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**NEW GEOMETRIC STRUCTURES DESIGNED ON MANIFOLDS  
MODELED ON FRECHET ALGEBRA WITHOUT NORM**

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IN  
MATHEMATICS**

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NEW GEOMETRIC STRUCTURES DESIGNED ON MANIFOLDS MODELED ON  
FRECHET ALGEBRA WITHOUT NORM

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August 2022

We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science

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**Zena Khairy Khudair AL-BAYATI**

## ABSTRACT

### NEW GEOMETRIC STRUCTURES DESIGNED ON MANIFOLDS MODELED ON FRECHET ALGEBRA WITHOUT NORM

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Master of Science in Mathematics

Advisor: Assoc. Prof. Dr. Ufuk ÖZTÜRK

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This thesis consists of four parts. In the first chapter, the basic definitions and theorems that will be used throughout the thesis are given. The second chapter is divided into two sections, in the first section we define a new class of topological algebra which we call convenient algebra by using and generalizing the method of convenient topological vector spaces, and in the second one designed a new manifold by considering these algebras as model space of these manifolds. In the third chapter, we studied and discussed the mappings between these manifolds that constructing in the second chapter. Chapter four deals with conclusions and future works.

**2022, 37pages**

**Keywords:** Frechet algebra, Seminorm, Mackey topology, Frechet space, Smooth manifold

## ÖZET

# NORMSIZ FRECHET CEBİRİNDE MODELLENEN MANİFOLDLAR ÜZERİNE TASARIMLANAN YENİ GEOMETRİK YAPILAR

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Bu tez dört bölümden oluşmaktadır. Birinci bölümde tez boyunca kullanılacak temel tanım ve teoremler verilmiştir. İkinci bölümde,  $p$ -büzülme tipi dönüşümler için metrik uzaylarda bazı en iyi periyodik yakınlık noktası sonuçlarını elde ediyoruz. Böylece literatürde var olan benzer sonuçları genelleştirip iyileştiriyoruz. Üçüncü bölümde, benzersiz olmayan daralma dönüşümleri için metrik uzaylarda bazı en iyi periyodik yakınlık noktası sonuçlarını kanıtıyoruz. Teklik özelliği olmayan daralma dönüşümleri için hem en iyi yakınlık noktası hem de sabit nokta sonuçlarının bazı genellemeleri elde edilmiştir.

**2022, 37 sayfa**

**Anahtar Kelimeler:** Frechet cebiri, Seminorm, Mackey topolojisi, Frechet uzayı,  
Düzgün manifold

## **PREFACE AND ACKNOWLEDGEMENTS**

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**Çankırı-2022**

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## LIST OF SYMBOLS

$C^{\infty}_{lip}(\mathbb{R}, \mathcal{E})$	The space of all smooth Lipschitz curves from $\mathbb{R}$ to $\mathcal{E}$
$\forall$	For all
$\exists$	There exists
$\ni$	Such that
$\cap$	The intersection
$\cup$	The union
$\Leftrightarrow$	If and only if
$\Rightarrow$	Implies that
$\mathbb{C}$	The set of all complex numbers
$\mathbb{R}$	The set of all real numbers



## LIST OF ABBREVIATIONS

B-algebra	Banach algebra
CA	Convenient algebra
CVS	Convenient vector space
LMC-algebra	Locally multiplicative convex algebra
<i>TVS</i>	Topological vector space
<i>inf</i>	The greatest of lower bound
<i>sup</i>	The least of upper bound



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## 1. INTRODUCTION

It is well known that any Banach space is Frechet space unless the converse is in general not true. Many category of differentiability between topological vector spaces were introduced and studied in many miners, see (Asplund 1968, BISTRÖM and Jaramillo 1994, Dodson *et al.* 2016, Frolicher 1988) also many differential and topological structures are discussed on these spaces, see (Alekseevsky and Michor 1994, Banaszczyk 2006, Biström 1993, Biström *et al.* 1993, Hong 1967, Kriegel *et al.* 2009, Kriegel *et al.* 2011, Kriegel *et al.* 2015, Kriegel and Michor 1991, Kriegel and Michor 1997, Leslie 1967, Leslie 1968), with the comparison with the topological algebras, the papers which studied the differentiability still few, see (Palmer 1994, Rudin 1991, Tosiek 2005, Trampus 1960, Waelbroeck 2006, Żelazko 1971). In the present work, we hope to design and study many new differentials and geometric structures defined on a manifold modeled on a Frechet algebra with topology non induced by the norm (i.e., non normable). This approach leads us to define a new class of differentiability between non-normable Frechet algebras and for this reason; we investigate the differential calculus defined by Frolicher and Kriegel in a more general context.

Topological algebra is the junction of two topical areas of modern geometry. The first is the theory of smooth manifolds modeled on Frechet spaces. In works on this topic, definitions, examples are given; differential-topological properties of Frechet manifolds are considered. At the same time, in many examples on the manifolds under study, there is an additional structure of the manifold over the algebra. Therefore, the second direction is the theory of spaces over algebras. This topic was not studied deeply. But so far these studies have been limited only to the case of spaces of finite dimension, although most of the definitions and results can be generalized to the infinite-dimensional case.

Therefore, it seems to us, of interest are any differential-geometric objects on infinite-dimensional manifolds that take into accounting the structure of a manifold over an algebra.

This work's aim is to foundation manifolds that are modeled on Frechet algebra which are not coming from Banach algebra, and which are smooth. To do this we begin carefully with Differential calculus in infinite dimensions using the methods of convenient vector spaces and generalize these spaces to a convenient algebra; in fact, differentiability in these algebras is more general than in Banach spaces and became stronger. The main difficulty appeared in the case when continuous linear mappings are composition to be jointly continuous and what are spaces of linear mappings is the exactly suitable topology in order to be continuous to solve this problem, we used the Mackey closure topology or a  $C^\infty$ - topology.

▼ A Fréchet algebras can be considered in two ways, first one, when these algebras are presented as Banach algebras projective limits (Alekseevsky and Michor 1994). Second, we will introduce by using the method of (Asplund 1968) that we will called convenient algebras, they (as example the convenient algebras) are characterized as dual of Fréchet algebras. The topology of these algebras is known as a topology of the Mackey closure. So, it is defined as a final topology with respect to all convergent Mackey sequences  $S: N_\infty := N \cup \{\infty\} \rightarrow E$ .

The topology of Mackey closure of the algebras  $E, F$  is coincide with the topologies defined by the norms when  $E, F$  are Banach algebras, the differential  $dh$  in this case will coincides with the Gâteaux differential where  $h: E \rightarrow F$  is a mapping between the underlying of  $E$  and  $F$  (i.e. the vector spaces on which these algebras are modeled) but, when  $E, F$  are Fréchet algebras that are not normable, the topology of Mackey closure on them coincides with respect to the topology of the Frechet algebra.

The derivative of the n-th order is defined as  $D^n h: x \in E \rightarrow D^n h(x) \in L_n(E, F), D^n h(x) = D(D^{n-1}h(x))$  where  $E, F$ , are convenient algebras, endowed with a topology of a locally convex space. If  $E, F$  are Banach spaces, though; this definition of the differential is coinciding with the definition of a Fréchet differential, therefore if  $E, F$  are Fréchet spaces, then  $L_n(E, F)$ , is although not, however it can be endowed with the topological algebra structures.

Moreover, differential  $Dh$  unlike  $dh$  having many properties with a classical differential: the differentiation chain rule of a complex mapping is performed, also the evolution mapping  $(x, f) \in F ( L(F, G) \rightarrow f(x) \in G$  is differentiable as there is an analogue with the finite increment theorem.

Chapter two deals with some of the basic definitions and results concerning topological linear spaces and topological algebras. For dealing with continuity questions, we have used largely types of topological algebras (Frechet algebra). We have included in the third chapter a construction for the convenience of a topological algebra using the method of (Kriegl and Michor 1997) also in this chapter, some differential analysis involving vector-valued functions is developed and extended on Frechet algebras. In chapter four, defined smooth manifolds defined on Frechet algebras as convenient algebras.



## 2. BASIC DEFINITIONS AND CONCEPTS

This chapter consists of two basic sections, the first one deal with basic definitions of the topological vector spaces and some properties of these spaces with their types and relations among them, and the second display fundamental concepts in the theory of topological algebras.

### 2.1 Basic Definitions of Topological Vector Spaces

**Definitions 2.1.1** (Dieudonné 1969) A vector spaces  $X$  over  $K$  is a set with two operations'  $+$  and  $\cdot$  (addition and scalar multiplication) these operations are defined as follows.

A- The operation  $+$  must satisfy the following some conditions:

- 1- Closure: If  $u, v \in X$ , then  $u + v \in X$ .
- 2- Commutative law: For any vectors,  $u$  and  $v$  in  $X$ ,  $u + v = v + u$ .
- 3- Associative law: For any vectors  $u, v$  and  $w$  in  $X$

$$u + (v + w) = (v + u) + w$$

- 4- Additive identity: There exists an additive identity element in  $X$ , denoted by zero, such tht for all vector  $v$  in  $X$ ,  $0 + v$ ,  $v + 0 = v$ .
- 5- Additive inverses: For any vectors  $u$  in  $X$ , the additive inverse of  $u$  is called a solution  $x$  of equations  $u + v = 0$  and  $x + u = 0$  and denoted by  $-u$ .

B- The operation  $(\bullet)$  (or called scalar multiplication) is defined between vectors, and satisfy the following conditions:

1- Closure: If  $u \in X$  and, for any  $a \in K$  (the field  $K$  is a field of real numbers or complex numbers), then  $a \bullet u \in X$ .

2- A- Distributive law: For any  $a \in K$  and all vectors  $u$  and  $v$  in  $X$ ,  $a \bullet (u + v) = a \bullet u + a \bullet v$ .

B-Distributive law: For any  $a, b \in K$  and the vector  $u$  in  $X$ ,  $(a + b) \bullet u = a \bullet u + a \bullet b$

3- Associative law: For any  $a, b \in K$  and the vector  $u$  in  $X$ ,

$$a \bullet (b \bullet u) = (a \bullet b) \bullet u$$

4- Unitary law: For any vector  $u$  in,  $1 \bullet u = u \bullet 1 = u$ .

**Definition 2.1.2** (Lindstrom 1986) A topological vector space  $(X, \tau)$  (in short *TVS*) is a vector space  $X$  over a field  $K$  ( may be real numbers ) and topology  $(\tau)$  such that: The two maps (addition and the scalar multiplication ) are continues.

That's mean:

1- For each  $a, b$  belong to  $X$ , and each open neighborhood  $W$  of  $a+b$ , there exist an open neighborhood  $U$  of  $b \in X$ , such that

$$U + V \subset W .$$

- 2- For each  $\lambda$  belong to  $K$ , and  $a \in X$  and each open neighborhood  $W$  in  $X$  containing  $\lambda \cdot a$ , there exist open neighborhood  $U$  of  $\lambda$  in  $K$  and open neighborhood  $V$  of  $a$  in  $X$  such  $U \cdot V \subset W$ .

**Examples 2.1.3**(Lindstrom 1986)

- 1- Let  $X$  be any vector space over  $K$ , then it is with trivial topology (indiscrete topology) that form a topological vector space.
- 2- Let  $X=\{0\}$  be a vector space over  $K$ , then it is with discrete topology that form a topological vector spaces.

**Definition 2.1.4** (Lindstrom 1986)A subset  $A$  of a vector space  $\Xi$  is called convex set if contains line segment between any two points in the set  $A$

(That is meaning that  $\forall x_1, x_2 \in A, \exists \gamma \in [0,1] \Rightarrow \gamma x_1 + (1 - \gamma)x_2 \in A$ ). Such that:

$$\{x = \gamma x_1 + (1 - \gamma)x_2, \gamma \in [0,1]\}$$

**Definition 2.1.5**(Tosiek 2005)Let  $\Xi$  be a vector space over the field  $K$ . A subset  $A$  of a vector space  $\Xi$  is called balanced if  $\gamma A \subseteq A$  for  $|\gamma| \leq 1, \gamma \in K$ .

**Definition 2.1.6**(Tosiek 2005)A subset of a vector space  $\Xi$  is called absorbing set if for every  $x \in \Xi$ , there exists  $\delta > 0$  such that  $tx \in A$  for  $\forall t, |t| < \delta$ .

**Definition 2.1.7**(Hong 1967)A cover of a  $\Xi$  is a family of sets  $\{u_\alpha\}_{\alpha \in A}$  such that,  $\Xi \subseteq \bigcup_{\alpha \in A} u_\alpha$ .

**Definition 2.1.8**(Rudin 1991)If  $\mathcal{E}$  is a vector space, a metric on  $\mathcal{E}$  is a map  $d: \mathcal{E} \times \mathcal{E} \rightarrow \mathbb{R}^+ \cup \{0\}$ , such that

- (i)  $d(x; y) = d(y; x) \forall x, y \in \mathcal{E}.$
- (ii)  $d(x; z) \leq d(x; y) + d(y; z) \forall x, y, z, \in \mathcal{E}.$
- (iii)  $X \neq y \leftrightarrow d(x; y) \neq 0 \forall x, y \in \mathcal{E}.$

A pair  $(\mathcal{E}, d)$  is called a metric space. If a metric space satisfies the condition:  $d(x; y) = d(x+z; y+z), \forall x, y, z \in \mathcal{E}$ , it is called translation invariant metric.

**Example 2.1.9**(Rudin 1991) Let  $\mathcal{E}$  be any vector space, a metric on  $\mathcal{E}$  is a map  $d: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ , such that

$$d(x; y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}, \text{ then } (\mathcal{E}, d) \text{ is translation metric space}$$

**Definition 2.1.10** (Beattie and Butzmann 2013) Let  $\mathcal{E}$  a vector space over  $\mathbb{R}$ . Then a map  $\|\cdot\|: \mathcal{E} \rightarrow \mathbb{R} \cup \{0\}, x \mapsto \|x\|$  is said to be a norm if the following properties are satisfied:

- 1- For all  $\alpha \in \mathbb{R} \Rightarrow \|\alpha \cdot x\| = |\alpha| \|x\|$  for all  $x \in \mathcal{E}$
- 2- For all  $x, y \in \mathcal{E}, \|x + y\| \leq \|x\| + \|y\|$ .
- 3-  $\|x\| \geq 0$  for all  $x \in \mathcal{E}$ .
- 4-  $\|x\| = 0 \leftrightarrow x = 0$ .

A pair  $(\mathcal{E}, \|\cdot\|)$  is called a normed space.

**Definition 2.1.11**(Öhman 2019) Let  $\mathcal{E}$  be a topological space.  $\mathcal{E}$  is called first countable if for each point in  $\mathcal{E}$  there exists a countable neighborhood local base of its filter of neighborhoods.

**Definition 2.1.12**(Öhman 2019) In any topological  $\mathfrak{E}$ -vector space  $\mathfrak{E}$  - the statement is fixed:

- 1- Every set of one point is 'bounded'.
- 2- Every neighborhood of 0, contains a neighborhood of 0 that is balanced.
- 3- For any neighborhood  $U$  of 0. Therefore, there exist a sub neighborhood  $N$  of 0 such that,  $N + N \subseteq U$ .

## 2.2 Fundamental Types of Topological Vector Spaces

Let  $(\mathfrak{E}, \tau)$  be a topological vector space, below are some types of topological vector spaces and some relation among them:

**Definition 2.2.1**(Loring 2011) Let  $(\mathfrak{E}, \tau)$  be a topological vector space, we say  $\mathfrak{E}$  is locally convex if any local base their components are convex.

**Definition 2.2.2**(Loring 2011) Let  $(\mathfrak{E}, \tau)$  be a topological vector space. Any subset  $B \subset \mathfrak{E}$  is bounded if for each neighborhood  $U$  of 0 in  $\mathfrak{E}$  there exists a scalar  $s > 0$  such that  $B \subset tU$  for all  $t > s$ .

**Definition 2.2.3**(Loring 2011) Let  $(\mathfrak{E}, \tau)$  be a topological vector space. We say  $(\mathfrak{E}, \tau)$  is locally bounded topological vector spaces if it admits a bounded neighborhood of 0.

**Definition 2.2.4**(Loring 2011) Any topological vector space which has a bounded neighborhood of zero is called locally bounded.

**Definition 2.2.5**(Lindstrom 1986) Any topological vector space which has a neighborhood of zero whose closure is called locally compact.

**Definition 2.2.6**(Lindstrom 1986) Let  $(\mathcal{E}, \tau)$  be a topological vector space. We say  $(\mathcal{E}, \tau)$  is metrizable if it is generated by some metric  $d$ .

**Definition 2.2.7**(Rudin 1991) Let  $(\mathcal{E}, \tau)$  be a topological vector space, then  $(\mathcal{E}, \tau)$  is said to be  $F$ -space if  $\tau$  is generated by a complete invariant metric  $d$ .

**Definition 2.2.8**(Rudin 1991) Let  $(\mathcal{E}, \tau)$  be a topological vector space, we say it is Frechet space if  $\mathcal{E}$  is satisfied the two conditions locally convex space and  $F$ -space.

**Definition 2.2.9**(Rudin 1991) Let  $(\mathcal{E}, \tau)$  be a topological vector space,  $\mathcal{E}$  has Heine – Borel property if any bounded and closed  $u \subseteq \mathcal{E}$  is compact set.

**Definition 2.2.10**(Öhman 2019) Let  $\mathcal{E}$  be a linear space. Then  $\mathcal{E}$  is said to be finite dimensional if there exists a finite subset  $\{x_1, x_2, \dots, x_n\} \subseteq \mathcal{E}$  such that  $\text{span}(x_1, x_2, \dots, x_n) = \mathcal{E}$ , the span of  $\{x_1, x_2, \dots, x_n\}$  is defined as:

$$\text{span}(x_1, x_2, \dots, x_n) = \{a_1x_1 + a_2x_2 + \dots + a_nx_n : a_1, a_2, \dots, a_n \in K\}.$$

**Definition 2.2.11**(Öhman 2019)A topological vector space  $(\mathcal{E}, \tau)$  is called normable if there exists a norm  $\|\cdot\|$  on  $\mathcal{E}$  such that the canonical metric  $(x, y) \mapsto \|y - x\|$  induces the topology  $\tau$  on  $\mathcal{E}$ .

**Theorem 2.2.12**(Öhman 2019)A topological vector space  $(\mathcal{E}, \tau)$  is normable if and only if there is a bounded convex neighborhood of 0, Figure 2.1 and Figure 2.2.

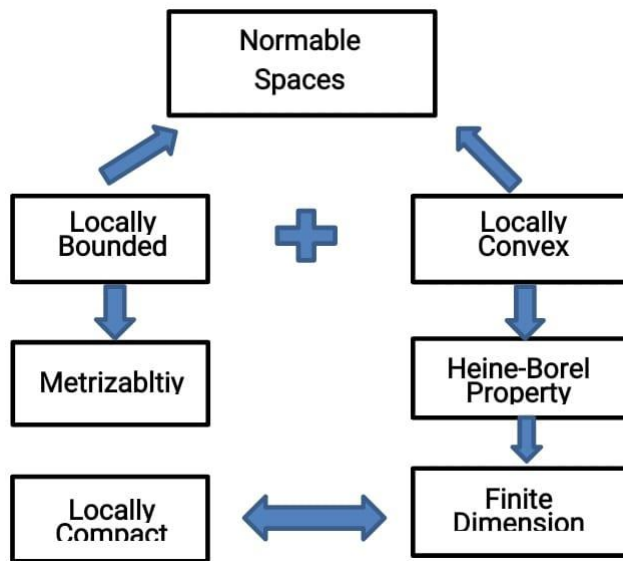


Figure 2.1 Relations among some spaces(Öhman 2019)

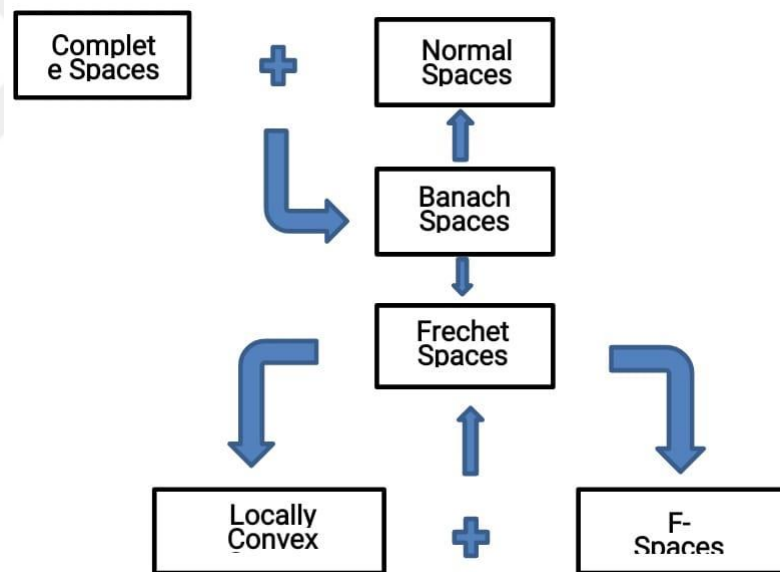


Figure 2.2 Relations among some spaces(Öhman 2019)

### 3. CONVENIENT ALGEBRAS AND CONVENIENT MANIFOLDS

Convenient vector spaces (CVS) constructed and studied by Kriegl, A., Michor, P. W., Rainer, A. in their work (Kriegl and Michor 1997), they are showing that there are locally convex vector spaces which are complete.

With finite-dimensional vector spaces analysis, as for Banach spaces; differential calculus is effective but beyond Banach spaces, difficulties appear, particularly, the continuity of compositions of continuous linear mappings are stopped for any considered topology on the spaces of continuous linear mappings. Mapping between these spaces is of class  $C^\infty$  if it maps, smooth curves to another smooth curves, and by this setting it will appear a new class of smoothness between  $C^\infty$ , open subsets of these spaces. These mappings calculus is called *convenient*. It is weaker, than any notion of differentiability, which can be applied, but not in general because this type of calculus is not useful in solving equations.

In this work, we will try to construct and study a new class of topological algebra by using this method and we so call a convenient topological algebra or for easy convenient algebra (CA).

#### 3.1 The Convenient Topology and Convenient Algebra

Let  $\mathcal{E}$  be a locally convex algebra and let  $\gamma : \mathbb{R} \rightarrow \mathcal{E}$  be a curve, then  $\gamma$  is so-called smooth if all derivatives are existed and continues.  $\gamma$  is called Lipschitz curve if  $\frac{\gamma(t)-\gamma(s)}{t-s} : t \neq s, |t|, |s| \leq \alpha$ . Let  $C^\infty_{lip}(\mathbb{R}, \mathcal{E})$  be the space of smooth curves. Then this space entirely does not depend on the topology of the locally convex space of, only on its bounded sets system. The final topologies with respect to the following sets of mappings into  $\mathcal{E}$  coincide,

1.  $C^\infty_{lip}(\mathbb{R}, \mathcal{E})$ .
2. The set of all Lipschitz curves is bounded in  $\mathcal{E}$ , for each  $\alpha$ .
3. The set of all Mackey convergent sequences  $y_n \rightarrow y$  (there exists a sequence  $0 < \mu_n \rightarrow \infty$  with  $\mu_n(y_n - y)$  is bounded).

This topology is called the *Mackey closure -topology* on  $\mathcal{E}$  and we write  $M_{cl} \mathcal{E}$  for the resulting topological spaces. This topology is finer than the given locally convex topology, if  $\mathcal{E}$  is a Fréchet space,  $M_{cl} \mathcal{E} = \mathcal{E}$ .

A locally convex algebra  $\mathcal{E}$  is said to be a convenient algebra (CA) if one of the following equivalent conditions holds (called  $c^\infty$ -completeness).

- i. For any  $c \in C^\infty(\mathbb{R}, \mathcal{E})$  the (Riemann-) integral  $\int_0^1 c(t) dt$  exists in  $\mathcal{E}$ .
- ii. Let  $c: \mathbb{R} \rightarrow \mathcal{E}$  be a curve. Then  $c$  is smooth if and only if the composition  $\lambda \circ c: t \mapsto \lambda(c(t))$  belong to the space  $C^\infty(\mathbb{R}, \mathbb{R})$  for all  $\lambda \in \mathcal{E}^*$  and  $\mathcal{E}^*$  is the dual of all continuous linear functions on  $\mathcal{E}$ .

By this setting the following facts are equivalent

- i. For all  $\lambda \in \mathcal{E}'$ , the dual of all bounded linear is functional.
- ii. For all  $\lambda \in V$ ;  $V$  is a subset of  $\mathcal{E}'$ , recognized as a bounded subsets in  $\mathcal{E}$ .
- iii. Any Mackey-Cauchy-sequence (i.e.,  $t_{nm}(x - x_m) \rightarrow 0$  for some  $t_{nm} \rightarrow \infty$  in  $\mathbb{R}$ ), converges in  $\mathcal{E}$ .

- iv. Let  $B$  be a closed absolutely convex. Then  $B$  is bounded iff  $E_B$  is a Banach space.
- v. If  $f: \mathbb{R} \rightarrow \mathcal{E}$  is scalar wise  $Lip^k$ , then  $f$  is  $Lip^k$ , for  $k > 1$ .
- vi. If  $f: \mathbb{R} \rightarrow \mathcal{E}$  is scalar wise  $C^\infty$ , then  $f$  is differential at 0.

Here, mapping  $f: \mathbb{R} \rightarrow \mathcal{E}$  is so-called  $Lip^k$  if all derivatives; up to order  $k$ ; is exist and are Lipschitz, locally on  $\mathbb{R}$ .

Let  $\mathcal{E}$  and  $F$  be CA, and let  $U \subseteq \mathcal{E}$  be  $M_{cl}$   $\mathcal{E}$ -open (open regarded to the Mackey closure topology on  $\mathcal{E}$ ). A mapping  $f: U \rightarrow F$  is called *of class  $C^\infty_{lip}$* , if the composition  $f \circ \gamma \in C^\infty_{lip}(\mathbb{R}, F)$  for all  $\gamma \in C^\infty_{lip}(\mathbb{R}, U)$ .

1. With maps on Fréchet algebras, this notion of smoothness is coinciding with all other sensible definitions.
2. Multilinear mappings are smooth if and only if they are bounded.
3. If  $f: U \rightarrow L(\mathcal{E}, F)$  is smooth where  $L(\mathcal{E}, F)$  denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space  $C^\infty_{lip}(U, F)$  of all smooth mappings  $U \rightarrow F$  is a convenient algebra since the following injection, as  $C^\infty_{lip}(\mathbb{R}, \mathbb{R})$  takes the compact convergence topology in each derivative separately.

$$C^\infty_{lip}(U, F) \rightarrow \prod_{\gamma \in C^\infty_{lip}(\mathbb{R}, U)} C^\infty_{lip}(\mathbb{R}, F), \quad f \mapsto (\ell \circ f \circ \gamma)$$

6. For  $M_{cl}$   $\mathcal{E}$ -open  $V \subseteq \mathbb{F}$ , therefore the mapping

$$C^\infty_{lip}(\mathbb{U}, C^\infty_{lip}(V, \mathbb{G})) \cong C^\infty_{lip}(\mathbb{U} \times V, \mathbb{G}), \quad f \mapsto g \quad f(u)(v) = g(u, v)$$

is a linear and diffeomorphism between convenient algebras.

7. Smooth uniform boundedness theorem.

A linear mapping  $f: \mathcal{E} \rightarrow C^\infty_{lip}(V, \mathbb{G})$  is said to be smooth (by (2) equivalent to bounded) if and only if  $ev_v \circ f: V \rightarrow \mathbb{G}$  is smooth for each  $v \in V$ .

8. The following canonical mappings are of class  $C^\infty_{lip}$ . This follows from the exponential law by simple categorical reasoning.

$$ev: C^\infty_{lip}(\mathcal{E}, \mathbb{F}) \times \mathcal{E} \rightarrow \mathbb{F}, \quad ev(f, x) = f(x)$$

$$ins: \mathcal{E} \rightarrow C^\infty_{lip}(\mathbb{F}, \mathcal{E} \times \mathbb{F}), \quad ins(x)(y) = (x, y)$$

$$(\theta)^\wedge: C^\infty_{lip}(\mathcal{E}, C^\infty_{lip}(\mathbb{F}, \mathbb{G})) \rightarrow C^\infty_{lip}(\mathcal{E}, \mathbb{G})$$

$$(\delta)^\vee: C^\infty_{lip}(\mathcal{E} \times \mathbb{F}, \mathbb{G}) \rightarrow C^\infty_{lip}(\mathcal{E}, C^\infty_{lip}(\mathbb{F}, \mathbb{G}))$$

$$comp: C^\infty_{lip}(f, \mathbb{G}) \times C^\infty_{lip}(\mathcal{E}, \mathbb{F}) \rightarrow C^\infty_{lip}(\mathcal{E}, \mathbb{G})$$

$$C^\infty_{lip}(\theta, \delta): C^\infty_{lip}(\mathbb{F}, \mathbb{F}_1) \times C^\infty_{lip}(\mathcal{E}_1, \mathcal{E}) \\ \rightarrow C^\infty_{lip}(C^\infty(\mathcal{E}, \mathbb{F}), C^\infty_{lip}(\mathcal{E}_1, \mathbb{F}_1)), \quad (f, g) \mapsto (h \mapsto f \circ h \circ g)$$

$$\prod: \prod C^{\infty}_{lip}(\mathcal{E}_i, \mathbb{F}_i) \rightarrow C^{\infty}_{lip}(\prod \mathcal{E}_i, \prod \mathbb{F}_i)$$

### 3.2 Smooth Convenient Manifolds

In this section, we use the convenient algebra as a model space to construct the concept of smooth convenient manifold or a manifold of class  $C^{\infty}_{lip}$  and we introduce the definitions of the  $C^{\infty}_{lip}$  – maps,  $C^{\infty}_{lip}$  – covering maps and  $C^{\infty}_{lip}$  – strongly bounded operator defined on smooth convenient manifold. In addition, we introduce the tangent vector of smooth convenient manifold, tangent space, tangent bundle of smooth convenient manifold.

**Definition 3.2.1** Let  $\mathcal{F}$  be a convenient algebra. A pair  $(V, \psi)$ , where  $V$  is  $C^{\infty}$ - open subset of  $\mathcal{F}$  so that the map  $V \rightarrow \psi(V)$  is a homeomorphism defined on a  $C^{\infty}$ -open set  $V$  of  $\mathcal{F}$  onto  $C^{\infty}$ -open  $\psi(V)$  of a convenient algebra.

In other word,  $\varphi$  maps  $V$  onto a  $C^{\infty}$ -open subset  $\psi(V)$  in convenient algebra, called a convenient chart on  $\mathcal{F}$ .

**Definition 3.2.2** Let  $\mathcal{F}$  be a convenient algebra. If  $(V_{\alpha}, \psi_{\alpha})$  and  $(V_{\gamma}, \psi_{\gamma})$  are convenient charts then

$$\psi_j \circ \psi_{\alpha}^{-1}: \psi_{\alpha}(V_{\alpha} \cap V_{\gamma}) \rightarrow \psi_j(V_{\alpha} \cap V_{\gamma})$$

is the transition map from  $\psi_{\alpha}$  to  $\psi_j$ . As well as,  $\psi_{\alpha}(V_{\alpha} \cap V_{\gamma})$  and  $\psi_j(V_{\alpha} \cap V_{\gamma})$  are  $C^{\infty}$ -open in convenient algebra .

**Definition 3.2.3** Let  $\mathcal{F}$  be a convenient algebra, also let  $(V_\alpha, \psi_\alpha), (V_\gamma, \psi_\gamma)$  are two convenient charts on  $\mathcal{F}$ , therefore  $(V_\alpha, \psi_\alpha)$  and  $(V_\gamma, \psi_\gamma)$  are said to be  $C^\infty_{lip}$  Compatible if the transition map  $\psi_j \circ \psi_\alpha^{-1}$  is diffeomorphism of class  $C^\infty_{lip}$ .

**Definition 3.2.4** Let  $\mathcal{F}$  be a convenient algebra. A collection  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  of convenient charts on  $\mathcal{F}$  such that  $V_\alpha, i \in I$  that cover  $\mathcal{F}$ , that is

$$\bigcup_{i \in I} V_\alpha = \mathcal{F}$$

Is called a convenient atlas on  $\mathcal{F}$ .

**Definition 3.2.5** A convenient algebra  $\mathcal{F}$  is said to convenient manifold if it admits a convenient atlas.

**Definition 3.2.6** A convenient atlas  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  on a convenient algebra  $\mathcal{F}$  is called smooth of class  $C^\infty_{lip}$  (in short,  $C^\infty_{lip}$ -atlas) if any two convenient charts in  $\mathbb{A}$  are  $C^\infty_{lip}$  - compatible ( also called  $C^\infty_{lip}$  - consides ).

**Definition 3.2.7** Two  $C^\infty_{lip}$ -atlas  $\mathbb{A}_1 = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  and  $\mathbb{A}_2 = \{(V_\gamma, \psi_\gamma)\}_{\gamma \in \delta}$  on a convenient algebra  $\mathcal{F}$  are said to be  $C^\infty_{lip}$  - consides if each convenient chart in  $\mathbb{A}_1$  is  $C^\infty_{lip}$  - compatible with any one of convenient charts in  $\mathbb{A}_2$ .

**Remark 3.2.8** It is not difficult to see, that the relation of  $C^\infty_{lip}$  - compatible between  $C^\infty_{lip}$  - atlases would be an equivalence relation.

**Definition 3.2.9** A  $C^\infty_{lip}$  - atlas  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  on a convenient algebra  $\mathcal{F}$  is called a maximal if each convenient chart is  $C^\infty_{lip}$  - consides with any convenient chart in  $\mathbb{A}$ ;

is already in  $\mathbb{A}$ . A maximal  $C^\infty_{lip}$  – atlas is also called  $C^\infty_{lip}$  – structure, or  $C^\infty_{lip}$  –differentiable structure.

**Definition 3.2.10** A convenient manifold of class  $C^\infty_{lip}$  ( in short,  $C^\infty_{lip}$  – manifold) is a pair  $(\mathcal{F}, \mathbb{A})$  where  $\mathcal{F}$  is convenient algebra and  $\mathbb{A}$  is a  $C^\infty_{lip}$  – structure. In other words, we can say just  $\mathcal{F}$  is  $C^\infty_{lip}$  – manifold.

**Remark 3.2.11** If  $\mathcal{F}$  is a  $C^\infty_{lip}$  – manifold, so any convenient chart at any maximal  $C^\infty_{lip}$  – atlas will called  $C^\infty_{lip}$  , and the corresponding map will be named  $C^\infty_{lip}$  – coordinate map.

**Lemma 3.2.12** Let  $\mathcal{F}$  be  $C^\infty_{lip}$  – manifold, let  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  be a convenient atlas on  $\mathcal{F}$ , therefore the convenient charts  $(U, \psi)$  and  $(W, \theta)$  are  $C^\infty_{lip}$  – compatible with the convenient atlas  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$ , rather then they are  $C^\infty_{lip}$  – cosides.

**Proof:** Let  $(U, \psi)$  ,  $(W, \theta)$  are convenient charts on  $C^\infty_{lip}$  – manifold  $\mathcal{F}$  that are  $C^\infty_{lip}$  – cosides with the convenient atlas  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  on  $\mathcal{F}$ , so we have to show that

$\theta \circ \psi^{-1}: \psi(U \cap W) \rightarrow \theta(U \cap W)$  is diffeomorphism of class  $C^\infty_{lip}$ .

Let  $z = \psi(q)$  be point in  $\psi(U \cap W)$ . then  $q$  will belong in the support of  $U \cap W$ . But  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  is convenient atlas on  $\mathcal{F}$ , then  $q$  will be in the support of  $V_\alpha$  with some  $i$ .

therefore  $q$  will be in the support of  $V_\alpha \cap U \cap W$  with some  $i$ . But  $(U, \psi)$  and  $(W, \theta)$  are both  $C^\infty_{lip}$  – cosides with the convenient atlas  $A = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$ , so  $\psi_\alpha \circ \psi^{-1}$  and  $\theta \circ \psi_\alpha^{-1}$  are  $C^\infty_{lip}$  with some  $i$ . Therefore,

$$\theta \circ \psi^{-1} = (\theta \circ \psi_\alpha^{-1}) \circ (\psi_\alpha \circ \psi^{-1})$$

is  $C^\infty_{lip}$  – map at  $z$ . But  $z$  is arbitrary, so  $\theta \circ \psi^{-1}$  is  $C^\infty_{lip}$  – map. But

$$(\theta \circ \psi^{-1})^{-1} = \psi \circ \theta^{-1}$$

So,  $(\theta \circ \psi^{-1})^{-1}$  is  $C^\infty_{lip}$ , also  $\theta \circ \psi^{-1}$  is diffeomorphism of class  $C^\infty_{lip}$ .

**Theorem 3.2.13** Let  $\mathcal{F}$  be  $C^\infty_{lip}$  – manifold,  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  are a convenient atlas on  $\mathcal{F}$ . The  $C^\infty$  – open sets family;  $W \subset \mathcal{F}$  so that  $W \subset V_\alpha, (V_\alpha, \psi_\alpha) \in \mathbb{A}$  and  $\psi_\alpha(W)$  is an  $C^\infty$  – open with some convenient algebra  $Z$  as a base for a proper  $C^\infty$  – topology on  $\mathcal{F}$ .

**Proof:** Let  $\mathcal{L} = \{W \subset \mathcal{F} \text{ so that } W \subset V_\alpha, (V_\alpha, \psi_\alpha) \in \mathbb{A} \text{ and } \psi_\alpha(W) \text{ is a } C^\infty \text{ – open in convenient algebra } Z\}$ .

Since for every  $(V_\alpha, \psi_\alpha) \in \mathbb{A}, V_\alpha \subset \mathcal{F}, V_\alpha \subset V_\alpha$  and  $\psi_\alpha(V_\alpha)$  is  $C^\infty$  – open set in some convenient algebra  $Z$  then each  $V_\alpha$  belong to  $\mathcal{L}$ .

Now, let  $U, W$  are two  $C^\infty$  – open sets in  $\mathcal{L}$  so there are two convenient charts  $(V_1, \psi_1)$  and  $(V_2, \psi_2)$  in  $\mathbb{A}$  such that  $U \subset V_1, W \subset V_2$  and  $\psi_1(U), \psi_2(W)$  are  $C^\infty$  – open  $Z$ .

Since  $U \subset V_1$  and  $W \subset V_2$  then  $U \cap W \subset V_1$  and  $U \cap W \subset V_2$  then

$$\begin{aligned} \psi_1(U \cap W) &= \psi_1(U \cap W \cap V_1 \cap V_2) \\ &= \psi_1(U) \cap \psi_1(W \cap V_1 \cap V_2) \\ &= \psi_1(U) \cap \psi_1 \circ \psi_2^{-1}(\psi_2(W) \cap \psi_2(V_1 \cap V_2)) \end{aligned}$$

Therefore  $\varphi_1(U \cap W)$  is an  $C^\infty$  – open in  $Z$ . Hence  $U \cap W \in \mathcal{L}$ .

Finally, for every  $0 \leq c < 1$ .  $W \in \mathcal{L}$ , there is a convenient chart  $(V_\alpha, \psi_\alpha)$  in  $\mathbb{A}$  so that  $W \subset V_\alpha$ ,  $\psi_\alpha(W)$  is a  $C^\infty$  – open in  $Z$ .

Let  $D = K_c \cap W$ , since  $D \subset W$  and  $W \subset V_\alpha$  then  $D \subset V_\alpha$  such that  $(V_\alpha, \psi_\alpha) \in \mathbb{A}$  and  $\psi_\alpha(W)$  is an  $C^\infty$  – open subset  $Z$ . And

$$\begin{aligned} \psi_\alpha(D) &= \psi_\alpha(W \cap K_c) \\ &= \psi_\alpha(W \cap K_c \cap V_\alpha) \\ &= \psi_\alpha(W) \cap \psi_\alpha(K_c \cap V_\alpha) \\ &= \psi_\alpha(W) \cap \psi_\alpha \circ \psi_\gamma^{-1}(\psi_\gamma(K_c) \cap \psi_\gamma(V_\alpha)) \end{aligned}$$

Thus,  $\psi_\alpha(D)$  is a  $C^\infty$  – open subset in  $Z$  and hence  $D \in \mathcal{L}$  and thus,  $\mathcal{L}$  define a base for a  $C^\infty$  – topology on  $\mathcal{F}$ . Therefore, a convenient atlas on  $\mathcal{F}$  induce a  $C^\infty$  – topology on  $\mathcal{F}$  and it is denoted by  $\tau_{\mathbb{A}}$ .

is a  $C^\infty$  – open in a convenient algebra  $Z$ . We get

**Theorem 3.2.14** Let  $\mathcal{F}$  be  $C^\infty_{lip}$  – manifold, let  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$  convenient atlas on  $\mathcal{F}$ . A  $C^\infty$  – open subset  $V$  of  $\mathcal{F}$  is a  $C^\infty$  –open subset in  $C^\infty$  – topology  $\tau_{\mathbb{A}}$  induced by convenient atlas  $\mathbb{A}$  ;if and only if;  $V$  intersect some convenient chart  $(V_\alpha, \psi_\alpha) \in \mathbb{A}$  and  $\psi_\alpha(V \cap V_\alpha)$  is a  $C^\infty$  – open subset in a convenient algebra  $Z$ .

**Proof:** Let  $V$  be a  $C^\infty$  – open of  $\mathcal{F}$  such that if  $V$  intersect some convenient chart  $(V_\alpha, \psi_\alpha) \in A$  and  $\psi_\alpha(V \cap V_\alpha)V = \cup_{i \in I}(V \cap V_\alpha)$

With each convenient chart  $(V_\alpha, \psi_\alpha) \in A$  intersect  $V$ ,  $V \cap V_\alpha \subset V_\alpha$ . By assumption,  $\psi_\alpha(V \cap V_\alpha)$  is a  $C^\infty$  – open subset in  $Z$ . Then, we get  $V \cap V_\alpha$  is  $C^\infty$  – open in the

$C^\infty$  – topology on  $\mathcal{F}$  and we get  $V$  is an  $C^\infty$  – open subset in the  $C^\infty$  – topology  $\tau_{\mathbb{A}}$  induced by convenient atlas  $\mathbb{A}$ .

Conversely, let  $V$  is an  $C^\infty$  – open in the  $C^\infty$  – topology  $\tau_{\mathbb{A}}$  and let  $(V_\alpha, \psi_\alpha)$  be a convenient chart in  $\mathbb{A}$ . then we have  $V \cap V_\alpha$  is a  $C^\infty$  – open in the  $C^\infty$  – topology  $\tau_{\mathbb{A}}$ . But  $\psi_\alpha$  are maps  $V_\alpha$  onto an  $C^\infty$  – open  $\psi_\alpha(V_\alpha)$  in  $Z$ ; then we have,  $\psi_\alpha(V \cap V_\alpha)$  is a  $C^\infty$  – open in  $Z$ .

**Lemma 3.2.15** Let  $\mathcal{F}$  be  $C^\infty_{lip}$  – manifold,  $\mathbb{A}_1 = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  be a convenient atlas on  $\mathcal{F}$  and Let  $\mathcal{R}$  be  $C^\infty_{lip}$  – manifold,  $\mathbb{A}_2 = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  be a convenient atlas on  $\mathcal{R}$ . Then the family of pairs  $(V_\alpha \times V_\alpha, \psi_\alpha \times \psi_\alpha)$  is a convenient atlas on  $\mathcal{F} \times \mathcal{R}$ .

**Proof:** By a convenient chart definition, we have the domains of the convenient charts  $(V_\alpha \times V_\alpha, \psi_\alpha \times \psi_\alpha)$  cover  $\mathcal{F} \times \mathcal{R}$ . though; the pairs family  $(V_\alpha \times V_\alpha, \psi_\alpha \times \psi_\alpha)$  is a convenient atlas on  $\mathcal{F} \times \mathcal{R}$ .

**Theorem 3.2.16** Let  $\mathcal{F}$  be a  $C^\infty_{lip}$  – manifold, so every  $C^\infty_{lip}$  – atlas on  $\mathcal{F}$  is contained in a maximal  $C^\infty_{lip}$  – atlas which is unique.

**Proof:** Let  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  be  $C^\infty_{lip}$  – atlas on  $\mathcal{F}$  and let  $D$  is the set for all convenient  $C^\infty_{lip}$  –compatible charts with each convenient chart in  $\mathbb{A}$ .

In order to show that  $D$  is  $C^\infty_{lip}$  – atlas on  $\mathcal{F}$ , we have to show any two convenient charts of  $D$  are  $C^\infty_{lip}$  –consides with each other.

Let  $(U, \xi)$  and  $(W, \chi)$  are convenient charts in  $D$ , on  $C^\infty_{lip}$  – manifold  $\mathcal{F}$  both  $C^\infty_{lip}$  –consides with  $A = \{(V_\alpha, \xi_\alpha)\}_{\alpha \in \Lambda}$ . So we show that

$$\chi \circ \xi^{-1}: \xi(U \cap W) \rightarrow \chi(U \cap W)$$

is diffeomorphism of class  $C^\infty_{lip}$ .

Let  $z = \xi(q)$  is an arbitrary point in  $\xi(U \cap W)$  so that  $q$  belong to the support of  $U \cap W$ . As  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  is convenient atlas on  $\mathcal{F}$ , so  $q$  belong to the support of  $V_\alpha$  with some  $\alpha$ .

Therefore  $q$  belong to the support of  $V_\alpha \cap U \cap W$  with some  $i$ . As  $(U, \xi)$  and  $(W, \chi)$  are both  $C^\infty_{lip}$  – considers with the convenient atlas  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$ , that  $\psi_\alpha \circ \xi^{-1}$  and  $\chi \circ \psi_\alpha^{-1}$  are  $C^\infty_{lip}$  with some  $\alpha$ . So

$$\chi \circ \xi^{-1} = (\chi \circ \psi_\alpha^{-1}) \circ (\psi_\alpha \circ \xi^{-1})$$

is  $C^\infty_{lip}$  –map at  $z$ . Since  $z$  is arbitrary, thus  $\chi \circ \xi^{-1}$  is  $C^\infty_{lip}$  –map.

And so

$$(\chi \circ \xi^{-1})^{-1} = \xi \circ \chi^{-1}$$

Thus,  $(\chi \circ \xi^{-1})^{-1}$  is  $C^\infty_{lip}$  –map, therefore;  $\chi \circ \xi^{-1}$  is  $C^\infty_{lip}$  –diffeomorphism.

Next step, we will show that  $D$  is maximal  $C^\infty_{lip}$  – atlas, so that each convenient chart is  $C^\infty_{lip}$  –considers with every convenient chart in  $D$ ; must be  $C^\infty_{lip}$  –considers with every convenient chart in  $\mathbb{A}$ , so it is belong to  $D$ . This meaning that existence of maximal  $C^\infty_{lip}$  – atlas containing  $\mathbb{A}$ .

At last, let  $E$  be another maximal  $C^\infty_{lip}$  – atlas that containing  $\mathbb{A}$ , so each convenient chart in  $D$  is  $C^\infty_{lip}$  – compatible with each convenient chart in  $\mathbb{A}$ , so  $E \subseteq D$ . By maximality of  $E$  we will have  $E = D$ .

## 4. MAPPINGS BETWEEN $C^\infty_{lip}$ – MANIFOLDS

In this chapter we study new class of mappings between  $C^\infty_{lip}$  – manifolds.

### 4.1 Mapping Between $C^\infty_{lip}$ – Manifolds Principles

**Definition 4.1.1** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be  $C^\infty_{lip}$  – manifolds and let  $S: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  be a map. We say that  $S$  is of class  $C^\infty_{lip}$  if for every  $z \in \mathcal{F}_1$  there is a convenient chart  $(W, \chi)$  at  $z \in \mathcal{F}_1$  and a convenient chart  $(V, \psi)$  at  $S(z) \in \mathcal{F}_2$  such that the mapping  $\psi \circ S \circ \chi^{-1}$  which maps  $\chi(W \cap S^{-1}(V))$  into  $\psi(V)$  is of class  $C^\infty_{lip}$  at  $\chi(z)$ .

**Theorem 4.1.2** Let  $\mathcal{F}_1$ ,  $\mathcal{F}_2$  and  $\mathcal{F}_3$  are  $C^\infty_{lip}$  – manifolds, Let  $S: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  and  $T: \mathcal{F}_2 \rightarrow \mathcal{F}_3$  are  $C^\infty_{lip}$  – maps then  $T \circ S: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is  $C^\infty_{lip}$  – map.

**Proof:** Let  $(W_1, \chi_1)$  be a convenient chart at  $z \in \mathcal{F}_1$ ,  $(W_2, \chi_2)$  be a convenient chart at  $S(z) \in \mathcal{F}_2$  and  $(W_3, \chi_3)$  be a convenient chart at  $T(S(z)) \in \mathcal{F}_3$ . Then,  $\chi_2 \circ S \circ \chi_1^{-1}$  which maps  $\chi_1(W_1 \cap S^{-1}(W_2))$  into  $\chi_2(W_2)$  is of class  $C^\infty_{lip}$ . Moreover,  $\chi_3 \circ T \circ \chi_2^{-1}$  which maps  $\chi_2(W_2 \cap T^{-1}(W_3))$  into  $\chi_3(W_3)$  is of class  $C^\infty_{lip}$ . Hence

$$(\chi_3 \circ T \circ \chi_2^{-1}) \circ (\chi_2 \circ S \circ \chi_1^{-1}) = \chi_3 \circ (T \circ S) \circ \chi_1^{-1}$$

which maps  $\chi_1(W_1 \cap S^{-1}(W_2) \cap S^{-1}(T^{-1}(W_3)))$  into  $\chi_3(W_3)$  is of class  $C^\infty_{lip}$ . Therefore

$$T \circ S: \mathcal{F}_1 \rightarrow \mathcal{F}_3$$

is a  $C^\infty_{lip}$  map.

**Definition 4.1.3** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be  $C^\infty_{lip}$  – manifolds a bijection map  $S: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is said to be  $C^\infty_{lip}$  – diffeomorphism if both  $S$  and  $S^{-1}$  are  $C^\infty_{lip}$  – maps.

**Definition 4.1.4** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be  $C^\infty_{lip}$  – manifolds, A  $C^\infty_{lip}$  – covering map  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – surjective map with the property, each point  $q \in \mathcal{F}_2$  has neighborhood  $V$  such that each component of  $\pi^{-1}(V)$  is  $C^\infty_{lip}$  – diffeomorphism onto  $V$  by  $\pi$ . We say that  $V$  is evenly covered by  $\pi$ .

**Remark 4.1.5** If  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – covering map,  $\mathcal{F}_1$  is said to be a covering space of  $\mathcal{F}_2$ , and  $\mathcal{F}_2$  is said to be the base of covering.

**Definition 4.1.6** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be  $C^\infty_{lip}$  – manifolds. If  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is any  $C^\infty_{lip}$  – map, A convenient section of  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – map  $\rho: \mathcal{F}_2 \rightarrow \mathcal{F}_1$  such that  $\pi \circ \rho = Id_{\mathcal{M}_2}$ .

**Definition 4.1.7** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be a  $C^\infty_{lip}$  – manifolds . If  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is any  $C^\infty_{lip}$  – map, A convenient local section of  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – map  $\rho: V \rightarrow \mathcal{F}_1$  defined on the support of some  $C^\infty_{lip}$  – open subset  $V$  of  $\mathcal{F}_2$  such that  $\pi \circ \rho = Id_V$ .

**Theorem 4.1.8** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are  $C^\infty_{lip}$  – manifolds. If  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – covering map then any point of  $\mathcal{F}_1$  lie in the form of a unique  $C^\infty_{lip}$  – local section  $\rho: V \rightarrow \mathcal{F}_1$  of  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ . Exactly, for each  $p \in \mathcal{F}_1$  there exists a neighborhood  $V$  of  $q = \pi(p)$  and a unique  $C^\infty_{lip}$  – local section  $\rho: V \rightarrow \mathcal{F}_1$  of  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  such that  $\rho(q) = p$ .

**Proof:** Since  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – covering map and since  $q \in \mathcal{F}_2$  then by definition of  $C^\infty_{lip}$  – covering map,  $q$  has a neighborhood  $V$  evenly covered by  $\pi$ .

If  $W$  is the component of  $\pi^{-1}(V)$  such that  $p$  lies in the support of  $W$ .

Then,

$$\pi_{/W}: W \rightarrow V$$

is a  $C^\infty_{lip}$  – diffeomorphism map. Let

$$\rho = \pi^{-1}_{/W}: V \rightarrow W$$

Since

$$\pi_{/W}: W \rightarrow V$$

is a  $C^\infty_{lip}$  – diffeomorphism map, then  $\rho = \pi^{-1}_{/W}: V \rightarrow W$

is a  $C^\infty_{lip}$  –local section of  $\pi$  such that  $\rho(q) = p$ .

For uniqueness, let  $\sigma$  be  $C^\infty_{lip}$  –local section of  $\pi$  different from  $\rho$  such that  $\sigma(q) = p$  and  $\pi \circ \sigma = Id_V$ . Since  $\pi \circ \rho = Id_V$  then  $\rho = \sigma$ .

Hence,  $\rho$  is a unique  $C^\infty_{lip}$  – local section  $\pi$ .

The following theorem is one of the best applications for convenient local sections. This theorem gives a very easy criterion for choosing whatever maps with base space covering of class  $C^\infty_{lip}$ .

**Theorem 4.1.9** Let  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_3$  are  $C^\infty_{lip}$  – manifolds. if  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – covering map then a map  $S: \mathcal{F}_2 \rightarrow \mathcal{F}_3$  is of class  $C^\infty_{lip}$  if and only if  $S \circ \pi: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is  $C^\infty_{lip}$  –map.

**Proof:** Let  $S: \mathcal{F}_2 \rightarrow \mathcal{F}_3$  be a  $C^\infty_{lip}$  – map. Since  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  –map, then

$$S \circ \pi: \mathcal{F}_1 \rightarrow \mathcal{F}_3$$

is  $C^\infty_{lip}$  –map.

For the converse, assume that  $S \circ \pi: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is  $C^\infty_{lip}$  – map and  $q \in \mathcal{F}_2$ , then there exists a neighborhood  $V$  of  $q$  and  $C^\infty_{lip}$  – local section  $\rho: V \rightarrow \mathcal{F}_1$  of  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  such that  $\pi \circ \rho = Id_V$ .

Therefore

$$S|_V = S|_V \circ Id_V = S|_V \circ (\pi \circ \rho) = (S \circ \pi) \circ \rho$$

We get a composition of two  $C^\infty_{lip}$  –maps. Hence  $S$  is  $C^\infty_{lip}$  –map.

The next theorem shows which covering space of  $C^\infty_{lip}$  –

manifold is itself  $C^\infty_{lip}$  – manifold.

**Theorem 4.1.10** If  $\mathcal{F}_2$  be  $C^\infty_{lip}$  –manifold and  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a covering map that  $\mathcal{F}_1$  has  $C^\infty_{lip}$  –manifold structure so that  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  –covering map.

**Proof:** Let  $\mathcal{F}_2$  be  $C^\infty_{lip}$  –manifold and  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  –covering map, thus by definition every point  $q \in \mathcal{F}_2$  has an evenly covered neighborhood  $V$ . We may suppose  $V$  is the domain of  $C^\infty_{lip}$  –chart  $(V, \psi)$  on  $\mathcal{F}_2$ . Let  $W$  be a component of  $\pi^{-1}(V)$ , Thus

$$\pi|_W: W \rightarrow V$$

is a  $C^\infty_{lip}$  – diffeomorphism map in particular, homeomorphism. Let

$$\chi = \psi \circ \pi_{/W}: W \rightarrow Z$$

where  $Z$  be a convenient algebra. Then  $(W, \chi)$  is convenient chart on  $\mathcal{F}_1$ .

Let  $(W_1, \chi_1)$  and  $(W_2, \chi_2)$  are two convenient charst on  $\mathcal{F}_1$ , then

$$\begin{aligned} \chi_2 \circ \chi_1^{-1} &= (\psi_2 \circ \pi_{/(W_1 \cap W_2)}) \circ (\psi_1 \circ \pi_{/(W_1 \cap W_2)})^{-1} \\ &= \psi_2 \circ (\pi_{/(W_1 \cap W_2)} \circ \pi_{/(W_1 \cap W_2)}^{-1}) \circ \psi_1^{-1} \\ &= \psi_2 \circ \psi_1^{-1} \end{aligned}$$

Since  $(V_1, \psi_1)$  and  $(V_2, \psi_2)$  are two convenient charts on  $C^\infty_{lip}$  – manifold  $\mathcal{F}_2$  then the transition map  $\psi_2 \circ \psi_1^{-1}$  is a  $C^\infty_{lip}$  – diffeomorphism so  $\chi_2 \circ \chi_1^{-1}$  is a  $C^\infty_{lip}$  – diffeomorphism map.

The family of all such convenient charts defines  $C^\infty_{lip}$  – structure on  $\mathcal{F}_1$ .

**Remark 4.1.11** Every  $C^\infty_{lip}$  – manifold is covered by itself.

We know that, If  $\mathcal{F}$  is a manifold modeled on convenient algebra  $Z$  then any subset of  $\mathcal{F}$  is locally homeomorphic to  $Z$ . If  $\mathcal{F}$  is considered to be a  $C^\infty_{lip}$  – manifold with a characteristic of convenient algebra then one can define linear operators between any two  $C^\infty_{lip}$  – manifolds.

## 4.2 Strongly Bounded Linear Operator on a $C^\infty_{lip}$ – manifold

**Definition 4.2.1** Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be  $C^\infty_{lip}$  – manifolds modeled on convenient algebras  $Z$  and  $H$  whose  $C^\infty$  – topologies  $\tau_\Lambda$  and  $\tau_\Gamma$  are induced by a countable

separating families of seminorms  $\Lambda = \{P_\lambda\}_{\lambda \in J}$  and  $\Gamma = \{P_\alpha\}_{\alpha \in I}$ , respectively. A linear operator  $T: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is said to be convenient strongly bounded on  $\mathcal{F}_1$  if and only if there is  $h \in \mathbb{R}^+$  such that  $\forall y \in \mathcal{F}_1$  and  $\forall r \in \mathbb{R}$

$$P_\alpha(T(y), r) \geq P_\lambda\left(y, \frac{r}{h}\right)$$

$\forall P_\lambda \in \Lambda$  and  $\forall P_\alpha \in \Gamma$ .

For easy we will use  $(C_{\text{sablo}})$  instead of convenient strongly bounded linear operators.

**Theorem 4.2.2** Let  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_3$  be  $C^\infty_{\text{lip}}$  – manifolds modeled on convenient algebras  $Z, H$  and  $G$  whose  $C^\infty$  – topologies  $\tau_\Lambda, \tau_\Gamma$  and,  $\tau_\aleph$  are induced by a countable separating families of semi norms as  $\Lambda = \{P_\lambda\}_{\lambda \in J}$ ,  $\Gamma = \{P_\alpha\}_{\alpha \in I}$  and  $\Delta = \{P_\kappa\}_{\kappa \in \aleph}$  respectively. If  $T: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  and  $S: \mathcal{F}_2 \rightarrow \mathcal{F}_3$  be a  $C_{\text{sablo}}$  then  $S \circ T: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is a  $C_{\text{sablo}}$ .

**Proof:** Let  $T: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  and  $S: \mathcal{F}_2 \rightarrow \mathcal{F}_3$  be a  $C_{\text{sablo}}$  then there exists positive real numbers  $h_1$  and  $h_2$ , such that

$\forall y \in \mathcal{F}_1, \forall r \in \mathbb{R}$

$$P_\alpha(T(y), r) \geq P_\lambda\left(y, \frac{r}{h_1}\right)$$

$\forall P_\lambda \in \Lambda$  and  $\forall P_\alpha \in \Gamma$ .

And  $\forall z \in \mathcal{F}_2, \forall r \in \mathbb{R}$

$$P_\kappa(S(z), r) \geq P_\alpha\left(z, \frac{r}{h_2}\right)$$

$\forall P_\lambda \in \Delta$  and  $\forall P_\alpha \in \Gamma$ .

Now, we will prove that  $S \circ T: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is a  $C_{\text{sblo}}$ .

Let  $T(y) = z$  and  $S(z) = w$  then we get

$$P_\alpha(z, r) \geq P_\lambda\left(y, \frac{r}{h_1}\right)$$

$\forall P_\lambda \in \Delta$  and  $\forall P_\alpha \in \Gamma$ .and

$$P_\alpha(w, r) \geq P_\alpha\left(z, \frac{r}{h_2}\right)$$

$\forall P_\lambda \in \Delta$  and  $\forall P_\alpha \in \Gamma$ . Since

$$(S \circ T)(y) = S(T(y)) = S(z) = w$$

Then

$$P_\alpha((S \circ T)(y), r) = P_\alpha(w, r)$$

And

$$P_\alpha(w, r) \geq P_\alpha\left(z, \frac{r}{h_2}\right) \geq P_\lambda\left(y, \frac{r}{h_1 h_2}\right)$$

Therefore

$$P_\alpha((S \circ T)(y), r) \geq P_\lambda\left(y, \frac{r}{h}\right)$$

where  $h = h_1 h_2$ . Hence  $S \circ T: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is a  $C_{\text{sblo}}$ .

**Definition 4.2.3** Let  $\mathcal{F}$  be  $C^\infty_{\text{lip}}$  – manifolds modeled on convenient algebras  $Z$  with  $C^\infty$  – topology  $\tau_\Lambda$  induced by a countable separating seminorms family  $\Lambda = \{P_\lambda\}_{\lambda \in J}$ . If  $T: \mathcal{F} \rightarrow \mathcal{F}$  is a  $C_{\text{sblo}}$  on  $\mathcal{F}$  such that

$\forall y \in \mathcal{F}$  and  $\forall r \in \mathbb{R}$

$$P_\lambda(T(y), r) \geq P_\lambda\left(y, \frac{r}{h}\right)$$

$\forall P_\lambda \in \Lambda, h \in \mathbb{R}^+$  then,  $T(y)$  and  $y$  are said to be related by the relation  $(\geq) \forall y \in \mathcal{F}$ .

**Theorem 4.2.4** Let  $\mathcal{F}$  be  $C^\infty_{\text{lip}}$  – manifolds , let  $T: \mathcal{F} \rightarrow \mathcal{F}$  is a  $C_{\text{sblo}}$  on  $\mathcal{F}$ , then the relation  $(\geq)$  is an equivalence relation.

**Proof:** Let  $T: \mathcal{F} \rightarrow \mathcal{F}$  is a  $C_{\text{sblo}}$  on  $\mathcal{F}$  such that  $\forall y \in \mathcal{F}$  and  $\forall r \in \mathbb{R}$

$$P_\lambda(T(y), r) \geq P_\lambda\left(y, \frac{r}{h}\right)$$

$\forall P_\lambda \in \Lambda, h \in \mathbb{R}^+$ .

First, we will show that the relation  $(\geq)$  is reflexive. It is trivial by identity a  $C_{\text{sblo}}$  on  $\mathcal{F}$  i.e.  $T(y) = y, \forall y \in \mathcal{F}$  then we have  $\forall y \in \mathcal{F}$  and  $\forall r \in \mathbb{R}$

$$P_\lambda(y, r) \geq P_\lambda\left(y, \frac{r}{h}\right)$$

$\forall P_\lambda \in \Lambda, h \in \mathbb{R}^+$ . Hence the relation  $(\geq)$  is reflexive.

Next, we will show that the relation  $(\geq)$  is symmetric. Since  $\forall y \in \mathcal{F}$  and  $\forall r \in \mathbb{R}$

$$P_\lambda(T(y), r) \geq P_\lambda\left(y, \frac{r}{h}\right)$$

$\forall P_\lambda \in \Lambda, h \in \mathbb{R}^+$ . Then

$\forall y \in \mathcal{F}$  and  $\forall r \in \mathbb{R}$

$$P_\lambda\left(y, \frac{r}{h_1}\right) \geq P_\lambda(T(y), r)$$

$\forall P_\lambda \in \Lambda, h \in \mathbb{R}^+$  and  $h_1 = \frac{1}{h} \in \mathbb{R}^+$ . Hence the relation  $(\geq)$  is symmetric.

Finally, we will show that the relation  $(\geq)$  is transitive. Let  $\forall y, z \in \mathcal{F}$  and  $\forall r \in \mathbb{R}$

$$P_\lambda(T(y), r) \geq P_\lambda\left(y, \frac{r}{h_1}\right)$$

and

$$P_\lambda(T(z), r) \geq P_\lambda\left(z, \frac{r}{h_2}\right)$$

$\forall P_\lambda \in \Lambda, h_1, h_2 \in \mathbb{R}^+$ . Let  $T(y) = z$  and  $T(z) = w$  then we get

$$P_\lambda(z, r) \geq P_\lambda\left(y, \frac{r}{h_1}\right)$$

and

$$P_\lambda(w, r) \geq P_\lambda\left(z, \frac{r}{h_2}\right)$$

Thus

$$P_\lambda(w, r) \geq P_\lambda\left(z, \frac{r}{h_2}\right) \geq P_\lambda\left(y, \frac{r}{h_1 h_2}\right)$$

i.e.

$$P_\lambda(T(z), r) \geq P_\lambda\left(y, \frac{r}{h}\right)$$

$\forall P_\lambda \in \Lambda, \forall y, z \in \mathcal{F}$  and  $h = h_1 h_2 \in \mathbb{R}^+$ . Hence the relation  $(\geq)$  is transitive and therefore it is an equivalence relation.

The discussion above shows that the  $C_{\text{sbl0}}$  on  $C^\infty_{\text{tip}}$  – manifolds  $\mathcal{F}$  defines a relation  $(\geq)$  as an equivalence relation that generates analytical structure  $(\mathcal{F}, \geq)$  on  $C^\infty_{\text{tip}}$  – manifolds  $\mathcal{F}$

## 5. CONCLUSION AND FUTURES WORKS

### 5.1 Conclusions

It is known that in the classical theory of differentiability between topological algebras stay has no big change if one replaces the given norms with other norms that induced the same topologies. Therefore it is natural to look for a theory without using the norms, but only the topologies of the considering vector spaces. In fact, leads to finding clarification and simplification methods allow a more general theory. Such a generalization is indeed desirable in certain spaces of function that have no natural norm but only a natural topology.

Since the whole theory we used, depends consistently on smooth curves, we start with some known facts concerning curves with Lipchitz conditions.

The conclusion of our work can be summarized by the main following results:

Let  $\mathcal{F}$  be  $C^\infty_{lip}$  – manifold, let  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  be a convenient atlas on  $\mathcal{F}$ . Then,

- 1- If convenient charts  $(U, \psi)$  and  $(W, \theta)$  on  $\mathcal{F}$  are both  $C^\infty_{lip}$  – considers with convenient atlas  $\mathbb{A} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$ , so they are  $C^\infty_{lip}$  – considers with one another.
- 2- The  $C^\infty$  – open sets family  $W \subset \mathcal{F}$  such that  $W \subset V_\alpha, (V_\alpha, \psi_\alpha) \in \mathbb{A}$  and  $\psi_\alpha(W)$  is an  $C^\infty$  – open subset in some convenient algebra  $Z$  is a base for a proper  $C^\infty$  – topology on  $\mathcal{F}$ .
- 3- A  $C^\infty$  – open subset  $V$  of  $\mathcal{F}$  is a  $C^\infty$  –open in the  $C^\infty$  – topology  $\tau_A$  induced by convenient atlas  $\mathbb{A}$  if and only if  $V$  intersect some convenient chart  $(V_\alpha, \psi_\alpha) \in A$  and  $\psi_\alpha(V \cap V_\alpha)$  is a  $C^\infty$  – open in a convenient algebra  $Z$ .

- 4- Let  $\mathcal{R}$  be  $C^\infty_{lip}$  – manifold,  $A_2 = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$  be a convenient atlas on  $\mathcal{R}$ . So that the family of pairs  $(V_\alpha \times V_\alpha, \psi_\alpha \times \psi_\alpha)$  is a convenient atlas on  $\mathcal{F} \times \mathcal{R}$ .
- 5- Every  $C^\infty_{lip}$  – atlas on  $\mathcal{F}$  is contained in a unique maximal  $C^\infty_{lip}$  – atlas.
- 6- Let  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_3$  are  $C^\infty_{lip}$  – manifolds , Let  $S: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  and  $T: \mathcal{F}_2 \rightarrow \mathcal{F}_3$  are  $C^\infty_{lip}$  – maps then  $T \circ S: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is  $C^\infty_{lip}$  –map.
- 7- Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are  $C^\infty_{lip}$  – manifolds. If  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  –covering map then any point of  $\mathcal{F}_1$  lie in the form of a unique  $C^\infty_{lip}$  – local section  $\rho: V \rightarrow \mathcal{F}_1$  of  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$ . Exactly, for each  $p \in \mathcal{F}_1$  there exists a neighborhood  $V$  of  $q = \pi(p)$  and a unique  $C^\infty_{lip}$  –local section  $\rho: V \rightarrow \mathcal{F}_1$  of  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  such that  $\rho(q) = p$ .
- 8- Let  $\mathcal{F}_1, \mathcal{F}_2$  and  $\mathcal{F}_3$  are  $C^\infty_{lip}$  – manifolds. if  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  – covering map then a map  $S: \mathcal{F}_2 \rightarrow \mathcal{F}_3$  is of class  $C^\infty_{lip}$  if and only if  $S \circ \pi: \mathcal{F}_1 \rightarrow \mathcal{F}_3$  is  $C^\infty_{lip}$  –map.
- 9- If  $\mathcal{F}_2$  be  $C^\infty_{lip}$  – manifold and  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a covering map then  $\mathcal{F}_1$  has  $C^\infty_{lip}$  –manifold structure, so that  $\pi: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  is a  $C^\infty_{lip}$  –covering map.

## 5.2 Future Works

One can define fuzzy structures on manifolds we studied ( $C^\infty_{lip}$  –manifold) such that introducing the tangent space notation of a  $C^\infty_{lip}$  – fuzzy manifold and the notation of tangent bundle on these manifolds. The following definitions and remark can be drawing inter of these structures.

**Definition 5.2.1** Let  $\mathcal{F}$  be  $C^\infty_{lip}$  – fuzzy manifold. Consider triplet  $(V_\alpha, \psi_\alpha, z)$  where  $(V_\alpha, \psi_\alpha)$  is a fuzzy convenient chart at  $q \in \mathcal{F}$  and  $z$  is a fuzzy point of convenient

algebra in which  $\psi_\alpha(V_\alpha)$  lies. Two such triplet  $(V_\alpha, \psi_\alpha, z), (V_\alpha, \psi_\alpha, w)$  are said to be related, written

$$(V_\alpha, \psi_\alpha, z) \sim (V_\alpha, \psi_\alpha, w)$$

if the fuzzy directional derivative of the transition map  $\psi_\alpha \circ \psi_\alpha^{-1}$  at  $\psi_\alpha(q)$  maps  $z$  into  $w$ . i.e.

$$D(\psi_\alpha \circ \psi_\alpha^{-1})(\psi_\alpha(q))z = w.$$

**Definition 5.2.2** An equivalence class of  $(V_\alpha, \psi_\alpha, z)$  is said to be a tangent vector of the  $C^\infty_{lip}$  – fuzzy manifold  $\mathcal{F}$  at  $q$  and denoted by  $[V_\alpha, \psi_\alpha, z]_q$ . the set of all tangent vectors at  $q \in \mathcal{F}$  called the tangent space and denoted by  $T_q(\mathcal{F})$ .

**Remark 5.2.3** If  $\mathcal{F}$  be  $C^\infty_{lip}$  – fuzzy manifold and  $(V, \psi)$  be fuzzy convenient chart at  $q \in \mathcal{F}$  then

$$T_q(\mathcal{F}) = T_q(V)$$

We can give the structures of a vector space on the  $T_q(\mathcal{F})$  by defining the sum and the product with scalar as follows,

$$[V_1, \psi_1, z]_q + [V_2, \psi_2, w]_q = [V_2, \psi_2, D(\psi_2 \circ \psi_1^{-1})(\psi_1(q))z + w]_q$$

$$\text{And } \beta \cdot [V_1, \psi_1, z]_q = [V_1, \psi_1, \beta z]_q.$$

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