T.C.
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INSTITUTE OF GRADUATE STUDIES IN SCIENCE AND ENGINEERING

Ph.D. THESIS

ON P- ARY LOCAL FROBENIUS RINGS AND THEIR HOMOGENEOUS WEIGHTS

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FOREWORD

First of all, I would like to express my heartfelt gratitude and appreciation to my supervisor, Assoc. Prof. Dr. Utku YILMAZTÜRK for the invaluable help, guidance, useful comments, suggestion and constructive criticisms throughout this study. I am also grateful to my committee members for their kind assistance and advice throughout the preparation of this study.

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LIST OF SYMBOLS AND ABBREVIATIONS

Symbol	Explanation
Rad ($\mathcal{R}$)	: Jacobson radical of the ring $\mathcal{R}$
Soc ($\mathcal{R}$)	: Socle of the ring $\mathcal{R}$.
$\mathbf{d_H}$	: Hamming distance.
$\boldsymbol{W_H}$	: Hamming weight.
$\boldsymbol{W_l}$	: Lee weight.
$\boldsymbol{W_{hom}}$	: Homogeneous weight.

ÖZET

DOKTORA TEZİ

P-Li LOKAL FROBENİOUS HALKALAR VE ONLARIN HOMOJEN AĞIRLIKLARI ÜZERİNE

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1990'lardan itibaren, cebirsel kodlama teorisindeki yeni rolünden ve önemli uygulamalarından dolayı, sonlu halkalar üzerindeki kodlara oldukça ilgi gösterildi. Bu halkalardan biri de Frobenius halkalarıdır. Bu çalışmada, p-li lokal Frobenius halkalarının çeşitli özellikleri ve üreteç karakterleri karakterize edildi.

Üreteç karakteri kullanılarak, bu tip halkaların homojen ağırlıkları için genel bir form bulundu. Özel olarak, homojen ağırlığın sıfırdan farklı iki değeri olduğu ispatlandı. Ayrıca, p-li yerel Frobenius halkalarının bazı sınıfları için, uzaklık koruyan lineer Gray dönüşümleri bulundu, ve bu Gray görüntü kullanılarak Griesmer sınırına erişen birçok lineer p-li kod inşa edildi.

Ekim 2017, 60 sayfa.

Anahtar kelimeler: Lokal Frobenius halkalar, Griesmer sınırı, Homojen ağırlık, genelleştirilmiş Reed-Muller kodlar.

SUMMARY

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Institute of Graduate Studies in Science and Engineering

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A great deal of attention has been given to codes over finite rings from the 1990s because of their new role in algebraic coding theory and their great application. One of the rings is called Frobenius ring. In this work, we characterize different properties of p-ary local Frobenius rings and their generating characters.

We find a general form for the homogeneous weights of such rings, using the generating character. In particular, we prove that the homogeneous weight has two non-zero values. We also find distance preserving, linear Gray maps for some classes of p-ary local Frobenius rings and using the Gray image, we construct many linear p-ary codes that attain the Griesmer bound.

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Keywords: Local Frobenius rings, Griesmer bound, homogeneous weight, Generalized, Reed-Muller codes.

1. INTRODUCTION

This Ph.D. thesis depends on the paper B.YILDIZ, M. ABDLJABBAR,"On p–ary Local Frobenius Rings and their Homogeneous weights, Appl. Math. Inf. Sci. 10, No. 6, 2109- 2116, (2016), http://dx.doi.org/10.18576/amis/100614.

Communication channels are susceptible to some amount of noise, which can lead to errors and erasures in the received data. It is important to be able to detect and correct these errors. This is especially relevant when small errors in the data can have a large effect, as when the data is highly compressed or encrypted. We achieve this by transmitting the data in the form of a code. Here are several important applications for the optimal code that can be listed as Error detection, Error correction, Communications channel, Cryptography, Image Compression, Combinatorial, Geometric and Numerical aspects. So we can say, finite fields have become increasingly important in cryptography. A number of cryptographic algorithms rely heavily on properties of finite fields. Finding optimal codes is one of the central problems in coding theory. The Griesmer bound which introduced in (Griesmer[13]), gives us an upper bound for the minimum distance of a linear code over finite fields. The codes for which the bound is attained are called optimal codes. For a linear [n, k,d] code of length n, dimension k, and minimum distance d over $\mathcal{F}_{q,}$ the field with q elements , the bound is given by

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil \qquad (1.a)$$

Where $\lceil x \rceil$denotes the smallest integer greater than or equal to x. In this work, we are going to be focusing on linear codes over $\mathcal{F}_{p}$. We will be constructing different classes of linear codes that are optimal in many of the special cases, i.e. they attain the Griesmer bound. In (Hamada [16]), (Aydin and Siap[5]), (Chen[8]) and (Bouyuklie and Simoins[7]), one can find different constructions or characterizations for optimal codes. Various constructions have been suggested for best-known codes with specific parameters in (Grassl[12]). Our construction will be a novel one in that we will construct our codes as the Gray images of linear codes over the ring $\mathcal{M}_{k}$, which is a nonfinite-chain ring of characteristic p, which mean that it is has ideals not included with other, for example when k=2 the ideals $r_1\mathcal{M}_2$ and $r_2\mathcal{M}_2$ are not included each other. In addition to that codes over rings have been an important field of research in algebraic coding

theory. In the last two decades, many works related to codes over different rings and their applications have appeared. An important work that in a sense defines this field is the work done by (Wood[39]) in which he described Frobenius rings and argued that they are the largest class of rings over which the MacWilliams identities and extensions work. Almost all the rings that have been studied recently in the context of codes have been Frobenius rings.

The homogeneous weight is an alternative to the Hamming weight that is defined over finite rings. While first introduced in (Constantinescu and Heise[10]), they were explicitly described in (Voloch and Walkar[36]) for any ring. They are related to such algebraic objects as exponential sums as was shown in (Voloch and Walker[36]). Different characterizations for homogeneous weights were suggested using different tools such as the Möbius function. However, in (Honold[20]) it was shown that all Frobenius rings are endowed with a homogeneous weight and an explicit characterization, using the generating character of the ring was given.

In this work, we focus on a special class of Frobenius rings, namely the p-ary local Frobenius rings. These are local Frobenius rings whose residue field is isomorphic to the basic prime field $\mathcal{F}_p \cong \mathbb{Z}_p$. We characterize these rings in detail, obtaining many of their properties and give many of the studied rings in coding theory as special examples. We then give a generating character for these rings, after which we prove that the homogeneous weight for all p-ary local Frobenius rings consists of two non-zero weights. We also define a distance preserving isometry for certain special cases and use the map to construct many Griesmer-optimal codes over several prime fields.

1.1. A PREVIEW OF SOME BASIC CONCEPTS IN CODING THEORY

We start with some required concepts about coding theory which refer to (Harold [14]), (Honold[20]), (Hou[21]), (Ling and Xing[25]) and (MacWilliams, Sloans[26]). Let $\mathcal{F}_p^n$ denote the vector space of all n-tuples over the finite field $\mathcal{F}_p$ of length n where p is prime . An (n, M) code C over $\mathcal{F}_p$ is a subset of $\mathcal{F}_p{}^n$ of size M, that is M = |C|, the number of all codewords of C. A code whose alphabet is $\mathbb{Z}_2 = \mathcal{F}_2 = \{0, 1\}$ is called a binary code, a code whose alphabet is $\mathbb{Z}_3 = \mathcal{F}_3 = \{0, 1, 2\}$ is called a ternary code, and a code whose alphabet is $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ is called quaternary code.

The Hamming Distance, $d_H(\bar{x}, \bar{y})$ between $\bar{x} = (x_1, x_2, \ldots, x_n)$ and $\bar{y} = (y_1, y_2, .., y_n)$ in C is given by,

$$d_H(\bar{x}, \bar{y}) = |\{ i : 1 \leq i \leq n, \ x_i \neq y_i \}| \quad (1.1.a)$$

in other words, the number of coordinates where $\bar{x}$ and $\bar{y}$ differ. The minimum distance of a non-trivial code C is given by,

$$d_H(C) = \min \{ d_H(\bar{x}, \bar{y}) : \bar{x} \in C, \bar{y} \in C, \bar{x} \neq \bar{y} \} . \quad (1.1.b)$$

The Hamming Weight $w_H(\bar{x})$ is the number of non-zero coordinates in $\bar{x}$. Clearly we have

$w_H(\bar{x}) = d_H(\bar{x}, 0)$. The minimum weight of a code C, denoted by $w_H(C)$ is defined as,

$$w_H(c) = \min\{ w_H(\bar{x}) : \bar{x} \in C, \bar{x} \neq 0 \} \quad (1.1.c)$$

A linear code C of length n is any linear subspace of the vector space $\mathcal{F}_p^n$, if C has dimension *k* and minimum distance *d*, we say that C is of parameter [n, *k*, *d*]. Note that a q-ary [n, M, *d*] code has cardinality M, while a q-ary [n, *k*, *d*] code is linear and has cardinality q^k.

The minimum distance of a linear code C is equal to the minimum weight of C. We can see this by the observation that if $\bar{x}, \bar{y} \in C$, then $(\bar{x} - \bar{y}) \in C$, and we have, $d(\bar{x}, \bar{y}) = d(\bar{x} - \bar{y}, 0) = w(\bar{x} - \bar{y})$. So, to determine the minimum distance of a linear code, we need just find the weight of all M -1 non-zero code words.

Definition 1.1.1. Distance preserving mapping can be defined as, let $M_1 = (A_1, d_1)$ and $M_2 = (A_2, d_2)$ be metric space. Let $\phi: M_1 \longrightarrow M_2$ be a mapping such that $\forall\, a, b \in M_1 : d_1(a, b) = d_2(\phi(a), \phi(b))$, then is called a distance preserving mapping.

The Griesmer Bound is one of the many bounds that relate to the parameters of codes, named after James Hugo Griesmer which can be defined as follows:

When C is a linear code over $\mathcal{F}_p$ with parameters [n, k, d], the Griesmer Bound is

$$n \geq \sum_{i=1}^{k-1} \left\lceil \frac{d}{p^i} \right\rceil \quad (1.1.d)$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. We will say C is a Griesmer-optimal code if C attains the Griesmer bound.

1.1.1. Overview of The Dissertation

The first section of our thesis is the Introduction, which it includes the following subsections: In the first subsection 1.1 we mentioned about some basic concepts of coding theory.

In subsection 1.2, we discuss the structural properties of p-ary local Frobenius rings and their examples, and describe the generating character for local Frobenius rings and then we introduce the structure of the ring $\mathcal{R}_{k,(p)} = \mathcal{M}_k$.

In subsection 1.3, using Honold's characterization of the homogeneous weight with the generating character, we find a form for the homogeneous weights of p-ary local Frobenius rings. In particular, we prove that all such weights have two non-zero values. We discuss the possible values for the average weight parameter λ that would allow us to define a distance preserving isometry. Using Generalized Reed-Muller codes, we find a linear map for certain examples of the rings.

In subsection 1.4, we construct many optimal p-ary linear codes that attain the Griesmer bound from the images of codes over p-ary local Frobenius rings. And in the last subsection 1.5, we summarize some theorems and definitions about general Frobenius rings.

The second section of thesis is materials and methods, the third section is results, the fourth section is discussion of our work and the last section includ the conclusion and recommendations.

1.2. FINITE LOCAL FROBENIUS RINGS

We will refer to some background material on finite local Frobenius rings and will study the properties and homogeneous weight of the general local Frobenius rings in subsection 1.5. The goal of this section is to characterize different properties of p-ary local Frobenius rings.

Let $\mathcal{R}$ be a finite commutative ring . If $\mathcal{R}$ is local which mean it has a unique maximal ideal, so a local ring is a commutative ring with an Identity element in which:

1) The set M of all non-unite is an ideal in $\mathcal{R}$.

2) Every ideal in $\mathcal{R}$ has finite basis.

By rad($\mathcal{R}$) we denote to the Jacobson radical of $\mathcal{R}$, which is the intersection of all maximal ideals of $\mathcal{R}$. Also, the socle of $\mathcal{R}$ is the sum of all simple ideals of $\mathcal{R}$ and will be referred to as

soc($\mathcal{R}$). In local Frobenius ring, rad($\mathcal{R}$) is the unique maximal ideal of $\mathcal{R}$, then soc($\mathcal{R}$) is the unique minimal ideal. In this case, $\mathcal{R}$/rad($\mathcal{R}$) is called the residue field of $\mathcal{R}$. From (Honold[20], Theorem 2) a finite ring $\mathcal{R}$ is Frobenius ring if and only if $\mathcal{R}$ /rad($\mathcal{R}$) ≅ soc($\mathcal{R}$). We define the residue field $\mathcal{F}_P \cong \mathcal{R}/\mathrm{M}$, p is a prime and M the maximal ideal of $\mathcal{R}$.

Defintion 1.2.1 Let A be a right module over a ring $\mathcal{R}$, given any subset X ⊆ A . The annihilator of X is the set, a_{nn} (X)={r∈ $\mathcal{R}$: xr = 0, ∀ x ∈ $\mathcal{R}$ } which is the right ideal of $\mathcal{R}$.

Remark 1.2.1 From [39, Theorem1.2 and Remark 1.3], if $\mathcal{R}$ is a finite commutative ring, the following conditions are equivalent:

(1) $\mathcal{R}$ is Frobenius,

(2) $\mathcal{R}$ is quasi-Frobenius,

(3) The $\mathcal{R}$ -module $\mathcal{R}$ is injective.

If $\mathcal{R}$ is a finite local ring with maximal ideal M and residue field *k*, these conditions are equivalent with

(4) $\dim_k A_{nn}(M) = 1$.

Example: Let us give examples of a local Frobenius ring over $\mathbb{Z}_4$.

Let $\mathcal{R} = \mathbb{Z}_4[x]/\prec x^2\succ = \{ a+b\,x +\prec x^2\succ : a,b \in \mathbb{Z}_4\}$, it is a finite local Frobenius ring with maximal ideal M. Let us now define and listed the elements of $\mathcal{R}$:

Let $x +\prec x^2\succ = u$, then $u^2=0$ and $\mathcal{R}= \{ a+b\,u : a,b \in \mathbb{Z}_4\}$, so the elememts of $\mathcal{R}$ = {0, 1, 2, 3, u, 2 u, 3 u, 1+ u, 3+ u, 2+ u, 1+2 u, 2+2 u, 3+2 u, 1+3 u, 2+3 u, 3+3 u}, then if we compute the square of the elements of $\mathcal{R}$ we get as:

$1^2 = 3^2 = (1 + 2\,u)^2 = (3 + 2\,u)^2 = 1.$

$(1 + u)^2 = (1 + 3\,u)^2 = (3 + u)^2 = (3 + 3\,u)^2 = 1 + 2\,x.$

$0^2 = 2^2 = u^2 = (2\,u)^2 = (3\,u)^2 = (2 + u)^2 = (2 + 2\,u)^2 = (2 + 3\,u)^2 = 0.$

Since according to about computation the square of the elements of the set {0, 2, u, 2 u, 3 u, 2+ u, 2+2 u, 2+3u } are all zero implies they are non-uints. So, the elements of all non-units in $\mathcal{R}$ are M={0, 2, u, 2 u, 3 u, 2 + u, 2 + 2 u, 2 + 3 u}= $\prec 2, u \succ$ such that $\prec 2, u \succ = \prec 2\succ + \prec u\succ$ = {2a + b u : a; b ∈$\mathbb{Z}_4$ }. From Remark 1.2.1, A_{nn}(M) ={0, 2u}, $dim_k(A_{nn}(\mathrm{M}))$=1, so M is the unique maximal ideal. Its clear that the residue field is $\mathcal{R}/\mathrm{M} \cong \mathcal{F}_2$.

We can define $\mathcal{R}$ as $\mathcal{R}=\mathbb{Z}_4+u\mathbb{Z}_4$ with $u^2=0$ which is studied by Yildiz and Karadeniz in [42]. As a generalization of this ring, we can introduce $\mathcal{T}_k = \mathbb{Z}_4[u] /\prec u^k \succ$ as an example of a family of rings that are local Frobenius ring. We can study a code over $\mathcal{T}_k$ when k=3 as following:

Let $\mathcal{T}_3=\mathbb{Z}_4 + u\,\mathbb{Z}_4 + u^2\,\mathbb{Z}_4= \{a + bu + cu^2 : u^3=0 \text{ and } a, b, c \in \mathbb{Z}_4\}$, with $|\mathcal{T}_3| = 4^3 = 64$ and its has 9 ideals. We can define these ideals as following:

Let $\mathcal{T}_3 = \mathbb{Z}_4 + u\mathbb{Z}_4 + u^2\,\mathbb{Z}_4$, so,

$I_{2u^2} = 2u^2(\mathcal{T}_3) = 2u^2\,\mathbb{Z}_4$,

$I_{u^2} = u^2(\mathcal{T}_3) = u^2\mathbb{Z}_4$,

$I_2 = 2(\mathcal{T}_3) = 2\mathbb{Z}_4 + 2u\mathbb{Z}_4 + 2u^2\,\mathbb{Z}_4$,

$I_u = u(\mathcal{T}_3) = u\,\mathbb{Z}_4 + u^2\,\mathbb{Z}_4$.

The elements of ideals are listed as following:

$I_1 = \{0\}$.

$I_{2u^2} = \{0, 2u^2\}$.

$I_{u^2} = \{0, u^2, 2u^2, 3u^2\}$.

$I_2 = \{0, 2, 2u, 2u^2, 2+2u, 2+2u^2, 2u+2u^2, 2+2u+2u^2\}$.

$I_u = \{0, 2, 2u, 3u, u^2, 2u^2, 3u^2, u+u^2, 2u+u2, 3u+u^2, u+3u^2, 2u+2u^2, 2u+3u^2, 3u+2u^2, u+2u^2, 3u+3u^2\}$.

$I_{2+u} = \{0, 2+u, 2u, u^2, 3u^2, 2u+u^2, 2+3u, 2u^2, 2u+3u^2, 2+3u+u^2, 2+u+2u^2, 2u+2u^2, 2+3u+3u^2, 2+2u+2u^2, 2+3u+2u^2, 2+u+3u^2, 2+u+u^2, 2+2u+3u^2, 2+u+2u^2\}$.

$I_{2+u^2} = \{0, 2u, 2u^2, 2+u^2, 2+3u^2, 2+2u+u^2, 2u+2u^2, 2+2u+3u^2, 2+u+u^2\}$.

$I_{\prec 2,u \succ} = \{0, 2, u, 2u, 3u, u^2, 2u^2, 3u^2, 2+u, 2+u^2, u+u^2, 2u+u^2, 3u+u^2, u+2u^2, u+3u^2, 2u+3u^2, 2+2u, 2+3u, 2u+2u^2, 2+3u^2, 3u+3u^2, 2+2u^2, 3u+2u^2, 2+3u+u^2, 2+u+3u^2, 2+u+u^2, 2+3u+3u^2, 2+2u+u^2, 2+2u+2u^2, 2+u+2u^2, 2+3u+2u^2, 2+3u+3u^2\}$.

We compute that $I_{\prec 2,u \succ}$ is the unique maximal ideal, its clear that $A_{nn}(I_{\prec 2,u \succ}) = \{0, 2u^2\}$, $dim_k(A_{nn})=1$, since it's not generated by a single element, so it's not principal local ring and $I_{2u^2} = \{0, 2u^2\}$ is the unique minimal ideal. The residue field of this ring is $T_3/I_{\prec 2,u \succ} \cong \mathcal{F}_2$. Finally, the set of units of $\mathcal{T}_3$ is $\{a+bu+cu^2$, if a+bu are unit in $\mathbb{Z}_4+u\mathbb{Z}_4$, a; b; c $\in \mathbb{Z}_4\}$. So the

units are {1, 3, 1+u, 1+2u, 1+3u, 3+u, 3+2u, 3+3u, $1+u+u^2$, $1+u+2u^2$, $1+u+3u^2$, $1+2u+u^2$, $1+2u+2u^2$, $1+2u+3u^2$, $1+3u+u^2$, $1+3u+2u^2$, $1+3u+3u^2$, $3+u+u^2$, $3+u+2u^2$, $3+u+3u^2$, $3+2u+u^2$, $3+2u+2u^2$, $3+2u+3u^2$, $3+3u+u^2$, $3+3u+2u^2$, $3+3u+3u^2$, $1+u^2$, $1+2u^2$, $1+3u^2$, $3+u^2$, $3+2u^2$, $3+3u^2$}.

The lattice of ideals of the ring is given by the following diogram

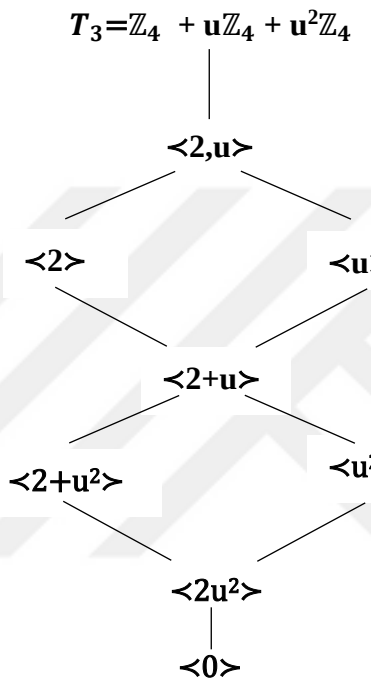

Figure 1.1: The ideal lattice of the ring $\mathcal{R}_3$

We can classify local Frobenius rings according to its characteristic, which will be described with the following section.

1.2.1. P-Ary Local Frobenius Rings And Their Properties

We start with a finite commutative ring $\mathcal{R}$ that is a local Frobenius ring. This means that there is a unique maximal ideal M and we further assume that $\mathcal{R}/\mathrm{M} \cong \mathcal{F}_p$. We call any such ring a p-ary local Frobenius ring. The following theorem will give some of the structural properties of all local p-ary local Frobenius rings.

Theorem 1.2.1.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring. Then

a) $\mathcal{R}$ has a unique minimal ideal m.

b) There is a $\in\mathcal{R}$ such that m = {0, a,…, (p - 1)a}.

c) If I is any non-zero ideal in $\mathcal{R}$, then $m \subseteq$ I.

d) We have $a^2 = 0$.

e) $x.a = \begin{cases} j\,.a\,, \; j = 1,2,\dots,p-1 & if\ x\ is\ a\ unit \\ 0 & if\ x\ is\ non-unit \end{cases}$.

Proof: This theorem will be proved in the result section as **Theorem 3.1.1**.

Theorem 1.2.2.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring. Then $|\mathcal{R}| = p^m$ for some suitable integer m.

Proof: This theorem will be proved in the result section as **Theorem 3.1.2**.
Many examples of rings that have been studied of lately in coding theory are of this form. We will distinguish the case when the characteristic is p and when it is not.

1.2.2. Examples

In this section would like to give some examples of p-ary local Frobenius rings, many of these examples are familiar rings in the context of coding theory in recent years. We will consider two separate cases in terms of the characteristic of the ring.

1.2.2.1. Characteristic p Rings

All these rings will have $\mathbb{Z}_p$ as a subring. In fact, it is easy to see that they will have the further structural property of being vector spaces over $\mathbb{Z}_p$. We give some examples below:

■ The finite chain rings of the form $\mathcal{F}_p[u]/\prec u^k \succ$.

■ $\mathcal{R}_2 = \mathcal{F}_2 + u\mathcal{F}_2 + v\mathcal{F}_2 + uv\mathcal{F}_2$ introduced first by Yildis and Karadeniz in [41], which has been extensively studied from many different aspects related to codes. $\mathcal{R}_2$ of characteristic 2 and its easy to see that is local Frobenius ring that is not finite chain principal because the ideals not included, which are described as,

$\{0\} \subseteq \prec uv \succ = \{0, uv\} \subseteq \prec u \succ , \prec v \succ , \prec u+v \succ \subseteq \prec u; v \succ \subseteq \mathcal{R}_2$ where,

$\prec u \succ = \{0,\ u,\ uv,\ u+uv\}$,

$\prec v \succ = \{0,\ u,\ uv, u+uv\}$,

$\prec u+v\succ = \{0; u+v; uv; u+v+uv\}$,

$\prec u, v\succ = \{0, u, v, u+v, uv, u + uv, u + v + uv\}$ is the unique maximal ideal .

Another observation about this ring if a $\in \mathcal{R}_2$, then,

$$a^2 = \begin{cases} 1 & if\ a\ is\ unit \\ 0 & otherwise \end{cases}$$

■ $\mathcal{R}_k = \mathcal{F}_2[u_1, u_2, \ldots, u_k] /(u^2_i = 0, u_iu_j = u_ju_i)$, which is an extension of $\mathcal{R}_2=\mathcal{F}_2 + u\mathcal{F}_2 + v\mathcal{F}_2 + uv\mathcal{F}_2$, that first appeared in (Steven T. et[33]), and has since been studied extensively, which is a finite commutative local Frobenius ring with characteristic 2 and maximal ideal $\prec u_1, u_2, \ldots, u_k\succ$, and it's neither a principal ring nor a chain ring when $k \geq 2$.

■ $\mathcal{R}_{k,m} = \mathcal{F}_2[u; v] /(u^k, v^m, uv = vu)$, studied by Tüfekci and Yildiz in [35] .

■ Another extension of $\mathcal{R}_k$ can be achieved by replacing $\mathcal{F}_2$ by any basic finite field $\mathcal{F}_p$. In other words, we consider rings of the form $\mathcal{R} = \mathcal{F}_p[u_1, u_2, \ldots, u_k] /(u^2_i = 0, u_iu_j = u_ju_i)$.

We denote this family of rings by $\mathcal{R}_{k,(p)}$The structural properties of these rings being exactly the same as $\mathcal{R}_k$.

1.2.2.2. Rings of non-prime Characteristic

The typical examples of these rings involve $\mathbb{Z}_p^m$ for some m > 1. Thus the following list can be given as a familiar list of rings that fall into this category:

■ $\mathbb{Z}_4$ is the first main example of such rings, and there is an extensive literature on these rings.

■ $\mathbb{Z}_{2^m}$ and $\mathbb{Z}_p^m$.

■ $\mathbb{Z}_4 + u\mathbb{Z}_4 = \mathbb{Z}_4[u]/\prec u^2\succ$, which was first studied byYildiz and Karadeniz in [42].

■ Different extensions of $\mathbb{Z}_p^m$ such as $\mathbb{Z}_p^m[u]/\prec u^t\succ$ or even such extensions as

$\mathbb{Z}_p^m[u_1, u_2, \ldots, u_k] /\prec u^2_i, u_i u_j = u_j u_i \succ$.

It is worth observing that p-ary local Frobenius rings of characteristic p^m will have a copy of $\mathbb{Z}_p{}^m$ as a subring and they will be of the form $\mathbb{Z}_p{}^m + s\,\mathbb{Z}_p{}^m + t\,\mathbb{Z}_p{}^m + \ldots$ for some suitable s, t.

1.2.3. The Generating Character For Local Frobenius Rings

A generating character is an important tool in finding the homogeneous weight and we know that all Frobenius rings possess a generating character. In this brief section, we review the essential ideas and properties of characters of the finite abelian group and of Frobenius rings.

Definition 1.2.2. Let G be a finite abelian group under addition. A character group of G is denoted by $\hat{G}$, is the set of Hom(G, $\mathbb{C}^*$) of all group homorphism from G to the multiplictive group $\mathbb{C}^*$ of the field $\mathbb{C}$,with respect to the addition, define by,

$(\chi_1 + \chi_2)(g) = \chi_1(g).\ \chi_2(g)$, $\forall g \in G$ and any $\chi_1,\ \chi_2 \in \hat{G}$.

The zero element is the principal character $\zeta \in \hat{G}$ given by $\zeta(a) = 1\ \forall\ a \in G$.

It's known that G and $\hat{G}$ are isomorphic. Now if $\mathcal{R}$ is a finite commutative rings with unity and let $\hat{\mathcal{R}}$ be the character group of the group $(\mathcal{R}, +)$, then $\hat{\mathcal{R}}$ can became a $\mathcal{R}$-module, if we define the module multiplicative as,

$\mathcal{R} \times \hat{\mathcal{R}} \longrightarrow \hat{\mathcal{R}}$, $(r, \chi) \longrightarrow \chi^r$ where $\chi^r : (\mathcal{R}, +) \longrightarrow (\mathbb{C}^*, \cdot)$. For $r \in \mathcal{R}$ and $\chi \in \hat{\mathcal{R}}$ define the character $\chi \mapsto \chi^r(x) = \chi(rx)$.

Definition 1.2.3 Let $\mathcal{R}$ be a finite commutative ring with unity. A character χ of $\mathcal{R}$ is generating character if the mapping $\phi: \mathcal{R} \longrightarrow \hat{\mathcal{R}}$, $\phi(r) = \chi^r$ is an isomorphism $\mathcal{R}$- modules.

We can see the fundamental properties of character by the following lemmas:

Lemma 1.2.1.[14] Let $\zeta \in \hat{\mathcal{R}}$, then

$$\sum_{a \in G} \chi(a) = \begin{cases} 0 & if\ \chi \neq \zeta \\ |\mathcal{R}| & if\ \chi = \zeta. \end{cases}$$

Proof: The second case is clear. Now for the first case, there exist $a_0 \in \mathcal{R}$ with $\chi(a_0) \neq 1$, we then re-index the summation

$$\sum_{a \in G} \chi(a) = \sum_{a \in G} \chi(a_0 + a) = \chi(a_0) \sum_{a \in G} \chi(a)$$

Since $\chi(a_0) \neq 1$, the sum be zero.■

Lemma 1.2.2.[39, Lemma 4.1] Let χ be a character of a finite ring $\mathcal{R}$, then χ is a generating character if and only if kerχ contains no non-zero ideal.

Proof: Because $|\widehat{\mathcal{R}}| = |\mathcal{R}|$, $\phi : \mathcal{R} \to \widehat{\mathcal{R}}$, defined by $r \to \chi^r$, is an isomorphism if and only if ϕ is injective, now $r \in \ker\phi$ if and only if, $\chi^r(x) = \chi(rx) = 1$, for all $x \in \mathcal{R}$ .i.e. if and only if the ideal $r\mathcal{R} \subset \ker\chi$.■

1.2.3.1. The Generating Character for p-ary Local Frobenius Rings

When defining the generating character for p-ary local Frobenius rings. We need to consider two separate cases depending on the characteristic of the ring.

■ Let $\mathcal{R}$ be a p-ary local Frobenius ring of characteristic p with minimal ideal m,

m ={0, a, 2a,..., (p-1)a}. In that case, $\mathcal{R}$ will be a vector space over $\mathbb{Z}_p$ and one of the basis elements will be a. Then we define the character χ as follows: let $r \in \mathcal{R}$, $r \cdot a \in \prec a \succ$, such that $r_a = c_a \cdot a$ for some $c_a \in \mathbb{Z}_p$, the character define as,

$$\chi(r) = e^{\frac{2\pi i c_a}{p}} \quad \text{where } c_a \text{ is the coefficient of a in } r \in \mathcal{R} \qquad (1.2.3.a)$$

Now we need to explian why χ is a generating character for $\mathcal{R}$.

Since $\mathcal{R}$ is a vector space over $\mathbb{Z}_p$, and $m = \prec a \succ$ is a minimal ideal, then there exist a basis B of $\mathcal{R}$ over $\mathbb{Z}_p$ containing a, such that $B = \{r_1, r_2, ..., r_s\}$, $r_1 = a$.

$\forall r \in \mathcal{R}$, we can write it as, $r = x_1r_1 + x_2r_2 + \cdots + x_sr_s$, $x_i \in \mathbb{Z}_p$, such that $\chi(r) = e^{\frac{2\pi i x_1}{p}}$.

And we can define $\ker\chi = \{x_2r_2 + \cdots + x_sr_s \mid x_2, \ldots, x_s \in \mathbb{Z}_p\}$.

We see that $|\ker\chi| = p^{s-1}$ and $|\mathcal{R}| = p^s$, and since $\ker\chi$ doese not contain any non-zero ideal, because otherwise it would contian m, and hence $a = r_1 \in \ker\chi$.

By Lemma 1.2.2. χ will be a generating character.

■ If $\mathcal{R}$ is a p-ary local Frobenius ring of characteristic p^m, then the minimal ideal will be of the form, $m = \{0, p^{m-1}b, 2p^{m-1}b, \ldots, (p-1)p^{m-1}b\}$ for some $b \in \mathcal{R}$.

Every element of $\mathcal{R}$ can be written in the form c_o+c_1 u_1 +…+c_su$_s$ with $c_i \in \mathbb{Z}_p{}^m$, one of the u$_i$s will be b. In that case, we define the character χ on $\mathcal{R}$ as,

$\chi(r) = e^{\frac{2\pi i c_b}{p^m}}$ where c_b is the coefficient of b in r$\in\mathcal{R}$ (1.2.3.b)

For example in $\mathbb{Z}_4$+u$\mathbb{Z}_4$ = $\mathbb{Z}_4$[u]/(u^2), the character can be defined as, $\chi(a+bu) = e^{\frac{2\pi i b}{4}} = (i)^b$.

In $\mathcal{F}_2$ + u$\mathcal{F}_2$ + v$\mathcal{F}_2$ + uv$\mathcal{F}_2$, the character is defined as, $\chi(a + bu + cv + duv) = e^{\frac{2\pi i d}{2}} = (-1)^d$.

Theorem 1.2.3.[46] The characters defined in eq.(1.2.3.a) and eq.(1.2.3.b) are generating characters for a p-ary local Frobenius rings of characteristic p and characteristic p^m, respectively.

Proof: This theorem will be proved in the result section as **Theorem 3.1.3**.

Example 1.2.3.1.1

Here are a number of examples of finite commutative Frobenius rings.

■ Let $\mathcal{R} = \mathcal{F}$ be a finite field. A generating character χ on $\mathcal{R} = \mathcal{F}$ is given by $\chi(x) = \zeta^{trx}$, $x \in \mathcal{F}$ and Tr : $\mathcal{F} \to \mathcal{F}_p$ is the trace map from $\mathcal{F}$ to its prime subfield $\mathcal{F}_p$ where $\zeta = e^{\frac{2\pi i}{p}}$ is a primitve pth root of 1 in T.

■ Let $\mathcal{R} = \mathbb{Z}_l$ be a finite commutative Frobenius ring with unity. Set $\chi = e^{\frac{2\pi i}{l}}$, then $\chi(x) = \zeta^x$, $x \in \mathbb{Z}_l$, is a generating character.

■ The finite direct sum of Frobenius rings is Frobenius. If $\mathcal{R}_1,\ldots,\mathcal{R}_n$ each has generating characters $\chi_1,\ldots,\chi_n$, then $\mathcal{R} = \oplus\mathcal{R}_i$ has generating character $\chi = \Pi\chi_i$.

■ Any Galois ring is Frobenius. A Galois ring $\mathcal{R}$ = G$\mathcal{R}$(p^n; r) $\cong \mathbb{Z}_p{}^n[x]/\prec f \succ$ is a Galois extension of $\mathbb{Z}_p{}^n$ of degree r, where f is a monic irreducible polynomial in $\mathbb{Z}_p{}^n[x]$ of degree r. Because f is monic, any element a of $\mathcal{R}$ is represented by a unique polynomial,

$a = \sum_{i=1}^{r} a_i\, x^{r-i}$ with $a_i \in \mathbb{Z}_p{}^n$. Set $\zeta = e^{\frac{2\pi i}{p^n}}$, then $\chi(a) = \zeta^{a_1}$ is a generating character.

■ Any finite chain ring is Frobenius. Let $\mathcal{R}$ be a finite chain ring with maximal ideal $\prec u \succ$ and let its residue field $\mathcal{R}/\prec u \succ$ be $\mathcal{F}_{p^n}$,

i.e. $\mathcal{R} = \mathcal{F}_q + u\mathcal{F}_q + \ldots + u_k\mathcal{F}_q$ with $q=p^n$ for some prime p. Any element r of $\mathcal{R}$ is represented by a unique polynomial,

$r = \sum_{i=1}^{r-1} a_i\, u^i$ with $a_i \in \mathcal{F}_q$, set $\zeta = e^{\frac{2\pi i}{q}}$, then $\chi_r = \zeta^{a_{r-1}}$.

■ $\mathcal{R} = \mathcal{F}_2[u_1, u_2, \ldots, u_k]/\prec u^2 = 0; u_i u_j = u_j u_i \succ$ is a Frobenius ring. Let

$$r_k = \sum_{A \subseteq \{1,2,\ldots,k\}} c_A u_A \in \mathcal{R},$$

then (c_A) can be thought of as a binary vector of length 2^k. Let $W_H(c_A)$ be the Hamming weight of this vector, Then $\chi(r_k) = (-1)^{W_H(c_A)}$.

Before defining a generating character for the main ring $\mathcal{M}_k$, let us refer to a structure and properties of it to a next section.

1.2.4. The Structure Of The Ring ($\mathcal{M}_k$)

We use the same method in (Steven[33]) when defining the structure of a ring $\mathcal{M}_k$:

$$\mathcal{M}_k = \mathcal{M}_{k-1}[r_k]/\prec r_k^2 = 0, r_k r_j = r_j r_k \succ = \mathcal{M}_{k-1} + r_k\mathcal{M}_{k-1},\ \text{j=1,2,..,k-1}. \qquad (1.2.4.a)$$

Where $\mathcal{M}_0 = \mathcal{F}_p$, and $\mathcal{M}_1 = \mathcal{M}_0 + r_1\mathcal{M}_0$.

The basis element of $\mathcal{M}_k$ can be viewed over $\mathcal{F}_p$- vector space by using subset $A \subseteq \{1, 2, \ldots, k\}$ and letting, $r_A = \prod_{i \in A} r_i$ with $r_\emptyset = 1$, any element of $\mathcal{M}_k$ can be represented as,

$$\sum_{A \subseteq \{1,2,..,k\}} c_A r_A,\ c_A \in \mathcal{F}_p. \qquad (1.2.4.b)$$

From above notation, we will simply observe that,

$$r_A r_B = \begin{cases} 0 & if\ A \cap B \neq 0 \\ r_{A \cup B} & if\ A \cap B = 0. \end{cases} \qquad (1.2.4.c)$$

So from this observation, we can represent the product of two elements as,

$$XY = \left(\sum_{A \subseteq \{1,2,..,k\}} c_A \, r_A \right) \left(\sum_{B \subseteq \{1,2,..,k\}} d_B \, r_B \right) = \sum_{\substack{A,B \subseteq \{1,2,..,k\} \\ A \cap B = \emptyset}} c_A d_B r_A \, r_B,$$

$$= \sum_{Z \subseteq \{1,2,..,K\}} \sum_{A \subseteq Z} c_A d_{Z-A} r_{Z,}$$

where $X = \sum_{A \subseteq \{1,2,..,k\}} c_A r_A$, $Y = \sum_{B \subseteq \{1,2,..,k\}} d_B r_B$.

It is clear that the element $\sum_{A \subseteq \{1,2,..,k\}} c_A \, r_A$ is unit if and only if $c_\emptyset \neq 0$.

The following lemmas summarizes the important structural properties in $\mathcal{M}_k$.

Lemma 1.2.3. The ring $\mathcal{M}_k$ is a commutative ring with $|\mathcal{M}_k| = p^{2^k}$ with p is prime, and k ≥ 2.

Proof: From the formula of elements of $\mathcal{M}_k$ in there are 2^k subsets of A = {1, 2,.., k}, and p choices for each coefficient c_A, so there are p^{2^k} elements in the ring $\mathcal{M}_k$. ■

Lemma 1.2.4. The ring $\mathcal{M}_k$ is a local ring with unique maximal ideal $\prec r_1, r_2, \ldots, r_k \succ$, this ideal consists of all non-units and has the cardinality $|\prec r_1, r_2, \ldots, r_k \succ| = p^{2^k - 1}$.

Proof: We know all non-units are in $\prec r_1, r_2, \ldots, r_k \succ$, those are the elements with $C_\phi = 0$, hence the number of subsets for A = {1, 2,.., k} is 2^k-1, means that there is the p^{2^k-1} cardinality of non-units. ■

We can write the units of this ring by the set $\sum_{A \subseteq \{1,2,..,k\}} c_A r_A$, $c_A \in \mathcal{F}_p$, and $C_\phi \neq 0$. It's clear that every non-unit is a zero-divisor and all zero divisors form an ideal. $\mathcal{M}_k$ is finite chain ring if and only if k = 1. When k ≥ 2 it's not a chain ring.

Let us take the ideal, for example, $\prec r_1 \succ$, $\prec r_2 \succ$ neither ideal contains the other. Moreover, the ring is not principal for k > 1, since the maximal ideal is not generated by a single element.

Lemma 1.2.5. The ideal $\prec r_1 r_{2 \ldots} r_s \succ$ has cardinality $p^{2^{k-s}}$. Moreover the minimal ideal has size p.

Proof: An element in $\prec r_1 r_2 \ldots r_s \succ$ can be written as $\sum_{A\subseteq\{1,2,..,k\}} c_A\, r_A$, $c_A \in \mathcal{F}_p$ where $r_\emptyset = 0$, means every r_A have {1, 2,.., s} ⊆A, so there are 2^{k-s} subsets of {1, 2,.., k} hence the cardinality of $\prec r_1 r_2 \ldots r_s \succ$ is $p^{2^{k-s}}$ for a ring $\mathcal{M}_k$, minimum ideal $\prec r_1 r_2 \ldots r_k \succ$, the cardinality is $p^{2^{k-k}} = p$.■

Theorem 1.2.4. $\mathcal{M}_k$ is a Frobenius ring.

Proof: For a ring $\mathcal{M}_k$, the minimal ideal $\prec r_1 r_2 \ldots r_k \succ$, such that m ={0, $r_1 r_2 \ldots r_k$, $2r_1 r_2 \ldots r_k$,…, (p-1) $r_1 r_2 .. r_k$} it's easy to see that $\mathcal{M}_k /$ rad($\mathcal{M}_k$) ≅ soc($\mathcal{M}_k$).

Hence $\mathcal{M}_k$ is Frobenius ring.■

1.2.4.1. Linear Code Over the Ring $\mathcal{M}_k$

A linear code of length n over the ring $\mathcal{M}_k$ is defined as $\mathcal{M}_k$-submodulo of $\mathcal{M}_k^n$. Define the standard Euclidean inner product on $\mathcal{M}_k$ by,

$\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$ for X =($x_1, x_2, \ldots, x_k$), and Y = ($y_1, y_2, \ldots, y_k$) in $\mathcal{M}_k$.

where the operation is performed in the ring $\mathcal{M}_k$. The duality for code over $\mathcal{M}_k$ then can be defined accordingly.

Definition 1.2.4. Let C be a linear code over $\mathcal{M}_k$ of length n, then the dual can be defined as,

$C^{\perp} = \{ \bar{x} \in \mathcal{M}_k^n : \prec \bar{x}, \bar{y} \succ = 0$, for all $\bar{y} \in C\}$.

Definition 1.2.5. Let C be a linear code over $\mathcal{M}_k$ of length n. C is said to be self-orthogonal if $C \subseteq C^{\perp}$ and self-dual if $C = C^{\perp}$.

Remark 1.2.2. If $\mathcal{R}$ is a finite local ring with maximal ideal M and residue field k, the condition (4) from Remark 1.2.1. gives immediately that the code M of length 1 satisfies $|M|\,|M^{\perp}| = |\mathcal{R}|$, since this relation is equivalent to $|A_{nn}(M)| = |k|$.

If $\mathcal{R}$ is not Frobenius, then $|M|\,|M^{\perp}|$ is not necessarily equal to $|\mathcal{R}|$ even if M is a self-dual code.

For example $\mathcal{R} = \mathcal{F}_2[u_1, u_2]/(u^2{}_1, u^2{}_2, u_1u_2) = \mathcal{F}_2[u_1, u_2]$ where $u^2{}_1 = u^2{}_2 = u_1u_2 = 0$.

The element of this ring are $\{0, 1, u_1, u_2, 1+u_1, 1+u_2, u_1+u_2, 1+u_1+u_2\}$ with maximal ideal M=$\{0, u_1, u_2, u_1+u_2\}$, the $dim_k A_{nn}(M) = 2$. From the same condition giving that the ring is not Frobenius.

We have $M^{\perp} = M = \{0, u_1, u_2, u_1+u_2\}$, so is self dual code of length 1. But $|M|\,|M^{\perp}| \neq |\mathcal{R}|$.

Lemma 1.2.6. For a,b ∈ $\mathcal{M}_k$, χ (a+b) = χ(a) · χ(b).

Proof: Let,

$$a = \sum_{A\subseteq\{1,2,..k\}} c_A r_A \ , \quad b = \sum_{B\subseteq\{1,2,..,k\}} c_B r_B$$

$$\chi(a+b) = \chi\left(\sum_{A\subseteq\{1,2,..k\}} c_A r_A + \sum_{B\subseteq\{1,2,..,k\}} c_B r_B\right)$$

$$= e^{\frac{2\pi i}{p}(d_1+d_2)}$$

where d_1 the coefficient of r_A in element a and d_2 the coefficient of r_B in element b, A={1,2,..,k} .We have:

$$\chi(a+b) = e^{\frac{2\pi i}{p}(d_1)+\frac{2\pi i}{p}(d_2)}$$

$$= \chi\left(\sum_{A\subseteq\{1,2,..k\}} c_A r_A\right)\cdot\chi\left(\sum_{B\subseteq\{1,2,..k\}} c_B r_B\right)$$

= χ(a) ·χ(b). ■

Theorem 1.2.5. Let χ be the character for $\mathcal{M}_k$ defined as

$\chi(\sum_{A\subseteq K} c_A r_A) = e^{\frac{2\pi i c_k}{p}}, K = \{1,2,\dots,k\}$. Then χ is a generating character of $\mathcal{M}_k$.

Proof: Let a,b ∈ $\mathcal{M}_k$, such that,

$$a = \sum_{A\subseteq K} c_A r_A \ , \ \ b = \sum_{A\subseteq K} d_A r_A$$

$$\chi\left(\sum_{A\subseteq K} c_A r_A + \sum_{A\subseteq K} d_A r_A\right) = \chi\left(\sum_{A\subseteq K} (c_A + d_A) r_A\right) = e^{\frac{2\pi i(c_k+d_k)}{p}} = e^{\frac{2\pi i c_k}{p}} . e^{\frac{2\pi i d_k}{p}}$$

$$= \chi\left(\sum_{A\subseteq K} c_A r_A\right) . \chi\left(\sum_{A\subseteq K} d_A r_A\right).$$

Hence χ is character. Now we need to show that this character is a generating character.

$$\chi = \left\{\sum_{A\subseteq K} c_A r_A \mid e^{\frac{2\pi i c_k}{p}} = 1\right\} = \left\{\sum_{A\subseteq K} c_A r_A \mid c_K = 0\right\},$$

from this we can get the following,

$\ker\chi \cap I_K = \{0\}$, where I_K is the unique minimal ideal, hence $\ker\chi$ does not contain any non-zero ideal, Therefore it is a generating character. ■

Lemma 1.2.7. Let χ be the generating character, such that $\chi\ (r_1\, r_2 \ldots r_k) = e^{\frac{2\pi i}{p}}$, then, $\sum_{x\in I}\chi(x) = 0$ for all non zero ideal I of $\mathcal{M}_k$.

Proof: Every element of the ideal I can be written in a unique way in the form $x = y + jr_k$ for some $y \in \ker\chi$ and j ={0,1,...,p-1} which is equal to $\mathbb{Z}_p$. Then

$$\sum_{x\in I}\chi(x) = \sum_{j=0}^{p-1}\sum_{y\in\ker\chi}\chi(y + jr_k)$$

$$= \sum_{j=0}^{p-1}\sum_{y\in\ker\chi}\chi(jr_k) = \sum_{j=0}|\ker\chi| . e^{\frac{2\pi ij}{p}}, where\ \zeta = e^{\frac{2\pi ij}{p}}$$

$$= |\ker\chi|\ (1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1})$$

$$= |\ker\chi|\ \frac{1-\zeta^p}{1-\zeta} = 0, since\ \zeta^p = 1.\blacksquare$$

Lemma 1.2.8. For all y ∈ $\mathcal{M}_k$, we have,

$$\sum_{y_i \in \mathcal{M}_k} \chi(y.y_i) = \begin{cases} p^{2^k} & if\ y = 0 \\ 0 & otherwise. \end{cases}$$

Proof: If y=0, then $\chi(yy_i) = 1$, for all $y_i \in \mathcal{M}_k$, $\sum_{y_i \in \mathcal{M}_k} 1 = |\mathcal{M}_k| = p^{2^k}$. If y ≠ 0, then { yy_i: $y_i \in \mathcal{M}_k$} form a nontrivial ideal of $\mathcal{M}_k$, and hence by previous Lemma the result is zero. ■

1.3. THE HOMOGENEOUS WEIGHT AND THEIR GRAY MAP FOR LOCAL FROBENIUS RINGS

Homogeneous weights have been introduced by Constantinescu and Heise in [10] as generalization of the Hamming weight for finite field and the Lee weight on the residue ring of integer module 4. In this subsection we will study the homogeneous weight over ring $\mathcal{R}_{k,(p)} = \mathcal{M}_k$ for any prime p. For all Frobenius rings, the homogeneous weight can be expresed in terms of a generating character χ. Let $\mathcal{M}^*$ denote the group of units of $\mathcal{M}_k$, where $I_{r_1,r_2,\ldots,r_k}$ is the non-principal maximal ideal containing all the zero divisors in $\mathcal{M}$, with a cardinality

$$\mathcal{M}^* = |\mathcal{M}_k| - |I_{r_1,r_2,\ldots,r_k}| = p^{2^k} - p^{2^k-1} = p^{2^k-1}(p-1). \quad (1.3.a)$$

The homogeneous weight is defined with two conditions for arbitrary finite rings as in (Greferath and Osullivan[27]).

Definition 1.3.1. A map w: $\mathcal{R} \to \mathbb{R}$ is called (a left) homogeneous weight if w(0) = 0, and the following two conditions hold:

a) For x, y ∈$\mathcal{R}$ if $\mathcal{R}\,x = \mathcal{R}\,y$, then w($x$) = w($y$).

b) There is a non negative λ ∈ ℝ such that for any non-zero x we have $\sum_{y \in \mathcal{R}x} w(y) = \lambda|\mathcal{R}x|$.

Similarly, we can define right homogenous weight. The condition(b) implies that the average of the homogeneous weight on nonzero principal ideal is λ which is independent of the choice of the principal ideal. It is easy to see that the Hamming weight on $\mathcal{R}$ is homogeneous if and only if $\mathcal{R}$ is a field. When the average λ=1, which we call the normalized homogeneous weight. It has been shown that all Frobenius rings are equipped with a homogeneous weight. Different

characterizations of the homogeneous weight for Frobenius rings have been given. Some of these use the Möbius function, and some use the generating character of Frobenius rings. In (Honold[20]) it is given a formula to the normalized homogeneous weight on Frobenius rings:

$$w(x) = 1 - \frac{1}{|\mathcal{R}^*|}\sum_{u\in\mathcal{R}^*}\chi(xu) \;\; for\; x \in \mathcal{R} \qquad (1.3.b)$$

$$w(x) = 1 - \frac{1}{|\mathcal{R}^*|}\sum_{u\in\mathcal{R}^*}\chi(ux) \;\; for\; x \in \mathcal{R} \qquad (1.3.c)$$

from two equations above we get,

$$\sum_{u\in\mathcal{R}^*}\chi(xu) = \sum_{u\in\mathcal{R}^*}\chi(ux) = |\mathcal{R}^*|\big(1 - w(x)\big), for\; x \in \mathcal{R}. \qquad (1.3.d)$$

It will be easier for determining the value of the homogeneous weight on our Frobenius ring $\mathcal{M}_k$, before defining the homogeneous weight over $\mathcal{M}_k$.

Let us give some properties by the following propositions from(Greferath and Osullivan[27]).

Proposition 1.3.1.[27] Let $\mathcal{R}$ be a finite ring with generating character χ, then every homogeneous weight on $\mathcal{R}$ is of the form,

$$w: \mathcal{R} \Longrightarrow \mathbb{R}, x \;\longmapsto\; \lambda[1 - \frac{1}{|\mathcal{R}^*|}\sum_{u\in\mathcal{R}^*}\chi(xu)] \qquad (1.3.e)$$

Proposition 1.3.2.[28] Let I be either a (left or right) ideal of a finite Frobenius ring $\mathcal{R}$, and let y $\in\mathcal{R}$, then,

$$\sum_{r\epsilon I+y} w(y) = \lambda\,|I|\,. \qquad (1.3.f)$$

Proof: By using the standard form of homogeneous weight over $\mathcal{R}$, we get

$$\sum_{r\epsilon I+y} w(r) = \lambda\,|I| - \frac{\lambda}{|\mathcal{R}^*|}\sum_{r\epsilon I+y}\sum_{u\epsilon\mathcal{R}^*}\chi(ru)$$

$$= \lambda\,|I| - \frac{\lambda}{|\mathcal{R}^*|}\sum_{u\epsilon\mathcal{R}^*}\sum_{r\epsilon I}\chi((r+y)u)$$

$$= \lambda\,|I| - \frac{\lambda}{|\mathcal{R}^*|}\sum_{u\epsilon\mathcal{R}^*}\chi(yu)\sum_{r\epsilon I}\chi(ru).$$

Since χ is a generating character, I is not in the kernel of χ, *so*, $\sum_{r\epsilon I}\chi(ru) = 0$. ■

The above proposition shows that for any coset of (left, right) ideal, the average is λ. This is equivalent to $\mathcal{R}$ being Frobenius. As is common in coding theory, a weight on finite ring $\mathcal{R}$ will be extended to a weight on the $\mathcal{R}$ -module as: $w: \mathcal{R}^n \Rightarrow \mathbb{R}, w(c_1, c_2, \dots, c_n) = \sum_i^n w(c_i)$.

1.3.1. The Homogeneous Weight For The Ring ($\mathcal{R}_{k,(p)}$)

In our work, we will use the Proposition (1.3.1), which describes the homogeneous weight in terms of the generating character of the ring. The λ that appears in the weight function is also called the average weight. Before proving the main result about the homogeneous weights of p-ary local Frobenius rings, we will obtain some auxiliary results. Let M be the unique maximal ideal of $\mathcal{R}$. Then M consists of all the non-units in $\mathcal{R}$ and since, $\mathcal{R}$/M ≅$\mathbb{Z}_p$, we have |M|= |$\mathcal{R}$|/p. Furthermore we can easily see that $\mathcal{R}^*$ = (1 +M) ∪ (2 +M) ∪…∪ ((p - 1) + M), gives us a partition of the units.

We first start with the following lemma, which will be useful in proving the next result:

Lemma 1.3.1.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring, χ its generating character as described before and x is an element in $\mathcal{R}$/m. Then there exists a unit λ ∈$\mathcal{R}$* such that $\chi(\lambda x) = 1$.

Proof: This lemma will be proved in the result section as **Lemma 3.1.1.**

Lemma 1.3.2.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring and x be an element in $\mathcal{R}$/ m. Then,

$$\sum_{a\in\mathcal{R}^*}\chi(\alpha x) = 0.$$

Proof: This lemma will be proved in the result section as **Lemma 3.1.2.**

We are now ready to describe the homogeneous weight for $\mathcal{M}_k = \mathcal{R}_{k,(p)}$.

Theorem 1.3.1.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring, with minimal ideal m. Then the homogeneous weight on $\mathcal{R}$ has the form:

$$w_{hom}(x) = \begin{cases} 0 & if\ x = 0 \\ \dfrac{p}{p-1}\lambda & if\ x \in m\backslash\{0\} \\ \lambda & otherwise. \end{cases}$$

Proof: This theorem will be proved in the result section as **Theorem 3.2.1.**

Example 1.3.1.1 We give some examples of finite rings with homogeneous weight functions.

■ The Hamming weight of finite field $\mathcal{F}_q$ is a homogeneous weight of average $\lambda = \frac{q}{q-1}$.

■ The Lee weight on $\mathbb{Z}_4$ is homogeneous with λ= 1, and in general, over chain ring $\mathbb{Z}_{pk}$ with minimal ideal (p^{k-1}) was defined as following,

$$w_{hom}(x) = \begin{cases} 0 & if\ u = 0 \\ p^{k-1} & if\ u \in\ p^{k-1}\ \mathbb{Z}_{p^k} \setminus \{0\} \\ (p-1)p^{k-2} & otherwise. \end{cases}$$

■The finite chain ring $\mathcal{R}$ with residue field G$\mathcal{F}$(q) has weight function

$$w_{hom}(x) = \begin{cases} 0 & if\ x = 0 \\ \dfrac{q}{q-1}\ \lambda & if\ x \in soc(\mathcal{R})\backslash\{0\} \\ \lambda & otherwise \end{cases}$$

which is a (left, right) homogeneous weight, for the Galois ring G$\mathcal{R}$(p^t;m) the homogeneous weight was defined as

$$w_{hom}(x) = \begin{cases} 0 & if\ u = 0 \\ p^{m(t-1)} & if\ u \in G\mathcal{R}(p^t, m)\{0\} \\ (p^m - 1) & otherwise. \end{cases}$$

■ The weight over the ring $\mathcal{F}_2$ + u$\mathcal{F}_2$ defined by, w (0, 0) = w (1,1) = 0, w (1, 0) = w(0, 1) = 2, is homogeneous with λ = 1.

■ The local ring $\mathcal{R}$ = $\mathcal{F}_2[x; y]/(x^2, y^2, xy)$ of order 8 has the weight function

$$w_{hom}(x) = \begin{cases} 0 & if\ x = 0 \\ 2\,\lambda & if\ x = u_1 u_2 \dots u_k \\ \lambda & otherwise. \end{cases}$$

1.3.1.1. The Ring $\mathcal{M}_2$

The ring $\mathcal{M}_2 = \mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1r_2\mathcal{F}_p$ is an extension of a ring $\mathcal{M}_1 = \mathcal{F}_p + r_1\mathcal{F}_p$ defined as characteristic p and $r_1^2 = r_2^2 = 0$, and $r_1r_2 = r_2r_1$. The units of it is given by $\{a + br_1 + cr_2 + dr_1r_2\}$,a, b, c, d $\in\mathcal{F}_p$, a $\neq 0$ in order we can see every non-unit is a zero-divisor which form a principal ideal. We can express these ideals as,

$r_1\mathcal{M}_2$, $r_2\mathcal{M}_2$, $(r_1+r_2)\mathcal{M}_2$, $(r_1+2r_2)\mathcal{M}_2,\dots$, $(r_1+(p-1)\,r_2)\mathcal{M}_2$, $(2r_1+r_2)\mathcal{M}_2,\dots$, $r_1r_2\mathcal{M}_2$. Its not finite chain ring, since $r_1\mathcal{M}_2$ and $r_2\mathcal{M}_2$ can not be compared, and its not principal ideal ring. Its local ring with maximal ideal $=\{ br_1 + cr_2 + dr_1r_2\}$, b, c, d $\in\mathcal{F}_p$, minimal ideal $r_1r_2\mathcal{M}_2 = \{0, r_1r_2, 2r_1r_2,\dots, (p-1)\,r_1r_2\}$ and cardinality p^3.

The homogenous weight of $\mathcal{M}_2$ was defined as,

$$w_{hom}(x) = \begin{cases} 0 & if\ x = 0 \\ p^3 & if\ x = i(r_1\,r_2), i \neq 0\text{, with } \lambda = (p-1)\,p^2\,. \\ (p-1)p^2 & otherwise \end{cases}$$

1.3.2. The Gray Map For The Homogeneous Weight

In the previous subsection, we proved that the homogeneous weight for each p-ary local Frobenius ring has two non-zero weights. There is a λ parameter, which is the average weight. In order to make use of the homogeneous weight in constructing good p-ary codes, we need a distance preserving Gray map from $\mathcal{R}$ to $\mathbb{Z}_p^s$ for some suitable s. This requires assigning a value for λ, as well as finding a suitable map. When the characteristic is not prime, i.e., when we are in the case of $\mathbb{Z}_p^m$ or Galois rings, it was shown that for a suitable λ and a suitable s such a map exists. The map was found using Affine Geometries in (Yildiz[45]). However, as in the case $\mathbb{Z}_4$, such a map generally is nonlinear. It is clear that when the characteristic is 2, first order binary Reed-Muller codes can be used to find linear distance-preserving Gray maps for any 2-ary local Frobenius ring. The main property of first order binary Reed-Muller codes that allow is that they have two non-zero weights, and the ratio of the second weight to the first weight is exactly the same as the case of homogeneous weights. In (Assmus and Key[4]), it was shown

that Generalized Reed-Muller codes (i.e. Reed-Muller codes over $\mathcal{F}_q$) of first order have the same type of properties. More precisely, G$\mathcal{R}$M$_q$(m, 1), is a linear code over $\mathcal{F}_q$ defined in a similar way to the Reed-Muller codes, which is of length q^m, dimension m+1 and the weight enumerator of the code is exactly $1+(q^{m+1}-q)z^{(q-1)q^{m-1}}+(q-1)z^{q^m}$. So, there are q-1 code words of full weight q^m (they are the non-zero multiples of all 1-vector) and all the remaining non-zero vectors have weight $(q-1)q^{m-1}$. Notice that, the ratio of the weights, the number of non-zero weights, the number of elements of the bigger weight are exactly in a match with the case of the homogeneous weight. Thus we can use Generalized Reed-Muller codes of the first order to find a linear distance-preserving Gray map for all p-ary local Frobenius rings of characteristic p. We will illustrate this on a specific ring family and then construct optimal codes using the Gray map.

1.3.2.1. The Gray map for The Homogeneous Weight on ($\mathcal{R}_{k,(p)}$)

We do not forget that $\mathcal{R}_{k,(p)}= \mathcal{M}_k$. Recall that $\mathcal{M}_k = \mathcal{F}_p[r_1, r_2, \dots, r_k]/(r_i^2; r_ir_j = r_jr_i)$, is a p-ary local Frobenius ring of characteristic p, with the unique maximal ideal $I=\prec r_1, r_2, \dots, r_k \succ$ and the unique minimal ideal $m = \prec r_1 r_2 \dots r_k \succ$, $\mathcal{M}_k$ can be viewed as an F_p-vector space with basis elements $\{1, r_1, r_2, \dots, r_1 r_2 \dots r_k\}$. Since to every subset of {1, 2,…, k}, there exists a basis element, the size of the ring is given by $|\mathcal{M}_k| = p^{2^k}$. To find a Gray map $\varphi : \mathcal{M}_k \Rightarrow \mathcal{F}_p^s$ for a suitable s, we will use the first order Generalized Reed-Muller codes over $\mathcal{F}_p$. The size of the ring forces us to use G$\mathcal{R}$M$_p$(2^k-1, 1). In this case, we need to assign the average weight $\lambda=(p-1)p^{2^k-2}$. Thus for $\mathcal{M}_k$, the homogeneous weight will be given by,

$$w_{hom}(x) = \begin{cases} 0 & if\ x = 0 \\ p^{2^k-1} & if\ x = i(r_1 r_2 \dots r_k), i \neq 0 \\ (p-1)p^{2^k-2} & otherwise. \end{cases}$$

Now we define φ by mapping $(r_1 r_2 \dots r_k)$ to (111…1) (which is a generator of G$\mathcal{R}$M$_p$(2^k-1,1)), and the remaining basis elements of $\mathcal{M}_k$ to the remaining generators of G$\mathcal{R}$M($\mathcal{M}_k$, 1) in a bijective way and then we extend φ linearly over $\mathcal{F}_p$ to all the ring $\mathcal{M}_k$. φ can be extended to $(\mathcal{M}_k)^n$ in an obvious way, by applying it to each component, i.e. $\varphi(c_1, c_2, \dots, c_n) = (\varphi(c_1), \varphi(c_2), \dots, \varphi(c_n)))$.

Now we explain how to define the Gray map for the ring $\mathcal{M}_k$, the parameter of first order Reed -Muller codes is [2^m, m+1, 2^{m-1}] and for first order Generalized Reed-Muller codes is [p^m, m+1, p^{m-1}]. Becuase the size of $\mathcal{M}_k$ is p^{2^k}, so we can choose m= 2^k-1. The first order Generalized Reed-Muller codes is $GRM_p(2^k-1,1)$ of parameter [$p^{2^k-1}, 2^k, p^{2^k-2}$], because that we assigned the average weight λ=(p-1) p^{2^k-2}.

Now let us define φ, $(r_1\, r_2 \ldots r_k) \longmapsto$ (111…1) which is the generator of $GRM_p(2^k-1,1)$, and the remaining basis of $\mathcal{M}_k \longmapsto$ to the remaining generators of $GRM_p(\mathcal{M}_k, 1)$ in a bijective way.

By using the basis generating for GRM_p of length p^m with,

$v_0 = \{1_{p^m}\}$,

$v_1 = \{0_{p^{m-1}}\}$,

$v_2 = \{1_{p^{m-2}}, 0_{p^{m-2}}, 1_{p^{m-2}}, 0_{p^{m-2}}\}$,

⋮

$v_m = \{0,1,\ldots,(p-1),0,1,\ldots,(p-1)$ of length p^m.

The properties of the Generalized Reed-Muller codes and defintion 1.1.1 then dictate the following theorem.

Theorem 1.3.2.[46] φ is a distance preserving linear isometry from ($\mathcal{M}_k{}^n$, homogeneous distance) to ($\mathcal{F}_p^{p^{2^k-1}\,n}$, hamming distance). Thus if C is a linear code over $\mathcal{M}_k$ of length n and minimum homogeneous weight d, then φ (C) is a binary linear code of length $p^{2^k-1}\,n$, and minimum hamming weight d. Moreover, the homogeneous weight distribution of C is the same as the hamming weight distribution of φ (C).

We can write down the Gray map explicitly in for $\mathcal{M}_2$. Here m=2^2-1= 3 , and with homogenous weight which is definied in subsubsection 1.3.1.1.

by letting

φ (0) = (0,0,0,…....,0),

$\varphi(1) = (\ \bar{0}_{p^2}\ , \bar{1}_{p^2}\ , \dots , \overline{(p-1)}_{p^2}\ )$,

$\varphi\ (r_1\ ) = (\ 0, 1, 2\ , \dots , (p-1), 0\ , 1\ 2, \dots , (p-1))$,

$\varphi(r_2) = (\ \bar{0}_p\ , \bar{1}_p\ , \dots , \overline{(p-1)}_p\ \dots , \bar{0}_p\ , \bar{1}_p\ , \dots , \overline{(p-1)}_p)$,

φ ($r_1\ r_2$) = (1, 1, 1,…, 1).

All these images of length p^3. The map is then extended to the ring $\mathcal{M}_k$ in an $\mathcal{F}_p$-linear way, i.e.

φ (a + r_1 b + r_2c + $r_1\ r_2$d) = a φ (1) + b φ (r_1) + c φ (r_2) + d φ ($r_1\ r_2$) (*)

For k =2 over p =3 , the homogeneous weghit is,

$$w_{hom}(x) = \begin{cases} 0 & if\ x = 0 \\ 27 & if\ x = i(r_1\ r_2), i \neq 0, \text{ with } \lambda = 18. \\ 18 & otherwise \end{cases}$$

Let φ: $\mathcal{M}_2 \longrightarrow \mathcal{F}_3^{27}$ and,

φ (0) = (0,0,0,…....,0) of length 27,

$\varphi(1) = (0,0,..,0_9, 1,1,..,1_9, 2,2,..,2_9)$ of length 27,

$\varphi\ (r_1\ ) = (\ 0, 1, 2\ , \dots , 0\ , 1\ 2, \dots , 0,1,2)$ of length 27,

$\varphi(r_2) = (0,0,0,1,1,1,2,2,2, \dots , 0,0,0,1,1,1,2,2,2)$ of length 27,

φ ($r_1\ r_2$) = (1, 1, 1,…, 1) of length 27.

This map extended by setting in (*), over p=2, this map under φ_{hom} is exactly RM(1,3) of parameter[8,4,4].

We then obtain the following corollary of Theorem 1.3.2.

Corollary 1.3.1.[46] If C is a linear (n, p^t, d)-code over $\mathcal{F}_p$ +$r_1\ \mathcal{F}_p$ + $r_2\mathcal{F}_p$ + $r_1\ \ r_2\mathcal{F}_p$, with p^t denoting the size of the code and d its homogeneous minimum distance, then φ (C) is a linear [p^3n, t, d] code over $\mathcal{F}_p$ with the usual Hamming minimum distance d.

1.4. CONSTRUCTION OF GOOD P-ARY CODES FROM CODES OVER P-ARY LOCAL FROBENIUS RINGS

1.4.1. Griesmer-Optimal Codes Over $\mathcal{F}_p$ From Codes Over $(\mathcal{R}_{2,(p)})$

The Griesmer bound introduced in (Griesmer[13]) gives us an upper bound for the minimum distance of a linear code over finite fields.The codes for which the bound is attained are called optimal codes. For a linear [n, k, d]-code over $\mathcal{F}_q$, the bound is given by

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil, \qquad (1.4.1.a)$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. By a Griesmer optimal code, we will denote a code that attains this bound. Finding Griesmer optimal codes over different alphabets have attracted a considerable amount of attention in coding theory. Among any works related to this problem, we can refer to (Hamada[16]) and (Helleseth[18]) and references there in. In what follows, we will construct Griesmer optimal codes over $\mathcal{F}_p$ for p = 2, 3 and 5 using the Gray homogeneous images of linear codes over $\mathcal{M}_2$.

1.4.2. $[p^3n, 4, (p- 1)p^2n]_p$ Codes

Let's take C to be the linear code over $\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p+r_1 r_2\mathcal{F}_p$ of length n generated by the vector (1,1,…, 1). Since every code word in C is of the form (a, a,.., a) with a$\in\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p+r_1 r_2\mathcal{F}_p$, we see that the minimum homogeneous weight of C is $(p - 1)p^2n$. Since the generating vector (1,1,.., 1) contains units, we see that $| C |= p^4$. Thus, by taking the Gray image of C, in the light of Theorem 1.3.2, we get, φ (C) is a linear code over $\mathcal{F}_p$ with parameters $[p^3n,4,(p-1)p^2n]$.

If we apply for cases p = 2,3,5 we get the following corollaries.

Corollary 1.4.1.[46] If we take p = 2, we see that we get a binary linear code with parameters [8n, 4, 4n] which are all optimal for $1\leq n \leq 6$ by the Griesmer bound.

Corollary 1.4.2.[46] If we take p = 3, we get a ternary linear code with parameters [27n, 4,18n] which are all optimal for $1\leq n \leq 9$ by the Griesmer bound.

Corollary 1.4.3.[46] If we take p = 5, we get linear codes over $\mathcal{F}_5$ with parameters [125n, 4,100n] which are all optimal for $1 \leq n \leq 15$ by the Griesmer boun.

1.4.3. $[p^4n, 5, (p - 1)p^3n]_p$ Codes

Theorem 1.4.1.[46] Let C be the linear code over $\mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1 r_2\mathcal{F}_p$, of length p generated by the vectors $\{(1, 1,1,\ldots, 1), (0, r_1 r_2, 2r_1 r_2,\ldots, (p - 1) r_1 r_2)\}$, then C is of size p^5 with minimum homogeneous distance $(p - 1)p^3$.

Proof: This theorem will be proved in the result section as **Theorem 3.3.1.**

Corollary 1.4.4.[46] By taking the repetitions of the generators in Theorem 1.4.1 and applying the Gray map, we get linear codes over $\mathcal{F}_p$ with parameters $[p^4n,5, (p-1)p3n]$.

Now, we apply for cases p=2,3,5 we get the following corollaries.

Corollary 1.4.5.[46] If we take p = 2, we obtain binary linear codes with parameters [16n, 5, 8n] which are optimal for $1 \leq n \leq 8$ by the Griesmer bound.

Corollary 1.4.6.[46] If we take p = 3, we obtain ternary linear codes with parameters [81n, 5, 54n] which are all optimal for $1 \leq n \leq 12$ by the Griesmer bound.

Corollary 1.4.7.[46] If we take p = 5, we obtain linear codes over $\mathcal{F}_5$ with parameters [625n, 5, 500n] which are all optimal for $1 \leq n \leq 20$ by the Griesmer bound.

1.4.4. $[p^5n, 6, (p - 1)p^4n]_p$ Codes

Theorem 1.4.2.[46] Let C be the linear code over $\mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1 r_2\mathcal{F}_p$, of length p^2 generated by the vectors $\{\bar{1} = (1, 1, 1,\ldots, 1), \bar{b} = (0, b_1, b_2,\ldots, b_{p^2-1})\}$, where $\{0, b_1,\ldots, b_{p^2-1}\} = r_1.(\mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1 r_2\mathcal{F}_p)$. Then C is of size p^6 with minimum homogeneous distance $(p - 1)p^4$.

Proof: This theorem will be proved in the result section as **Theorem 3.3.2.**

Remark 1.4.4.1. Note that, by taking the repetitions of the generators in the previous theorem, and then applying the Gray map one can obtain linear codes over $\mathcal{F}_p$ with parameters $[p^5n, 6, (p - 1)p^4n]$.

Again, we apply for cases p=2,3,5 we get the following corollaries.

Corollary 1.4.8.[46] If we take p = 2, we obtain binary linear codes with parameters [32n, 6, 16n] which are all optimal when $1 \leq n \leq 10$ by the Griesmer bound.

Corollary 1.4.9.[46] If we take p = 3, we obtain ternary linear codes with parameters [243n, 6, 162n] which are all optimal when $1 \leq n \leq 15$ by the Griesmer bound.

Remark 1.4.4.2. We do actually obtain similar results for p = 5 that are optimal for even more values of n, but since the lengths of the codes must be multiples of 3125; it is not very practical to study them.

1.4.5. $[p^6n,7,(p - 1)p^5n]_p$ Codes

With a similar argument, we can find linear codes over $\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p+r_1 r_2\mathcal{F}_p$ of length p^3, size p^7 and minimum homogeneous weight $(p - 1)p^5$. The generators one can take to construct this code would be (1, 1, 1,…, 1) and (0, b_1,…, b_{p^3-1}) where the set { 0, b_1,…, b_{p^3-1} } is the set of all zero-divisors of the ring $\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p+r_1 r_2\mathcal{F}_p$. This code would have minimum homogeneous weight $(p-1)p^5$, the proof of which, is similar to the previous ones, will be omitted. Then, taking repetitions of the generators and applying the Gray map, we would obtain linear codes over $\mathcal{F}_p$ with parameters $[p^6n, 7,(p - 1)p^5n]$.

We could analyze the optimality of these codes using the Griesmer bound but since the lengths of the codes are quite restrictive (the length being multiple of 64 for binary codes, of 729 for ternary codes, etc.), we will omit that for practical reasons.

1.5. FROBENIUS RINGS

1.5.1. Local Frobenius Rings

In subsection 1.2 we mentioned about a definition of p-ary local Frobenius rings, and in this section, we speak about a general finite local Frobenius ring and its homogeneous weights and some examples. Now we start with the following theorem.

Theorem 1.5.1.[21,Theorem 15] Let $\mathcal{R}$ be a finite ring. The following statements are equivalent.

a) $\mathcal{R}$ is Frobenius and local.

b) $\mathcal{R}$ is QF(quasi-Frobenius) and local.

c) $\mathcal{R}$ has a unique minimal left ideal.

d) $\mathcal{R}$ has a unique minimal right ideal.

Proof: (a) $\Longrightarrow$ (b) clearly.

(b) $\Longrightarrow$ (c) and (d). Since $\mathcal{R}$ is local, $\mathcal{R}$ has a unique maximal right ideal M. Since $\mathcal{R}$ is QF, $\mathcal{m}$ is the unique minimal left ideal of $\mathcal{R}$. So we have (c). (d) by the same. (c) $\Longrightarrow$ (d) and (a). By the same, we assume (c).

Let $\mathcal{m}$ be the unique minimal left ideal of $\mathcal{R}$. We choose a character χ of ($\mathcal{R}$,+) such that χ is nontrivial on $\mathcal{m}$. Then ker χ does not contain any nonzero left ideal of $\mathcal{R}$, so χ is a generating character of $\mathcal{R}$. By (J.Wood[39]), $\mathcal{R}$ is Frobenius. Since $\mathcal{R}$ is QF, r ($\mathcal{m}$) is the unique maximal right ideal of $\mathcal{R}$. Thus $\mathcal{R}$ is local.■

Remark 1.5.1 We summarize the following properties about Frobenius rings, some of which characterize the Frobenius property for details can be found in wood [39], Greferath [28] and Honold [20], for afinite Frobenius rings the following are true:

a) $\mathcal{R}$ and $\hat{\mathcal{R}}$ are isomorphic left $\mathcal{R}$ -modules and isomorphic right $\mathcal{R}$ -modules. More precisely, there exists a character $\chi \in \hat{\mathcal{R}}$ such that $\mathcal{R} \longrightarrow \hat{\mathcal{R}}$, r $\longrightarrow$ r$\cdot \chi$, resp. $\mathcal{R} \longrightarrow \hat{\mathcal{R}}$, r $\longrightarrow \chi \cdot$r is an isomorphism of a left (resp. right) $\mathcal{R}$- modules. Any such χ is called a generating character of $\mathcal{R}$. Obviously, any two generating characters χ, χ' differ by a unit,

i.e., $\chi' = u \cdot \chi$ and $\chi' = \chi \cdot u'$ for some u, u′ $\in \mathcal{R}^*$. More generally, for each n, the maps,

$\alpha_l : \mathcal{R}^n \longrightarrow \hat{\mathcal{R}}^n$, v $\longmapsto \chi(\prec - , v \succ)$, and $\alpha_r : \mathcal{R}^n \longrightarrow \hat{\mathcal{R}}^n$, v $\longmapsto \chi(\prec v, - \succ)$ are left (resp. right) $\mathcal{R}$-module isomorphism (here $\prec v,w \succ$ denotes the standard inner product on $\mathcal{R}^n$).

b) Let χ be a character of $\mathcal{R}$. Then χ is a generating character of $\mathcal{R}$ if and only if the only left (resp. right) ideal contained in ker χ = {a $\in \mathcal{R}$: χ(a) = 1} is the zero ideal.

c) $\mathcal{R}$ satisfies the double annihilator property, i.e.,$ann_l(ann_r(I)) = I$ for each left ideal I of $\mathcal{R}$ and $ann_r(ann_l(I))=I$ for each right ideal I, where ann_l (resp. ann_r) denotes the left (resp. right) annihilator ideal. In particular, $ann_l(rad(\mathcal{R})) = soc(\mathcal{R}) = annr(rad(\mathcal{R}))$.

d) Soc($\mathcal{R}$) is a principal ideal.

e) If $\mathcal{R}$ is a commutative finite Frobenius ring, then $\mathcal{R} = \mathcal{R}_1 \times \ldots \times \mathcal{R}_t$ for suitable local Frobenius rings $\mathcal{R}_i$.

f) If $\mathcal{R}$ is local, that is, rad($\mathcal{R}$) is the unique maximal left (resp. right) ideal of $\mathcal{R}$, then soc($\mathcal{R}$) is the unique minimal ideal. In this case $\mathcal{R}$/rad($\mathcal{R}$) is called the residue field of $\mathcal{R}$.

Example 1.5.1.1 Many standard examples of rings are Frobenius. For details, we refer to (Wood[39]).

a) Every finite field is Frobenius, and every non-principal character is generating.

b) Finite chain rings (e.g., Galois rings), finite group rings over a Frobenius ring and direct products of Frobenius rings are Frobenius.

1.5.2. Generating Characters of Finite Frobenius Rings And Möbius Functions

Let $\mathcal{R}$ be a finite ring. A character χ of $\mathcal{R}$ is a right (resp., left) generating character if the mapping $\varphi: \mathcal{R} \longrightarrow \hat{\mathcal{R}}$, $\varphi(r) = \chi^r$ (resp.,$\varphi(r) = r_\chi$ is an isomorphism of right (resp., left) $\mathcal{R}$-modules.

A finite ring is Frobenius if and only if it admits a right or a left generating character. The phrase generating character comes from (Klemm[24]).

(Wood[39]) proved that a finite ring $\mathcal{R}$ is a Frobenius ring if and only if $\mathcal{R}$ has a generating character. For details you can refere to [21, Theorem 11]. In subsection 1.3 we used a character theoretic methods.

In this section we define the Möbius function and show that the Möbius function is related to generating characters.

For a finite poset P consider the function $\mu : P \times P \longrightarrow \mathbb{C}$ implicitly defined by $\mu(x, x) = 1$ and,

$$\sum_{y \le t \le X} \mu(t, x) = 0 \qquad (1.5.3.a)$$

If $y < x$, and $\mu(y, x) = 0$ if $y \not\leq x$. It is called the Möbius function on P and induces for arbitrary pairs of real-valued functions f, g on P the following equivalence, referred to as Möbius inversion,

$$g(x) = \sum_{y \le x} f(y). \qquad (1.5.3.b)$$

For all $x \in P \iff f(x) = \sum_{y \le x} g(y)\mu(y, x)$ for all x∈P.

Let $\mathcal{R}$ be a finite ring and μ the Möbius function on the set { $\mathcal{R}$ x | x ∈$\mathcal{R}$ } of its principal left ideals (partially ordered by set inclusion). Further, let $\mathcal{R}^*$ denote the set of units of $\mathcal{R}$.

Lemma 1.5.1.[47] Let $\mathcal{R}$ be a finite Frobenius ring and χ be a generating character of $\mathcal{R}$. Then for any $x \in \mathcal{R}$, we have,

$$\mu(\mathcal{R}x, 0) = \sum_{y \in \check{x}} \chi(y).$$

Proof: We prove it by induction on the cardinality $|\mathcal{R}\, x|$. If $|\mathcal{R}\, x| = 1$, then $\mathcal{R}\, x = 0$ is the zero-ideal and $x = 0$, hence $\chi(0) = 1 = \mu(0, 0)$. In the following, we assume that $x \neq 0$ and set $I = \mathcal{R}\, x$. For J ∈P, we denote the set of principal generators of J by $\check{J}$, i.e. $\check{J}$ = {y ∈ J | $\mathcal{R}$ y = J}, which is just the corresponding association class of J.

Note that I is the disjoint union of association classes contained in I. So,

$$\sum_{y \in I} \chi(y) = \sum_{J \in P, I \supseteq J} \sum_{y \in \check{J}} \chi(y) = \sum_{y \in \check{I}} \chi(y) + \sum_{J \in P, I \supsetneq J} \sum_{y \in \check{J}} \chi(y).$$

Since the restriction χ |I is a non-unity character of I, we get $\sum_{y \in I} \chi(y) = 0$.

By induction, for J ∈ P with J $\subsetneq$ I, we have $\sum_{y \in J} \chi(y) = \mu(J, 0)$, thus

$$\sum_{y \in \check{I}} \chi(y) + \sum_{J \in P, I \supset J} \mu(J, 0) = 0.$$

On the other hand, as I ≠ 0, in the partial order interval [I, 0] we have,

$$\mu(I, 0) + \sum_{J \in P, I \supset J} \mu(J, 0) = \sum_{J \in P, I \supseteq J} \mu(J, 0) = 0.$$

Comparing the above two qualities, we obtained,

$$\sum_{y\in\tilde{I}} \chi(y) = \mu\,(I, 0).\ \blacksquare$$

The following theorem shows the existence and uniqueness of homogeneous weights on finite rings.

Theorem 1.5.2.[20] A real-valued function *w* on the finite ring $\mathcal{R}$ is a homogeneous weight if and only if there exists a real number λ such that

$$w_{hom}(x) = \lambda\left[1 - \frac{\mu(0,\mathcal{R}x)}{|\mathcal{R}^{*}x|}\right] \quad \text{for all } x \in \mathcal{R}.$$

This formula used in (YildiZ and Kelebek[43]) for finding the homogeneous weight of $\mathcal{R}_k$.

Lemma 1.5.2 [29] For all a finite ring $\mathcal{R}$ the following are eguivalent :

(a)Soc($\mathcal{R}x$) is left principal

(b)For all non-zero $I \subseteq \mathcal{R}$ holds $\sum_{\mathcal{R}x\subseteq I} \mu(0,\mathcal{R}x) = 0$.

For details about prove of Theorem 1.5.2 and Lemma 1.5.3, you can see in [20], [29].

1.5.3. Explicit Values of the Homogeneous Weight

Let $\mathcal{R}$ be a finite Frobenius ring with group of units $\mathcal{R}^{*}$, and fix a generating character χ. The following definition is taken from Greferath and Schmidt [29],

Definition 1.5.1. The (left) homogeneous weight on $\mathcal{R}$ with average value λ is a function

w : $\mathcal{R} \to$ Q such that

(1) *w* (0) = 0.

(2) *w* (x) = *w* (y) for all x, y $\in \mathcal{R}$ such that $\mathcal{R}$ x = $\mathcal{R}$ y.

(3) $\sum_{y\in\mathcal{R}x} w(y) = \lambda\,|\mathcal{R}\,x|$ for all x $\in \mathcal{R} \setminus \{0\}$.

in other words, the average weight over each nonzero principal ideal is λ. In [29, Greferath and Schmidt] proved the existence and uniqueness of the homogeneous weight with given average

value for any finite ring, and they show that (3) is satisfied by the homogeneous weight for all nonzero ideals of $\mathcal{R}$.

It is easy to see that the Hamming weight on $\mathcal{R}$ is homogeneous if and only if $\mathcal{R}$ is a field, in which case it has average value $\frac{q-1}{q}$, where q = | $\mathcal{R}$|.

Without loss of generality we restrict ourselves to the homogeneous weight with average value $\lambda = 1$, which we call the normalized homogeneous weight.

Proposition 1.5.1.[27] Let $\mathcal{R}$ be a finite Frobenius ring with generating character χ. Then every homogeneous weight on $\mathcal{R}$ is the form $w: \mathcal{R} \to \mathbb{R}, x \mapsto \lambda [1 - \frac{1}{|\mathcal{R}^*|}\sum_{u\in\mathcal{R}^*} \chi(xu)]$.

We mentioned about this formula in subsection 1.3, but now we prove it.

Proof: By the uniqueness of the homogeneous weight on a fixed ring $\mathcal{R}$, we only need to check that *w* as given in the statement, satisfies the criteria of definition 2.3.1 As $-1 \in \mathcal{R}^*$, we see that *w* is real-valued, and certainly *w*(0)= 0. If $\mathcal{R}\, x = \mathcal{R}\, y$ then $y = vx$ for some $v \in \mathcal{R}^*$. We know that every left generating character is also a right generating character. For this reason, there exists $v \in \mathcal{R}^*$ such that χ (vx)= χ(x v$^{/}$) and we obtain $\sum_{u\in\mathcal{R}^*} \chi(vxu) = \sum_{u\in\mathcal{R}^*} \chi(xu)$, which finally shows that *w*(x)=*w*(y).

In order to obtain the remaining part of the claim, we observe that for all $x \neq 0$ the expression $\chi(\mathcal{R}\, xu)$ can not be trivial since the kernel of a generating character does not contain nonzero ideals of $\mathcal{R}$. Therefore

$$\sum_{y\in\mathcal{R}^*} \frac{1}{|\mathcal{R}^*|} \sum_{u\in\mathcal{R}^*} \chi(yu) = \frac{1}{\mathcal{R}^*} \sum_{u\in\mathcal{R}^*} \sum_{y\in\mathcal{R}^*} \chi(yu) = \frac{1}{\mathcal{R}^*} \sum_{u\in\mathcal{R}^*} \sum_{y\in\mathcal{R}xu} \chi(y) = 0 \blacksquare$$

Example 1.5.1 This is example of the normalized homogeneous weight, let $\mathcal{R}$ be a local Frobenius ring with residue field $\mathcal{R}/rad(\mathcal{R})$ of order q, then

$$\omega(a) = \begin{cases} o & if\ a = 0 \\ q & \\ \dfrac{}{q-1} & if\ a \in soc(\mathcal{R}) \setminus \{0\} \\ 1 & otherwise. \end{cases}$$

2. MATERIALS AND METHODS

In our work, we did not use any of the practical methods. Our work is theoretical, when we proved the homogeneous weight for p- ary local Frobenius rings, we have two descriptions :one of them was Mobius function and the another is definition of homogeneous weight which its terms of generating characters. The second one which we used in our work.

3. RESULTS

3.1. STRUCTURAL PROPERTIES OF ALL LOCAL P-ARY LOCAL FROBENIUS RINGS

In this work, we focus on a special class of Frobenius rings, namely the p-ary local Frobenius rings. These are local Frobenius rings whose residue field is isomorphic to the basic prime field, i.e. $\mathcal{F}_p \cong \mathbb{Z}_p$. We characterize these rings in detail, obtaining many of their properties and give many of the studied rings in coding theory as special examples. We then give a generating character for these rings after which we prove that the homogeneous weight for all p-ary local Frobenius rings consists of two non-zero weights. We also define a distance preserving isometry for certain special cases and use the map to construct many Griesmer-optimal codes over several prime fields. Some of the structural properties of all local p-ary local Frobenius rings.

Theorem 3.1.1.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring then

a) $\mathcal{R}$ has a unique minimal ideal m.

b) There is a $\in\mathcal{R}$ such that m = {0, a,…, (p - 1)a}.

c) If I is any non-zero ideal in $\mathcal{R}$, then $m \subseteq$ I.

d) We have $a^2 = 0$.

e) $x.a = \begin{cases} j\,.\,a\,, & j = 1,2,\dots,p-1 \quad if\ x\ is\ a\ unit \\ 0 & if\ x\ is\ non-unit \end{cases}$

Proof: a) This follows from the fact that $\mathcal{R}$ is a p-ary local Frobenius ring. By the definition of Frobenius rings, $\mathcal{R}/\mathrm{J}(\mathcal{R})$ is isomorphic to soc($\mathcal{R}$) as a module. But $\mathcal{R}/\mathrm{J}(\mathcal{R})$ being isomorphic to $\mathbb{Z}_p$ as a ring, we must have soc($\mathcal{R}$) $\cong \mathbb{Z}_p$ as an additive group. Now, soc($\mathcal{R}$) is the sum of minimal ideals, so every minimal ideal must be an additive subgroup of soc($\mathcal{R}$). But since $\mathbb{Z}_p$ does not have any non-trivial subgroups, we see that there has a to be a unique minimal ideal m.

b) By (a), we know that m = soc($\mathcal{R}$), which is isomorphic as an additive group to $\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is cyclic, there exists a $\in\mathcal{R}$ such that $m = \prec a \succ$. Clearly,

then $p^a = 0$ and we have $\mathcal{m} = \{0, a, 2a, \ldots, (p-1)a\}$.

c) Let I be any nonzero ideal in $\mathcal{R}$. Since $\mathcal{R}$ is a finite ring, it is Artinian and so every ideal must contain a minimal ideal. Since the minimal ideal is unique, I must contain $\mathcal{m}$.

d) $a^2 \in \mathcal{m}$, since $\mathcal{m}$ is an ideal. Now suppose $a^2 = ra$ for r = 1, 2,…, p - 1. This means that a(a-r .1) = 0. Now, a is a non-unit but r.1 is a unit in $\mathcal{R}$. Since $\mathcal{R}$ is local, the set of all non-units form the maximal ideal and everything else is a unit. Thus the sum of a unit and a non-unit must be a unit which implies a-r.1 must be a unit. This is a contradiction to a(a - r . 1) = 0. Thus we must have $a^2 = 0$.

e) Suppose d is a non-unit in $\mathcal{R}$. Since $\mathcal{m}$ is an ideal, we must have $ad \in \mathcal{m}$. Now, suppose ad = ra for some r = 1, 2,…, p – 1. Then we would have a(d – r.1) = 0. But then, by the same reason as in (d), we must have d-r .1 as a unit, which would contradict the equality. So we must have ad = 0 if d is a non-unit. On the other hand, if u is a unit in $\mathcal{R}$, then again we must have $au \in \mathcal{m}$. Now if au = 0, this would imply that u is a zero-divisor, which is a contradiction.■

An important property of p-ary local Frobenius rings is that their size should be a power of p.

Theorem 3.1.2.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring. Then $|\mathcal{R}| = p^m$ for some suitable integer m.

Proof: The socal of $\mathcal{R}$ is the minimal ideal $\mathcal{m}$. Since $\mathcal{R}/M \cong \mathbb{Z}_P$ as additive group to $\mathbb{Z}_P$, so $\mathcal{m}$ is the subgroup of $\mathcal{R}$ mean that $p||\mathcal{R}|$. Now suppose that there is another prime q dividing $|\mathcal{R}|$. Then additive group of $\mathcal{R}$ has an element x such that $|\prec x \succ| = q$, i.e q x =0 by Cauchys theorem, but $qx = (q.1_{\mathcal{R}}).x = 0, x \neq 0$, we see that $q.1_{\mathcal{R}}$ is a non-unit. As every element of $\mathcal{R}/M$ is a unit, we see that $q.1_{\mathcal{R}} \in$ M. Inparticular M= $q.1_{\mathcal{R}}$+ M= q($1_{\mathcal{R}}$+ M). This implise that order of $1_{\mathcal{R}}$+ M must divide q, i.e p = q. ■

Theorem 3.1.3.[46] The characters which were defined in eq. (1.2.3.a) and eq. (1.2.3.b) are generating characters for a p-ary local Frobenius ring of characteristic p and characteristic p^m, respectively.

Proof: We will prove the characteristic p case. The proof of the other case, being similar, will be omitted. We first note that if r, s $\in \mathcal{R}$ such that

r = …. + c_aa and s = …. + d_aa, then r + s = …. + (c_a + d $_a$)a. Hence we have $\chi(r+s) = e^{\frac{2\pi i(c_a+d_a)}{p}} = e^{\frac{2\pi i c_a}{p}} . e^{\frac{2\pi i d_a}{p}} = \chi(r).\chi(s)$.

To show this character is generating character, we can put r as a general form of p- ary local frobenius ring, $r = \sum_{A\subseteq K} c_A r_A$, $K = \{1,2,\dots,K\}$. $\ker\chi = \{\sum_{A\subseteq K} c_A r_A \mid e^{\frac{2\pi i c_k}{p}} = 1\}$ = $\{\sum_{A\subseteq K} c_A r_A \mid c_K = 0\}$, from this we can get the following, $\ker\chi \cap I_K = \{0\}$, where I_K is the unique minimal ideal, hence $\ker\chi$ does not contain any non-zero ideal, Therefore it is a generator. ■

Lemma 3.1.1.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring, χ its generating character as described before and x is an element in $\mathcal{R}/\mathcal{m}$. Then there exists a unit λ ∈$\mathcal{R}$* such that χ(λ x) = 1.

Proof: We will proceed by considering different cases, depending on x. If x is a unit, then choosing λ = x^{-1}, we get λ x = 1, for which we know χ(1) = 1. Now, suppose x is a non-unit that does not contain a multiple of a in it. Then we can choose λ = 1, and we will have χ(λ x) = 1. For the last case, let us assume that x is a non-unit that contains a multiple of a. Without loss of generality, we can assume that x = y + a, where y is a non-unit that is not in $\mathcal{m}$. (Otherwise, x = y + a would also be in $\mathcal{m}$). Thus y is non-zero, and (y) is a nonzero ideal. Since $\mathcal{m}$ ⊂(y), we see that (p - 1)a = ry for some r ∈$\mathcal{R}$. Note that r must also be a non-unit. Because otherwise, y would be in (a) = $\mathcal{m}$.

Now, since r is a non-unit and $\mathcal{R}$ is a local ring, we must have 1 + r as a unit. Hence choosing λ= 1 + r, we see that λx = (1 + r)(y + a) = y + a + ry + ra = y + a + (p - 1)a + 0 = y, since ra = 0, by Theorem 1.2.1. and with $\mathcal{m}$ = {0, a, …, (p - 1)a}, we have pa = 0. But now, since y does not contain any positive multiple of a, we see that χ(λ x) =χ(y) = 1.■

Lemma 3.1.2.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring and x be an element in $\mathcal{R}/\mathcal{m}$. Then, $\sum_{a\in\mathcal{R}^*} \chi(ax) = 0.$

Proof: Since χ is a generating character, it is non-trivial when restricted to any nonzero ideal. Since x≠0, the ideal generated by x is a nonzero ideal.Thus we have,

$$\sum_{\alpha\in\mathcal{R}} \chi(\alpha x) = 0. \qquad (3.1.a)$$

Now, by the above partition, we know that M,1+M,2+M,…,(p+1)+M gives a partition of $\mathcal{R}$. Thus we have, $\sum_{\alpha\epsilon M} \chi(\alpha x) + \sum_{\alpha\epsilon\mathcal{R}^*} \chi(\alpha x) = 0$. On the other hand, considering that

$\mathcal{R}^*$= (1+M)∪(2+M)∪….∪((p -1)+M), and the fact that χ ((s + α) x) = χ(sx)χ(αx), we see that,

$$\sum\nolimits_{\alpha\in\mathrm{M}} \chi(\alpha x)(1 + \chi(x) + \chi(2x) + \cdots + \chi\big((p-1)x\big) = 0.$$

Since $\sum_{\alpha\in\mathcal{R}^*} \chi(\alpha x) = -\sum_{\alpha\in\mathrm{M}} \chi(\alpha x)$, this last equation reduces to,

$$\sum_{\alpha\in\mathcal{R}^*} \chi(\alpha x)(1 + \chi(x) + \cdots + \chi\big((p-1)x\big) = 0. \qquad (3.1.b)$$

Now, since χ(2x) = χ(x + x) = χ(x)^2, χ(3x) = χ(x)^3, and so on, we see that the above equation turns to,

$$(1 + \chi(x) + \chi(x)^2 + \cdots + \chi(x)^{p-1}) \sum_{\alpha\in\mathcal{R}^*} \chi(\alpha x) = 0. \qquad (3.1.c)$$

Now, if χ(x)≠ 1, then 1 +, That is why we can not immediately conclude from (3.1.d) that $\sum_{\alpha\in R^*} \chi(\alpha x) = 0$. However, if χ($x$) = , $1 + \chi(x) + \cdots + \chi(x)^{p-1} = \frac{1-\chi(x)^p}{1-\chi(x)} = 0$ then the sum in the parenthesis is not zero. To get around this difficulty, let us label this sum,

F(x)m= $\sum_{\alpha\in R^*} \chi(\alpha x)$.

Let λ be any unit in $\mathcal{R}$. As α runs through all the units in $\mathcal{R}$, so does λα. Thus it is clear that F(x) = F(λx), for all λ∈$\mathcal{R}^*$. Thus replacing x by λx in (3.1.c), we see that we have $(1+\chi(\lambda x)+\chi(\lambda x)^2+...+\chi(\lambda x)^{p-1})F(x) = o$ for all $\lambda\in\mathcal{R}^*$ (3.1.e)

Now, using Lemma 1.3.1. we know that χ(λx) = 1, for some λ∈$\mathcal{R}^*$ which would imply that

F(x) = 0.■

3.2. DESCRIPTION OF THE HOMOGENEOUS WEIGHT FOR $\mathcal{M}_K = \mathcal{R}_{K,(P)}$.

After that, we described the homogeneous weight for $\mathcal{M}_k = \mathcal{R}_{k,(p)}$.

Theorem 3.2.1.[46] Let $\mathcal{R}$ be a p-ary local Frobenius ring, with minimal ideal m. Then the homogeneous weight on $\mathcal{R}$ has the form:

$$w_{hom}(x) = \begin{cases} 0 & if\ x = 0 \\ \dfrac{p}{p-1}\lambda & if\ x \in m\backslash\{0\} \\ \lambda & otherwise. \end{cases}$$

Proof: Suppose $x = 0$. Then $\chi(\alpha x) = 1$ for all $\alpha \in \mathcal{R}^*$, thus by proposition 1.3.1. we have

$$w_{hom}(0) = \lambda\,(1 - \frac{1}{|\mathcal{R}^*|}\sum_{\alpha\in\mathcal{R}^*} 1\,) = 0.$$

Now assume $x \in$ m\{0}. Without loss of generality, assume that x = a. Then as α runs through all the units in $\mathcal{R}$, αa will take the values a, 2a,…, (p-1)a equally often. Hence we have

$$\sum_{\alpha\in\mathcal{R}^*} \chi(\alpha\, a) = \frac{|\mathcal{R}^*|}{p-1}\ (\chi(a) + \chi(2a) + \cdots + \chi\big((p-1)a\big).$$

Of course, since χ, restricted to m is not a trivial character, we have χ(0)+χ (a)+…+ χ ((p -1)a) = 0, which implies that χ(a) + χ(2a) +...+χ((p - 1)a) = -1. Putting this into the above equation, and again using proposition 1.3.1, we obtain

$$w_{hom}(a) = \lambda\left(1 - \frac{1}{|R^*|}\sum_{\alpha\in R^*}\chi(\alpha a)\right) = \lambda\left(1 - \frac{1}{|R^*|}\,\frac{|R^*|}{P-1}\,(-1)\right) = \frac{p}{p-1}\,\lambda.$$

Clearly, this is true for 2a, 3a,…, (p- 1)a as well. Finally, assume that $x \in \mathcal{R}\backslash$ m. Then by Lemma 1.3.2, we know that $\sum_{\alpha\in\mathcal{R}^*}\chi(\alpha x) = 0$, then by proposition 1.3.1. we have,

$$w_{hom}(x) = \lambda\,(1 - \tfrac{1}{|\mathcal{R}^*|}\textstyle\sum_{\alpha\in\mathcal{R}^*}\chi(\alpha x)) = \lambda\,(1-0) = \lambda\,.\ \blacksquare$$

3.3. CONSTRUCTION OF GRIESMER OPTIMAL CODES OVER $\mathcal{F}_p$ FOR p = 2, 3 AND 5

We constructed Griesmer optimal codes over $\mathcal{F}_p$ for p=2, 3 and 5 by using the Gray-homogeneous images of linear codes over $\mathcal{M}_2$.

3.3.1. $[p^3n, 4, (p-1)p^2n]_p$ Codes

Let's take C to be the linear code over $\mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1r_2\mathcal{F}_p$ of length n generated by the vector (1,1,…, 1), since every code word in C is of the form (a, a,.., a) with $a \in \mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1r_2\mathcal{F}_p$, we see that the minimum homogeneous weight of C is $(p-1)p^2n$. Since the generating vector (1,1,.., 1) contains units, we see that $|c| = p^4$ .Thus, by taking the Gray image of C. In the light of Theorem 1.3.2, we note that $\varphi(C)$ is a linear code over $\mathcal{F}_p$ with parameters $[p^3n,4,(p-1)p^2n]$.

3.3.2. $[p^4n, 5, (p-1)p^3n]_p$ Cods

Theorem 3.3.1.[46] Let C be the linear code over $\mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1r_2\mathcal{F}_p$, of length p generated by the vectors $\{(1, 1,1,\ldots, 1); (0, r_1r_2, 2r_1r_2,\ldots, (p-1)r_1r_2)\}$, then C is of size p^5 with minimum homogeneous distance $(p-1)p^3$.

Proof: The code words in C are of the form,

$a(1, 1,\ldots, 1)+b(0, r_1r_2, 2r_1r_2,\ldots, (p-1)r_1r_2)$ where $a \in \mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1r_2\mathcal{F}_p$ and $b \in \mathcal{F}_p$. Linear independence can be seen from the first coordinates, so $|C| = p^4 . p = p^5$. Since $\mathcal{F}_p$ is a field, for a nonzero $b \in \mathcal{F}_p$, $b(0, r_1r_2, 2r_1r_2,\ldots, (p-1)r_1r_2)$ is just a permutation of $(0, r_1r_2, 2r_1r_2,\ldots, (p-1)r_1r_2)$. This means every nonzero code word c of C is a permutation of $(a, a + r_1r_2; a + 2r_1r_2,\ldots, a + (p-1)r_1r_2)$ for some $a \in \mathcal{F}_p + r_1\mathcal{F}_p + r_2\mathcal{F}_p + r_1r_2\mathcal{F}_p$. Now if $a \notin \{0, r_1r_2,\ldots, (p-1)r_1r_2\}$, then C contains all nonzero coordinates which means $w_{hom}(\bar{C}) \geq p(p-1)p^2 = (p-1)p^3$.

If $a \in \{0, r_1r_2,\ldots, (p-1)r_1r_2\}$, then C is a permutation of $(0, r_1r_2, 2r_1r_2,\ldots, (p-1)r_1r_2)$ and hence $w_{hom}(\bar{C}) = (p-1)p^3$.■

3.3.3. [p^5n, 6, (p - 1)p^4n]p Codes

Theorem 3.3.2.[46] Let C be the linear code over $\mathcal{F}_p+r_1\mathcal{F}_p+r_2\mathcal{F}_p+r_1\,r_2\mathcal{F}_p$, of length p^2 generated by the vectors $\{\bar{1} = (1, 1, 1,\ldots, 1), \bar{b}= (0, b_1, b_2,\ldots, b_{p^2-1})\}$, where $\{0, b_1,\ldots, b_{p^2-1}\} = r_1$. ($\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$). Then C is of size p^6 with minimum homogeneous distance $(p - 1)p^4$.

Proof: Every code word in C is of the form,

$x.\bar{1}$ + y .$\bar{b}$, where $x\in \mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$, y $\in\mathcal{F}_p + r_2\mathcal{F}_p$. Since linear independence is obvious from the first coordinates, we see that | C |= p^4. $p^2 = p^6$. Now, because of the structure of the ring $\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$, every nonzero code word in C is either of the form, $(x,x,x,\ldots,x)$ which has weight $\geq p^2 . (p - 1)p^2 = (p - 1)p^4$, or a permutation of (0, b_1, b_2,…, b_{p^2-1}), which has weight, $(p - 1) . p^3 + (p^2- p)(p - 1)p^2 = (p - 1)p^4$, or is of the form C =$(x,x$ +yb_1,..,$x +yb_{p^2-1}$) where $x\in \mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$ and y $\in \mathcal{F}_p +r_2\mathcal{F}_p$. Now, if y is a unit and $x\notin r_1$ ($\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$), then C is just a permutation of (x,x +b_1,…,$x +b_{p^2-1}$) which has no zero coordinates, and hence $w_{hom}(\bar{C})\geq p^2.(p-1)p^2 = (p-1)p^4$. If $x\in r_1$ ($\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$) and y is a unit, then C will have exactly one zero coordinate and p-1 coordinates that are multiples of $r_1\,r_2$. Thus $w_{hom}(\bar{C}) = (p-1) . p^3+(p^2-p)(p-1)p^2 = (p-1)p^4$.

Finally, if y is a multiple of r_2, then (0, yb_1,…, yb_{p^2-1}) will have exactly p zero coordinates and p^2-p coordinates that are multiples of $r_1\,r_2$ which have weight $(p^2 - p)p^3 = (p-1)p^4$. The case when $x\neq0$ with this last case is very similar to the case handled before. Note that, by taking the repetitions of the generators in the previous theorem, and then applying the Gray map one can obtain linear codes over $\mathcal{F}_p$ with parameters [p^5n, 6, (p - 1)p^4n].

3.3.4. [p^6n,7,(p - 1)p^5n]p Codes

With a similar argument, we can find linear codes over $\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$ of length p^3, size p^7 and minimum homogeneous weight $(p - 1)p^5$. The generators one can take to construct this code would be (1, 1, 1,…, 1) and (0, b_1,…,b_{p^3-1}) where the set { 0, b_1,…, b_{p^3-1} } is the set of all zero-divisors of the ring $\mathcal{F}_p+ r_1\mathcal{F}_p+r_2\mathcal{F}_p +r_1\,r_2\mathcal{F}_p$. This code would have minimum homogeneous weight $(p-1)p^5$, the proof of which, is similar to the previous ones, will be omitted. Then, taking repetitions of the generators and applying the Gray map, we would obtain linear codes over $\mathcal{F}_p$ with parameters [p^6n, 7,(p - 1)p^5n]. We could analyze the optimality of these codes using the Griesmer bound but since the lengths of the codes are quite restrictive

(the length being multiple of 64 for binary codes, of 729 for ternary codes, etc.), we will omit that for practical reasons.

4. DISCUSSION

The ring $\mathcal{M}_k$, labeled by $\mathcal{R}_{k,(p),}$ is a non-chain ring and non-principal ideal ring. It has not been considerd so far in the context of coding theory. We studied some theoretical aspects of this ring when p=2, k=2 and linear codes over it in [41]. In our work, we tried to focus on some paractical advantages of working on this ring. To this extent, we were able to define homogeneous weight as well as Gray maps which allowed us to come up with new constructions of some families of codes that turned out to be optimal in many of the cases over $\mathcal{R}_{2,(p)} = \mathcal{M}_2$ when p=2, p=3 or p=5. The results can be generalized to any prime p, however for practical purposes we only confined ourselves to the aforementioned cases. As can be seen from subsection 1.4 the binary and ternary codes that were constructed were optimal for a considerable number of cases, i.e the Griesmer bound was met by the parameters. Even when the codes were not optimal, they were close to being optimal since they came close to the Griesmer bound.

It is quite interesting that for n=1, the families of codes that we obtain are equivalent to Reed-Muller codes of first order. In [41], it was proved that binary Reed – Muller codes of any order are linear over the ring $F_2 + uF_2 + vF_2 + uvF_2$ with respect to the Gray map defined in [41]. But, here we see that something very similar is true even when the weight and the map are drastically different than the ones defined in [41].

There are some disadvantages of our construction. Due to the choice of the homogeneous weight, in case $\mathcal{M}_2$ every weight is divisible by p^2, so the image code must always have minimum Hamming distance a multiple of p^2. The construction of the Gray map forces the length of the code to be a multiple of p^3.

A possible direction for research could be the modifications of these optimal codes, which might allow the constructions of some more families of optimal or near-optimal codes. Since this ring is new to coding theory, some of its structure could lead to different weight and Gray maps which could allow for better bounds and less restrictions on the parameters.

5. CONCLUSION AND RECOMMENDATIONS

Codes over Frobenius ring make up an important field of study in the literature of coding theory. The ring $\mathcal{R}_{k,(p)}$being a non-chain and non-principal ideal ring, has not been considered so far with regards to coding theory. We studied some theoretical parts of this ring, in this work attempted to define homogeneous weights as well as Gray map which enabled us to come up with new constructions of some families of codes that ended up being optimal in many cases when p=2, p=3, and p= 5. The results can be generalized to any prime p. We have shown that the homogeneous weight for any p-ary local Frobenius ring has two non-zero values. The weight of an element simply depends on whether it is in the minimal ideal or not. We believe that the same is true for any local Frobenius ring. In that case, the generating character has to change in order to accommodate for the change of the residue field.

An important property of the homogeneous weight is that we get divisible codes as a result. Because the two non-zero values are described in terms of powers of the prime p, many of the codes that we have obtained in subsubsection 1.4.1 fall into the category of divisible codes, described by Ward in [37]. This makes the codes we obtained even more special because they are Griesmer-optimal and divisible.

In [43] the authors used theoretical results about the homogeneous weights for Frobenius rings, and described the homogeneous weight for the ring family $\mathcal{R}_k$, and in particular they used a generating character for the ring $\mathcal{R}_k$. Then they found a Gray map, ψ_k, by using the first order Reed-Muller codes.

In subsection 1.4 we have seen that the binary and ternary codes that were constructed were optimal for a considerable number of cases, from this, we can understand the Griesmer bound was met by the parameters, even when the codes were not optimal, they were close to being optimal since they nearer the Griesmer bound. The relative ease with which we obtained Griesmer-optimal codes, which are otherwise hard to obtain, increases the relevance of the tools we have discussed. For applications, we focused on a rather special class of p-ary local Frobenius rings. We believe that similar results can be obtained for other examples of such rings. We also think that many of the results and characterizations can be carried over to more general classes of a ring.

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