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Opportunities Provided in Mathematics Methods Textbooks for Pre-Service Teachers to Develop Mathematical Knowledge for Teaching Fractions

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Opportunities Provided in Mathematics Methods Textbooks for Pre-Service Teachers to Develop
Mathematical Knowledge for Teaching Fractions

by

Irem Ercan

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in Curriculum and Instruction
with a concentration in Mathematics Education
Department of Teaching and Learning
College of Education
University of South Florida

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Keywords: Fractions Task, Pedagogical Content Knowledge, Subject Matter Knowledge,
Textbook Analysis, Qualitative Content Analysis

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DEDICATION

To my son, Deniz- may you always believe in yourself!



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As my “7-year long journey” has come to an end, I want to mention that this dissertation would not be possible without the support, encourage, guidance, and advise of many individuals. I owe the first and the deepest thanks to M. Kemal Ataturk, the founder of the Republic of Turkey, for providing equal rights and opportunities to Turkish women and for acknowledging the value of women’s education nearly a hundred years ago.

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ABSTRACT

I explored using qualitative content analysis what type of opportunities in mathematics methods textbooks are provided for pre-service elementary teachers to develop *Mathematical Knowledge for Teaching Fractions* (MKTF). Specifically, I examined all fractions tasks that are located in fraction chapters of mathematics methods textbooks in order to explore the ways they support pre-service teachers' *Subject Matter Knowledge* and *Pedagogical Content Knowledge* of fractions.

I used qualitative content analysis of three mathematics methods textbooks to analyze how MKTF is offered for pre-service elementary teachers. In particular, I investigated the chapters that are designated to fraction concepts using both deductive and inductive approaches in developing a framework with the guide of the Domains of MKT (Ball et al., 2008). First, I used a deductive approach to generate main categories and subcategories of the framework, and then I continued to develop the framework inductively based on the fraction concepts presented in three mathematics methods textbooks. Finally, I used the developed framework to examine the extent to which fraction concepts presented in three textbooks provide opportunities for pre-service teachers to gain/enhance their MKTF.

After analyzing the fraction chapters of the three mathematics method textbooks, I was able to discover that pre-service teachers are offered various types of opportunities to develop mathematical knowledge for teaching fractions through the fraction tasks given in those chapters. Findings revealed that all three methods textbooks offer the greatest opportunities for pre-service

teachers to develop *Subject Matter Knowledge*. The greatest number of tasks in each textbook required pre-service teachers to possess *Common Content Knowledge*, followed by *Specialized Content Knowledge*. The results indicated that fraction tasks in mathematics methods textbooks provide only limited opportunities for pre-service teachers to improve *Pedagogical Content Knowledge*. The results of this inquiry contribute to teacher education, where pre-service teachers should be offered opportunities to learn fraction concepts and how to teach fractions.



CHAPTER 1: INTRODUCTION

Fractions have been recognized one of the most essential topics for both students and teachers due to being the key element of success not only in higher mathematics – algebra, geometry, etc. (Booth & Newton, 2012; Fennell, Kobett, & Wray, 2014; National Mathematics Advisory Panel, 2008; Wu, 2001), but also in science, technology, engineering and mathematics (STEM) as well as everyday life (Jordan, Carrique, Hansen, & Resnick, 2016; Lamon, 2007; Siegler et al., 2012). Although young children start using fractions in daily life before they enter the school by sharing a cake or cookies among their friends (Bezuk, 1988), according to Common Core State Standards for Mathematics (CCSSM), major emphasis is given to part-whole fraction concept and its language for the first time under the Number and Operations Content Standards when students start 3rd grade (Wu, 2011). According to the National Council of Teachers of Mathematics (NCTM, 2000), “The Number and Operations Standard describes deep and fundamental understanding of, and proficiency with, counting, numbers, and arithmetic, as well as an understanding of number systems and their structures ” (p. 32). The NCTM (2000) stressed the importance of numbers in the mathematics curriculum by describing it as a “cornerstone” (p.32), and suggested that the understanding of fractions needs to be developed as “division of numbers” (p.33); which makes fraction concepts as the sine qua non of the mathematics curriculum (Zhou, Peverly, & Xin, 2006). Therefore, the importance of fractions in mathematics students’ and teachers’ lives is undeniable.

Students' Understanding of Fractions

Despite its importance, over the past decades, issues with both teachers' and students' understanding of fractions persist (Ball, 1990a; Ma, 1999; Marchionda, 2006). The National Mathematics Advisory Panel (NMAP) indicated in its final report, "difficulty with the learning of fractions is pervasive and is an obstacle to further progress in mathematics and other domains dependent on mathematics, including algebra" (NMAP, 2008, p. 28). Students' and teachers' difficulty with understanding fractions is described by Siebert and Gaskin (2006) as follows:

When we add or subtract fractions, we have to find a common denominator, but not when we multiply or divide. And once we get a common denominator, we add or subtract the numerators, but not the denominators, despite the fact that when we multiply, we multiply both the numerators and the denominators, and when we divide, we divide neither the numerators nor the denominators. (p. 394)

As one of the six principles in the Principles and Standards for School Mathematics (NCTM, 2000), *Learning*, claims that all students must learn mathematics with understanding. However, Hanson (2001) reported that due to having five interpretations (part-whole, ratio, operator, quotient, and measurement) for fractions, students prefer memorizing methods rather than comprehending the essential concepts behind fractions, and this makes fractions appear nonsensical and mysterious. Research has shown that students need to master the five interpretations in order to develop full understanding (Charalambous & Pitta-Pantazi, 2007; Lamon, 2012; National Research Council, 2001). Results from the Rational Number Project (RNP), the National Assessment of Educational Progress (NAEP), and Strategies and Errors in Secondary Mathematics (SESM) show that most of students' misconceptions are due to learning fractions through rote memorization (as cited in Mack, 1990). For instance, it is stated in

Benchmarks for Science Literacy that many middle-school students have difficulty adding $7\frac{1}{6} + 3\frac{1}{2}$ due to not having “essential concepts about fractions and also applying memorized procedures”, which are not retrieved correctly (AAAS, 1994, p. 359). Since teachers’ mathematical knowledge appears to be an important predictor for students’ learning and understanding, “regardless of the diversity in conceptualizations, measurement tools, and methods, recent findings connecting teacher knowledge to the quality of instruction and consequently to student learning” (Charalambous & Hill, 2012, p. 446); further attention should be given to teacher preparation programs and pre-service teachers’ (PSTs’) mathematical knowledge of fractions.

PSTs’ understanding of fractions

Teacher education programs play an essential role in helping PSTs acquire a deep understanding of fractions (Borko et al., 1992; Ma, 1999; Newton, 2008; Toluk-Uçar, 2009; Zhou et al., 2006). Even though PSTs have completed K-12 education, and attended mathematics content and/or mathematics methods courses during their teacher preparation program, many of them still have fragmented understanding of fractions (Marchionda, 2006). For example, although PSTs were able to divide fractions, they were neither able to explain why the invert and multiply algorithm works (lack of specialized content knowledge, details provided in Chapter 2) nor the reasons behind students’ errors (knowledge of content and students, details given in Chapter 2) related to division of fractions (Borko et al., 1992; Tirosh, 2000). This issue has directed many educators and researchers’ attention to future teachers and teacher education programs, in which teachers are provided opportunities to gain foundational pedagogical knowledge and subject matter knowledge (Feuer, Floden, Chudowsky, & Ahn, 2013).

Since a) it is widely agreed that students should learn fractions with understanding, b) teachers need to understand a topic deeply in order to meaningfully teach the topic to their students (Shulman, 1986), and c) many teachers still have fragmented understanding of fractions (Marchionda, 2006), great emphasis should be given to elementary in-service and PSTs' understanding of fractions and fraction operations. Therefore, it is critical to explore the opportunities given to elementary PSTs to develop their knowledge of fractions, since their knowledge impacts students' learning and understanding of fractions (Fennema & Franke, 1992; Nillas, 2003).

Mathematical Knowledge for Teaching

The quality of mathematics teaching depends on teachers' mathematical knowledge (Ball, Hill, & Bass, 2005). Nevertheless, many teachers in the United States do not possess strong mathematical understanding and skill (Ball et al., 2005; Ma, 1999). This could be due to the result of being graduates of the education system, which needs to be improved (Ball et al., 2005; Hill & Lubienski, 2007).

Throughout the years, *Subject Matter Knowledge* (SMK) has been considered as the primary qualification of a teacher, and the knowledge of educational theories and teaching methods have been undervalued (Shulman, 1986). However, possessing content knowledge alone is not sufficient to teach for understanding (Shulman, 1986; Silverman & Thompson, 2008).

Teachers must not only be capable of defining for students the accepted truths in a domain. They must also be able to explain why a particular proposition is deemed

warranted, why it is worth knowing, and how it relates to other propositions, both within the discipline and without, both in theory and in practice (Shulman, 1986, p. 9).

Therefore, in order to learn to think from the perspective of the learners, Shulman proposed three categories of content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. According to Shulman, subject matter content knowledge is based on knowing basic principles and rules and defining those accepted facts to students. However, having adequate subject matter content knowledge is not enough to teach, teachers should be capable of explain why students need to know these accepted facts and how they are related to other facts. In Shulman's understanding, content knowledge and pedagogy cannot be considered mutually exclusive. Therefore, Shulman developed *Pedagogical Content Knowledge* (PCK), which is a combination of content knowledge (teachers' content knowledge) and pedagogical knowledge (knowledge of instructional methods). In other words, pedagogical knowledge states what teachers know about teaching, whereas content knowledge shows what teachers know about what they teach. Thus, pedagogical content knowledge refers to teaching the subject with appropriate teaching techniques (using proper teaching models, creating an effective learning environment based on student's prior knowledge, knowing common students' misconceptions, etc.) to make it comprehensible and clear to students. Therefore, Shulman suggested that pedagogical content knowledge should be in teacher education programs at all levels.

The most useful forms of representing of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations- in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others... Pedagogical content knowledge also includes an understanding of what makes the

learning of specific topic easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (Shulman, 1986, p. 9).

The last type of knowledge, curricular knowledge, refers to knowledge of programs designed for teaching particular subjects and topics with the help of using appropriate instructional materials.

Shulman's (1986) categories of teacher content knowledge shed light into variety of studies that focus on mathematics and many other content areas. Ball, Thames, and Phelps (2008) presented a new term, *Mathematical Knowledge for Teaching* (MKT), using Shulman's work as a framework. Ball and her colleagues developed MKT, building on Shulman's categories of teacher content knowledge, and it combines SMK and PCK. In the new developed framework, SMK was divided into three categories: *Common Content Knowledge*, *Horizon Content Knowledge*, *Specialized Content Knowledge*. Common Content Knowledge (CCK) is the mathematical knowledge that every well-educated adult (teachers and nonteachers alike) should possess. For instance, a teacher should be able to find equivalent fractions, but also engineers, architects, bakers, etc. need to have that knowledge to pursue their job. Horizon Content Knowledge (HCK) is the "mathematical peripheral vision needed in teaching" (Hill, Rowan, & Ball, 2005, p. 70). For instance, a teacher should have HCK to be able to response whether the student is right or his statements mathematically significant when he claims that the quotient is smaller than dividend when a whole number is divided by a fraction. And, Specialized Content Knowledge (SCK) is the mathematical knowledge that not every well-educated adult should possess but is specifically required for teaching. In this case, an architect might be able to find equivalent fractions, and do not need to know and explain the reasoning behind the concept;

however, a teacher should be able to explain the reasoning underlying of finding equivalent fractions.

Further, Ball and her colleagues divided PCK into three categories: *Knowledge of Content and Students*, *Knowledge of Content and Teaching*, and *Knowledge of Content and Curriculum*. Knowledge of Content and Students (KCS) allows teachers to be able to anticipate students' thinking and misunderstandings. A teacher with KCS identifies the common misconceptions or errors that students have when dividing fractions. Knowledge of Content and Teaching (KCT) is considered as a combination of knowing about teaching and mathematics, and addresses knowing proper representations to teach mathematics. For instance, a teacher needs to be able to decide which representations she is going to use when introducing division of fraction concepts (Ball, 2016). Finally, Knowledge of Content and Curriculum (KCC) is knowledge of curriculum programs. For instance, a teacher should be able to know at what grade level they introduce division of fractions to students. In general, each domain of MKT (CCK, SCK, HCK, KCS, KCT, KCC) is critical for teachers to be able to teach mathematics effectively. In following section, I provide the importance of mathematics methods courses offered to PSTs to develop their MKT while they are in teacher preparation programs.

Mathematics Methods Courses

Many have argued that there is a link between K-6 students' obstacles to learn mathematics and their teachers' inadequate MKT (Beckmann et al., 2004; Burton, Daane, & Giesen, 2008; Clarke & Clarke, 2004). To improve elementary school mathematics education, developing PSTs' understanding of the mathematics they will teach and of how to teach mathematics effectively have become the main focus (Burton et al., 2008). According to the

standards recommended by the National Council on Teacher Quality (NCTQ), “appropriate and adequate content coursework, good pedagogy instruction, and coordination between mathematics content and mathematics methods courses are vital to good teacher preparation which is urgently needed” (Greenberg & Walsh, 2008, p. 4).

The mathematics courses offered to PSTs are determined by program faculty in colleges and universities (Rakes, 2015). Ball, Sleep, Boerst and Bass (2009) pointed out an important problem in mathematics education: “no shared professional curriculum to prepare teachers to teach mathematics” (p. 459). Ball and her colleagues stated that the lack of shared curriculum is problematic for both PSTs and instructors. Further some of the elementary mathematics teacher preparation programs, such as the one at the University of South Florida (USF) offer only mathematics methods courses for elementary PSTs. As it is stated in *The Standards for Preparing Teachers of Mathematics* (SPTM), “mathematics methods courses in effective mathematics teacher preparation programs present multiple opportunities for candidates to reconsider and deepen their mathematical understandings as both learners of mathematics and as mathematics teachers (AMTE, 2017, p. 33). In these courses, PSTs are offered with mathematics content and best practices for teaching mathematics effectively to elementary students. Therefore, mathematics methods courses provide the primary context where PSTs develop *Mathematical Knowledge for Teaching Fractions* (MKTF), along with other mathematics topics taught at the elementary school.

According to Hill and Lubienski (2007), “most K-8 teachers take mathematics content and methods course work as part of their preparation, yet the curricula of these courses are highly variable across institutions” (p. 765). Taylor and Ronau (2006) analyzed 58 mathematics methods course syllabi that were shared by the Association of Mathematics Teacher Educators

(AMTE) members. The analysis of these syllabi revealed that the objectives of the methods courses centered on improving PSTs' competence on MKT (content knowledge, pedagogical knowledge and PCK). Since textbooks are considered as a primary course components for the teaching and learning mathematics in higher education (Mesa & Griffiths, 2012), it is critical to explore to what extent the textbooks used in mathematics methods courses meet the course objectives, which is improving PSTs' MKTF.

The Importance of Textbooks

Charalambous and Hill (2012) argued that textbooks shape learning by impacting what is taught and how it is taught in classrooms even though teachers and instructors do not follow it verbatim. Further, "textbooks remain a ubiquitous course component with various implications for teaching and learning of mathematics at a tertiary level" (Mesa & Griffiths, 2012, p. 86). Although, there are studies that focus on mathematics content textbooks at the university level (McCrary & Stylianides, 2014; Mesa & Griffiths, 2012) less is known about the textbooks used in mathematics methods courses. Ball (1990b) has described methods courses as "mainstay of traditional teacher education programs. Prospective (pre-service) teachers typically look forward to them because they expect that they will learn how to teach specific things" (p. 12). Many PSTs have high expectations from mathematics methods courses, where is especially they are given opportunities to learn how to teach fractions and how students understand fractions. And, textbooks are used as a teaching material by many mathematics methods course instructors (Harkness & Brass, 2017), which gives textbooks a crucial role in developing PSTs' MKTF. Therefore, I conducted this study that adds to the growing body of literature in which PSTs' fraction understanding is the main attention by exploring opportunities provided in mathematics methods textbooks to develop the MKTF.

Statement of the Problem

It has been seen that not only students but also teachers have difficulty understanding of fraction concepts (Wu, 2001; Zhou et al., 2006). PSTs who have not been able to improve their subject matter knowledge by the time they get into teacher preparation programs; mathematics content and/or mathematics methods courses can be considered as a great but last formal opportunity for them to improve, since “teachers who do not acquire mathematical competency during schooling are unlikely to have another opportunity to acquire it” (Zhou et al., 2006, p.450).

Ball proposed that when preparing PSTs we need to identify the important content knowledge needed for teaching, how that knowledge needs to be understood, and how that knowledge is actually learnt in the classroom (as cited in Marshman & Porter, 2013). The findings of Hill et al.'s (2005) study supported that teachers with strong MKT create high quality of mathematical instruction.

Teachers with stronger MKT made fewer mathematical errors, responded more appropriately to students, and chose examples that helped students construct meaning of the targeted concepts and processes; teachers with weaker MKT were not successful at selecting and sequencing examples, presenting and elaborating upon textbook definitions, and using representations. (Charalambous & Hill, 2012, p. 447)

It has been shown by several researchers that there is a correlation between curricula and students' learning, since some teachers use textbooks as a main resource for planning instruction and for structuring the course (Reys, Reys, & Chavez, 2004). According to Begle (1973), the textbook is “ the only variable that on the one hand we can manipulate and on the other hand does affect student learning” (p. 209). More particularly, novice teachers rely on their textbooks

more frequently due to their absence of self-confidence to design their own lesson (Ball & Feiman-Nemser, 1988). According to Robitaille and Travers's (1992):

Teachers of mathematics in all countries rely heavily on textbooks in their day-to-day teaching, and this is perhaps more characteristic of the teaching of mathematics than of any other subject in the curriculum. Teachers decide what to teach, how to teach it, and what sorts of exercises to assign to their students largely on the basis of what is contained in the textbook authorized for their course (p. 706).

The role of mathematics methods courses is essential, since they give creative ideas about how to teach math and how students learn it (Ball, 1990b). Based on the Conference Board of the Mathematical Sciences' (CBMS) recommendations in the Mathematical Education of Teachers II (MET II, 2012), the number of semester-hours of mathematics courses that PSTs need to take was increased to 12 from 9, which was the original MET Report's recommendation in 2001. However, many post-secondary institutions require only two elementary education mathematics courses (Marchionda, 2006). The required number of courses for PSTs may vary from one institution to another, but certain mathematics content standards, such as those addressed in NCTM's Principles and Standards for School Mathematics (2000) and CCSS, should be covered in the elementary mathematics teacher education curriculum. The mathematics content standards are common in NCTM and CCSSM as follows: numbers and operations, geometry, measurement and data analysis, and probability; and many publishers claim that their elementary mathematics textbooks are aligned with these standards. Fractions are a very important topic taught at the elementary school under the "numbers and operations" standard and, as such, textbooks used in teacher preparation programs address fractional concepts and operations with fractions. However, little is known about the opportunities offered in mathematics methods textbooks for

PSTs to improve MKTF. Therefore, I employed qualitative content analysis to shed light into the ways PSTs are given chances to obtain MKTF through mathematics methods textbooks.

Research Questions

In order to analyze opportunities provided for PSTs to develop MKTF by the elementary mathematics methods textbooks, I used the following research questions:

- 1) To what extent and in what ways do textbooks used in mathematics methods courses in teacher preparation programs provide PSTs opportunities to develop Mathematical Knowledge for Teaching Fractions?
 - a. What opportunities are provided in the textbooks for PSTs to develop subject matter knowledge for fractions?
 - i. What opportunities are provided in the textbooks for PSTs to develop common content knowledge for fractions?
 - ii. What opportunities are provided in the textbooks for PSTs to develop horizon content knowledge for fractions?
 - iii. What opportunities are provided in the textbooks for PSTs to develop specialized content knowledge for fractions?
 - b. What opportunities are provided in the textbooks for PSTs to develop pedagogical content knowledge for fractions?
 - i. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and students?
 - ii. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and teaching?

- iii. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and curriculum?

Purpose of the Study

Proficiency with fractions is considered as a major goal for K-8 mathematics education since “such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped” (NMAP, 2008, p. xvii). However, fractions are described as “the most protracted in terms of development, the most difficult to teach, the most mathematically complex, the most cognitively challenging, the most essential to success in higher mathematics and science, and one of the most compelling research sites” (Lamon, 2007, p. 629). Thus, it is not surprising for students to have difficulties in understanding fraction concepts.

Charalambous and Hill (2012) pointed two crucial instructional resources: teacher knowledge and curriculum materials. And, in another study Hill and Charalambous (2012) stated teachers, who have weak MKT, can implement high-quality instruction if they are provided with supportive curriculum. For this study, I only focused on textbooks, as they are an important aspect of written curriculum. Additionally, it has been shown by studies that teachers with strong MKT has a greater impact on students’ learning gains (Hill et al., 2005). Thus, teacher preparation programs can be considered as a threshold for PSTs to improve their MKT before they get into the field of teaching. Because of that, I aimed to explore to what extent textbooks used in mathematics methods courses provide opportunities for PSTs to develop MKTF.

Significance of the Study

The Number and Operation standard “describes deep and fundamental understanding of, and proficiency with, counting, numbers, and arithmetic, as well as an understanding of number

systems and their structures” (NCTM, 2000, p. 32). Students start building fractional understanding from early grades and continue to build on it from then on. “While the focus of the Number and Operation standard, including rational numbers, occurs within the earlier grades, it is important to note that NCTM’s standards are interconnected” (Marchionda, 2006, p.12). Thus having strong understanding of rational numbers is fundamental, and needed by other areas of mathematics as well (NCTM, 2000).

Despite the importance of fractions, many students and teachers (in-service and pre-service) have difficulties with fractional concepts. Researchers have noted that PSTs’ poor understanding of fractional concepts is due to their own K-12 educational experiences that have emphasized on rote-memorization (CBMS, 2001, 2012; Wu, 2001). As stated in MET II, “many PSTs, learning mathematics has meant only learning its procedures and, they may, in fact, have been rewarded with high grades in mathematics for their fluency in using procedures” (CBMS, 2012, p. 11). Thus, this issue puts a greater emphasis on mathematics courses in teacher preparation programs, in which PSTs should be given opportunities to relearn mathematical concepts with understanding.

Textbooks are commonly used curriculum material in methods courses (Harkness & Brass, 2017). In order to foster PSTs’ understanding of fractions, teacher preparation programs can help future teachers strengthen their knowledge by providing mathematics methods textbooks that are conducive to doing so. Furthermore, insight into the ways in which mathematics methods textbooks provide PSTs with opportunities to develop MKTF can help guide faculty/instructors in teacher education programs as well as curriculum developers in regard to the selection and implementation of mathematics methods textbooks.

Researcher's Background

In addition to the significance of fraction concepts and teacher education programs in PSTs' knowledge, I have had personal reasons to pursue this inquiry. I taught *Intermediate Algebra* at the University of South Florida for two semesters, where undergraduate students from different fields were given opportunities to develop their algebraic knowledge needed in their studies. And I realized that many students have had weak understanding of fractions when we started working on operations with algebraic fractions. Later, I started teaching mathematics methods courses. At the University of South Florida, mathematics methods courses are given in two semesters and titled as *Teaching Elementary School (K-6) Mathematics I* and *Teaching Elementary School (K-6) Mathematics II*. The purposes of these courses are to give PSTs the opportunities to develop knowledge and skills that will be needed as a teacher of mathematics in elementary schools. Specifically, *Teaching Elementary School (K-6) Mathematics I* course covers the meaning of doing mathematics, problem-based learning, guides for equitable mathematics teaching, and exploring number sense and operations. *Teaching Elementary School (K-6) Mathematics II* course's main purpose is to improve fraction, decimal and ratio understanding, as well as to develop the concepts of geometry, measurement, data, statistics and probability. I taught *Teaching Elementary School (K-6) Mathematics II* course for 4 semesters and *Teaching Elementary School (K-6) Mathematics I* course for 3 semesters. And every semester I taught two sections of classes, which gave me a great chance to work with many PSTs and to cover fraction concepts with them over the years. During this time, I observed and assessed PSTs' understanding of fraction concepts in different ways and had many opportunities to have class discussions about the topic. Based on their feedback, I found that many PSTs I have worked with were introduced and learned fractions through rote memorization. For instance, they

were giving examples of “butterfly method” to divide fractions, but never thought about the reasoning behind this method. Thus, I created the course materials in a way that support PSTs’ critical thinking on “hows” and “whys”, and I relied on the course textbook for choosing fractional tasks to be completed in classes. Also, I observed my colleagues who had taught the same lesson, and witnessed the value was given the mathematics methods textbooks. We have used different editions of *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle, Karp and Bay-Williams as our primary course textbook. As previously mentioned, the teacher preparation programs at the University of South Florida, like many others, offer only two mathematics methods courses, therefore methods courses and methods textbooks play very critical role in supporting/ improving PSTs’ knowledge. Therefore, this inquiry grows from my experiences with PSTs to understand what sort of opportunities were given in the methods textbooks, since the textbooks were commonly used curriculum materials for my courses over the years.

Definitions of Terms

Content Knowledge: The amount and organization of knowledge per se in the mind of the teacher (Shulman, 1986, p.9).

Mathematical Knowledge for Teaching (MKT): Mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students (Ball et al., 2008, p.399). A kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations, such as engineering, accounting, or carpentry (Ball et al., 2005, p. 17).

Mathematical Task: A classroom activity, the purpose of which is to focus students’ attention on a particular idea (Stein, Grover, & Henningsen, 1996; p. 460).

Methods Courses: Teacher education courses that focus on the teaching of specific content areas

in elementary grades (Smitherman, 2006, p. 1).

Mathematics Methods Courses: are specific types of courses that are neither mathematics courses nor general methods courses but instead lie at the intersection and focus on pedagogy associated with teaching mathematics (Association of Mathematics Teacher Educators [AMTE], 2017, p. 33).

Pedagogical Content Knowledge: The ways of representing and formulating the subject that make it comprehensible to others (Shulman, 1986, p. 9).

Teacher preparation programs (TPPs): are where prospective teachers gain a foundation of knowledge about pedagogy and subject matter, as well as early exposure to practical classroom experience (Feuer et al., 2013).

Teaching: Everything that teachers must do to support the learning of their students, such as planning for the lesson, evaluating students' work, writing and grading assessments, etc. (Ball et al., 2008, p. 395).

Qualitative Content Analysis: An approach of empirical, methodological controlled analysis of texts within their context of communication, following content analytic rules and step-by-step models, without rash quantification (Mayring, 2000, p. 2).

Delimitations

Although the study aims to add to the growing body of the literature, there are some delimitations as well. First, I merely focused on analyzing mathematics methods course textbooks. However, there are many PSTs who also take mathematics content courses in addition to methods courses. Thus, the opportunities offered in the content course textbooks to improve MKTF could be different than opportunities provided in the methods textbooks used in the study. However, this delimitation might provide a foundation for later work that focuses on content

textbooks as well. Second, I applied qualitative content analysis using both inductive and deductive approach. Therefore, another researcher might explore, interpret and describe the data differently due to having different point of view, background, understanding, and so forth. Accordingly, a different coding frame could be developed and that might have an impact on findings in different ways. Third, I selected textbooks in the study based on a survey that was completed by other researchers in 2013. Therefore, the sample of textbooks could be different if the data were collected recently. Finally, the participants of the survey (mathematics methods instructors) provided the name of textbooks they used in their methods courses were members of AMTE (one professional organization in the United States to improve mathematics teacher education). Although, the survey was sent to all AMTE members (over 900 at that time), the return rate to the survey was low. As known, the survey's response rate is one of the important indicators of survey quality, therefore, the survey outcomes could be different with higher response rate.

Summary

Professional Standards for Teaching Mathematics (NCTM, 1991) supported that PSTs need to be equipped with appropriate subject knowledge and pedagogical content knowledge while they are being educated in teacher education program. Additionally, Hill and Charalambous (2012) showed in their study that MKT (SMK and PCK) greatly influences the quality of instruction. There have been many studies focusing on in-service and PSTs' knowledge about fractions. The results of these studies have shown that elementary teachers in the U.S. (in-service and pre-service) have impoverished knowledge of fractions (Ball, 1990a; Zhou et al., 2006). Ball (1990a) revealed that PSTs' knowledge about mathematics is mainly focused on remembering rules and using standard procedure. And, more than two decades after

Ball's work, Utley and Reeder (2012) found that PSTs start their elementary education program with a weak understanding of fractions. And, carrying these problems into their professional teaching careers would become unavoidable (Huang, Liu, & Lin, 2009).

Research has shown that “how content is presented in textbooks is how it will likely be taught in the classroom” (Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002, p. 125); thus it is crucial to examine the mathematics methods textbooks in order to understand to what extent these textbooks provide opportunities to learn MKTF. This study provides an insight of the mathematics methods textbooks (written curriculum) and the opportunities provided in them to develop PSTs' MKTF. The results of this study could guide curriculum developers to make necessary changes, if needed, in any textbooks to ensure that there are opportunities to develop MKT; as well as methods course instructors to select the textbook that meets the objectives of their courses and the needs of their students.

CHAPTER 2: LITERATURE REVIEW

The purpose of this study is to explore the opportunities for elementary pre-service teachers (PSTs) to develop *Mathematical Knowledge of Teaching Fractions* (MKTF) through mathematics methods textbooks. In the first chapter, I provided the importance and purpose of the inquiry, and research questions. In this chapter, I discuss the literature that guides my study. First of all, I describe what fractions are, the importance of fractions, meanings of fractions, and fractions in CCSSM. Later, I discuss why fractions are fundamental for students' mathematical success and daily life. Following this section, I discuss the teacher knowledge, theoretical considerations on MKT, and PSTs' knowledge of fractions. Next, I present the types of curriculum, and the role and use of textbooks. Finally, I conclude the chapter with a brief summary.

To identify the literature for synthesis, I searched the following databases for relevant articles and books: Google Scholar, ERIC, the University of South Florida's library databases, ProQuest Dissertations and Theses, and Journal Storage (JSTOR). And, I used the following subject headings and key terms to find related articles for inclusion: PSTs' knowledge of fractions, learning fractions, fraction concepts, *Mathematical Knowledge for Teaching* (MKT), mathematics methods courses, mathematics textbooks in higher education, and textbook analysis. Additionally, I reviewed the reference lists of selected articles and books. I intentionally sought for the most recent literature, however some older but seminal studies included to shed light into literature review.

Fractions

The term “fractions” comes from the Latin word “fractio,” which means to break. Around 1800 BC, Babylonians and Egyptians were the first two cultures that used fractions in order to answer how many and how much quantities they have (Chapin & Johnson, 2006). They used fractions when they had to express amounts, which are bigger than zero but less than one, e.g., dividing an acre of land between two people, or sharing three apples among four children.

However, their computational practices were limited with unit fractions, which are fractions of the form $\frac{1}{n}$, where n is natural number (Burton, 2011). Thus, for instance, when they want to

express $\frac{2}{7}$, they used multiple unit fractions with different denominators $\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$. The use of

fractions became much more common after the Renaissance, and its form has been changed from

$\frac{1}{n}$ to $\frac{a}{b}$, where a is called the numerator (from the Latin word “number”) and b is called the

denominator (from the Latin word “namer”). And in the CCSSM, fraction $\frac{a}{b}$ is described as a part of equal size $\frac{1}{b}$.

Meanings of Fractions

Even though fractions were invented to make people’s lives easier, they become one of the most challenging topics in most students and teachers’ lives year after year. The one important reason that makes fractions important as well as complicated is the variety of interpretations/meanings of fractions in mathematics curricula (Chapin & Johnson, 2006; Lamon, 2001; Van de Walle, Karp, & Bay-Williams, 2019). Chapin and Johnson (2006) listed five main

constructs/interpretations for fractions: “part-whole: fractions as parts of wholes or parts of sets; division: fractions as the result of dividing two numbers; ratio: fractions as the ratio of two quantities; operator: fractions as operators; and measurement: fractions as measures” (p. 100). And many researchers and educators believe that students could have better understanding of fractions when they are provided opportunities to place greater emphasis across all meanings of fractions (Clarke, Roche, & Mitchell, 2008; Siebert & Gaskin, 2006; Van de Walle et al., 2019). Furthermore, it is suggested by Number and Operation Standard for grades 3-5, students need to advance all of these meanings for fractions (NCTM, 2000).

As a first interpretation, the part-whole construct is considered an important foundation for developing understanding of fractions. In the part-whole construct, the denominator shows the total number of equal parts in the whole whereas the numerator represents how many parts of that whole are being included. With this interpretation, the fraction $\frac{2}{5}$ means two out of five equal parts. Part-whole construct can be applied to different contexts such as quantities, length, volume, time, or region. As a next interpretation of fractions, fraction can be seen as a division. For instance, the fraction $\frac{2}{5}$ can be considered as dividing two cakes among five people, and each person get two-fifth ($\frac{2}{5}$) of the cake. Fractions may also be used as a ratio, which compares any two quantities or measures of the same type. There are two types of ratios, which compare measures. First, “part–part comparisons compare the measure of part of a set to the measure of another part of the set” (Lamon, 2012, p. 225). For instance, a bag of marbles that contain two red marbles and five blue marbles, the ratio $\frac{2}{5}$ (2 to 5) could be the ratio of red marbles to blue

marbles (red to blue, part-part comparison). Second, “part-whole comparisons are ratios that compare the measure of part of a set to the measure of the whole set” (Lamon, 2012, p. 225). In this comparison, the ratio $\frac{2}{7}$ (2 to 7) could be the ratio of red marbles to all of the marbles in the bag (red to whole, part- whole comparison), or the ratio $\frac{5}{7}$ (5 to 7) could be the ratio of blue marbles to whole marbles in the bag (blue to whole, part- whole comparison). Furthermore, fractions can be used as operators, in other words fraction plays functions, which serve as a function to carry out a process (Lamon, 2012). For instance, in order to find $\frac{2}{5}$ of something, students may perform different type of operations; e.g., dividing by 5, then multiplying by 2; or multiplying by 2, then dividing by 5. As a last construct/interpretation, fractions can be used to indicate a measure. In order to use measurement construct in a one-dimensional space, one needs to determine a distance from zero to particular point on a number line to represent a fraction. However, in a two-dimensional space, a fraction measures the area (Lamon, 2012).

Moreover, Battista (2012) identified three critical components of understanding fractions: partitioning, fractions (symbol), and iteration. First, students must understand partitioning in order to understand fractions in depth, however it is also suggested that if students are able to partition, that does not mean they fully understand fractions (Battista, 2012). For example, a student might be able to partition a pie equally among four people, but not make the connection between the pieces (parts) and the whole. Second, students also need to be able to understand how fractional quantities are shown mathematically. As a general representation of a fraction $\frac{a}{b}$, students should understand the denominator (b) represents how many equal-sized parts are in the

whole, while the numerator (a) symbolizes the number of these equal-sized parts used to represent the fractional quantity by $\frac{a}{b}$. Finally, as opposed to partitioning, which starts with the whole and divides it into equal-sized parts, iteration starts with a part and repeat that part number of times to find how many parts in the whole. For instance, students need to be able to think $\frac{2}{5}$ as *fifths* is counted by two times.

According to Battista (2012), understanding the relationship between the process of partitioning and iteration may have a positive impact on students' substantive understanding of fractions. For instance, it is stated that iteration could be used to understand unit fractions and gave one visual representation to show that one shaded rhombus could be iterated 4 times to make the whole rhombus, then the shaded rhombus became $\frac{1}{4}$ of the whole shape. Furthermore, the author stated that once students internalize unit fractions, iteration again could be used to understand non-unit fractions. Then, whenever students understand proper fractions using iteration, this could be appropriate time for them to move on understanding improper fractions or mixed numbers. In the next section, I provide when and which fraction concepts introduced in elementary grades according to the CCSSM.

Fractions in the CCSSM

In 3rd grade, the CCSSM introduces fractions first time formally under the “Number and Operations- Fractions” category by building on students' previous informal practices for instance cutting a cake equally among five friends (CCSSI, 2010). And students continue building their fraction knowledge formally under the Number and Operations-Fraction standards through 5th

grade and under the Number System in 6th grade. In Table 1, I present the main standards for each grade.

Table 1. Fractions Standards under the CCSSM

Grade Level	Standards
3	<i>Develop understanding of fractions as numbers.</i> - CCSS.MATH.CONTENT.3.NF.A.1 Understand a fraction $1/b$ as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size $1/b$. - CCSS.MATH.CONTENT.3.NF.A.2 Understand a fraction as a number on the number line; represent fractions on a number line diagram. - CCSS.MATH.CONTENT.3.NF.A.3 Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.
4	<i>Extend understanding of fraction equivalence and ordering.</i> - CCSS.MATH.CONTENT.4.NF.A.1 Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions. - CCSS.MATH.CONTENT.4.NF.A.2 Compare two fractions with different numerators and different denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as $1/2$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model. <i>Build fractions from unit fractions.</i> - CCSS.MATH.CONTENT.4.NF.B.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$. - CCSS.MATH.CONTENT.4.NF.B.4 Apply and extend previous understandings of multiplication to multiply a fraction by a whole number. <i>Understand decimal notation for fractions, and compare decimal fractions.</i> - CCSS.MATH.CONTENT.4.NF.C.5 Express a fraction with denominator 10 as an equivalent fraction with denominator 100, and use this technique to add two fractions with respective denominators 10 and 100.² For example, express $3/10$ as $30/100$, and add $3/10 + 4/100 = 34/100$.

Table 1. (Continued)

Grade Level	Standards
	• CCSS.MATH.CONTENT.4.NF.C.6 Use decimal notation for fractions with denominators 10 or 100. <i>For example, rewrite 0.62 as 62/100; describe a length as 0.62 meters; locate 0.62 on a number line diagram.</i> • CCSS.MATH.CONTENT.4.NF.C.7 Compare two decimals to hundredths by reasoning about their size. Recognize that comparisons are valid only when the two decimals refer to the same whole. Record the results of comparisons with the symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual model.
5 Number and Operations- Fractions	<i>Use equivalent fractions as a strategy to add and subtract fractions.</i> • CCSS.MATH.CONTENT.5.NF.A.1 Add and subtract fractions with unlike denominators (including mixed numbers) by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators. For example, $\frac{2}{3} + \frac{5}{4} = \frac{8}{12} + \frac{15}{12} = \frac{23}{12}$. (In general, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.) • CCSS.MATH.CONTENT.5.NF.A.2 Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g., by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. For example, recognize an incorrect result $\frac{2}{5} + \frac{1}{2} = \frac{3}{7}$, by observing that $\frac{3}{7} < \frac{1}{2}$. <i>Apply and extend previous understandings of multiplication and division.</i> • CCSS.MATH.CONTENT.5.NF.B.3 Interpret a fraction as division of the numerator by the denominator ($\frac{a}{b} = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers, e.g., by using visual fraction models or equations to represent the problem. • CCSS.MATH.CONTENT.5.NF.B.4 Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction. • CCSS.MATH.CONTENT.5.NF.B.5 Interpret multiplication as scaling (resizing) • CCSS.MATH.CONTENT.5.NF.B.6 Solve real world problems involving multiplication of fractions and mixed numbers, e.g., by using visual fraction models or equations to represent the problem. • CCSS.MATH.CONTENT.5.NF.B.7 Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

Table 1. (Continued)

Grade Level	Standards
6	<i>Apply and extend previous understandings of multiplication and division to divide fractions by fractions.</i>
The Number System	- CCSS.MATH.CONTENT.6.NS.A.1 Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In general, $(a/b) \div (c/d) = ad/bc$.) How much chocolate will each person get if 3 people share $1/2$ lb of chocolate equally? How many $3/4$-cup servings are in $2/3$ of a cup of yogurt? How wide is a rectangular strip of land with length $3/4$ mi and area $1/2$ square mi? <i>Compute fluently with multi-digit numbers and find common factors and multiples.</i> - CCSS.MATH.CONTENT.6.NS.B.3 Fluently add, subtract, multiply, and divide multi-digit decimals using the standard algorithm for each operation.

Note. Adapted from “Common Core State Standards Initiative, Number and Operations-Fractions and The Number System” (n.d.). Retrieved from <http://www.corestandards.org/Math/Content/NF/>

According to the CCSSM, students are expected to develop an understanding of a fraction as a number in 3rd grade. In grade 4, students begin exploring decimals by changing a fraction with denominator 10 into a fraction with denominator 100. Later than, they start using decimal notation for fractions with denominator 10 and 100 and compare these decimals. In 6th grade, students are expected to complete four operations with multi-digit decimals. Although, students start building decimal understanding in elementary school, introducing conversion of fractions to decimals before students have mastered in conceptual understanding of fractions is considered as the greatest mistake done by fifth and sixth grade teachers (Collins & Dacey, 2010 as cited in Neagoy, 2017). According to Wu (2001), “decimals are merely a shorthand notation for special type of fractions” therefore “the understanding of decimals is founded on an

understanding of fractions” (p.7). Thus, in this study I merely focused on fraction concepts and the opportunities provided for PSTs to develop MKTF.

Importance of Fractions for Students’ Success

Proficiency with fractions is essential for students to be successful in mathematics and also in daily lives. There are several reasons why fractions play a key role in the mathematical life of a student. First, fractions are considered as “the first mathematical stumbling block” (Neagoy, 2017, p.2). Because, students begin having negative feelings toward mathematics when they stop making sense of fractions and rely on rote memorization. Second, fractions are not just essential for school mathematics but also for everyday life. Although students need fractions to build understanding of complex mathematical topics, e.g., rates, ratios, percent, slope etc. (Son, 2011), they need fluent fraction understanding for their everyday life to complete such tasks: to calculate discounts, follow recipes, compare rates, etc. (Neagoy, 2017). Next, fractions are fundamental to be successful in algebra (Brown & Quinn, 2007; Neagoy, 2017; NMAP, 2008; Wu, 2001). NMAP’s (2008) final report, *Foundations for Success*, indicated U.S. students’ poor proficiency with fractions result in failure in algebra. Thus, it is hard to accomplish the goal of “algebra for all” without “fractions for all” (Neagoy, 2017, p.2).

Teacher Knowledge of Mathematics

There might be a variety of different factors that have impact on the teaching of mathematics, but teachers’ mathematical knowledge has been seen as one of the most important influences on teaching process for decades (Fennema & Franke, 1992; Shulman, 1986; Stylianides & Ball, 2004). In 1986, Shulman defined three types of content knowledge: *Subject Matter Content Knowledge*, *Pedagogical Content Knowledge*, and *Curricular Knowledge*.

Content knowledge is about knowing the subject, which involves knowing the structure of the subject. For instance, a mathematics teacher needs to know set of rules (e.g., multiply by the reciprocal when dividing divisions), and also must be able to explain why those particular rules work. Pedagogical content knowledge includes “the ways of representing and formulating the subject that make it comprehensible to others”; besides it contains “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (Shulman, 1986, p. 9). Shulman stressed the third kind of content knowledge, curricular knowledge, as important as pedagogical knowledge and stated:

The curriculum is represented the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to those programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances. (Shulman, 1986, p. 10)

Additionally, Shulman pointed to two additional aspects of curricular knowledge, lateral curriculum knowledge and vertical curriculum knowledge. Lateral knowledge underlies the teachers’ ability to relate the curriculum being taught to the curriculum that students learn in other classes (in other subjects). Vertical knowledge refers to “familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school, and the materials that embody them” (Shulman, 1986, p. 10).

Several years later, Fennema and Franke (1992) built on the work of Shulman by suggesting that teachers’ knowledge consists of four components: knowledge of mathematics, knowledge of mathematical representations, teachers’ knowledge of students, and general

knowledge of teaching and decision-making. In brief, knowledge of mathematics is fundamental for a teacher who helps students to learn mathematics. Fennema and Franke stated the importance of knowing mathematics representations in instruction, and mentioned it is not very different than content knowledge since one needs to take content knowledge and translate it into representations that students can make sense of it. Moreover, they added that mathematics consists of highly related abstractions, and “if teachers do not know how to translate those abstractions into a form that enables learners to relate the mathematics to what they already know, they will not learn with understanding” (Fennema & Franke, 1992, p. 153). Next, teachers’ knowledge of students has been seen as another important components of teacher knowledge, since it is useful for teachers to understand how learners acquire knowledge and to structure the learning environment for individuals. The last component of teacher knowledge is general knowledge of teaching and decision-making. What students learn in classrooms is mostly determined by teachers’ decisions, which are highly affected by teachers’ beliefs, thoughts, judgments, and knowledge (Fennema & Franke, 1992). Knowledge of mathematics and knowledge of representations are connected to content knowledge, whereas knowledge of students and knowledge of teaching are linked to pedagogical content knowledge.

Over the years, many researchers have proposed different categories of teachers’ knowledge in the literature. And, some of these categories of teachers’ knowledge might probably not be right, but that’s not very important; what is important is that teachers’ knowledge for teaching mathematics is multidimensional (Hill, Schilling, & Ball, 2004), and categories of teacher knowledge need to be revised and improved (Ball et al., 2008). Likewise, many researchers revised the dimensions of teacher knowledge, but have made a limited progress on developing a theoretical framework for content knowledge for teaching (Ball et al., 2008).

With this need, Ball and her colleagues developed a term “MKT”, which means “the mathematical knowledge needed to carry out the work of teaching mathematics” (Ball et al., 2008, p. 395). I discuss MKT under the theoretical considerations section in more detail.

Theoretical Considerations on MKT

The term *Mathematical Knowledge for Teaching* (MKT) developed by Ball et al. (2008), and describes the knowledge that needed by teachers to complete tasks for teaching mathematics to students. Ball and her colleagues stressed that they elaborate and built on Shulman’s (1986) two initial categories of knowledge: *Subject Matter Knowledge* (SMK) and *Pedagogical Content Knowledge* (PCK). They divided SMK into three categories: *Common Content Knowledge* (CCK), *Specialized Content Knowledge* (SCK), and *Horizon Content Knowledge* (HCK). Similarly, PCK consists of three dimensions as follows: *Knowledge of Content and Students* (KCS), *Knowledge of Content and Teaching* (KCT) and *Knowledge of Content and Curriculum* (KCC). Figure 1 represents all dimensions of MKT. The following instance explains the way Ball and her colleagues used Shulman’s work as a foundation. Their description of KCT and KCS overlap with Shulman’s two dimensions of pedagogical content knowledge: “the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” and “the ways of representing and formulating the subject that make it comprehensible to others” (Shulman, 1986, p. 9).

MATHEMATICAL KNOWLEDGE FOR TEACHING

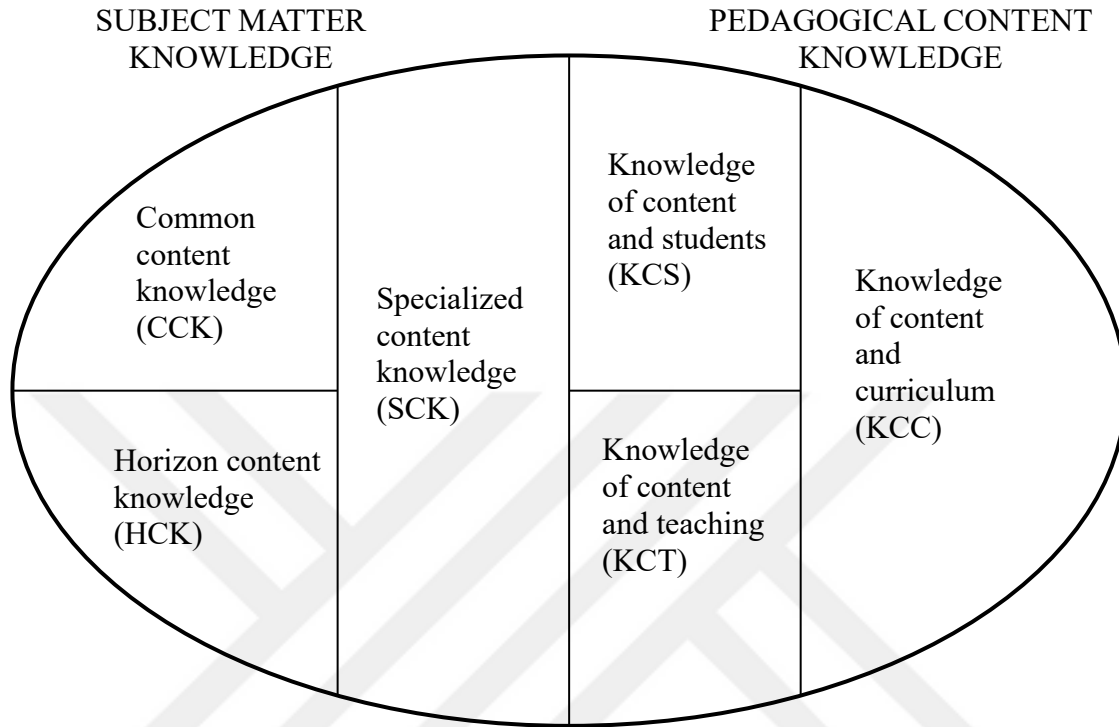


Figure 1. Domains of Mathematical Knowledge for Teaching (MKT)

Note. From “Content Knowledge for Teaching: What Makes It Special?” by D. L. Ball, H. M. Thames, and G. Phelps, 2008, *Journal of Teacher Education*, 59, p. 403. Copyright 2008 by Sage Publications. Reprinted with permission.

The first domain of the SMK is CCK. Although the domain includes “common” term, it is not expected for everyone to have this knowledge. Ball et al. (2008) defined it as knowledge and skill that are not unique to teaching. For instance, being able to order fractions from least to greatest is very essential but does not require specialized understanding and can be completed by any others who know and use mathematics. Ball and her colleagues stated that teachers need CCK to be able to complete the work that they are assigning their students or recognize incorrect answers.

The second domain, HCK, is newer to the theory of MKT. HCK is considered to be less developed and has not reached clarity, since it has not received much attention unlike other sub-domains of MKT (Jakobsen, Thames & Ribeiro, 2013; Mosvold & Fauskanger, 2014;

Wasserman & Stockton, 2013). Ball et al. described it as “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (p. 403), and Hill and Ball (2009) defined it as “a kind of mathematical ‘peripheral vision’ needed in teaching, that is, a view of the larger mathematical landscape that teaching requires” (p. 70). Besides, Sleep (2009) indicated that teachers might need HCK to “recognize concepts for which a topic is foundational and to unpack the relationship to big mathematical ideas” (p. 223). For instance, teachers need to know that fractions they teach in third grade set the mathematical foundation to percent and ratio concepts students would learn at sixth grade. It is important to distinguish HCK and KCC. Since the “horizon” in HCK refers to mathematical horizon, not to curricular horizon as in KCC (Jakobsen et al., 2013). And, the last domain of the SMK is SCK, which is defined as the mathematical knowledge and skill, which is unique to teaching. Being able to answer whether a method works in general and explain why and how it works is a demand not needed for purposes other than teaching, but it is essential for teachers.

The first domain of the PCK is KCS, which involves knowing about students and knowing about mathematics. It refers to the idea that “teachers must anticipate what students are likely to think and what they will find confusing. Central to these tasks is knowledge of common student conceptions and misconceptions about particular mathematical content” (Ball et al., 2008, p. 401). For instance, teachers should be able to know what the common students’ conceptions/misconceptions are when learning fraction divisions. The second domain under PCK is KCT, which combines knowing about teaching and knowing about mathematics (Ball et al., 2008). Teachers need to have this mathematical knowledge to design their instructions to complete most of the mathematical tasks of teaching (e.g., altering a task to be either easier or harder, deciding which examples to begin teaching fraction multiplication, knowing when to

pause for clarification during class discussion, etc.). For instance, this type of knowledge supports teachers to know whether a set model would be an ideal model to support or confound students who have fragmented understanding of fraction addition.

The last domain of PCK is KCC, which shares the same meaning of Shulman's curricular knowledge. With the help of this specific type of knowledge, teachers might know at what grade level they introduce division of fractions, or instructional materials available for multiplying fractions. It should be noted that KCC has been a part of Shulman's categories of PCK since 1986, it was not a part of MKT when it was first developed by Ball and her colleagues. According to Sleep (2009), both HCK and KCC are still not well developed like other domains of MKT, and there are not many studies focus on these specific domains. I used MKT as my framework to ground this inquiry. In the next section, I provide studies on PSTs' MKT on general and specific to fractions, and their findings.

PSTs' Knowledge of Fractions

According to the theoretical framework for MKT elementary teachers should possess variety types of mathematical knowledge. Through years, research has proved the positive relationship between teachers' MKT and student achievement (Hill et al., 2005; NMAP, 2008) and instructional quality (Hill et al., 2008). Due to its proven impact in student achievement and instruction quality, a significant amount of research has focused on in-service teachers' MKT (Hill et al., 2008, Hill et al., 2004; Hill et al., 2005). For instance, in 2008, Hill and her colleagues analyzed 90 lessons that were taught by 10 teachers. They measured teachers' MKT using an instrument that was rigorously developed by Hill, Rowan and Ball in 2005. Besides, they developed and used a framework for observing mathematical quality of instruction (MQI).

The results showed that teachers' MKT related to the mathematical quality of instruction. Teachers who had stronger MKT made less mathematical errors, especially language errors. Teachers with weaker MKT were less successful at choosing examples that support students to construct meanings of those particular concepts, and relied on textbooks as a support compared to teachers with stronger MKT.

Lately, researchers have turned their interests on examining PSTs' mathematical knowledge (Ball, 1990c; Isiksal & Cakiroglu, 2011; Li & Kulm, 2008; Newton, 2008). In Newton's study, 85 PSTs' CCK for all four operations with fractions were analyzed with a pre-test and with a post-test after completing a course. The instrument for the pre-test and post-test was identical, and examined PSTs' knowledge with 20 computation, basic concepts, word problems, flexibility, and transfer items in total. The results of the study showed an improvement on PSTs' computational skill, knowledge of basic concepts and solving word problems. However, there were no significant change in flexibility and transfer. Specifically, PSTs in the study were procedurally proficient in addition and subtraction with fraction, but their procedures lacked flexibility. For instance, 72 out of 85 PSTs needed to find a common denominator to solve the problem $\frac{2}{4} - \frac{3}{6}$, whereas only 7 of them able to solve the problem by converting each fraction to $\frac{1}{2}$. When it comes to computation skills, most of the PSTs in the study had difficulties with working fraction division the most, followed by fraction subtraction, multiplication and addition. They had common misconception for cross-multiply procedures, for instance they cross-multiplied when they were multiplying fractions (instead of multiplying across), and they added both denominators and both numerators when they supposed to find a

common denominator. Although they had fewer misconceptions at the post-test, their lack of flexibility proved that they did not possess deep knowledge of fractions.

A study conducted by Isiksal and Cakiroglu (2011) involved 17 PSTs, and examined their PCK, specifically their KCS of multiplication of fractions. For that purpose, a questionnaire consisted of three items that assess PSTs' knowledge of students' common conceptions and misconceptions about fraction multiplication were given to PSTs. PSTs were asked to write a mathematical expression for a given word problem for fraction multiplication, to provide possible mistakes that sixth or seventh grade students might have when performing the task and the strategies to overcome difficulties. Upon completing the questionnaire, the researchers conducted semi-structured interviews related to PSTs' responses to the questionnaire. Based on the outcomes of the study, there were five main categories of misconceptions suggested by PSTs for students' common mistakes about fraction multiplication: algorithmically based mistakes, intuitively based mistakes, mistakes based on formal knowledge of fraction operations, misunderstanding of the symbolism of a fraction, and misunderstanding of the problem. For instance, one pre-service teacher pointed out that students might not be able to perform $7 \frac{1}{3}$ because of not having underlying meaning of the fraction symbol. This explanation was classified under "misunderstanding of the symbolism of a fraction", which connects to Battista's (2012) statement about the importance of understanding how fractions are symbolized. To overcome students' misconceptions, PSTs in the study emphasized the importance of using multiple representations and different teaching methods, providing variety of practices to students, and encouraging students to share their reasoning strategies.

Ball (1990c) completed a study with ten elementary and nine secondary PSTs. The focus of the study was on PSTs' understanding of division. The participants were asked to solve a fraction division task ($1\frac{3}{4} \div \frac{1}{2}$) and to write an appropriate real-world problem for that task, which requires PSTs to have SCK. Almost all of the participants provided correct calculation, which shows they possess strong CCK. However, there was no elementary PSTs who could be able to pose an appropriate problem that reflects the fraction division task. Three elementary PSTs presented an inappropriate story, which was dividing by two instead of by one-half.

In 2008, 46 PSTs who were enrolled mathematics methods course at the time of the study participated Li and Kulm's study. The purpose of the study was to explore PSTs' own beliefs and perceptions about their knowledge preparation and their mathematics knowledge for teaching fraction divisions. The researchers developed a survey for exploring PSTs' beliefs and perceptions; and it consisted of four tasks to assess PSTs' mathematics knowledge for teaching fraction division. In the instrument, there were questions dedicated to assessing CCK, SCK, KCS, and KCT. More than half of the participants were able to answer questions that require CCK such as finding the value of division of two proper fractions. Besides, none of the participants were able to explain the reasoning behind "flip and multiply" strategy when dividing fractions; and this result suggested that participants of the study had limited SCK, which is considered as critical component for teaching. Li and Kulm revealed that PSTs had insufficient content knowledge and pedagogical content knowledge on fraction division. In this section, PST's knowledge, conceptions, misconceptions and beliefs about fractions were discussed; in the following section I provide recommendations for teaching fractions.

Recommendations for Teaching Fractions

Research has been showing the importance of fraction (Siegler et al., 2012), students' misconceptions of fractions (Mack, 1990) and teachers' fragmented knowledge of fractions (Charalambous & Pitta-Pantazi, 2007; Newton, 2008). To provide some recommendations for teaching fractions, I searched variety of literature about teaching fraction concepts. First, Siegler et al. (2010) recommended teachers to use equal-sharing activities to introduce fraction concepts since children would have developed initial understanding of sharing some objects equally among two or three people before they begin school; and that initial knowledge would provide a foundation for teaching fractions. The equal-sharing activities can also help students to understand some of the basic interpretations of fractions such as ratio (when sharing two pizzas by three people, the ratio of the number of pizzas to the number of people is 2:3) and division (when sharing 10 cupcakes into five equally numerous groups). Second, Fennell et al. (2014) and Siegler and his colleagues stressed the importance of understanding fractions as numbers. Therefore, teachers should give strong emphasis on introducing fractions as numbers in order to prevent misconceptions such as adding both numerators and denominators when adding two fractions or believing $\frac{4}{9} > \frac{4}{7}$ since 9 is bigger than 7 (Siegler et al., 2010). For instance, Fennell and his colleagues stated many students consider a fraction as being one number over another, or part of a whole (which might be possible) but usually don't recognize that a fraction is a number itself too. They suggested using different representations (drawings, models, manipulatives, technological tools, etc.) when introducing fraction concept would play key role for understanding fractions as a number. Specifically, number lines can be used as a key representational tool for teaching common, proper and improper fractions, equivalent and

nonequivalent fractions, mixed numbers, decimals, etc. (Siegler et al., 2010). They also suggested that students should be given opportunities to compare and locate fractions on number lines in order to consider fractions as numbers.

Third, research has shown the use of manipulatives when teaching fractions has positive impact on students' understanding of fraction concepts (Bezuk & Cramer, 1989; Hudson Hawkins, 2007). Therefore, it is recommended to include variety of manipulatives (cuisenaire rods, fraction strips, paper folding, color counters, etc.) when teaching fractions. Next, Fennel and his colleagues described the “understanding of and proficiency with equivalence” as “the cornerstone of developing fraction sense” (2014, p. 489). Thus, it is important for teachers to provide opportunities for students to become comfortable representing equivalent fractions in different ways (mentally, using regions and number lines). For instance, students should be given chances to locate three or more equivalent fractions for $\frac{2}{3}$ in a number line, or to use bar models

to show fractions that are equivalent to $\frac{6}{9}$. In the late 1980s, it was suggested to introduce fraction operations after students establish clear understanding of order and equivalent of fractions to allow students internalize the concept (Bezuk & Cramer, 1989). Additionally, students should be given opportunities to represent fractions as percent and decimal, which support them to understand fractions, decimals, and percent are different ways to represent the same number. For instance, $\frac{7}{10}$ is equivalent to 70% and 0.7 (Siegler et al., 2010).

Studies have shown that part-whole interpretation is mostly emphasized interpretations of fractions when fractions are introduced in the United States (Ni & Zhou, 2005; Thompson & Saldanha, 2003, as cited in Siegler, Thompson, & Schneider, 2011). For instance, students who

are introduced fractions focusing on part-whole interpretation only, tend to interpret $\frac{1}{4}$ as one of four slices of pizza, but missed the opportunity to think of it as “the distance from zero to one on a number line” (Moseley, Okamoto, & Ishida, 2007; as cited in Siegler et al., 2011, p. 293). Siegler et al. (2011) stated that teaching only one interpretation of fraction is not similar to the approach to teaching fractions in Japan, China and other countries where students’ understanding of fractions are better. Siegler and his colleagues stressed that many U.S. teachers can only make sense of part-whole interpretation unlike Chinese and Japanese teachers who emphasize measurement (number line) and other interpretations; and this limited knowledge of fraction interpretations has some drawbacks. For instance, one cannot represent negative fractions using part-whole interpretation. Besides, it might be very confusing for students to use part-whole interpretation when they work on fractions with large numerators and denominators and improper fractions. Therefore, Siegler and his colleagues emphasized the importance of involving different interpretations of fractions into teaching fractions, especially measurement interpretation (e.g., number lines).

Finally, Son's (2011) study compared Korean and the U.S.’ curricula based on their perspective on teaching and learning of fractions. And, since Korean students mostly perform better in international assessments compared to U.S. counterparts, Son aimed to compare Korean and U.S. curricula and explore how Korean and U.S. mathematics textbooks introduce fractions. The study revealed that U.S. mathematics textbooks mainly use area models (pies and pizzas) to develop part-whole understanding, whereas Korean textbooks provide opportunities to develop part-whole understanding with area, discrete (set) and measurement (linear) models. Although using different models to introduce fraction concepts cannot be one reason to explain Korean

students' high performance in international tests; the importance of using variety of models can be undeniable (Zhang, Clements, & Ellerton, 2015). In their study, Zhang et al. (2015) suggested using different activities when teaching fractions with different models. They suggested developing number lines using paper strips and human number lines as an example of linear model. As an activity to embody discrete (set) models, they offered using twelve identical blocks, ask students to consider them as “precious diamonds” and share them equally among two, three and then four friends (p. 144). Finally, pouring water activity was suggested to internalize area model. Students are able to consider full glass of water as a whole, and then share water equally among two and three other glasses. In this activity, students are also given opportunities to demonstrate their process by drawing each step they complete. In this section, recommendations for teaching fractions were provided. The following part delivers brief description of teacher candidates' preparation process to become in-service teachers.

Elementary Teacher Preparation

According to Ball (2003), “improving the mathematics learning of every child depends on making central the learning opportunities of our teachers. Teachers cannot be expected to know or do what they have not had opportunities to learn” (p. 9). We, as educators, cannot expect PSTs to teach quality of mathematics to their future students without providing them opportunities to develop their mathematical knowledge (Lee, Meadows, & Lee, 2003). Additionally, Ball and Forzani (2009) indicated that the work of teaching is not natural, so it needs to be learned and taught. *The Standards for Preparing Teachers of Mathematics* (SPTM) stressed solid mathematics content knowledge and pedagogical content knowledge are essential to the success of a beginning teacher of mathematics (AMTE, 2017). Therefore, teacher preparation programs play an essential role for PSTs to develop mathematical competence.

The role of mathematics courses in teacher preparation programs would be undeniable, and should focus on “life-long learning of mathematics, rather than to teach PSTs all they will need to know” (The Mathematical Education of Teachers II [MET II], Conference Board of the Mathematical Sciences, 2012, p. 5). For that purpose, the recommended number of semester hours of mathematics courses that PSTs should take was raised from 9 to 12 hours (MET II, CBMS, 2012). There are mathematics content courses and mathematics methods courses that are offered to PSTs to develop their mathematical knowledge. Content courses’ focus is addressing mathematics concepts and procedures (Burton et al., 2008), whereas the aim of methods courses is to support PSTs learn to teach mathematics (Ball, 1990b).

According to CBMS (2001), “teaching elementary mathematics requires both considerable mathematical knowledge and a wide range of pedagogical skills” (p.55). Although content courses and methods courses are generally taught separately, there are some studies that focused on the effects of integration of mathematics content and methods courses (Burton et al., 2008; Steele & Hillen, 2012). The results of these studies proved that even limited amount of mathematical content integrated methods courses could make a positive impact on teachers’ content knowledge for teaching.

According to the Oxford English Dictionary, the methods word comes from the Greek word “methodos”, which means “pursuit of knowledge” (as cited in Smitherman, 2006, p. 13). Also, in her book, *The Making of a Teacher*, Grossman (1990) indicated that methods courses develop PSTs’ PCK, since it provides a combination of content and pedagogical understanding while PSTs learn to teach a subject (as cited in Kinach, 2002). Therefore, PSTs should be given opportunities to develop not only their subject matter knowledge but also their pedagogical content knowledge in teacher preparation programs.

Curriculum Types

Curriculum materials and teacher knowledge are considered two key resources in the instructional system (Charalambous & Hill, 2012). Previously I have presented the importance of teacher knowledge, and in this section, I describe the curriculum materials in teacher education. The “curriculum” term comes from a Latin verb “currere”, and refers to a “course or track to be followed” (van den Akker, Bannan, Kelly, Nieveen, & Plomp, 2010, p. 37). In the education context, where learning is the main purpose, Taba (1962) provided the description of curriculum as “plan for learning” (as cited in van den Akker et al., 2010, p. 37), and Stein, Remillard and Smith (2007) stated that “curriculum refers to the substance or content of teaching and learning – the “what” of teaching and learning (as distinguished from the 'how' of teaching)” (p. 321).

There are multiple meanings of curriculum. Some researchers provided three forms of curriculum as the intended curriculum, the implemented curriculum, and the attained curriculum (Mesa & Griffiths, 2012; van den Akker et al., 2010). The intended curriculum refers to “the intensions, goals, and objectives for mathematics that are envisioned for learning at a national, regional, or local level (e.g., state standards, textbook content... and skills that we want students to learn)” (Mesa & Griffiths, 2012, p. 86). The implemented curriculum refers to the curriculum, which is interpreted by its users, especially teachers who design discussions or activities for students to learn the material. The attained curriculum refers to the learning outcomes of students that were measured via standardized tests or other kind of assessments. Instead, Stein et al. (2007) used different terms to define various meanings of curriculum, which are written curriculum, intended curriculum, and enacted curriculum. They use “written” term to refer to what other researchers call the “intended” curriculum. Basically, written curriculum refers to curriculum as given in “the printed page”; intended curriculum differs from the written

curriculum since it also includes “teachers’ plans for instructions”; and enacted curriculum indicates implemented curriculum in classrooms (Stein et al., 2007, p. 321).

In this study, I focused on the written curriculum, since my aim was to examine the opportunities provided for PSTs to develop their MKTF through the given fraction tasks in mathematics methods textbooks. Specifically, I examined the “tasks as they appear in curriculum”, which is the first of three different phases that tasks pass through in the Mathematical Tasks Framework (Stein & Smith, 1998, p. 270). Mesa and Griffiths (2012) describes textbooks as “intermediaries in turning intentions, aims, and goals, into implementations, thus affording probabilistic rather than deterministic opportunities to learn mathematics” (p. 86). Therefore, it is critical to examine textbooks in teacher preparation programs to discover opportunities to develop mathematical knowledge. In the following section, I provide the role of textbooks in teacher education.

Role and Use of Textbooks in Teacher Preparation

There were not adequate studies, which have focused merely on mathematics methods textbooks; therefore, I discuss the use and the importance of textbooks in teacher preparation programs in general. Several studies have been showing that the importance of textbooks in higher education is undeniable (Harkness & Brass, 2017; Mesa & Griffiths, 2012; Weinberg, Wiesner, Benesh, & Boester, 2012). As Mesa and Griffiths (2012) stated, “textbooks remain a ubiquitous course component with various implications for the teaching and learning of mathematics at tertiary level” (p. 86). In 2017, Harkness and Brass completed a study with mathematics methods instructors based on their textbook selection. Out of 132 participants, only 20 participants indicated that there were no required texts in their courses. The remaining

participants stated that they made textbook selection for their courses considering recommendations, course objectives, and institution decisions. More than 100 of participants mentioned they use textbooks “to stimulate in-class discussions and as sources for activities for PSTs to explore” (Harkness & Brass, 2017, p. 95). In their study, Weinberg et al. (2012) surveyed 1156 undergraduate students, including students in elementary education, who were enrolled mathematics courses. Their survey results showed that all students in the study owned their own copy of the required text. Nearly 90% of participants reported they focus on worked examples to complete homework, get ready for exams, and gain better understanding. The study revealed that students value textbooks that consist of many examples to support understanding of the concept.

The National Council on Teacher Quality (NCTQ) revealed a report on what extent research-proven instructional strategies are given in the textbooks used in PSTs’ educational psychology and general and subject-specific methods courses (Pomerance, Greenberg, & Walsh, 2016). Specifically, they reviewed 48 textbooks used in elementary and secondary teacher preparation programs focusing on strategies identified by the Institute of Education Sciences (IES), the research arm of the U.S. Department of Education, that promote student learning. The six fundamental instructional strategies are as follows: pairing graphics with words, linking abstract concepts with concrete representations, posing probing questions, repeatedly alternating solved and unsolved problems, distributing practice, and assessing to boost retention. According to Pomerance, Greenberg and Walsh (2016), “textbooks underpin most teacher preparation coursework, and they provide insights into the instructional approaches used by teacher educators” (p. vii); however only one strategy (posing probing questions) were mostly emphasized in textbooks whereas PSTs were not given any chances to practice two out of six

standards (repeatedly alternating solved/unsolved problems and assessing to boost retention) in any textbooks. None of the textbooks in the sample provided more than two strategies for PSTs. Therefore, NCTQ's report concluded that textbooks used in educational psychology and methods courses do not provide enough opportunities for PSTs to develop instructional strategies they need to support student learning. In the next section, I present studies on content analysis in teacher preparation.

Research on Content Analysis in Teacher Preparation

According to Cole, content analysis is a research method that is used to analyze “written, verbal or visual communication messages” (as cited in Elo & Kyngäs, 2008, p. 107). Krippendorff (2004) also described it as “a research technique for making replicable and valid inferences from texts (or other meaningful matter) to the contexts of their use. It is a scientific tool” (p. 18), and it can be conducted with quantitative or qualitative approaches (Elo & Kyngäs, 2008; Saeli, Perrenet, Jochems, & Zwaneveld, 2012). Qualitative content analysis are completed systematically to describe the meaning of qualitative material (Schreier, 2012). Quantitative approaches are mostly performed to provide information about the number of concepts, pages, or other quantities (Saeli et al., 2012). However Krippendorff (2004) emphasized that since “reading of texts is qualitative, even when certain characteristics of a text are later converted into numbers” (p. 16), all content analysis can be considered qualitative in nature.

There have been several studies used content analysis as a method to explore, describe, or make sense of data in different fields such as nursing studies (Elo & Kyngäs, 2008) and information and library science (ILS) (Zhang & Wildemuth, 2009). There is limited research has been done using content analysis in teacher preparation, and they mostly focused on analysis of syllabi or coursework. Gorski (2009) used qualitative content analysis to examine 45 syllabi

from multicultural teacher education (MTE) courses taught across the United States. Specifically, Gorski collected data in a deductive way by following couple of existing typologies of MTE, and also in inductive way by seeking patterns in course descriptions, goals and objectives. Results revealed “most of these syllabi appeared crafted to prepare teachers with cultural sensitivity, tolerance, and multicultural competence. Most of the courses were not designed to prepare teachers to identify or eliminate educational inequities or to create equitable learning environments” (Gorski, 2009, p. 316).

More recently, Polkinghorne (2015) used qualitative content analysis to examine the coursework required in 94 undergraduate business teacher education programs. According to Polkinghorne, there were some business programs that followed the National Association for Business Teacher Education (2010)’s standards; but many business programs’ coursework requirements were different. The researcher developed an instrument in a deductive way by following the Darling-Hammond and Bransford’s framework. The instrument consisted of three categories: subject matter, instructional context, and instructional strategies. Data revealed that business teacher education programs in the study were not aligned with recommendations of National Association for Business Teacher Education. Specifically, the programs in the study provided less hours for subject matter competency whereas they required more coursework in instructional context compared to recommendations of the Association.

There were limited studies conducted using content analysis in general teacher preparation, however no content analysis has been conducted to explore textbooks used in mathematics methods courses. Hence, this study aims to fill this gap within the literature by exploring fraction concepts given in mathematics methods courses using qualitative content

analysis. This section provided insights of content analysis completed in teacher preparation. In the following section, I provide the summary of the chapter.

Summary of Literature Review

In this chapter, I provided the review of the literature that guided my study. Firstly, I discussed what fractions are, the importance and the meanings of fractions, and how fractions are presented in CCSSM. Next, I provided teacher knowledge of mathematics, theoretical considerations on MKT, and research related to PST's knowledge of fractions. And then, I discussed types of curriculum and the role and use of textbooks in teacher preparation.

Babylonians and Egyptians were the first two cultures invented and used unit fractions to express amounts that are less than one but bigger than zero (Chapin & Johnson, 2006). After the Renaissance, fractions have become more common and been used in the form of $\frac{a}{b}$. Although, there is one form to represent fractions, there are five main interpretations of fractions: part-whole, division, ratio, operator, and measurement (Chapin & Johnson, 2006; Van de Walle et al., 2019). Providing opportunities for students to practice all meanings of fractions could result better understanding of fractions (Clarke et al., 2008; NCTM, 2000; Siebert & Gaskin, 2006; Van de Walle et al., 2019). According to the CSSM, formal introduction to fractions starts at 3rd grade, and student are expected to develop understanding of fractions as numbers. Fennell et al. (2014) and Siegler et al. (2012) stressed the importance of understanding fractions as numbers. They mentioned introducing fractions as numbers prevent common misconceptions (e.g., adding both numerators and denominators when adding two fractions), and stated the use of different representations/models plays a key role for deep understanding of fractions as numbers.

There is a positive correlation between teachers' knowledge and students achievement (Hill et al., 2005; NMAP, 2008) and instructional quality (Hill et al., 2008), therefore teachers should possess strong mathematical knowledge of fractions to support students for deep understanding of fractions. In 1986, Shulman identified three types of teacher knowledge: subject matter content knowledge, pedagogical content knowledge and curricular knowledge. Couple decades later, Ball et al. (2008) developed a term "MKT" building on Shulman's two dimensions of knowledge: subject matter knowledge and pedagogical knowledge. MKT was describes as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p. 395), and consists of six domains as follows: CCK, HCK, SCK, KCS, KCT, and KCC.

Several studies have investigated PSTs' mathematical knowledge, and found that PSTs have fragmented mathematical knowledge of fractions (Isiksal & Cakiroglu, 2011; Li & Kulm, 2008; Newton, 2008). As known, one cannot expect PSTs to teach quality of mathematics when they are not provided opportunities to develop mathematical knowledge (Lee et al., 2003), thus teacher preparation programs play a critical role for PSTs for developing mathematical knowledge. Due to lack of shared curriculum in teacher preparation programs, PSTs' learning opportunities differ across institutions (Ball et al., 2009; Hill & Lubienski, 2007). Whereas many PSTs take both content and methods courses (Hill & Lubienski, 2007); there are some institutions only offer methods courses to teachers as part of their preparation. Hence, methods courses are essential for providing opportunities for PSTs to acquire mathematical competency for teaching.

The curriculum is divided into three levels: written curriculum, intended curriculum, and enacted curriculum (Stein et al., 2007). Textbooks are part of written curriculum since they

reflect the state standards and national objectives for mathematics. Although, textbooks are widely used curriculum material for teaching and learning of mathematics in higher education (Mesa & Griffiths, 2012) there hasn't been much attention given textbook analysis in teacher preparation, especially elementary mathematics teacher preparation.

This review of the literature sheds light upon the importance of fractions and the need of investigation of PSTs' mathematics textbooks. Therefore, I conducted a qualitative content analysis to add to the body of literature on PSTs' fraction knowledge, by exploring opportunities provided in mathematics methods textbooks to develop MKTF. In the next chapter, I present the methods that I used to conduct this qualitative content analysis.

CHAPTER 3: METHOD

In this study, I applied qualitative content analysis to examine how mathematical knowledge of teaching fractions (MKTF) is presented in the pre-service elementary teachers' mathematics methods textbooks. In particular, I investigated how mathematics methods textbooks provide opportunities for pre-service teachers (PSTs) to gain/enhance their MKTF. This chapter is divided into five sections. First, I present the research questions. Second, I present the sample of textbooks that I analyzed. Next, I describe the study's research design, the coding framework, and data collection procedures. Then, I present the results of the pilot study that I conducted to test my coding frame. Finally, I present the reliability and validity measures, main analysis procedure, and I conclude the chapter with a brief summary of the research design and methods I used.

Research Questions

In order to analyze opportunities provided for PSTs to develop their MKTF by the elementary mathematics methods textbooks, the following research questions guided my study:

- 1) To what extent and in what ways do textbooks used in mathematics methods courses in teacher preparation programs provide PSTs opportunities to develop MKTF?
 - a. What opportunities are provided in the textbooks for PSTs to develop subject matter knowledge for fractions?

- i. What opportunities are provided in the textbooks for PSTs to develop common content knowledge for fractions?
 - ii. What opportunities are provided in the textbooks for PSTs to develop horizon content knowledge for fractions?
 - iii. What opportunities are provided in the textbooks for PSTs to develop specialized content knowledge for fractions?
- b. What opportunities are provided in the textbooks for PSTs to develop pedagogical content knowledge for fractions?
 - i. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and students?
 - ii. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and teaching?
 - iii. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and curriculum?

Sample Selection

The importance of mathematics methods textbooks that PSTs are using is undeniable (Harkness & Brass, 2017). Thus, I examined how three mathematics methods textbooks, which PSTs are assigned, provided opportunities to develop their MKTF.

The textbooks selected for this study did not reflect a random sample among mathematics methods textbooks available to all elementary PSTs in teacher preparation programs. To identify the most widely used textbooks in elementary mathematics methods courses in the United States., I conducted a literature search on popular mathematics methods textbooks and met with a librarian. However, no source with any kind of information related to my search was available.

Therefore, I selected the textbooks based on the survey conducted by Harkness and Brass (2017). Harkness and Brass created the survey in 2013 and sent it to the members of Association of Mathematics Teacher Educators (AMTE) in an electronic format. In total, 941 AMTE members received the survey electronically, and there were 132 AMTE members who volunteered to participate the survey. The participants were the instructors of mathematics methods courses across the U.S. The survey consisted of 10 questions regarding mathematics methods courses in general, and one of the questions in the survey was: “Please name the text(s) that you use in your methods course(s)” (Harkness & Brass, 2017, p. 104). After having e-mail exchanges with Dr. Brass, she was very generous to share their data with me. Based on the survey outcomes, there was an overwhelming majority of the participants mentioned that they use the *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle, Karp and Bay-Williams and *Teaching Student-Centered Mathematics, Developmentally Appropriate Instruction* by Van de Walle, Karp, Lovin, & Bay-Williams. Some participants did not provide a specific reference to a book, but only mentioned “Van de Walle” as the author of the textbook. While other participants cited *Mathematics for Elementary Teachers with Activities* (Beckmann), *Helping Children Learn Mathematics* (Reys, Lindquist, Lambdin, & Smith), and *Teaching and Learning Middle Grades Mathematics* (Rubenstein, Beckmann, & Thompson) as their primary textbooks. Based on the participants’ responses, it was not clear which edition they used for the given textbooks, and Table 2 shows the survey results.

Table 2. Survey Outcomes for Textbooks

Authors	Title
Van de Walle, et al.	Elementary and Middle School Mathematics
Van de Walle, et al.	Teaching Student-Centered Mathematics
	Grade-Band Specific
	K-3
	3-5
	6-8
	PreK-2
	5-8
	No Specific Grade-Band
Van de Walle	No Specific Textbook Reference
Beckmann	Mathematics for Elementary Teachers
Reys et al.	Helping Children Learn Mathematics
Rubenstein et al.	Teaching and Learning Middle Grades Mathematics

According to the survey results, first, I considered the textbooks in Table 2 as my potential sample. Since the fraction concepts are introduced and developed in elementary mathematics, I only targeted the elementary mathematics methods textbooks that cover fractions concepts. Therefore, I disregarded some of the textbooks due to the fact that their focus on middle grades mathematics. Besides, I found considerable similarity between *Elementary and Middle School Mathematics* (Van de Walle, et al.) and *Teaching Student-Centered Mathematics Grades 3-5* (Van de Walle, et al.) since they are both written by the same authors. Specifically, I compared both textbooks' table of contents and given fractional tasks in chapters and found most of the tasks were alike. Therefore, I decided to not include *Teaching Student-Centered Mathematics Grades 3-5* (Van de Walle, et al.) into my sample. I finalized my sample considering these circumstances and selected the textbooks for this inquiry (see Table 3). As previously stated, the survey took place in 2013 and the participants of the survey did not

specifically mentioned which edition they were using at that time; for this study, I included the most recent edition of each textbook as my sample (which can be seen in the Table 3) to reflect the mathematics methods textbooks used in today’s teacher education programs.

Table 3. Selected Textbooks for Analysis

Authors	Year of Publication	Title
Van de Walle, Karp, & Bay-Williams	2019	Elementary and Middle School Mathematics, Teaching Developmentally, 10 th Edition (EMSM)
Beckmann	2018	Mathematics for Elementary Teachers with Activities, 5 th Edition (META)
Reys, Lindquist, Lindquist, Lambdin, & Smith	2015	Helping Children Learn Mathematics, 11 th Edition (HCLM)

Structure of the Textbooks and Chapters

Elementary and Middle School Mathematics, Teaching Developmentally 10th Edition (EMSM; Van de Walle et al., 2019) focused on a problem-based approach, and aimed to support PSTs to learn how to teach mathematics by “doing mathematics”. When I did an analysis of the table of contents, I found that the chapters were given under two different sections. The first section consisted of six chapters, which focused on challenging tasks to support students learn mathematics. The authors stressed that in order to know what it means to “do mathematics” teachers should learn how to teach mathematics effectively. Thus, it is critical for PSTs to be able to understand how students learn, so they can support and challenge all students by learning how to apply variety of instructional and assessment strategies.

The second section contained 16 chapters and gave main mathematics content areas suggested for the Pre-K-8 as addressed in CCSSM. All chapters in the second section start with

the list of big ideas related to the chapters, and the authors stated planning an instruction around big ideas instead of skills or concepts encourages student-centered approach. Besides, the focus of the second section was to support PSTs to understand how students grasp the content through many problem-based activities. The activities are designed to engage students in mathematics in addition to supporting PSTs to see possible challenges students might have and ways to help them. Through the textbook, the chapters are supported with the figures and notes, such as technology notes, formative assessment notes, and also the eight Standards for Mathematical Practices from CCSS are given if there is connection to the content. In total, EMSM has 22 chapters, and fraction chapters introduced after the middle of the textbook, which were Chapter 14 and Chapter 15, and titled as *Developing Fraction Concepts* and *Developing Fraction Operations*.

Mathematics for Elementary Teachers with Activities, 5th Edition (META; Beckmann, 2018) designed to support PSTs to develop deep understanding of mathematics by not only knowing the mathematical contents they need when teaching, but also being able to explain the reasoning behind them. And, specifically the author mentioned, “knowing why requires a much deeper understanding than knowing how” (p. xii), therefore the author designed the textbook focusing on explaining why. Also, the textbook addresses CCSSM by providing the standards with the grade levels through the chapters. The chapters are aimed to support PSTs to recognize how mathematical concepts develop across grade levels. Each chapter begins with the brief overview of the chapter and follows with Standards for Mathematical Content in CCSSM and Standards for Mathematical Practice in CCSSM. The author stressed that the textbook is great source for mathematical preparation of teachers following the recommendations of the Conference Board of the Mathematical Sciences (CBMS).

After analyzing the table of the contents, I found the concepts in META are organized around the operations (addition, subtraction, multiplication, and division). Thus, fraction concepts were presented under each operation repeatedly throughout the textbook, which Beckmann (2018) stated that is a key advantage for PSTs to internalize reasoning with fractions. META also provided a chapter of problem solving specifically focusing on fractions, which gave opportunities for PSTs to work with challenging problems and to think about explanations. There were 16 chapters in total, and fraction concepts were covered in Chapters 2, 3, 5 and 6. However, these chapters consist of lessons on different topics, which means chapters are not merely focused on fraction concepts. For instance, Chapter 3 is titled *Addition and Subtraction*, and includes lessons on adding and subtracting whole numbers, negative numbers, and fractions.

Fraction concepts are first introduced in the 2nd chapter (*Fraction and Problem Solving*) under the three lessons (*Defining and Reasoning About Fractions, Reasoning About Equivalent Fractions, Reasoning to Compare Fractions*). Fraction lessons were then given under the operation chapters. In the Chapter 3 (*Addition and Subtraction*), the lesson was called *Reasoning About Fraction Addition and Subtraction*, and when it comes to the Chapter 5 (*Multiplication of Fractions, Decimals, and Negative Numbers*) the lesson was titled as *Making Sense of Fraction Multiplication*. Lastly, fractions were given under the Chapter 6 (*Division*), and lessons were named *Division and Fractions and Division with Remainder, Fraction Division from the How-Many-Groups Perspective, Fraction Division from the How-Many-Units-in-1-Group Perspective*. Overall, fraction concepts were given under eight lessons in four different chapters, which were located toward the beginning of the textbook.

Helping Children Learn Mathematics, 11th Edition (HCLM; Reys et al., 2015) focused on three main ideas: “helping children make sense of mathematics, incorporating practical

experiences and using research to guide teaching” (p. iv). There are two main parts in the textbook. The first part focuses on understanding how students learn mathematics, and also planning instructions and evaluations that align with the CCSSM (Chapters 1-6). The authors stated that the main focus is given to the problem solving and assessment due to their significant effect on mathematics teaching. The second part focused on teaching techniques and strategies, included learning practices for specific topics (Chapters 7-18). The attention also is given to make mathematics meaningful for diverse learners. The authors mentioned that the textbook addresses the recommendations from professional associations in addition to the research on teaching mathematics. There were 18 chapters in total and all fraction content were grouped together in one chapter, *Chapter 12: Fractions*, which was after to the middle of the textbook.

Table 4. Chapters and Number of Pages Devoted to Fractions

Textbooks	Total No. of Chapters	Chapters Devoted to Fractions	Total No. of Instruction Pages	No. of Fraction Content Pages	Fraction Content Pages Percent	Total No. of Tasks
EMSM	22	Chapter 14 Chapter 15	631	68	10.7%	162
META	16	Chapter 2 Chapter 3 Chapter 5 Chapter 6	752	78	10.3%	300
HCLM	18	Chapter 12	394	21	5.3%	32

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

In this section, I provided the number of chapters, pages, and tasks devoted to fraction concept in three mathematics methods textbooks. As previously stated, fraction concepts introduced in lessons under four different chapters in META, whereas chapters in the other two textbooks were devoted to fraction concepts only. Therefore, it is critical to present the number and percentage of pages that fraction concepts are given in each textbook. Table 4 displays the total number of chapters and pages in each textbook and the total number and percent of the fraction chapters (units of analysis), and total number of fraction tasks (units of coding) that I examined for this inquiry. In the following section, I describe my research design.

Research Design

For this inquiry, I employed qualitative content analytical methods. Qualitative content analysis is one of several appropriate research methods for analyzing texts such as interview transcripts, emails, textbooks, diaries, newspaper articles and so on (Hsieh & Shannon, 2005; Mayring, 2000; Schreier, 2012; Zhang & Wildemuth, 2009). Mayring (2000) defined it as “an approach of empirical, methodological controlled analysis of texts within their context of communication, following content analytic rules and step-by-step models, without rash quantification” (p. 2). Krippendorff (2004) claimed that all content analysis is qualitative in nature by stating “all reading of texts is qualitative, even when certain characteristics of a text are later converted into numbers” (p. 16). There are three important features of qualitative content analysis. First, qualitative content analysis is systematic. Researchers follow the same order of steps regardless of research questions and materials. Second, qualitative content analysis is a very flexible method and allows researchers to modify their framework (in qualitative content analysis this is called a coding frame) until it matches their material. Using a flexible method allows researchers to develop their coding frame as concept-driven (deductively - developing the

framework based on the prior knowledge, theory, experiences, etc.) and/or data-driven (inductively - developing the framework based on the data). Thus, in qualitative content analysis, coding frames are developed as a partly data-driven to match the material (Schreier, 2012).

Hsieh and Shannon (2005) described three approaches to qualitative content analysis based on the degree of inductive approach involvement: conventional, directed, and summative. In conventional qualitative content analysis, coding categories are emerged directly and inductively from raw data. In directed content analysis, initial coding starts with an existing theory or prior research, and additional themes/codes emerge from the data when researcher starts to immerse in the data. In summative content analysis starts with identifying and counting of words (manifest meanings) and extends including more latent meanings to discover underlying meanings of the words in inductive approach. For this study, I employed directed qualitative content analysis. Besides being flexible and systematic, data reduction is the third important feature of qualitative content analysis. Qualitative content analysis helps researchers to concentrate their analysis on a particular perspective, and this feature reduces data if necessary (Schreier, 2012). These were the three key points of qualitative content analysis, which lead me to select qualitative content analysis as a method to analyze my data.

Qualitative content analysis is mostly used when a research study has a more descriptive focus (Drisko & Maschi, 2016; Schreier, 2012). As such, I used qualitative content analysis to describe how my data relate to my research question. As described above, qualitative content analysis is systematic in nature, and it involves same sequence of steps regardless of the data and research questions. I utilized the steps provided by Schreier (2012, p. 6) for analyzing my data. These steps are listed in Table 5.

Table 5. Steps in Qualitative Content Analysis

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1. Deciding on your research question
 2. Selecting your material
 3. Building a coding frame
 4. Dividing your material into units of coding
 5. Trying out your coding frame
 6. Evaluating and modifying your coding frame
 7. Main analysis
 8. Interpreting and presenting your findings
-

Note. Adapted from *Qualitative Content Analysis in Practice* by M. Schreier, 2012, p.6

Previously, I described how I decided on my research questions and selected the textbooks I analyzed, as the first and the second steps provided in Table 5. In the next sections, I provide the details on how I developed my coding frame and divided the material into units of coding. I tried out my coding frame by conducting a pilot study as the fifth step given in Table 5.

Coding Frame, Data Collection, and Analysis

In qualitative research, data collection and analysis processes can be overwhelming for researchers since there are numerous amounts of material to discover. However, qualitative content analysis prevents researchers from getting lost in this phase, since it allows researchers to develop a coding frame based on the key aspects, which researchers desire to focus on (Schreier, 2012). In the following sections, I explain the developing phases of the coding frame, and then procedures of data collection and data analysis that guided me to answer my research questions.

As noted previously, qualitative content analysis is a flexible method, which allows researchers to build their coding frame using concept-driven strategies, data-driven strategies or frequently both (Schreier, 2012). Using concept-driven strategies to build a coding frame means that you come up with a coding frame based on a theory, previous research, everyday

knowledge, etc. without looking at your data. Using data-driven strategies allow researchers to build their coding frame based upon their data. Researchers use this strategy to build their coding frame when the purpose of their study is to describe the material they analyze in detail (Schreier, 2012).

For this study, I use both concept- driven and data-driven strategies to form my coding frame. First, I used concept-driven strategies to build my main categories (dimensions) and subcategories. For that purpose, I used the *Domains of Mathematical Knowledge for Teaching* (MKT) as my main categories and subcategories emerged. The framework of MKT was developed by Ball et al. (2008), as illustrated in Figure 1. Since I analyzed the given opportunities for PSTs to develop specific domains of SMK and PCK, I considered SMK and PCK as my main categories. I also considered each domain of SMK and PCK as subcategories of my study. Specifically, I had two main categories: SMK and PCK; and I had three subcategories for each main category. CCK, HCK and SCK are the subcategories of SMK. As previously stated, CCK is the mathematical knowledge, which is not unique to teaching, and most mathematically knowledgeable adults possess it. In short, SCK is the mathematics content knowledge that is specific to teaching, and HCK encompasses understanding how topics in mathematics are related.

There are three subcategories of PCK: KCS, KCT, and KCC. Briefly, KCS is the knowledge that teachers need to possess in order to understand students' mathematical thinking and reasoning. The knowledge that combines understanding mathematical content and teaching is KCT. Knowing available curriculum materials and programs for teaching mathematics is defined as KCC. See Table 6 for a brief description of each domain.

Table 6. Brief Description of Domains of MKT

SMK	PCK
CCK: Mathematical content knowledge that is not unique to teaching. Teachers and non-teachers need to possess it.	KCS: Knowledge that is necessary to understand students' mathematical thinking and reasoning.
HCK: Mathematical knowledge that is needed to understand how topics in mathematics are related.	KCT: Knowledge that combines mathematical content and teaching.
SCK: Mathematical content knowledge that is specific to teaching, and it is not needed for purposes other than teaching.	KCC: Knowledge of available curriculum materials and programs for teaching mathematics.

Building A Coding Frame

As demonstrated in Table 5, there are steps to be followed in qualitative content analysis. And building a coding frame is the third step in qualitative content analysis, which comes after deciding the research question and materials. To build my coding frame, I used concept-driven and data-driven strategies to develop my codes (subcategories of my subcategories). First, I created some codes with the help of the prior researches, theories and also CCSS on fractions, and these codes make the concept-driven part of my framework. I created concept-driven codes based on CCSS on fractions (see Table 1 in Chapter 2), because CCSS provides clear demonstration of what students need to learn at each grade, therefore teachers should possess those high-quality mathematics standards to understand and support students' learning. Besides creating codes based on prior research and considering CCSSM, I also developed some of the codes based on the descriptions of each domain even though there is no content analysis has been done using MKT as a framework. Specifically, without looking at the data, I came up with some codes that define the corresponding domain. For instance, the codes *Compare*, and *Simplifying* could fall under the CCK subcategory, since most well educated adults can compare and simplify

fractions. Then definition of the code *Compare* could be given as “Teachers are expected to compare fractions by finding common denominators or cross-multiplying or converting to decimals” and the code *Simplify* as “Teachers are expected to know how to simplify fractions”. After I developed some codes based on the knowledge I gained from the literature and theories, I used a data-driven strategy to develop additional codes to describe my materials in detail (Schreier, 2012). In qualitative content analysis, data-driven coding frame can be created based on a part of the material, and then it is repeatedly revised as more material is coded. For this manner, I started developing my partly data-driven coding frame based on 10% of the material first.

Qualitative content analysis shares some common techniques with other types of qualitative research (Drisko & Maschi, 2016; Schreier, 2012). For instance, open coding, first step of grounded theory, might be used to build the data-driven coding frame in qualitative content analysis. Open coding is appropriate for building coding frame inductively (data-driven) in qualitative content analysis (Drisko & Maschi, 2016; Elo & Kyngäs, 2008; Lune & Berg, 2017; Schreier, 2012). Strauss and Corbin (1998) describe open coding as “working on a puzzle” (p. 223), and state that researcher has to organize and categorize the pieces by color and build a whole picture by putting each piece back together.

According to Strauss and Corbin (1998), there are three steps for open coding: conceptualizing, defining categories, and developing categories. I first used conceptualizing as the process of “abstracting” the data (Strauss & Corbin, 1998, p. 105) by breaking down the data based on the relevant concept. Then I defined categories by grouping the concepts, which have emerged during the first step based on their similarity. In the last step, I arranged the categories according to how they relate to each other. As previously stated when building coding frame

partly data-driven, it is not required to go through all of the data, instead it can be build based on either some part of the data, which makes qualitative content analysis more flexible than other methods (Schreier, 2012). Therefore, I developed my partly data-driven coding frame based on 10% of the material. According to Mayring (2000, para 11, Figure 11), coding frame should be revised after 10% to 50% of the materials are coded for a “formative” reliability check.

Therefore, I did revise my coding frame by discarding hardly used codes, which don’t address the research questions. However, the coding frame in qualitative content analysis has to be adapted to fit all of the materials, so I continued to revise it by discarding or adding more codes through the analyses of whole data.

I also followed some requirements while developing my coding frame. One requirement is *unidimensionality*, which means that each category in the coding frame needs to capture one aspect of the material. Besides, the subcategories of the coding frame should mutually exclude each other, which means each unit of coding could be assigned one subcategory within a given dimension. However, it does not prevent assigning a unit of coding to subcategories belonging to different dimensions. For this study, there are some tasks consisted of multiple steps since they demand multi-dimensional knowledge; therefore, I considered each step in the units of coding as sub-units of coding to meet this requirement. Another requirement is exhaustiveness. The coding frame can count as exhaustive when each unit of coding in the material is assigned to at least one subcategory, which requires all materials to be captured by the coding frame. Additionally, saturation plays important role to ensure each subcategory is used at least once and no subcategory remains empty. Therefore, I considered these requirements while I developed the coding frame, and Table 7 shows the codes for Subject Matter Knowledge Domain.

Table 7. The Coding Frame for Subject Matter Knowledge

Subject Matter Knowledge
CCK <u>Benchmarks</u>: Teachers are expected to know benchmarks for fractions, the use of them, and be able to place fractions between benchmarks. <u>Compare</u>: Teachers are expected to compare two fractions by finding common denominators/cross-multiplying/converting to decimals etc. and explain their reasoning. <u>Convert</u>: Teachers are expected to know how to convert fractions to decimals, percents, and ratios, or vice versa. <u>Equal Share</u>: Teachers are expected to know and understand the concept of equal share (parts) or equal-sized groups <u>Equation</u>: Teachers are expected to write/make one or more equations for a given fractional tasks, and annotate/explain those equations <u>Equivalence</u>: Teachers are expected to know/use/find equivalent fractions, interpret/use common denominators and explain their reasoning <u>Estimate</u>: Teachers are expected to estimate/predict fractions (i.e., using whole numbers or benchmarks) and explain their thinking <u>Find</u>: Teachers are expected to find (name) fraction using the visuals/models, and/or find the part when given the whole and the fraction; and/or find the whole when given the part and the fraction, and/or find fractions between two fractions. (Tasks might require using models/drawings and/or explanation) <u>Fraction Notation</u>: Teachers are expected to understand the meaning of the fraction symbol and notation/representation (e.g., knowing what a denominator or numerator in fraction tells), and/or be able to provide example of fractions used in daily life. <u>Number Line</u>: Teachers are expected to draw a number line and/or plot given fractions on the number line and/or perform fraction operations by using number line and/or find a fraction between two other fractions on number line. The tasks might require explanation. <u>Operations</u>: Teachers are expected to know and explain definition/meaning/relations of four operations and/or to be able to perform/calculate/give examples of operations with fractions. (Tasks might require the use of models/drawings/explanations/calculator) <u>Order</u>: Teachers are expected to know how to order fractions from least to greatest, and/or from greatest to least, and explain their answer <u>Properties</u>: Teachers are expected to know/explain/use properties of mathematics/arithmetic, and be able to solve fraction tasks require this knowledge <u>Simplify</u>: Teachers are expected to know/interpret simplifying fractions <u>Solve</u>: Teachers are expected to solve fraction tasks (Tasks might be word problem and require an explanation on how to solve them and/or to show their work)
HCK <u>Connections</u>: Teachers are expected to know fractions are connected to the broader mathematical concepts. (This knowledge may help to decide if a student's statement is mathematically significant)

Table 7. (Continued)

Subject Matter Knowledge
SCK
<u>Alternative:</u> Teachers are expected to provide alternative/different solution methods/examples/ways and be able to explain their methods.
<u>Describe/Discuss/Interpret:</u> Teachers are expected to describe/ discuss/interpret/write solution methods/terms/procedures/data and their differences.
<u>Determine:</u> Teachers are expected to determine and write the fractional parts/expressions of a given shapes and explain their reasoning.
<u>Illustrate:</u> Teachers are expected to illustrate/draw/show/sketch a given fraction task to solve a problem and be able to explain/justify how the illustration/drawings/answer help them to solve the problem.
<u>Meanings:</u> Teachers are expected to know and describe all interpretations/definitions/meanings of fractions and models used for all meanings and explain (possible) limitations of the meanings.
<u>Models:</u> Teachers are expected to know/interpret/use/make sense of representations and models to find/show fractions and justify their answers.
<u>Problem Posing:</u> Teachers are expected to pose/propose/write/select/determine/modify a word problem/sentence that reflects fraction task (e.g., computation), and be able to explain why it make sense and solve it.
<u>Reasoning:</u> Teachers are expected to use/explain/justify their reasoning strategies for a given task; i.e., why the procedure make sense/what is the logic (rationale) behind the procedure/does it work in general
<u>Select:</u> Teachers are expected to select appropriate illustrations/models/representations for given fractions, and explain their selection

Following the same requirements, I mentioned previously, I created codes for Pedagogical Content Knowledge Domain as seen in the Table 8.

Table 8. The Coding Frame for Pedagogical Content Knowledge

Pedagogical Content Knowledge
KCS <u>Misconceptions</u>: Teachers are expected to know and understand students' misconceptions, errors, limited conceptions and thinking, and difficulties toward fractions concepts, and be able to explain why and how these misconceptions occur. <u>Representations</u>: Teachers are expected to know alternative/appropriate representations/models that students might use when solving fractional tasks <u>Strategies</u>: Teachers are expected to know and/or explain strategies of students when solving fraction tasks <u>Students Reasoning</u>: Teachers are expected to understand/know/explain students' reasoning with fractions, and to be aware of the possible students' responses/solutions/drawings/methods and determine whether or under what circumstances they are valid or not valid. <u>Understanding</u>: Teachers are expected to know/decide what skills/understanding may be needed for students to complete tasks, and what steps to be completed for student's understanding of fractions.
KCT <u>Activities</u>: Teachers are expected to select/identify/develop appropriate activities/models to use when teaching fractional concepts, and to know the advantages and disadvantages of these activities/models <u>Assessment</u>: Teachers are expected to know/use appropriate assessment strategies when teaching fractions to assess students' understanding <u>Before/After</u>: Teachers are expected to know what to do before/after (follow up) when they teach fraction concepts, or which concepts should be taught separately, together, first, last, etc. <u>Support</u>: Teachers are expected to provide context/methods/questions/explanations to support/expand students' understanding, and/or to overcome difficulties/confusion/misconceptions, and/or to work/modify students' work to make it correct. <u>Teaching Techniques</u>: Teachers are expected to know/use/explain/select appropriate techniques/instruction methods/sequence/strategies/learning experiences for teaching fraction concepts
KCC <u>Standards</u>: Teachers are expected to know the Standards (Number and Operations) and Objectives for Fractions and be able to decide when to introduce/reintroduce these standards.

Developing the initial coding frame was the third step in qualitative content analysis; in the next part I describe the fourth step, which is dividing the material into units of coding.

Dividing Your Material into Units of Coding

As previously noted, there are sequential steps in qualitative content analysis, and dividing the material into units of coding comes after building the coding frame. Dividing the material into small pieces is called “segmentation” in qualitative content analysis, and it is important for different reasons (Schreier, 2012, p. 126). Segmentation ensures that researchers consider all relevant material segment by segment; which prevent them from overlooking anything. Furthermore, it allows researchers to check for consistency of coding. And, checking consistency only makes sense if different coders code the same segment of text (Schreier, 2012). Overall, segmentation allows researchers to divide research material into units. For this inquiry, units of analysis refer to all chapters, which are dedicated to fraction concepts in each textbook. And, “units of coding are those parts of the units of analysis that can be interpreted in a meaningful way with respect to your categories” (Schreier, 2012, p. 131). I specified units of coding as mathematical tasks in chapters, which are devoted to fraction concepts in each textbook. Specifically, I considered each mathematical task that is provided to PSTs to be completed as a unit of coding. However, some mathematical tasks require several steps to be completed, such as a task could ask PSTs to multiply fractions, and then illustrate the computation with math drawings. These kinds of tasks consist of more than one step, and as previously mentioned I considered these steps as sub-units of coding. For each sub-units of coding there was one code assigned, therefore the number of tasks in textbooks would not reflect the number of codes assigned.

Structure of The Tasks

I analyzed each fraction chapter in each textbook and found that the tasks (units of coding) were given in different types and purposes (see Table 9). In the first textbook, EMSM,

hundreds of tasks were provided for PSTs to “actively engage in your learning about students learning mathematics” (Van de Walle et al., 2019, p. xvi). *Pause and Reflect* tasks are open-ended questions, and their purpose is to make PSTs to check their knowledge about the concept mostly before it’s introduced. For instance, a task, “Beyond shading a region of a shape, how else are fractions represented? Try to name three ideas”, was offered to PSTs before introducing all of the fraction meanings. There were plenty of tasks embedded in the chapters, which were given through the concept, for instance *Content Problems* and *Activities*. Although, the authors did not indicate the type of the task, I called those tasks “Content Problems”, since they were given to support PSTs’ understanding of the content. All activities provide standards from CCSS, and some of them provide adaptations and instructions for students with special needs and for English Learners. Most of the *Activities* provide PSTs to develop their understanding through hands-on learning. They give PSTs some directions to complete these tasks in their future classes, and some of the *Activities* require handouts, which can be located in the online resource (MyLab Education). Besides providing different types of tasks through the chapters in the textbook, PSTs are also offered online learning experiences. All tasks under *Self-Checks*, *Application Exercises* and *Math Practices* were given in the online resource, and they were designed to support PSTs pedagogical and content knowledge. However, PSTs need to have an access code in order to work on the tasks provided in the e-Text. *Self-checks* are multiple-choice items, which were connected to learning outcomes of each chapter. PSTs are provided an explanations/a rationale when they select correct answers. The authors mentioned that these items are designed for PSTs to self-assess how well they have grasped the concept. *Application Exercises* are open-ended questions, which require PSTs to watch a short video clip to answer accompanying questions. These tasks provide opportunities for PSTs to reflect their knowledge

on real classroom situations. PSTs are also given expert feedback when they submit their responses. Lastly, *Math Practices* are a combination of fill-in-the-blank and multiple-choice questions and are offered at the end of the chapter to help PSTs practice their skills with the content covered in chapters. EMSM provides 162 tasks in total, and almost half of them (75 tasks) were given in its online resource.

In the second textbook, META, the chapters were divided into lessons (sections), and *Class Activities* are given through the lessons. The tasks consisted of different sort of items that offer opportunities for critiquing reasoning and using alternative and nonstandard methods. For instance, PSTs are given chances to explore students' misconceptions or multiply fractions mentally. *Practice Exercises* are open-ended questions and can be found at the end of each lesson. PSTs are given solutions right after the *Practice Exercises* to check themselves. Besides, each lesson also concludes with tasks that are called *Problems*. *Problems* are also open-ended, but unlike *Practice Exercises*, they do not offer solutions, and they focus on the concept covered in each section.

The third textbook, HCLM, is the only textbook that offers tasks before introducing the concept. The chapters start with tasks called *Reflection on the Lesson*. These tasks can be answered by watching a short video, which can be easily accessed online (no access code needed), and the videos offer real classroom situations. The authors mentioned that working on these tasks give PSTs a good preparation for teacher certification tests. Besides, *Chapter Learning Outcomes* are given before the chapter introduced, and focus on major issues related to chapter. The purpose of these tasks is to guide PSTs to read and encourage discussion. *In the Classroom* activities are provided throughout the chapter, and mostly require the use of variety of models. *Common Errors* are provided through the chapters, which can be made by students

and as well as teachers. The authors provided “why” these common errors occur and offered “what” can be done to prevent these errors. There are also problems embedded in the chapter, however authors did not specifically label these tasks, so I called them “*Content Problem*”, since their focus was to help PSTs make sense of the content covered. Each chapter concludes with *Things to Do* tasks, which were given in two parts. The first part is called “From what you’ve read”, and tasks offer practices on the concept introduced in the chapter. The second part “Going beyond this book” combines different activities that can be completed working with students, using additional resources such as articles, books and technology and writing down PSTs’ personal experiences as students and teachers. Overall, each textbook provided fractional tasks in different ways, and Table 9 provides the structure of fractional tasks.

Table 9. Structure of Tasks in Three Textbooks

Textbook	EMSM	META	HCLM
Types of Tasks	Pause and Reflect Content Problem Activity Self-Check Application Exercise Math Practice	Class Activity Practice Exercises Problems	Reflecting on the Lesson Chapter Learning Outcomes In the Classroom (Activity) Common Error Content Problem Things to Do

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

The number of the units of coding per each unit of analysis is provided in the Table 10. There were some mathematical tasks that required more than one step to be completed, therefore there were some units of coding that contained more than one code. For instance, META provides many fractional tasks that have sub-questions, e.g., 1.a, 1.b, 1.c, and 1.d. Although these

mathematical tasks require PSTs to complete more than one step; I labeled the task as one unit of coding, but I coded each step considering them as sub-units of coding.

Table 10. The Total Number of Units of Analysis and Units of Coding

Textbooks	No. of Units of Analysis	No. of Units of Coding
EMSM	UoA 1	81
	UoA 2	81
	UoA 3	122
META	UoA 4	45
	UoA 5	47
	UoA 6	86
HCLM	UoA 7	32

Note. UoA = Unit of Analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

Overall, I analyzed 162 tasks from EMSM, 300 tasks from META, and 32 tasks from HCLM to seek this inquiry.

Trying Out the Coding Frame (Pilot Study)

Once I developed my initial coding frame and segmented my material into units of coding, I tried out the coding frame as the fifth step of qualitative content analysis. The purpose of this step is to “discover the shortcomings of the coding frame at an early stage”, before the main coding (Schreier, 2012, p. 147), and also to help the coders to get familiar with categories. For the pilot phase, I selected 20% of the material, and made sure to have variability and practicability on my selection. Specifically, I randomly selected 20% of the material from each unit of analysis to ensure I included enough variability, and also to make sure I could apply most of the codes in the coding frame to my material. While I conducted the trial coding, I numbered my units of coding to keep record of my codes. I followed the order of textbooks as shown in

Table 4, and numbered chapters devoted to fraction concepts (units of analysis) from 1 to 7 since there were two chapters in EMSM, four chapters in META and one chapter in HCLM. Thus, in this recording system and the following the orders of textbook in Table 4, the first textbook, EMSM, has 1st and 2nd units of analysis; the second textbook, META, has 3rd, 4th, 5th and 6th units of analysis, and the last textbook, HCLM, has the 7th unit of analysis. As an example of the recording system, the first unit of coding in the first chapter of the first textbook was referred to as 1.1, the fifth unit of coding in the fourth chapter of the second textbook as 6.5, the sixth unit of coding in the chapter of the third textbook as 7.6, etc. This recording system was helpful for the other coders involved in the coding process for the reliability issues. The revised coding frame based on 20% of material can be seen in Table 7 for SMK and Table 8 for PCK. And, Table 11 provides several mathematical tasks (units of coding) from textbooks to illustrate some of the codes and their definitions.

The first task was selected from HCLM (Reys et al., 2015, p. 262), the second task was from META (Beckmann, 2018, CA-90), and the last two tasks were from EMSM (Van de Walle et al., 2019 p. 375, p. 365-Self-Check 15.4). I coded the first task (the unit of coding) as *Illustrate* and *Meanings*, since PSTs are expected to know different meanings of a fraction and to be able to illustrate these meanings. These codes fall under SCK, because the knowledge that is necessary to complete this task is not needed for purposes other than teaching. The second task was coded as *Find*, *Equation*, and *Equivalent*. PSTs are asked to find the part when given the whole and the fraction, then to write an equation and to use equivalent fractions. Teachers and non-teachers alike can do this task; thus, it falls under CCK.

Table 11. Sample Mathematical Tasks with Codes and Definitions

Task	Code
1. Illustrate three different meanings of $\frac{3}{4}$. (Reys et al. (2015), p. 262)	<u>Illustrate (SCK)</u>: Teachers are expected to illustrate/draw/show/sketch a given fraction task to solve a problem and be able to explain/justify how the illustration/drawings/answer help them to solve the problem. <u>Meanings (SCK)</u>: Teachers are expected to know and describe all interpretations/definitions/meanings of fractions and models used for all meanings and explain (possible) limitations of the meanings.
2. If 1 serving of dozen slurpy is $\frac{4}{5}$ of a liter, then how many liters are in $\frac{2}{3}$ of a serving? Complete the math drawing and write and annotate an equation for this situation. Don't forget to use equivalent fractions. 1 serving $\longrightarrow \frac{4}{5}$ liter $\frac{2}{3}$ serving $\longrightarrow$ _____ liters (Beckmann (2018), p. CA-90)	<u>Find (CCK)</u>: Teachers are expected to find (name) fraction using the visuals/models, and/or find the part when given the whole and the fraction; and/or find the whole when given the part and the fraction, and/or find fractions between two fractions. <u>Equation (CCK)</u>: Teachers are expected to write/make one or more equations for a given fractional tasks, and annotate/explain those equations. <u>Equivalence (CCK)</u>: Teachers are expected to know/use/find equivalent fractions, interpret/use common denominators and explain their reasoning.
3. Try to think of two ways that students might solve the following problem without using a common-denominator symbolic approach. Jack and Jill ordered two medium pizzas, one cheese and one pepperoni. Jack ate $\frac{5}{6}$ of a pizza, and Jill ate $\frac{1}{2}$ of a pizza. How much pizza did they eat together? (Van de Walle et al. (2019), p. 378)	<u>Strategies (KCS)</u>: Teachers are expected to know and/or explain strategies of students when solving fraction tasks. <u>Solve (CCK)</u>: Teachers are expected to solve fraction tasks (Tasks might be word problem and require an explanation on how to solve them and/or to show their work).

Table 11. (Continued)

Task	Code
4. Which of the following is not a way to illustrate equivalent fractions?	<u>Illustrate (SCK)</u> : Teachers are expected to illustrate/draw/show/sketch a given fraction task to solve a problem and be able to explain/justify how the illustration/drawings/answer help them to solve the problem.
a) Cut a paper strip, shade part of the strip, and ask students to use paper folding to describe what fraction of the strip is shaded	
b) Place a pile of 24 two-color counters with $\frac{1}{4}$ showing red under the document camera and ask students to tell you different ways to tell what fraction is red	<u>Equivalence (CCK)</u> : Teachers are expected to know/use/find equivalent fractions, interpret/use common denominators and explain their reasoning.
c) Draw a rectangle on grid paper with part of it shaded and ask students to determine the fraction that is shaded while giving different possible answers	<u>Teaching Techniques (KCT)</u> : Teachers are expected to know/use/explain/select appropriate techniques/instruction methods/sequence/strategies/learning experiences for teaching fraction concepts.
d) Show an algorithm of multiplying the numerator and denominator by the same number	
(Van de Walle et al. (2019), p. 367-Self-Check 14.3.1)	

The third task was coded as *Strategies* and *Solve*, since the task requires knowledge of pedagogy and subject matter. In other words, PSTs are expected to know the content and students, in other words they need to think of different students' strategies to complete a fraction task. In order for PSTs to answer the last task, they need to possess different kind of knowledge. First, PSTs are expected to know what equivalent fractions are (coded as *Equivalence*, which falls under CCK). Then, they need to know how to illustrate equivalent fractions (coded as *Illustrate*), which is unique to teaching (SCK). Then they are expected to select appropriate teaching technique (coded as *Teaching Techniques*) to show equivalent fractions, which fall under the category of KCT.

In qualitative content analysis it is essential to recode all the material if there are any changes on the coding frame. During the trial coding, I realized two of my codes were sharing definitions, thus I altered some codes under the subcategories. For that reason, I started the coding process over one more time by coding 20% of the material again. It is important to take notes during the coding process; therefore, I took notes of any difficulties and concerns I encountered. For the trial coding, I did not use a second coder, therefore when I was done with the initial coding; I waited around 2 weeks for the second coding to assess the coding frame's quality, which I discuss the details in the following section.

Evaluating and Modifying the Coding Frame

According to Schreier (2012), after trying out the coding frame (the pilot phase), it is important to evaluate it and check its quality as the sixth step given in Table 5. This is where reliability and validity measures get involved. I discuss reliability and validity in the following sections in more detail; but generally, a coding frame can be considered reliable if it produces error free data. In order to do so, two or more coders use the coding frame to analyze the same units of coding independently (blind coding). According to Weber (1990), in order “to make valid inferences from the text, it is important that the classification procedure be reliable in the sense of being consistent: Different people should code the same text in the same way” (p. 12). Using the same coding frame, one coder analyzes the same units of coding over time to get stable analysis results (Schreier, 2012). Furthermore, a coding frame can be considered valid, if categories in the coding frame sufficiently represent the concepts in research questions. In the following sections, I present how I evaluated my coding frame and checked its quality.

Reliability

In qualitative content analysis, it is important to go beyond your individual understanding and interpretation in order to check consistency, which refers to reliability (Schreier, 2012). To ensure reliability, I did check stability and reproducibility. For checking stability (intra-rater reliability), following the pilot phase I reanalyzed the some of the tasks after two weeks and checked whether I get the stable analysis results. I used comparison-coding sheet as shown in Table 12 and that helped me track my first, second, and the final coding to ensure consistency.

Table 12. Comparison-Coding Sheet

No. of Units of Coding	First Code	Second Code	Final Code
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I found that there were some units of coding that I interpreted differently at the two points in time. To check consistency on my codes, I calculated the coefficient of agreement, which is a method to assess reliability in qualitative content analysis. Calculating the percentage of agreement is one of the most common ways to determine coefficients of agreement. I calculated the percentage of agreement as dividing number of units of coding that share the same codes by the total number of units of coding, and then multiply it by 100. If the percentage of agreement is close to 100%, then the study can be called highly reliable. I had 87% of agreement on my first and second codes, which can be considered as reliable. I was able to identify why I chose one code at the first time and an another at the second time; thus, differences between the first and second codes helped me to think more about the coding frame and improve my understanding of it. As previously stated, the pilot phase in qualitative content analysis focuses on modifying and improving the coding frame, therefore it helped me to finalize the coding frame and made it ready for the main analysis.

For checking reproducibility (inter-rater reliability), I used a second independent coder's support. The second coder recently received her doctoral degree in mathematics education and had conducted a content analysis for the dissertation. The second coder was trained until she met the reliability requirements. During the training, I met with the second coder and discussed the description of the main categories and subcategories, the codes and their definitions in the revised coding frame. Then I randomly selected some units of coding, and we coded those mathematical tasks together. This step helped the second coder get familiar with the coding frame and data. To ensure that the second coder meets the reliability requirements, I randomly selected and shared 20 mathematical tasks (approximately 4% of total tasks) with the second coder to analyze. The second coder did double-coding (recoding the same tasks after a while) to check stability, which helped me to see if the second coder got the same results when she reanalyzed the tasks at different times. The second coder had 90% agreement on her first and second coding.

After the second coder met the reliability requirements, I randomly selected 10% of units of coding (mathematical tasks) and shared with the second coder to check whether the revised coding frame lead us to code the same units of coding under the same category when coding independently of each other (blind coding) (Schreier, 2012). To determine consistency between the second coder and I, I calculated coefficient of agreement as well. To calculate percentage of agreement, I divided the number of units of coding in which the codes agrees by the total number of units of coding, then multiplied it by 100. There was 93% of agreement between my coding and second coder's coding on tasks. It is essential to indicate that most of the tasks involved more than one code; therefore, I focused on the agreement on tasks (units of coding) as mentioned in the calculation process of percentage of agreement.

Validity

Validity is another important consideration that concerns the quality of the coding frame besides the reliability in qualitative content analysis. According to Krippendorff (2004), a measuring instrument can be considered as valid if it captures what it claims to capture. In other words, the coding frame can be stated as valid if its categories represent the concepts in the research question (Schreier, 2012). Although there are variety of validity types, such as face, content, construct, social, empirical, etc., most researchers who conduct content analysis rely on face validity since content analysis primarily focus on texts (Krippendorff, 2004). Generally “face validity refers to the extent to which your instrument gives the impression of measuring what it supposed to measure” (Schreier, 2012, p. 185), and Neuendorf (2002, p. 115) called it as “WYSIWYG (what you see is what you get)” validity. Besides, content validity measures whether the instrument covers all dimensions of concept being measured. Overall, face validity and content validity concern the instrument and the concept being measured. In order to validate the coding frame, I assessed face validity (when dealing with data-driven (inductive) coding frames) and content validity (when dealing with concept-driven (deductive) coding frames). Specifically, I assessed the face validity of inductive coding frame by checking if the coding frame “measures what its user claims it measures” (Krippendorff, 2004, p. 313). Besides, there is another consideration to assess face validity. As mentioned earlier, qualitative data allow researchers to reduce data. However, if the categories in the coding frame are too abstract, the results do not have sufficient face validity. Thus, I went over my material several times in order to not miss any aspect of my data.

Furthermore, Schreier (2012) stressed that content validity is useful to assess the validity of deductive (concept-driven) coding frame. And, having another researcher, who is familiar

with the concept, to take a look at the coding frame is a good way to assess content validity (Elo & Kyngäs, 2008; Schreier, 2012). In this study, I used the Domains of Mathematical Knowledge for Teaching (MKT) from Ball et al. (2008) to develop the coding frame in a deductive way; thus, I discussed the coding frame with another researcher who knows the dimensions of MKT. Generally, the coding frame in qualitative content analysis is developed inductively through the coding process. According to Schreier (2012), “your coding frame can be regarded as valid to the extent that your categories adequately represent the concepts in your research question, and to achieve this you have to adapt your frame so as to fit your material” (p. 7). Therefore, I kept checking whether my coding frame reflects the material I have and covers the concepts in my research questions.

Overall, I created the coding frame based on an existing framework (MKT), then I received feedback from another researcher, then I made changes on the framework by going through it carefully. Over the time, I revised the framework systematically after analyzing the data. Therefore, all these procedures did enhance the validity of the framework. Based on the steps in qualitative content analysis (see Table 5), main analysis comes after evaluating the coding frame. Therefore, I discuss the main analysis phase of this study in the next section.

Main Analysis

Building the coding frame and trying it out by checking its reliability and validity helped me to have modified coding frame to start the main analysis phase. In the main analysis phase, there are four steps to follow: preparation for main analysis, doing the main analysis, making sense of the material (comparing codes and discussing), and preparation for results (Schreier, 2012). As the first step, being prepared for main analysis, I went through the revised (modified) version of the coding frame to be more familiar with it. As a next step, I did read all units of

analysis one after another and identified relevant parts by marking each unit of coding. As mentioned earlier, some of the mathematical tasks require more than one steps to be completed by PSTs; so, reading each task carefully and marking where each unit of coding begins, and ends was helpful. Afterwards, I started the main coding by using the final version of the coding frame. At this step, qualitative content analysis does not allow making any changes in the coding frame, since any changes would cause coding all material once more (Schreier, 2012). It is very critical to develop the coding frame that fits whole material. Thus, if there were a need to alter the coding frame, I would have considered the main coding as a second trial coding (second pilot phase). Then, after completing the second trial coding, I would have coded all the material once again. Fortunately, however, I did not have to make any changes on the coding frame, therefore the second pilot phase did not occur in this inquiry. When I completed coding all material, qualitative content analysis requires recoding some parts of the material after about two weeks. According to Schreier (2012), it is important to compare codes during the main analysis to check the quality to the coding frame and also to decide on meanings of units of coding. Thus, I made sure to recode some units of coding from each unit of analysis after two weeks. In order to do so, I randomly selected around 10% of tasks (53 tasks in total) from each unit of analysis and recoded them. Similar to the step I completed in pilot phase, I created comparison-coding sheet to record the first, second and the final coding of the units of coding that were double coded. The comparison-coding sheet consisted of three columns: first code, second code, and final code. The example of the comparison-coding sheet can be found in Table 13.

Table 13. Example of Comparison-Coding Sheet for SMK

Units of Coding	First Code	Second Code	Final Code
1.3	CCK (Find)	CCK (Find)	CCK (Find)
4.35	SCK (Select)	SCK (Illustrate)	SCK (Illustrate)
7.24	CCK (Operations)	CCK (Operations)	CCK (Operations)

When there were differences between the first and second code, I had to decide between them to determine the final code; and second unit of coding (4.35) illustrated in the Table 13 provides an example of decision process. During this process, acquiring the second coder's opinion could be an option; therefore, I had received another researcher's opinion on some of the tasks to determine the final code. I also defined the coding consistency by calculating percentage of agreement for those units of coding, which were coded twice, in order to get information about the quality of the coding frame. The percentage of agreement was 92% following the main coding, which was 87% following the pilot phase; thus, it indicates that I was able to improve the coding frame slightly after the pilot phase.

After comparing codes and deciding the final meanings of the units of coding, the last step of the main analysis is the preparation of the results. In order to do so, qualitative content analysis requires focusing on units of analysis instead of units of coding if units of analysis are larger than units of coding, which was the case for this inquiry (Schreier, 2012). Specifically, I transformed the results to the level of the units of analysis. In previous sections, I defined my units of analysis as the chapters devoted to fraction concepts, and there are seven units of analysis in the study. Since there were two fraction chapters available for PSTs in the first textbook, the 1st and 2nd units of analysis belong to EMSM. The second textbook, META,

includes the 3rd, 4th, 5th, and 6th units of analysis. The third textbook, HCLM, contains the 7th unit of analysis.

To answer my research questions, I created a data matrix, which included all of my units of analysis (which are given in the lines of matrix) and all of my main categories (given in the columns of the matrix). As stated previously, I present the results in percentages rather than frequencies, however data matrix helped me to track on how many times (if any) a subcategory was coded for a given unit of analysis. An example of the data matrix can be found in Table 14 (please note that the numbers in the matrix below does not reflect the actual results).

Table 14. Partial Data Matrix Example

No. of UoA	SMK			PCK		
	CCK	HCK	SCK	KCS	KCT	KCC
UoA.1	3	2	5	2	6	1
UoA.3	4	1	3	4	2	5
UoA.6	6	4	3	4	5	2

Creating the data matrix, by combining all of the coding sheets, was the last step of main analysis. In this section, I provided details on how I completed the main analysis. In the following section, I provide the summary of the chapter.

Summary of Method

In this chapter, I described the research design and methodology that I used during this study. Specifically, I presented and discussed the research questions, sample selection process, research design method, the coding frame, and the data collection. Then I discussed the

reliability and validity measures and concluded the chapter with main analysis. In the next chapter, I present results of the main analysis.



CHAPTER 4: RESULTS

The purpose of this inquiry was to explore the kind of opportunities that are provided in mathematics methods textbooks for elementary pre-service teachers (PSTs) to develop Mathematical Knowledge for Teaching Fractions (MKTF). To do so, I selected and examined three elementary mathematics methods textbooks that are used in mathematics methods courses.

Research Questions

The following research questions were addressed in this inquiry:

- 1) To what extent and in what ways do textbooks used in mathematics methods courses in teacher preparation programs provide PSTs opportunities to develop MKTF?
 - a. What opportunities are provided in the textbooks for PSTs to develop subject matter knowledge for fractions?
 - i. What opportunities are provided in the textbooks for PSTs to develop common content knowledge for fractions?
 - ii. What opportunities are provided in the textbooks for PSTs to develop horizon content knowledge for fractions?
 - iii. What opportunities are provided in the textbooks for PSTs to develop specialized content knowledge for fractions?
 - b. What opportunities are provided in the textbooks for PSTs to develop pedagogical content knowledge for fractions?

- i. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and students?
- ii. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and teaching?
- iii. What opportunities are provided in the textbooks for PSTs to develop knowledge of content and curriculum?

In previous chapters, I provided the steps in qualitative content analysis (see Table 5) that I followed to address my research questions. Following the main analysis, the next and the last step is to interpret and present the findings. In qualitative content analysis, there are two ways to present results: in a qualitative style or in a quantitative style (Schreier, 2012). Since I sorted the material using data matrix and coding sheets as described previously, I present the results using a quantitative style. Presenting the results in a quantitative style typically means that the focus would be on the categories instead of cases (Schreier, 2012). Schreier (2012) provided three strategies to present findings in quantitative way: “providing absolute frequencies, doing descriptive group comparisons, and using inferential statistics” (p. 231). In this study, I used descriptive group comparisons, since I collected data from different sources and compared coding frequencies among textbooks. As mentioned, some of the textbooks had more units of analysis as well as units of coding than others, so their data size differed; therefore, Schreier suggested to present results by reporting percentages instead of absolute frequencies. For instance, the 2nd textbook, *Mathematics for Elementary Teachers with Activities, 5th Edition* (META; Beckmann, 2018), consists of 300 fractional tasks, and could have 8 KCC tasks. The 3rd textbook, *Helping Children Learn Mathematics, 11th Edition* (HCLM; Reys et al., 2015), includes 32 tasks in total, and 5 of them could demand KCC. Although, 8 tasks are more than 5

tasks, the ratio of 8 tasks to 300 tasks (the total number of tasks in the 2nd textbook) would be very small compared to the ratio of 5 tasks to 32 tasks (the total number of tasks in the 3rd textbook). The results would make sense if there were same number of units of analysis as well as units of coding in each textbook, which was not the case for the sample of this inquiry. Therefore, I transformed the frequencies into percentages to answer my research questions. According to Schreier, calculating frequencies and reporting the results with percentages “does not make the method any less qualitative” (p. 239) since presenting results comes after conducting and completing the qualitative content analysis.

This chapter is divided into three main sections that are aligned with the research questions, and first two sections are divided into four parts. In the first section, I report the findings on opportunities provided in each textbook to improve *Subject Matter Knowledge*. In order to do so, I report the results related to *Common Content Knowledge* in the first part. Then, in the second part I present findings for *Horizon Content Knowledge*. Next, I report the findings related to *Subject Matter Knowledge*. In the last part, I combine the findings from the three sub-questions to present Subject Matter Knowledge in general. In the second section, I present findings on to what extent each textbook offer PSTs to develop their *Pedagogical Content Knowledge*. And this section is divided into four parts as well. First, I report findings for *Knowledge of Content and Student*, and then for *Knowledge of Content and Teaching*. In the third part, I present findings related to *Knowledge of Content and Curriculum*. Then I combine the findings from the three sub-questions to reflect results for Pedagogical Content Knowledge in general. In the final section, I report results combining both Subject Matter Knowledge and Pedagogical Content Knowledge to reflect Mathematical Knowledge for Teaching Fractions. Then, I conclude the chapter with a brief summary of the results.

Opportunities of Subject Matter Knowledge

In this section, I present the results related to the opportunities provided for PSTs to develop their Subject Matter Knowledge (SMK) through mathematical tasks in three mathematics methods textbooks. According to Ball et al. (2008), SMK consisted of three categories: Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK). Therefore, I first report the findings for CCK, HCK, and then SCK. Lastly, I combine all the three categories to represent SMK in general.

Common Content Knowledge Findings

I examined all units of coding (fraction tasks) to explore to what extent they provide opportunities to support and develop PSTs' CCK. I used the CCK codes that I developed in the coding frame to determine if a task demands a common content knowledge to be completed (see Table 15). As previously mentioned, I developed these codes in a deductive and inductive way and eliminated unused codes to make sure the coding frame matches the data I have. Therefore, the number of codes shows differences in each domain.

I evaluated a total of 494 fraction tasks within the three mathematics methods textbooks. Specifically, there were a total of 162 tasks (81 tasks in Unit of Analysis 1 (UoA1), and 81 tasks in UoA2) in the *Elementary and Middle School Mathematics, Teaching Developmentally*, 10th Edition (EMSM; Van de Walle et al., 2019); 300 tasks (122 tasks in UoA3, 45 tasks in UoA4, 47 tasks in UoA5, and 86 tasks in UoA6) in the *Mathematics for Elementary Teachers with Activities*, 5th Edition (META; Beckmann, 2018); and 32 tasks (all of them given in UoA7) in *Helping Children Learn Mathematics*, 11th Edition (HCLM; Reys et al., 2015).

Table 15. CCK Codes and Labels

Codes	Labels
Benchmarks	B
Compare	CMP
Convert	CNV
Equal Share	ES
Equation	EQN
Equivalence	EQU
Estimate	EST
Find	F
Fraction Notation	FN
Number Line	NL
Operations	OP
Order	O
Properties	P
Simplify	SF
Solve	SL

For a detailed description of the CCK codes please refer to Chapter 3. There were tasks that contained more than one type of CCK, such as a task could require both knowledge of operation (OP) and simplifying (SF). In this section I considered a task supports CCK if it consisted of at least one type of CCK. Since the data size was not identical across textbooks,

some of the textbooks had more units of analysis as well as units of coding than others, I present findings in percentages as well as absolute frequencies in this inquiry.

Table 16. The Number and Percentage of Tasks Support CCK in Units of Analysis and Textbooks

Textbook	No. of UoA	Total Tasks	No. of CCK Tasks	No. of CCK Tasks	CCK Tasks in Textbook
EMSM	UoA1	81	61	75%	78%
	UoA2	81	65	80%	
META	UoA3	122	80	66%	76%
	UoA4	45	26	58%	
	UoA5	47	43	91%	
	UoA6	86	79	92%	
HCLM	UoA7	32	20	63%	63%

Note. UoA = Unit of Analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

As shown in Table 16, the total number of fractional tasks in seven units of analysis ranged from 32 to 122. UoA7 (HCLM) contained the least number of fractional tasks, whereas UoA3 had the greatest number of tasks. The number of tasks in UoA1 and UoA2 (EMSM) had almost tripled the quantity of tasks in UoA7, whereas the numbers of tasks in UoA3 (META) had almost quadrupled the number of tasks in UoA7. In terms of the percent of fraction tasks that supported CCK across all seven units of analysis, the range was from 58% to 92% (I rounded the data to the nearest whole number). Figure 2 provides a graphical representation of the percentage of CCK tasks across all units of analysis.

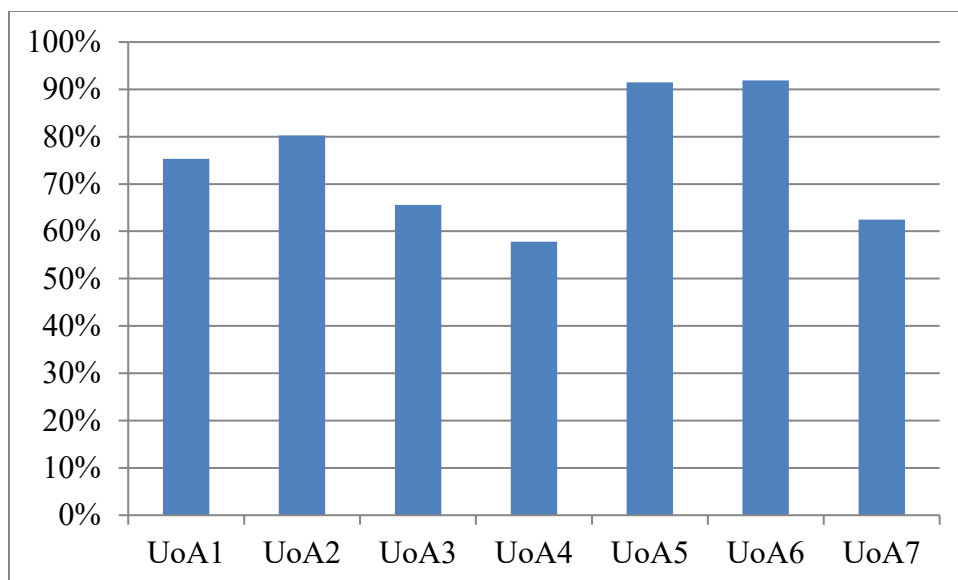


Figure 2. The Percentage of CCK Tasks in Each Unit of Analysis

Note. UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in Mathematics for Elementary Teachers with Activities by Beckmann (2018); UoA7 in Helping Children Learn Mathematics by Reys et al. (2015).

UoA6 offered the highest percentage of tasks that address CCK with 92% followed by UoA5 with 91%. However, UoA4 had the least percentage of fraction tasks that provide opportunities for CCK. The three-fourths of the fraction tasks in UoA1 offered opportunities to develop CCK. Slightly more than three-fifths of the fraction tasks in UoA3 and UoA7, and four-fifths of the tasks in UoA2 encouraged PSTs to develop CCK. Notice that although UoA3 had the highest number of fractional tasks devoted to CCK, it was not the case in terms of percentage due to the size of the data.

Overall, at least three-fourths of tasks in four out of seven units of analysis were provided to support CCK, where the three-fifths of tasks in the remaining units of analysis offered opportunities to develop CCK. The following figure provides a visual representation of the percent of CCK tasks within each methods textbook.

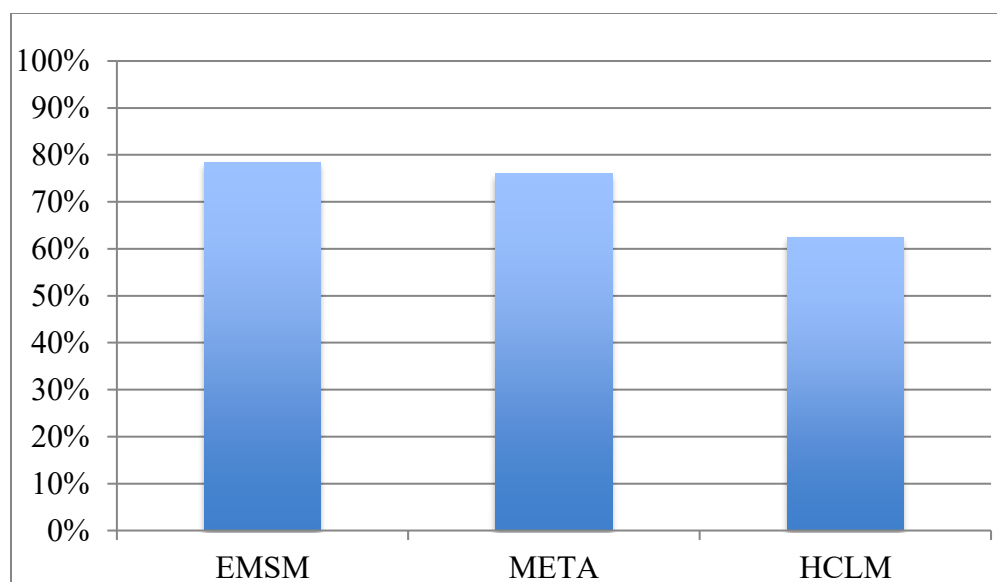


Figure 3. The Percentage of CCK Tasks Across Textbooks

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

A closer examination of Figure 2 and Figure 3 revealed that although the two units of analysis from META had the highest percentage of CCK tasks, overall EMSM textbook contained the largest proportion of fraction tasks demand CCK followed by META. It is important to note that EMSM offered almost half of the fraction tasks in the online resource (MyLab Education), and more than one-third of CCK tasks were given in the e-text. Due to having only one unit of analysis in HCLM, the findings of UoA7 were identical for HCLM textbook. Approximately two-thirds of tasks from HCLM supported CCK.

As previously stated, there were some tasks that involved only one type of CCK, whereas others involved more than one. I recorded each type of codes during the main analysis. The following table represents findings related to what type of CCK was needed to complete tasks in each unit of analysis (UoA) and in textbooks in general. There were 61 tasks that supported CCK in UoA1, and in nearly half of these tasks PSTs were expected to find (F) fractions using visuals or models and fractional parts when the whole was given, or vice versa. There were no chances

given to PSTs to convert (CNV) fractions and to use properties of mathematics (P) to complete a fraction task. Besides, less than 10% of the CCK tasks in UoA1 supported experiencing four operations with fractions (OP), the use of benchmarks (B), to simplify (SF), to compare (CMP), to estimate (EST), to order (O), and to find equivalent (EQU) fractions. In the UoA2, a total number of 65 tasks provided opportunities to develop CCK, and almost 60% of these tasks supported the understanding of four operations with fractions (OP). There were no chances given to experience to convert (CNV), find equivalence (EQU), use fraction notation (FN), order fractions (O), use properties of mathematics (P), and simplify fractions (SF). Nearly one third of the CCK tasks supported solving fraction tasks (SL). Less than 10% of CCK tasks in UoA2 focused on the use of benchmarks (B), fraction comparison (CMP), equal share (ES), writing fraction equations (EQN), and the number line (NL).

As seen in Table 17 and Figure 4, almost half of the CCK tasks in UoA3 provided experiences to support the understanding and the use of the number line (NL) and equivalence of fractions (EQU). Nearly 20% of CCK tasks were given to practice comparison of fractions (CMP) and finding fractions (F). Besides these four types, there were very little, or no chances given to PSTs to experience the rest of the types of CCK, such as benchmarks (B), equation (EQN), estimation (EST), finding (F), operations (OP), etc. UoA4 contained a little more than 80% of CCK tasks that focused on operations (OP). Besides that, 10% of the CCK tasks gave PSTs the opportunities to convert (CNV) and solve (SL) fraction tasks. The remaining types of CCK received little or no attention.

Table 17. The Number of CCK Codes Discovered in Each Unit of Analysis

Textbook	UoA	B	CMP	CNV	ES	EQN	EQU	EST	F	FN	NL	OP	O	P	SF	SL
EMSM	UoA1	2	5	0	8	4	11	3	26	4	10	3	1	0	1	7
	UoA2	4	1	0	3	6	0	8	15	0	2	38	0	0	0	23
META	UoA3	0	13	2	1	0	39	0	15	1	33	0	1	0	1	1
	UoA4	0	1	4	0	0	0	1	0	0	1	21	0	0	0	3
	UoA5	0	1	0	0	9	2	0	2	0	0	37	0	2	0	4
	UoA6	0	0	0	1	4	5	0	2	0	21	65	0	0	0	34
HCLM	UoA7	1	6	0	1	0	6	0	5	2	1	4	1	0	0	1

Note. UoA = unit of analysis; B = benchmarks; CMP = compare; CNV = convert; ES = equal share; EQN = equation; EQU = equivalence; EST = estimate; F = find; FN = fraction notation; NL = number line; OP = operations; O = order; P = properties; SF = simplify; SL = solve; EMSM = Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); META = Mathematics for Elementary Teachers with Activities by Beckmann (2018); HCLM = Helping Children Learn Mathematics by Reys et al. (2015).

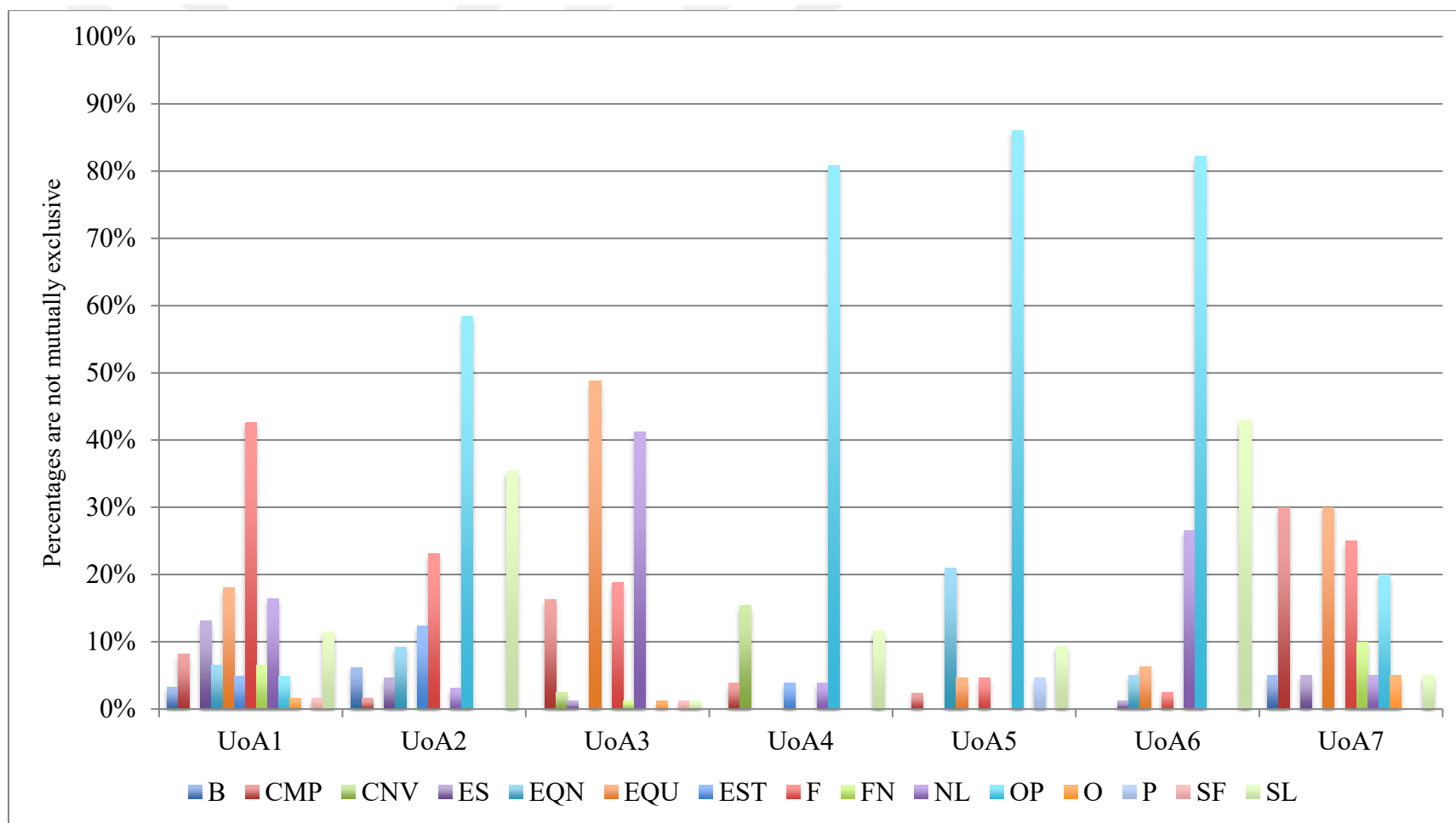


Figure 4. The Percentage of CCK Codes in Each Unit of Analysis

Note. UoA = unit of analysis; B = benchmarks; CMP = compare; CNV = convert; ES = equal share; EQN = equation; EQU = equivalence; EST = estimate; F = find; FN = fraction notation; NL = number line; OP = operations; O = order; P = properties; SF = simplify; SL = solve; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

Similar to the UoA4, operations with fractions (OP) had the highest percentage among other types of CCK in the UoA5. Specifically, 86% of CCK tasks in UoA5 supported four operations with fractions. One-fifth of the CCK tasks were focused on writing and/or explaining equations for fractional tasks. Slight number of tasks supported comparison (CMP), equivalence (EQU), finding (F), properties of mathematics (P), and solving (SL). As seen in the Table 17 and Figure 4, there were no opportunities provided to experience the understanding of the rest of the types.

A greater emphasis was given to operation with fractions (OP) in UoA6, specifically more than 80% of the CCK tasks were devoted to four operations. Nearly half of the CCK tasks supported solving fractional tasks (SL), whereas approximately one-fourth of the CCK tasks were focused on the use of the number line (NL). The opportunities provided for the remaining types of CCK were either very little or not given at all. In the last unit of analysis, UoA7, there were equal amounts of the CCK tasks (about one-third) given to compare fractions (CMP), and to use equivalent fractions understanding (EQU). One-fourth of the CCK tasks demanded knowledge of finding fractions (F), while one-fifth of them required the understanding of fraction operations (OP). The understanding and the use of benchmarks (B), fraction estimations (ES), fraction notations (FN), number line (NL), ordering fractions (O), and solving fractional tasks (SL) were given in the less than one-tenth of CCK tasks. However, no tasks demanded the knowledge of converting fractions (CNV), writing equations for fractions (EQN), estimating fractions (EST), properties of mathematics (P), and simplifying fractions (SF).

Table 18 and Figure 5 display the total number of CCK codes in each mathematics methods textbook. In EMSM, the most needed types of CCK were finding (F), four operations (OP), and solving fractions (SL).

Table 18. The Total Number of CCK Codes Discovered in Each Textbook

Textbook	B	CMP	CNV	ES	EQN	EQU	EST	F	FN	NL	OP	O	P	SF	SL
EMSM	6	6	0	11	10	11	11	41	4	12	41	1	0	1	30
META	0	15	6	2	13	46	1	19	1	55	123	1	2	1	42
HCLM	1	6	0	1	0	6	0	5	2	1	4	1	0	0	1

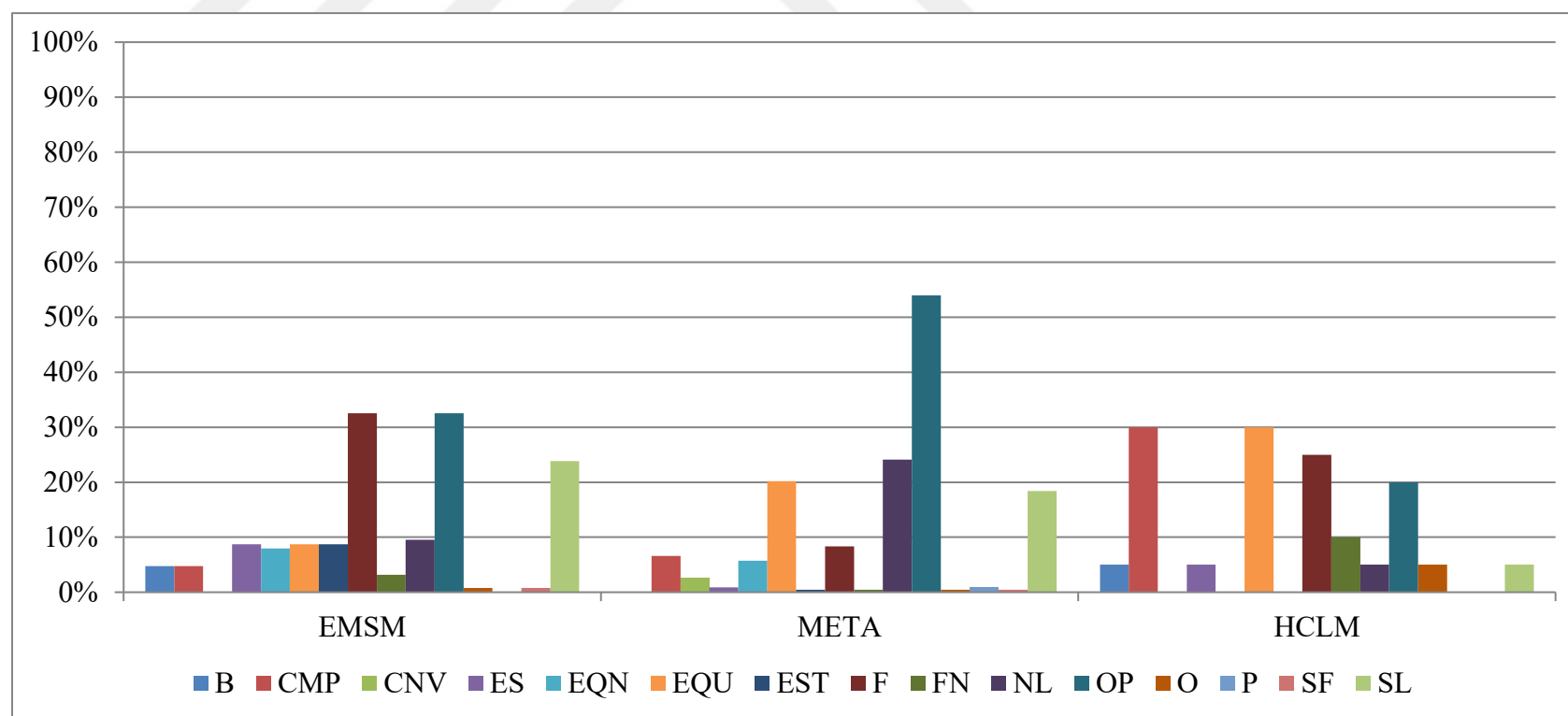


Figure 5. The Percentage of CCK Codes in Each Textbook

Note. B = benchmarks; CMP = compare; CNV = convert; ES = equal share; EQN = equation; EQU = equivalence; EST = estimate; F = find; FN = fraction notation; NL = number line; OP = operations; O = order; P = properties; SF = simplify; SL = solve; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

PSTs were not given any chances to support their knowledge of converting fractions (CNV) and properties of mathematics (P). Besides, opportunities were very limited to practice the use of benchmarks (B) and fraction notation (FN), and comparing (CMP), ordering (O) and simplifying (SF) fractions. In META, more than 50% of CCK tasks were focused on fraction operations (OP). Specifically, there were 228 CCK tasks in total, and 123 of them focused on operations with fractions. Nearly one fourth of CCK tasks included opportunities to experience equivalent fractions (EQU), number line (NL), and a little less than one fifth of CCK tasks supported solving fractional tasks (SL). The chances given to develop the rest of the types of CCK were either very limited or not given at all. The results of HCLM were identical to the results of UoA7, therefore fraction comparison (CMP) and equivalent fractions (EQU) were the mostly used types of CCK with 30%, followed by finding fractions (F) with 25%.

Overall, opportunities to develop CCK differed across the three mathematics methods textbooks. However, a little or no emphasis was given to the knowledge of benchmarks (B), conversion (CNV), equal sharing (ES), equation (EQN), estimation (EST), ordering (O), properties of mathematics (P) and simplifying fractions (SF) among all three textbooks. The operations with fractions (OP) was the only type of CCK, which was offered by high percentages (between 20% to 54%) among three textbooks.

Horizon Content Knowledge

To explore the kind of opportunities provided for PSTs to develop their HCK, I examined all units of coding. In order to find if a task required HCK to be completed, I used the framework that I developed (see Table 19). There was only one type of HCK code, which was Connections (CON). Chapter 3 provides more detailed description of the HCK code.

Table 19. HCK Codes and Labels

Codes	Labels
Connections	CON

As previously stated, there were a total of 494 fraction tasks within the three mathematics methods textbooks. More precisely, EMSM consisted of 162 tasks, META had 300 tasks, and HCML had 32 tasks in total. As stated in the previous section, some tasks consisted of more than one type of CCK, however this was not the case for HCK since there was only one type of HCK code. Table 20 displays the number of HCK tasks in seven units of analysis ranged from 0 to 4. There were no chance given in the UoA1 and UoA2 to support HCK, while UoA3, UoA4 and UoA5 provided only one HCK task. UoA7 had the highest number of HCK tasks followed by the UoA6.

Table 20. The Number and Percentage of Tasks Support HCK in Units of Analysis and Textbooks

Textbook	No. of UoA	Total Tasks	No. of HCK Tasks	No. of HCK Tasks	HCK in Textbook
EMSM	UoA1	81	0	0%	0%
	UoA2	81	0	0%	
META	UoA3	122	1	1%	2%
	UoA4	45	1	2%	
	UoA5	47	1	2%	
	UoA6	86	3	3%	
HCLM	UoA7	32	4	13%	13%

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

A closer examination of Table 20 and Figure 6 showed the differences in terms of the percentages of tasks support HCK across all units of analysis. UoA1 and UoA2 did not contain any HCK tasks, thus the percentage of HCK tasks of these units of analysis was zero. Although the number of HCK tasks in UoA3, UoA4 and UoA5 was the same, the percentage of HCK tasks slightly differed among these units of analysis. They all consisted of one HCK task, but the percentages varied between 1% and 2% since their data size was different. In terms of tasks that addressed HCK, almost one out of seven tasks in UoA7 supported PSTs to develop HCK, while the percentage was 3% in UoA6.

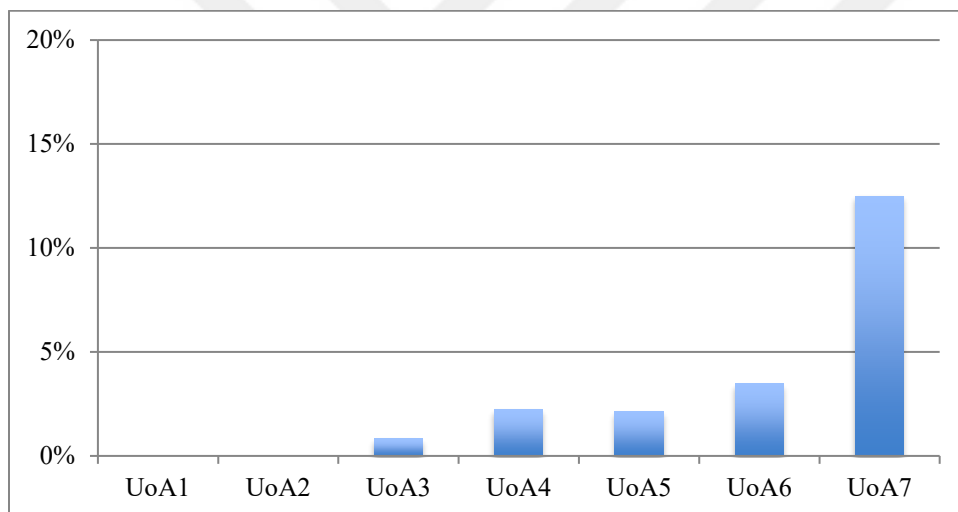


Figure 6. The Percentage of HCK Tasks in Each Unit of Analysis

Note. UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

In general, opportunities differed across the seven units of analysis. Two out of seven units of analysis did not provide any HCK tasks, whereas four units of analysis consisted of less than 5% of tasks supporting HCK. The following figure provides a visual representation of the percentage of HCK tasks within each mathematics methods textbook.

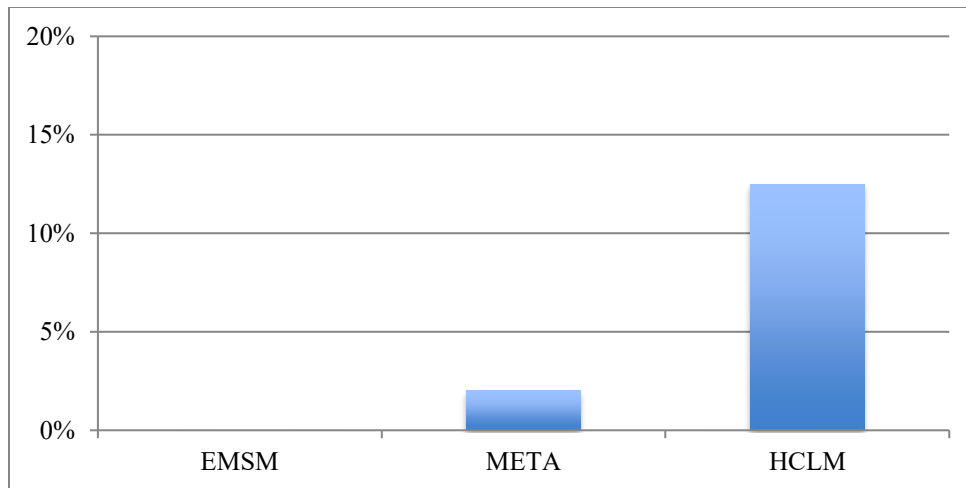


Figure 7. The Percentage of HCK Tasks Across Textbooks

Note. EMSM = Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); META = Mathematics for Elementary Teachers with Activities by Beckmann (2018); HCLM = Helping Children Learn Mathematics by Reys et al. (2015).

As seen in the Figure 7, EMSM did not provide any HCK tasks. Although, the total number of HCK tasks in META was higher than the amount in HCLM, HCLM had the highest percentage of HCK tasks due to its size. Specifically, there were six HCK tasks out of 300 tasks in META, and four out of 32 in HCLM. Figure 8 presents the percentage of HCK codes given in each unit of analysis.

EMSM consisted of first two units of analysis, and there were no chances given to develop HCK in these two units of analysis. However, there was one HCK task provided in UoA3, UoA4 and UoA5. The HCK task in these three units of analysis demanded PSTs to know how fractions are connected to the broader mathematical concepts (CON). There were three HCK tasks given in UoA6, and four HCK tasks in the last unit of analysis. All of these HCK tasks supported PSTs to develop the knowledge of fraction connections with wider mathematics concepts (CON). Figure 8 displays a visual representation of percentages of HCK codes given in each unit of analysis.

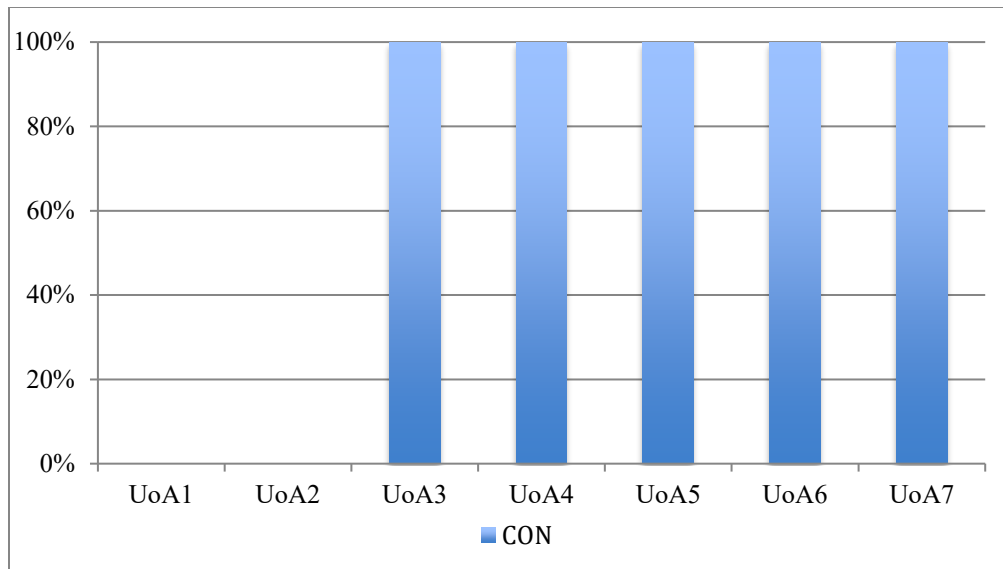


Figure 8. The Percentage of HCK Codes in Each Unit of Analysis

Note. UoA = unit of analysis; CON = connections; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

The following figure presents the percentages of HCK code found in each mathematics methods textbook. Again, there were no HCK tasks in EMSM, therefore it did not contain any HCK code. There was a total of six HCK tasks in META, and all of them supported the knowledge of connecting fractions to the broader mathematical concepts (CON). As the UoA7 represented HCLM, the results were the same for HCLM. All HCK tasks given in textbooks supported PSTs to know fraction connections to the broader mathematical concepts (CON).

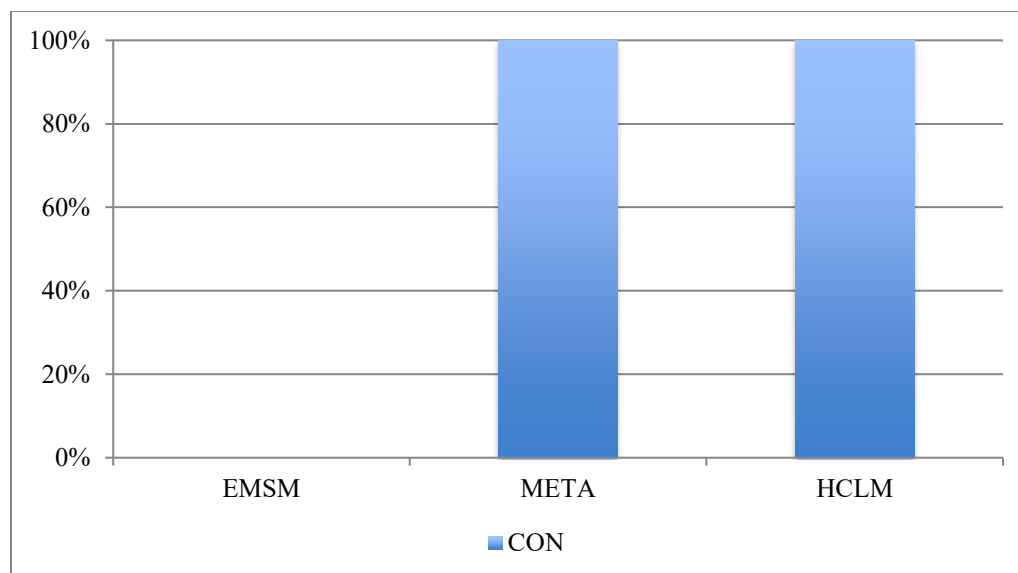


Figure 9. The Percentage of HCK Codes in Each Textbook

Note. CON = connections, EMSM = Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); META = Mathematics for Elementary Teachers with Activities by Beckmann (2018); HCLM = Helping Children Learn Mathematics by Reys et al. (2015).

In general, PSTs were offered very limited opportunities to develop their HCK in these three mathematics methods textbooks. While EMSM did not provide any HCK tasks, META offered only six HCK tasks out of a total of 300 tasks, which was approximately 2%. Besides, 13% of tasks in HCLM supported PSTs to develop HCK. The purpose of HCK tasks was to support PSTs to know how fractions are connected to the wider mathematical concepts (CON). In the following section, I present findings related to the last domain of Subject Matter Knowledge.

Specialized Content Knowledge

In order to explore chances provided for PSTs to develop their Subject Matter Knowledge (SMK), I analyzed the three types of knowledge. Specialized Content Knowledge (SCK) was the third domain of SMK. Similar to the other two types of knowledge, I examined all fraction tasks in terms of how they support PSTs to improve their SCK. In order to do it, I

used SCK codes that I developed in the coding frame (see Table 21; for further information related to SCK codes please refer to Chapter 3).

Table 21. SCK Codes and Labels

Codes	Labels
Alternative	AL
Describe/Discuss/Interpret	DDI
Determine	D
Illustrate	I
Meanings	MEA
Models	MOD
Problem Posing	PP
Reasoning	R
Select	SE

I discovered nine different types of SCK in mathematics methods textbooks after evaluating 162 tasks from EMSM; 300 tasks from META; and 32 tasks from HCLM. Table 22 displays the number and percentage of SCK tasks in each unit of analysis as well as in three mathematics methods textbooks. The total number of SCK tasks in seven units of analysis ranged from 16 to 61. UoA3 and UoA6 were the two units of analysis, which had the highest number of SCK task. By way of contrast, UoA2 and UoA7 contained the least amount of SCK tasks.

Table 22. The Number and Percentage of Tasks Support SCK in Units of Analysis and Textbooks

Textbook	No. of UoA	Total Tasks	Number SCK Tasks	No. of SCK Tasks	SCK Tasks in Textbook
EMSM	UoA1	81	24	30%	26%
	UoA2	81	18	22%	
META	UoA3	122	50	41%	60%
	UoA4	45	34	76%	
	UoA5	47	35	74%	
	UoA6	86	61	71%	
HCLM	UoA7	32	16	50%	50%

Note. UoA = unit of analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

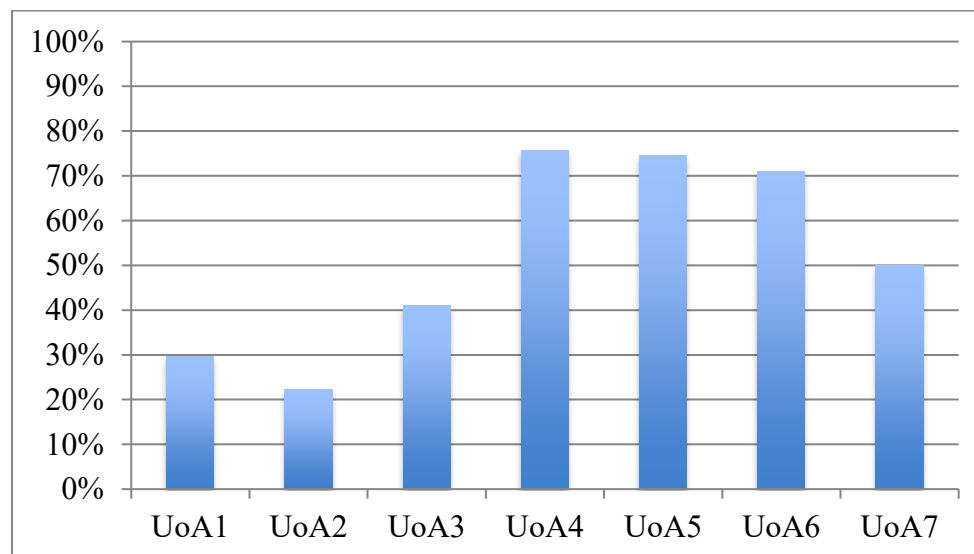


Figure 10. The Percentage of SCK Tasks in Each Unit of Analysis

Note. UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

As seen in the Table 22 and Figure 10, the percentage of SCK tasks in seven units of analysis ranged from 22% to 76%. UoA4 had the highest percentage of SCK tasks closely followed by UoA5 and UoA6. Although the number of SCK tasks was higher in UoA2 than UoA7, due to their size differences, UoA2 appeared to have the least percentage of SCK tasks among seven units of analysis. Opportunities to develop SCK differed across seven units of analysis. Four out of seven units of analysis supported PSTs' SCK by offering at least 50% of tasks that demand SCK, whereas two out of seven units of analysis contained less than one-third of total tasks focus on SCK. A visual representation of the percentage of SCK tasks within each mathematics methods textbook can be seen in the following figure.

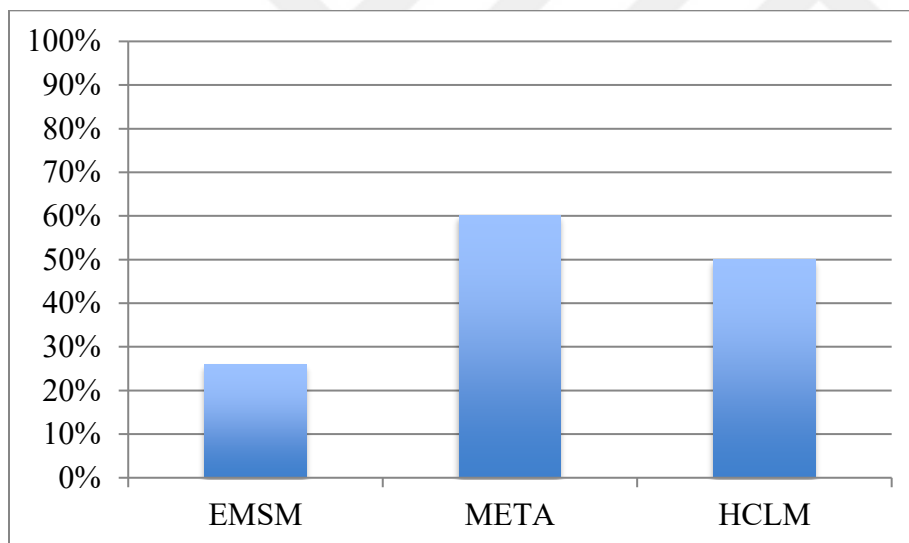


Figure 11. The Percentage of SCK Tasks Across Textbooks

Note. EMSM = Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); META = Mathematics for Elementary Teachers with Activities by Beckmann (2018); HCLM = Helping Children Learn Mathematics by Reys et al. (2015).

As illustrated in Figure 11, EMSM appeared to have the least percentage of SCK tasks among the three mathematics textbooks. Approximately, one-fourth of the total tasks in EMSM offered chances to support SCK. Additionally, 40% of SCK tasks in EMSM were given in the online resource. Although, most of the units of analysis from META consisted of more than 70%

SCK task, nearly three-fifths of the tasks in META demanded SCK to be completed in general. Around 50% of fraction tasks in HCLM supported PSTs to develop their SCK.

In the following table, I present what types of SCK were offered in each unit of analysis. It is important to note that there were some tasks that demand two or more types of SCK to be completed. As displayed in the Table 22, there were 24 SCK tasks in UoA1, but the total number of SCK codes determined in UoA1 was more than 30 as seen in the Table 23. Therefore, please note that the number of SCK tasks would not equal to the total number of SCK codes recorded in each unit of analysis.

Table 23. The Number of SCK Codes Discovered in Each Unit of Analysis

Textbook	UoA	AL	DDI	D	I	MEA	MOD	PP	R	SE
EMSM	UoA1	0	3	0	8	3	12	2	7	2
	UoA2	3	3	0	5	0	3	4	2	0
META	UoA3	3	6	0	20	10	8	5	12	0
	UoA4	4	3	3	4	0	1	20	6	0
	UoA5	0	6	4	19	0	2	18	4	0
	UoA6	4	10	0	24	2	0	31	16	0
HCLM	UoA7	0	1	0	5	3	6	2	1	1

Note. UoA = unit of analysis; AL = alternative, DDI = describe/discuss/interpret; D = determine; I = illustrate; MEA = meanings; MOD = models; PP = problem posing; R = reasoning; SE = select; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

A closer examination of Table 23 and Figure 12 reveals that, half of the SCK tasks in UoA1 focused on the use of representations and models to find fractions (MOD). Almost one-third of SCK tasks supported PST to use and explain their reasoning strategies (R) and also to illustrate a given fraction to complete the task (I). There were no chances given to provide alternative solution methods (AL) or to determine fractional parts form a given shapes (D). The opportunities to know the meanings of fractions (MEA), to describe and discuss solution

methods (DDI), to select appropriate models for a given fractions (SE), and to pose or select word problem that reflects fraction task (PP) were given approximately in 10% of SCK tasks. There were 18 SCK tasks in UoA2, and none of these tasks involved practicing the meanings of fractions (MEA), selecting appropriate representations (SE) and determining the fractional parts from a given shapes (D). Illustrating a given fraction to complete a task (I) was the most used type of SCK in UoA2. Similar to UoA2, UoA3 also supports PSTs by providing more chances to illustrate a given fraction to solve a problem (I) compared to other types of SCK. Two types of SCK were not discovered in UoA3, which are selecting appropriate models for a given fractions (SE) and determining fractional parts form a given shapes (D).

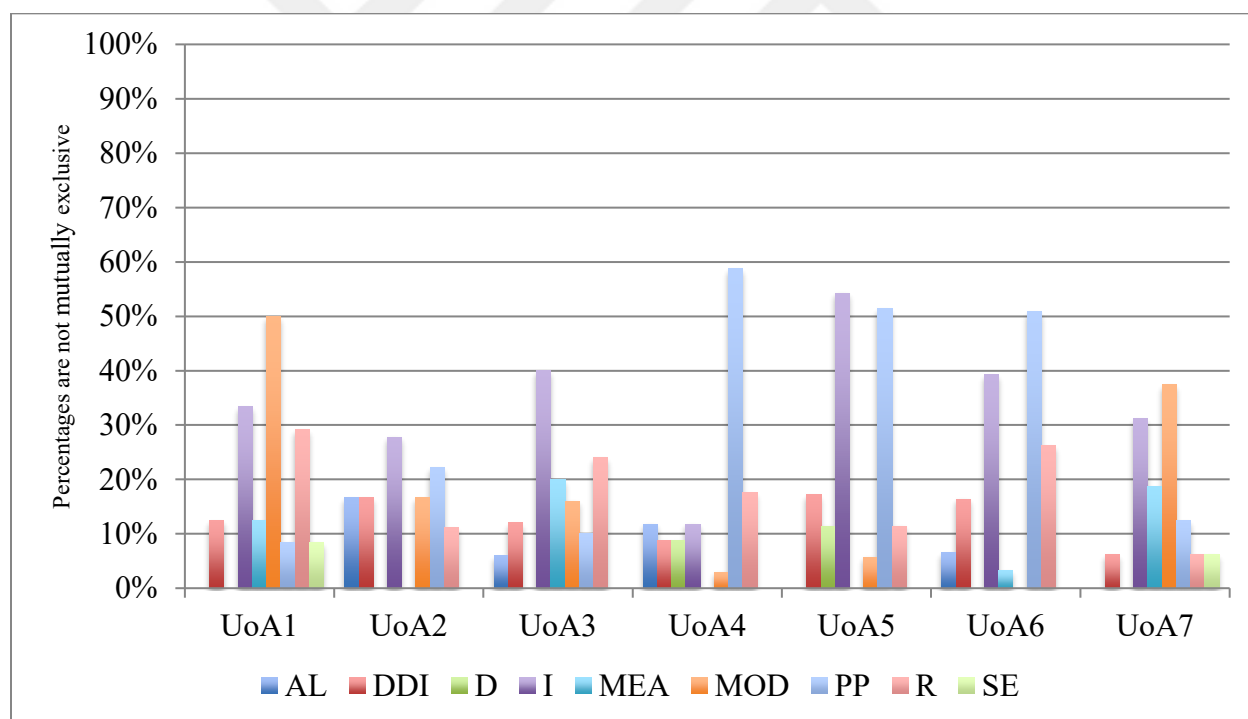


Figure 12. The Percentage of SCK Codes in Each Unit of Analysis

Note. UoA = unit of analysis; AL = alternative, DDI = describe/discuss/interpret; D = determine; I = illustrate; MEA = meanings; MOD = models; PP = problem posing; R = reasoning; SE = select; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

Figure 12 provides a visual comparative analysis between and within units of analysis in terms of SCK codes frequencies. Almost 60% of the SCK tasks in UoA4 were devoted to

problem posing (PP) for 20 out of 34 tasks. Notice that similar to UoA2, UoA4 did not offer any SCK tasks that address fraction meanings (MEA) and representation selection for a given fraction (SE). UoA5 offered an equal proportion of SCK tasks that demand PSTs to illustrate (I) and to pose a word problem (PP). There were no chances given to practice providing alternative solution methods (AL), knowing different meanings of fractions (MEA) and selecting appropriate models for a given task (SE).

In the UoA6, there were 61 SCK tasks, and half of these tasks demanded PSTs to pose or select a word problem, which reflects a fraction task (PP). Besides, nearly 40% of SCK tasks offered chances to illustrate fractions for a task completion. There were not any task offering the use of fraction models (MOD) and the selection of appropriate fraction representations (SE). On the contrary, the use of fraction representations (MOD) code had the highest percentage of SCK tasks in UoA7, followed by a fraction illustration (I).

Overall, practices to select appropriate representations for a given fraction (SE) were only offered in UoA1 and UoA7. UoA4, UoA5 and UoA6 placed a greater emphasis on problem posing (PP) than any other unit of analysis. UoA4 and UoA5 were the only two units of analysis, that provided chances to determine the fractional expressions of a given shapes (D). Notice that, UoA6 was the only unit of analysis that did not provide any opportunities for PSTs to use models to find/show fractions (MOD). The following table displays the results of total number of SCK codes found in each textbook.

Table 24. The Total Number of SCK Codes Discovered in Each Textbook

Textbook	AL	DDI	D	I	MEA	MOD	PP	R	SE
EMSM	3	6	0	13	3	15	6	9	2
META	11	25	7	67	12	11	74	38	0
HCLM	0	1	0	5	3	6	2	1	1

Note. AL = alternative, DDI = describe/discuss/interpret; D = determine; I = illustrate; MEA = meanings; MOD = models; PP = problem posing; R = reasoning; SE = select; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

As indicated in Table 24, META placed a greater emphasis on most types of SCK, such as describing and discussing the solution methods (DDI), illustration (I), problem posing (PP) and reasoning (R). The knowledge of determining and writing the fractional parts of a given shapes was only offered in META. While, EMSM and HCLM provided limited opportunities related to the knowledge of appropriate model selection for a given fraction (SE); META did not offer any task. EMSM contained a greater number of tasks requiring the knowledge of fraction models to be completed (MOD) compared to the other two textbooks.

Figure 13 provides a graphical representation of the frequency of SCK codes among three textbooks. Notice that HCLM had the lowest frequency in terms of SCK codes. The number of SCK codes given in META was nearly 10 times greater than the amount in HCLM. Closer examination of Table 24 and Figure 13 revealed that opportunities are significantly different in the three mathematics methods textbooks. The knowledge of fraction models (MOD) was the mostly needed SCK in EMSM and HCLM, whereas, META gave more emphasis on illustration (I) and problem posing (PP).

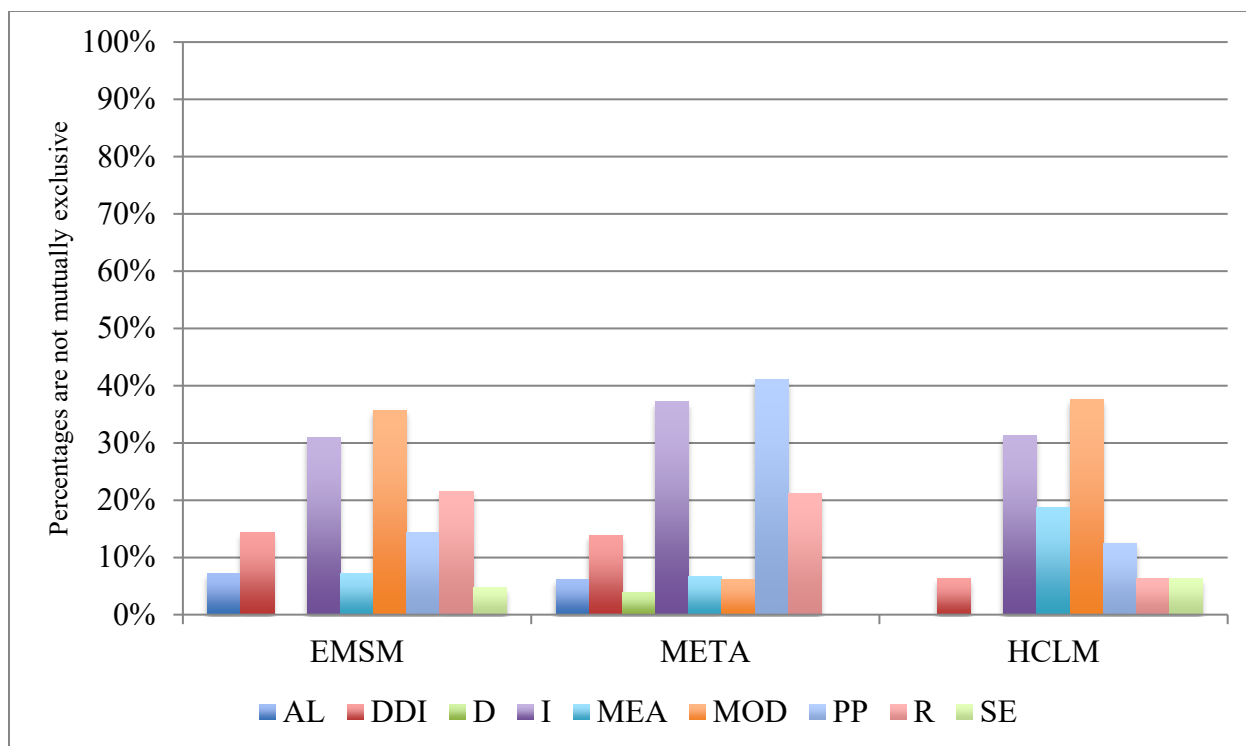


Figure 13. The Percentage of SCK Codes in Each Textbook

Note. AL = alternative, DDI = describe/discuss/interpret; D = determine; I = illustrate; MEA = meanings; MOD = models; PP = problem posing; R = reasoning; SE = select; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

In sum, META appeared to offer the highest percentage of SCK tasks than the other two mathematics methods textbooks. Specifically, there were 180 tasks out of 300 (60%) providing chances to improve SCK in META, followed by HCLM with a percentage of 50%. However, HCLM consisted of 32 tasks in total, therefore there were only 16 tasks providing opportunities to develop SCK. Although, EMSM contained more SCK tasks than HCLM did, the percentage of SCK task in EMSM was less than HCLM due to their data size difference. In the following section, I report the findings of three domains (CCK, HCK and SCK) as a combination in order to present results related to Subject Matter Knowledge (SMK).

Combination of Subject Matter Knowledge Domains

In the previous sections, I presented each domain's results separately. In this section, I combine findings of each domain to answer my research question related to Subject Matter Knowledge, because SMK contains domains of CCK, HCK and SCK. In particular, I first report the number of tasks in each unit of analysis and textbooks that offer chances to support any domains of SMK. Later, I present these combined findings to show to what extent the three mathematics methods textbook support PSTs to develop their SMK.

To discover whether a task supports any domain of SMK or not, I analyzed each task under each unit of analysis. Specifically, I considered a task support PSTs' SMK if it demands CCK, HCK or SCK. As Table 25 indicated, the percentages of tasks demand any domain of SMK was high across seven units of analysis, with a range from 84% to 100%. In more detail, all of the tasks in UoA5 and UoA6 required PSTs to have CCK, HCK or SCK to complete tasks. UoA7 had the least percentage of tasks offering SMK opportunities, followed by UoA2. Around 90% of tasks in the rest of the units of analysis offered chances to support SMK by providing at least one domain of SMK.

Table 25. The Number and Percentage of SMK Tasks in Units of Analysis and Textbooks

		No. of Tasks	SMK Tasks	SMK % in UoA	SMK % in Texts
EMSM	UoA1	81	73	90%	88%
	UoA2	81	69	85%	
META	UoA3	122	109	89%	95%
	UoA4	45	42	93%	
	UoA5	47	47	100%	
	UoA6	86	86	100%	
HCLM	UoA7	32	27	84%	84%

Note. UoA = unit of analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

The analysis indicated that 95% of tasks in META supported PSTs' SMK by providing opportunities to practice any three domains of SMK. HCLM appeared to have the least percentage of SMK tasks, while almost 90% of tasks in EMSM gave PSTs chances to improve their SMK. In the following paragraphs I present the findings related to the number of tasks demanding different type of subject matter knowledge in each unit of analysis and also in each mathematics methods textbook.

Table 26. The Number of Tasks Demand Combination of SMK in Units of Analysis

	CCK+HCK	HCK+SCK	CCK+SCK	CCK+SCK+HCK
UoA1	0	0	12	0
UoA2	0	0	14	0
UoA3	1	0	21	0
UoA4	0	0	19	0
UoA5	1	1	31	1
UoA6	0	0	57	0
UoA7	3	2	10	1

Note. UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

Table 26 displays the total number of tasks and Table 27 presents the percentage of tasks in each unit of analysis that demand more than one type of SMK. As revealed in both tables, four out of seven units of analysis only contained a combination of CCK and SCK. UoA3, UoA5 and UoA7 were the only three units of analysis that provided chances in the tasks to develop both CCK and HCK. However, the amount of these tasks was very low on all three units of analysis, with a range from 1 to 3 tasks. UoA7 had the highest percentage of tasks require CCK and HCK. The tasks required both HCK and SCK were found in only two units of analysis, which were UoA5 and UoA7. The number and the percentage of these tasks were low on both units of analysis.

All units of analysis contained a good number of tasks demanding CCK and SCK at the same time. Notice that UoA6 had the greatest quantity of tasks in terms of opportunities given to support both CCK and SCK, however both UoA5 and UoA6 had the highest percentage among

seven units of analysis. UoA7 had the least amount of CCK and SCK tasks, but UoA1 had the least percentage followed by both UoA2 and UoA3 due to the data size differences. When it comes to the combination of three domains of SMK, only two units of analysis, UoA5 and UoA7, provided opportunities for PSTs to practice CCK, HCK and SCK. Both units of analysis provided insignificant amount of task, only one, but the percentages of the tasks differed between 2% and 3%.

Table 27. The Percentage of Tasks Demand Combination of SMK in Units of Analysis

	CCK+HCK	HCK+SCK	CCK+SCK	CCK+HCK+SCK
UoA1	0%	0%	15%	0%
UoA2	0%	0%	17%	0%
UoA3	1%	0%	17%	0%
UoA4	0%	0%	42%	0%
UoA5	2%	2%	66%	2%
UoA6	0%	0%	66%	0%
UoA7	9%	6%	31%	3%

Note. UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

The following figure presents a visual representation of the percentages of tasks in three textbooks that require combination of SMK domains. EMSM only offered one combination of SMK, which is CCK and SCK, in 16% of tasks. META and HCLM provided at least one task for each combination, although the percentages of these combinations were insignificant in META. In particular, both of the textbooks offered only one task that demands CCK, HCK and SCK to be completed. Since META had 300 tasks in total and HCLM had only 32, the percentages

showed slight differences. Again, due to the data size differences, HCLM had the highest percentage of tasks supporting CCK and HCK, HCK and SCK, and CCK, HCK and SCK combinations. All of the textbooks provided the largest proportion of tasks that support PSTs' both CCK and SCK among all four combinations.

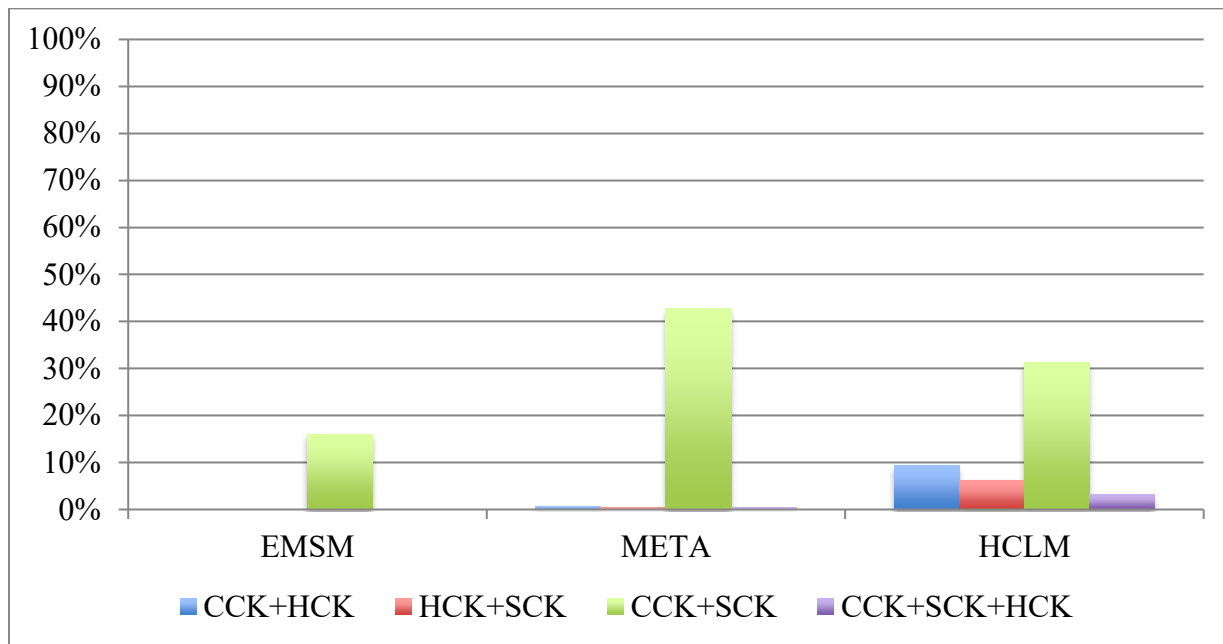


Figure 14. The Percentage of Tasks Demand Combination of SMK in Textbooks

Note. EMSM = Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); META = Mathematics for Elementary Teachers with Activities by Beckmann (2018); HCLM = Helping Children Learn Mathematics by Reys et al. (2015).

Summary of Subject Matter Knowledge Opportunities

In sum, opportunities given in fraction tasks to improve PSTs' SMK differed among three mathematics methods textbooks. EMSM provided the greatest percentage of CCK tasks, followed by META. One-third of CCK tasks in EMSM offered chances to practice finding fractions (F) and knowing four operations (OP). Likewise, more than half of the CCK tasks in META focused on the knowing four operations (OP). Around 60% of tasks in HCLM supported CCK, and one-third of those tasks were on comparing fractions (CMP) and on finding equivalence (EQU). In terms of opportunities provided to support HCK, HCLM had the highest

percentage of HCK tasks, whereas EMSM did not provide any. All of the HCK tasks given in two textbooks focused on how fractions are connected to the broader mathematics concepts (CON). As the third domain of SMK, META supported PSTs' SCK with a greater opportunity than other two textbooks (60% of SCK tasks). Almost half of those tasks demand posing word problems (PP). Around 40% of SCK tasks in EMSM and HCLM required PSTs to use models to find fractions (MOD).

The majority of fraction tasks in all textbooks contained at least one domain of SMK. However, some tasks in textbooks demanded more than one domain to be completed. Therefore, I explored the combination of knowledge needed in tasks, which are CCK and HCK, HCK and SCK, CCK and SCK, and CCK, HCK and SCK. The tasks in EMSM only offered a combination of CCK and SCK. As Figure 13 displays, the percentages of tasks supporting both CCK and SCK across all textbooks was the greatest compared to other knowledge combinations. META provided insignificant number of tasks in each combination excluding CCK and SCK; whereas HCLM offered the highest percentage of tasks in combinations of CCK and HCK, HCK and SCK, and CCK, HCK and SCK among three textbooks.

Opportunities of Pedagogical Content Knowledge

In this section, I report the findings on the given opportunities in fractional tasks for PSTs to develop their Pedagogical Content Knowledge (PCK) through three mathematics methods textbooks. Similar to the steps followed previously in SMK sections, I again used the framework that I developed considering MKT framework as a foundation. In MKT framework, PCK consisted of three domains: Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). I present the findings

for KCS, KCT, and then KCC. Next, I combine all the three domains to represent PCK in general.

Knowledge of Content and Students

To answer my research question related to the opportunities given to develop PST's KCC, I examined all fraction tasks under each unit of analysis (UoA). There were approximately 500 fraction tasks within the three mathematics methods textbooks. Of those tasks, 162 were given in the first two units of analysis (EMSM), 300 were located in the following four units of analysis (META), and 32 were found in the last unit of analysis (HCLM).

I used the corresponding codes for KCS under the coding frame to examine whether a task required KCS. There were five types of KCS present in the data, and Table 28 provides the codes and their labels (a detailed description of codes can be found in Chapter 3).

Table 28. KCS Codes and Labels

Codes	Labels
Misconceptions	MI
Representations	REP
Strategies	STG
Student Reasoning	SR
Understanding	U

It is important to mention here that some tasks required more than one type of KCS, which I present in the following sections. However, in this section I considered a task demanding KCS if it consisted of at least one type of KCS. Table 29 shows the number and percentage of tasks supporting KCS in each unit of analysis and also in three mathematics methods textbooks.

The total number of KCS tasks in unit of analysis ranged from 1 to 21. UoA3 had the highest number of SCK task followed by UoA1. UoA5 contained only one KCS task, which was the least amount among seven units of analysis.

Table 29. The Number and Percentage of Tasks Support KCS in Units of Analysis and Textbooks

Textbook	No. of UoA	Total Tasks	No. of KCS Tasks	% of KCS Tasks	KCS Tasks in Textbook
EMSM	UoA1	81	11	14%	12%
	UoA2	81	9	11%	
META	UoA3	122	21	17%	11%
	UoA4	45	7	16%	
	UoA5	47	1	2%	
	UoA6	86	3	3%	
HCLM	UoA7	32	6	19%	19%

Note. UoA = unit of analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

Again, although the number of KCS tasks were given in each unit of analysis was important, comparing the number of tasks to another could not make sense without considering the data size of units of analysis or textbooks. Specifically, the number of KCS tasks in UoA1 was almost doubled the amount in UoA7; however, the percentage of KCS tasks in UoA7 was greater than UoA1's. Therefore, it is important to focus on percentages rather than number of tasks when there is a comparison among units of analysis or textbooks.

As seen in Table 29 and Figure 15, UoA5 had the least percentage of KCS tasks. Only 2% of tasks were devoted to supporting PSTs' KCS. Besides, UoA6 offered limited chances to

develop KCS by providing 3% KCS tasks. Although, UoA7 did not contain the greatest number of KCS tasks, it appeared to have the highest percentage of KCS tasks among seven units of analysis.

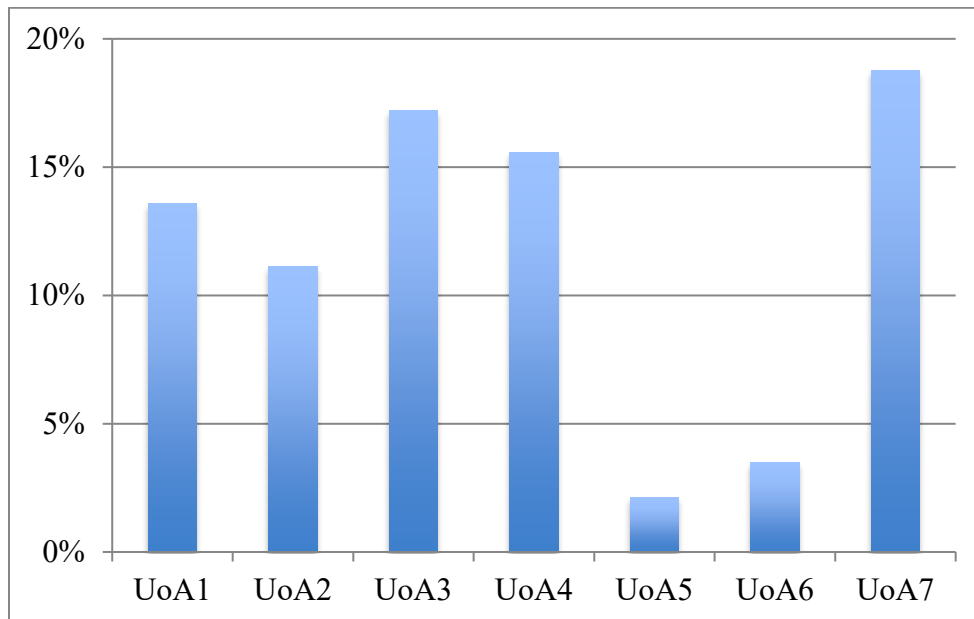


Figure 15. The Percentage of KCS Tasks in Each Unit of Analysis

Note. UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

Overall, the percentages of KCS tasks in the majority of units of analysis differed between 10% and 20%. In particular, around one-fifth of tasks in UoA7, one-sixth of tasks in UoA3 and UoA4, and one-seventh of tasks in UoA1 supported KCS, while the rate was lower than one-ninth in UoA2. The following figure demonstrates a visual representation of the percentage of KCS tasks provided in each mathematics methods textbook.

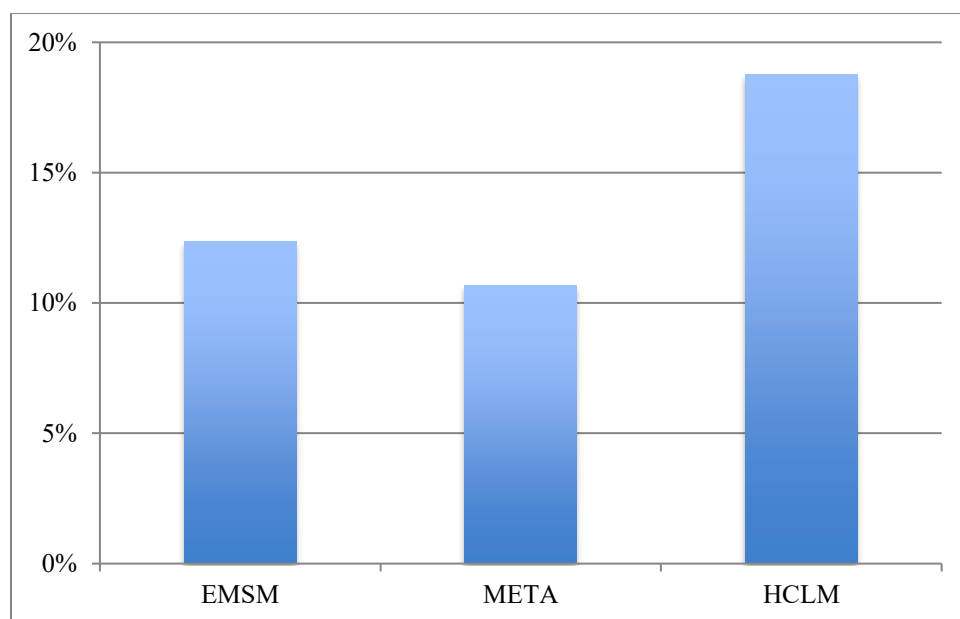


Figure 16. The Percentage of KCS Tasks Across Textbooks

Note. EMSM = Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); META = Mathematics for Elementary Teachers with Activities by Beckmann (2018); HCLM = Helping Children Learn Mathematics by Reys et al. (2015).

Since there was only one fraction chapter in HCLM, UoA7 represented the whole HCLM textbook. Therefore, HCLM had the highest percentage of KCS tasks among three mathematics methods textbooks with 19%. As META consisted of UoA3, UoA4, UoA5 and UoA6, and two of these units of analysis had very low percentages of KCS tasks; the percentage of KCS tasks in META decreased to 11%, which was the lowest across three textbooks. EMSM textbook supported PSTs' KCS by providing opportunities in 12% of fraction tasks. It is important to note that EMSM provided around 90% of KCS tasks in the online resource.

In the next table, I present the types of knowledge needed for PST to complete KCS tasks in each unit of analysis and in three methods textbooks. Remember that some of the tasks demand more than one type of specific knowledge to be completed. Specifically, one KCS task might require the knowledge of student reasoning (SR) and misconceptions (MI) at the same time. Therefore, the total number of KCS tasks is not expected to be the same with the total number of KCS codes explored in each unit of analysis.

Table 30. The Number of KCS Codes Discovered in Each Unit of Analysis

Textbook	UoA	MI	REP	STG	SR	U
EMSM	UoA1	5	1	1	2	3
	UoA2	2	0	3	1	3
META	UoA3	6	0	0	16	0
	UoA4	4	0	0	6	0
	UoA5	1	0	0	0	0
	UoA6	2	0	0	3	0
HCLM	UoA7	4	0	0	0	2

Note. UoA = unit of analysis; MI = misconceptions; REP = representations; STG = strategies; SR = student reasoning; U = understanding; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

A closer examination of Table 29 and Table 30 reveals that UoA1 supported PSTs to know and understand students' misconceptions and to be able to explain these misconceptions (MI) more than other types of KCS. There were 11 tasks supporting KCS in UoA1, and in a little less than one fourth of these tasks, PSTs were expected to know what skills and understanding students may need to complete the task (U). Knowing alternative representations (REP) and students' strategies to solve fraction tasks (STG) were the least needed type of KCS in UoA1. However, the opportunity to know alternative representations (REP) was only given in the UoA1 among seven units of analysis. The knowledge of students' understanding needed for a task (U) and students' strategies to complete a task (STG) were the most demanded type of KCS in UoA2. Surprisingly, these two types of knowledge were not given in any tasks in UoA3, UoA4, UoA5 and UoA6. A little more than three-fourths of KCS tasks in UoA3 supported PST to understand student reasoning (SR), whereas the percentage of this sort of knowledge is higher, around 85%, in UoA4.

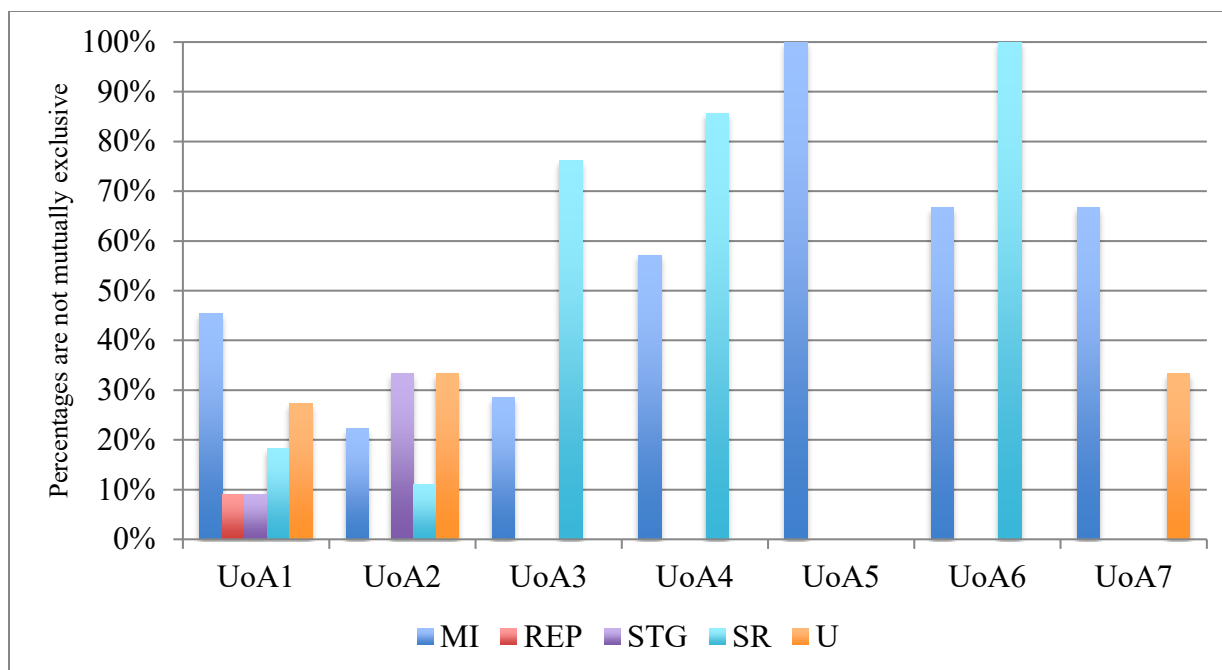


Figure 17. The Percentage of KCS Codes in Each Unit of Analysis

Note. MI = misconceptions; REP = representations; STG = strategies; SR = student reasoning; U = understanding; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

There was only one type of KCS, understanding students' misconceptions and being able to explain these misconceptions (MI), provided in UoA5. And this type of knowledge was required in a single task. In UoA6, every KCS task supported PST to understand student reasoning (SR), and two-third of KCS tasks provided chances to know students' misconceptions (MI). There were six KCS tasks in UoA7, and two-thirds of them supported the knowledge of students' misconceptions (MI), whereas one-third required PSTs to know the skill and understanding students need to complete a task (U).

Overall, the understanding of student reasoning and being aware of the possible students' responses (SR) was the most demanded type of KCS in three out of seven units of analysis. The percentage of the knowledge of students' misconceptions and being able to explain these misconceptions (MI) was the highest in UoA1, UoA5, and UoA7. UoA1 was the only unit of analysis, which consisted of all five types of KCS, and UoA2 contained tasks that demand four

different types of KCS. There were four units of analysis that contained only two types of KCS, while one unit of analysis contained only one types of KCS.

The following table displays the total number of KCS codes in each textbook. In EMSM and HCLM, the mostly asked types of KCS were misconceptions (MI) followed by understanding (U). The opportunities to know students' alternative representations (REP) were very low in EMSM, however it was the only textbook that demanded this type of knowledge. META offered highest opportunities for PSTs to practice students' reasoning with fractions (SR).

Table 31. The Total Number of KCS Codes Discovered in Each Textbook

Textbook	MI	REP	STG	SR	U
EMSM	7	1	4	3	6
META	13	0	0	25	0
HCLM	4	0	0	0	2

Note. MI = misconceptions; REP = representations; STG = strategies; SR = student reasoning; U = understanding; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

Figure 18 presents the percentages of KCS codes found in KCS tasks in each mathematics methods textbook. It is important to note that the sum of these percentages may not add up to or may exceed 100% due to some tasks consisting of more than one KCS code. As seen in Table 31 and Figure 18, EMSM was the only textbook, which supported PSTs' KCS by providing tasks that demand five different types of KCS. META and HCLM provided two types of KCS, one of them (Misconceptions) was common on both textbooks. META provided the highest percentage of KCS tasks that demand the knowledge of students' reasoning with fractions (SR), whereas this type of knowledge was not offered in HCLM. However, two types of KCS given in HCLM, misconceptions (MI) and understanding (U), appeared to have the

highest percentages in comparison to the percentages of the same codes in the other two textbooks.

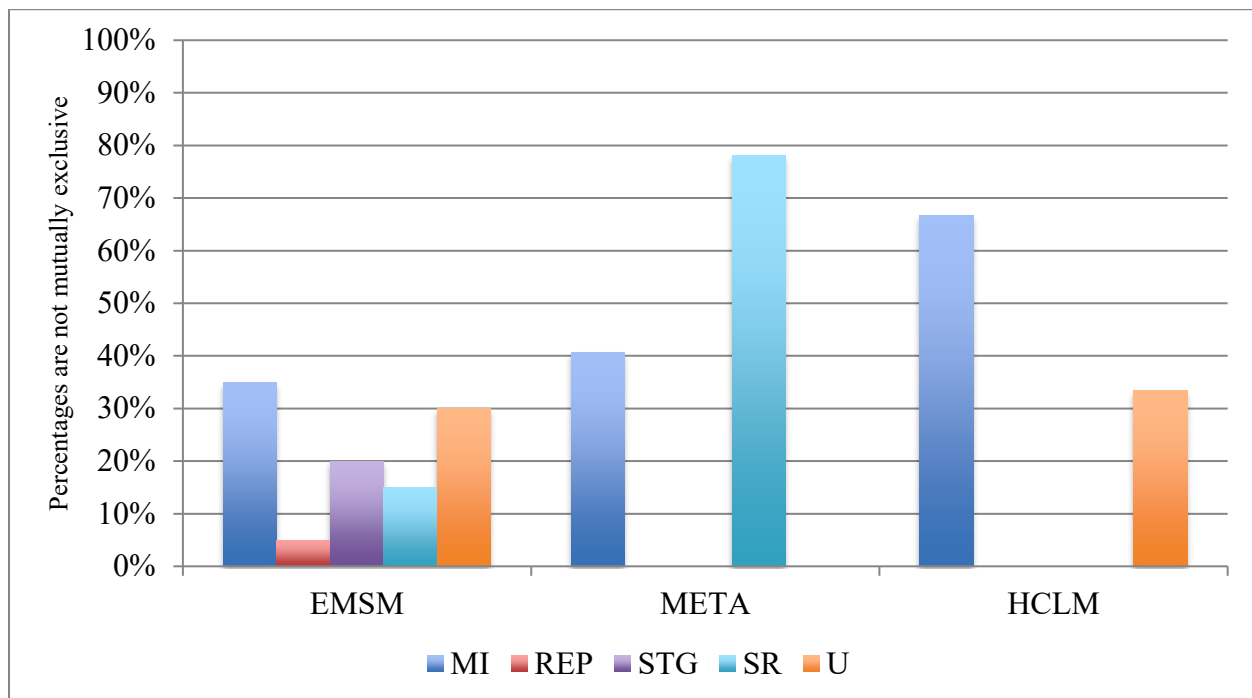


Figure 18. The Percentage of KCS Codes in Each Textbook

Note. MI = misconceptions; REP = representations; STG = strategies; SR = student reasoning; U = understanding; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

Overall, the opportunities provided for PSTs to develop their KCS differed across the three mathematics methods textbooks. Although the number of KCS tasks found in META was nearly six times greater than the number of KCS task in HCLM, HCLM provided the highest percentage of KCS tasks, whereas META had the least due to the difference on their data size. The majority of KCS tasks in EMSM and in HCLM focused on understanding students' misconceptions (MI), however META provided greater chances to understand students' reasoning with fractions (SR) among five types of KCS. In the next section, I present findings related to second domain of Pedagogical Content Knowledge, which is Knowledge of Content and Teaching.

Knowledge of Content and Teaching

In order to explore the opportunities given for PSTs to develop the Knowledge of Content and Teaching (KCT) in the three mathematics methods textbooks, I evaluated a total of 494 fraction tasks. Specifically, EMSM consisted of first two units of analysis, and each had 81 fraction tasks. META contained the 3rd, 4th, 5th, and 6th unit of analysis, and there was a total of 300 tasks in these units of analysis. The last textbook, HCLM, had the last unit of analysis, which consisted of 32 tasks in total. As Table 32 displays, I used KCT codes from the coding frame I developed to analyze whether a task demanded knowledge of content and teaching to be completed. Please refer to Chapter 3 for a detailed description of KCT codes.

Table 32. KCT Codes and Labels

Codes	Labels
Activities	AC
Assessment	AS
Before/After	BA
Support	SUP
Teaching Techniques	TT

In qualitative content analysis it is important to have the codes that match with the data. As previously stated, I developed these codes in a deductive and inductive way. After analyzing all of the fraction tasks, I disregarded some codes that did not explain the data. Therefore, I explored five types of Knowledge of Content and Teaching (see Table 32), which describe the data in this inquiry. In the following table, I present the findings related to the number and

percentage of task demand KCT to be completed in units of analysis and in mathematics methods textbooks.

Table 33. The Number and Percentage of Tasks Support KCT in Units of Analysis and Textbooks

Textbook	No. of UoA	Total Tasks	No. of KCT Tasks	% of KCT Tasks	% of KCT Tasks in Textbook
EMSM	UoA1	81	9	11%	12%
	UoA2	81	10	12%	
META	UoA3	122	4	3%	3%
	UoA4	45	3	7%	
	UoA5	47	1	2%	
	UoA6	86	2	2%	
HCLM	UoA7	32	7	22%	22%

Note. UoA = unit of analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

As shown in Table 33, the total number of KCT tasks in seven units of analysis ranged from 1 to 10. Four out of seven units of analysis contained four or less KCT tasks. While Uoa5 had only one KCT task, UoA2 had the greatest amount of tasks support PSTs to develop KCT, followed by UoA1. In terms of the percentage of fraction tasks that demanded KCT across all seven units of analysis, the range was from 2% to 22% (I rounded the data to the nearest whole percent). Figure 19 provides a graphical representation of percentages of KCT tasks across the seven units of analysis.

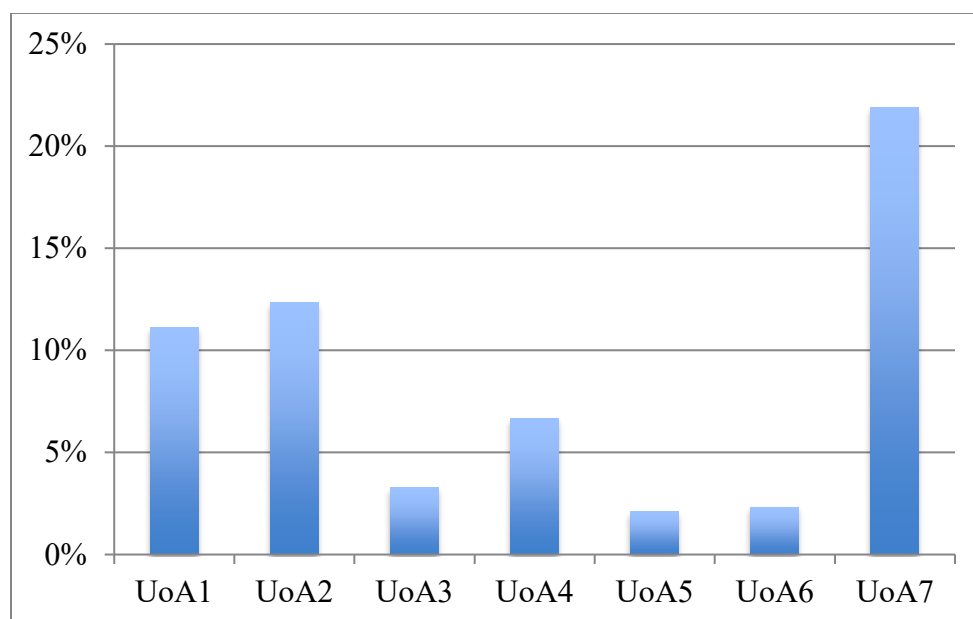


Figure 19. The Percentage of KCT Tasks in Each Unit of Analysis

Note. UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

As seen in the Figure 19, UoA7 had the highest percentage of tasks that address KCT, followed by UoA2 and then UoA1. Notice that the difference of the data size played a crucial role when it comes to the percentage of tasks. Specifically, both UoA1 and UoA2 had more KCT tasks than UoA7 has. Since their data size was larger than UoA7's, both had fewer than 22%, which was the percentage of KCT tasks in UoA7. Three units of analysis out of seven (UoA3, UoA5, and UoA6) contained less than 5% of KCT tasks.

Figure 20 presents the percentages of KCT tasks in each mathematics methods textbook. As META consisted of UoA3, UoA4, UoA5 and UoA6, and there were low chances given to develop KCT in these units of analysis, META offered the least opportunity for PSTs to improve KCT through fraction tasks. Although the number of KCT tasks in EMSM was nearly tripled the amount of KCT tasks in HCLM, HCLM had the highest percentage of KCT tasks with 22%

because of differences in the data size. Besides, it is important to mention that all of the KCT tasks given in EMSM were provided in the online resource.

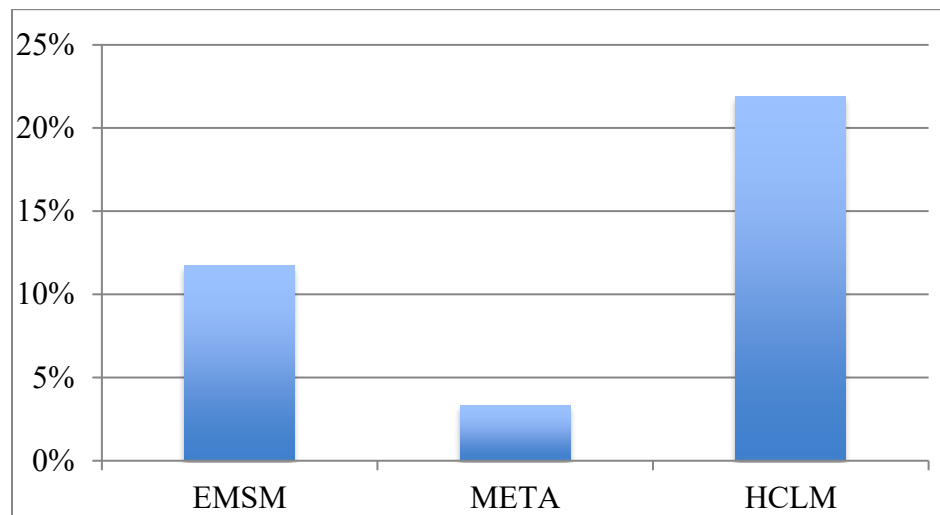


Figure 20. The Percentage of KCT Tasks Across Textbooks

Note. EMSM = Elementary and Middle School Mathematics, Teaching Developmentally by Van de Walle et al. (2019); META = Mathematics for Elementary Teachers with Activities by Beckmann (2018); HCLM = Helping Children Learn Mathematics by Reys et al. (2015).

When presenting the findings for previous domains, I mentioned that some tasks involve more than one type of knowledge. However, it was not the case for this domain, because every KCT task demanded only one type of knowledge. Therefore, the number of recorded codes was equal to the number of KCT task. Table 34 represents findings related to the types of KCT discovered in each unit of analysis. There were nine tasks supporting KCT in UoA1, and two-thirds of them supported the knowledge of appropriate teaching techniques for teaching fractions (TT) and one-third demanded explanations to support students' understanding to overcome their misconceptions (SUP). UoA2 was the only unit of analysis that consisted of all five types of KCT, activities (AC), assessment (AS), before/after (BA), support (SUP) and teaching techniques (TT). Of these five types, support (SUP) and teaching techniques (TT) were the most demanded types, followed by before/after (BA).

Table 34. The Number of KCT Codes Discovered in Each Unit of Analysis

Textbook	UoA	AC	AS	BA	SUP	TT
EMSM	UoA1	0	0	0	3	6
	UoA2	1	1	2	3	3
META	UoA3	0	0	0	3	1
	UoA4	0	0	0	3	0
	UoA5	0	0	0	1	0
	UoA6	0	0	0	2	0
HCLM	UoA7	1	0	1	3	2

Note. UoA = unit of analysis; AC = activities; AS = assessment; BA = before/after; SUP = support; TT = teaching techniques; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

UoA3 had four KCT tasks in total, and three of them asked PSTs to provide methods to support students' understanding (SUP), and one task supported PSTs to know appropriate teaching techniques (TT). There were very limited KCT tasks in UoA4, UoA5 and UoA6, and all of these tasks offered opportunities for PSTs to provide explanations to support students' understanding (SUP). In seven KCT tasks given in UoA7, four types of KCT codes were discovered. Almost half of these tasks offered opportunities to provide explanations for supporting students' understanding (SUP), and less than one-third of KCT tasks demanded to know appropriate teaching techniques to teach fractions (TT).

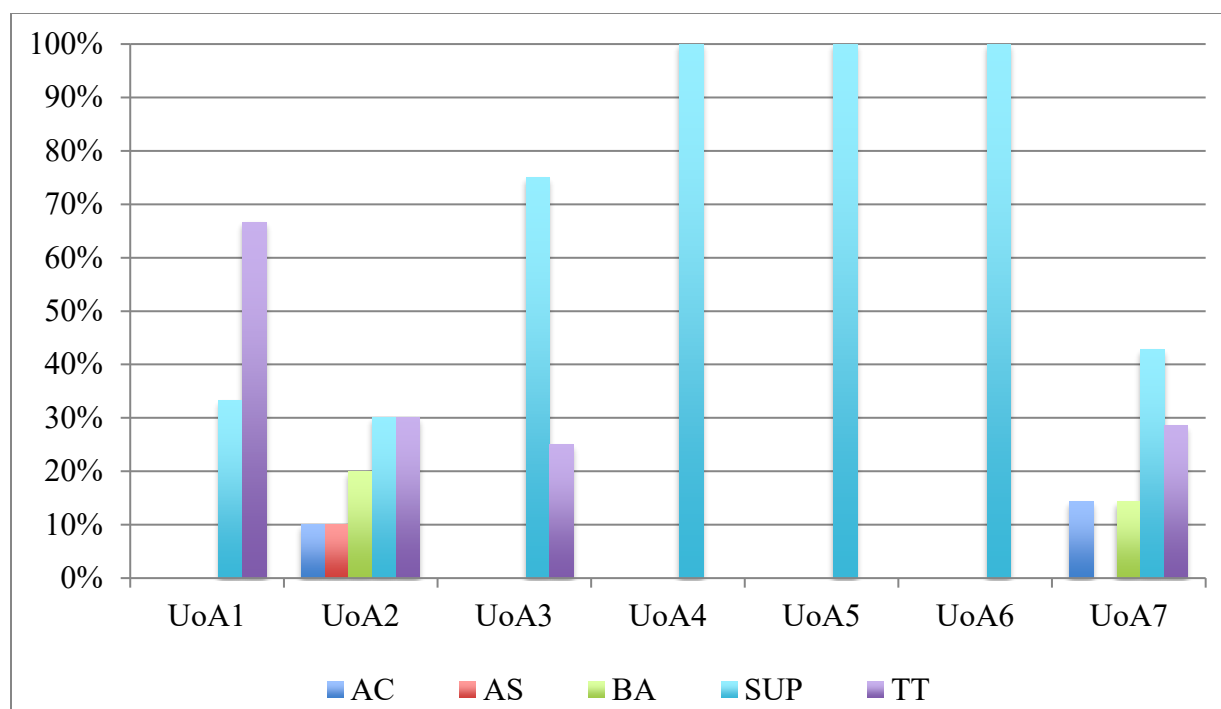


Figure 21. The Percentage of KCT Codes in Each Unit of Analysis

Note. UoA = unit of analysis; AC = activities; AS = assessment; BA = before/after; SUP = support; TT = teaching techniques; UoA = unit of analysis; UoA1 and UoA2 are fraction chapters in *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); UoA3, UoA4, UoA5 and UoA6 in *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); UoA7 in *Helping Children Learn Mathematics* by Reys et al. (2015).

Since each KCT task involved one KCT code, the percentages of KCT codes in each unit of analysis added up to 100%. Closer examination of Table 34 and Figure 21 revealed that providing explanations to support students' understanding (SUP) was the most demanded type of KCT in most of the units of analysis, whereas most of the KCT tasks in UoA1 required the knowledge of appropriate teaching techniques (TT) to be completed. By the way of contrast, the knowledge and use of appropriate assessment strategies (AS) was the least offered type of KCT among units of analysis. The following figure represents the percentage of KCT codes in each mathematics methods textbook.

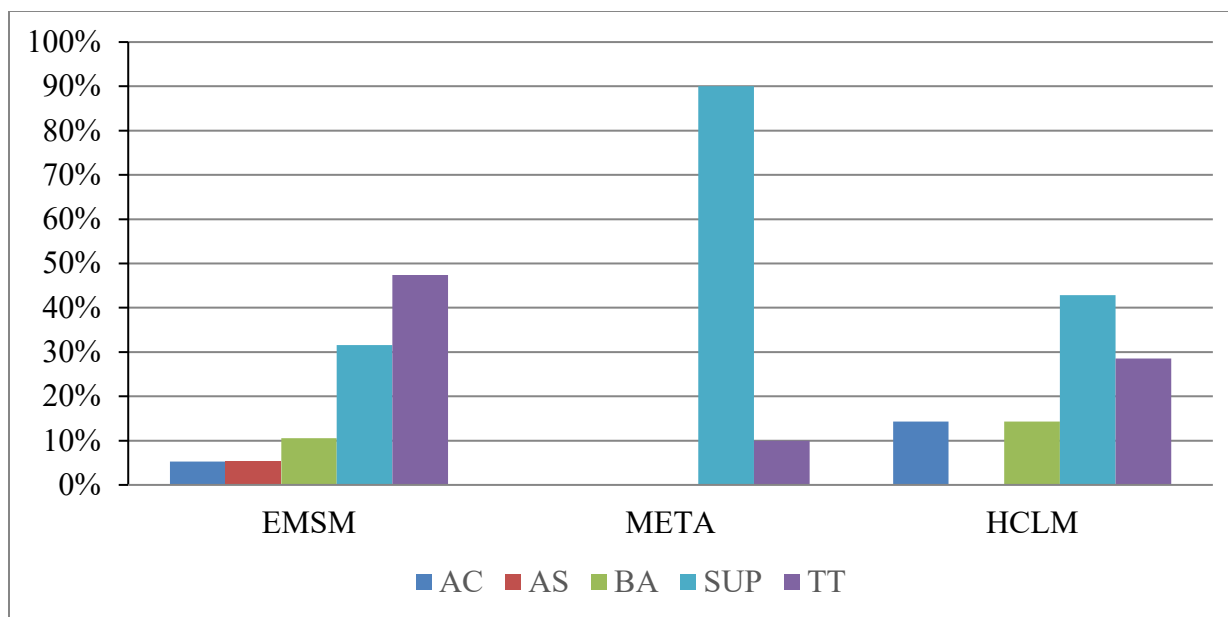


Figure 22. The Percentage of KCT Codes in Each Textbook

Note. AC = activities; AS = assessment; BA = before/after; SUP = support; TT = teaching techniques; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

As Figure 22 displays, EMSM was the only textbook that offered tasks demanding five different types of KCT. There were in total of 19 KCT tasks, and almost 50% of these tasks provided PSTs opportunities to develop the knowledge of teaching techniques (TT), while one-third demanded methods and/or explanations to support students' understanding (SUP). META consisted of 10 KCT tasks, and most of them were asked to support students' understanding (SUP), and only one of them focused on teaching techniques (TT). HCLM shared the same results with UoA7, since it had only one chapter. Thus, HCLM contained seven KCT tasks in total, and nearly half of them demanded explanations to support students' understanding (SUP).

Overall, the opportunities provided to support PSTs' KCT were limited. EMSM offered the greatest amount of KCT tasks among three textbooks. Most of these tasks focused on the knowledge of teaching techniques (TT). On the contrary, META offered 10 KCT tasks and in most them PSTs were expected to support students to overcome their misconceptions (SUP). The knowledge of assessment strategies was the least demanded KCT among three textbooks.

Although the number of KCT tasks in HCLM was less than the amount in EMSM and in META, HCLM had the highest percentage of KCT tasks due to its data size. In the next section I provide the findings related to the last domain of PCK, which is Knowledge of Content and Curriculum (KCC).

Knowledge of Content and Curriculum

To explore the opportunities provided in three mathematics methods textbooks to support PSTs' KCC, I examined all units of coding. Again, I used the coding frame to determine if a task demands a KCC to be completed (see Table 35). It is important to mention here that I discarded the unused codes after the pilot phase; therefore, the codes represent the data. As seen in the table, after evaluating nearly 500 fraction tasks within the three mathematics methods textbooks, only one KCC code was located. For a detailed description of the KCC code, please refer to Chapter 3.

Table 35. KCC Codes and Labels

Codes	Labels
Standards	STD

Table 36 displays the results for KCC in both units of analysis and in each mathematics methods textbooks. However, there were no chances given to improve/support PSTs' KCC in six out of seven units of analysis. The last unit of analysis included only one KCC task, and it represented the whole HCLM textbook.

Table 36. The Number and Percentage of Tasks Support KCC in Units of Analysis and Textbooks

Textbook	No. of UoA	Total Tasks	No. of KCC Tasks	% of KCC Tasks	% of KCC Tasks in Textbook
EMSM	UoA1	81	0	0%	0%
	UoA2	81	0	0%	
META	UoA3	122	0	0%	0%
	UoA4	45	0	0%	
	UoA5	47	0	0%	
	UoA6	86	0	0%	
HCLM	UoA7	32	1	3%	3%

Note. UoA = unit of analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

Remember that, the number of total fraction tasks ranged from 32 to 300 among three mathematics textbooks. HCLM offered 32 tasks, whereas META provided 300 tasks to develop the understanding of fraction concepts. Although the number of tasks was very low in HCLM compared to other two textbooks, it was the only textbook that offered opportunity to practice KCC. Specifically, as the following figure presents 3% of tasks in HCLM demanded the KCC.

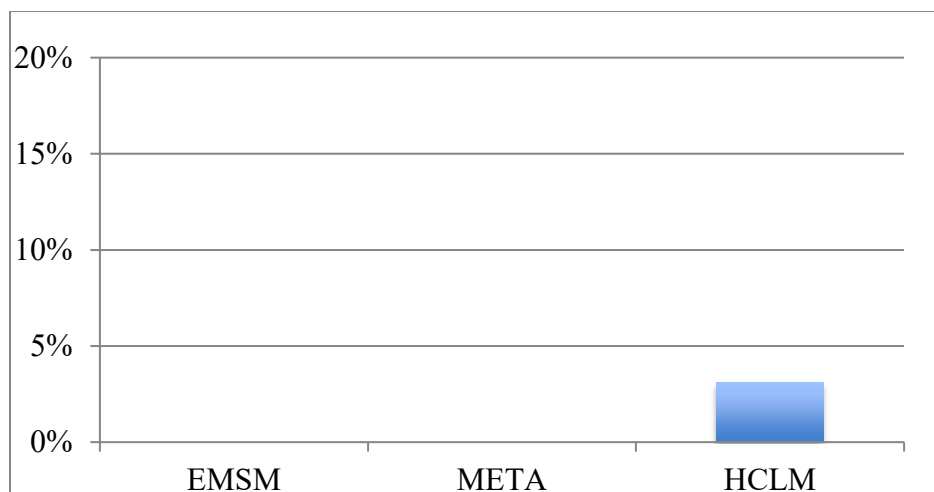


Figure 23. The Percentage of KCC Tasks Across Textbooks

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

As mentioned earlier, there was only one type of KCC located in textbooks. And, it was located in the UoA7. Since HCLM consisted of one fraction chapter (unit of analysis), UoA7 represented the whole textbook. One KCC task located in UoA7 (HCLM) demanded one type of KCC, which was the knowledge of standards and objectives for fractions and as well as the ability to decide when to introduce/reintroduce the standards (STD). The following figure provides a visual representation of the percentage of KCC tasks within each mathematics methods textbooks.

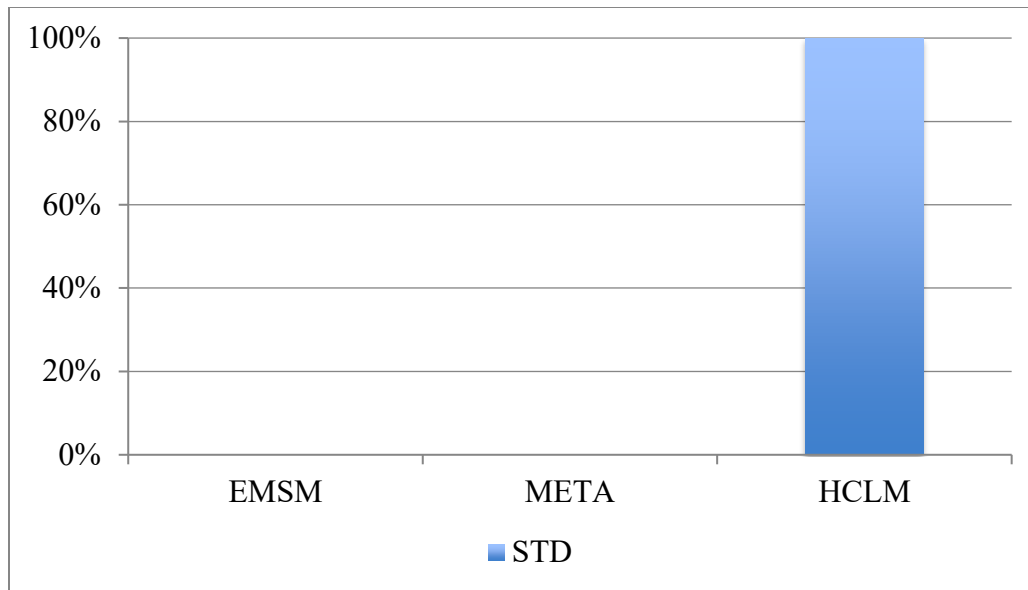


Figure 24. The Percentage of KCC Codes in Each Textbook

Note. STD = standards; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

Overall, the opportunities provided for PSTs to develop their KCC were very limited in three mathematics textbooks compared to other domains of Mathematical Knowledge for Teaching (MKT). HCLM was the only textbook that offered a KCC task, and the task demanded the knowledge of standards and objectives (STD). In the following section, I report the combinations of findings of three domains (KCS, KCT and KCC) in order to present results related to Pedagogical Content Knowledge (PCK).

Combination of Pedagogical Content Knowledge Domains

Earlier, I presented the results of KCS, KCT and KCC separately. In this section, I combine the results of each domain to answer my research question related to PCK. In order to do that I first report the number of tasks in each unit of analysis and textbooks demand the knowledge of at least one domain of PCK. Then, I provide the combined results to determine whether the three mathematics methods textbook support PSTs to develop their PCK.

To discover to what extent the textbooks contain domains of PCK, I analyzed each task under each unit of analysis. Precisely, I considered a task developing PSTs' PCK if it demands one of three sub-domains (KCS, KCT, KCC). The following table presents the number and percentages of tasks demanding any domain of PCK among seven units of analysis as well as the three mathematics textbooks. The number of PCK tasks ranged from 1 to 23. UoA5 had the least number of tasks supporting PCK followed by UoA6. On the contrary, UoA3 had the greatest number of PCK tasks, closely followed by UoA1. It is important to remind the impact of data sizes when it comes to find the percentages of tasks.

The UoA5 appeared to have the least percentage of PCK tasks, whereas UoA7 had the greatest percentage. In more detail, nearly one-third of the tasks in UoA7 and one-fourth all the tasks in UoA1 and UoA2 demanded PSTs to have KCS, KCT or KCC to be completed.

Table 37. The Number and Percentage of PCK Tasks in Units of Analysis and Textbooks

		No. of Tasks	PCK Tasks	% of PCK in UoA	% of PCK in Texts
EMSM	UoA1	81	20	25%	23%
	UoA2	81	18	22%	
META	UoA3	122	23	19%	12%
	UoA4	45	7	16%	
	UoA5	47	1	2%	
	UoA6	86	4	5%	
	UoA7	32	10	31%	

Note. UoA = unit of analysis; EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

The analysis indicated that almost 25% of tasks in EMSM offered PSTs opportunities to practice at least one domain of PCK. HCLM appeared to have highest percentage of PCK tasks, while around 10% of tasks in META provided PSTs chances to improve any of three domains of

PCK. In the following paragraphs I present the findings related to the number of tasks demand more than one domain of Pedagogical Content Knowledge (PCK) in three mathematics methods textbooks.

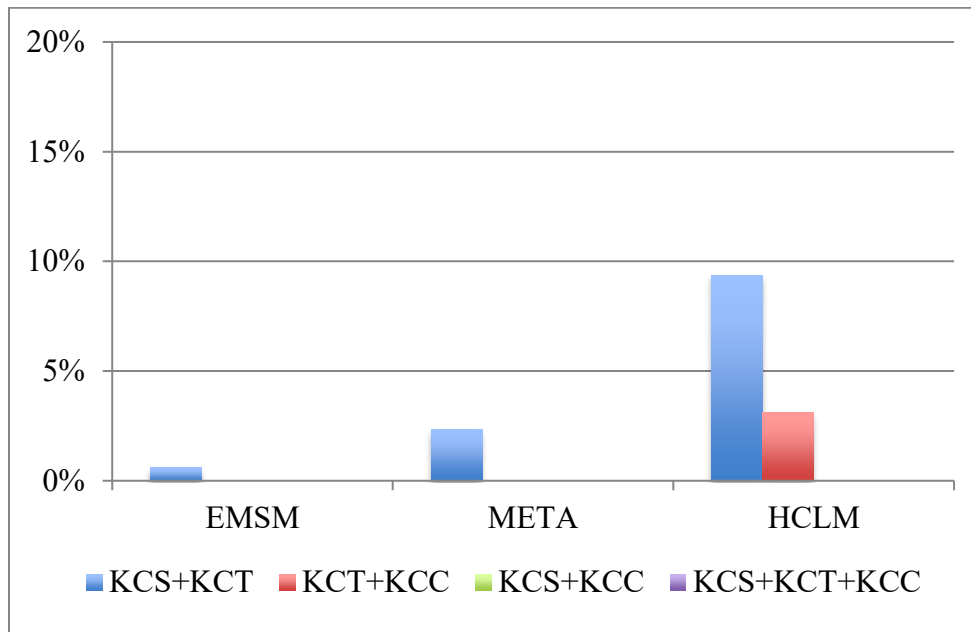


Figure 25. The Percentage of Tasks Demand Combination of PCK in Textbooks

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

As revealed in the figure, there was no textbook consisting of tasks that demand three knowledge domains at the same time. The only combination of PCK exist in EMSM and META was KCS and KCT. There was only one task in EMSM and seven tasks in META that demanded both KCS and KCT to be completed. HCLM was the only textbook that consisted of a task demand a combination of KCT and KCC to be completed. It contained one task requiring KCT and KCC at the same time and also three tasks demanding both KCS and KCT to be completed. Again, it is important to mention that HCLM had the least number of fractional tasks compared to other two mathematics method textbooks; however, it was the only textbook offered opportunities to support PSTs' KCC.

In general, I presented the findings related to the three domains of PCK in this section. Specifically, EMSM provided tasks that PSTs need either KCS or KCT to complete and offered almost the same number of tasks for these two sub-domains. The number of tasks that PSTs were offered to develop KCS had tripled the number of KCT tasks in META. Additionally, HCLM was the only textbook that provided an opportunity to develop KCC, and the chances to improve KCT was higher compared to other two domains in HCLM. In the last part, I present results combining both SMK and PCK to reflect Mathematical Knowledge for Teaching Fractions (MKTF).

Opportunities of Mathematical Knowledge for Teaching Fractions

Previously, I presented findings related to the sub-questions to discover opportunities offered in three mathematics methods textbooks for PSTs to develop SMK and PCK for fractions. MKT consists of two domains, SMK and PCK. Therefore, in order to reflect the main research question, which addresses the extents and ways the three mathematics methods textbooks provide PSTs chances to develop MKTF, I combined the findings of SMK and PCK. In the following paragraph, I offer the results of each mathematics methods textbooks in terms of the opportunities provided for PSTs to support their MKTF in general.

The following table displays the number and the percentage of tasks and Figure 26 provides a visual representation of percentages of tasks that demand each domain of MKT in each mathematics methods textbook. All three mathematics methods textbooks placed a greater emphasis on developing PSTs' SMK rather than PCK. Specifically, CCK was the most demanded knowledge that PSTs need to possess in order to complete the fraction tasks compared to other domains of MKT. More than three-fourths of tasks supported CCK in EMSM and META, whereas the ratio was a little less than that in HCLM. EMSM supported PSTs' CCK by

providing tasks that focused on finding fractions and four operations. Similar to EMSM, CCK tasks in META supported the understanding of four operations, whereas HCLM relied on the knowledge of comparison and equivalence of fractions. As a second most required knowledge to complete the fraction tasks in three textbooks was SCK. Sixty percent of the tasks in META, half of the tasks in HCLM, and around one-fourth of the tasks in EMSM required PSTs to have SCK. Of those SCK tasks in EMSM and HCLM, most of them focused on the use of models to show fractions. META stressed the knowledge of problem posing by proving more tasks about it. EMSM offered equal amount of opportunities for PSTs to develop KCS and KCT, whereas KCS was the third most demanded knowledge in META. EMSM mostly supported the understanding of students' misconceptions, whereas META provided more tasks that demand the knowledge of students' reasoning with fractions. While two textbooks, EMSM and META, placed equal or more emphasis on KCS than KCT, it was the other way around for HCLM. Based on the findings, KCT was the third most needed knowledge in HCLM. Additionally, KCC was the least demanded knowledge in the mathematics methods textbooks among six domains of MKT.

There were no opportunities to develop KCC in EMSM, and it was the only textbook that did not provide opportunities to support PSTs' HCK through completing fraction tasks. Similar to EMSM, META did not offer any chance to support PSTs' KCC, whereas it provided very limited opportunities to develop KCT.

Table 38. The Number and Percentage of Tasks Support Each Domain of MKT in Textbooks

Textbook	No. of Tasks	SMK						PCK					
		No. of CCK	% of CCK	No. of HCK	% of HCK	No. of SCK	% of SCK	No. of KCS	% of KCS	No. of KCT	% of KCT	No. of KCC	% of KCC
EMSM	162	126	78%	0	0%	42	26%	20	12%	19	12%	0	0%
META	300	228	76%	6	2%	180	60%	32	11%	10	3%	0	0%
HCLM	32	20	63%	5	16%	16	50%	6	19%	7	22%	1	3%

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

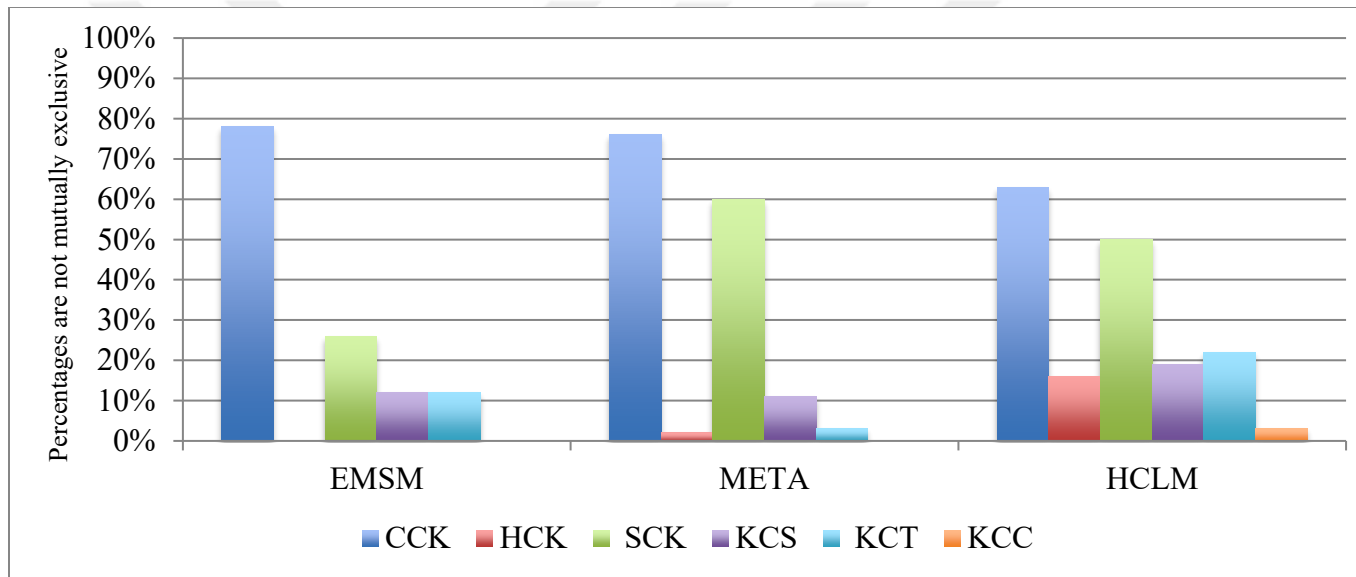


Figure 26. The Percentage of Tasks Support Each Domain of MKT in Textbooks

Note. EMSM = *Elementary and Middle School Mathematics, Teaching Developmentally* by Van de Walle et al. (2019); META = *Mathematics for Elementary Teachers with Activities* by Beckmann (2018); HCLM = *Helping Children Learn Mathematics* by Reys et al. (2015).

Moreover, META offered more fraction tasks that emphasis on HCK than other two methods textbooks. Specifically, it provided highest amount of HCK tasks within three textbooks. It is important to mention that META consisted of the highest amount of fraction tasks (units of coding) compared to other textbooks, and it placed a greater emphasis on CCK tasks and SCK tasks.

In terms of offering the variety of knowledge opportunities, HCLM supported PSTs' both SMK and PCK (all six domains) even though it provided a small amount of fraction tasks of some domains. Specifically, HCLM was the only textbook in this sample, which offers a task that demands KCC. Although, the number of HCK, KCS and KCT tasks was not the greatest among textbooks, HCLM offered the highest percentages of tasks demand HCK, KCS and KCT among three textbooks. Overall, HCLM showed a greater emphasis on CCK and SCK.

Summary of Results

In this chapter, I presented the results of the opportunities given in three mathematics methods textbooks to support PSTs' MKT. In particular, I first examined the extent fraction tasks given in mathematics methods textbooks support PSTs to develop their SMK using the coding frame I developed. Then I followed the same criteria to explore the chances given to support PSTs' PCK through fraction tasks within three mathematics textbooks. During the analysis, I observed similarities and differences in terms of given opportunities to develop PSTs' MKT through fraction tasks. First, all textbooks placed a greater emphasis on SMK than PCK. Specifically, CCK was the most common knowledge provided for PSTs to practice followed by SCK. Second, all textbooks provided tasks that demand CCK and SCK at the same time to be successfully completed. However, HCLM was the only textbook that provided task that demand

different combination of two and three domains of SMK, such as CCK and HCK, HCK and SCK, etc. Moreover, all three textbooks offered either zero or only one task that demanded KCC. And, there were no chances given in any fraction tasks among the three textbooks to possess a combination of three domains of PCK at the same time to be completed. As a significant difference within three textbooks, I found that although HCLM had the least amount of fraction tasks, it was the only textbook that offered tasks demanding KCC and also a combination of two domains of PCK, which were KCT and KCC. On the contrary, META had the greatest amount of fraction tasks but offered insignificant amount of KCT tasks. In the last chapter, I present the conclusions that can be drawn from the results, limitations, significance, and implications of this inquiry.

CHAPTER 5: DISCUSSION

The purpose of this inquiry was to explore the ways fraction tasks in three mathematics methods textbooks provide PSTs with learning opportunities to develop *Mathematical Knowledge for Teaching Fractions* (MKTF). Particularly, I examined all fraction tasks within mathematics methods textbooks to assess the extent to which they support PSTs' *Subject Matter Knowledge* (SMK) and *Pedagogical Content Knowledge* (PCK) for teaching fractions.

In this chapter, I discuss what can be taken away from this inquiry and how the findings can inform mathematics educators, teacher preparation programs, and curriculum developers. I begin with the brief summary. Then I provide the discussion of the findings for the research question that the inquiry aims to answer and limitations of the study. Next, I address importance and implications of the study for the teacher education. I conclude the chapter with suggestions for future study.

Summary of the Study

Fractions are considered one of the most important content of mathematics (Siegler et al., 2012). Neagoy (2017) provided three fundamental reasons on fraction importance: "fractions play a key role in students' feelings about mathematics, fractions are fundamental to school mathematics and daily life, and fractions are foundational to success in algebra" (p. 2). In addition, other studies have found that a positive relation between success in algebra and students' understanding of fractions (Brown & Quinn, 2007; NMAP, 2008; Wu, 2001).

Although the importance of fractions is undeniable, many students have fragmented understanding of fractions (Mack, 1990). And, as far as students' understanding of fractions is concerned, one of the key impacts on this issue is teacher knowledge. As Shulman (1986) stressed teachers cannot teach the content that they don't know; therefore, it is unexpected for teachers to teach fractions meaningfully unless they possess strong understanding of it. Studies have shown that PSTs' limited understanding of fractions persists (Ball, 1990a; Marchionda, 2006; Newton, 2008; Utley & Reeder, 2012). One of the main roots of this issue is learning fractions in their own K-12 education through rote memorization (Ball, 1990a; CBMS, 2001, 2012). This problem leads mathematics courses in teacher preparation programs to a place where PSTs should be offered with the essential opportunity to learn fractions with understanding. Also, a study that analyzed methods course syllabi showed that objectives of the course is to improve PSTs' both content knowledge and pedagogical content knowledge (Taylor & Ronau, 2006), which align with the SPTM (AMTE, 2017), NCTM Standards (1991) and recommendations in MET I and MET II (CBMS, 2001, 2012).

There is no shared curriculum in teacher preparation programs, therefore opportunities offered for PSTs differ across institutions (Ball et al., 2009; Hill & Lubienski, 2007). However, textbooks are the commonly used curriculum material for teaching and learning of mathematics in higher education (Mesa & Griffiths, 2012). As noted by Harkness and Brass (2017), mathematics methods instructors use textbooks for many reasons, but mostly as sources of class activities and class discussions. Textbooks play an important role in learning by impacting what is taught and how it is taught (Charalambous & Hill, 2012). Therefore, the opportunities for PSTs to develop MKTF differ as a result of differences in the fraction concept covered in methods textbooks.

In this inquiry, I analyzed fraction tasks of three elementary mathematics methods textbooks in order to explore PSTs' opportunities to improve MKTF using qualitative content analysis. I used the following research questions to guide this inquiry:

- 1) To what extent and in what ways do textbooks used in mathematics methods courses in teacher preparation programs provide PSTs opportunities to develop Mathematical Knowledge for Teaching Fractions?
 - a) What opportunities are provided in the textbooks for PSTs to develop Subject Matter Knowledge for fractions?
 - i. What opportunities are provided in the textbooks for PSTs to develop Common Content Knowledge for fractions?
 - ii. What opportunities are provided in the textbooks for PSTs to develop Horizon Content Knowledge for fractions?
 - iii. What opportunities are provided in the textbooks for PSTs to develop Specialized Content Knowledge for fractions?
 - b) What opportunities are provided in the textbooks for PSTs to develop pedagogical content knowledge for fractions?
 - i. What opportunities are provided in the textbooks for PSTs to develop Knowledge of Content and Students?
 - ii. What opportunities are provided in the textbooks for PSTs to develop Knowledge of Content and Teaching?
 - iii. What opportunities are provided in the textbooks for PSTs to develop Knowledge of Content and Curriculum?

Discussion of Findings

This study set out to examine the types of opportunities offered for PSTs to develop their MKTF in mathematics methods textbooks. Therefore, I examined all fraction tasks given under fraction concepts within three mathematics methods textbooks to explore the type of opportunities provided in fraction tasks to support PSTs to improve their MKTF. To answer the research questions, I started developing the coding framework in a deductive way based on the well-known framework (MKT) developed by Ball and her colleagues (2008). The MKT framework consists of two main domains, SMK and PCK. And, each main domain consists of three sub-domains; *Common Content Knowledge* (CCK), *Specialized Content Knowledge* (SCK) and *Horizon Content Knowledge* (HCK) are part of SMK, and *Knowledge of Content and Students* (KCS), *Knowledge of Content and Teaching* (KCT), and *Knowledge of Content and Curriculum* (KCC) belongs to PCK. Then, I continued to develop the coding frame inductively by describing and making sense of the data. Using the developed coding frame, I conducted a qualitative content analysis to reveal the data outcomes. In this section, I discuss the findings in regard to the opportunities offered for PSTs to develop MKTF.

Opportunities to Develop Mathematical Knowledge for Teaching Fractions

As stated earlier, I sought to answer the type of opportunities mathematics methods textbooks provide for PSTs to improve MKTF. Since, the MKT framework consists of two main domains, SMK and PCK, I report and discuss the findings of SMK first, and then PCK individually to provide a clear discussion.

Opportunities to Develop Subject Matter Knowledge

It is widely known that robust subject matter knowledge is essential for teaching mathematics. Therefore, I now discuss the findings related to sub-domains of SMK in the following paragraphs.

Opportunities to Develop Common Content Knowledge

First, I discuss findings related to CCK. It has been well indicated by many researchers that teachers should know the content they teach (Ball et al., 2008; Shulman, 1986). Ball et al. (2008) mentioned that CCK is very essential for teachers, and its importance becomes more clear when they make a calculation error or would not be able to correctly solve a problem. Among all different types of knowledge, CCK was the mostly demanded knowledge for PSTs to complete fraction tasks in three mathematics methods textbooks. Both EMSM and META provided the highest percentage of CCK tasks that focus on operations. And around half of these operation tasks (19 out of 41) in EMSM and a little more than 50% of operation tasks (65 out of 123) in META requested PSTs to divide fractions. There were only one out of four operation tasks focus on fraction division in HCLM. Studies have shown that PSTs have fragmented understanding of division of fractions over the years (Ball, 1990c; Newton, 2008; Nillas, 2003). It is suggested to provide “variety of problem-solving situations in teaching division of fractions” would be effective when learning fraction division with understanding (Nillas, 2003, p.110). Therefore, this finding could suggest the authors of HCLM to consider implementing the recommendations.

The use of number lines was suggested by researchers (Fennel et al., 2014; Siegler et al., 2010) for different reasons such as teaching equivalency, proper and improper fractions, comparing and ordering fractions, etc. Siegler et al. (2010) suggested teachers to “use number lines as a central representational tool in teaching this and other fraction concepts from the early

grades on” (p.1). However, I found limited opportunities for PSTs to develop their understanding of number line in EMSM and HCLM, whereas META offered opportunities to use number lines in nearly one-fourth of CCK tasks. Besides, Fennel et al. (2014) stressed that the understanding of equivalent fractions is the basis for developing fraction sense. The findings revealed that the textbooks offered a variety of opportunities to develop an understanding of fraction equivalence. The highest percentage of equivalence fractions were given in HCLM (around 30% of CCK tasks) followed by META (around 20%). EMSM provided less than 10% of CCK tasks that demand the knowledge of equivalent fractions. Moreover, Siegler et al. (2010) suggested introducing fractions using equal sharing practices. However, the findings do not coincide with this suggestion, because META and HCLM provided almost no opportunities whereas EMSM offered less than 10% of CCK tasks focus on equal sharing.

These findings showed that there are different kind of opportunities given for PSTs to develop their CCK. It should be noted that among three textbooks, META had the highest number of fraction tasks (a total of 300) because of providing repetitive questions for PSTs to internalize the concepts, which was not the case for EMSM and HCLM. It is important to emphasize that EMSM offered more than one-third of CCK tasks in the online resource, which required PSTs to purchase an access codes to have access. In my experiences with PSTs, almost all of the PSTs have preferred the print version instead of e-text. Therefore, they would not have accessed to those tasks that are offered in the online resource.

Opportunities to Develop Horizon Content Knowledge

As the second domain of SMK, I discuss findings related to HCK. EMSM did not offer any chances for PSTs to develop HCK, whereas there were only 6 tasks out of 300 in META. Although HCLM contained the least number of tasks among three textbooks, a little less than

20% of the tasks demanded HCK to be appropriately completed. It has been emphasized that connecting fractions with numbers when introducing fractions could prevent misconceptions (Fennell et al., 2014; Siegler et al., 2010). However, there were not adequate opportunities for PSTs to make this connection among three textbooks, but especially in EMSM. This might be due to not having clear definition and understanding of HCK (Wasserman & Stockton, 2013), or not having sufficient evidence that show the impact of HCK on teaching mathematics, teachers' instruction quality and students' achievement (Mosvold & Fauskanger, 2014). The findings shed light on the lack of opportunities for PSTs to develop HCK. It also showed the need to develop measuring items for HCK, and to explore if and how teachers' HCK influences their mathematics teaching and students' achievement.

Opportunities to Develop Specialized Content Knowledge

I now turn to discuss the findings related to the last domain of SMK, which is SCK. Overall, SCK was the second most demanded knowledge amongst three mathematics methods textbooks. META offered highest percentage of SCK tasks (60%), but as I mentioned previously it contains many repetitive tasks. The half of the tasks in HCLM demanded PSTs to possess SCK, whereas the percentage of SCK tasks was around 25 % for EMSM.

Many studies stressed the importance of providing opportunities to practice all meanings of fractions (Clarke et al., 2008; Lamon, 2012; NCTM, 2000; Siebert & Gaskin, 2006). EMSM was the only mathematics methods textbook that provided all five meanings of fractions (part-whole, division, measurement, operator, and ratio), and stressed it is essential to know all meanings of fractions to understand fractions in depth. However, there were three SCK tasks in EMSM that demanded PSTs to know the meanings of fractions, and only one of them required knowing meanings other than part-whole. META provided only part-whole interpretation in the

lessons and did not offer any tasks that demand any other meanings of fractions. Whereas, three meanings of fractions (part-whole, quotient, and ratio) were introduced in the fraction chapter in HCLM, and PSTs were given opportunities to practice these three meanings in two SCK tasks. These findings revealed that PSTs were offered with limited opportunities to practice all meanings of fractions, which cause some drawbacks (Siegler et al., 2011).

More than one-third of SCK tasks in EMSM and HCLM demanded PSTs to know/use different models (area, set, and length) and manipulatives, which was the most demanded type of SCK in both textbooks. It has been stated that there is a positive relationship between the use of manipulatives and fraction understanding (Hudson Hawkins, 2007). Therefore, the findings displayed that these two mathematics methods textbooks supported PSTs' fraction understanding by offering a significant number of tasks that required the use of manipulatives. Results showed that PSTs were also not offered opportunities to work with models other than area models in HCLM. This is similar to findings from studies on student textbooks that showed the frequent use of area models (Son, 2011; Zhang et al., 2015). Although many tasks involved the use of number lines, most of them did not demand knowing its connection with the length model. For instance, there were several tasks in META requested to plot fractions on the number line placing a tick mark by eye. I operationalized these types of tasks as a support to the use of number lines, but they did not clearly make connections to help PSTs internalize number lines as a length model. Specifically, I anticipated a task demands the knowledge of models as a part of SCK if it demands PSTs to understand the link between representations and models, such as fraction circles for area model, number lines for length model, two-color counters for set models.

While EMSM and HCLM offered limited opportunities for PSTs to pose a word problem, META gave the most attention to problem posing among other types of SCK. Around 40% of

SCK tasks in META required PSTs to write a word problem corresponding to a fraction expression. The author of META mentioned that PSTs will need to write word problems for their students in the future, so making practices is helpful to see how little changes in wording can result big changes in meaning. Using problem posing as a teaching technique had a positive impact on PSTs' fraction understanding (Toluk-Uçar, 2009) and, it is suggested to include the problem-posing tasks in teaching (Nillas, 2003). Thus, the findings pointed the need to provide more problem posing opportunities for PSTs in EMSM and HCLM textbooks.

Opportunities to Develop Pedagogical Content Knowledge

Through the years, many studies have focused on teachers' content knowledge and considered it as the most important impact on students' understanding. However, Shulman (1986) indicated that teacher knowledge is complex and multidimensional, and content knowledge is not sufficient for teaching effectively. Shulman stressed the necessity to involve PCK in teacher education programs. Therefore, I discuss the findings related to sub-domains of PCK in the following sections.

Opportunities to Develop Knowledge of Content and Students

As the first domain of PCK, I discuss findings related to KCS. Although, KCS was the third most demanded type of knowledge in two methods textbooks (EMSM and META), the percentage of KCS tasks was low with a range between 11% and 19%. According to Ball et al., (2008), the knowledge of understanding students' misconceptions and errors is the central type of KCS, and it was the most demanded KCS type in both EMSM and HCLM. It is important to emphasize due to the very limited number of KCS tasks, none of the textbooks provided sufficient opportunities for PSTs to improve their knowledge related to students' misconception. Moreover, almost all of KCS tasks (18 out of 20) in EMSM were offered in the online resource

(MyLab Education). Although, the authors mentioned that materials in the online source were designed to support PSTs' SMK and PCK; offering nearly all of the KCS tasks in the online resource could cause a lack of opportunities for PSTs who do not have access to the online materials. META offered opportunities to focus on students' reasoning more than other types of KCS. Even though, students' misconceptions might occur due to their reasonings, I have operationalized them as two different types in this inquiry because there were some tasks that require to discuss students' logical reasoning.

Opportunities to Develop Knowledge of Content and Teaching

The second domain of PCK is KCT, and the importance of this knowledge indicated by Ball and her colleagues (2008), "the demands of teaching require knowledge at the intersection of content and teaching" (p. 402). The findings related to KCT revealed that there are limited opportunities for PSTs to develop this type of knowledge through fraction tasks. Being able to support students to overcome their misconceptions related to fractions, and to use appropriate teaching techniques for teaching fractions were the most common type of KCT given among textbooks. In their study, Li and Kulm (2008) mentioned that due to the lack of teaching experience PSTs would not be expected to possess this type of knowledge, thus this could be a reason for offering limited opportunities. On the contrary, one cannot expect students (PSTs in this case) to learn content if they are not exposed (Stein et al., 2007), therefore textbook authors might consider including more tasks that demand KCT. Research on PST' knowledge of fractions indicates that PSTs might have same misconceptions as students (Newton, 2008; Tirosh, 2000), which gives an emphasis on the importance of both KCS and KCT. Additionally, it is critical to stress that EMSM offered all of the KCT tasks in the online resource, which restrains PSTs, who purchase only printed format of textbook, to practice this type of knowledge.

Opportunities to Develop Knowledge of Content and Curriculum

The findings related to KCC, the last domain of PCK, revealed that it is the least offered knowledge type in fraction tasks. One of the main reasons of this outcome could be KCC being one of the less developed domains of MKT (Sleep, 2009). Among three textbooks, there was only one task demands KCC, and it was offered in HCLM. The task was requesting PSTs find the reason why objectives from 3rd grade standards are presented to 5th grade students. Even though, EMSM and META did not provide any task that demands PSTs' KCC to be completed, they included CCSS references for tasks that address specific standards, and that did not occur in HCLM. The findings of this inquiry suggested information in terms of the strengths and weaknesses of providing opportunities to develop each domain of MKT. The findings could be helpful for mathematics methods instructors, curriculum and teacher preparation program developers to make decisions and adjustments to meet PSTs' need.

Overall, the results of this inquiry revealed the differences regarding opportunities provided for PSTs to develop SMK and PCK among three mathematics methods textbooks. While all three textbooks provided sufficient number of tasks that demand SMK, especially CCK and SCK, there were very limited opportunities given for PSTs to develop PCK through fraction tasks.

Limitations of the Study

While this study provided significant results and important implications, there are some limitations, which have impacted the research findings and should be acknowledged to make suggestions for further research. With regard to the sample of this study, there are some shortcomings. First, I selected three mathematics methods textbooks based on the survey, that was conducted by other researchers in 2013. Although the sample of the study is widely accepted

in elementary mathematics courses now, these three mathematics textbooks are not a representative sample of all textbooks used in teacher preparation courses. Therefore, I acknowledge that the results could be different if I have included other mathematics methods textbooks. The survey took place in 2013 and around 130 AMTE members who teach mathematics methods courses responded the survey. There might be different outcomes if it is conducted recently and also included larger population of mathematics methods instructors. Next, I first developed the framework in this study based on the mathematics standards and the research of the U.S and then finalized it to reflect and describe the data I analyzed. Even though, the textbooks used in this study are not a representative sample of all textbooks used in mathematics methods courses, they are only used in the United States. However, there might be different perspectives on the knowledge needed by PSTs for teaching mathematics in other countries. Therefore, the framework used in this inquiry might not be appropriate to analyze written curricula used across different countries. Lastly, I merely examined the fraction tasks given under fraction chapters in mathematics methods textbooks, and the findings might have been different if I have examined fraction tasks in mathematics content textbooks.

Importance of the Study

It has been emphasized by many studies and researchers that strong understanding of fractions is fundamental for both students and teachers (Booth & Newton, 2012; Fennel et al., 2014; Siegler et al., 2012). Despite its importance, issues with students' and teachers' fraction understanding is persistent (Ma, 1999; Siebert & Gaskin, 2006). Studies indicated that teachers' fragmented knowledge of fractions might be caused by their own poor K-12 education, which rewards rote learning (CBMS, 2001, 2012; Wu, 2001). It has been noted in the literature that PSTs start their teacher preparation programs with a fragmented understanding of fractions

(Uitley & Reeder, 2012). However, teachers need to possess strong mathematical knowledge and pedagogical skills in order to teach elementary mathematics (CBMS, 2001). The objectives of mathematics methods courses focus on improving PSTs' both content knowledge and pedagogical content knowledge. Thus, mathematics methods courses in teacher preparation programs play a key role providing learning opportunities for future teachers to strengthen their knowledge prior to teaching fractions to their future students.

This inquiry seeks out to examine mathematics methods textbooks in order to explore PSTs' opportunities to develop subject matter knowledge and pedagogical content knowledge. The results highlighted both advantages and disadvantages of three mathematics methods textbooks in terms of chances to improve MKTF. Hence, the results of this study add the growing body of literature regarding the opportunities provided for PSTs to develop MKTF in curriculum materials. Teacher educators and curriculum developers can benefit from the findings of the study in regard to the selection and implementation of mathematics methods textbooks. For instance, being aware of fractions tasks in textbooks might not be significant source to develop PCK, especially KCC, might help mathematics methods instructors to make instructional decisions to ensure their course meets its learning objectives.

I conducted a qualitative content analysis in this study to explore textbooks used in mathematics methods courses. There was a gap in the literature regarding mathematics methods textbooks analysis, thus this study filled the gap by conducting a qualitative content analysis of methods textbooks. I applied qualitative content analysis to analyze and make sense of the data, which is a flexible method and allows researchers to make modifications on their framework until it explains their data. Therefore, this method can be used and adapted in a future study to explore and describe different concepts covered in methods textbooks.

Although I finalized the framework based on the sample textbooks, I started developing it based on the literature and research on MKT. Specifically, the types of knowledge still represent and describe each domain of MKTF. Therefore, the framework might be helpful for teacher educators and PSTs to analyze and assess all type of learning opportunities given in mathematics methods courses to develop MKTF. For instance, this framework might help mathematics methods instructors to consider selecting different curriculum materials to meet PSTs' need if there are limited opportunities to develop such knowledge in textbooks they use.

Implications for Teacher Education

Textbooks play a key role determining the concepts covered in mathematics classrooms, and mathematics teachers use textbooks as a main resource to make instructional decisions (Reys et al., 2004). Although, there are not enough studies, which focus on curriculum materials in teacher education programs; most of the mathematics methods instructors in Harkness and Brass's (2017) study implied that they use textbooks for different instructional purposes. Also, I have personally observed how mathematics methods instructors are influenced by textbooks when making instructional decisions. Therefore, the role of textbooks is also critical in teacher education programs that prepare PSTs with content and pedagogical knowledge and skills. The findings of this inquiry have several implications for teaching and teacher education. Teacher preparation programs should offer opportunities for PSTs to develop their understanding of content and teaching skills and practices to teach such content. The results of the study indicated that differences exist in textbooks in terms of opportunities to develop mathematical knowledge for teaching (SMK and PCK). These differences in textbooks signify unequal opportunities to develop MKTF. For instance, textbooks mainly focus on improving two domains of SMK (common content knowledge and specialized content knowledge), whereas the

purpose of methods courses is to improve both content knowledge and pedagogical content knowledge.

Various recommendations and standards for teaching fractions demand specific types of subject matter knowledge. For instance, using equivalent fractions as a foundation to have a strong fraction sense (Fennel et al., 2014) and to perform operations (Common Core State Standards for Mathematics [CCSSM]). Connecting fractions and numbers (Fennell et al., 2014; Siegler et al., 2010; CCSS-M), and practicing all meanings of fractions (NCTM, 2000; Siebert & Gaskin, 2006) are also important in supporting fraction understanding. Although, all three textbooks offered decent number of tasks that demand subject matter knowledge, the aforementioned recommendations were barely presented in tasks. Therefore, teacher educators might consider including more fraction tasks to follow recommendations and standards.

Curriculum developers and teacher educators must be aware of the disconnection between the objectives of mathematics methods courses and opportunities provided by the methods textbooks. Specifically, methods courses should offer opportunities for PSTs to learn how to teach (Ball, 1990b) and support PSTs' PCK since it provides a combination of content and pedagogical understanding during the process of learning to teach (Grossman, 1990, as cited in Kinach, 2002, p. 53). Unfortunately, the results of this study indicate that none of the textbooks offer sufficient opportunities for PSTs to develop PCK. For instance, it is critical for teachers to know students' reasoning and misconceptions (Ball et al., 2008), and to support students to overcome misconceptions. Despite the importance, there were inadequate number of tasks that demand PSTs to know students' common misconceptions and to provide explanations to support student understanding. Given these findings, teacher educators and curriculum

developers might consider increasing PSTs' opportunities to learn how to teach, since it is critical for PSTs to develop these types of knowledge prior teaching fractions.

The results of this study shed light on different opportunities provided for PSTs to develop SMK and PCK in textbooks, since there is no shared curriculum in teacher education (Ball et al., 2009). This awareness can guide curriculum developers and teacher educators to use these findings as a groundwork for developing a shared curriculum, which offer equal opportunities for PSTs to improve MKTF.

Recommendations for Future Research

In this inquiry, I explored the opportunities provided for PSTs to improve their MKTF through the fraction tasks in three mathematics methods textbooks. The results revealed that PSTs are offered with a variety of learning opportunities to develop their MKTF via fraction tasks. Since the inquiry focused on the tasks as they appear in written curriculum, a possible extension to this study would be an analysis on how mathematics methods instructors set up the tasks. This extension can be supportive for mathematics educators, mathematics methods instructors and curriculum developers in teacher preparation programs to understand whether mathematics methods instructors set up fraction tasks as the way they are intended by the authors or provide different opportunities to fulfill PSTs' needs.

Another extension to this study can be an investigation on how these fraction tasks are implemented by PSTs. As Stein and Smith (1998) indicated that all phases of tasks are important, but especially implication phase has an important effect on student learning. According to Stein and Smith, "the nature of tasks often changes as they pass from one phase to another" (p. 270); therefore, a study on the enacted curriculum could be beneficial to see if there is any alteration among the tasks that are implemented by PSTs, tasks appear in the textbook

(written curriculum) and tasks that are set-up by mathematics methods instructors (intended curriculum). This can shed light on the transformations from one curriculum to another regarding to the opportunities provided for PSTs to develop their MKTF, because textbooks can be used by instructors and students as they are intended by the authors completely, slightly, or not at all (Mesa & Griffiths, 2012). An interview with mathematics methods instructors and multiple class observations might provide valuable information on whether they use the textbook as a source when they select fraction tasks, because fraction tasks appear in textbooks do not mean they are covered or highlighted by the instructors.

In this inquiry, I examined only fraction tasks in three mathematics methods textbooks. Since knowing fraction with understanding is the foundation for understanding decimals (Wu, 2001), an extension to this study could be inclusion of decimal tasks as well in order to investigate any differences or similarities in terms of opportunities to develop PSTs' MKTF. A further research can also be done by examining tasks in different content areas such as geometry, number sense, measurement etc. to determine the type of opportunities offered in the methods textbooks to develop PSTs' MKT. For instance, there might be a potential to see differences between chances provided to support PSTs' HCK in fraction and number sense tasks.

The sample of the inquiry is limited to three mathematics methods textbooks. Since these textbooks are not a representative sample of all textbooks used in elementary mathematics methods courses, I recommend conducting a study that includes other mathematics methods textbooks. As stated previously, there is no shared curriculum in teacher preparation programs, thus this type of study might shed light upon whether textbooks, which are used by PSTs in the U.S., offer equal opportunities to improve MKTF or not. Some teacher preparation programs in the U.S. offer only mathematics content courses for PSTs to develop their fraction

understanding, therefore an analysis of mathematics content course textbooks can provide us better insight of the differences and similarities in terms of opportunities given to PSTs to develop their MKTF through fraction tasks.

Concluding Remarks

Teacher mathematical knowledge has an impact on instructional quality and student learning. Therefore, it is important for teachers to have a strong understanding of fractions, which is considered to be one of the most complex mathematical topics for students to learn and teachers to teach. Thus, teacher preparation programs play a key role in providing opportunities for PSTs to develop their MKTF. Since textbooks are used as one of the curriculum materials in teacher preparation programs, conducting a content analysis to assess opportunities offered for PSTs to learn how to teach fractions.

In this inquiry, I examined the opportunities provided for PSTs to develop mathematical knowledge for teaching fractions by examining fraction tasks in three mathematics methods textbooks. This study highlighted important aspects of mathematics methods textbooks and fills gaps in the elementary mathematics education literature in terms of opportunities provided for PSTs to develop mathematical knowledge for teaching fractions in written curriculum. Considering the fact that students' and teachers' difficulties with fractions persist, the findings indicate the need for additional content analysis of textbooks used in teacher preparation programs to further explore this difficulty and also offer a foundation for potential research in different areas.

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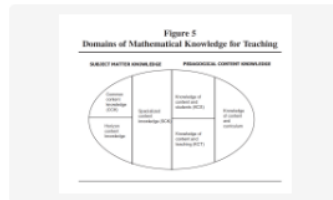
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