ISOMETRIES OF LENGTH 1 IN FREE KLENIAN GROUPS AND TRACE INEQUALITIES

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ABSTRACT

ISOMETRIES OF LENGTH 1 IN FREE KLENIAN GROUPS AND TRACE INEQUALITIES

In this thesis, a generalization of a discreteness criteria for a large class of subgroups of $\mathrm{PSL}_2(\mathbb{C})$ is proven. In particular, given a finitely generated, purely loxodromic, free Kleinian group $\Gamma = \langle \xi_1, \xi_2, \dots, \xi_n \rangle$ for $n \geq 2$, the inequality

$$|\text{trace}^2(\xi_i) - 4| + |\text{trace}(\xi_i \xi_j \xi_i^{-1} \xi_j^{-1}) - 2| \geq 2\sinh^2\left(\tfrac{1}{4}\log \alpha_n\right)$$

holds for some ξ_i and ξ_j for $i \neq j$ in Γ provided that certain conditions on the hyperbolic displacements given by ξ_i, ξ_j and their length 3 conjugates formed by the generators are satisfied. Above, the constant α_n turns out to be the real root strictly larger than $(2n-1)^2$ of a fourth degree, integer coefficient polynomial obtained by solving a family of optimization problems via Karush-Kuhn-Tucker theory. The use of this theory in the context of hyperbolic geometry is another novelty of this work.

ÖZET

ÖZGÜR KLENIAN GRUPLARINDA 1 UZUNLUĞUNDA İSOMETRİLER VE İZ EŞİTSİZLİKLERİ

Bu tezde, $\mathrm{PSL}_2(\mathbb{C})$'nin geniş bir altgrup sınıfı için geçerli olan ayrıklık kriterinin bir genellemesi isplatlanmıştır. Özellikle, $n \geq 2$ durumunda sonlu üreticili, saf loxodromik, özgür Kelinian grupu $\Gamma = \langle \xi_1, \xi_2, \ldots, \xi_n \rangle$ için ξ_i ve ξ_j ile 3 uzunluğunda eşleniklerinin kaydırım fonksiyonlarının belli şartları sağlaması durumunda aşağıdaki eşitsizlik doğrudur;

$$|\mathrm{trace}^2(\xi_i) - 4| + |\mathrm{trace}(\xi_i \xi_j \xi_i^{-1} \xi_j^{-1}) - 2| \geq 2 \sinh^2\left(\tfrac{1}{4} \log \alpha_n\right)$$

Yukarıda bahsi geçen sabit α_n'nin, Karush-Kuhn-Tucker teorisi kullanılarak çözülmüş bir optimizasyon sorusu ailesinden çıkan dördüncü dereceden, tam sayı katsayılı bir polinomun $(2n-1)^2$'den büyük tek kökü olduğu ortaya çıkar. Bu teorinin hiperbolik geometri bağlamında kullanımı, yazılan tezin bir diğer özgünlüğüdür.

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LIST OF SYMBOLS/ABBREVIATIONS

$\mathbb{R}$	The set of real numbers.
n	Number of generators.
$\mathrm{PSL}(2,\mathbb{C})$	Projective special linear group.
ξ_i	Elements of $\mathrm{PSL}(2,\mathbb{C})$.
Γ	Purely loxodromic subgroup of $\mathrm{PSL}(2,\mathbb{C})$, freely generated by ξ_i for $i = 1,\ldots,n$.
Ψ	$\{\xi_i^{2t},\ \xi_i^t\xi_j^{2s},\ \xi_i^t\xi_j^s\xi_k^p \mid i,j,k \in \{1,2,\ldots,n\},\ i \neq j,\ j \neq k,\ t,s,p \in \{-1,+1\}\}$
Γ_*	$\{1,\ \xi_i^t,\ \xi_i^t\xi_j^s\xi_i^{-t},\ \mid i,j \in \{1,2,\ldots,n\},\ i \neq j,\ t,s \in \{-1,+1\}\}$
Γ_1	$\{1\} \cup \Xi \cup \Xi^{-1}$
r	Group theoretical relation defined by the triple $\gamma_r \in \Gamma_*,\ \psi_r \in \Psi,\ \Psi_r \subseteq \Psi,$ satisfying $\gamma_r J_{\psi_r} = \Gamma - (\{\cdot\} \cup J_{\Psi_r})$.
$\mathbb{R}^\Psi$	The vector space of all functions from Ψ to $\mathbb{R}$.
$\mathbb{R}_{++}$	The set of strictly positive real numbers.
x	An element of $\mathbb{R}^\Psi$.
x_r	The ψ_r coordinate of the vector x.
X_r	The sum of all coordinates in Ψ_r of the vector x.
X	The sum of all the coordinates of the vector x.
f_r	The displacement function defined by the relation r.
$\mathcal{P}$	A special polynomial of degree 4 whose coefficients are dependent on n.
α	The unique root of $\mathcal{P}$ greater than $(2n-1)^2$.

1. INTRODUCTION

By the work of Thurston [1,2], Jorgensen and Gromov [3], it is known that volume is a geometric invariant of finite-volume hyperbolic 3-manifolds. Furthermore, by the Mostow-Prasad Rigidity Theorem (see [4,5]), it is also a topological invariant. The interrelation between the volume and the classical invariants of topology is therefore, one of the main topics of research in the study of finite-volume hyperbolic 3-manifolds.

One of the approaches practiced in the investigation of these interrelations is to search for discreteness criteria such as Jorgensen's Inequality [6] that can be used not only to calculate estimates for the volume but also for geometric quantities like the injectivity radius, the diameter, etc. Some seminal results in this direction can be found in [7]. A brief survey of such a quantitative analysis that also motivated this paper can be obtained in [8]. Jorgensen's Inequality [6] implies that if $\Gamma = \langle \xi_1, \xi_2 \rangle$ is a Kleinian group, then

$$|\mathrm{trace}^2(\xi_1) - 4| + |\mathrm{trace}(\xi_1\xi_2\xi_1^{-1}\xi_2^{-1}) - 2| \geq 1. \tag{1.1}$$

The lower bound given above is the best possible. It is conceivable that if one considers the class of purely loxodromic and free Kleinian groups (eg, all finitely generated Schottky groups are in this class [9]), a larger lower bound may be achieved. Consequently, the implications of this lower bound on the geometric and topological invariants of finite-volume hyperbolic 3-manifolds can be investigated, in particular, for finite-volume hyperbolic 3-manifolds whose fundamental groups are k-free. A Kleinian group Γ is called k-free if every finitely generated subgroup of Γ of rank at most k is free.

Let $d_\gamma z$ denote the hyperbolic distance between $z \in \mathbb{H}^3$ and $\gamma(z)$ for an isometry γ of $\mathbb{H}^3$. With the motivation above, the following statement was proven in [10] .

Theorem 1.1. *Let $\Gamma = \langle \xi_1, \xi_2 \rangle$ be a purely loxodromic free Kleinian group so that (i) $d_\gamma z_2 < 1.6068...$ for every $\gamma \in \{\xi_2, \xi_1^{-1}\xi_2\xi_1, \xi_1\xi_2\xi_1^{-1}\}$ and (ii) $d_{\xi_2\xi_1\xi_2^{-1}} z_2 \leq d_{\xi_2\xi_1\xi_2^{-1}} z_1$ for the midpoints z_1 and z_2 of the shortest geodesic segments connecting the axis of ξ_1 to the* *axes of* $\xi_2\xi_1\xi_2^{-1}$ *and* $\xi_2^{-1}\xi_1\xi_2$ *then,*

$$|\mathrm{trace}^2(\xi_1) - 4| + |\mathrm{trace}(\xi_1\xi_2\xi_1^{-1}\xi_2^{-1}) - 2| \geq 1.5937.... \tag{1.2}$$

Also, in the same paper [10], a generalization to an arbitrary number of generators has been conjectured. This conjecture, which has been proven as the main result of this thesis, can be stated as fallows:

Theorem 1.2. *Let* $\Gamma = \langle \xi_1, \xi_2 \ldots, \xi_n \rangle$ *be a purely loxodromic free Kleinian group. If there exist generators* ξ_i *and* ξ_j *for* $i \neq j$ *such that; for the midpoints* z_1 *and* z_2 *of the shortest geodesic segments connecting the axis of* ξ_i *to the axes of* $\xi_j\xi_i\xi_j^{-1}$ *and* $\xi_j^{-1}\xi_i\xi_j$*, respectively,*

(i) $d_\gamma z_2 < \frac{1}{2}\log\alpha_n$ *for every* γ *a generator or a length* 3 *conjugate formed by the generators except for* ξ_i*,* $\xi_j^{-1}\xi_i\xi_j$ *and* $\xi_j\xi_i\xi_j^{-1}$ *and,*

(ii) $d_{\xi_j\xi_i\xi_j^{-1}} z_2 \leq d_{\xi_j\xi_i\xi_j^{-1}} z_1$ *then,*

$$|\mathrm{trace}^2(\xi_i) - 4| + |\mathrm{trace}(\xi_i\xi_j\xi_i^{-1}\xi_j^{-1}) - 2| \geq 2\sinh^2\left(\tfrac{1}{4}\log\alpha_n\right),$$

where α_n *is the only root greater than* $(2n-1)^2$ *of the polynomial*

$$\begin{aligned}\mathcal{P}(\lambda) = {} & (8n^3 - 12n^2 + 2n + 1)\,\lambda^4 + \\ & (-64n^6 + 192n^5 - 192n^4 + 64n^3 + 4n^2 + 2n - 4)\,\lambda^3 + \\ & (-96n^5 + 224n^4 - 168n^3 + 52n^2 - 18n + 6)\,\lambda^2 + \\ & (32n^5 - 112n^4 + 128n^3 - 68n^2 + 22n - 4)\,\lambda + \\ & 16n^4 - 32n^3 + 24n^2 - 8n + 1.\end{aligned} \tag{1.3}$$

The general approach to obtain the proof of the above theorem is very reminiscent to the techniques used in [10] with one major difference, the generalization to an arbitrary number of generators makes the necessary calculations much more complicated. A direct analogue of the previous paper for even $n = 3$ would have been technically very cumbersome. Therefore, it was clear that a new notational system for the decompositions, group theoretical relations and even the displacement functions was necessary. Such a system was created and is presented in Section 2 together with the proof of necessary properties of the polynomial $\mathcal{P}$.

The plan of attack employed to tackle the increase in the number of generators, aside from the new notational system, was to use the inherent symmetries of the provided decomposition from the start rather than towards the end as opposed to the way it was done in [10]. This simplified the entire process as the symmetries did not get encrypted into the displacement functions. The use of these symmetries is achieved via classifications of the decomposition elements and group theoretical relations as they are discussed and presented with tables in Section 4.

In Section 3, we summarize the relevant part of Culler-Shalen machinery to underline the importance and applications of these group theoretical relations. Section 5 is largely a simplified and generalized version of the efforts in [10] as we reap the fruits of our efforts in classification. Almost all the ideas that are used in this chapter are from [10] as they were presented in the previous paper but they are translated to the language of our classification. The only exception to this is the use of Karush-Kuhn-Tucker theorem which simplified the proof even further than that of the $n = 2$ case. Finally, in Section 6, the main results Theorem 6.3 and Theorem 6.4 together with some auxiliary lemmas are proven.

2. BASIC DEFINITIONS AND NOTATIONS

Throughout this text, we will work with a fixed subset $\Xi = \{\xi_1,\ \xi_2,\ \ldots,\ \xi_n\}$ of $PSL(2,\mathbb{C})$ such that $\Gamma = \langle\Xi\rangle$ is a purely loxodromic group freely generated by Ξ. Of particular interest to us will be the special subsets Ψ, Γ_* and Γ_1 of Γ defined below:

$$\Psi = \{\xi_i^{2t},\ \xi_i^t\xi_j^{2s},\ \xi_i^t\xi_j^s\xi_k^p \mid i,j,k \in \{1,2,\ldots,n\},\ i \neq j,\ j \neq k,\ t,s,p \in \{-1,+1\}\}, \quad (2.1)$$

$$\Gamma_* = \{1,\ \xi_i^t,\ \xi_i^t\xi_j^s\xi_i^{-t},\ \mid i,j \in \{1,2,\ldots,n\},\ i \neq j,\ t,s \in \{-1,+1\}\}, \quad (2.2)$$

$$\Gamma_1 = \{1\} \cup \Xi \cup \Xi^{-1}. \quad (2.3)$$

In general, we will consider i,j,k,t,s,p to be "free" and use sub-indexes (such as i_0, i_1) to represent fixed elements. We will also refrain from stating obvious equalities/inequalities (such as $i \neq j$ and $t = \pm 1$). This notational compromise will have the following relaxations

$$\Psi = \{\xi_i^{2t},\ \xi_i^t\xi_j^{2s},\ \xi_i^t\xi_j^s\xi_k^p\}, \quad \text{and} \quad \Gamma_* = \{1,\ \xi_i^t,\ \xi_i^t\xi_j^s\xi_i^{-t}\} \quad (2.4)$$

of the definitions above. For $\psi \in \Psi$, let J_ψ represents the elements of Γ that start (in their simplified form) with ψ. Notice that $(J_\psi)_{\psi\in\Psi}$ gives a collection of disjoint subsets of Γ. Moreover, we have the following partition of Γ:

$$\Gamma = \{1\} \sqcup \Xi \sqcup \Xi^{-1} \sqcup \bigsqcup_{\psi\in\Psi} J_\psi. \quad (2.5)$$

The setting (Ψ, Γ_*) together with the partition above is said to be a decomposition of Γ. A group theoretical relation r of this decomposition is defined to be a triple $(\gamma_r,\ \psi_r,\ \Psi_r)$ satisfying the following:

$$r\ :\ \gamma_r \in \Gamma_*,\ \psi_r \in \Psi,\ \Psi_r \subseteq \Psi, \quad \text{and} \quad \gamma_r J_{\psi_r} = \Gamma - (\{\cdot\} \cup J_{\Psi_r}). \quad (2.6)$$

To better understand this definition, consider r with $\gamma_r = \xi_1^{-1}$ and $\psi_r = \xi_1\xi_2\xi_3$. It is easy to see that $\gamma_r J_{\psi_r}$ will consist of all the elements of Γ starting with $\xi_2\xi_3$. The term $(\{\cdot\} \cup J_{\Psi_r})$ in the relation equation should be all elements **not** starting with $\xi_2\xi_3$. If we take $\Psi_r = \Psi - \{\xi_2\xi_3,\ \xi_2\xi_3\xi_i^t\}$, then $J_{\Psi_r} = \bigcup_{\psi\in\Psi_r} J_\psi$ will exactly be what needs to be removed from Γ.

Hence, the followings

$$r:\gamma_r=\xi_1^{-1},\ \psi_r=\xi_1\xi_2\xi_3,\ \Psi_r=\Psi-\{\xi_2\xi_3^2,\ \xi_2\xi_3\xi_i^t\},\ \{\cdot\}=\Gamma_1 \tag{2.7}$$

define a group theoretical relation. Including the relation above, all relations are quite symmetrical and we can generalize them into families of relations as follows

$$r:\gamma_r=\xi_{i_0}^{-t_0},\ \psi_r=\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{k_0}^{p_0},\ \Psi_r=\Psi-\{\xi_{j_0}^{s_0}\xi_{k_0}^{2p_0},\ \xi_{j_0}^{s_0}\xi_{k_0}^{p_0}\xi_i^t\},\ \{\cdot\}=\Gamma_1. \tag{2.8}$$

Next, we shall simplify the notation for displacement functions. Let $x\in\mathbb{R}^\Psi$ and r a relation. We define

$$x_r=x(\psi_r),\ X_r=\sum_{\psi\in\Psi_r}x(\psi)\text{ and }X=\sum_{\psi\in\Psi}x(\psi). \tag{2.9}$$

Although this notation has been described above only for x, we shall use capital letters almost exclusively for this purpose. Accordingly, the formula of the displacement function $f_r:\triangle\longrightarrow\mathbb{R}_{++}$ of the relation r can be given as:

$$x\longmapsto\frac{1-x_r}{x_r}\cdot\frac{1-X_r}{X_r}, \tag{2.10}$$

where $\triangle$ will denote the simplex $\{x\in\mathbb{R}^\Psi_{++}\mid X=1\}$. To define the α used in Section 1, we use the polynomial $\mathcal{P}$ in (1.3) and introduce the lemma below:

Lemma 2.1. *The polynomial $\mathcal{P}$ has a unique root larger than $(2n-1)^2$ and all other roots are smaller than* 1.

Proof. All four roots of $\mathcal{P}$ are real and distinct for $n\geq 2$. Therefore, it suffices to find certain intervals such that the evaluation on either side have opposite signs.

λ	$\mathcal{P}(\lambda)$	sign of $\mathcal{P}$
-2	$(2n-3)(256n^5-608n^4+424n^3-36n^2+18n-17)$	positive
$-1/n$	$-(16n^8-48n^7+72n^6-84n^5+33n^4+10n^3+8n^2-6n-1)/n^4$	negative
$-1/(2n-1)$	$32n^4(n-1)/(2n-1)^2$	positive
1	$-64n^4(n-1)^2$	negative
$(2n-1)^2$	$-128n^4(2n^2-3n+1)^3(4n^2-8n+5)$	negative
$(2n-1)^3$	$16(2n-1)^4(n-1)^2(32n^7-80n^6+56n^5+4n^4-22n^3+16n^2-5n+1)$	positive

Table 2.1. Sign of $\mathcal{P}$ at various points

The first 4 rows of the table above proves that $\mathcal{P}$ has three roots between -2 and 1. The last two rows prove that $\mathcal{P}$ has a root larger than $(2n-1)^2$. Hence, we have located all four roots. $\square$

By Lemma 2.1, we can formally give the definition of α : the unique root of $\mathcal{P}$ larger than $(2n-1)^2$.

3. CULLER-SHALEN MACHINERY

We will use S_∞ to denote the boundary of the canonical compactification of $\mathbb{H}^3$. The decomposition of Γ and its relations defined in the previous section allow us to decompose the area measure of S_∞ also and, connect the relations with these measures. This has been done in an almost identical way in [13, Lemma 5.3], [16, Lemma 3.3, Theorem 3.4] and [12, Theorem 2.1]. Therefore, we present it without any proofs.

Theorem 3.1. *Let $z \in \mathbb{H}^3$ and A_z be the area measure of S_∞ at z. If Γ is geometrically infinite, then there exists a family of Borel measures $\{\nu_\psi\}_{\psi\in\Psi}$ on S_∞ such that:*

i. $A_z = \sum_{\psi\in\Psi} \nu_\psi$

ii. $A_z(S_\infty) = 1$

iii. $\int_{S_\infty} \lambda^2_{\gamma_r,z} d\nu_{\psi_r} = 1 - \sum\limits_{\psi\in\Psi_r} \nu_\psi(S_\infty)$*, for any relation r.*

With this theorem, we can define a special element

$$m(\psi) = v_\psi(S_\infty) \in \mathbb{R}^\Psi \tag{3.1}$$

for all $\psi \in \Psi$ for any given $z \in \mathbb{H}^3$. We take $\{\nu_\psi\}_{\psi\in\Psi}$ as in Theorem 3.1. We also use the notations of [11] for the hyperbolic displacement of z. If γ is an isometry of $\mathbb{H}^3$ with respect to the hyperbolic metric ρ, then

$$d_\gamma z = \rho(z, \gamma z). \tag{3.2}$$

The crucial part of Culler-Shalen machinery for this paper is that it gives lower bounds for the displacements given by the hyperbolic isometries. This is due to [13, Lemma 5.5] and its improved version [14, Lemma 2.1]. We present the latter:

Lemma 3.2. *Let a and b be numbers in $[0,1]$ which are not both equal to 0 and are not both equal to 1. Let γ be a loxodromic isometry of $\mathbb{H}^3$ and let $z \in \mathbb{H}^3$. Suppose that ν is a measure on S_∞ such that*

$$v \leq A_z,\ v(S_\infty) \leq a \text{ and } \int_{S_\infty} \lambda^2_{\gamma,z} d\nu \geq b. \tag{3.3}$$

Then, we have

$$a > 0,\ b < 1 \textit{ and } d_\gamma z \geq \frac{1}{2}\log\frac{b(1-a)}{a(1-b)}. \tag{3.4}$$

Finally, we can merge the results of the preceding theorems to obtain the lower bounds for the hyperbolic displacements that we will need in Section 6.

Proposition 3.3. *Let $z \in \mathbb{H}^3$ and m be the relevant vector in $\mathbb{R}^\Psi$. If Γ is geometrically infinite, then for any $\gamma \in \Gamma_*$ and for any relation r defined by Ψ we have:*

i. $m \in \triangle$,

ii. $e^{2d_\gamma z} \geq f_r(m)$ *for any relation r with $\gamma_r = \gamma$.*

Proof. First, we shall prove that $m(\psi) \neq 0$ for all $\psi \in \Psi$. We do so by starting with the conjugate elements of Ψ and then generalizing to the rest of the elements.

Assume $m(\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}) = 0$. Let r_0 be the relation defined by:

$$r_0 : \gamma_0 = \xi_{i_0}^{t_0}\xi_{j_0}^{-s_0}\xi_{i_0}^{-t_0},\ \psi_0 = \xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0},\ \Psi_0 = \{\xi_{i_0}^{2t_0},\ \xi_{i_0}^{t_0}\xi_j^{2s},\ \xi_{i_0}^{t_0}\xi_j^{s}\xi_k^{p}\}. \tag{3.5}$$

Since $m(\psi_0) = 0$, we have $M_0 = \sum_{\psi\in\Psi_0} m(\psi) = 1$ by Theorem 3.1(*iii*). This implies that there exists a $\psi_1 \in \Psi_0$ such that $m(\psi_1) \neq 0$. Take $\psi_2 = \xi_{i_2}^{t_2}\xi_{j_2}^{s_2}\xi_{i_2}^{-t_2}$, where $i_2 \neq i_0$ so that $\psi_2 \notin \Psi_0$. Notice that $M = \sum_{\psi\in\Psi} m(\psi) = 1$ by Theorem 3.1 (*i*) and (*ii*). Now, as both M and M_0 are equal to 1, we have $\sum_{\psi\in\Psi-\Psi_0} m(\psi) = 0$. This implies that $m(\psi_2) = 0$. Using the same trick again, let r_2 be the relation defined by:

$$r_2 : \gamma_2 = \psi_2^{-1},\ \psi_2 = \psi_2,\ \Psi_2 = \{\xi_{i_2}^{2t_2},\ \xi_{i_2}^{t_2}\xi_j^{2s},\ \xi_{i_2}^{t_2}\xi_j^{s}\xi_k^{p}\} \tag{3.6}$$

so that $M_2 = 1$ and $M - M_2 = 0$. This gives that $m(\psi) = 0$ for all $\psi \in \Psi - \Psi_2$, which contradicts with the existence of ψ_1 as $\Psi_0 \subset \Psi - \Psi_2$. Hence, we have the following fact

$$m(\xi_i^t\xi_j^s\xi_i^{-t}) \neq 0. \tag{3.7}$$

Next, assume that $m(\xi_{i_0}^{2t_0}) = 0$. Take r_0 to be the relation:

$$r_0 : \gamma_0 = \xi_{i_0}^{-t_0},\ \psi_0 = \xi_{i_0}^{2t_0},\ \Psi_0 = \Psi - \{\xi_{i_0}^{2t_0},\ \xi_{i_0}^{t_0}\xi_j^{2s},\ \xi_{i_0}^{t_0}\xi_j^{s}\xi_k^{p}\}. \tag{3.8}$$

By Theorem 3.1, we obtain $m(\psi) = 0$ for all $\psi \in \Psi - \Psi_0$. Thus, $m(\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}) = 0$, contradicting (3.7). This proves that

$$m(\xi_i^{2t}) \neq 0. \tag{3.9}$$

Using identical arguments to the previous arguments, we will get the contradictions to (3.7) listed in Table 3.1. Therefore, we derive that

$$m(\xi_i^t\xi_j^{2s}) \neq 0,\ m(\xi_i^t\xi_j^s\xi_i^t) \neq 0,\ \text{and}\ m(\xi_i^t\xi_j^s\xi_k^p) \neq 0. \tag{3.10}$$

We have cycled over all element types in Ψ showing $m \in \mathbb{R}_{++}^{\Psi}$. Since we already know that $M = 1$, we have in fact proven (i) of the current proposition.

To prove (ii) fix any relation r with $\gamma_r = \gamma$. Take $a = \nu_{\psi_r}(S_\infty) = m(\psi_r)$ and $b = \int_{S_\infty} \lambda_{\gamma,z}^2 d\nu_{\psi_r}$. Notice that a and b satisfy the conditions of Lemma 3.2. Thus,

$$d_\gamma z \geq \frac{1}{2}\log\left(\frac{1-a}{a}\cdot\frac{b}{1-b}\right). \tag{3.11}$$

Moreover, if $a = m_r$ and $b = 1 - \sum_{\psi\in\Psi_r} m(\psi) = 1 - M_r$ (by Theorem 3.1(iii)), then the inequality above turns into

$$d_\gamma z \geq \frac{1}{2}\log\left(\frac{1-m_r}{m_r}\cdot\frac{1-M_r}{M_r}\right) = \frac{1}{2}\log f_r(m). \tag{3.12}$$

Taking the exponential yields (ii). □

The proposition above shows that we can use the displacement functions of the relations and m to find lower bounds for hyperbolic displacements. We could sharpen this bound by searching for the maximal value of $f_r(m)$ over a family of relations. Obviously, the larger the family, the better the lower bound for max$\{f_r(m) \mid r\}$ will be.

Unfortunately, calculating values for m for every $z \in \mathbb{H}^3$ is not feasible. Therefore, we look at all possible values of m within the bounds set by Proposition 3.3(i). As we don't know which element of $\triangle$ is actually m, we have to look at the infimum over all elements. Ultimately, we shall prove in Section 6 that what matters is the infimum of the max function of a collection of displacement functions over $\triangle$. This optimization is performed in Section 5.

We have juggled the elements of Ψ and in fact a large collection of relations in the proof of Propostion 3.3. This type of juggling will only get more complicated as we try to solve the infimum problem over $\triangle$. In this regard, we hit two birds with one stone in the next chapter. Not only do we define the collection of relations that we need, but we also classify them in such a way that almost all arguments in Section 5 collapse into the language of relation types.

$m(\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0}) = 0$ $r_0 :\ \gamma_0 = \xi_{i_0}^{t_0}\xi_{j_0}^{-s_0}\xi_{i_0}^{-t_0},\ \psi_0 = \xi_{i_0}^{t_0}\xi_{j_0}^{2s_0},\ \Psi_0 = \Psi - \{\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0},\ \xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_k^p\}$	implies	$m(\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}) = 0$
$m(\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}) = 0$ $r_0 :\ \gamma_0 = \xi_{i_0}^{-t_0},\ \psi_0 = \xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{l_0}^{p_0},\ \Psi_0 = \Psi - \{\xi_{j_0}^{s_0}\xi_{i_0}^{2t_0},\ \xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_k^p\}$	implies	$m(\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_{j_0}^{-s_0}) = 0$
$m(\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{k_0}^{p_0}) = 0$ $r_0 :\ \gamma_0 = \xi_{i_0}^{-t_0},\ \psi_0 = \xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{k_0}^{p_0},\ \Psi_0 = \Psi - \{\xi_{j_0}^{s_0}\xi_{k_0}^{2p_0},\ \xi_{j_0}^{s_0}\xi_{k_0}^{p_0}\xi_k^p\}$	implies	$m(\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}) = 0$

Table 3.1. List of contradictions

4. THE MAX FUNCTIONS AND CLASSIFICATION OF RELATIONS

In this section, we mainly present some tables illuminating the mechanics of the displacement functions for the relevant decompositions. We also discuss how these tables are used in later sections and the inspirations behind their discovery. Table 4.1 below is used to count and classify elements of Ψ into 5 types.

	ψ	count
type 1	ξ_i^{2t}	$2n$
type 2	$\xi_i^t \xi_j^{2s}$	$4n(n-1)$
type 3	$\xi_i^t \xi_j^s \xi_i^t$	$4n(n-1)$
type 4	$\xi_i^t \xi_j^s \xi_i^{-t}$	$4n(n-1)$
type 5	$\xi_i^t \xi_j^s \xi_k^p$	$8n(n-1)(n-2)$
$\lvert\Psi\rvert$	$\lvert\{i,\ j,\ k\}\rvert = 3$	$2n + 4n(n-1) + 8n(n-1)^2$

Table 4.1. Types of elements in Ψ

While reading this table, remember the notational relaxations that we have discussed in Section 2 and also assume $|\{i,\ j,\ k\}| = 3$ at any instance. The above classification (with the counts) will be useful in simplifying the calculations for the optimization question that was discussed in Section 1. However, what we really need is a set of coherent relations that will yield many symmetries of the simplex $\triangle$.

Notice that for the purposes of finding a lower bound for $f(m)$ as discussed in Section 3, one can work with as small or as large a set of relations as they desire. The catch here is that if the set is small, the resulting lower bound runs the risk of being too small as well and not be useful in any shape for future applications. On the opposite side, if the set of relations being considered is too large (or badly curated) the resulting optimization question will become too complicated to solve.The trick that was used here, is to work with and optimize a midsize collection of displacement functions but to choose that collection in such a way that the resulting lower bound is actually identical to that of a much larger collection.

Table 4.2 lists, classifies and counts all relations in our large collection $\mathcal{G}$. In fact, $\mathcal{G}$ is the largest possible collection. This table has been prepared in a ψ_r centric way, in the sense that to generate a relation, first ψ_r is fixed and then a suitable γ_r is chosen. The calculation of Ψ_r and $\{\cdot\}$ at every type is very similar to the example done in Section 2. Now, with this collection at hand we can define the function $G : \triangle \longrightarrow \mathbb{R}_{++}$ as follows

$$G(x) = \max_{r \in \mathcal{G}} f_r(x). \tag{4.1}$$

It is the infimum of this function that we will ultimately obtain in Section 5. The midsize collection $\mathcal{F}$ is the subset of $\mathcal{G}$ consisting of Type 1a, 2b, 3a, 4b and 5a relations. The relevant function $F : \triangle \longrightarrow \mathbb{R}_{++}$ of this collection is defined as follows

$$F(x) = \max_{r \in \mathcal{F}} f_r(x). \tag{4.2}$$

In Section 5, we will mostly be optimizing F.

The choice of $\mathcal{F}$ is simple in some aspects. For example, for any type 1b relation there exists a type 1a relation such that the type 1a relation's displacement function is larger at every point of the simplex. Implying that the displacement functions for all the type 1b relations are unnecessary inside the max function. Alas, most of the other "cancellations" aren't this straightforward and ultimately unnecessary as once the unique optimal solution for the infimum of F is calculated, it is straight forward to show that it must also be the unique optimal point for the infimum of G. The collection $\mathcal{F}$ has been chosen largely thanks to the patterns observed in the $n = 2$ case by Yüce in [10] and in small part to the computer models observed by Narman for the larger values of n.

Table 4.3 is listing the elements of $\mathcal{F}$ for ease of access. In Table 4.3, we have also omitted the set $\{\cdot\}$ and γ_r columns (as they are irrelevant to the calculation of displacements) and expanded the description of Ψ_r as to distinguish (and count) type 3, 4 and 5 elements of Ψ_r better (see Table 4.1). You might also notice that $|\mathcal{F}| = |\Psi|$. This is not a coincidence and discussed further in Section 5.

In this section, we shall also calculate the gradients of the displacement functions defined in Section 2. They will be needed in Section 5. It is a good exercise for a better understanding of the definition of displacement functions. Recall that for a relation r, the definition of the displacement function of r is $f_r(x) = (1 - x_r)(1 - X_r)/(x_r X_r)$, where $x_r = x(\psi_r)$ and $X_r = \sum_{\psi \in \Psi_r} x(\psi)$. Thus, the partial derivative of f_r with respect to a coordinate $\psi \in \Psi$ depends on whether the coordinate is in Ψ_r and if it is equal to ψ_r or not. To further complicate the issue, we have $\psi_r \in \Psi_r$ for some displacements and $\psi_r \notin \Psi_r$ for some displacements. Therefore, we present the gradient of f_r for a given relation r in two cases. For relations of type 1a, 2b, we have

$$\begin{gathered}\psi_r \notin \Psi_r \text{ then } \frac{df_r}{d\psi} = \begin{cases} \frac{1-x_r}{x_r} \cdot \frac{-1}{X_r^2} & \text{if } \psi \in \Psi_r \\ \frac{-1}{x_r^2} \cdot \frac{1-X_r}{X_r} & \text{if } \psi = \psi_r \\ 0 & \text{if } \psi \notin \Psi_r \cup \{\psi_r\} \end{cases} \\ \text{and } \nabla f_r(x) = -\frac{1-x_r}{x_r X_r^2} \cdot \sum_{\psi \in \Psi_r} e_\psi - \frac{1-X_r}{x_r^2 X_r} \cdot e_{\psi_r}.\end{gathered} \tag{4.3}$$

For relations of type 3a, 4b, 5a, we have

$$\begin{gathered}\psi_r \in \Psi_r \text{ then } \frac{df_r}{d\psi} = \begin{cases} \frac{1-x_r}{x_r} \cdot \frac{-1}{X_r^2} & \text{if } \psi \in \Psi_r - \{\psi_r\} \\ \frac{-1}{x_r^2} \cdot \frac{1-X_r}{X_r} + \frac{1-x_r}{x_r} \cdot \frac{-1}{X_r^2} & \text{if } \psi = \psi_r \\ 0 & \text{if } \psi \notin \Psi_r \cup \{\psi_r\} \end{cases} \\ \text{and } \nabla f_r(x) = -\frac{1-x_r}{x_r X_r^2} \cdot \sum_{\psi \in \Psi_r} e_\psi - \left(\frac{1-X_r}{x_r^2 X_r} + \frac{1-x_r}{x_r X_r^2} \right) \cdot e_{\psi_r},\end{gathered} \tag{4.4}$$

where $(e_\psi)_\Psi$ is the canonical basis for $\mathbb{R}^\Psi$.

	γ_r	ψ_r	Ψ_r	$\{\cdot\}$	count
type 1a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{2t_0}$	$\Psi - \{\xi_{i_0}^{2t_0},\ \xi_{i_0}^{t_0}\xi_j^{2s},\ \xi_{i_0}^{t_0}\xi_j^{s}\xi_k^{p}\}$	$\Gamma_1 - \{\xi_{i_0}^{t_0}\}$	$2n$
type 1b	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{2t_0}$	$\Psi - \{\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\}$	Γ_1	$4n(n-1)$
type 2a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0}$	$\Psi - \{\xi_{j_0}^{2s_0}\}$	Γ_1	$4n(n-1)$
type 2b	$\xi_{i_0}^{t_0}\xi_{j_0}^{-s_0}\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0}$	$\Psi - \{\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0},\ \xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_k^{p}\}$	Γ_1	$4n(n-1)$
type 3a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}$	$\Psi - \{\xi_{j_0}^{s_0}\xi_{i_0}^{2t_0},\ \xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_k^{p}\}$	Γ_1	$4n(n-1)$
type 3b	$\xi_{i_0}^{t_0}\xi_{j_0}^{-s_0}\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}$	$\Psi - \{\xi_{i_0}^{2t_0}\}$	Γ_1	$4n(n-1)$
type 4a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}$	$\Psi - \{\xi_{j_0}^{s_0}\xi_{i_0}^{-2t_0},\ \xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}\xi_k^{p}\}$	Γ_1	$4n(n-1)$
type 4b	$\xi_{i_0}^{t_0}\xi_{j_0}^{-s_0}\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}$	$\{\xi_{i_0}^{2t_0},\ \xi_{i_0}^{t_0}\xi_j^{2s},\ \xi_{i_0}^{t_0}\xi_j^{s}\xi_k^{p}\}$	$\{\xi_{i_0}^{t_0}\}$	$4n(n-1)$
type 5a	$\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{k_0}^{p_0}$	$\Psi - \{\xi_{j_0}^{s_0}\xi_{k_0}^{2p_0},\ \xi_{j_0}^{s_0}\xi_{k_0}^{p_0}\xi_k^{p}\}$	Γ_1	$8n(n-1)(n-2)$
type 5b	$\xi_{i_0}^{t_0}\xi_{j_0}^{-s_0}\xi_{i_0}^{-t_0}$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{k_0}^{p_0}$	$\Psi - \{\xi_{i_0}^{t_0}\xi_{k_0}^{2p_0},\ \xi_{i_0}^{t_0}\xi_{k_0}^{p_0}\xi_k^{p}\}$	Γ_1	$8n(n-1)(n-2)$
$\lvert\mathcal{G}\rvert$					$2(2n+4n(n-1)+8n(n-1)^2)+2n(2n-3)$

Table 4.2. Relations in the collection $\mathcal{G}$

	ψ_r			Ψ_r			count
1a	$\xi_{i_0}^{2t_0}$	$\Psi - \{\xi_{i_0}^{2t_0},$	$\xi_{i_0}^{t_0}\xi_j^{2s},$	$\xi_{i_0}^{t_0}\xi_j^{s}\xi_{i_0}^{t_0},$	$\xi_{i_0}^{t_0}\xi_j^{s}\xi_{i_0}^{-t_0},$	$\xi_{i_0}^{t_0}\xi_j^{s}\xi_k^{p}\ \}$	$2n$
		1	$2(n-1)$	$2(n-1)$	$2(n-1)$	$4(n-1)(n-2)$	
2b	$\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0}$	$\Psi - \{$	$\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0},$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{t_0},$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0},$	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_k^{p}\ \}$	$4n(n-1)$
			1	1	1	$2(n-2)$	
3a	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}$	$\Psi - \{$	$\xi_{j_0}^{s_0}\xi_{i_0}^{2t_0},$	$\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_{j_0}^{s_0},$	$\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_{j_0}^{-s_0},$	$\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_k^{p}\ \}$ in	$4n(n-1)$
			1	1	1	$2(n-2)$	
4b	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}$	$\{\xi_{i_0}^{2t_0},$	$\xi_{i_0}^{t_0}\xi_j^{2s},$	$\xi_{i_0}^{t_0}\xi_j^{s}\xi_{i_0}^{t_0},$	$\xi_{i_0}^{t_0}\xi_j^{s}\xi_{i_0}^{-t_0},$	$\xi_{i_0}^{t_0}\xi_j^{s}\xi_k^{p}\ \}$	$4n(n-1)$
		1	$2(n-1)$	$2(n-1)$	$2(n-1)$	$4(n-1)(n-2)$	
5a	$\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{k_0}^{p_0}$	$\Psi - \{$	$\xi_{j_0}^{s_0}\xi_{k_0}^{2p_0}$	$\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_{j_0}^{s_0},$	$\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_{j_0}^{-s_0},$	$\xi_{j_0}^{s_0}\xi_{i_0}^{t_0}\xi_k^{p}\ \}$	$\frac{8n!}{(n-3)!}$
			1	1	1	$2(n-2)$	
$\lvert\mathcal{F}\rvert$							$2n+4n(n-1)+8n(n-1)^2$

Table 4.3. Relations in the collection $\mathcal{F}$

5. OPTIMIZING THE INFIMUM OF THE MAX FUNCTION

The purpose of this section is to optimize $\inf_{x\in\triangle}(G(x))$, but will mostly deal with $\inf_{x\in\triangle}(F(x))$. Our plan can be summarized in three phases. The first phase is to show the attainment of optimality (for both G and F), the second phase is to prove the uniqueness of optimality and the final phase is to calculate the point of optimality using the Karush-Kuhn-Tucker theorem. We would like to point out that the problem with the infimum is not that of existence, as the image is a bounded below subset of $\mathbb{R}$ that part is obvious, but attainment in $\triangle$. In fact, from here on we shall denote the infimums with α^* and β^*, ie,

$$\alpha^* = \inf_{x\in\triangle}(G(x)), \text{ and } \beta^* = \inf_{x\in\triangle}(F(x)). \tag{5.1}$$

Lemma 5.1. *There exists $x^* \in \triangle$ such that $G(x^*) = \alpha^*$.*

Proof. As α^* is the infimum of the set $G(\triangle)$, there must be a sequence $(x^n)_n$ in $\triangle$ such that $G(x^n) \to \alpha^*$. Moreover, $(x^n)_n$ must have a convergent subsequence since the closure of $\triangle$ is a bounded subset of $\mathbb{R}^\Psi$.

Without loss of generality, assume $x^n \to \bar{x}$ for some $\bar{x} \in cl(\triangle)$. Notice that $cl(\triangle) = \{x \in \mathbb{R}_+^\Psi \mid X = 1\}$. The fact that the max function of continuous functions is also continuous is a simple exercise ([15] Exercise 2.3.1). Our aim is to prove $\bar{x} \notin cl(\triangle) - \triangle$. We shall do so by assuming the opposite and obtaining a contradiction.

Let $K = \{\psi \in \Psi \mid \bar{x}_\psi = 0\}$. The assumption that $\bar{x} \in cl(\triangle) - \triangle$ implies $K \neq \emptyset$. Now, if there exists a relation r such that $\psi_r \in K$ and $\overline{X}_r \in [0,1)$, then $x_r^n \to \bar{x}_r = 0$ and $\frac{1-X_r^n}{X_r^n}$ either converges or diverges to ∞, which shows that $f_r(x^n) = \frac{1-x_r^n}{x_r^n} \cdot \frac{1-X_r^n}{X_r^n} \to \infty$. However, $f_r(x) \leq G(x)$ for any x. Thus, we also have $\lim f_r(x^n) \leq \lim G(x^n) = \alpha^*$, contradicting the existence of α^*.

The above paragraph shows that for all $r \in \mathcal{G}$ with $\psi_r \in K$, we have $\overline{X}_r = 1$. However, we know that $\overline{X}_r + \sum_{\psi\notin\Psi_r} \bar{x}(\psi) = \overline{X} = 1$ proving $\bar{x}(\psi) = 0$ for all $\psi \in \Psi - \Psi_r$, in any such relation. In summary, what we have shown is for a relation $r \in \mathcal{G}$,

$$\text{If } \psi_r \in K \text{ then } \Psi - \Psi_r \subseteq K. \tag{5.2}$$

The contradiction that we are seeking will soon come in the form of $K = \Psi$ as that would imply $\overline{X} = 0$, contradicting the fact that $\bar{x} \in cl(\triangle)$. Fortunately, the result we have obtained in (5.2) will help immensely and turn this endeavor into a cross puzzle. We already know $K \neq \emptyset$ by our assumption. We shall start with a random element of K and cycling over all possible types in Table 4.1, use (5.2) together with Table 4.2 to show that $K = \Psi$.

Now, if $\xi_{i_0}^{2t_0} \in K$ the relevant type 1a relation r_0 will satisfy $\psi_{r_0} \in K$, and by (5.2), we get $\Psi - \Psi_{r_0} \subseteq K$, implying $\{\xi_{i_0}^{t_0}\xi_j^s\xi_{i_0}^{-t_0}\} \subseteq K$. Repeating the same process for the type 4b relation r_j with $\psi_{r_j} = \xi_{i_0}^{t_0}\xi_j^s\xi_{i_0}^{-t_0}$ we get $\{\xi_i^{2t}\} - \{\xi_{i_0}^{2t_0}\} \subseteq K$. Since we already know that $\xi_{i_0}^{2t_0} \in K$, we actually have $\{\xi_i^{2t}\} \subseteq K$. The arguments so far in this paragraph have shown that for any type 1a relation r, $\psi_r \in K$. Merging all this information together, we get $\Psi = \bigcup_{i=1}^{n}(\Psi - \Psi_{r_i}) \subseteq K$, proving what we wanted. We summarize this result below:

$$\text{If there exists a type 1 element in } K \text{ then } \Psi = K. \tag{5.3}$$

If $\xi_{i_0}^{t_0}\xi_{j_0}^{2s_0} \in K$, the relevant type 2b relation r_0 will satisfy $\psi_{r_0} \in K$ and so by (5.2) we get $\Psi - \Psi_{r_0} \subseteq K$, implying $\{\xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}\} \subseteq K$. Repeating the same process for the type 4b relation r_1 with $\psi_{r_1} = \xi_{i_0}^{t_0}\xi_{j_0}^{s_0}\xi_{i_0}^{-t_0}$, we get $\{\xi_i^{2t}\} - \{\xi_{i_0}^{2t_0}\} \subseteq K$. This shows that K has a type 1 element and $K = \Psi$ by (5.3). We summarize this result below:

$$\text{If there exists a type 2 element in } K \text{ then } \Psi = K. \tag{5.4}$$

If K has an element of the other types, we have the same conclusion $\Psi = K$. More specifically, we see that

- K has a type 3 element and the relevant type 3a relation together with (5.4) implies $K = \Psi$,
- K has a type 4 element and the relevant type 4b relation together with (5.3) implies

$K = \Psi$,

- K has a type 5 element and the relevant type 5a relation together with (5.4) implies $K = \Psi$.

This gives that as long as K has an element, no matter what type, $K = \Psi$. □

Notice that in the proof above, we have only used the relations in $\mathcal{F}$. Therefore, the following lemma has an identical proof.

Lemma 5.2. *There exists $x^* \in \triangle$ such that $F(x^*) = \beta^*$.*

The next lemma gives us a rough location for β^* and α^* which proves very useful for the proofs in the second phase.

Lemma 5.3. *The infimums of $F(x)$ and $G(x)$ on $\triangle$, β^* and α^* are greater than 1 and less than or equal to α. Moreover, $\beta^* \leq \alpha^*$.*

Proof. First of all, we know that $1 < \alpha$ by Lemma 2.1. Therefore, at the very least the lemma makes sense. Also, since $\mathcal{F} \subset \mathcal{G}$, we have $F(x) \leq G(x)$ for any $x \in \triangle$. This proves $\beta^* \leq \alpha^*$.

To show that $1 < \beta^*$, let r be any type 1a relation. As $\psi_r \notin \Psi_r$ we have $1 = X > x_r + X_r$ for all $x \in \triangle$. Dividing both sides of this inequality with $x_r X_r$, we obtain:

$$\frac{1}{x_r X_r} > \frac{1}{x_r} + \frac{1}{X_r} \text{ which gives that } \frac{1}{x_r X_r} - \frac{1}{x_r} - \frac{1}{X_r} > 0. \tag{5.5}$$

Moreover, we have

$$f_r(x) = \frac{1 - x_r}{x_r} \cdot \frac{1 - X_r}{X_r} = \frac{1}{x_r X_r} - \frac{1}{x_r} - \frac{1}{X_r} + 1, \tag{5.6}$$

which must be greater than 1 by the first inequality. Since $F(x) \geq f_r(x)$ for any relation (and specifically any type 1a relation) in $\mathcal{F}$, we see that $\beta^* = F(x^*) \geq f_r(x^*) > 1$ for x^* in Lemma 5.2.

To show that $\alpha^* \leq \alpha$, we shall need a special element $y \in \triangle$. Define

$$y(\psi) = \begin{cases} \dfrac{1}{(2n-1)\alpha+1} & \text{if } \psi \text{ is type } 1, \\ \dfrac{(2n-1)(\alpha-1)}{(4n^2-4n-1)(2n-1)\alpha^2+(4n^2-2)\alpha-(2n-1)} & \text{if } \psi \text{ is type 2,3,5,} \\ \dfrac{(2n-1)}{(2n-1)+\alpha} & \text{if } \psi \text{ is type } 4. \end{cases} \tag{5.7}$$

All the coordinates of y are obviously greater than 0 and it is straight forward to show that $Y = 1$ by using Table 4.1 and the counts of element types therein. The resulting expression can easily be simplified using $\mathcal{P}(\alpha) = 0$. Similarly, one can also show that $f_r(y) = \alpha$ for all $r \in \mathcal{F}$ and $f_r(y) \leq \alpha$ for all $r \in \mathcal{G}$ by using Table 4.3 and the counts of element types within every Ψ_r. Therefore $\alpha^* = \inf(G(x)) \leq G(y) = \alpha$. In summary, we have $1 < \beta^* \leq \alpha^* \leq \alpha$, completing the proof. □

The previous lemma marks the end of phase one. The vector y given in the proof is in fact the unique optimization point for our problem. Unfortunately, proving that statement is not as easy as writing it. What we shall do next is to prove the uniqueness of the optimal point using convexity arguments via a bank of lemmas.

Lemma 5.4. *For all $x^* \in \triangle$ such that $F(x^*) = \beta^*$, there exists a type 4b relation r_0 such that $f_{r_0}(x^*) = \beta^*$.*

Proof. If we consider $\triangle$ as a submanifold of $\mathbb{R}^\Psi$, the tangent space $T_x\triangle$ at any point $x \in \triangle$ is the set of vectors whose coordinates sum to 0. Moreover, every displacement function is smooth on an open set containing $\triangle$ which implies that the directional derivative for $\vec{u} \in T_x\triangle$ will be $\nabla f(x) \cdot \vec{u}$.

Our plan of attack is to find a common direction of decrease for type 1a, 2b, 3a and 5a relations, which will give us a very nice contradiction if the infimum was not attained by any type 4b relation at an optimal point. As the common direction of decrease, consider the vector $\vec{u}$ described below.

$$u(\psi) = \begin{cases} 1 & \text{if } \psi \text{ is not type } 4, \\ -\dfrac{1+4(n-1)^2}{2(n-1)} & \text{if } \psi \text{ is type } 4 \end{cases} \tag{5.8}$$

It is easy to check that the sum of coordinates of $\vec{u}$ is 0 with the help of Table 4.1, in turn, this shows that $\vec{u} \in T_x\triangle$. Moreover, we present the calculations below using (4.3), (4.4) and Table 4.3 proving that directional derivatives of type 1a, 2b, 3a and 5a relation displacements are all negative in the direction $\vec{u}$.

- For r of type 1a, we see that

$$\nabla f_r(x) \cdot \vec{u} = -\frac{1-X_r}{x_r^2 X_r} \cdot u_{\psi_r} - \frac{1-x_r}{x_r X_r^2} \sum_{\psi \in \Psi_r} u_\psi \tag{5.9}$$

$$= -\frac{1-X_r}{x_r^2 X_r} \cdot 1 - \frac{1-x_r}{x_r X_r^2} \left(\sum_{\psi \in \Psi_r - \text{type 4}} u_\psi + \sum_{\psi \in \Psi_r \cap \text{type 4}} u_\psi \right) \tag{5.10}$$

$$= -\frac{1-X_r}{x_r^2 X_r} - \frac{1-x_r}{x_r X_r^2} \left(\sum_{\psi \in \Psi_r - \text{type 4}} 1 + \sum_{\psi \in \Psi_r \cap \text{type 4}} -\frac{1+4(n-1)^2}{2(n-1)} \right) \tag{5.11}$$

$$= -\frac{1-X_r}{x_r^2 X_r} \tag{5.12}$$

$$- \frac{1-x_r}{x_r X_r^2} \left((2n-1)(1+4(n-1)^2) - (1+4(n-1)^2) \cdot (2n-1) \right) \tag{5.13}$$

$$= -\frac{1-X_r}{x_r^2 X_r} \tag{5.14}$$

$$< 0. \tag{5.15}$$

For the last inequality, recall that $x_r,\ X_r \in (0,1)$.

- For type 2b, 3a and 5a, we find that

$$\nabla f_r(x) \cdot \vec{u} = -\frac{1-X_r}{x_r^2 X_r} - \frac{1-x_r}{x_r X_r^2} \cdot \left(1 - \frac{1}{2(n-1)}\right) < 0. \tag{5.16}$$

Assume that there exists an optimal point x^* such that $f_r(x^*) < \beta^*$ for all r of type 4b. Let $\varepsilon > 0$ be small enough so that $x^* + \varepsilon\vec{u} \in \triangle$ and $f_r(x^* + \varepsilon\vec{u}) < \beta^*$ for all r of type 4b. Such a choice is possible by the continuity of displacement functions. Under these conditions, we have $F(x^* + \varepsilon\vec{u}) < \beta^*$ as $f_r(x^* + \varepsilon\vec{u}) < f_r(x^*) \leq F(x^*) = \beta^*$ for all r of type 1a, 2b, 3a, 5a and $f_r(x^* + \varepsilon\vec{u}) < \beta$ for all r of type 4b, contradicting the minimality of β^*. $\square$

Unfortunately, the displacement functions are not convex on the entire simplex. Therefore, neither are the max functions. Any argument of uniqueness of the optimal solution must first be defended on a local level, displacement by displacement. Fortunately, each displacement function is strictly convex on a relatively large set.

Lemma 5.5. *For $r \in \mathcal{F}$ define $C_r = \{x \in \triangle \mid x_r + X_r - x_r X_r < 3/4\}$. The displacement function f_r is strictly convex on C_r.*

Proof. Consider the function $g : (0,1)^2 \to \mathbb{R}$ defined by $g(z_1, z_2) = \frac{1-z_1}{z_1} \cdot \frac{1-z_2}{z_2}$. As g is twice differentiable, g is strictly convex, where the Hessian of g is positive definite by [15, Theorem 3.1.11 and Exercise 3.1.12(c)].

$$\text{Hess}(g) = \begin{pmatrix} 2(1-z_2)/z_1^3 z_2 & 1/z_1^2 z_2^2 \\ 1/z_1^2 z_2^2 & 2(1-z_1)/z_1 z_2^3 \end{pmatrix} \tag{5.17}$$

Moreover, a 2×2 matrix is positive definite if and only if $\det(H) > 0$ and $\text{trace}(H) > 0$, where

$$\det(\text{Hess}(g)) = \frac{4(1-z_1)(1-z_2)}{z_1^4 z_2^4} - \frac{1}{z_1^4 z_2^4} \text{ and,} \tag{5.18}$$

$$\text{trace}(\text{Hess}(g)) = \frac{2(1-z_1)}{z_1 z_2^3} + \frac{2(1-z_2)}{z_1^3 z_2}. \tag{5.19}$$

Since the trace of the Hessian is always positive in our domain, g is strictly convex, where the determinant in positive. Hence, g is strictly convex on $C_g = \{(z_1, z_2) \in (0,1)^2 \mid z_1 + z_2 - z_1 z_2 < 3/4\}$. Now, notice that $f_r = g(x_r, X_r)$. Since the map $x \to (x_r, X_r) \in (0,1)^2$ satisfies the conditions listed in [15, Exercise 1.1.9], we get f_r is strictly convex on $C_r = \{x \in \triangle \mid x_r + X_r - x_r X_r < 3/4\}$. □

We already know that some type 4b displacement attains β^* at a point of optimality by Lemma 5.4. It is also true that any optimal point of a strictly convex function is unique [15, Exercise 2.1.8(a)]. Therefore, proving any optimal point must be in the domain of convexity of all type 4b relation displacements will take us most of the way to the uniqueness.

Theorem 5.6. *If r is a type 4b relation, then $x^* \in C_r$ for all $x^* \in \triangle$ such that $F(x^*) = \beta^*$.*

Proof. Assume there exists a point of optimality x^* and a type 4b relation r_0 such that $x^* \notin C_{r_0}$, ie,

$$r_0 : \;\; \psi_0 = \xi_{i_0}^{t_0} \xi_{j_0}^{s_0} \xi_{i_0}^{-t_0}, \quad \Psi_0 = \{\xi_{i_0}^{2t_0},\ \xi_{i_0}^{t_0}\xi_j^{2s},\ \xi_{i_0}^{t_0}\xi_j^{s}\xi_k^{p}\}, \tag{5.20}$$

where

$$x_0^* + X_0^* - x_0^* X_0^* \geq 3/4. \tag{5.21}$$

Our aim is to obtain a contradiction under these conditions. Let T be the collection of type 4b relations such that $\psi_r = \xi_i^t \xi_{j_0}^{s_0} \xi_i^{-t}$ and $\psi_r = \xi_{j_0}^{s} \xi_{i_0}^{t_0} \xi_{j_0}^{-s}$. Notice that $(\Psi_r)_{r\in T}$ gives a disjoint partition of Ψ. This means that

$$\Psi = \bigsqcup_{r\in T} \Psi_r, \tag{5.22}$$

which implies that for any $x \in \triangle$, we get

$$1 = \sum_{r\in T} X_r. \tag{5.23}$$

Based on the possible location of x^*, we will separate the proof into the following three cases

$$(A)\ \frac{1}{2} \geq X_0^*, \tag{5.24}$$

$$(B)\ \frac{1}{2n+2} \geq X_{r_1}^* \text{ for some } r_1 \in T, \tag{5.25}$$

$$(C)\ \frac{1}{2} < X_0^* \text{ and } \frac{1}{2n+2} < X_r^* \text{ for all } r \in T. \tag{5.26}$$

Each one of these cases will end up with a contradiction, proving the theorem.

Assume (A) is the case. Then, (5.21) gives us the following chain of inequalities:

$$X_0^*(1 - x_0^*) + x_0^* \geq 3/4 \tag{5.27}$$

$$\text{or } \ \frac{1}{2}(1 - x_0^*) + x_0^* \geq 3/4 \tag{5.28}$$

$$\text{or } \ x_0^* \geq 1/2. \tag{5.29}$$

However, as $\psi_0 \in \Psi_0$, we get the contradiction $1/2 \geq X_0^* > x_0^* \geq 1/2$.

Assume (B) is the case. Let $\psi_{r_1} = \xi_{i_1}^{t_1} \xi_{j_1}^{s_1} \xi_{i_1}^{-t_1}$. Consider any type 4b relation r with $\psi_r =$

$\xi_{i_1}^{t_1}\xi_j^s\xi_{i_1}^{-t_1}$. By Lemma 5.3, we have

$$\frac{1-x_r^*}{x_r^*}\cdot\frac{1-X_r^*}{X_r^*}=f_r(x^*)\le\beta^*\le\alpha. \tag{5.30}$$

This, after a bit of re-arrangement, turns into the following inequality

$$\frac{1-X_r^*}{X_r^*(\alpha-1)+1}\le x_r^*. \tag{5.31}$$

Notice that $\Psi_{r_1}=\Psi_r$ and so $1/(2n+1)\ge X_{r_1}^*=X_r^*$. In the above inequality, the left side is a decreasing function of X_r^*. Thus, by the case assumption, we preserve the inequality by replacing X_r^* with $1/(2n+2)$. We obtain

$$\frac{2n+1}{\alpha+2n+1}=\frac{1-\frac{1}{2n+2}}{\frac{1}{2n+2}(\alpha-1)+1}\le x_r^*=x^*(\xi_{i_1}^{t_1}\xi_j^s\xi_{i_1}^{-t_1}). \tag{5.32}$$

The following

$$\frac{1}{2n+2}\ge X_{r_1}^*>\sum_{j\neq i_1,\ s=\pm1}x^*(\xi_{i_1}^{t_1}\xi_j^s\xi_{i_1}^{-t_1})\ge 2(n-1)\cdot\frac{2n+1}{\alpha+2n+1} \tag{5.33}$$

is obtained by the previous inequality and the fact that $\xi_{i_1}^{t_1}\xi_j^s\xi_{i_1}^{-t_1}\in\Psi_{r_1}$ for all $j\neq i_1$, $s=\pm1$. Again, after a bit of re-arrangement, we get

$$\alpha>(2n-1)^3+8n^2-12n-4. \tag{5.34}$$

However, we have $8n^2-12n-3>0$ for $n\ge 2$ implying $\alpha>(2n-1)^3$, contradicting the proof of Lemma 2.1.

Assume (C) is the case. By using (5.23) and the fact that $n\ge 2$, we derive that

$$1=X_0^*+\sum_{r\in T-\{r_0\}}X_r^*>\frac{1}{2}+(2n-1)\cdot\frac{1}{2n+2}=1+\frac{1}{2}-\frac{3}{2n+2}\ge 1, \tag{5.35}$$

which is the final contradiction that we needed. □

It is a simple exercise to show that the max function of strictly convex functions is strictly convex. Hence, we have the fact that F is strictly convex on $\cap_{r\in\text{Type 4b}}C_r$ which is a non empty set as it contains all the optimal points at the very least. As optimal points of strictly convex functions are unique, we have proven the following corollary.

Corollary 5.7. *The optimal point of $inf_{x\in\triangle}F(x)$ is unique.*

Henceforth, we shall use x^* to denote the unique optimal point. In other words, we let

$$x^* : \quad \text{the unique point in } \triangle \text{ with } F(x^*) = \beta^*,$$

which marks the end of phase two.

We start phase three by simplifying x^*. The uniqueness of x^* will be heavily used in this simplification. Then, we apply the Karush-Kuhn-Tucker theorem.

Lemma 5.8. *There exists $a^*,\ b^*,\ c^* \in (0,1)$ such that*

$$x^*(\psi) = \begin{cases} a^* & \textit{for } \psi \in \textit{type 1} \\ b^* & \textit{for } \psi \in \textit{type 2 or type 3 or type 5} \\ c^* & \textit{for } \psi \in \textit{type 4.} \end{cases} \tag{5.36}$$

Proof. Let $i_1, i_2 \in \{1, 2, \ldots, n\}$ and $t_1, t_2 \in \{-1, +1\}$ such that $i_1 \neq i_2$. Consider the symmetry $\tau \in \text{Sym}(\Psi)$, which switches $\xi_{i_1}^{t_1}$ with $\xi_{i_2}^{t_2}$ and vice versa in every element of Ψ. For clarification, we have given the details of the action of τ in Table 5.1.

One can prove that τ can also be considered as an element of $\text{Sym}(\mathcal{F})$ in the sense that $\psi_{\tau(r)} = \tau(\psi_r)$ and $\Psi_{\tau(r)} = \tau(\Psi_r)$ by observing the effects of τ on the elements of $\mathcal{F}$ and using Table 4.3. As a result, we have the following

$$F(\tau(x)) = \max_{r\in\mathcal{F}} f_r(\tau(x)) = \max_{r\in\mathcal{F}} f_{\tau(r)}(x) = \max_{r\in\mathcal{F}} f_r(x) = F(x). \tag{5.37}$$

In particular, we have $F(\tau(x^*)) = F(x^*)$ and hence $\tau(x^*) = x^*$ by Corollary 5.7. By repeated application of τ (with different values for (i_1, t_1) and(i_2, t_2)), we can show that any coordinate of x^* is equal to any other coordinate of the same type. We give the following

type 1	$\xi_{i_1}^{\pm 2t_1} \leftrightarrow \xi_{i_2}^{\pm 2t_2}$	
type 2	$\xi_{i_1}^{t_1}\xi_{i_2}^{\pm 2t_2} \leftrightarrow \xi_{i_2}^{t_2}\xi_{i_1}^{\pm 2t_1}$	$\xi_{i_1}^{-t_1}\xi_{i_2}^{\pm 2t_2} \leftrightarrow \xi_{i_2}^{-t_2}\xi_{i_1}^{\pm 2t_1}$
	$\xi_i^t\xi_{i_1}^{\pm 2t_1} \leftrightarrow \xi_i^t\xi_{i_2}^{\pm 2t_2}$	$\xi_{i_1}^{\pm t_1}\xi_j^{2s} \leftrightarrow \xi_{i_2}^{\pm t_2}\xi_j^{2s}$
type 3	$\xi_{i_1}^{t_1}\xi_{i_2}^{\pm t_2}\xi_{i_1}^{t_1} \leftrightarrow \xi_{i_2}^{t_2}\xi_{i_1}^{\pm t_1}\xi_{i_2}^{t_2}$	$\xi_{i_1}^{-t_1}\xi_{i_2}^{\pm t_2}\xi_{i_1}^{-t_1} \leftrightarrow \xi_{i_2}^{-t_2}\xi_{i_1}^{\pm t_1}\xi_{i_2}^{-t_2}$
	$\xi_i^t\xi_{i_1}^{\pm t_1}\xi_i^t \leftrightarrow \xi_i^t\xi_{i_2}^{\pm t_2}\xi_i^t$	$\xi_{i_1}^{\pm t_1}\xi_j^s\xi_{i_1}^{\pm t_1} \leftrightarrow \xi_{i_2}^{\pm t_2}\xi_j^s\xi_{i_2}^{\pm t_2}$
type 4	$\xi_{i_1}^{t_1}\xi_{i_2}^{\pm t_2}\xi_{i_1}^{-t_1} \leftrightarrow \xi_{i_2}^{t_2}\xi_{i_1}^{\pm t_1}\xi_{i_2}^{-t_2}$	$\xi_{i_1}^{-t_1}\xi_{i_2}^{\pm t_2}\xi_{i_1}^{t_1} \leftrightarrow \xi_{i_2}^{-t_2}\xi_{i_1}^{\pm t_1}\xi_{i_2}^{t_2}$
	$\xi_i^t\xi_{i_1}^{\pm t_1}\xi_i^{-t} \leftrightarrow \xi_i^t\xi_{i_2}^{\pm t_2}\xi_i^{-t}$	$\xi_{i_1}^{\pm t_1}\xi_j^s\xi_{i_1}^{\mp t_1} \leftrightarrow \xi_{i_2}^{\pm t_2}\xi_j^s\xi_{i_2}^{\mp t_2}$
type 5	$\xi_{i_1}^{t_1}\xi_{i_2}^{\pm t_2}\xi_l^p \leftrightarrow \xi_{i_2}^{t_2}\xi_{i_1}^{\pm t_1}\xi_l^p$	$\xi_{i_1}^{-t_1}\xi_{i_2}^{\pm t_2}\xi_l^p \leftrightarrow \xi_{i_2}^{-t_2}\xi_{i_1}^{\pm t_1}\xi_l^p$
	$\xi_{i_1}^{t_1}\xi_j^s\xi_{i_2}^{\pm t_2} \leftrightarrow \xi_{i_2}^{t_2}\xi_j^s\xi_{i_1}^{\pm t_1}$	$\xi_{i_1}^{-t_1}\xi_j^s\xi_{i_2}^{\pm t_2} \leftrightarrow \xi_{i_2}^{-t_2}\xi_j^s\xi_{i_1}^{\pm t_1}$
	$\xi_i^t\xi_{i_1}^{t_1}\xi_{i_2}^{\pm t_2} \leftrightarrow \xi_i^t\xi_{i_2}^{t_2}\xi_{i_1}^{\pm t_1}$	$\xi_i^t\xi_{i_1}^{-t_1}\xi_{i_2}^{\pm t_2} \leftrightarrow \xi_i^t\xi_{i_2}^{-t_2}\xi_{i_1}^{\pm t_1}$
	$\xi_i^t\xi_j^s\xi_{i_1}^{\pm t_1} \leftrightarrow \xi_i^t\xi_j^s\xi_{i_2}^{\pm t_2}$	$\xi_i^t\xi_{i_1}^{\pm t_1}\xi_l^p \leftrightarrow \xi_i^t\xi_{i_2}^{\pm t_2}\xi_l^p$
	$\xi_{i_1}^{\pm t_1}\xi_j^s\xi_l^p \leftrightarrow \xi_{i_2}^{\pm t_2}\xi_j^s\xi_l^p$	

Table 5.1. Actions of the symmetry $\tau \in \mathrm{Sym}(\Psi)$

example of type 5 coordinates for clarification.

$$x^*(\xi_{i_1}^{t_1}\xi_{j_1}^{s_1}\xi_{l_1}^{p_1}) \underset{(i_1,t_1)\to(i_2,t_2)}{\overset{\tau}{=}} x^*(\xi_{i_2}^{t_2}\xi_{j_1}^{s_1}\xi_{l_1}^{p_1}) \underset{(j_1,s_1)\to(j_2,s_2)}{\overset{\tau}{=}} x^*(\xi_{i_2}^{t_2}\xi_{j_2}^{s_2}\xi_{l_1}^{p_1}) \tag{5.38}$$

$$\underset{(l_1,p_1)\to(l_2,p_2)}{\overset{\tau}{=}} x^*(\xi_{i_2}^{t_2}\xi_{j_2}^{s_2}\xi_{l_2}^{p_2}) \tag{5.39}$$

This proves that there are $a^*,\ b_2^*,\ b_3^*,\ b_5^*,\ c^* \in (0,1)$ such that:

$$x^*(\psi) = \begin{cases} a^* & \text{for } \psi \in \text{type 1} \\ b_2^* & \text{for } \psi \in \text{type 2} \\ b_3^* & \text{for } \psi \in \text{type 3} \\ b_5^* & \text{for } \psi \in \text{type 5} \\ c^* & \text{for } \psi \in \text{type 4.} \end{cases} \tag{5.40}$$

It remains to prove $b_2^* = b_3^* = b_5^*$. Fortunately, the above simplification makes it much easier.

Consider $\sigma_1 = (\xi_i^t \xi_j^{2s} \leftrightarrow \xi_j^s \xi_i^t \xi_j^s \mid \forall i, j, t, s) \in \text{Sym}(\Psi)$. It is not true that in general $F(\sigma_1(x)) = F(x)$. However, by using the previous simplification for x^* and Table 4.3, it is easy to see that $F(\sigma_1(x^*)) = F(x^*)$. Thus, again by Corollary 5.7, we get

$$b_2^* = x^*(\xi_1 \xi_2^2) = x^*(\xi_2 \xi_1 \xi_2) = b_3^*. \tag{5.41}$$

Similarly, $\sigma_2 = (\xi_1 \xi_2^2 \leftrightarrow \xi_1 \xi_2 \xi_3) \in \text{Sym}(\Psi)$ gives us $b_2^* = b_5^*$ which completes the proof. $\square$

Now that we have simplified x^* as much as possible. We shall use the optimization question with the Karush-Kuhn-Tucker theorem to get further equalities on x^* with a target of calculating exact values for its coordinates.

Theorem 5.9. *For all $r_1, r_2 \in \mathcal{F}$, we have $f_{r_1}(x^*) = f_{r_2}(x^*)$.*

Proof. We will obtain the proof by applying the Karush-Kuhn-Tucker theorem to the optimization question

$$\min F(x) \text{ for } x \in \triangle. \tag{5.42}$$

By Lemma 5.8, we can restrict our search for an optimal solution to those elements of $\triangle$

showing the same coordinate distribution as x^*. In other words, we solve the problem below

$$\min F(x) \text{ for } x \in \tilde{\triangle}, \text{ where} \tag{5.43}$$

$$\tilde{\triangle} := \{x \in \triangle \mid \exists a, b, c \in (0,1),\ x(\psi) = \begin{cases} a & \text{for } \psi \in \text{type 1} \\ b & \text{for } \psi \in \text{type 2 or type 3 or type 5} \\ c & \text{for } \psi \in \text{type 4} \end{cases} \}. \tag{5.44}$$

A more illuminating observation comes from the fact that $X = 1$ for all $x \in \tilde{\triangle}$ and the counts in Table 4.1. For any $x \equiv (a, b, c)$, we have

$$1 = 2na + 8n(n-1)^2 b + 4n(n-1)c. \tag{5.45}$$

The displacement functions are simplified as it no longer matters what specific coordinate Ψ_r contains, but the number of types of coordinates. As an example, for any type 1a relation r and $x \equiv (a, b, c) \in \tilde{\triangle}$, we have (with the help of Table 4.3),

$$x_r = a \text{ and } X_r = 1 - (a + 4(n-1)^2 b + 2(n-1)c). \tag{5.46}$$

Moreover using (5.45) to replace a, we obtain $X_r = (2n-1)/(2n)$. This simplifies $f_r(x) = \frac{1-x_r}{x_r} \cdot \frac{1-X_r}{X_r}$ into $f_r(x) = \frac{1-a}{a} \cdot \frac{1}{2n-1}$. We summarize these simplifications for all types of relations in $\mathcal{F}$ in Table 5.2.

r	x_r	X_r	$f_r(x)$
Type 1a	a	$\frac{2n-1}{2n}$	$\frac{1-a}{a} \cdot \frac{1}{2n-1}$
Type 2b, 3a and 5a	b	$1 - (2(n-1)b + c)$	$\frac{1-b}{b} \cdot \frac{1-2na}{4n(n-1)-1+2na}$
Type 4b	c	$\frac{1}{2n}$	$\frac{1-c}{c} \cdot (2n-1)$

Table 5.2. $f_r(x)$ for $x \equiv (a, b, c) \in \tilde{\triangle}$

By Lemma 5.4, we know that there exists a Type 4b relation r_0 such that $\beta^* = f_{r_0}(x^*) \geq f_r(x^*)$ for all $r \in \mathcal{F}$. Applying this knowledge in light of Table 5.2, we get two new inequalities further narrowing the search for the optimal solution. We obtain

$$\frac{1-a}{a} \cdot \frac{1}{2n-1} \leq \frac{1-c}{c} \cdot (2n-1), \tag{5.47}$$

$$\frac{1-b}{b} \cdot \frac{1-2na}{4n(n-1)-1+2na} \leq \frac{1-c}{c} \cdot (2n-1). \tag{5.48}$$

This simplifies the optimization question to its ultimate form as follows

Target:	$f(a,b,c) = \frac{(2n-1)(1-c)}{c}$ (minimize)
Constraints:	$g_1(a,b,c) = \frac{1-a}{a} \cdot \frac{c}{1-c} - (2n-1)^2 \leq 0$
	$g_2(a,b,c) = \frac{1-2na}{4n(n-1)-1+2na} \frac{1-b}{b} \frac{c}{1-c} - (2n-1) \leq 0$
	$h(a,b,c) = 2na + 8n(n-1)^2 b + 4n(n-1)c = 1$
	$(a,b,c) \in U = (0,1)^3$.

From hereon in, we shall apply the optimization techniques detailed in [15, Chapter 7.2]. Our goal is to prove that the Lagrange multipliers for g_1 and g_2 can not be 0, which (by KKT as in [15, Theorem 7.2.9]) will imply that the relevant inequalities in the constraints have to be active (equal) at the optimal point, proving our theorem.

We shall first discuss the three prerequisites of KKT. The existence of the optimal solution is proven in Lemma 5.2. The target and constraint functions are obviously differentiable on U. We also need to show that the Mangasarian-Fromovitz constraint qualification [15, Assumption 7.2.3] holds. The qualification for our question can be summarized as follows

- The function $p \mapsto \nabla h(x^*) \cdot p$ from $\mathbb{R}^3$ to $\mathbb{R}$ is onto and,
- $\{p \in \mathbb{R}^3 \mid \nabla h(x^*) \cdot p = 0$ and $\nabla g_i(x^*) \cdot p < 0$ for $i = 1, 2\} \neq \emptyset$.

The first one is trivially true as ∇h is constant and nonzero. The second one also holds since $p = (2(n-1), 0, -1)$ is in the set. Hence, by KKT, there exists $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and $\lambda_3 \in \mathbb{R}$

such that:

$$(\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \lambda_3 \nabla h)(x^* \equiv (a^*, b^*, c^*)) = \vec{0} \tag{5.49}$$

This gives us the following linear system of λ_1, λ_2 and λ_3

$$-\frac{c^*}{(a^*)^2(1-c^*)}\lambda_1 + \frac{8n^2(n-1)}{A}\lambda_2 + 2n\lambda_3 = 0, \tag{5.50}$$

$$-\frac{(1-2na^*)c^*}{A(b^*)^2(1-c^*)}\lambda_2 + 8n(n-1)^2\lambda_3 = 0, \tag{5.51}$$

$$\frac{1-a^*}{(a^*)(1-c^*)^2}\lambda_1 + \frac{(1-2na^*)(1-b^*)}{Ab^*(1-c^*)^2}\lambda_2 + 4n(n-1)\lambda_3 = \frac{2n-1}{(c^*)^2}, \tag{5.52}$$

where $A = 4n(n-1) - (1-2na)$. It is easy to see that $1 - 2na \in (0,1)$ by using the constraint $h = 1$. Thus, every parenthesis in this system is positive. This implies that the signs in front of the coefficients are the signs of the coefficients.

If $\lambda_3 = 0$, then by (5.51) $\lambda_2 = 0$ as well, which implies $\lambda_1 = 0$ by (5.50). But, this is a contradiction to (5.52). Thus, $\lambda_3 \neq 0$. The fact that λ_3 is non-zero implies $\lambda_2 > 0$ by (5.51), which in turn, again via (5.51), shows that $\lambda_3 > 0$ as well. Finally, the positiveness of λ_2 and λ_3 on (5.50) shows that $\lambda_1 > 0$. Therefore, the Lagrange multipliers of g_1 and g_2 are greater than 0, proving what we wanted. □

Theorem 5.10. *If $x^* \equiv (a^*, b^*, c^*)$ is as in Lemma 5.8 and α is as defined after Lemma 2.1, then we have*

i. $a^* = \dfrac{1}{(2n-1)+\alpha}$,

$b^* = \dfrac{(2n-1)(\alpha-1)}{(4n^2-4n-1)(2n-1)\alpha^2 + 2(2n^2-1)\alpha - (2n-1)}$,

$c^* = \dfrac{2n-1}{2n-1+\alpha}$,

ii. $\beta^* = \alpha^* = \alpha$.

Proof. By Theorem 5.9, we know that $f_r(x^*) = \beta^*$ for all $r \in \mathcal{F}$. By the proof of the same theorem, we also know that there are really three functions at the point of optimality.

Therefore, we have the following equations

$$\frac{1-a^*}{a^*}\cdot\frac{1}{2n-1}=\beta^*, \tag{5.53}$$

$$\frac{1-b^*}{b^*}\cdot\frac{1-2na^*}{4n(n-1)-1+2na^*}=\beta^*, \tag{5.54}$$

$$\frac{1-c^*}{c^*}\cdot(2n-1)=\beta^*. \tag{5.55}$$

Simply solving this system for a^*, b^* and c^* yields that

$$a^*=\frac{1}{(2n-1)+\beta^*}, \tag{5.56}$$

$$b^*=\frac{(2n-1)(\beta^*-1)}{(4n^2-4n-1)(2n-1)(\beta^*)^2+2(2n^2-1)\beta^*-(2n-1)}, \tag{5.57}$$

$$c^*=\frac{2n-1}{2n-1+\beta^*}. \tag{5.58}$$

Putting these formulas into the equation (5.45) gives $\mathcal{P}(\beta^*)=0$, where $\mathcal{P}$ is the polynomial defined in Lemma 2.1. But, by Lemma 5.3, we know that $1<\beta^*$. Since α is the only root of $\mathcal{P}$ greater than 1, we find $\beta^*=\alpha$. Applying that to $\beta^*\leq\alpha^*\leq\alpha$ from Lemma 5.3 gives us $\alpha=\beta^*\leq\alpha^*\leq\alpha$, also proving that $\alpha^*=\alpha$. □

6. PROOF OF THE MAIN THEOREM

We will first prove two lemmas that will simplify the proof of the main theorem.

Lemma 6.1. *Let g, h in $PSL(2, \mathbb{C})$ and $G = \langle g, h\rangle$. If G is freely generated by $\{g, h\}$, purely loxodromic and Kleinian, then G is nonelementary.*

Proof. Assume G is elementary. By the classification of discrete elementary groups done in [11, Chapter 5.1], there exists a two element subset of $\hat{\mathbb{C}}$ that is invariant under the action of G. Without loss of generality, take the invariant subset to be $\{0, \infty\}$.

If the fixed points of g were not 0 and ∞, we would have $g(0) = \infty$ and $g(\infty) = 0$. This implies that $g(z) = \frac{a}{z}$, contradicting with the freeness of G as $g^2 = id$. Similarly, the fixed points of h must be 0 and ∞ as well. Hence, the fixed point of g and h are the same. By [11, Theorem 4.3.5] we conclude that either $[g, h] = id$ or $[g, h]$ is parabolic. The first case contradicts with the freeness of G, and the second one contradicts the lemma hypothesis that G is purely loxodromic. □

Lemma 6.2. *Let $g, h \in PSL(2, \mathbb{C})$ be noncommuting and loxodromic. If z_0 is the midpoint of the shortest geodesic segment connecting the axes of g and $h^{-1}gh$, then $d_g z_0 < d_{hgh^{-1}} z_0$.*

Proof. For $\gamma \in PSL(2, \mathbb{C})$, let $Z_\lambda(\gamma) = \{z \in \mathbb{H}^3 \mid d_\gamma z \leq \lambda\}$, the displacement cylinder of γ with radius λ. By [11, 5.4.11], $d_\gamma z$ is an increasing function of $\rho(z, A_\gamma)$. Thus, $Z_\lambda(\gamma)$ really is a cylinder with axis A_γ.

Fix $\lambda = d_g z_0$ so that $z_0 \in Z_\lambda(g)$. By lemma hypothesis, we have $\rho(z_0, A_g) = \rho(z_0, A_{h^{-1}gh})$. Moreover, the terms T_g and θ_g in [11, 5.4.11] are invariant under conjugation. Hence, $\rho(z_0, gz_0) = \rho(z_0, h^{-1}ghz_0)$ which proves that $z_0 \in Z_\lambda(h^{-1}gh)$.

We claim that z_0 is the only element in $Z_\lambda(g) \cap Z_\lambda(h^{-1}gh)$. If it was not, then there would be an element $z \neq z_0$ such that $\rho(z, gz) = \rho(z, h^{-1}ghz)$ which (again by [11, 5.4.11]) would imply that $\rho(z, A_g) = \rho(z, A_{h^{-1}gh})$. By lemma hypothesis, we would have $\rho(z_0, A_g) < \rho(z, A_g)$ so that $\lambda = \rho(z_0, gz_0) < \rho(z, gz)$, a contradiction.

Now that we know $Z_\lambda(g) \cap Z_\lambda(h^{-1}gh) = \{z_0\}$, one can easily show that $Z_\lambda(g) \cap$

$Z_\lambda(hgh^{-1}) = \{hz_0\}$. Because $z_0 \in Z_\lambda(g)$, we can derive that $z_0 \notin Z_\lambda(hgh^{-1})$. Therefore, $\lambda = d_g z_0 < d_{hgh^{-1}} z_0$. □

The following theorem will be the principal effect of the optimization that was done in Section 5 to our main result. The geometrically infinite case is quite straightforward thanks to the Culler-Shalen machinery. However, the generalization to the geometrically finite case requires some high level theorems which we refer to inside the proof.

Theorem 6.3. *Let* Γ *be a Kleinian, purely loxodromic subgroup of* $PSL(2, C)$ *that is freely generated by* $\Xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$*. If* $\Gamma_* = \{1, \xi_i^t, \xi_i^t \xi_j^s \xi_i^{-t}\}$ *and* α *is the unique root of the polynomial* $\mathcal{P}$ *greater than* $(2n-1)^2$*, then for any* $z \in \mathbb{H}^3$ *we have*

$$\max\{d_\gamma z \mid \gamma \in \Gamma_*\} \geq \frac{1}{2} \log \alpha. \tag{6.1}$$

Proof. Fix $z \in \mathbb{H}^3$ and let $\lambda = \max\{d_\gamma z \mid \gamma \in \Gamma_*\}$. Assume Γ is geometrically infinite. By Proposition 3.3(ii), we have

$$\lambda \geq d_{\gamma_r} z \geq \frac{1}{2} \log f_r(m), \text{ for any relation } r \tag{6.2}$$

so that $\lambda \geq \frac{1}{2} \log G(m)$, where G is defined in Section 4. Moreover, by Proposition 3.3(i) and Theorem 5.10, we derive

$$\lambda \geq \frac{1}{2} \log G(m) \geq \frac{1}{2} \log \left(\inf\{G(x) \mid x \in \triangle\}\right) = \frac{1}{2} \log \alpha. \tag{6.3}$$

Assume that Γ is geometrically finite. Our plan of attack is to prove that the infimum of all such lower boundaries over all possible free groups is attained at a special place close to the geometrically infinite case. We will have to use yet another optimization and some prior knowledge. Consider the function f_z defined below

$$f_z : \mathrm{PSL}(2, \mathbb{C})^n \longrightarrow \mathbb{R}^+ \tag{6.4}$$

$$\xi \longmapsto \max\{d_\gamma z \mid \gamma \in \Gamma_*(\xi)\}. \tag{6.5}$$

Let $\mathfrak{G}\mathfrak{F}$ denote the subset of elements of $\mathrm{PSL}(2, \mathbb{C})^n$ whose components form a set of free generators of a purely loxodromic free subgroup of $\mathrm{PSL}(2,\mathbb{C})$ that is geometrically finite. As the image $f_z(\mathfrak{GF})$ is bounded below and f_z is continuous, there exists a $\overline{\xi} \in \overline{\mathfrak{GF}}$ such that

$$f_z(\overline{\xi}) = \inf\{f_z(\xi) \mid \xi \in \mathfrak{GF}\}. \tag{6.6}$$

However, it has been shown in the proofs of [16, Theorem 5.1], [12, Theorem 4.1] and finally in [10, Theorem 4.2] that any such optimal point must satisfy

$$\overline{\xi} \in \overline{\mathfrak{GF}} - \mathfrak{GF}.$$

Moreover, the subset of elements of $\mathrm{PSL}(2,\mathbb{C})^n$ whose components are generating purely loxodromic and geometrically infinite groups is dense in $\overline{\mathfrak{GF}} - \mathfrak{GF}$ as was shown in [13, Propositions 8.2 and 9.3], [17, Main Theorem] and [18]. Hence, the geometrically finite case folds into the geometrically infinite case, completing the proof. □

The final theorem of this paper is more general and in appearance more complicated than its counterpart in [10]. However, most of the complications in the proof have been handled by the previous lemmas and theorems in this paper and the arguments inside the proof is largely identical to that of the counterpart.

Theorem 6.4. *Let Γ and α be as in Theorem 6.3. Assume that there exist i_0, j_0 such that the inequalities $d_{\xi_{i_0}\xi_{j_0}\xi_{i_0}^{-1}} z_2 \leq d_{\xi_{i_0}\xi_{j_0}\xi_{i_0}^{-1}} z_1$ and $d_\gamma z_2 < \frac{1}{2}\log\alpha$ for every $\gamma \in \Phi = \Gamma - \{\xi_{i_0},\ \xi_{j_0}^t \xi_{i_0}^s \xi_{j_0}^{-t}\}$ hold, where z_1 and z_2 are the midpoints of the shortest geodesic segments connecting the axis of ξ_{j_0} to the axes of $\xi_{i_0}\xi_{j_0}\xi_{i_0}^{-1}$ and $\xi_{i_0}^{-1}\xi_{j_0}\xi_{i_0}$ respectively. Then, we have*

$$|\mathrm{trace}^2(\xi_{j_0}) - 4| + |\mathrm{trace}(\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}\xi_{j_0}^{-1}) - 2| \geq 2\sinh^2\left(\frac{1}{4}\log\alpha\right). \tag{6.7}$$

Proof. We shall make use of the following facts. The first one is a straightforward exercise to the teachings of [11] and the second one is due to Lemma 6.1.

- The theorem remains invariant under conjugation.
- The subgroup $\langle \xi_{i_0},\ \xi_{j_0}\rangle$ must be non-elementary as it is free, Kleinian and purely loxodromic.

Since the theorem remains invariant under conjugation, we may assume without loss of generality that ξ_{i_0} fixes 0 and ∞. This gives us some useful matrix presentations as follows

$$\xi_{i_0} = \begin{bmatrix} u & 0 \\ 0 & \frac{1}{u} \end{bmatrix} \text{ for some } u = |u|e^{i\theta}, \tag{6.8}$$

$$\xi_{j_0} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ so that } ad - bc = 1 \tag{6.9}$$

so that we can write

$$|\text{trace}^2(\xi_{i_0}) - 4| + |\text{trace}([\xi_{i_0}, \xi_{j_0}] - 2)| = (u - \frac{1}{u})^2(1 + |bc|) \tag{6.10}$$

$$= 4\sinh^2(\frac{1}{2}T_{\xi_{i_0}}) + 4\sin(\theta), \tag{6.11}$$

where the last equation is due to [11, equations 5.4.8 and 5.4.10].

By [11, proof of Theorem 5.1.3 case(i)], ξ_{i_0} and $\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}$ have no common fixed point. This implies $a, b, c, d \neq 0$. Consider the cross-ratio equation $[1, -1, w, -w] = [0, \infty, b/d, a/c]$ which is true for any w that satisfies the following equation

$$bc = \frac{(1-w)^2}{4w}. \tag{6.12}$$

This equation has a unique solution for w with norm greater than 1. Let $w = e^{2z_0}$ (for $z_0 = x_0 + iy_0$) be that solution. Replacing w in (6.12) yields $bc = \sinh^2 z_0$ which we can use to obtain the following chain of (in)equalities

$$4|bc|^2 = |\cosh^2 z_0 - 1|^2 \tag{6.13}$$

$$= (\cosh 2x_0 - \cos 2y_0)^2 \tag{6.14}$$

$$\geq (\cosh 2x_0 - 1)^2 \tag{6.15}$$

$$\geq (\cosh^2 x_0 - 1)^2. \tag{6.16}$$

Using the above inequality and a bit of trimming gives

$$1 + |bc| \geq \frac{\cosh^2 x_0}{2}. \tag{6.17}$$

Let $\mathcal{A}$ be the translation axis of ξ_{i_0} and $\mathcal{B}$ be the translation axis of $\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}$. Also notice that $\mathcal{B} = \xi_{j_0}\mathcal{A}$. Since trace squared, translation length and the square of the sine square of translation angle are all invariant under conjugation, we have the following inequalities for all $z \in \mathbb{H}^3$

$$\sin^2 \frac{1}{2} d_{\xi_{i_0}} z = \sinh \frac{1}{2} T_{\xi_{i_0}} \cosh d(z, \mathcal{A}) + \sin^2 \theta_{\xi_{i_0}} \sinh^2 d(z, \mathcal{A}) \tag{6.18}$$

$$\leq \left(\sinh^2 \frac{1}{2} T_{\xi_{i_0}} + \sin^2 \theta_{\xi_{i_0}} \right) \cosh^2 d(z, \mathcal{A}), \tag{6.19}$$

$$\sin^2 \frac{1}{2} d_{\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}} z = \sinh \frac{1}{2} T_{\xi_{i_0}} \cosh d(z, \mathcal{B}) + \sin^2 \theta_{\xi_{i_0}} \sinh^2 d(z, \mathcal{B}) \tag{6.20}$$

$$\leq \left(\sinh^2 \frac{1}{2} T_{\xi_{i_0}} + \sin^2 \theta_{\xi_{i_0}} \right) \cosh^2 d(z, \mathcal{B}). \tag{6.21}$$

By using the above inequalities and the fact that the hyperbolic sine and cosine are increasing for $x \geq 0$, we conclude that

$$\sinh^2 \frac{1}{2} \max\{d_{\xi_{i_0}} z, d_{\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}} z\} \leq \frac{1}{4} \left| u - \frac{1}{u} \right|^2 \cosh^2 \max\{d_z \mathcal{A}, d_z \mathcal{B}\}. \tag{6.22}$$

Let $\Psi \in \mathcal{M}$ be defined by taking $[0, \infty, h0, h\infty]$ to $[1, -1, w, -w]$, where w is defined in (6.12). Under this setting $\Psi\mathcal{A}$ will be the geodesic from -1 to 1 and $\Psi\mathcal{B}$ will be the geodesic from $-w$ to w so that we have

$$d(\mathcal{A}, \mathcal{B}) = d(j, |w|j) = \log |w| = 2x_0. \tag{6.23}$$

By the above equality and the definition of z_1 given in theorem hypothesis, we can establish the fact that $d(z_1, \mathcal{A}) = x_0 = d(z_1, \mathcal{B})$. Applying this fact to the first equations in (6.18) and (6.20) gives us another fact $d_{\xi_{i_0}} z_1 = d_{\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}} z_1$. These two facts can be used to simplify (6.22) into the first of the following inequalities. Using (6.17) gives us the second one, which is

$$\sinh^2 \frac{1}{2} d_{\xi_{i_0}} z_1 \leq \frac{1}{4} \left| u - \frac{1}{u} \right|^2 \cosh^2 x_0 \leq \frac{1}{2} \left| u - \frac{1}{u} \right|^2 (1 + |bc|). \tag{6.24}$$

Finally, assume that the opposite of the theorem is true. In other words, assume that we have the inequality

$$|\mathrm{trace}^2(\xi_{j_0}) - 4| + |\mathrm{trace}(\xi_{j_0}\xi_{i_0}\xi_{j_0}^{-1}\xi_{j_0}^{-1}) - 2| < 2\sinh^2\left(\frac{1}{4}\log\alpha\right). \tag{6.25}$$

Because we have $d_{\xi_{i_0}\xi_{j_0}\xi_{i_0}^{-1}} z_2 \leq d_{\xi_{i_0}\xi_{j_0}\xi_{i_0}^{-1}} z_1$ and $d_\gamma z_2 < \frac{1}{2}\log\alpha$ for every $\gamma \in \Phi$ by the hypothesis, we get $d_\gamma z_2 < \frac{1}{2}\log\alpha$ for all $\gamma \in \Gamma$ by (6.24) and Lemma 6.2. This is a contradiction to Theorem 6.3. $\square$

7. CONCLUSIONS

The work done here gives an even finer criteria for discreteness of Klenian groups. However, the main body of work is the invention of the new notational system used in the description of group theoretical relations and in calculation of the displacement functions. Furthermore the application of convex optimization techniques such as the Karush-Kuhn-Tucker theorem simplifies the process much further than the efforts shown in [10]. This level of simplification encourages the use of even larger collections for Ψ which in turn would yield an even finer result. So, in future studies, one not only has the opportunity to apply the criteria obtained here to more geometrical examples but can also adjust the result and generalize them even further to fit a specific purpose.

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