.

(ii) $\alpha(-e) = -\alpha(e)$.

(iii) $\alpha(e_1 - e_2) = \alpha(e_1) - \alpha(e_2), \forall e_1, e_2 \in E$.

Definition 2.37: (Kasch 1982, Mahima and Avishek 2014)

A homomorphism $\alpha : E \rightarrow F$ of S – modules E and F is called

(i) a monomorphism if α is injective (*one to one*).

(ii) an epimorphism if α is surjective (*onto*).

(iii) an isomorphism if α is bijective (*one to one and onto*).

(iv) an endomorphism if $E = F$.

(v) an automorphism if $E = F$ and α is an isomorphism.

Definition 2.38: (Kasch 1982, Mahima and Avishek 2014)

Let $\alpha : E \rightarrow N$, then :

i) The kernel of α is denoted by $\text{Ker}\alpha$ and is defined by $\text{Ker}\alpha = \{x \in E : \alpha(x) = 0\}$, where 0 is additive identity of N .

ii) The image of α is denoted by $\text{Im}(\alpha)$ and is defined by $\text{Im}(\alpha) = \{\alpha(x) : x \in E\}$.

Proposition 2.39: (Kasch 1982, Mahima and Avishek 2014)

If $\alpha: E \rightarrow N$ be an S -homomorphism . Then we have

i) $\text{Ker}\alpha$ is a sub module of E .

ii) $\text{Im}(\alpha)$ is a sub module of N .

Proof :

i) Let $\alpha: E \rightarrow N$.

Since $\alpha(0) = 0$, it follows that $0 \in \text{Ker}\alpha$ implies $\text{Ker}\alpha \neq \emptyset$.

Let $k_1, k_2 \in \text{Ker}\alpha$. Then $\alpha(k_1) = 0$, $\alpha(k_2) = 0$ and

$$\alpha(k_1 - k_2) = \alpha(k_1) - \alpha(k_2) = 0 .$$

Thus , $(k_1 - k_2) \in \text{Ker}\alpha$.

Now, let $r \in S$ and $k \in \text{Ker}\alpha$ implies $\alpha(k) = 0$

$$\text{Now, } \alpha(rk) = r\alpha(k) = r \cdot 0 = 0$$

So , $rk \in \text{Ker}$.

Therefore, $\text{Ker}\alpha$ is a submodule of E . □

(ii) Let $\alpha(x_1), \alpha(x_2)$ be any two elements of $\text{Im}(\alpha)$, where $x_1, x_2 \in E$.

Now, $\alpha(x_1) - \alpha(x_2) = \alpha(x_1 - x_2) \in \text{Im}(\alpha)$, since $x_1 - x_2 \in E$.

Again, let $s \in S$ and $\alpha(x) \in \text{Im}(\alpha)$.

Then $\alpha(sx) = s\alpha(x) \in \text{Im}(\alpha)$, since $sx \in E$.

Hence $\text{Im}(\alpha)$ is a submodule of N . □

Theorem 2.40: (Lam 1999)

Let E and N be two S -modules and $\beta : E \rightarrow N$ be an S -homomorphism. Then β is an S -isomorphism if and only if $\text{Ker}\beta = \{0\}$.

Proof: let $\text{Ker}\beta = \{0\}$.

We have to prove that β is one to one.

Now, if $e_1, e_2 \in E$, then $\beta(e_1) = \beta(e_2)$

$$\Rightarrow \beta(e_1) - \beta(e_2) = 0$$

$$\Rightarrow \beta(e_1 - e_2) = 0$$

$$\Rightarrow (e_1 - e_2) \in \text{Ker}\beta$$

$$\Rightarrow e_1 - e_2 = 0$$

$$\Rightarrow e_1 = e_2$$

This means that β is one to one.

Conversely, let β be one to one.

Then $m \in \text{Ker}\beta \Rightarrow \beta(m) = 0$

But by Remark (2.36) $\beta(0) = 0$.

And since β is one – one , we have $\beta(m) = \beta(0) = 0$.

Thus , $\text{Ker}\beta = \{0\}$. □

Definition 2.41 : (Kasch 1982)

An R -monomorphism (-epimorphism) $f: E \rightarrow N$ is called split, if $\text{Im}f(\text{ker}f)$ is a direct summand of N (E) respectively .

Lemma2.42: (Kasch 1982) *Let $\alpha: E \rightarrow B$ be R -monomorphism . Then α is called split monomorphism , if and only if there exists an R -homomorphism $f: B \rightarrow E$ such that $f\alpha = I_E$. It follows that f is epimorphism, and $B = \text{Im}\alpha \oplus \text{Ker}f$. In this case f is called a split epimorphism.*

2.5 Singular and Non-Singular Modules

Definition 2.43 : (Sharpe and Vamos 1972) *Let E be an S -module and H be submodule of E . The right annihilator of H in S will denoted by $r_S(H)$ is defined by $r_S(H) = \{r \in S \mid xr = 0, \text{ for all } x \in H\}$. Also, the left annihilator of H in S will denoted by $l_S(H)$ is defined by $l_S(H) = \{r \in S \mid rx = 0, \text{ for all } x \in H\}$.*

Let $\mathbb{S}(S)$ be the set of all essential right ideals of the ring S . Given any S -module E , we set $Z(E) = \{m \in E \mid mI = 0, \text{ for some } I \in \mathbb{S}(S)\}$. It is clear that $Z(E)$ is submodule of E . $Z(E)$ is called the singular submodule of E (Goodearl 1976).

Definition 2.44: (Goodearl 1976) An S -module E is called a singular provided $Z(E) = E$. At the other extreme, we say that E is a nonsingular provided $Z(E) = (0)$.

For example Z as Z -module is non-singular, while Z_n as Z -module is singular.

The following propositions give a description of singular and nonsingular modules.

Proposition 2.45: (Goodearl 1976)

An S -module E is nonsingular if and only if $\text{Hom}(N, E) = 0$ for all singular S -modules N .

Remark 2.46: (Goodearl 1976)

Let E and H be two S -modules. Then E/H is singular whenever $H \leq_e E$.

Proposition 2.47: (Goodearl 1976)

Let E be nonsingular, and let N be submodule of E . Then E/N is singular if and only if $N \leq_e E$. \square

The following results give some properties of singular and nonsingular modules.

Proposition 2.48: (Goodearl 1976)

(i) All submodules, direct products and essential extensions of a nonsingular modules are nonsingular.

(ii) All submodules, factor modules and direct sum of singular modules are singular.

\square

Proposition 2.49: (Goodearl 1976)

Let S is a non-singular ring. Then all essential extension of singular S -module are singular.

2.6 Rational Submodules and Monofrom Modules

In this part, we review the notion of rational submodules and state basic properties and some results which are use in our work. It is worth mentioning here that the rational submodule will be needed as key to obtain the new generalizations of P-extending and injectivity which will be introduced in the following chapters.

Definition 2.50: (Goodearl 1976), *Let E be any an S -module. A submodule H is called rational submodule in E , if $\text{Hom}(N/H, E) = 0$, whenever $H \leq N \leq E$. If H is rational in E , then we write $H \leq_r E$. Equivalently $H \leq_r E$ if and only if for each $x, y \in E$, with $x \neq 0$, there exists $0 \neq r \in S$, such that $yr \in H$ and $xr \neq 0$. Following, [Lam, 1999] some time the rational submodule is called dense submodule.*

In this case, E is also called a rational extension of H . Note that we always have $E \leq_r E$

An R -monomorphism $\alpha: N \rightarrow E$ is said to be rational monomorphism if $\text{Im}\alpha \leq_r E$ (Goodearl, 1976).

Proposition 2.51: (Lam 1999) (i) *If $N \leq_r E$ then $N \leq_e E$.*

(ii) *If E is nonsingular and $N \leq_e E$ then $N \leq_r E$.* □

In general by (Lam 1999), essential extensions need not be rational. For example, let $S = Z$, if $E = Z/p^{n+1}Z$ and $L = pZ/p^{n+1}Z$ (p is prime, $n \geq 1$), then $L \leq_e E$, but $L \not\leq_r E$

We now list basic properties of rational submodules.

Proposition 2.52: (Goodearl 1976, Lam 1999)

Let H and F are non-zero submodules of an S -module E .

- (i) If $H \leq F$, then $H \leq_r E$ if and only if $H \leq_r F$ and $F \leq_r E$.
- (ii) If $K \leq_r B$ and $H \leq_r E$, then $K \cap H \leq_r B \cap E$ (where B any S -module).

Clearly, from (ii) we can conclude the following results.

- (iii) If $H, F \leq_r E$, then $H \cap F \leq_r E$.
- (iv) If $H \leq_r E$, then $F \cap H \leq_r F$ □

Lemma 2.53: A submodule L is rational and direct summand of a module E if and only if $L = E$.

Proof: Follows from proposition (2.51) and lemma (2.30). □

Remarks and Examples 2.54: (1) Let E and B be two S -modules and $f: E \rightarrow B$ be S -homomorphism. If $H \leq_r B$, then does not imply $f^{-1}(H) \leq_r E$ (Lam 1999).

As shown in the following example:

Let $B = Z$ and $H = 2Z$ as Z -modules it is clear that $H \leq_r B$. Now, let $E = Z \oplus Z_2$ and $f: E \rightarrow B$ define by $f(n, \bar{x}) = n$; $n \in Z$ and $\bar{x} \in Z_2$, it is easy to show that f is Z -homomorphism but not Z -monomorphism. Let $H' = f^{-1}(H) = 2Z \oplus Z_2$, $0 \neq x =$

$(0, \bar{1}) \in E$ and $y = (1, \bar{0}) \in E$. For each $r \in S$ we have $yr \in H'$ implies r is even, and hence, $xr = 0$. Thus H' is not rational in E .

(2) The direct sums of rational submodules are not rational. This means that, $L_i \leq_r E_i$ ($i = 1, 2$) does not imply $L_1 \oplus L_2 \leq_r E_1 \oplus E_2$ (Lam 1999). As shown in the following example

Consider $S = Z$, put $L_1 = Z, E_1 = \mathbb{Q}$, it is clear that $L_1 \leq_r E_1$, and take $L_2 = E_2 = Z_4$. Now, let $0 \neq x = (0, \bar{1}) \in E_1 \oplus E_2$ and $y = (1/4, \bar{0}) \in E_1 \oplus E_2$, if $yr \in L_1 \oplus L_2$ we must have $r = k4$, for some $k \in Z$, but, this implies $xr = (0, \bar{r}) = 0$. This show that $L_1 \oplus L_2$ is not rational in $E_1 \oplus E_2$.

(3) Let E and N be two S -modules and $\alpha: E \rightarrow N$ be R -homomorphism. If $L \leq_r E$, then $\alpha(L)$ may not be true is rational in N . As in the following example:

Let $E = Z$ and $N = Z_6$ as Z -modules and $\alpha: E \rightarrow N$ defined by $\alpha(x) = 3x$, for each $x \in E$, it is clear that α is Z -homomorphism. Now, let $L = 3Z \leq_r Z = E$, then we have $\alpha(L) = \{0, 3\}$ be a non-zero submodule of N . But $\alpha(L) \not\leq_r N$, so by lemma (2.53), $\alpha(L)$ is not rational in N .

In the following proposition we prove that the remark (2.54)(1) is true when f is S -monomorphism.

Proposition 2.55 : (Mahdi 2015) *Let E and N be two S -modules. If $H \leq_r B$. Then $f^{-1}(H) \leq_r E$, for each S -monomorphism $f: E \rightarrow B$.*

Proof: suppose that $f: E \rightarrow B$ be S -monomorphism and $H \leq_r B$. Let $x, y \in E$, with $x \neq 0$, then $f(x), f(y) \in B$ and since f monomorphism we have $f(x) \neq 0$. Now, since $H \leq_r B$ then there exists $r \in S$ such that $f(y)r \in H$ and $(f(x)r) \neq 0$.

Follows that, $f(yr) \in H$ and $f(xr) \neq 0$ implies $yr \in f^{-1}(H)$ and $xr \neq f^{-1}(0) = \ker(f) = 0$. Therefore, by definition (2.50), $f^{-1}(H) \leq_r E$. \square

The following result is appeared in [Lam 1999, p.284], as Exercises and proved by (Mahdi 2015).

Proposition 2.56 : *If $E' \leq E$ and $K \leq_r E$. For any S -homomorphism $h: E' \rightarrow E$ then $h^{-1}(K) \leq_r E'$.*

Proof: Assume that $h^{-1}(K)$ is not rational submodule of E' . Then, for each $r \in S$ there exist $x, y \in E'$, with $x \neq 0$ such that $yr \notin h^{-1}(K)$ or $xr = 0$.

Follows that $h(yr) \notin h(h^{-1}(K))$ or $h(xr) = h(0)$ implies $h(y)r \notin K$ or $h(x)r = 0$, this lead to $mr \notin K$ or $nr = 0$, for some $h(y) = m \in E$ and $h(x) = n \in E$, a contradiction since $K \leq_r E$. Therefore, $h^{-1}(K) \leq_r E'$. \square

Definition 2.57: (Zelmanowits 1986) *Anon-zero S -module E is called monofom (some times termed strongly uniform), if each non-zero submodule H of E is rational.*

It is clear that, every monofom module is uniform, but uniform modules need not be monofom.

Example 2.58 (1) the Z -module Z_p is monofom for every prime number p

(2) The Z -module Z_9 is uniform module but it is not monofom since the submodule $\langle 3 \rangle$ is not rational.

3. EXTENDING AND INJECTIVE MODULES WITH SOME THEIR GENERALIZATIONS

In this chapter, follows (Dung and *et al.* 1994, Mohamed and Muller 1990) we recall the definition of extending module and some of basic properties of extending module. Also, we present some of their generalizations such as P-extending follows (Kamal and Elmnophy 2005) and rationally extending follows (Abbas and Ahmed 2011), which are required for our study in this work. In last part of this chapter we discuss the concepts of injectivity namely (injective, quasi-injective, c-injective, rationally injective and e-injective) modules.

3.1 Extending Modules (CS modules)

Definition 3.1 : (Dung *et al.* 2019, Mohamed and Muller 1990)

An S-module E is said to be extending (or CS module), if every submodule of M is essential in a direct summand of M.

Proposition 3.2: (Mohamed and Muller 1990, Dung *et al.* 2019)

An R-module E is extending if and only if every closed submodule is a direct summand.

Lemma 3.3: (Dung *et al.* 2019, Mohamed and Muller, 1990)

Any direct summand of an extending module is also extending.

Proof:

Let H be a direct summand of E. Hence $E = H \oplus B$ for some $B \leq E$.

Let K be any closed submodule of H .

Since H is a direct summand of E , then H is a closed submodule of E by lemma (2.27).

So by proposition (2.23) K is closed in E .

Then by Proposition (2.29), K is a direct summand in E .

Hence $E = K \oplus L$ for some $L \leq E$.

Thus $A = (K \oplus L) \cap H$, and by [Lemma 2.34], $H = K \oplus (H \cap L)$.

So K is a direct summand of H .

Hence A is an extending module. \square

Examples and Remark 3. 4:

(1) Every uniform module is extending.

(2) Any semi simple module is extending and hence any simple module is extending.

(3) Any closed submodule of a CS-module is always CS.

(4) The \mathbb{Z} -module $E = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$, where p is a prime number is not CS.

(5) The \mathbb{Z} -module $E = \mathbb{Z}_8 \oplus \mathbb{Z}$ is CS-module.

Remark 3.5: (Dung *et al.* 2019) *A finite direct sum of CS-module and homomorphic image of CS-modules may not be CS.*

Example 3.6:

$E = \mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ as a \mathbb{Z} -module is not CS-module for any prime number p .

But \mathbb{Z}_p and \mathbb{Z}_{p^3} are CS-modules for any prime number p .

Proposition 3.7: (Lam 1999, Mohamed and Muller 1990) *Let M be an R -module, then M is uniform if and only if M is indecomposable and extending.*

3.2 Principally-Extending (P-Extending) Modules

Definition 3.8: (Kamal and Elmnophy 2005)

An S -module E is called P-extending, if each cyclic sub module of " E is essential in direct summand.

Equivalently, E is P-extending iff each EC-closed sub module of " E " is direct summand.

Remarks and Examples 3.9:

- (i) Every extending module is P-extending.
- (ii) Every uniform module is P-extending.
- (iii) Every semi simple (and hence simple) module is P-extending.
- (iv) By all above we have that the \mathbb{Z} -module \mathbb{Z}_2 , \mathbb{Z}_6 , and \mathbb{Z} are all P-extending.

Definition 3.10: (Kamal and Elmnophy 2005) *A sub module B of an S-module E, said to be EC-(closed) sub module, if B is (closed) submodule which contains essentially a cyclic sub module; i.e $\exists x \in B$ such that $x S \leq_e B$.*

Proposition 3.11: (Kamal and Elmnophy 2005) *Every summand of an EC- submodule of E is EC-submodule.*

Proof. Let $x.S \leq_e E$ be an EC-submodule of E, and $H \leq_{\oplus} E$, then $E = H \oplus T$

for some sub module T in E. Let $e = h + t$, where $h \in H$ and $t \in T$.

It is not hart to explain that $S \leq_e H$.

Therefore, H is an EC-sub module of E.

Corollary 3.12: *Each summand of an EC-closed sub module of E is EC-closed.*

Proposition 3.13: *The following are equivalent for an indecomposable module E:*

1. *E is extending;*
2. *E is P-extending;*
3. *E is uniform.*

Proof:

1) \Rightarrow 2) \Rightarrow 3) it is clear and the converse follows (Kamal and Elmnophy 2005).

Kamal and Elmnohy (2005) was introduced the following definition which using it to show that when the direct sum of P-extending also P-extending.

Definition 3.14: Let $E = E_1 \oplus E_2$ be a module. The module E_2 is called E_1 -E C-injective, if for every E C-(closed) submodule H of E_1 , and every homomorphism from H to E_2 can be extended to E_1 .

Remark and Examples 3.15::

(1) Each module over a regular ring S is S -E C-injective.

(2) Following (1), we can present the next example, Let $R = Z_6$, it is well known that S is semi simple ring hence S is regular. Thus, the module $E = Z_4$ over a ring Z_6 is Z_6 - E C - injective.

Proposition 3.16: (Kamal and Elmnohy 2005) Let $E = E_1 \oplus E_2$ and E_2 be E_1 -E C-injective. Then:

1. E_2 is K -E C-injective, for all $K \leq E_1$.
2. H is E_1 -E C-injective, for all $H \leq_{\oplus} E_2$.
3. H is K -E C-injective, for all $K \leq_{\oplus} E_1$, and $H \leq_{\oplus} E_2$.

In the following two propositions (Kamal and Elmnohy 2005) was explained when the direct sum of p-extending module is p-extending.

Proposition 3.17: Let $E = V \oplus U$, where V is a semi simple module. Then E is P-extending if and only if every E C-closed sub module H of E with $H \cap V = 0$, is a summand.

Follows, (Nicholson and Yousif 1995) For any an A -modules F and E . F is called E -principally injective (for short E - P -injective) if each A -homomorphism from a cyclic submodule of E to F can be extended to F .

Proposition 3.18: Let $E = V \oplus U$, where V is P -extending and U is V - P -injective. If U is nonsingular, then every E C -closed submodule H of E , with $H \cap U = 0$, is a summand of E .

3.3 Rationally Extending Modules

Abbas and Ahmed (2011) was introduced the concept rationally extending R -modules. An S -module E is called rationally extending (or RCS-module), if each submodule of E is rational in a direct summand.

Definition 3.19: (Abbas and Ahmed 2011) Let N be a submodule of an S -module E . Then N is called rationally closed in E if and only if N has no a proper rational extension in E .

Lemma 3.20: (Abbas and Ahmed 2011) Let E be an S -module. For each submodule H of E there exists a submodule P of E such that H is rational in P and P has no proper rational extension. □

Remark 3.21: It is clear that every closed submodule (and hence direct summand) of an S -module is rationally closed, but the opposite is not true in general. For example, the submodule $N = 2Z$ of the Z -module $E = Z_4$ is rationally closed but not closed. However for non-singular modules they are equivalent.

Here, we introduce the following two lemmas which are needed later.

Lemma 3.22: (Mahdi 2015) Let U be a submodule of E and V be a complement of U in E . Then

(i) $U \oplus V \leq_r E$

(ii) V is rationally closed in E .

Proof: (i) By lemma (2.20)(ii), we have $U \oplus V \leq_e E$. Now, suppose that $U \oplus V$ is not rational submodule of E .

Thus, there exist $0 \neq x, y \in E$ such that for each $r \in S$, either $yr \notin U \oplus V$ or $xr = 0$. It is clear that, in the both cases $U \oplus V \not\leq_e E$, which a contradiction. Thus, $U \oplus V \leq_r E$.

(ii) By lemma (2.20)(iii), we have V is closed submodule of E implies V is rationally closed submodule of E . \square

Lemma 3.23: (Mahdi 2015) *If B is rationally closed submodule in L and L is rationally closed submodule in S -module E , then B is rationally closed in E .*

Proof: Let B is rationally closed in L and L is rationally closed in E .

Suppose that, B is not rationally closed submodule in E . Then there exists a proper submodule K of E such that $B \not\leq_r K \not\leq E$, and since $L \leq_r L$, then $B \cap L \leq_r K \cap L$. This implies that $B \leq_r K \cap L \leq L$, such that $K \cap L \neq L$, if not, then $L \leq K \leq E$.

This implies that $\leq_r K \cap L = L \leq K$. But, $B \not\leq_r K$, hence by proposition (2.52(i)), $L \not\leq_r K$. This contradiction, since L is rationally closed in E . Thus $K \cap L \neq L$ and this implies that B has a proper rational extension in L .

But, B is rationally closed in L , this contradiction. Hence B is rationally closed in E .

\square

Corollary 3.24: (Abbas and Ahmed 2011) *Let E be an S -module and let $H \leq E$. If H is rationally closed in a direct summand of E . Then $H \leq_{rc} E$.*

Proof: Let L be a direct summand of E and H is rationally closed in L . Follows that, we have $H \leq_{rc} L \leq_{rc} E$ and so by Lemma (3.23), we get $H \leq_{rc} E$. \square

Definition 3.25 (Abbas and Ahmed 2011) *An S -module E is said to "be rationally extending (or RCS module), if every submodule of E is rational in direct summand".*

Remarks and Examples 3.26: (i) It is clear that every RCS modules is CS modules, but the converse is not true in general.

For example, Z -module $E = Z_{p^\infty}$ is trivially extending S -module, but it is not RCS .

- (ii) Every monofrom modules is trivially rationally extending.
- (iii) Non-singular extending modules are rationally extending.

The next two propositions appear in (Abbas and Ahmed 2011).

Proposition 3.27 : (i) *An S -module E is RCS if and only if each rationally closed submodule of E is summand.*

(ii) *Each direct summand of RCS module is RCS.*

Proposition 3.28: *An E -module E is monofrom if and only if E is indecomposable RCS module.*

3.4 Injective Modules and Injective Envelope

In this section , we list some definitions and well-known fundamental properties of injective modules and some of their generalizations which are required for our study in this work. Also, in part of this section we introduce some results which will be needed in our work.

Definition 3.29 : (Faith 1967) *An S -module E is said to be injective, if for every S -monomorphism $\alpha:U \rightarrow V$ (where U and V are any S -modules) and every S -homomorphism $\beta:U \rightarrow E$, there is an S -homomorphism $f:V \rightarrow E$ such that $f \circ \alpha = \beta$.*

A ring S is called right self-injective, if it is a right injective S -module.

It is well-known that (*Baer's Condition*), an S -module E is injective if and only if for each right ideal L of S and each S -homomorphism $f:L \rightarrow E$, there is an element $m \in E$ such that $f(x) = mx$ for all x in L .

A ring S is called principal ideal domain if it is an integral domain and each right ideal in S is principal.

Kasch (1982) Recall that, an S -module E is divisible, if $E = Er$ for each non-zero divisor $r \in S$. It is well known that each injective S -module is divisible.

The converse is true in case that the ring S is a principal ideal domain. Then concepts on injectivity and divisibility are coincided over such rings.

Examples 3.30 :

(1) *The group of rational numbers Q is an injective Z -module.*

Proof: Let $H = nZ$ be a right ideal of Z , and let $f: H \rightarrow Q$ be a Z -homomorphism, if $H=0$, then the situation is trivial.

Now suppose that $H \neq (0)$. For each non-zero element $a \in H$, there is an integer x such that $a = nx$, provided that $n \neq 0$. Now $f(a) = f(nx) = f(n)x = (f(n)/n)(nx) = m(nx) = ma$.

Where $m \in Q$, then by Bear's condition Q is injective as Z -module. \square

Definition 3.31: (Mohamed and Muller 1990) *Let F be S -module. An S -module E is said to be injective relative to F (or F -injective), if for every submodule H of F , any S -homomorphism $\alpha: H \rightarrow E$ can be extended to an S -homomorphism $f: F \rightarrow E$. Its clear that an S -module E is injective if E is F -injective for all S -module F .*

A family of modules $\{E_i \mid i \in I\}$ is relatively injective if E_i is E_j -injective, for every $j, i \in I, i \neq j$ (Lam 1999).

Some fundamental properties of injectivity follow below.

Proposition 3.32: (Sharpe and Vamos 1972) *An S -module E is injective if and only if satisfies any one of the equivalent conditions:*

(i) *For any S -module X , every S -monomorphism $E \rightarrow X$ splits.*

(ii) *E has no proper essential extensions.* \square

Mohamed and Muller (1990) we have the following fundamental properties of relative injectivity.

Proposition 3.33: (i) *An S -module E_1 is E_2 -injective if and only if E_1 is mS -injective for every $m \in E_2$*

(ii) Let E_1 and E_2 are two S -modules. If E_1 is E_2 -injective, then, for every submodule H of E_2 , E_1 is H -injective and (E_2/H) -injective.

(iii) If T is E -injective, then any monomorphism $T \rightarrow E$ splits. \square

Proposition 3.34 : Let $\{E_i \mid i \in I\}$ and F be modules. Then F is $(\bigoplus_{i \in I} E_i)$ -injective if and only if F is E_i -injective, for every $i \in I$. \square

Proposition 3.35: Let E and $\{F_i \mid i \in I\}$ be modules. Then $\prod_{i \in I} F_i$ is E -injective if and only if F_i is E -injective, for each $i \in I$. \square

Proposition 3.36: (Clara 1998) Let F_1 and F_2 be R -modules, let D be a submodule of F_1 and $F = F_1 \oplus F_2$. The following conditions are equivalent.

(i) F_2 is (F_1/D) -injective;

(ii) For every submodule K of F such that $K \cap F_2 = 0$ and $\pi_1(K) \cap D \leq K$, there exists a submodule K' of F such that $K \leq K'$ and $F = K' \oplus F_2$.

Proposition 3.37: (Sharpe and Vamos 1972) Let F be an S -module. Then there is an S -module satisfying the next equivalent conditions:

(i) E is an essential injective extension of F ;

(ii) E is a maximal essential extension of F ;

(iii) E is a minimal injective extension of F . \square

Definition 3.38 (Sharpe and Vamos 1972) *Let F is S -module, the S -module E which satisfies the conditions of proposition (4.1.9) is called injective hull (or envelope) of F , and denoted by $E(F)$.*

Definition 3.39: (Johnson and Wong 1961) *An S -module E is called quasiinjective if it is E -injective.*

Proposition 3.40: (Lam 1999) *The S -module F is quasi-injective iff $h(F) \subseteq F$. For every $h \in \text{End}_S(E(F))$. \square*

For more properties of quasi-injective see (Faith and Utumi 1964, Jain and Singh 1975, Osofsky 1968, Sharpe and Vamos 1972). Hence we have the following remarks

Remark and Examples 3.41: (1) An S -module F is injective iff $E(F) = F$.

(2) If F and H be two S -modules and $H \leq_e F$, then $E(H) = E(F)$.

(3) Every injective S -module is quasiinjective, but the opposite is not hold in general. Such as the Z -module Z_2 is quasiinjective which is not injective.

(4) Every semi-simple R -module is quasi-injective.

(5) It is not hard to prove that Z is not quasi-injective Z -module.

Proposition 3.42: (Mohamed and Muller 1990) *Every (quasi-)injective module E is rationally extending (and hence extending).*

While the opposite of above proposition not hold, where we have the Z -module Z is rationally extending (in fact Z is monofrom), but not quasi injective.

Thus, the following implications hold:

Injective \rightarrow *quasi-injective* \rightarrow *rationally extending* \rightarrow *extending* \rightarrow *P-extending*

Moreover, the above implications are not reversing.



4. MAIN RESULTS

4.1 Principally Rationally Extending Modules

In this section, we will take advantage of the concept of rationally extending which is provided by (Abbas and Ahmed 2011) together with work (p-extending) of (Kamal and Elmnophy 2005), to introduce a new class of modules namely principally rationally extending modules (for short, RP-extending) which is stronger properly than P-extending modules and weak than rationally extending modules. Also we study some properties of those concepts and discuss some connections between those concepts and some concepts of well-known modules.

We will start this section by introduce our main definition

Definition 4.1: *An S -module E is said to be Rationally Principally extending module (for shortly, RP- extending); if every cyclic sub module of E is rational in summand of E .*

Some notes with examples 4.2:

i): *Each RP-extending module is P-extending.*

Proof: Suppose that E be a module and $K \leq_{cy} E$, then by definition (1), $K \leq_r D \leq E$ for some summand D in E . So, by proposition (2.51), $K \leq_e D \leq E$, hence, E is P-extending module. However, the converse of (1) is not true in general. For example, let $E = Z_4$ as Z -module, it is clear E is P-extending (In fact, E is uniform) and hence extending but, E is not RP-extending since $\langle 2 \rangle$ is cyclic sub module that is not rational in direct summand in E .

ii): *Every rationally-extending module is P-extending.*

Proof: Follows Remark[3.9(i)] and Remarks[3.26(i)]. □

iii): In general, we can explain that the converse of (ii) is not hold, for example,

Let $E = Z_9$ as Z -module, it is clear E is extending (In fact, E is uniform) but, E is not rationally extending since $\langle 3 \rangle$ be rationally closed sub module in E but not direct summand of E .

iv): It's easy one can show that every rationally extending (and hence semi simple module) is RP-extending.

v): *Every monofrom module is RP-extending.*

Proof: Let E be monofrom module then by Remark[3.26(ii)], E is rationally-extending. Thus, by part (iv) E is RP-extending. □

vi): Let $E = Z$ as Z -module, since E is monofrom, then by (v) E is RP-extending.

vii): Let $E = Z_6$ as Z -module, since E is semi simple then by (iv) E is RP-extending.

Thus, by all above we get the following implication:

Rationally-Extending \Rightarrow RP- Extending

↓ ↓

Extending \Rightarrow P- Extending

In the upcoming results, we will present the sufficient condition that makes those concepts are equivalent.

Proposition 4.3: For a nonsingular S -module E , the following are equivalent:

- i- E is RP -Extending module.
- ii- E is P -Extending module.

Proof:

(\Rightarrow) It is clear, by [Note (i)].

(\Leftarrow) It follows, [proposition(2.51(ii))]. \square

Proposition 4.4: If every submodule of E is cyclic, then the following is equivalent.

- i) E is rationally extending-module
- ii) E is RP -Extending- module

Proof:

i) \Rightarrow (ii) follows the [Note (ii)].

ii) (\Rightarrow) (i) Suppose that E is RP -extending module, then by definition (4.1) every cyclic submodule of E is rational in direct summand of E . But by hypothesis, we have every submodule of E is cyclic, so that E is rationally extending. \square

Lemma 4.5 : Let E be indecomposable module such that every submodule of E is cyclic submodule. Then E is monofree if and only if E is RP -extending.

Proof: The first direction: Is clear by [Note (vii)].

The second direction: Suppose that E is RP-extending, then for each $(0) \neq H \leq_{cy} E$ implies $H \leq_r D \leq_{\oplus} E$ (for some non-zero sub module D of E). But by hypothesis each submodule of E is cyclic, thus we get for, $(0) \neq H \leq E$ implies $H \leq_r D \leq_{\oplus} E$. But E is indecomposable module and $D \neq (0)$. Thus $D=E$, this mean E is monoform. \square

Now, we are ready to present the following theorem.

Theorem 4.6: *Let E be indecomposable nonsingular S -module such that every sub module of E is cyclic sub module. Then following are equivalent:*

- i- *E is monoform module.*
- ii- *E is rationally Extending.*
- iii- *E is RP-Extending module.*
- iv- *E is P-Extending module.*
- v- *E is Extending module.*
- vi- *E is uniform modules.*

Proof: i) \Leftrightarrow (ii): Follows [Abbas and Ahmad 2011 , proposition(3.3)].

ii) \Rightarrow (iii) : Directly by [proposition 4.5].

iii) \Rightarrow (iv) : It is clearly, by [proposition 4. 4].

iv) \Rightarrow (v) \Rightarrow (vi) : Since E is indecomposable module then, by [Proposition 3.13] the results done.

vi) \implies (i): Since E is nonsingular and uniform module, then by [Proposition 2.51], E is monofrom. \square

Definition 4.7: A (closed) sub module H of S -module E is called *Rc-(closed) sub module*, and symbolizes it by $(H \leq_{Rcc} E) \ H \leq_{Rc} E$, if H contains properly a rationally cyclic sub module, i.e $\exists x \in H$ such that $x.S \leq_r H$.

Lemma 4.8: Every *Rc-(closed) submodule* of E is *Ec- (closed) submodule*.

Proof: Let $H \leq_{Rcc} E$, then by [definition5.7], there exist $h \in H$ such that $S.h \leq_r H$ and hence by [Remark 3.9(i)] $S.h \leq_e H$. Therefore, H is *Ec-closed submodule*. \square

Lemma 4.9: For nonsingular S -module E . A submodule H is *Rc-(closed) submodule* of E iff H is *Ec- (closed) submodule*.

Proof: Follows lemma (4. 8) and the opposite we can get it by Proposition (2.51).

Notes and Examples 4.10:

(1) The converse of corollary(5.9) is not true in general, as following example, let $E = Z_8$ as Z - module, then $N = 2Z \leq_{Ec} E$ since $\exists 4 \in N$ and $4Z \leq_e N$. But N is not *RC-sub module*, since $N \not\leq_{Rc} E$, since $4Z \not\leq_r N$.

(2) We known that, each summand is closed, but in the following example, we show that, no every summand is *Rc-closed*.

Now, let $W = Z_6$ as Z -module and $N=3Z$ is direct summand of W , but N is not *Rc-closed*, because no there is element $y \in N$ such that $yZ \leq_r N$.

(3) Every Rc-closed is closed sub module, but the opposite is not correct in general, as shown, let $E = Z/6Z$ as Z -module and $N = \langle 2 + 6Z \rangle$. It is clear N is closed (in fact N is summand in E), But N is not Rc-closed because $\langle 0 + 6Z \rangle$ is only properly cyclic sub module of N which it is not rational in N .

(4) Not necessary every sub module of uniform S -module is Ec- sub module.

As example: let $E = Z_4$ as Z -module it is easy to explain E is uniform, and $H = 2Z$ is sub module of E but not Ec-sub module, since no there exist element $x \in H$ such that $\langle x \rangle \leq_e H$.

Proposition 4.11: *Let B be a sub module of S -module E such that $B = B_1 \oplus B_2$ (for some sub modules B_1 and B_2 of B). If B is Rc-sub module in E then B_1 is Rc-sub module in B .*

Proof: Suppose that, $B = B_1 \oplus B_2$ is Rc-sub module in E then for some $b \in B$, we have $b.S \leq_r B \leq_{RC} E$, and $B_1 \leq_{\oplus} E$.

Now, let $b = b_1 + b_2$, where $b_1 \in B_1$ and $b_2 \in B_2$.

Since $b.S \leq_r B$

Implies $(b_1 + b_2).S \leq_r B_1 + B_2$

Implies $b_1.S + b_2.S \leq_r B_1 + B_2$

So, we get $b_1.S \leq_r B_1$

This means, B_1 is Rc-sub module of E . □

Now, directly from proposition (4.11) and Lemma (2.32), we get the proof of the following results.

Proposition 4.12: *Let H be semi simple sub module of S -module E . If H is Rc -sub module in E then every sub module of H is Rc -sub module.*

Corollary 4.13: *Let E be a semi simple S -module E . If H is Rc -sub module in E then every sub module of H is Rc -sub module.*

4.2 Direct Sum of RP-Extending Modules and RCC-Modules

In the next section, we will introduce the definition of RCC-module and discuss when the direct sum of RP-Extending is also RP-Extending.

Definition 4.14: *An S -module E is said to be $RC(C)$ -module and $EC(C)$ -module , if every non-zero submodule of E is Rc -(closed)sub module and Ec -(closed)sub module (respectively).*

Notes 4.15: (1) It is surely, by corollary(5.9), every Rc -module is Ec -module.

(2) In general, the opposite of (1) is not satisfying. As shows:

Let $E = Z_{16}$ as Z -module, it is not hard to show that E is Ec -module.

But in this time E is not Rc -module, since $4Z \leq_{Ec} E$. But $4Z$ is not Rc -sub module, because only cyclic non-zero sub module of $4Z$ is $8Z$ which is not rational in $4Z$.

Proposition 4.16: *Let $E = E_1 \oplus E_2$, where E_2 is a semi simple S -module. Then every Rc -closed sub module H of E with $H \cap E_2 = 0$, is a direct summand if and only if E is RP -extending.*

Proof: Suppose that the first direction hold, now to prove that E is RP-extending. Let B be any Rc-closed sub module of E . Then by [definition 4.7], there exist $b \in B$ such that $.b.S \leq_r B$ but by assumption $B \leq_{\oplus} E$. Thus, we get $.b.S \leq_r B \leq_{\oplus} E$. So that, by [definition 4.1], E is RP- extending.

Now, for second direction, let B be an Rc-closed submodule of E . If $E_2 \cap B = (0)$, then the proof is done on out this case, since E_2 is semi simple, we obtain

$$E_2 \cap B \leq_{\oplus} E_2 \text{ and for some } \hat{B} \leq B.$$

$$\text{We get } B = (E_2 \cap B) \oplus \hat{B}.$$

$$\text{Thus, by [Proposition 4.11], } \hat{B} \leq_{Rcc} E$$

$$\text{and } E_2 \cap \hat{B} = (0),$$

$$\text{then } \hat{B} \leq_{\oplus} E.$$

$$\text{There for, } B \leq_{\oplus} E. \quad \square$$

Corollary 4.17: *Let $W=W_1 \oplus W_2$ such that, W_2 is simple module. Then W is RP-extending if and only if every Rc-closed submodule V of W with $V \cap W_2 = (0)$, is a direct summand.*

We need the following definition, (Lam 1999), "Let E is an injective hull of the module F ". "Then submodule Q of E is said to be the rational hull of F in E , if $F \leq_r Q$ and Q contains each rational extension of F which is contained in E .

Proposition 4.18: *Let $E = E_1 \oplus E_2$ and $(B \cap E_2) \leq_{Rcc} E$, for each $B \leq_{Rcc} E$. Then E is RP-extending if and only if every Rc-closed sub module B , with $E_1 \cap B = (0)$, or $E_2 \cap B = (0)$, is a direct summand.*

Proof: Let $b.S \leq_r B \leq_{Rcc} E$ (for some $b \in B$). If $E_1 \cap B = (0)$ then the proof is done.

Otherwise, $E_1 \cap B \leq_{Rcc} E$. "Now, let B_1 , be rational hull of $(E_1 \cap B)$ in B , thus by above definition, $(E_1 \cap B) \leq_r B_1$. Hence, $B_1 \leq_{Rcc} E$ ". So by our assumption,

$B_1 \leq_{\oplus} E$, Now, we can write $E = B_1 \oplus B_2$ and by using modular Law [Lemma 2.34] we get, $B = B_1 \oplus (B \cap B_2)$. Then by [proposition 4.11] we get $(B \cap B_2) \leq_{Rcc} E$ with $(B \cap B_2) \cap E_1 = (0)$, this implies to $B \cap B_2 \leq_{\oplus} E$ and hence by [Lemma 2.14] $B \leq_{\oplus} E$. Therefore, E is RP- extending.

The Second direction is clearly. \square

Following, (Nicholson and Yousif 1995) "Let S be a ring and V, U are two S -modules. Then V is say that, U "-prin cipally injective ("for short U - P -injective) if every S -homomorphism from a cyclic sub module of U to V can be extended to V ".

Proposition 4.19: *Let $E = E_1 \oplus E_2$ where E_2 is RP-extending and E_1 is E_2 - P -injective. If E_1 is nonsingular then every Rc-closed submodule H of E with $H \cap E_1 = 0$, is a direct summand of E .*

Proof: Let $\langle h \rangle \leq_r H$ (for some $h \in H$) and $H \leq_{Rcc} E$ with $H \cap E_1 = (0)$ and write $h = h_1 + h_2$, where $h_1 \in E_1$, and $h_2 \in E_2$. Since E_1 is E_2 - P -injective, then follows (Kamal 1995).

$$\langle h \rangle = \langle h_2 \rangle = H_2^* = [h_2 a + \alpha(h_2)a : a \in S] \subseteq E_2^*$$

where $E_2^* = \{e_2 + \alpha(e_2) : e_2 \in E_2\} \cong E_2$ and such that $E = E_1 \oplus E_2^*$, where $\alpha \in \text{Hom}_S(E_2, E_1)$. Now, let $x \in H$ and write $x = e_1 + y$ where $y \in E_2^*$ and $e_1 \in E_1$. Since $S \cdot h \leq_r H$ then $\exists I \leq_r S$ such that $e_1 I = 0$ (where, I is an ideal of a ring S). Since E_1 is nonsingular, then $e_1 = 0$. So that $H \subseteq E_2^*$. Now, since E_2^* is RP-extending we have $H \leq_{\oplus} E_2^* \leq \oplus E$.

4.3 Rcc-Injective and Rcc-Quasi-Injective Modules

In this section we will introduce a proper generalization of W -injective and quasi-injective modules namely (Rcc- W -injective and Rcc-quasi injective) respectively.

Definition 4.20: Let W and F are S -modules. Then F is said to be Rcc- W -injective if every homomorphism $h: L \rightarrow F$, (where $L \leq_{rcc} W$), can be extended to a homomorphism $t: W \rightarrow F$.

An S -module F is called Rcc-injective, if F is Rcc- W -injective, for every S -module W . An S -module F is called Rcc-quasi-injective or Rcc(-self)-injective, if F is Rcc- F -injective.

Remarks and Examples 4.21:

(1) Every injective S -module is Rcc-injective.

(2) The opposite of above is not hold always, where if we take:

$F = Z/2Z$ as Z -module. Since F has no Rcc-closed submodule, so F is Rcc-injective. But, in the same time F is not divisible Z -module. Thus, F is impossible injective.

3) Every uniform (and so, each monofrom) module is Rcc-injective module.

Proof:- Suppose that F be a uniform module, then F is only Rc -closed submodule in F since every submodule of F is large submodule. So, that, F is Rcc -quasi-injective. \square

(4) Every quasi-injective module is Rcc -quasi-injective module.

(5) The converse of (4) is not true in general. Take $F = Z$ as Z -module. It is surly that, Z is not quasiinjective. But, since F is uniform (where F is monofrom), thus follows (3), we get Z is Rcc -quasiinjective.

Note from the remarks (4) and (5), the class of Rcc -quasi-injective modules is actually contains the class of quasi-injective modules.

(6) It is clear that, C -injectivity (Clara 1998) is stronger than Rcc -injectivity. This means that, if F_2 is F_1 - c -injective then F_2 is also $F_1 - Rcc$ -injective.

So we have the following diagram :

Injective \Rightarrow quasiinjective $\Rightarrow C$ -quasiinjective $\Rightarrow Rcc$ -quasiinjective

In the next results we will discuss when the classes of Rcc -injective, c -injective and ec -injective are equivalent.

Proposition 4.22: *Let W be non-singular S -module. F is $Ec - W$ -injective iff F is $Rcc - W$ -injective.*

Proof: *First diraction:* Suppose that, F is $Ec - W$ -injective and let $L \leq_{Rcc} W$ with a homomorphism $i: L \rightarrow F$. Now by lemma (4.8.), L is Ec -closed submodule of W . Thus, by $Ec - W$ -injectivity of F , there exists a homomorphism $h: W \rightarrow F$ such that $\theta = h \circ i$, (where i is the inclusion map of L). Thus, F is $Rcc - W$ -injective.

Second diraction: Let, F be nonsingular $Rcc - W$ -injective and $E \leq_{ec} W$ with a homomorphism $\theta: E \rightarrow F$. Since W be nonsingular module then by lemma [4.9] $E \leq_{rcc} W$.

Now by assumption we have, F is $Rcc - W$ -injective, then $\exists \mu: W \rightarrow F$ as S -homomorphism such that $\theta = \mu \circ i$, (where i is the inclusion map of E). Thus, F is $ec - W$ -injective. \square

Corollary 4.23: *For a nonsingular module W . Then W is $Ec - quasi$ -injective iff W is $Rcc - quasi$ -injective.*

Proof: The proof is similar to the proof of proposition (4.22), just replace $F = W$. \square

Proposition 4.24: *Let W is RCC module. Then F is $W - njective$ iff F is $Rcc - W$ -injective.*

Proof: (\rightarrow) Follows the remarks (4.21)(1).

(\leftarrow) Assume that, F is $Rcc - W$ -injective. Now, let $T \leq W$ and $\tau: T \rightarrow F$ is homomorphism. Because W be RCC module so, $T \leq_{rcc} W$.

Now, by $Rcc - W$ -injectivity of F , there is a homomorphism $f: W \rightarrow F$ such that $\tau = f \circ i_T$, (where i_T is the inclusion of T).

Therefor, F is W -injective. \square

Corollary 4.25: *For RCC module W . Then W is quasi injective if and only if W is $Rcc - quasi$ injective.*

Proof: Just replace $F = W$ in proposition (4.24), Then in the same way we can done it.

Theorem 4.26: *Let W be RCC nonsingular module and F is any S -module. Then the next statement are equivalent:*

(a) F is W -injective

(b) F is c - W -injective

(c) F is ec - W -injective

(d) F is Rcc - W -injective.

Proof:

(a) \Rightarrow (b) \Rightarrow (c): Follows (Kamal and Elmenophy 2005, Canan and Tercan 2007).

(c) \Rightarrow (d) : By proposition (4.22).

(d) \Rightarrow (a): By proposition (4.24).

Corollary 4.27: *Let F be non-singular RCC module. Then the next are equivalents:*

(a) F is quasi- injective

(b) F is c - quasi -injective

c) F is ec - quasi -injective

(d) F is Rcc - quasi -injective.

From the following proposition we can get many examples of Rcc-injectivity.

Proposition 4.28: *Let S be a regular ring. Then every S -module W is Rcc- S -injective.*

Proof: Take S is regular ring. Then by (Kamal and Elmenophy 2005), we have that every S -module (where S is regular ring) is S -EC-injective, so that by proposition (4.22) the proof done.

In the next results we present basic properties of Rcc –injectivity

Proposition 4.29: *Let H and L are two isomorphic modules with W is any module. If H is Rcc – W –injective, then L is Rcc – W - injective.*

Proof: The proof is clear. □

Proposition 4.30: *Let W and $F_i (i \in I)$ are modules. Then $F = \prod_{i \in I} F_i$ is Rcc – F - injective if and only if F_i is Rcc – W -injective, for every $i \in I$.*

Proof: (\Rightarrow): Let, $\prod_{i \in I} F_i$ is Rcc – W - injective. Let $f: V \rightarrow F_i$ is homomorphism(where $V \leq_{Rcc} W$).

Now, we can determine the natural projection and injection mapping respectively: as follows

$$\pi_i: \prod_{i \in I} F_i \rightarrow F_i \text{ and } \sigma_i: F_i \rightarrow \prod_{i \in I} F_i \text{ for every } i \in I, .$$

Since $\sigma_i \circ f: V \rightarrow F$ is a homomorphism and by hypothesis, F is Rcc – W - injective, so, there is S -homomorphism $\theta: W \rightarrow F, \exists \sigma_i \circ f = \theta \circ \omega$ (where ω is inclusion $\omega: V \rightarrow F$).

Claim that, there is an S-homomorphism $h: W \rightarrow F_i$ such that $\mu = h \circ \omega$, to prove this, for every $x \in V, f(x) = I_{F_i} \circ f(x) = ((\pi_i \circ \sigma_i) \circ f)(x)$

$$= (\pi_i \circ ((\sigma_i \circ f)(x)))$$

$$= (\pi_i \circ ((\theta \circ \omega)(x)))$$

$$= ((\pi_i \circ \theta) \circ \omega)(x) = (h \circ \omega)(x), \text{ (where } I_{F_i} \text{ is identity map of } F_i \text{ for every } i \in I).$$

Therefore, this show that, F_i is $Rcc - W$ -injective, for every $i \in I$.

(\Leftarrow) Let, F_i is $Rcc - W$ -injective, for every $i \in I$. Let $\varphi: H \rightarrow F = \prod_{i \in I} F_i$ is an S-homomorphism, (where $H \leq_{Rcc} W$).

Then, for any homomorphism $\pi_i \circ \varphi: H \rightarrow F_i$, there is an S-homomorphism $h_i: W \rightarrow F_i$ such that $\pi_i \circ \varphi = h_i \circ \mu$.

Thus by (Kasch 1982) there is an S-homomorphism $h: W \rightarrow F$ with $\pi_i \circ h = h_i$. It is easy to show that $\varphi = h \circ \mu$.

Therefore, $F = \prod_{i \in I} F_i$ is $Rcc - W$ - injective. \square

As a direct application of the proposition (4. 30) we will present following corollary.

Corollary 4.31: Let $F = H \oplus H'$. If F is $Rcc - W$ -injective module then H is $Rcc - W$ -injective. \square

Lemma 4.32: Let $D \leq_{Rcc} W$. If D is Rcc - W -injective, then $D \leq_{\oplus} W$.

Proof: By Rcc - W -injectivity of D , there is a homomorphism $f: W \rightarrow D$ which extend

identity map $I : D \rightarrow D$. It is not hard to see that, $W = D \oplus \text{Ker}f$, so that $D \leq_{\oplus} W$.

Proposition 4.33: *Let W is RCC module and E is any module with E is Rcc – W -injective. Then the next statements holds.*

(i) *If $V \leq_{Rcc} W$, then E is Rcc – V -injective.*

(ii) *If $V \leq_{Rcc} W$ and $U \leq_{Rcc} W$ then E is Rcc – $V \cap U$ -injective.*

Proof:(i) Let $V \leq_{Rcc} W$, $L \leq_{Rcc} V$ and $\varphi: B \rightarrow E$ be homomorphism.

By hypothesis W is RCC module so, we have $B \leq_{Rcc} W$, hence by Rcc – W -injectivity of E , there is a homomorphism $h: W \rightarrow E$ such that $h i_V i_B = \varphi$ where $i_B: B \rightarrow V$ and $i_V: V \rightarrow W_1$ are inclusion maps.

Let $f = h \circ i_V$. Clearly, f is homomorphism, and $f = h i_B = h i_V i_B = \varphi$.

Therefore, E is Rcc – V -injective. □

ii)) Let $V \leq_{Rc} W$ and $U \leq_{Rc} W$ and since W is RCC module then $V \cap U \leq_{Rc} W$.

Thus by part (i) we get, E is Rcc – $V \cap U$ -injective. □

Now, directly from prop.(4.3.14) we can get the proof of the following tow results

Corollary 4.34: *Let W_1 and W_2 be any RCC modules. If $W_1 \oplus W_2$ is Rcc – quasi injective, then W_1 and W_2 are both Rcc – quasi injectives .* □

Corollary 4.35: *An RCC module W is Rcc – quasiinjective if W is Rcc – V -injective for every Rc-sub module V of W .* □

Proposition 4.36: *Let W is RCC module. Then every direct summands of Rcc-quasi injective is also Rcc-quasi.*

Proof: Suppose that W be RCC module with W is Rcc-quasi injective and $H \leq_{\oplus} W$.

So $M = H \oplus H'$ for some submodule H' of W . Now, let $X \leq_{Rc} H$ and $\mu: X \rightarrow H$ is any S -homomorphism.

Since, W is RCC module, implies $X \leq_{Rc} W$ then by Rcc – quasi-injectivity of W , there is S -homomorphism $l: W \rightarrow W$ with $li_H i_X = j_H \mu$, where (j_H is injection map) $j_H: H \rightarrow W$ and (i_X is inclusion) $i_X: X \rightarrow H$.

Let $\rho: W \rightarrow H$ is projection map. Define $f: H \rightarrow H$ by $f(h) = \rho(l(h))$ for any $h \in H$. Follows that, for each $x \in X$, $f(x) = f i_X(x)$

$$= \rho l i_X(x)$$

$$= \rho j_H \circ \mu(x) = \mu(x).$$

Hence H is Rcc – quasi injective. □

Proposition 4.37: *Let W be any module and F be an RCC module. Then W is Rcc – F -injective if and only if each, $U \leq_{\oplus} W$ is Rcc – L -injective for each, $L \leq_{Rcc} F$.*

Proof: Suppose that, W is Rcc – F -injective. Let $U \leq_{\oplus} W$ then by [Corollary (4.31)], U is Rcc – F -injective.

Now, let $L \leq F$ but by hypothesis F is RCC module then, $\leq_{Rc} F$, So that by [Proposition(4.33)], U is Rcc – L -injective.

Conversely, since we have $W \leq_{\oplus} W$ and $F \leq_{Rc} F$ then by above the result is done. \square

The characterization of $Rcc - W$ -injective module is given in the next result .

Proposition 4.38: *Let $W = W_1 \oplus W_2$, where W_1 and W_2 are two modules. Then the following conditions are equivalent:*

(i) W_2 is $Rcc - W_1$ -injective

(ii) For every $(Rc$ -closed) submodule P of M such that $P \cap W_2 = 0$ and $\pi_1(P) \leq_{Rcc} W_1$ (where π_1 is the natural projection of W into W_1), there exists a submodule P' of W such that $P \leq P'$ and $W = P' \oplus W_2$.

Proof: (i) \Rightarrow (ii) Let, $P \leq W$ such that $P \cap W_2 = 0$ and let W_2 is $Rcc - W_1$ -injective.

Let $\pi_i: W \rightarrow W_i$, ($i = 1, 2$) be the projective mapping and $\pi_1(P) \leq_{Rc} W_1$.

As $P \cap W_2 = 0$, the restriction of π_1 to P is an isomorphism between P and $\pi_1(P)$.

Define $t: \pi_1(P) \rightarrow W_2$ such that $t(x) = \pi_2[(\pi_1)^{-1}(x)]$ (where, $\forall x \in \pi_1(P)$), $\exists p \in P$ such that $x = \pi_1(p)$).

It is not hard to show that t is well-define S -homomorphism.

Since $\pi_1(P) \leq_{Rc} W_1$ and W_2 is $Rcc - W_1$ -injective, so t can be extended to an homomorphism $f: W_1 \rightarrow W_2$.

Define, $P' = \{p + \theta(p), p \in W_1\}$. Clearly, P' is a submodule of W and $W = P' \oplus W_2$.

For every $p \in P$ we have $p = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

So, since $\pi_1(p) = w_1$ and $\pi_2(p) = w_2$ with $f(\pi_1(p)) = t(\pi_1(p)) = \pi_2(p)$.

Hence, we get $p = \pi_1(p) + \pi_2(p) = \pi_1(p) + f(\pi_1(p)) \in P'$.

Therefore, $P \leq P'$.

(i) \Leftarrow (ii), Suppose that, the part (ii) holds. Let $X \leq_{Rc} W_1$ and let $\varphi: X \rightarrow W_2$ be an R -homomorphism. Define $P = \{x - \varphi(x), x \in X\}$.

Clearly, P is a sub module of M such that $P \cap W_2 = 0$. It is easily to show that $\pi_1(P) = X$ and so that $\pi_1(P) \leq_{Rcc} W_1$.

Then by part (ii), $\exists P' \leq W \ni P \leq P'$ and $W = P' \oplus W_2$.

Let $\pi_2: W \rightarrow W_2$ is the projection with $\ker \pi_2 = P'$ and let $f: W_1 \rightarrow W_2$ is the restriction of π to W_1 .

For every $x \in X$, $f(x) = \pi(x) = \pi[x - \varphi(x) + \varphi(x)] = \pi(x - \varphi(x)) + \pi(\varphi(x)) = \varphi(x)$ and, hence, f extends φ .

Thus W_2 is $Rcc - W_1$ -injective. \square

Recall an S -module W is projective, if for any epimorphism $f: N \rightarrow H$ (H and N be any two S -module) each homom. $g: W \rightarrow H$ there is a homom. $t: W \rightarrow N$ with $g = ft$ (Kasch 1982).

Theorem 4.39: Let E and V are two modules such that E is $Rcc - V$ - injective for any projective S -module V . Then every Rc -submodule of V is projective iff $\frac{E}{H}$ is $Rcc - V$ - injective (where H is any submodule of E).

Proof: Let $Y \leq_{Rc} V$, then by hypothesis Y is projective. Let $\frac{E}{H}$ and E are two S -modules with an S -homomorphism $\tau: E \rightarrow \frac{E}{H}$.

Consider $l: Y \rightarrow \frac{E}{H}$ is an R -homomorphism and E is $Rcc - V$ - injective.

Then by projectivity of Y , there is an S -homomorphism $f: Y \rightarrow E$ lifts l .

Since E is $Rcc - V$ - injective, so there is an homomorphism $\beta: V \rightarrow E$ extends f . Hence, $f\beta: V \rightarrow \frac{E}{H}$ extends l .

Conversely: Suppose that $H \leq E$ such that $\frac{E}{H}$ is $Rcc - V$ - injective.

Now, take N is injective and $\frac{N}{K}$ is a factor module of N with an epimorphism $t: N \rightarrow \frac{N}{K}$.

Clearly that, $\frac{N}{K}$ is $Rcc - V$ - injective. let that Y is an Rc -submodule of V , then by V - Rcc - injectivity of $\frac{N}{K}$, we get each homomorphism $\beta: Y \rightarrow \frac{N}{K}$ can be extended to a homomorphism $\tau: V \rightarrow \frac{N}{K}$. Since V is a projective, so there is a homomorphism $f: V \rightarrow N$ lifts τ . So obviously, $f|_Y$ lift β .

Hence, Y is projective sub module of V . \square

If each submodule of E is a projective then S -module E is said to be a hereditary (Dung *et al.* 2019). Then, by theorem (4.39) we have the next result.

Corollary 4.40: *Let E is a hereditary S -module .Then each factor S -module of an $R_{cc} - E$ -injective S -module is $R_{cc} - E$ -injective.*



5. CONCLUSIONS AND RECOMMENDATION

In our work we get the following conclusions. Every RP – extending is P-extending and every rationally extending module is RP-extending, but the converse may be not holds in general. The concepts RP-extending module and P-extending module are equivalent when the module is nonsingular. Every Rc-module is Ec-module. Every injective S-module is Rcc-injective, but the opposite is not hold always. Every uniform (and so, each monofrom) module is Rcc-injective module. Every quasi-injective module is Rcc-quasi-injective module and the conversly is not true. C-injectivity (Clara 1998) is stronger than Rcc-injectivity. Finaly the concepts(W-injectivity, C-W-injectivity, EC-W-injectivity and Rcc-W-injectivity) are equivalent when W be RCC nonsingular module.

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