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UNIVERSAL COVERING OF A LIE GROUP

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Merve ERSOY

Signature

Dedicated to my mom who is my inspiration, my best friend, my role model

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Merve ERSOY

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LIST OF SYMBOLS

$\mathbb{S}^1$	Circle
$\bar{A}$	Closure of the set A
$\mathbb{S}^n$	n-Sphere
$\mathbb{T}^n$	n-Torus
$\mathbb{R}^n$	Real n-space
$\mathbb{Z}$	Set of integers
$\mathbb{R}_+$	Set of positive real numbers

LIST OF ABBREVIATIONS

iff	If and only if
LSC	Locally simply connected
nhood	Neighborhood
SLSC	Semi-locally simply connected
s.t.	Such that
w.r.t.	With respect to

LIST OF FIGURES

ABSTRACT

Universal Covering of a Lie Group

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Master of Science Thesis

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It is well known that any Lie algebra homomorphism extends to a Lie group homomorphism in the simply connected case. This result combined with the (Lie) group structure on the universal covering space of a Lie group provides an explicit form of connected Lie groups from their corresponding simply connected Lie groups. In this thesis, we mainly intend to explain how such a structure is obtained on the simply connected covering space of a given Lie group.

Keywords: Homotopy**,** covering space, universal cover, Lie group, Lie homomomorphism

ÖZET

Lie Gruplarının Evrensel Örtüsü

Merve ERSOY

Matematik Bölümü

Yüksek Lisans Tezi

Danışman: Doç. Dr. Eyüp KIZIL

Tanım bölgesi basit bağlantılı ise, Lie cebirleri arasındaki herhangi bir homomorfizmin karşılık gelen Lie gruplarına genişlediği iyi bilinmektedir. Bu sonuç, bir Lie grubunun evrensel örtü uzayı üzerindeki Lie grubu yapısı inşasıyla birlikte, bunlara karşılık gelen basit bağlantılı Lie gruplarından bağlantılı Lie gruplarının açık bir tanımını sağlar. Bu tezde, esas olarak, bir Lie grubunun evrensel örtü uzayı üzerindeki Lie grubu yapısının nasıl elde edildiğini açıklamayı amaçlıyoruz.

Anahtar Kelimeler: Homotopi, örtü uzayı, evrensel örtü uzayı, Lie grubu, Lie grubu homomorfizmi

YILDIZ TEKNİK ÜNİVERSİTESİ

FEN BİLİMLERİ ENSTİTÜSÜ

1
INTRODUCTION

1.1 Literature Review

Covering spaces are important in many areas like homotopy theory, geometric group theory, harmonic analysis, differential topology, Riemannian geometry, etc. In particular, they allow us to study the structure of groups through their actions on certain spaces.

The goal of the present work is to study the universal covering of a Lie group.

We have organized our work as follows: An historical overview together with the necessary hypothesis of the main topic of the thesis are presented in the first chapter.

In the second chapter, a basic knowledge on general topology, and in particular, some fundamental notions from homotopy theory are exposed in detail.

In the third chapter, we introduce the covering space and consider in detail the necessary requirements to ensure that universal covering space exists.

The fourth chapter contains a brief information about Lie groups, Lie homomorphisms and Lie algebra homomorphisms since we take into account the covering of a Lie group.

Last chapter studies the extensions of Lie algebra homomorphisms to Lie group homomorphisms since this is closely related with the main subject of our work. Finally, we use the existence theorem and Monodromy principle to show that the universal covering of a connected Lie group exists and is unique.

1.2 Objective of the Thesis

The goal of this thesis is to study the simply connected universal covering of a Lie group.

1.3 Hypothesis

We claim that a Lie group possesses a universal covering if it is connected.

2
GENERALITIES

This chapter introduces fundamental definitions, concepts that will be useful later.

Definition 2.1 (Topological Space). A collection τ consisting of subsets of a non-empty set T is a topology on T if:

1. $\emptyset$ and T are both members of τ.
2. Arbitrary unions of sets in τ are in τ.
3. The intersection of any finite number of members in τ are again τ.

T equipped with such a collection is called a topological space that we denote by (T, τ) or just by T if there is no need to emphasize the collection. Members of τ will be nominated as open sets. A subset $D \subset T$ will be called closed if its complement $T \setminus D$ belongs to τ.

Definition 2.2 (Basis and Subbasis for a Topology). A subcollection $\mathcal{B}$ of a topology τ on T is a basis for τ if it fulfills the criterias:

1. The elements of $\mathcal{B}$ cover T.
2. Finite intersections of elements of $\mathcal{B}$ appear as unions of elements of $\mathcal{B}$.

Whenever we are given such a basis for τ we will prefer the notation $\tau = \langle \mathcal{B} \rangle$.

A subbasis for τ is a subcollection $\mathcal{S} \subset \tau$ such that $\langle \mathcal{S} \rangle = \mathcal{B}$.

Definition 2.3 (2nd Countable Space). We call a space 2nd countable if the topology of this space has a countable basis.

Definition 2.4 (Product Space). Let (T_1, τ_1), (T_2, τ_2), ..., (T_n, τ_n) be topological spaces. The product topology on $T = \prod T_i$ is a topology having as basis

$$\mathcal{B} = \{\prod V_i : V_i \in \tau_i\}. \tag{2.1}$$

Definition 2.5 (Quotient Space). Let $\sim$ be an equivalence relation on a topological space T and let $T/\sim$ be the set of all equivalence classes in T. There exists a natural surjective map

$$\pi : T \to T/\sim \tag{2.2}$$

that associates to any t its class $[t]$, called the canonical map.

The set $T/\sim$ provided with the quotient topology (for which open sets are subsets of $T/\sim$ that have an open preimage under π) is called the quotient space under $\sim$.

Definition 2.6 (Cover). By a covering of a topological space T we mean a collection $\mathfrak{U}$ consisting of subsets of T s.t. elements of $\mathfrak{U}$ under union is equal to T. The covering $\mathfrak{U}$ is called open if it is formed by only open subsets.

Definition 2.7 (Compact Space). A space T is compact if every open covering of T possesses a finite subcovering.

Definition 2.8 (Compact-Open Topology). Let $\mathcal{C}(T, S)$ denote the set of all continuous functions between T and S. Let $K \subset T$ be a compact subspace and $V \subset S$ be an open subset. Define

$$\mathcal{S}(K, V) = \{f : f \in \mathcal{C}(T, S) \text{ and } f(K) \subset V\}. \tag{2.3}$$

The collection of the sets $\mathcal{S}(K, V)$ constructs a subbasis for the compact-open topology on $\mathcal{C}(T, S)$. [1]

Definition 2.9 (Connectedness, Path Connectedness). A space T is connected if it cannot be expressed as a union of two disjoint non-empty open sets. Otherwise, T is disconnected. We say T is path connected if points or states of T might be linked to each other via a path.

The above definition might also be given in local setting if each state of T has a nhood with the required property.

Remark 2.10. It is clear that path connectedness yields connectedness. Nonetheless, a connected space may fail to be path connected (e.g. topologist's sine curve). If one adds local path connectedness property, then the converse implication is also true.

The following facts might be encountered in any standart topology textbook:

- (Path) connectedness is preserved under continuity.
- Finite products of (path) connected spaces are (path) connected. The situation is distinguished if one considers arbitrary products.
- Quotients spaces of connected, path connected, locally connected, locally path connected etc. keep the same characteristic properties.
- A space is connected iff the only open and closed sets are the empty set and the space itself.

Definition 2.11 (Connected Component). The connected subset of a space T which is maximal with this property is a connected component of T.

Remark 2.12. A typical component is always a closed set. Nonetheless, it might be also open if they are finite in number. Also, a connected space has only one connected component which is the space itself.

Definition 2.13 (Open Map). A function between topological spaces is an open mapping if it maps open sets to open sets in the obvious way.

Definition 2.14 (Homeomorphism). A homeomorphism is a continuous open bijection between topological spaces. In this case, the two spaces are called homeomorphic.

In particular, a homeomorphism $h : T \to S$ between open sets of T and S is said to be a local homeomorphism.

Remark 2.15. Homeomorphic spaces are topologically the same.

We find it convenient to mention now some elementary notions from group theory since we will need them in the sequel.

Definition 2.16 (Group Homomorphism, Isomorphism). A map $\varphi : L \to K$ between two groups is called a homomorphism if

$$\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y), \forall\, x, y \in L. \tag{2.4}$$

In particular, a homomorphism is an isomorphism if it is a bijection. [2]

Note that, we have $\varphi(1_L) = 1_K$ and $\varphi(x^{-1}) = \varphi(x)^{-1}$ for every $x \in L$.

Definition 2.17 (Kernel). The kernel of a homomorphism $\varphi : L \to K$ is the set

$$\ker\varphi = \{x \in L : \varphi(x) = 1_K\} \subset L. \tag{2.5}$$

Definition 2.18 (Coset). Let $N \subset L$ be a subgroup. Given $x \in L$, the left cosets of N in L are defined by

$$xN = \{xn : n \in N\}. \tag{2.6}$$

Similarly, the right cosets are denoted by $Nx = \{nx : n \in N\}$ for $x \in L$. We say N is a normal subgroup and write $N \lhd L$ if $xN = Nx$ for every $x \in L$.

Definition 2.19 (Quotient Group). Take $N \lhd L$ and define the set $L/N = \{xN : x \in L\}$ to be the set of all cosets of N in L. This set forms a quotient group under a product operation: $(xN)(yN) = xyN$. In this case, there is a natural projection

$$\pi : L \to L/N, \pi(x) = xN \tag{2.7}$$

such that $\ker\pi = N$.

Definition 2.20 (Center). By the center of a group L we understand the set

$$\mathcal{Z}(L) = \{z \in L : \forall\, x \in L, xz = zx\} \subset L \tag{2.8}$$

which is a normal subgroup. One can easily see that L is commutative iff $\mathcal{Z}(L) = L$.

Definition 2.21 (Central Subgroup). A subgroup $N \subset L$ is central if it lies inside the center of the group.

Remark 2.22. Central Subgroups are abelian normal subgroups.

Theorem 2.23. Let $\varphi : L \to K$ be a homomorphism.

1. $\ker\varphi \lhd L$.
2. φ is injective iff $\ker\varphi = \{1_L\}$.
3. The image $\mathrm{im}\varphi \subset K$ is a subgroup.
4. φ is trivial if it maps every element of L to 1_K.
5. $\mathrm{im}\varphi$ is always isomorphic to a quotient of L. Particularly, $\mathrm{im}\varphi \cong L/\ker\varphi$.

Definition 2.24 (Identity Component). The identity component of a topological group L is the connected component $L_0 \ni 1_L$.

Theorem 2.25. A connected topological group L is generated by any open nhood of its identity 1_L.

Proof. Given a nhood V of 1_L, consider the set $V^n = \{v_1 \ldots v_n : v_i \in V\}$ for $n \in \mathbb{N}$. Define now $U \coloneqq \bigcup_{n \in \mathbb{N}} V^n$. That U is open follows at once since V^n is open for each $n \in \mathbb{N}$. To see U is also a closed, take $x \in \overline{U}$. Since xV^{-1} is a nhood of x, it follows that $xV^{-1} \cap U \neq \emptyset$. So, take $y \in xV^{-1} \cap U$.

$\Rightarrow \exists\, v \in V, y = xv^{-1}$ and $\exists\, n \in \mathbb{N}, y = v_1 \ldots v_n$.

Then we get $x = v_1 \ldots v_n v \in V^{n+1} \subset U$, showing that U is closed.

Finally, $U = L$. [1] ■

Now we mention some facts from elementary homotopy theory that we will use for later references.

2.1 Homotopy of Paths

Definition 2.26. Two continuous maps $\alpha, \beta : T \to S$ between topological spaces are homotopic if there is a continuous map $F : T \times I \to S$ given by

$$F(t, 0) = \alpha(t) \ \& \ F(t, 1) = \beta(t), \forall\, t \in T. \tag{2.9}$$

We call this map a homotopy between α and β; in this case we write $\alpha \simeq \beta$. Moreover, if β is a constant map, we say that α is nullhomotopic. [2]

We might restrict the definition above to the particular class of continuous maps, namely, paths. For convenience, we use the closed unit interval as the domain I for a path.

Definition 2.27. Two paths $\alpha, \beta : I \to T$ are path homotopic if their departure and terminal states coincides (i.e., $\alpha(0) = \beta(0) = t_0$ and $\alpha(1) = \beta(1) = t_1$), and if there is a continuous map $F : I \times I \to T$ given by

$$F(x, 0) = \alpha(x) \ \& \ F(x, 1) = \beta(x), \tag{2.10}$$

$$F(0, y) = t_0 \ \& \ F(1, y) = t_1, \tag{2.11}$$

for every $x, y \in I$. Then, F is called a path homotopy; $\alpha \simeq_p \beta$. See Figure 2.1. We will denote the path homotopy class of α by $[\alpha]$. [2]

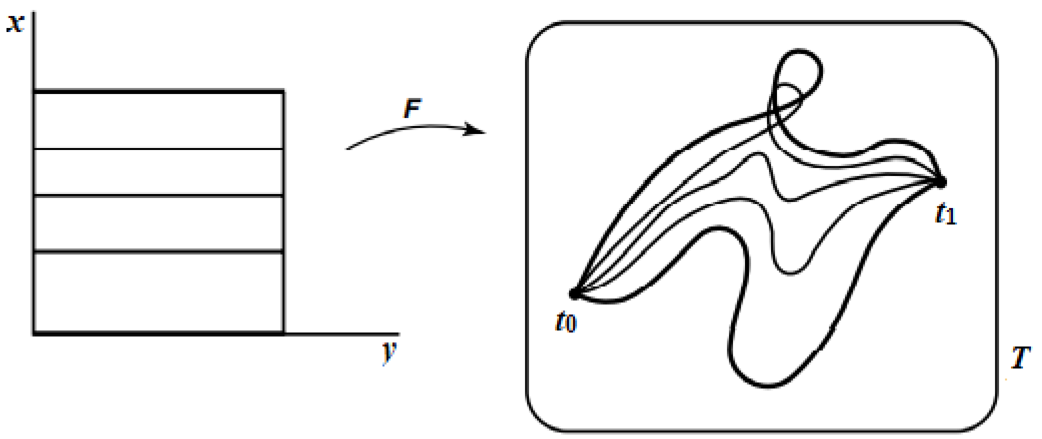

Figure 2.1 Path homotopy [2]

Lemma 2.28. The relations $\simeq$ and $\simeq_p$ are equivalence relations. [2]

Example 2.29. Any two paths having the same initial state and final state in a convex subspace $A \subseteq \mathbb{R}^n$ are path homotopic, as pictured in Figure 2.2.

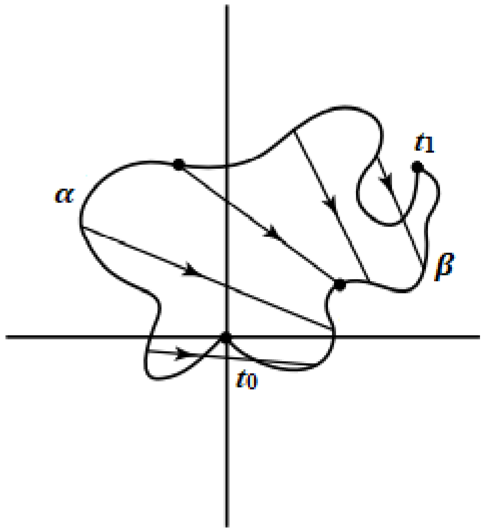

Figure 2.2 Linear homotopy between the paths α and β [2]

Definition 2.30 (Concatenation). Given a path α in T with initial state t_0 and final state t_1, and another path β also in T starting at t_1 and ending at t_2. The concatenation (or product) $\alpha * \beta$ is a path γ in T with $\gamma(0) = t_0$ and $\gamma(1) = t_2$, given by

$$\gamma(x) = \begin{cases} \alpha(2x) & \text{for } x \in [0, ½], \\ \beta(2x-1) & \text{for } x \in [½, 1]. \end{cases} \tag{2.12}$$

By the gluing lemma, the map γ satisfies the necessary requirements. [2]

This concatenation provides a corresponding operation on homotopically equivalent paths by means of $[\alpha] * [\beta] = [\alpha * \beta]$.

Remark 2.31. Note that concatenation is only defined in case the termination of the first path is the origin of the second one. It is associative which means

$$[\alpha] * ([\beta] * [\gamma]) = ([\alpha] * [\beta]) * [\gamma]. \tag{2.13}$$

Moreover, it satisfies the identities:

$$[\alpha] * [c_{t_1}] = [\alpha], \tag{2.14}$$

$$[c_{t_0}] * [\alpha] = [\alpha], \tag{2.15}$$

$$[\alpha] * [\bar{\alpha}] = [c_{t_0}], \tag{2.16}$$

$$[\bar{\alpha}] * [\alpha] = [c_{t_1}]. \tag{2.17}$$

where c_{t_0} and c_{t_1} are constant paths at t_0 and t_1, separately, and $\bar{\alpha}(x) = \alpha(1-x)$ is the inverse path of α.

2.2 The Fundamental Group

We constrain ourselves to loops emanating from a fixed point of T to obtain a well-defined group called the fundamental group.

Definition 2.32. The set of path homotopy classes of loops originating at a fixed state $t_0 \in T$ is the fundamental group of T under the concatenation, and is denoted by $\pi_1(T, t_0)$. [2]

The importance of the fundamental group is that it gives a topological invariant of the space T. We give below a quite simple example for the fundamental group of a space.

Example 2.33. Let $A \subseteq \mathbb{R}^n$ be any convex subset. Then $\pi_1(A,\, t_0)$ consists of the neutral element only. The linear homotopy is a path homotopy between c_{t_0} and any loop based at t_0. [2]

Definition 2.34. Select a path α in T with initial state t_0 and terminal state t_1. Define

$$\hat{\alpha} : \pi_1(T, t_0) \to \pi_1(T, t_1),\ \hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]. \tag{2.18}$$

This map is well-defined because $*$ is well-defined. [2]

Remark 2.35. The map $\hat{\alpha}$ given in (2.18) is a group isomorphism. [2]

Remark 2.36. The fundamental group does no longer depend on the state if the space under consideration is assumed to be path connected.

Definition 2.37 (Induced Homomorphism). Define for a continuous $h:(T,t_0)\to(S,s_0)$ the map

$$h_*:\pi_1(T,t_0)\to\pi_1(S,s_0),\ h_*([f])=[h\circ f] \tag{2.19}$$

which is called the induced homomorphism. [2]

Remark 2.38. For distinct base points t_0, t_1 of T, the homomorphism in (2.19) is denoted by $(h_{t_0})_*$, $(h_{t_1})_*$, respectively.

Theorem 2.39. The composition of the continuous maps $h:(T,t_0)\to(S,s_0)$ and $g:(S,s_0)\to(R,r_0)$ induces the homomorphism

$$(g\circ h)_*=g_*\circ h_*. \tag{2.20}$$

See [1, Th. 52.4.] for the proof of the above statement. An immediate corollary of it also says that if $h:(T,t_0)\to(S,s_0)$ is a homeomorphism, h_*: $\pi_1(T,t_0)\to\pi_1(S,s_0)$ is an isomorphism.

Definition 2.40. A path connected space T is simply connected (or 1-connected) if each loop in T is nullhomotopic in T.

In particular, a space T is locally simply connected (abrev. LSC) if each state in T has a nhood which is simply connected.

Example 2.41. Although the circle is not 1-connected, locally it is.

Remark 2.42. A LSC space is both locally path connected and locally connected.

Example 2.43. Every topological manifold is immediately locally connected, locally path connected, and LSC, since it is locally Euclidean. Connectedness of a manifold is equivalent to its path connectedness.

Definition 2.44. A space T is a semi-locally simply connected (abrev. SLSC) if each state in T has a nhood V s.t. every loop in V is nullhomotopic in T. Highlight the fact that the contraction is not required to be within the nhood V.

Remark 2.45. It is obvious by definition that both 1-connected spaces and LSC spaces are SLSC, however its converse does not necessarily hold.

Lemma 2.46. In 1-connected topological spaces any pair of paths with the same extreme points are path homotopic.

Proof. Take two paths α and β in T linking t_0 and t_1. Then $\alpha * \bar{\beta}$ is a loop originated at t_0. By 1-connectivity of T, it follows that $\alpha * \bar{\beta}$ is nullhomotopic. Then

$$[\alpha * \bar{\beta}] * [\beta] = [c_{t_0}] * [\beta] \Rightarrow [\alpha] = [\beta] \tag{2.21}$$

is obtained. [2] ■

3
COVERING SPACES

Covering spaces are closely related with the study of fundamental groups. This relationship will be established in the first section.

Definition 3.1. Let $q : T \to S$ be a continuous onto map. An open set $U \subset S$ is evenly covered by the map q if

$$q^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha \tag{3.1}$$

where for each $\alpha \in I$ the open set $V_\alpha \subset T$ is disjointed and $q|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism (Figure 3.1). [2]

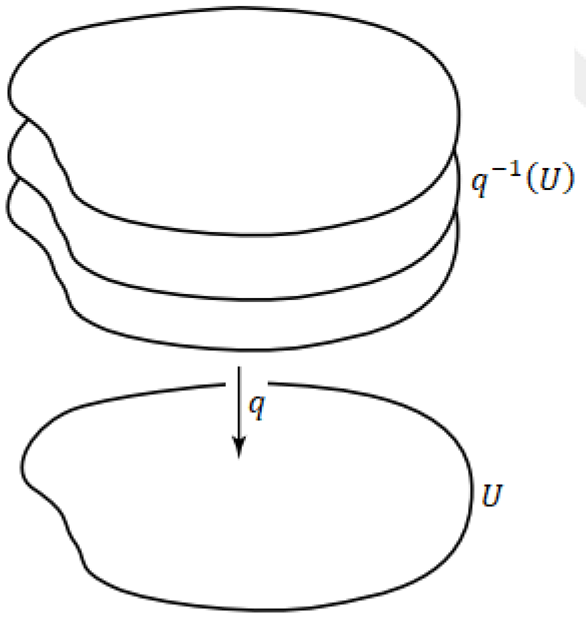

Figure 3.1 A partition of $q^{-1}(U)$ [2]

Definition 3.2. A continuous onto map $q : T \to S$ is a covering map if each state in S has an evenly covered nhood by the map q. Also, T is called a covering space of the base space S. [2]

Remark 3.3. If q is a covering map, it might be noted that for each state of S the topology of the fiber $q^{-1}(s) \subset T$ is discrete and that q is both an open map and a local homeomorphism. It is also evident that any space may be viewed as a trivial cover.

Example 3.4. The map $q : \mathbb{R} \to \mathbb{S}^1$ by $p(t) = (\cos 2\pi t, \sin 2\pi t)$ is a covering.

Example 3.5. The map $q : \mathbb{S}^1 \to \mathbb{S}^1$ by $p(z) = z^2$ is a covering.

Theorem 3.6. Let $q : T \to S$ be a covering map and $S_0 \subset S$ be a subspace. Then, the restricted map $q_0 : q^{-1}(S_0) \to S_0$ is a covering map. [2]

Theorem 3.7. If $q : T \to S$ and $q' : T' \to S'$ are covering maps, $q \times q' : T \times T' \to S \times S'$ is a covering map. [2]

Example 3.8. Consider the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. It follows that the map $q \times q : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{S}^1$ is a covering of $\mathbb{T}^2$, where q represents the covering map in Example 3.4. [2]

1.1 Homotopy Lifting Property

Definition 3.9. Let $q : T \to S$ be a map. The lifting of a continuous map $f : R \to S$ is a map $\tilde{f} : R \to T$ s.t. $q \circ \tilde{f} = f$: [2]

$$\begin{tikzcd} & T \arrow[d, "q"] \\ R \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & S \end{tikzcd}$$

Figure 3.2 The lifting diagram [2]

We note that under a covering map paths and path homotopies can be lifted as we will show in the example taken from [2].

Example 3.10. Consider the covering $q : \mathbb{R} \to \mathbb{S}^1$ given in Example 3.4. The path $\alpha(x) = (\cos\pi x, \sin\pi x)$ lifts to the path $\tilde{\alpha}(x) = x/2$ such that $\tilde{\alpha}(0) = 0$ and $\tilde{\alpha}(1) = 1/2$. The path $\beta(x) = (\cos\pi x, -\sin\pi x)$ lifts to the path $\tilde{\beta}(x) = -x/2$ such that $\tilde{\beta}(0) = 0$ and $\tilde{\beta}(1) = -1/2$. The lifting of the path $\gamma(x) = (\cos 4\pi x, \sin 4\pi x)$ is $\tilde{\gamma}(x) = 2x$ such that $\tilde{\gamma}(0) = 0$ and $\tilde{\gamma}(1) = 2$. These paths are shown in Figure 3.2.

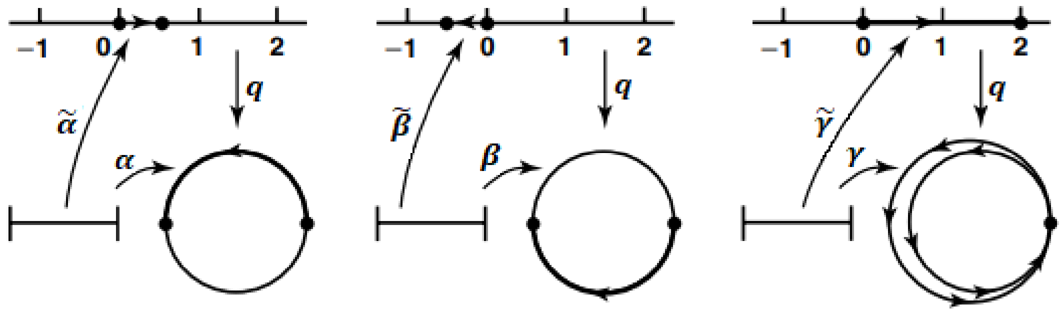

Figure 3.3 The liftings of the paths α, β, and γ [2]

Lemma 3.11. Let $q : T \to S$ be a covering map and $\alpha : I \to S$ be a path with $\alpha(0) = s_0$. Then for each $t_0 \in q^{-1}(s_0)$, there exists a unique lift to a path $\tilde{\alpha} : I \to T$ with $\tilde{\alpha}(0) = t_0$.

Lemma 3.12. Let $q : (T, t_0) \to (S, s_0)$ be a covering map. Any continuous map $F : I \times I \to S$ with $F(0, 0) = s_0$ lifts to a unique continuous map $\tilde{F} : I \times I \to T$ with $\tilde{F}(0, 0) = t_0$. If F is a path homotopy, so is $\tilde{F}$. [2]

Theorem 3.13. Let $q : (T, t_0) \to (S, s_0)$ be a covering map. If any pair of paths in S from s_0 to s_1 are path homotopic, then their liftings beginning at t_0 have the same terminal point. [2]

Definition 3.14. Let $q : (T, t_0) \to (S, s_0)$ be a covering map. Given $[\alpha] \in \pi_1(S, s_0)$, let $\tilde{\alpha}$ be the lifting of α to a path in T such that $\tilde{\alpha}(0) = t_0$. Consider the following well-defined mapping

$$\sigma : \pi_1(S, s_0) \to q^{-1}(s_0), \tag{3.2}$$

where $\sigma([\alpha])$ stands for the terminal point $\tilde{\alpha}(1)$ of $\tilde{\alpha}$. This map is called the lifting correspondence through q which is dependent on the choice of t_0. [2]

Theorem 3.15. The lifting correspondence in (3.2) is onto if T is path connected. Moreover, it is both onto and one-to-one if T is not only path connected but also 1-connected. [2]

Proof. Let us first take a look at the path connected case. For any $t_1 \in q^{-1}(s_0)$, there is a path $\tilde{\alpha}$ in T with $\tilde{\alpha}(0) = t_0$ and $\tilde{\alpha}(1) = t_1$. Then $\alpha = q \circ \tilde{\alpha}$ is a loop in S originated at s_0, so $\sigma([\alpha]) = t_1$.

Now assume T is 1-connected. Take $[\alpha], [\beta] \in \pi_1(S, s_0)$ with $\sigma([\alpha]) = \sigma([\beta])$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be their respective liftings emanating at a state t_0. Then, the same applies to their ending states, that is,

$$\tilde{\alpha}(1) = \tilde{\beta}(1). \tag{3.3}$$

Lemma 2.46 guarantees existence of a path homotopy $\tilde{F}$ (in T) between $\tilde{\alpha}$ and $\tilde{\beta}$. Then its image under q yields a path homotopy in S between α and β. ■

The following theorem and its proof which we present here as an example is quoted from [2].

Example 3.16. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof. Take the covering map $q : \mathbb{R} \to \mathbb{S}^1$ in Example 3.4 and $q(0) = s_0 = (1, 0)$. The fiber $q^{-1}(s_0)$ is $\mathbb{Z}$ and the lifting correspondence $\sigma : \pi_1(\mathbb{S}^1, s_0) \to \mathbb{Z}$ is bijective since $\mathbb{R}$ is 1-connected.

To see that σ is a homomorphism, select two classes $[\alpha]$ and $[\beta]$ from the fundamental group of the circle and denote by $\tilde{\alpha}$ and $\tilde{\beta}$ their liftings to paths in the real line starting at 0, respectively. Now let $a = \sigma([\alpha]) = \tilde{\alpha}(1)$ and $b = \sigma([\beta]) = \tilde{\beta}(1)$. Let $\tilde{\tilde{\beta}}$ be the path $\tilde{\tilde{\beta}}(x) = a + \tilde{\beta}(x)$ on $\mathbb{R}$. Because $q(a + t) = q(t)$ for $\forall\ t \in \mathbb{R}$, $\tilde{\tilde{\beta}}$ is a lifting of β starting at a. Then $\tilde{\alpha} * \tilde{\tilde{\beta}}$ is the lifting of $\alpha * \beta$ starting at 0. Then,

$$\sigma([\alpha] * [\beta]) = a + b = \sigma([\alpha]) + \sigma([\beta]) \tag{3.4}$$

is obtained by definition. ■

Theorem 3.17. Let $q : (T, t_0) \to (S, s_0)$ be a covering map. [2]

1. The induced homomorphism q_* is one-to-one.
2. Let $N = q_*(\pi_1(T, t_0))$. The lifting correspondence σ induces a one-to-one map
$$\sigma : \pi_1(S, s_0)/N \to q^{-1}(s_0) \tag{3.5}$$
which is also onto in case T is path connected.
3. Let $[\alpha] \in \pi_1(S, s_0)$. Then $[\alpha] \in N$ iff $[\tilde{\alpha}] \in \pi_1(T, t_0)$.

Throughout the following sections, we will take into account covering maps with both path connected and locally path connected total and base space, unless otherwise stated. [2]

3.2 Equivalent Coverings

A space may have more than one covering. The following definition states under which condition two coverings of a space are equivalent.

Definition 3.18. Two covering maps $q : T \to S$ and $q' : T' \to S$ are equivalent if there exists a homeomorphism $h : T \to T'$ satisfying $q = q' \circ h$. [2]

$$\begin{tikzcd} T \arrow[rr, "h"] \arrow[dr, "q"'] & & T' \arrow[dl, "q'"] \\ & S & \end{tikzcd}$$

Figure 3.4 Equivalence of two coverings [2]

We present without proof a series of statements from [2] for further references:

Lemma 3.19. Suppose $q : (T, t_0) \to (S, s_0)$ is a covering map and $f : (R, r_0) \to (S, s_0)$ is a continuous map. Let R be path connected and locally path connected. There exists a unique lift $\tilde{f} : (R, r_0) \to (T, t_0)$ iff

$$f_*(\pi_1(R, r_0)) \subset q_*(\pi_1(T, t_0)) \tag{3.6}$$

holds.

Theorem 3.20. The covering maps $q : (T, t_0) \to (S, s_0)$ and $q' : (T', t_0') \to (S, s_0)$ are equivalent iff the subgroups $q_*(\pi_1(T, t_0))$and $q_*'(\pi_1(T', t_0'))$ of $\pi_1(S, s_0)$ are equal.

3.3 The Universal Covering Space

Definition 3.21. Let $q : T \to S$ denote a covering map. If T is 1-connected, we say that T is a universal covering space of S. [2]

Remark 3.22. Any two universal coverings of a space T are equivalent.

Example 3.23. $\mathbb{R}$ is the universal cover of $\mathbb{S}^1$.

Example 3.24. A 1-connected space is its own universal cover. For instance, $\mathbb{S}^n$ is its own universal cover for $n \geq 2$.

The following result is interesting because it says how a covering behaves under restrictions.

Lemma 3.25. Let $q : T \to S$ be a covering map such that T is not necessarily path connected. Let S be path connected and locally path connected. Then for any path component T_0 of T, the restricted map $q|_{T_0} : T_0 \to S$ is a covering map. [2]

Lemma 3.26. Let $q : (T, t_0) \to (S, s_0)$ be universal covering. Then s_0 has a nhood V such that the homomorphism

$$i_* : \pi_1(V, s_0) \to \pi_1(S, s_0) \tag{3.7}$$

induced by $i : V \hookrightarrow S$ is trivial. [2]

3.4 Existence Theorem

It is not true that any space possesses a universal covering. The following theorem states under which circumstances a space has a universal covering. To keep the manuscript more clear, we only give a sketch of the proof from [2, Th. 82.1.].

Theorem 3.27. Assume that a space S is path connected and locally path connected, which is at the same time SLSC. Let $N \subset \pi_1(S, s_0)$ be a subgroup for some $s_0 \in S$. There is a covering $q : (T, t_0) \to (S, s_0)$ such that

$$q_*(\pi_1(T, t_0)) = N. \tag{3.8}$$

Proof. Define an equiv. relation $\sim$ on $\Omega(S, s_0)$ the set of all paths in S beginning at s_0 as follows:

$$\alpha \sim \beta \text{ if } \alpha(1) = \beta(1) \text{ and } [\alpha * \bar{\beta}] \in N. \tag{3.9}$$

Denote by $\alpha^{\#}$ the equiv. class of α and by T the family of all equiv. classes of the members of $\Omega(S, s_0)$. Now consider the map $q : T \to S$ that sends a class to the terminal point of its representative:

$$q(\alpha^{\#}) = \alpha(1). \tag{3.10}$$

It follows from the assumption on S that q is onto. Moreover, we have the following useful equations:

1. If $[\alpha] = [\beta]$, then $\alpha^{\#} = \beta^{\#}$.
2. If $\alpha^{\#} = \beta^{\#}$, then $(\alpha * \delta)^{\#} = (\beta * \delta)^{\#}$ for any path δ in S beginning at $\alpha(1)$.

One topologizes T so that q becomes a covering map. In fact, one of the most convenient topologies on function spaces, namely, compact-open topology, is used here to accomply with topological purposes. Let $\alpha \in \Omega(S, s_0)$, and take a path connected nhood V of $\alpha(1)$. Consider the collection of the sets

$$\mathbb{U}_{(V,\alpha)} = \{(\alpha * \delta)^{\#} \mid \delta \in \Omega(V, \alpha(1))\}. \tag{3.11}$$

that construct a basis as in Definition 2.2 for the topology on T.

That q is open follows from the observation that $q(\mathbb{U}_{(V,\alpha)}) = V \subset S$. Standart continuity arguments guarantee continuity of q. To prove q is more than a continuous open map but a covering we have to guarantee that (i) any point of S has an evenly covered nhood, (ii) any path in S lifts to a path in T. We refer the reader to [1, Th. 82.1.] for further details of this part of the proof.

Finally, $N = q_*(\pi_1(T, t_0))$. Let $[\alpha] \in \pi_1(S, s_0)$ whose lifting $\tilde{\alpha}$ is a curve in T starting at t_0. It follows that $[\alpha] \in N$ iff $[\tilde{\alpha}] \in \pi_1(T, t_0)$. Observe that $\tilde{\alpha}(1) = \alpha^{\#}$, and $\alpha^{\#} = t_0$ iff $\alpha \sim c_{s_0}$ which occurs when $[\alpha] \in N$. ■

Corollary 3.28. A path connected, locally path connected and SLSC space admits a universal covering. [2]

4
LIE GROUPS

By definition, a Lie group admits a differentiable manifold structure. So, we give below the definition of a topological manifold and say further what the differentiable structure on such a manifold is.

4.1 Manifolds

Definition 4.1. A topological manifold (m-manifold) M is a locally Euclidean Hausdorff space, i.e.,

$$\forall\, p \in M \;\; \exists\, V_P \in \tau \;\exists\, \psi : V \subset M \to \psi(V) \subset \mathbb{R}^m \tag{4.1}$$

such that ψ is a homeomorphism. (V, ψ) is called a chart on M. [3]

Definition 4.2. A C^k-atlas on an m-manifold M is a family $\mathcal{A} = \{(V_\alpha, \psi_\alpha) : \alpha \in I\}$ of charts fulfilling the criterias: [3, 4]

1. $\bigcup_{\alpha \in I} V_\alpha = M$.
2. For each $(V_\alpha, \psi_\alpha), (V_\beta, \psi_\beta) \in \mathcal{A}$ the transition maps

$$\psi_{\alpha\beta} : \psi_\alpha(V_\alpha \cap V_\beta) \to \psi_\beta(V_\alpha \cap V_\beta) \tag{4.2}$$

$$\psi_{\beta\alpha} : \psi_\beta(V_\alpha \cap V_\beta) \to \psi_\alpha(V_\alpha \cap V_\beta) \tag{4.3}$$

are of class C^k.

Remark 4.3. A member of C^k is a k times continuously differentiable function. If there is no limitation on derivative we say the function is infinitely differentiable (smooth) and hence belongs to the class C^∞. From now on we will be considering the smooth case only.

Definition 4.4. A Hausdorff and 2nd countable space is called a differentiable (or smooth) manifold if it admits a maximal differentiable atlas. [3]

Theorem 4.5. Any open subset of a manifold receives a manifold structure (w.r.t. the subspace topology) of the equal dimension. [3]

Proof. Take $p \in O \in \tau$. Since $p \in M$,

$$\exists\, V_p \in \tau \text{ and } \exists\, \psi : V_p \subset M \to \psi(V_p) \subset \mathbb{R}^m. \tag{4.4}$$

Since $p \in O \cap V_p \in \tau_O$,

$$\varphi = \psi|_{O \cap V_p} : O \cap V_p \subset O \to \psi(O \cap V_p) \subset \mathbb{R}^m \tag{4.5}$$

is the homeomorphism we are looking for. ■

Example 4.6. a) Product of manifolds remains a manifold.

b) $\mathbb{R}^n$ is a smooth manifold with $\mathcal{A} = \{(\mathbb{R}^n, id_{\mathbb{R}^n})\}$.

c) The n-sphere $\mathbb{S}^n$ and the n-torus $\mathbb{T}^n$ are smooth manifolds.

d) The Klein bottle and Mobius strip are 2-dimensional manifolds.

Definition 4.7. Let M and N be smooth manifolds with dim$M = m$ and dim$N = n$. A map $f : M \to N$ is smooth iff $\varphi \circ f \circ \psi^{-1} \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^n)$ for some charts (V_p, ψ_p), $(U_{f(p)}, \varphi_{f(p)})$ and for every state p of M. A bijection $f : M \to N$ $(m = n)$ is a smooth diffeomorphism if f, $f^{-1}$are smooth. In particular, a map $f : M \to \mathbb{R}$ is smooth iff $f \circ \psi^{-1} \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^n)$ for every chart (V, ψ). [3]

Definition 4.8. A vector $\mathcal{X}_p$ tangent to M at a state p is a map $\mathcal{X}_p : \mathcal{C}^\infty(M) \to \mathbb{R}$ which is linear and satisfies for all $f, g \in \mathcal{C}^\infty(M)$ the Leibnitz law: [3]

$$\mathcal{X}_p(fg) = \mathcal{X}_p(f) \cdot g(p) + f(p) \cdot \mathcal{X}_p(g). \tag{4.6}$$

Definition 4.9. The set $T_pM = \{\mathcal{X}_p : p \in M\}$ is the tangent space at the state p (Figure 4.1). Furthermore, we call the family of such spaces the tangent bundle (TM). Note that, $\dim T_pM = \dim M$.

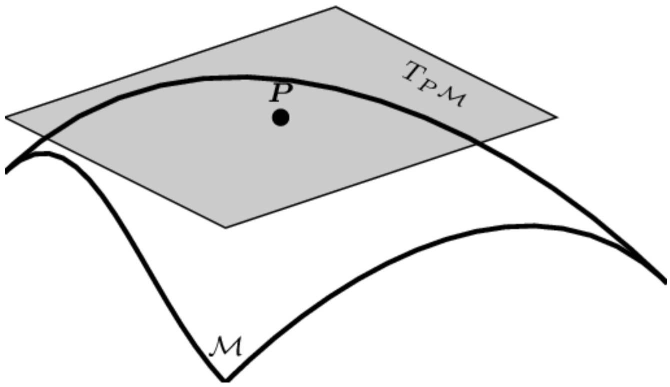

Figure 4.1 The tangent space T_pM [5]

Definition 4.10. A vector field $\mathcal{X}$ on a smooth manifold M is a mapping $\mathcal{X} : M \to TM$ given by $p \to \mathcal{X}_p$. Alternatively, $\mathcal{X}$ might be viewed as a derivation $\mathcal{X} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ s.t. for every $f, g \in \mathcal{C}^\infty(M)$

$$\mathcal{X}(fg) = \mathcal{X}(f) \cdot g + f \cdot \mathcal{X}(g). \tag{4.7}$$

Definition 4.11. The Lie bracket of some $\mathcal{X}, \mathcal{Y} \in \mathcal{C}^\infty(M, TM)$ is defined as

$$[\mathcal{X}, \mathcal{Y}]_p(f) = (\mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X})_p(f) = (\mathcal{X}_p\mathcal{Y} - \mathcal{Y}_p\mathcal{X})f = \mathcal{X}_p(\mathcal{Y}f) - \mathcal{Y}_p(\mathcal{X}f) \tag{4.8}$$

for $p \in M, f \in \mathcal{C}^\infty(M)$.

Definition 4.12. The differential of a mapping $f \in \mathcal{C}^\infty(M, N)$ is the map $df_p : T_pM \to T_{f(p)}N$ which is defined by

$$df_p(\mathcal{X}_p)(g) = \mathcal{X}_p(g \circ f) \tag{4.9}$$

for every state p, $\mathcal{X}_p \in T_pM$ and $g \in \mathcal{C}^\infty(N)$. [3]

4.2 Lie Groups

Definition 4.13. A topological group L is a Lie group if it is a smooth manifold and the operation

$$\mu : L \times L \to L, \mu(x, y)=xy^{-1} \tag{4.10}$$

is smooth. Below we give a short list of examples for Lie groups. [3]

Example 4.14. a) $(\mathbb{R}^n, +)$.

b) Product of Lie groups.

c) Any closed subgroup of a Lie group.

d) $\mathbb{S}^n$ (under complex multiplication for n = 0, 1 and 3).

e) Classical matrix groups $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{O}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$ etc.

Defintion 4.15. [6] Given a group L, the right translation by $x \in L$ is the bijection

$$\mathcal{R}_x : L \to L, y \to \mathcal{R}_x(y) = yx. \tag{4.11}$$

The left translation by $x \in L$ is the bijection

$$\mathcal{L}_x : L \to L, y \to \mathcal{L}_x(y) = xy. \tag{4.12}$$

Remark 4.16. (4.11) and (4.12) are smooth diffeomorphisms if L is a Lie group. [6]

Remark 4.17. Given $x, y \in L$: [7]

1. $\mathcal{R}_x \circ \mathcal{R}_y = \mathcal{R}_{yx}$,
2. $\mathcal{L}_x \circ \mathcal{L}_y = \mathcal{L}_{xy}$,
3. $\mathcal{L}_x \circ \mathcal{R}_y = \mathcal{R}_y \circ \mathcal{L}_x$,
4. $(\mathcal{R}_x)^{-1} = \mathcal{R}_{x^{-1}}$,
5. $(\mathcal{L}_x)^{-1} = \mathcal{L}_{x^{-1}}$.

Remark 4.18. Let $\varphi : L \to K$ be a group homomorphism. Then for every $x \in L$,

1. $\varphi \circ \mathcal{L}_x = \mathcal{L}_{\varphi(x)} \circ \varphi$;
2. $\varphi \circ \mathcal{R}_x = \mathcal{R}_{\varphi(x)} \circ \varphi$.

Definition 4.19. A vector field $\mathcal{X}$ on a Lie group L is a left-invariant vector field if

$$\mathcal{X} \circ \mathcal{L}_x = (d\mathcal{L}_x) \circ \mathcal{X} \tag{4.13}$$

for all $x \in L$. [6]

Definition 4.20. Denote by $\mathfrak{l}$ the linear space of left-inv. vect. fields on L. The mapping

$$\alpha : \mathfrak{l} \to T_1L, \mathcal{X} \to \mathcal{X}_1 \tag{4.14}$$

defines a linear isomorphism so that $\mathfrak{l}$ is usually identified with T_1L. Thus, $\dim\mathfrak{l} = \dim T_1L = \dim L$. [7]

Definition 4.21. The linear space $\mathfrak{l}$ provided with a bilinear operation $[\cdot,\cdot] : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{l}$ is a Lie algebra if:

1. $[\mathcal{X}, \mathcal{X}] = 0$,
2. $[\mathcal{X}, \mathcal{Y}] = -[\mathcal{Y}, \mathcal{X}]$,
3. $[\mathcal{X}, [\mathcal{Y}, \mathcal{Z}]] + [\mathcal{Y}, [\mathcal{Z}, \mathcal{X}]] + [\mathcal{Z}, [\mathcal{X}, \mathcal{Y}]] = 0$,

for all $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \mathfrak{l}$. [6]

Definition 4.22. The Lie algebra of L which we either denote by $\mathrm{Lie}(L)$ or $\mathfrak{l}$ is the space of left-inv. vect. fields.

Example 4.23. The set $\mathfrak{gl}(n, \mathbb{R})$ of real matrices of order n is a Lie algebra whose bracket operation is by the standart commutator rule:

$$[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X} \tag{4.15}$$

for $\mathcal{X}, \mathcal{Y} \in \mathfrak{gl}(n, \mathbb{R})$. [6]

Example 4.24. The linear space $C^\infty(M, TM)$ together with the Lie bracket operation is an (infinite-dimensional) Lie algebra. [6]

Remark 4.25. Let Lie(L) = $\mathfrak{l}$ and Lie(K) = $\mathfrak{k}$. Then, Lie$(L \times K)$ = $\mathfrak{l} \times \mathfrak{k}$. [6]

Definition 4.26. A subspace $\mathfrak{l}' \subset \mathfrak{l}$ closed under the Lie bracket is a Lie subalgebra.

Theorem 4.27. Let Lie(L) = $\mathfrak{l}$ and let $\mathfrak{l}' \subset \mathfrak{l}$ be a Lie subalgebra. There exists a unique connected Lie subgroup $L' \subset L$ with Lie(L') = $\mathfrak{l}'$. [6]

Definition 4.28. Let Lie(L) = $\mathfrak{l}$. Denote by $\gamma : \mathbb{R} \to L$ the 1-parameter subgroup of L whose derivative at the identity is $\mathcal{X}$. The exponential mapping is defined as the following smooth function:

$$\exp : \mathfrak{l} \to L, \exp(\mathcal{X}) = \gamma(1) \tag{4.16}$$

is called the exponential map.

Observe that $\exp(t\mathcal{X}) = \gamma(t)$ is valid because of the chain rule.

4.3 Lie Homomorphisms

Definition 4.29. Let L, K be Lie groups. A homomorphism $\varphi : L \to K$ (resp. isomorphism) is a Lie group homomorphism (resp. isomorphism) if it is smooth. [6]

Definition 4.30. A linear map $\theta : \mathfrak{l} \to \mathfrak{k}$ is a Lie algebra homomorphism if it preserves the Lie bracket. Furthermore, θ is a Lie algebra isomorphism if it is bijective. [6]

Definition 4.31. Let $\varphi : L \to K$ be a Lie group homomorphism. Then, $d\varphi_1 : \mathfrak{l} \to \mathfrak{k}$ is a Lie algebra homomorphism. [6]

Remark 4.32. As a result from the previous definition, the diagram below commutes:

$$\begin{tikzcd}
\mathfrak{l} \arrow[r, "d\varphi_1"] \arrow[d, "\exp"'] & \mathfrak{k} \arrow[d, "\exp"] \\
L \arrow[r, "\varphi"'] & K
\end{tikzcd}$$

Figure 4.2 Commutative diagram of homomorphisms [8]

Example 4.33. The determinant function $|\cdot| : \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}_+$ is a Lie group homomorphism whose derivative is the trace function of matrices. [7]

5
UNIVERSAL COVERING OF A LIE GROUP

5.1 Homomorphisms Revisited

5.1.1 Immersions and Submersions

Definition 5.1 (Immersion, Submersion). A differentiable map between differentiable manifolds is an immersion (resp. submersion) if its differential is injective (resp. surjective) everywhere.

Remark 5.2. A differentiable homomorphism φ becomes an immersion (resp. submersion) iff $d\varphi_1$ is one-to-one (resp. onto). Equivalently, one might say that φ is an immersion iff its kernel is discrete. [6]

Remark 5.3. Let $\varphi : L \to K$ be a Lie group homomorphism.

1. $\mathrm{im}\varphi \subset K$ is a Lie subgroup if φ is injective.
2. $\ker\varphi \subset L$ is a closed Lie subgroup with Lie algebra $\ker d\varphi_1$.
3. If φ is injective (resp. surjective, bijective), then φ is an immersion (resp. submersion, diffeomorphism).

Proposition 5.4. Let L be a Lie group and N be a topologically closed normal subgroup. Then, L/N is a quotient Lie group of dimension $\dim L/N = \dim L - \dim N$ such that $\mathrm{Lie}(L/N) \cong \mathfrak{l}/\mathfrak{n}$. [6]

Proposition 5.5. Let $\varphi : L \to K$ be a Lie group homomorphism. Then, $L/\ker\phi$ is a Lie group and the homomorphism $\tilde{\varphi} : L/\ker\varphi \to K$ given by $\tilde{\varphi}(x \ker\varphi) = \varphi(x)$ commutes the diagram:

$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathrm{im}\varphi \subset K \\ L/\ker\varphi \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

Figure 5.1 Commutative diagram of homomorphisms [6]

Moreover, $\tilde{\varphi}$ is an injective immersion, so $\mathrm{im}\varphi \subset K$ is a Lie subgroup isomorphic to $L/\ker\varphi$. It follows that, $\dim(\mathrm{im}\varphi) = \dim L - \dim(\ker\varphi)$. Furthermore, if L is connected, then $\mathrm{im}\varphi = \langle \exp(\mathrm{im} d\varphi_1) \rangle$. [6]

Proposition 5.6. Given a locally connected topological group L and a discrete normal subgroup $\Gamma \subset L$, the standart surjective projection $\pi : L \to L/\Gamma$ is a covering map. [6]

Proof. Pick an open set V that intersects Γ at $\{1_L\}$ and select now a connected open set $U \ni 1_L$ with $U^{-1} = U$ and $U^2 \subset V$. Given $x \in L$, $\pi(xU)$ is a connected nhood of $x\Gamma$ and

$$\pi^{-1}(\pi(xU)) = \bigcup_{\gamma \in \Gamma} xU\gamma. \tag{5.1}$$

The open sets $xU\gamma$ are connected and $xU\gamma_1 \cap xU\gamma_2 \neq \emptyset$ iff $\gamma_1 = \gamma_2$. If $y \in xU\gamma_1 \cap xU\gamma_2$, then $u_1\gamma_1 = x^{-1}y = u_2\gamma_2$, with $u_1, u_2 \in U$. Hence, $u_2^{-1}u_1 = \gamma_2\gamma_1^{-1} \in V \cap \Gamma = \{1\}$ and this implies that $u_1 = u_2$ and $\gamma_1 = \gamma_2$. So, (5.1) is homeomorphic to $\pi(xU)$. [6] ■

Proposition 5.7. Let $\varphi : L \to K$ be an onto Lie group homomorphism and let $d\varphi_1 : \mathfrak{l} \to \mathfrak{k}$ be an isomorphism of the corresponding Lie algebras. Then, φ is a covering map. [6]

Proof. Since φ is onto, $L/\ker\varphi \cong \mathrm{im}\varphi = K$ by Proposition 5.5. Furthermore, $\ker\varphi$ is a discrete subgroup by Remark 5.2. Hence, φ is a covering map from Proposition 5.6.

■

Corollary 5.8. Let $\varphi : L \to K$ be a differentiable homomorphism. If φ is an immersion, then $\varphi : L \to \mathrm{im}\varphi$ is a covering map. [6]

5.1.2 Graph of a Function and Differentiability

Definition 5.9 (Graph of a Function). Given a function $\varphi : L \to K$, we understand by the graph of φ the subset

$$\mathrm{gr}(\varphi) = \{(x, \varphi(x)) : x \in L\} \subset L \times K. \tag{5.2}$$

Proposition 5.10. [6] A map $\varphi : L \to K$ is a (group) homomorphism iff $\mathrm{gr}\varphi \subset L \times K$ is a subgroup which in turn implies that $\mathrm{gr}\varphi \cong L$ via the projection

$$p : \mathrm{gr}\varphi \to L,\ p(x, \varphi(x)) = x. \tag{5.3}$$

Proposition 5.11. That a homomorphism $\varphi : L \to K$ (with K being Hausdorff) is continuous is equivalent to say that $\mathrm{gr}\varphi \subset L \times K$ is a closed subgroup. Analogously, if L and K are Lie groups, a map $\varphi : L \to K$ is a Lie group homomorphism iff $\mathrm{gr}\varphi \subset L \times K$ is a closed Lie subgroup diffeomorphic to L via (5.3). [6]

Remark 5.12. A differentiable function is always continuous on its domain. But the converse need not be true.

Theorem 5.13. It follows at once that if a homomorphism between Lie groups is continuous then it is also smooth. [6]

5.2 Extensions of Homomorphisms

In this section, we study the extensions of Lie algebra homomorphisms to Lie group homomorphisms bearing in mind that this is closely related to the universal covering of a Lie group.

Proposition 5.14. A map $\theta : \mathfrak{l} \to \mathfrak{k}$ is a Lie algebra homomorphism iff its graph $\mathfrak{l}(\theta) \subset \mathfrak{l} \times \mathfrak{k}$ is a Lie subalgebra. [6]

Definition 5.15 (Local Homomorphism). A map $\varphi : V \subset L \to K$ with $1_L \in V$ between Lie groups L and K is a local homomorphism if it verifies $\varphi(v_1 \cdot v_2) = \varphi(v_1) \cdot \varphi(v_2)$ for every $v_1, v_2 \in V$ such that $v_1 \cdot v_2 \in V$. [6]

Definition 5.16. Two Lie groups L and K are called locally isomorphic if there are nhoods $U \subset L$ and $V \subset K$ of the respective identity elements and a diffeomorphism $\varphi : U \to V$ such that $\varphi(u_1 \cdot u_2) = \varphi(u_1) \cdot \varphi(u_2)$ for all $u_1, u_2 \in U$ with $u_1 \cdot u_2 \in U$.

We understand that extensions of Lie algebra homomorphisms are not necessarily unique unless the domain is connected, as stated in [6, Prop. 7.8]. We now show in detail how one obtains an extension of a Lie algebra homomorphism in the proposition below from [6, Prop. 7.10].

Proposition 5.17. For any Lie algebra homomorphism $\theta : \mathfrak{l} \to \mathfrak{k}$, there exists a differentiable local homomorphism $\varphi : V \subset L \to K$ with $d\varphi_1 = \theta$.

Proof. *Step1.* Let $\theta : \mathfrak{l} \to \mathfrak{k}$ be a Lie algebra homomorphism. Hence, $\mathfrak{l}(\theta) \subset \mathfrak{l} \times \mathfrak{k}$ is a subalgebra as we mentioned before. Then by Theorem 4.27, there exists a unique connected Lie subgroup $L(\theta) \subset L \times K$ with $\mathrm{Lie}[L(\theta)] = \mathfrak{l}(\theta)$. If we restrict the projection map $p : L \times K \to L$ to $q = p|_{L(\theta)} : L(\theta) \to L$, we obtain a differentiable homomorphism. Its differential dq_1: $\mathfrak{l}(\theta) \to \mathfrak{l}$ is the restriction of the projection $\mathfrak{l} \times \mathfrak{k} \to \mathfrak{l}$. And since $\mathfrak{l}(\theta)$ is the graph of θ, we have that dq_1 is a Lie algebra isomorphism. Note that the image of q is the identity component L_0 of L (since connectedness is preserved under continuity). Particularly, q is onto if L is connected.

Step2. The Lie groups $L(\theta)$ and L are locally isomorphic as shown in (*Step1*). Then by Definition 5.19, there exists nhoods $U_1 \subset L(\theta)$ and $V_1 \subset L$ of the respective identities such that $\psi = q|_{U_1} : U_1 \to V_1$ is a diffeomorphism. Note that, $\psi(\psi^{-1}(x)) = x, \forall\, x \in V_1$. We define ψ^{-1} as follows:

$$\psi^{-1}(x) = (x, \varphi(x)) \text{ for } x \in V_1, \varphi(x) \in K. \tag{5.5}$$

This gives rise to a map $\varphi : V_1 \to K$ where gr$\varphi = U_1$, by definition.

Now, take a nhood $U \subset U_1$ of the identity with $U^2 \subset U_1$ and define $V = q(U)$. That φ is a local homomorphism follows from restrictions of the maps to U and V. Check [7, Prop. 7.10] for the details. Differentiability of φ follows because gr$\varphi = U \subset L \times K$ is a submanifold. ■

Corollary 5.18. Two Lie groups L and K are locally isomorphic iff Lie$(L) \cong$ Lie(K).

Proof. By Proposition 5.17, one extends a Lie algebra isomorphism to a local isomorphism of the corresponding Lie groups. Reciprocally, if φ is a local isomorphism, then $d\varphi_1$ is an isomorphism. [7] ■

Example 5.19. The n-torus $\mathbb{T}^n$ and $\mathbb{R}^n$are locally isomorphic as Lie groups.

Proof. It holds that Lie$(\mathbb{T}^n) \cong \mathbb{R}^n \cong$ Lie$(\mathbb{R}^n)$. So, the Lie algebras of $\mathbb{T}^n$ and $\mathbb{R}^n$are isomorphic even though $\mathbb{T}^n$ and $\mathbb{R}^n$are not isomorphic. ■

We have seen that extensions are indeed possible (locally) as we have proved in Proposition 5.17. Now we discuss the necessary condition for the global case in the following theorem and give the proof quoted from [6, Th. 7.13].

Theorem 5.20 (Monodromy Principle). Any Lie algebra homomorphism $\theta : \mathfrak{l} \to \mathfrak{k}$ extends to a unique differentiable homomorphism $\varphi : L \to K$ of the corresponding Lie groups in a way that $\theta = d\varphi_1$, if L is 1-connected.

Proof. Let Lie$[L(\theta)] = \mathfrak{l}(\theta)$ as mentioned before. The differentiable homomorphism (projection) $q : L(\theta) \to L$ is onto because L is connected. In addition, dq_1 is an isomorphism by (*Step1*) of Proposition 5.17. It follows that q is a covering map by Proposition 5.7. Moreover, q becomes a homeomorphism under the assumption that L is 1-connected. Hence, we obtain a bijective differentiable homomorphism (projection) making the Lie groups diffeomorphic. So, it is safe to say that $L(\theta)$ is the graph of a Lie group homomorphism $\varphi : L \to K$ by Proposition 5.12. Conclusively, $d\varphi_1 = \theta$, since $d[L(\theta)]_1 = \mathfrak{l}(\theta)$ is the graph of θ. φ is unique by Proposition 5.15. ■

Corollary 5.21. If L and K are 1-connected Lie groups with $\mathrm{Lie}(L) \cong \mathrm{Lie}(K)$, then $L \cong K$.

Proof. Let $\theta : \mathfrak{l} \to \mathfrak{k}$ be an isomorphism where $\mathrm{Lie}(L) = \mathfrak{l}$ and $\mathrm{Lie}(K) = \mathfrak{k}$. Denote by $\varphi : L \to K$ and $\phi : K \to L$ the homomorphisms with $d\varphi_1 = \theta$ and $d\phi_1 = \theta^{-1}$. Uniqueness arguments of extensions imply that $\phi^{-1} = \varphi$ is more than a homomorphism but an isomorphism. [6] ■

5.3 Universal Covering

Since the goal of this work is to study the universal cover of a Lie group, we finally discuss in Theorem 5.23 under what condition it exists and show the Lie group structure on it by using the proof from [6, Th. 7.16].

Remark 5.22 (Covering Group). A covering homomorphism is a covering map $\varphi : L \to K$ of topological groups such that φ is an (continuous onto) group homomorphism. In this setting, L is called the covering group of K. Note that, φ is a local isomorphism.

Theorem 5.23. For a connected Lie group L there exists a unique 1-connected Lie group $\tilde{L}$ and a covering map $\pi : \tilde{L} \to L$ which is also a Lie group homomorphism. Furthermore, L and $\tilde{L}$ have isomorphic Lie algebras.

Proof. L is a connected manifold satisfying the conditions in Corollary 3.28, and hence, has a universal cover $\pi : \tilde{L} \to L$, where $\tilde{L}$ is a 1-connected manifold. It only remains to show that $\tilde{L}$ is a (Lie) group and π is a Lie group homomorphism.

We obtain a unique multiplication on $\tilde{L}$ simply by lifting the group multiplication μ on L as follows: Since π is a surjective map, for $1 \in L$, $\exists\, \tilde{1} \in \tilde{L}$ with $\pi(\tilde{1}) = 1$. Take into account the differentiable map $\lambda : \tilde{L} \times \tilde{L} \to L$ given as

$$\lambda(\tilde{x}, \tilde{y}) = \mu\big(\pi(\tilde{x}), \pi(\tilde{y})\big). \tag{5.6}$$

By simple connectivity of $\tilde{L} \times \tilde{L}$, we can lift λ to a unique differentiable map $\tilde{\mu} : \tilde{L} \times \tilde{L} \to \tilde{L}$, with $\lambda = \pi \circ \tilde{\mu}$ and $\tilde{\mu}\big(\tilde{1}, \tilde{1}\big) = \tilde{1}$:

$$\begin{tikzcd}
\tilde{L} \times \tilde{L} \arrow[r, "\tilde{\mu}"] \arrow[d, "\pi"] \arrow[dr, "\lambda"] & \tilde{L} \arrow[d, "\pi"] \\
L \times L \arrow[r, "\mu"'] & L
\end{tikzcd}$$

Figure 5.2 Lifting diagram [6]

Thanks to this multiplication defined on $\tilde{L}$ through $\tilde{\mu}$, we conclude (without further details) that $\tilde{L}$ is a Lie group since $\tilde{\mu}$ is smooth. It is obvious from the construction of the maps that

$$\pi\big(\tilde{\mu}(\tilde{x}, \tilde{y})\big) = \lambda(\tilde{x}, \tilde{y}) = \mu\big(\pi(\tilde{x}), \pi(\tilde{y})\big). \tag{5.7}$$

Hence, π is a homomorphism. Moreover, π is a Lie group homomorphism from Theorem 5.13. Uniqueness of $\tilde{L}$ is guaranteed by the Monodromy principle and its Corollary 5.21. Remember that uniqueness of $\tilde{\mu}$ follows from that of liftings. Finally, $\mathrm{Lie}(L) \cong \mathrm{Lie}(\tilde{L})$ since the covering homomorphism π is a local isomorphism. ■

Remark 5.24. The covering homomorphism $\pi : \tilde{L} \to L$ mentioned in Theorem 5.23 is a local diffeomorphism.

Remark 5.25. Particular case of liftings: Let T be a connected smooth manifold and $\pi : (\tilde{T}, \tilde{t}) \to (T, t)$ be its universal cover. Since $\tilde{T}$ is 1-connected, we can lift the continuous map π to a unique map $\tilde{\pi} : (\tilde{T}, \tilde{t}) \to (\tilde{T}, \tilde{t})$ such that $\pi \circ \tilde{\pi} = \pi$. In fact, any map $\theta : (\tilde{T}, \tilde{t}) \to (\tilde{T}, \tilde{t})$ with $\pi \circ \theta = \pi$ is a lifting of π. The set of these liftings, which we will denote by Λ, forms a group under composition of maps.

Note that these liftings are homeomorphisms in this specific case. In addition, these liftings are differentiable since π is differentiable.

Now let us show that $\Lambda \cong \pi_1(T)$: Given $t \in T$ and $\tilde{t} \in \pi^{-1}\{t\}$, the liftings $\tilde{\pi}(\tilde{t}) \in \pi^{-1}\{t\}$ exhaust the group Λ. So, Λ is in bijection with $\pi^{-1}\{t\}$. If $[\gamma] \in \pi_1(T,t)$, then $\tilde{\gamma} \in \Omega(\tilde{T}, \tilde{t})$. On that account, the well-defined map

$$[\gamma] \in \pi_1(T,t) \to \tilde{\pi}_{t,\tilde{t},\tilde{\gamma}(1)} \in \Lambda \tag{5.8}$$

is an isomorphism. [6, Appendix 7.4]

We give below an important statement from [6, Prop. 7.17].

Proposition 5.26. Let $\tilde{L}$ be a 1-connected Lie group and $\Gamma \subset \tilde{L}$ be a discrete normal subgroup (that is, $\Gamma \subset Z(\tilde{L})$). Then, $\Gamma \cong \pi_1(L)$ if $L \cong \tilde{L}/\Gamma$.

Proof. *Step1*. Since the universal cover $\pi : \tilde{L} \to L$ is onto, we have that $L \cong \tilde{L}/\ker\pi$ by Proposition 5.5. Having this in mind, the following results complete the first part of the proof:

a) $\Gamma = \ker\pi$ is a discrete normal subgroup, since π is a local isomorphism.
b) If $\Gamma \subset \tilde{L}$ is a discrete normal subgroup, then $\Gamma \subset Z(\tilde{L})$. To prove this, take $\alpha \in \Gamma$ and an open set $V \subset \tilde{L}$ with $\alpha V \cap \Gamma = \{\alpha\}$. From the continuity of the map $\tilde{x} \in \tilde{L} \to \tilde{x}\alpha\tilde{x}^{-1} \in \tilde{L}$ we have $\tilde{x}\alpha\tilde{x}^{-1} \in \alpha V$ for $\tilde{x} \in U$ where U is a nhood of the identity. Since Γ is normal, $\tilde{x}\alpha\tilde{x}^{-1} = \alpha$, which means $\tilde{x}$ commutes with α if $\tilde{x} \in U$. Recall that $\tilde{L}$ is generated by U (Theorem 2.25). So, α commutes with every element of $\tilde{L}$. Hence, $\alpha \in Z(\tilde{L})$.

Step2. Now, we know by Remark 5.25 that $\pi_1(L) \cong \Lambda$ (the group of continuous liftings of π). If $\alpha \in \Gamma$, then the right translation $\mathcal{R}_\alpha \in \Lambda$. So logically we define the homomorphism

$$\alpha \in \Gamma \to \mathcal{R}_\alpha \in \Lambda. \tag{5.9}$$

The homomorphism is injective, since $\alpha = \tilde{1}$ if $\mathcal{R}_\alpha = id$. To confirm surjectivity, take $\theta \in \Lambda$. Then $\pi\theta(\tilde{1}) = \pi(\tilde{1})$, which means that $\theta(\tilde{1}) \in \Gamma$. It follows that $\mathcal{R}_{\theta(\tilde{1})}(\tilde{1}) = \theta(\tilde{1})$, for $\mathcal{R}_{\theta(\tilde{1})}$, $\theta \in \Lambda$. By uniqueness of liftings, $\mathcal{R}_{\theta(\tilde{1})} = \theta$. Consequently we have, $\Gamma \cong \Lambda \cong \pi_1(L)$. ■

Corollary 5.27. $\pi_1(L)$ is commutative for any connected Lie group L. [6]

Proof. Connectedness assumption on L means it is isomorphic to $\tilde{L}/\Gamma$ and hence $\pi_1(L)$ is indeed commutative, since $\Gamma \cong \pi_1(L)$ is central. ■

Corollary 5.28. Connected Lie groups are locally isomorphic iff their universal coverings are isomorphic. [6]

6
RESULTS AND DISCUSSION

In this study, the existence of the universal covering space of a Lie group is discussed. As a result, we show that a connected Lie group which is a differentiable manifold admits a universal covering space since it is path connected, locally path connected, and semi-locally simply connected.

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