ALMOST LOCAL-GLOBAL RINGS

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ABSTRACT

ALMOST LOCAL-GLOBAL RINGS

The main purpose of this thesis is to investigate the Invariant Factor Theorem for Prüfer domains. In accordance with this aim, we give a survey of necessary and sufficient conditions on a Prüfer domain to satisfy the Invariant Factor Theorem. In this process, almost local-global rings have important role since they satisfy the USC-property. Regarding to the UCS-property, BCS-rings together with their properties are also investigeted.

ÖZET

NEREDEYSE YEREL-BÜTÜNSEL OLAN HALKALAR

Bu tezde Prüfer tamlık bölgeleri üzerinde Invariant Factor Teoremi incelenmiştir. Bu amaç doğrultusunda, Prüfer tamlık bölgesi üzerine bu teoremi sağlayabilmesi için gerekli ve yeterli koşullar verilmiştir. Bu süreçte, neredeyse yerel-bütünsel halkalar UCS özelliğine sahip oldukları için büyük önem taşımaktadırlar. UCS özelliğiyle bağlantılı olarak BCS halkaları ve özellikleri de incelenmiştir.

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

R	a commutative ring with identity 1
R^*	$R - \{0\}$
$GL(N, R)$	general linear group of R
R_M	localization of R at a maximal ideal M
$I^{(t)}$	(external) direct sum of t copies of I
I^t	ideal product in a ring
I^{-1}	inverse of an ideal I
$\sqrt{I}$	radical of an ideal I
K	quotient field of R
$Pic(R)$	Picard group of R
$I(R)$	invertible ideals of R
$P(R)$	principal ideals of R
$\subseteq$	submodule
$\subset$	proper submodule
$\cong$	isomorphic
$\mathbb{Z}$	the ring of integers
$\bigoplus_{i\in I} M_i$	direct sum of R- modules M_i
$\prod_{i\in I} M_i$	direct product of R- modules M_i
$Ann(X)$	annihilator of the set X
$Ker(\phi)$	the kernel of the map ϕ
$Im(\phi)$	the image of the map ϕ

CHAPTER 1

INTRODUCTION

In Chapter 2 we give the definitions of some basic tools about commutative algebra and its properties which are useful for our further studies.

In Chapter 3 we define reachable systems and related property called GCU-property. We examine under which condition GCU-property holds for domains which have 2-generator property and Prüfer domains. In this process we mostly use the notion of the Picard group which is a multiplicative abelian group consisting of the invertible fractional ideals modulo the principal fractional ideals. We present Simulteneuos Basis property and related Invariant Factor theorem, which are useful for Prüfer domains. Then we define local-global rings, and give some examples of them together with their properties. After giving definition of "content" for different concepts such as vectors, matrices and submodules, we define a related property called UCS-property. This property is satisfied if, for each matrix of unit content, the column space of the matrix contains a rank one projective summand of the containing free module. We observe that every almost local-global ring has the UCS-property. Moreover, we show that UCS-property is equivalent to the Simulteneous Basis property for Prüfer domains. Afterwards, we give the main theorem of this chapter, shown by ((Brewer & Klingler, 1987)): Let R be a Prüfer domain such that every proper homomorphic image of R is a local-global ring. Then R satisfies Invariant Factor Theorem, so that R has the GCU-property if and only if the Picard group of R is torsion-free.

Chapter 4 consists of some special tools which we mostly use in the chapter 5. First, we remember some topological concepts, and then examine topological groups and their some properties to understand inverse systems and their limits, in particularly in a ring R. We show that $(R, +)$ is a topological ring with $\{I^n \mid n \geq 0\}$ as a fundamental system of neighorhoods at 0, where I is ideal of R and construct inverse sytem of quotient rings:

$$R/I \longleftarrow R/I^2 \longleftarrow R/I^3 \longleftarrow \dots \quad .$$

We denote the completion of R, the inverse limit of this sytem, by $\tilde{R}$ and say R is I-adically complete if $R \cong \tilde{R}$. We examine for which condition on I a ring R is I-adically complete. Then we work out idempotent lifting, which is closely associated with the I-adic completion of a ring R. If I is an ideal in a ring R, we say that an idempotent $x \in R/I$ can be lifted to R if there exists an idempotent element $e \in R$ whose image under the natural map $R \longrightarrow R/I$ is x. For an arbitrary ideal I, we certainly do not expect every idempotent $x \in R/I$ to be liftable. We give some sufficent conditions on $I \subseteq R$ which guarentee the liftability of idempotents by Proposition 4.1. Afterwards, we explain the relation between idempotent matrices and projective modules.

In chapter 5 we define BCS-rings closely associated with the UCS-property on submodules and give a sufficent condition on ideal I of a ring R for R to be a BCS-ring, while so is the quotient ring R/I. Then we examine some properties of Von Neumann regular rings and reduced rings in order to help us to work out the main theorem of this chapter.

In chapter 6 we include some results over almost local-global rings which might improve the research on comparisons of some "weaker" forms of isomorphism of modules, done by B. Ay Saylam and L. Klingler.

In Conclusion we summarize the main results obtained in this thesis.

CHAPTER 2

PRELIMINARIES

This chapter consists of some basic tools about commutative algebra that are used in this thesis. All rings mentioned in this thesis are commutative with identity.

2.1. Rings and Ideals

Definition 2.1 *A proper ideal P is a* ***prime ideal*** *if $xy \in p$, then $x \in P$ or $y \in P$. A proper ideal M is a* ***maximal ideal*** *if there is no ideal I such that $M \subset I \subset < 1 >$. Equivalently:*

$$P \ \ is \ \ prime \ \ \Leftrightarrow \ \ R/P \ \ is \ \ an \ \ integral \ \ domain.$$
$$P \ \ is \ \ maximal \ \ \Leftrightarrow \ \ R/P \ \ is \ \ a \ \ field.$$

Definition 2.2 *A prime ideal is said to be a* ***minimal prime ideal*** *if it is a minimal prime ideal containing the zero ideal. In an integral domain, the only minimal prime ideal is the zero ideal.*

Definition 2.3 *The* ***Krull dimension*** *of a ring R is the maximum number $n \geq 0$ such that there is a chain of prime ideals*

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq ... \subsetneq P_n$$

of lenght n in R. We also say that R is n-dimensional and write $dim(R) = n$. If there is no such an integer n, we say that the ring has infinite Krull dimension or has no Krull dimension. Evidently, $dim(R) = 0$ for an integral domain if and only if it is a field.

Definition 2.4 *The Jacobson Radical of R, denoted by* ***Rad(R)****, is defined to be the intersection of all maximal ideals of R. It can be characterized as follows.*

Proposition 2.1 *((Atiyah & Macdonald, 1969), Proposition 1.9) $x \in Rad(R)$ if and only if $1 - xy$ is unit for all $y \in R$.*

Definition 2.5 *A non-zero ring R with exactly one maximal ideal is called a **local ring**.*

Proposition 2.2 *((Atiyah & Macdonald, 1969), Proposition 1.6) Let R be a ring and M a proper ideal of R such that every $x \in R - M$ is unit in R. Then R is a local ring and M is its maximal ideal.*

Definition 2.6 *A ring R with finite number of maximal ideals is called **semi-local ring**.*

Definition 2.7 *Two ideals I, J of a ring R said to be **coprime** if $I + J = R$.*

Remark 2.1 *For coprime ideals I and J, the following can be seen easily:*

(i) *$I \cap J = IJ$, and there exist $a \in I$ and $b \in J$ such that $a + b = 1$.*

2.2. Modules

2.2.1. Modules of Fractions and Localizations

This section includes the definitions and basic properties of the formulation of fractions.

Let S be a submonoid of R^* (i.e., a multiplicatively closed subset such that $0 \notin S$ and $1 \in S$). The set $S^{-1}R$ of equivalence classes of pairs (r, s), $r \in R$, $s \in S$, under the equivalence relation

$$(r, s) \sim (r', s') \text{ if and only if } rs' = r's,$$

becomes a ring. If the equivalence class of (r, s) is denoted by $\frac{r}{s}$, then the ring operations are

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2} \text{ and } \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2},$$

where $r_i \in R$, $s_i \in S$. $S^{-1}R$ is called the localization of R at S or the quotient ring of R with respect to S.

Example 2.1 *If S consists of all the non-zero elements of R, $S^{-1}R$ coincides with K.*

Let M be an R-module. We form the pairs (m, s), where $m \in M$ and $s \in S$ (m, s) and say that (m', s') are equivalent if there exists a $t \in S$ such that

$$t(s'm - sm') = 0.$$

The equivalence class containing (m, s) is denoted by $\frac{m}{s}$. The R-module $S^{-1}M$ consisting of the equivalence classes is called the module of quotients of M with respect to S or the localization of M at S. It becomes an $S^{-1}R$-module by setting

$$\frac{r}{t} \cdot \frac{m}{s} = \frac{rm}{ts} \quad (r \in R,\ s, t \in S,\ m \in M).$$

The canonical homomorphism $\phi : M \to S^{-1}M$ which sends m to $m/1$ need not be monic.

2.2.2. Free and Projective Modules

Definition 2.8 *The rank of a R-module M, where R is an integral domain, is the dimension of the $M \otimes K$, where K is the quotient field of R.*

Definition 2.9 *A free R-module is one that is isomorphic to an R-module of the form $\oplus_{i \in I} M_i$, where each $M_i \cong R$ (as an R-module). Therefore, a finitely generated free R-module is isomorphic to $R^{(n)}$ for some positive integer n, and the rank of a finitely generated free R-module is defined as the number of elements in the basis of the free module, which is unique for a commutative ring.*

Definition 2.10 *A module P is* ***projective*** *if there is a free module F such that $P \oplus Q = F$, for some $Q \subseteq F$.*

Definition 2.11 *We shall say that projective R-module P has a* ***constant rank*** *if the rank of P_M is the same in each localization of R where R is an integral domain and P_M is localization of P at a maximal ideal M.*

Proposition 2.3 *An R-module P is projective if, whenever $\phi : B \longrightarrow P$ is an R-module epimorphism, then B decomposes into an R-module direct sum :*

$$B = Ker(\phi) \oplus X.$$

Remark 2.2 *A free module is a projective module, but the converse holds for local rings and PIDs.*

2.2.3. Exact sequences and Five lemma

Definition 2.12 *A sequence of R-modules and R-homomorphisms*

$$... \to R_{i-1} \xrightarrow{f_i} R_i \xrightarrow{f_{i+1}} R_{i+1} \to ...$$

*is said to be exact at R_i if $Im(f_i) = Ker(f_{i+1})$. The sequence is **exact** if it is exact at each R_i. In particular,*

- $0 \to R_1 \xrightarrow{f} R_2$ *is exact* $\Leftrightarrow$ f *is injective,*
- $R_1 \xrightarrow{g} R_2 \to 0$ *is exact* $\Leftrightarrow$ g *is surjective.*

We give a special commutative diagram with exact rows, which will be useful Chapter 3.

Lemma 2.1 *((Alizade & Pancar, 1999), Five Lemma) Let $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5$ and $B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow B_4 \longrightarrow B_5$ be exact sequences, and suppose the diagram*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "h_1"] & A_2 \arrow[r] \arrow[d, "h_2"] & A_3 \arrow[r] \arrow[d, "h_3"] & A_4 \arrow[r] \arrow[d, "h_4"] & A_5 \arrow[d, "h_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

commutes.

(i) If h_1 is an epimorphism, h_2 and h_4 are monomorphisms, then h_3 is a monomorphism.

(ii) If h_5 is a monomorphism, h_2 and h_4 are epimorphisms, then h_3 is an epimorphism.

(iii) If h_1 is an epimorphism, h_5 is a monomorphism and h_2 and h_4 are isomorphisms, then h_3 is an isomorphism.

2.3. Fractional Ideals

Definition 2.13 *Let R be an integral domain with the quotient field K. A **fractional ideal** of an integral domain R, is an R-submodule J of K such that $rJ \leq R$ for some non-zero $r \in R$.*

Remark 2.3 *The followings can be seen easily:*

(i) *A R-submodule of K is a fractional ideal if and only if it is isomorphic to an ideal of R.*

(ii) *The ideals of R are clearly fractional ideals.*

(iii) *A finitely generated submodule of K is a fractional ideal.*

For R-submodules I and J of K, we have a binary operation which is called the product:

$$IJ = \{\sum_{i=1}^{n} a_i b_i | a_i \in I, b_i \in J, n < \omega\}.$$

Definition 2.14 *A fractional ideal I of an integral domain R is said to be **invertible** if there exists a fractional ideal J of R such that $IJ = R$.*

Definition 2.15 *If R is an integral domain, the **Picard group** of R is the (multiplicative) abelian group consisting of the invertible fractional ideals of R modulo the principal fractional ideals of R, denoted by $Pic(R)$.*

2.4. Valuation Rings and Prüfer Domains

Definition 2.16 *A ring R said to be a **valuation ring** if the ideals of R totally ordered by inclusion. Equivalently, if $a, b \in R$, then either $a \in Rb$ or $b \in Ra$. A valuation ring that is a domain is called **valuation domain**.*

Remark 2.4 *A valution ring R contains a unique max ideal, which is the Jacobson Radical of R, and it consits of all non-ivertible elements of R.*

Definition 2.17 *An integral domain R is a **Prüfer domain** if all its localizations at maximal ideals are valuation domains; thus, Prüfer domains are those domains which are locally valuation domains.*

Remark 2.5 *Clearly, if R is a Prüfer domain and P is a non-zero prime ideal of R, then R_P is a valuation domain.*

Theorem 2.1 *((Fuchs & Salce, 2001), Theorem 1.1) For a domain R, the following conditions are equivalent.*

(i) *R is a Prüfer domain.*

(ii) *Every finitely generated non-zero fractional ideal is invertible.*

(iii) *The lattice of the fractional ideals of R is distributive: for fractional ideals A, B, C of R,*

$$A \cap (B + C) = (A \cap B) + (A \cap C).$$

Remark 2.6 *We note that if R is a Prüfer domain, then each of its finitely generated ideals are projective since invertible ideals are projective.*

CHAPTER 3

INVARIANT FACTOR THEOREM IN PRUFER DOMAINS

3.1. Good Modules and Steinitz Property

Definition 3.1 *Given a pair of matrices F and G of sizes $n \times n$ and $n \times m$, respectively, over a commutative ring R, the n-dimensional system (F, G) is said to be* ***reachable*** *if the columns of the matrix $[G, FG, F^2G, ..., F^{n-1}G]$ span $R^{(n)}$.*

Definition 3.2 *Given an R -module E, we say that E is* ***good*** *if and only if there exists a sequence*

$$R^{(m)} \xrightarrow{G} R^{(n)} \xrightarrow{F} R^{(n)}$$

of R-homomorphisms G and F such that the system (F, G) is reachable and E is isomorphic to the image of G.

Theorem 3.1 *((Brewer & Klingler, 1987),Theorem 1) Let R be an integral domain with a fractional ideal I of R.*

(i) *If I is a good R module, then there exists a positive integer t such that I^t is isomorphic to finitely generated free R-module. In particular, I^t is principal, so that I is invertible.*

(ii) *Suppose that I is generated by 2 elements. If I^n is principal for some positive integer n, then I is a good R module.*

Proof **(i)** Suppose that $R^{(m)} \xrightarrow{G} R^{(n)} \xrightarrow{F} R^{(n)}$, where the image of G is rank one and

$$< Im(G), Im(FG), ..., Im(F^{n-1}G) >= R^{(n)}.$$

If $Im(G) \cong I$, then $Im(G) = Ix_1$ for $x_1 \in K$, where K is the quotient field of R. Let $F(x_i) = x_{i+1}$ for $1 \leq x \leq n-1$. Then we have;

$$\begin{aligned} Im(G) &= Ix_1 \\ Im(FG) &= F(Ix_1) = Ix_2 \\ Im(F^2G) &= F(Ix_2) = Ix_3 \\ \ldots\ldots\ldots\ldots\ldots & \\ Im(F^nG) &= F(Ix_{n-1}) = Ix_n. \end{aligned}$$

Then $\{x_1, x_2, .., x_n\}$ is linearly independent over K since (F, G) is reachable. Consequently,

$$< Im(G), Im(FG), Im(F^2G), ..., Im(F^{n-1}G) >= Ix_1 \oplus Ix_2 \oplus ... \oplus Ix_n = R^{(n)}.$$

So, $R^{(n)} \cong I^{(n)}$. This implies that $I^n \cong R^n = R$ (see, (Kaplansky,1952), Lemma 1) .Therefore, I^n is principal, so it is invertible. There exists a fractional ideal J such that $I^nJ = II^{n-1}J = R$, which means I is invertible.
(ii) Since I^n is assumed to be principal, it is invertible. Thus, I is invertible. Moreover, since I is generated by 2 elements, we can write $1 = ax+by$, where $a, b \in I$ and $x, y \in I^{-1}$. Hence, $I =< a, b >$ and $I^{-1} =< x, y >$. Then $I^{-t} =< x^t, y^t >$ for any positive integer t (see, (Gilmer, 1972),Theorem 6.5). Thus, we can write $1 = a^tx^t + b^ty^t$, where $a^t, b^t \in I^t$. Define $\phi : I \oplus I^t \longmapsto R$ by $\phi(u, v) = xu + y^tv$. Then $\phi(a^tx^{t-1}, b^t) = 1$, and so ϕ is surjective. By the projectivity of R, $R \oplus Ker(\phi) \cong I \oplus I^t$. Hence, we have these exact squences:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & I \cap I^t \arrow[r] \arrow[d] & I \oplus I^t \arrow[r] & I + I^t \\ 0 \arrow[r] & 0 \arrow[r] & Ker(\phi) \arrow[r] & I \oplus I^t \arrow[r, "\phi"] & R \end{tikzcd}$$

Then by Lemma 2.1 we must have, $Ker(\phi) \cong I \cap I^t = II^t = I^{t+1}$ since I and I^t are coprime. Hence

$$R \oplus I^{t+1} \cong I \oplus I^t. \tag{3.1}$$

Using this and induction on t, it follows that $I^{(t)} \cong R^{(t-1)} \oplus I^t$ for each positive integer $t \geq 2$.

For $t = 2$, $I^{(2)} \cong R \oplus I^2$ can be obtained by writing $t = 1$ in (3.1). Suppose it holds for $t = n$,

$$I^{(n)} \cong R^{(n-1)} \oplus I^n. \tag{3.2}$$

Now consider $t = n + 1$.

$$\begin{array}{rcll}
R^{(n)} \oplus I^{n+1} & = & R^{(n-1)} \oplus R \oplus I^{n+1} & \\
R^{(n)} \oplus I^{n+1} & \cong & R^{(n-1)} \oplus I^n \oplus I & (by\ \ (3.1)) \\
I^{(n)} \oplus I & \cong & R^{(n-1)} \oplus I^n \oplus I & (by\ \ (3.2)) \\
I^{(n+1)} & \cong & R^{(n-1)} \oplus R \oplus I^{n+1} & (by\ \ (3.1)) \\
I^{(n+1)} & = & R^{(n)} \oplus I^{n+1}. &
\end{array}$$

And since $I^n \cong R$, we get that $I^{(n)} \cong R^{(n)}$.

We claim that if P is a (projective) module such that $P^{(m)}$ is a finitely generated free module, then P is good. We may assume that $P^{(m)} = R^{(m)}$.

Let $G : R^{(m)} \longrightarrow R^{(n)}$ defined by $G(r_1, r_2, ..., r_n, r_{n+1}, ..., r_m) = (r_1, 0, ..., 0)$, and $F : R^{(n)} \longrightarrow R^{(n)}$ is defined by $F(r_1, r_2, ..., r_n) = (r_n, r_1, r_2, ..., r_{n-1})$. Then,

$$\begin{array}{rcl}
Im(G) & = & (r_1, 0, ..., 0, 0) \\
Im(FG) & = & (0, r_1, 0, ..., 0) \\
Im(F^2G) & = & (0, 0, r_1, ..., 0) \\
\ldots\ldots\ldots & & \\
Im(F^{n-1}G) & = & (0, 0, ..., 0, r_1).
\end{array}$$

Therefore, we clearly have,

$$< Im(G), Im(FG), Im(F^2G), ..., Im(F^{n-1}G) >= R^{(n)} = P^{(n)} and\ \ Im(G) \cong P.$$

$\square$

Definition 3.3 *Let R be an integral domain. We shall say that R has* $1\frac{1}{2}$***-generator property*** *if and only if, given an invertible fractional ideal A and a non-zero element* $a \in A$*, there exists an element* $b \in A$ *such that* $A=< a,b >$*. We shall say that R has* ***2-generator property*** *if each invertible fractional ideal of R can be generated by* 2 *elements.*

Remark 3.1 *It is clear that* $1\frac{1}{2}$*-generator property implies* 2*-generator property. And it should be noted that both the* $1\frac{1}{2}$*-generator and* 2*-generator properties are usually defined only for Prüfer domains.*

Definition 3.4 *Let R be an integral domain. We shall say that R has the* ***Steinitz property*** *if, whenever A and B are invertible fractional ideals of R, it follows that*
$R \oplus AB \cong A \oplus B$*.*

Proposition 3.1 *((Brewer & Klingler, 1987), Proposition 1) Let R be an integral domain and consider the following conditions on R.*

- ***(i)*** *R has* $1\frac{1}{2}$*-generator property.*
- ***(ii)*** *For all invertible fractional ideals A and all integral ideals B,* $A/AB \cong R/B$*.*
- ***(iii)*** *R has the Steinitz property.*
- ***(iv)*** *R has 2-generator property.*
- ***(v)*** *For each invertible fractional ideal A of R ,* $A \oplus A^{-1} \cong R \oplus R$*.*
- ***(vi)*** *For each invertible fractional ideal A of R and each positive integer t,*
  $A^{(t)} \cong R^{(t-1)} \oplus A^t$*.*

Then (i) $\Leftrightarrow$ *(ii)*$\Rightarrow$ *(iii)*$\Rightarrow$ *(iv)* $\Leftrightarrow$ *(v)* $\Leftrightarrow$ *(vi) and (iii)*$\nRightarrow$ *(ii).*

Proof **(i)**$\Rightarrow$ **(ii)** Choose any non-zero $x \in AB \subseteq A$. Then there exists $y \in A$ such that $A =< x, y >= AB + Ry$ since R has $1\frac{1}{2}$-generator property. Define $\phi : R \longrightarrow A/AB$ by $\phi(a) = ay + AB$, so that ϕ is surjective since $A = AB + Ry$ and $ay \in Ry$. For any $b \in B$, $\phi(b) = by + AB = 0_{A/AB}$, so that $B \subseteq Ker\phi$. If $d \in Ker(\phi)$, then $\phi(d) = dy + AB = 0_{A/AB} = AB$, which implies $dy \in AB$, and certainly $dx \in dAB \subseteq AB$. Thus $dA = dRx + dRy \subseteq AB$. Then multiplying both side by A^{-1}, we obtain $dR \subseteq RB$ which implies $d \in B$. Hence, $Ker(\phi) = B$. By the first isomorphism theorem, $R/B \cong A/AB$.

(ii)$\Rightarrow$ **(i)** Let A be invertible fractional ideal and $x \in A$ arbitrary non-zero element. Then $Rx = xA^{-1}A = BA$ for $B = xA^{-1} \subset R$ an integral ideal. By assumption, $A/AB \cong R/B$ is cyclic. So, $A/AB = (Ry + AB)/AB$ for some $y \in A$. Hence, $A = AB + Ry = Rx + Ry$, and

R has $1\frac{1}{2}$-generator property.

(i)$\Rightarrow$ (iii) If A and B invertible fractional ideals of R, then it suffices to find a surjective homomorphism from $A \oplus B$ to R. For this, it suffices to find scalars α and β such that $\alpha A \oplus \beta B = R$. Choose $\alpha \neq 0$ such that $\alpha A \subseteq R$. We can do this since A is fractional ideal of R . Then $\alpha A + \beta B = R$ if and only if $\alpha AB^{-1} + \beta BB^{-1} = RB^{-1}$, that is $\alpha AB^{-1} + \beta R = B^{-1}$. Let $\alpha^{'} \in \alpha AB^{-1} \subseteq B^{-1}$ with $\alpha^{'} \neq 0$. By the $1\frac{1}{2}$-generator property, there exists $\beta \in B^{-1}$ such that $B^{-1} =< \alpha^{'}, \beta >$, that is $\alpha^{'} R + \beta R = B^{-1}$. So, $\alpha AB^{-1} + \beta R = B^{-1}$ since $\alpha^{'} R \subseteq \alpha AB^{-1} \subseteq B^{-1}$. Thus, $\alpha A + \beta B = R$ for this choice of α and β. We showed that there is a surjective homomorphism, say ϕ, from $A \oplus B$ to R. By the projectivity of R, $A \oplus B \cong R \oplus Ker(\phi)$. And we have these exact squences:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & A \cap B \arrow[r] \arrow[d] & A \oplus B \arrow[r] & A + B \\
0 \arrow[r] & 0 \arrow[r] & Ker(\phi) \arrow[r] & A \oplus B \arrow[r, "\phi"] & R
\end{tikzcd}$$

Then by Lemma 2.1, we must have $Ker(\phi) \cong A \cap B = AB$ since A and B are coprime. Thus $A \oplus B \cong R \oplus AB$, R has the Steinitz property.

(iii)$\Rightarrow$ (iv) Let A be an invertible fractional ideal of R. By the Steinitz property, $A \oplus A^{-1} \cong R \oplus R$. This gives homomorphism from $R \oplus R$ onto A. Thus, A is generated by 2 elements.

(iv)$\Rightarrow$ (v) Let A be an invertible fractional ideal of R. Since R has 2-generator property, $A =< a, b >$ for some $a, b \in A$. Then $ax + by = 1$, where $x, y \in A^{-1}$. Define $\phi : A \oplus A^{-1} \longrightarrow R$ by $\phi(u, v) = ux + vb$. Since $\phi(a, y) = 1$, ϕ is surjective. By projectivity of R, $A \oplus A^{-1} \cong R \oplus Ker(\phi)$. And again by Lemma 2.1, we get $Ker(\phi) \cong AA^{-1} = R$. Thus $A \oplus A^{-1} \cong R \oplus R$.

(v)$\Rightarrow$ (iv) This is just an argument of (iii) $\Rightarrow$ (iv). Choosing $B = A^{-1}$ in (iii), it follows.

(iv)$\Rightarrow$ (vi) This was the first part of Theorem1 part (ii).

(vi)$\Rightarrow$ (iv) Let A be an invertible fractional ideal of R. For $t = 2$, $A \oplus A \cong R \oplus A^2$. This gives a homomorphism from $A \oplus A$ onto R, and hence $Ax + Ay = R$ for some $x, y \in A^{-1}$. Then writing $ax + by = 1$ for some $a, b \in A$, we get that $A =< a, b >$, so R has 2-generator property. This completes the proof of the equivalence of (iv),(v),(vi).

We note that a domain R satisfies condition (ii), if for every ideal $B \subseteq R$, R/B has the property that rank one projectives are free. Hence such a domain has the $1\frac{1}{2}$-generator property, and consequently, the Steinitz property. $\square$

3.2. Picard Group and GCU-property

In this section we define GCU-property and we will examine under which condition GCU-property holds for domains with 2-generator property and Prüfer domains.

Definition 3.5 *A vector $x \in R^{(n)}$ is* ***unimodular*** *if and only if the ideal generated by the coordinates of x is R.*

Definition 3.6 *Let R be a commutative ring. We shall say that R has* ***GCU-property*** *(Good Contains Unimodular Property) if for every reachable system (F, G) over R, the matrix G has a unimodular vector in its image. If $G : R^{(m)} \longrightarrow R^{(n)}$, then this condition is equivalent to the image of G containing a rank one free summand.*

Proposition 3.2 *((Brewer & Klingler, 1987), Proposition 2) Let R be an integral domain with 2-generator property. If R has the GCU-property, then the Picard group of R is torsion-free.*

Proof Let I be invertible fractional ideal of R with I^t principal for some positive integer t. By Theorem 3.1 (ii), I is good. So, there exist a squence

$$R^{(m)} \xrightarrow{G} R^{(n)} \xrightarrow{F} R^{(n)}$$

of R-module homomorphisms G and F such that the system (F, G) is reachable and I is isomorphic to image of G. Since R has GCU-property, $Im(G) \cong I$ contains rank one free summand. So, $xR \subseteq I$ for some $x \in R$. And since I is fractional ideal of R, there exists $r \in R$ such that $rI \subseteq R$. We get $R \cong rxR \subseteq rI \subseteq I$. Thus, $rI = R$, which means $I =< \frac{1}{r} >$ is principal. Hence, $Pic(R) = I(R)/P(R) = P(R)/P(R) \cong 1$, is torsion-free. □

Consequently, torsion-free picard group is necesssary condition for domains having 2-generator property for the GCU-property to hold. Now we examine that under which condition having torsion free Picard group is sufficent for the GCU-property to hold.

Definition 3.7 *If R is a Prüfer domain with the quotient field K. We shall say that R has the* ***Simultaneous Basis property*** *if and only if, given finitely generated projective R-modules $N \subseteq M$, there exists vectors $x_1, x_2, ..., x_m$ in $KM \cong K^{(m)}$ and invertible fractional* *ideals $J_1, J_2, ..., J_n$ and invertible integral ideals $A_1 \supseteq A_2 \supseteq ... \supseteq A_m$ $(n \leq m)$ such that*

$$\begin{aligned} M &= J_1x_1 \oplus J_2x_2 \oplus J_nx_n \oplus .. \oplus J_mx_m \\ N &= A_1J_1x_1 \oplus A_2J_2x_2 \oplus .. \oplus A_nJ_nx_n. \end{aligned}$$

Definition 3.8 *We say that R satisfies* ***Invariant Factor Theorem*** *if and only if R has the Simultaneous Basis Property, and, for each pair of finitely generated projective R-modules $N \subseteq M$, it is possible to decompose M and N simultaneously (as above) with the added condition that $J_1 = J_2 = ... = J_{n-1} = R$.*

Lemma 3.1 *((Heitmann & Levy, 1975), Lemma 4.2) The following are equivalent for finitely generated non-zero ideals A, B of a Prüfer domain.*

(i) *$A \oplus B \cong R \oplus AB$.*

(ii) *$\exists \acute{A} \cong A$ and $\acute{B} \cong B$ such that $\acute{A} + \acute{B} = R$.*

Theorem 3.2 *((Brewer & Klingler, 1987), Theorem 2) Let R be a Prüfer domain satisfying the Invariant Factor Theorem. Then R has the GCU-property if and only if the Picard group of R is torsion-free.*

Proof We first claim that R has the Stenitz property. By the Invariant Factor Theorem, we can take $M = A \oplus B = Rx_1 \oplus Jx_2$, where A, B and J are invertible fractional ideals of R and x_1, x_2 in KM. Then $Rx_1 \cong R$ is a direct summand of $A \oplus B$, and there exists onto map from $A \oplus B$ to R, which means $A + B = R$. Then by Lemma 3.1, $A \oplus B \cong R \oplus AB$ so R has the Stenitz property. Thus, by Proposition 3.1, R has 2-generator property, so that, by Proposition 3.2, the Picard group of R is torsion-free.

Conversely, suppose that the Picard group of R is torsion-free, and let E be a good R-module. This means that there exists a squence of R-modules and homomorphisms

$$R^{(k)} \xrightarrow{G} R^{(n)} \xrightarrow{F} R^{(n)}$$

such that the colums of $[G, FG, F^2G, ..., F^{n-1}G]$ span $R^{(n)}$ and $E \cong Im(G) \subseteq R^{(n)}$. There are two cases to consider.

case(i) Suppose that E has rank one, so $E \cong I$ for some fractional ideal I of R. Then I^t is principal for some positive integer t, and I is invertible by Theorem 3.1 (i). Since the

Picard group of R is torsion-free, I is principal. Hence $I \cong R$. Moreover, the proof of Theorem 3.1 (i) shows that $E \cong Ix_1$ for some x_1 in $K^{(n)}$ and

$$R^{(n)} = Ix_1 \oplus Ix_2 \oplus ... \oplus Ix_n.$$

Thus, $E \cong Ix_1$ is a rank one free summand of $R^{(n)}$. This proves the result in case the rank of E is rank one, and it does not require the Invariant Factor Theorem. □

case(ii) Suppose that the rank of E is $m \geq 2$. By the Invariant Factor Theorem, choosing $M = R^{(n)}$ and $N = E$ we can write,

$$\begin{aligned} R^{(n)} &= Rx_1 \oplus Rx_2 \oplus ... \oplus Rx_m \oplus ... \oplus Rx_n \\ E &= A_1Rx_1 \oplus ... \oplus A_{m-1}Rx_{m-1} \oplus A_mRx_m, \end{aligned}$$

where $A_1 \supseteq A_2 \supseteq ... \supseteq A_m$ are invertible integral ideals of R. If $A_1 \neq R$, then $Im(G) \subseteq A_1R^{(n)}$ and

$$< G, FG, F^2G, ..., F^{n-1}G > \subseteq A_1R^{(n)} \neq R^{(n)}$$

contradicting with (F, G) is reachable. This proves that $A_1 = R$, and so $Im(G) \cong E$ contains a rank one free summand.

3.3. Almost Local-Global Rings and UCS-property

In this section we define a special ring and new property to establish a sufficent condition for Prüfer domains to satisfy Invariant Factor theorem.

Definition 3.9 *A ring R is said to be* ***Local-Global*** *if every polynomial over R in finitely many indeterminates which represents units locally, also represents units globally.*

Remark 3.2 *R is local-global if and only if $R/Rad(R)$ is local-global.*

Proof It follows from the fact that the elements are units in R if and only if they are units modulo the $Rad(R)$. To prove this observation, first let x be a unit in R. Then there exists $y \in R$ such that $xy = 1$. Consider $(x + Rad(R))(y + Rad(R)) = xy + Rad(R) = 1 + Rad(R) = 1_{Rad(R)}$, where $(x+Rad(R)), (y+Rad(R))$ are units of $R/Rad(R)$. Conversely, if we take units of $R/Rad(R)$ such that $(x + Rad(R))(y + Rad(R)) = 1 + Rad(R)$ then

$xy + Rad(R) = 1 + Rad(R)$, so $1 - xy \in Rad(R)$. By Proposition 2.1, $1 - (1 - xy)$ is invertible. So, there exists an element $z \in R$, $xyz = zxy = 1$, which implies that x and y are units of R.

Example 3.1 *Fields are trivially local-global.*

Example 3.2 *Local rings are local-global since localization of a local ring is again itself, every polynomial which represents units locally, also represents units globally.*

Example 3.3 *Semilocal rings are local-global. To prove this statement, let R be semilocal ring with maximal ideals $M_1, M_2, .., M_n$. Define $\phi : R \longrightarrow R/M_1 \oplus R/M_2 \oplus ... \oplus R/M_n$ by $\phi(r) = (r + M_1, r + M_2, ..., r + M_n)$. Clearly ϕ is onto. And*
$Ker(\phi) = \{r \in R | f(r) = 0_{R/M_1 \oplus .. R/M_n}\} = \bigcap_{i=1}^{n} M_i = Rad(R)$. By the first isomorphism theorem,

$$R/Rad(R) \cong R/M_1 \oplus R/M_2 \oplus ... \oplus R/M_n.$$

Note that R/M_i is field for each $i = 1, 2, ..., n$, so $R/Rad(R)$ is direct sum of fields. Since fields are local-global and finite direct sum of local-global rings again local-global, $R/Rad(R)$ is local global, so by Remark 3.2 R is local-global.

Example 3.4 *Domains of Krull-dimension zero are local-global. It follows from the fact that in such rings every element is either a zero-divisor or a unit.*

Definition 3.10 *A ring R is said to be* ***Almost Local-Global*** *if each of its proper factor ring is local-global.*

Example 3.5 *Domains of Krull-dimension one are almost local-global. Since dim(R) = 1, 0 and the maximal ideals of R are the only prime ideals of R. Consider R/I for any ideal I of R. Since prime ideals of R/I are of the form J/I, where J is prime ideal of R containing I, dim$(R/I) = 0$. Thus R/I is local-global.*

Definition 3.11 ***(i)*** *For $x \in R^{(n)}$ content of x defined as the ideal of a ring R generated by coordinates of x, denoted $c(x)$.*

(ii) *If M is a submodule of $R^{(n)}$, then content of M means the ideal generated by all $c(x)$ where $x \in M$, denoted $C(M)$.*

(iii) *For matrix X content of X is the ideal generated by its entries, denoted $C(X)$.*

Lemma 3.2 *((Brewer & Klingler, 1987), Lemma 1) Let R be a local-global ring, $R^{(n)}$ a finitely generated free R-module, and x a non-zero element of $R^{(n)}$. If c(x)+c(y)= R for some $y \in R^{(n)}$, then $c(x+ey)$=R for some $e \in R$.*

Proof Let $x = (a_1, a_2, ..., a_n)$, $y = (b_1, b_2, ..., b_n)$ and $f = (z_1, z_2, ..., z_n, w)$ be the polynomial $\sum_{i=1}^{n} z_i(a_i + wb_i)$. Since $c(x)$+$c(y)$= R ,we have $\sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} r_i b_i = 1$. So, each prime ideal M of R, there is some a_i or b_j not in M. Thus, f represents units locally, so that by assumption on R, f represents units globally, that is for some $r_1, r_2, ..., r_n, e \in R$, $f(r_1, r_2, ..., r_n, e) = \sum_{i=1}^{n} r_i(a_i + eb_i)$ is unit in R. But this implies $c(x + ey)$=R . □

Definition 3.12 *We say that a commutative ring R has the* ***UCS-Property*** *(Unit Contains Unimodular Property) if for each matrix G of unit content, there exists matrix V such that GV has unit content and all 2×2 minors of GV are zero. This is equivalent to the property that, for each matrix G of unit content, the column space of G contains a rank one projective summand of the containing free module (see, (Gilmer & Heitmann, 1980).*

Theorem 3.3 *((Brewer & Klingler, 1987), Theorem 3) Let R be a ring such that every homomorphic image of R is local-global ring. Then R has the UCS-property.*

Proof Let $G = [a_{ij}]_{n \times n}$ be a matrix of unit content, that is $\sum_{i,j=1}^{n} r_{ij}a_{ij} = 1$, and let B denote the column space of G, which is a submodule of $R^{(n)}$. Choose $x = (x_1, x_2, ..., x_n) \neq 0$ in B. Then $c(x) =< x_1, x_2, ..., x_n >= \sum_{i=1}^{n} r_i x_i$ is an ideal of R. By assumption, $\bar{R} = R/c(x)$ is a local-global ring. Reducing the entries of G modulo $c(x)$, denote new matrix by $\bar{G} = [a_{ij} + c(x)]_{n \times n}$ and its column space by $\bar{B} \subseteq \bar{R}^{(n)}$. Then, $\bar{G}$ still has unit content since $\sum_{i=1}^{n} r_{ij}(a_{ij} + c(x)) = \sum_{i=1}^{n} r_{ij}a_{ij} + c(x) = 1 + c(x) = 1_{\bar{R}}$. Then, there is some $\bar{y} \in \bar{R}$ such that $c(\bar{y}) = R$ (see, (Brewer & Katz & Ullery, 1987) Proposition 3). Thus, if $y = (y_1, y_2, ..., y_n) \in B$ any pre-image of $\bar{y}$, then $c(x) + c(y) = R$. Let B_0 be submodule of B generated by 2 vectors x and y.

Let I be the ideal of R generated by the 2×2 minors of the $n \times 2$ matrix $[x, y]$, as below:

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ . & . \\ . & . \\ x_n & y_n \end{pmatrix}$$

$$I =< (x_1y_2 - x_2y_1), (x_2y_3 - x_3y_2), ..., (x_{(n-1)}y_n - x_ny_{(n-1)}) > .$$

If $I = 0$, then;

$$x_1y_2 - x_2y_1 = 0 \;\Rightarrow x_1y_2 = x_2y_1 \;\Rightarrow \frac{x_1}{x_2} = \frac{y_1}{y_2}$$

$$x_2y_3 - x_3y_2 = 0 \;\Rightarrow x_2y_3 = x_3y_2 \;\Rightarrow \frac{x_2}{x_3} = \frac{y_2}{y_3}$$

....

$$x_{n-1}y_n - x_n - y_{n-1} = 0 \;\Rightarrow x_{n-1}y_n = x_ny_{n-1} \;\Rightarrow \frac{x_{n-1}}{x_x} = \frac{y_{n-1}}{y_n}.$$

That means x and y are linearly dependent, then B_0 itself is a rank one projective summand of $R^{(n)}$ contained in B.

If $I \neq 0$, then, again by assumption, R/I is local-global ring. Reducing $c(x)+c(y) = R$ in modulo I, we get that $c(\bar{x}) + c(\bar{y}) = \bar{R}$, so that by Lemma 3.2, $c(\bar{x} + \bar{e}\bar{y}) = \bar{R}$ for some $\bar{e} \in \bar{R}$. Thus, if e is a pre-image of $\bar{e}$ in R, we get $c(x+ey)+I = R$. Suppose $c(x+ey) \subseteq M$ for some maximal ideal M of R. Then $c(x + ey) =< x_1 + ey_1, x_2 + ey_2, ..., x_n + ey_n > \subseteq M$, that means $x_i + ey_i \in M$ for each $i = 1, ..., n$. So, $x + ey$ becomes zero in $(R/M)^{(n)}$. Then x and y become linearly dependent in $(R/M)^{(n)}$. Since I is generated by 2×2 minors of the matrix $[x, y]$, $I = 0 \bmod M$. Hence $I \subseteq M$. Then $c(x + ey) + I \subseteq M$ contradicting with the fact that $c(x + ey) + I = R$. Thus, $c(x + ey) = R$, so that $x + ey$ generates a rank one free summand of $R^{(n)}$ contained in $B_0 \subseteq B$. □

We obtain two useful instances when the hypothesis of Theorem 3.3 is satisfied. They are satisfied for any one-dimensional integral domain. They are also satisfied if R is a ring having the property that each nonzero element of R belongs to only finitely many maximal ideals. In particular, such rings have the UCS-property (see, ((Hautus & Sontag, 1986), Corollary 2)).

Theorem 3.4 *((Brewer & Katz & Ullery, 1987), Theorem 6) Let R be a Prüfer domain. Then R has UCS-property if and only if R has the Simultaneous Basis property.*

Proof First let us suppose that R has the UCS-property, and let M be a non-zero finitely generated submodule of $R^{(n)}$. Set $J_1 = c(M)$, which is an ideal of R. Since M is finitely generated, so is J_1, and it is invertible since R is a Prüfer domain. Then $M = J_1(J_1{}^{-1}M)$, $(J_1{}^{-1}M)$ is finitely generated. Then, multiplying $J_1 = c(M)$ by J_1^{-1}, we have $R = J^{-1}C(M) = C(J^{-1}M)$, which means $J_1{}^{-1}M$ has a unit content. So, by the UCS-property, $J^{-1}M$ contains a rank one free summand P_1 of $R^{(n)}$. Let us write $R^{(n)} = P_1 \oplus N_1$. Thus, $J^{-1}M = P_1 \oplus M_1$, where $M_1 = J^{-1}M \cap N_1$ and multiplying $J^{-1}M = P_1 \oplus M_1$ by J_1, we have $M = J_1P_1 \oplus J_1M_1$. If $J_1M_1 \neq 0$, then set $J_2 = c(M_1)$. As above J_2 is invertible, and $J_2{}^{-1}M_1$ contains a rank one projective summand P_2 of $R^{(n)}$. Therefore, $R^{(n)} = P_1 \oplus P_2 \oplus N_2$. Thus $J_2^{-1}M_1 = P_2 \oplus M_2$, where $M_2 = J_2{}^{-1}M_1 \cap N_2$, and

$M_1 = J_2P_2 \oplus J_2M_2$. Then $M = J_1P_1 \oplus J_1M_1$, so we have $M = J_1P_1 \oplus J_1J_2P_2 \oplus J_1J_2M_2$. Continuing in this way, we eventually reach $k \leq n$ with $J_1J_2...J_kM_k = 0$ since M is finitely generated. So, $R^{(n)} = P_1 \oplus P_2 \oplus ...P_k \oplus N_k$ and $M = J_1P_1 \oplus J_1J_2P_2 \oplus ... \oplus J_1J_2...J_kP_k$. Setting $I_r = J_1J_2...J_r (1 \leq r \leq k)$, $M_1 = I_1P_1 \oplus I_2P_2 \oplus ... \oplus I_kP_k$, and we observe that (since R is a Prüfer domain) $N_k = 0$ or direct sum of rank one projectives. We conclude that R has the Simultaneous Basis property.

Conversely, suppose R has the Simultaneous Basis property, and let G be $n \times n$ matrix over R with unit content. Let $M \subseteq R^{(n)}$ be the submodule generated by columns of G. So, M has unit content. Thus, there exists rank one projective summands $P_1, P_2, ..., P_k$ of $R^{(n)}$ and $I_1 \supseteq I_2 \supseteq ... \supseteq I_k$ are ideals of R such that $M = I_1P_1 \oplus I_2P_2 \oplus ... \oplus I_kP_k$. Since M has unit content, $I_1 \supseteq I_2 \supseteq ... \supseteq I_k$ implies that $I_1 = R$, and so $I_1P_1 = P_1$. Therefore, M contains rank one projective summand of $R^{(n)}$. □

By adjusting the proof of Theorem 3.3 and using Theorem 3.4, we obtain sufficent condition for a Prüfer domain to satisfy the Invariant Factor Theorem.

Theorem 3.5 *((Brewer & Klingler, 1987), Theorem 4) Let R be a Prüfer domain such that every proper homomorphic image of R is a local-global ring. Then R satisfies the Invariant Factor Theorem, so that R has the GCU-property if and only if the Picard group of R is torsion-free.*

Proof If R is a Prüfer domain such that every proper homomorphic image of R is a local-global ring, then, by Theorem 3.3, R has the UCS-property. Let $N \subseteq M$ be finitely generated projective R modules. By Theorem 3.4, R has the Simultaneous Basis Property. Thus, with K the quotient field of R, there exist vectors $x_1, x_2, ..., x_m$ in $KM \cong K^{(m)}$, and invertible fractional ideals $J_1, J_2, ..., J_m$ and invertible integral ideals $A_1 \supseteq A_2 \supseteq ... \supseteq A_n$ such that

$$M = J_1x_1 \oplus J_2x_2 \oplus ... \oplus J_nx_n \oplus .. \oplus J_mx_m$$

$$N = A_1J_1x_1 \oplus A_2J_2x_2 \oplus .. \oplus A_nJ_nx_n.$$

To get the Invariant Factor Theorem, we must show that the decomposition can be choosen in such a way that $J_1 = J_2 = ... = J_{n-1} = R$.

By an obvious inductive argument, we can suppose that $n = m = 2$. Then multiplying $N = A_1J_1x_1 \oplus A_2J_2x_2$ by $A_1{}^{-1}$, we have $A_1{}^{-1}N = J_1x_1 \oplus A_1{}^{-1}A_2J_2x_2 \subseteq M$ (since $A_1 \supseteq A_2$, $R \supseteq A_1{}^{-1}A_2$). Since $A_1{}^{-1}N$ is a submodule of M, we can replace N by $A_1{}^{-1}N$, and we can suppose $A_1 = R$. This yields the decompositions,

$$M = J_1x_1 \oplus J_2x_2,$$

$$N = J_1x_1 \oplus A_2J_2x_2,$$

where $A_2 \subseteq R$.

Since every proper homomorphic image of R assumed to be local-global ring, it follows from the proof of Theorem 3.3 that every homomorphic image of R has the property that rank one projectives are free. Thus, as noted at the end of the proof of Proposition 3.1, R has $1\frac{1}{2}$-generator property, and hence Steinitz property.

If $A_2 = R$ in $N = J_1x_1 \oplus A_2J_2x_2$, then by the Steinintz property;

$$M = N = J_1x_1 \oplus J_2x_2 \cong R \oplus J_1J_2. \tag{3.3}$$

So $M = N = Ry_1 \oplus J_1J_2y_2$ for some $y_1, y_2 \in KM$, and we are finished in this case.

Thus, we can suppose that $A_2 \neq R$. In order to use the ideas of Theorem 3.3, we view M as a direct summand of free module. By (3.3), a matrix G with column space equal to N must be of unit content since N contains a rank one projective summand of the containing free module. As it was in the proof of Theorem 3.3, choose $x \neq 0$ in N and, consider the local-global ring $\bar{R} = R/c(x)$. Reducing the entries of G modulo $c(x)$, denote the new matrix by $\bar{G}$ and its column space by $\bar{N}$. Then $\bar{G}$ still has unit content. There is some $\bar{y} \in \bar{N}$ such that $c(\bar{y}) = \bar{R}$. Thus, if $y \in R$ pre-imge of $\bar{y}$, then $c(x) + c(y) = R$. We let I be the ideal generated by the 2×2 minors of the $n \times 2$ matrix $[x, y]$. If $I \neq 0$, then the proof of Theorem 3.3 shows that N contains rank one projective summand of the containing free module, hence of M. Say $Ry_1 \subseteq N \subseteq M$ is direct summand of M with complimentary direct summand X in the free module. Then $M = Ry_1 \oplus (X \cap M)$, where $X \cap M = J_2'y_2$ for some $y_2 \in KM$ and invertible fractional ideal J_2' of R. And similarly it follows that $N = Ry_1 \oplus (X \cap N)$ where $X \cap N \subseteq X \cap M = J_2'y_2$. So that $X \cap N = A_2'J_2'y_2$ for some invertible integral ideal A_2' of R. Thus $M = Ry_1 \oplus J_2'y_2$ and $N = Ry_1 \oplus A_2'J_2'y_2$, and so we are finished if $I \neq 0$.

It remains to show that we can choose $x \in N$ so that I is not zero. Let x be any nonzero element of $A_2J_2x_2$ in (3.3). As above, we get similary $y \in N$ such that $c(x)+c(y) = R$ since N has unit content. Let I be the ideal generated by 2×2 minors of the $n \times 2$ matrix $[x, y]$. If $I = 0$, then x and y are linearly dependent in $Kx \subseteq KN = Kx_1 \oplus Kx_2$. So, $x, y \in Kx \cap N = A_2J_2x_2$. But, then $x, y \in A_2M$, which is an integral ideal. Thus $c(x) + c(y) \subseteq A_2$, contradicting with the assumption $A_2 \neq R$. Hence, $I \neq 0$. □

Corollary 3.1 *((Brewer & Klingler, 1987), Corollary 1) If R is a Prüfer domain with the property that every non-zero element of R belongs to only finitely many maximal ideals,* *or if R is a Prüfer domain of dimension one, (in particular if R is a Dedekind domain), then R satisfies Invariiant Factor Theorem. Consequently, such a Prüfer domain R has the GCU-property if and only if Picard group of R is torsion-free.*

Proof If R satisfies either hypothesis of Theorem 3.5, then R is a Prüfer domain such that every proper homomorphic image of R is a local-global ring. The result follows from Theorem 3.5. □

CHAPTER 4

IDEMPOTENT LIFTING

4.1. Topologies and Completions

In this chapter we will examine some property of topological groups to understand inverse systems and their limits, in particular, in a ring R. First, let us remember some topological concepts.

A topological space is an ordered pair (X, τ), where X is a set and τ is a collection of subsets of X satisfying :

(i) The empty set and X belong to τ.

(ii) τ is closed under finite intersections and arbitrary unions.

Definition 4.1 *The sets in τ are called **open sets** and complements of open sets are **closed**.*

Definition 4.2 *A topological space is said to be **Hausdorff space** if and only if given two point $x \neq y \in X$ there exist open subsets such that $U \cap V$ is empty and $x \in U$ and $y \in V$.*

Definition 4.3 *A continuous function between topological spaces which has a continuous inverse function is called **homeomorphism**.*

Let G be a topological abelian group (written additively), not necessarily Hausdorff; thus G is both a topological space and an abelian group. These structures on G are compatible in the sense that the mappings $f_1 : G \times G \longrightarrow G$ defined by $f_1(x, y) = x + y$ and $f_2 : G \longrightarrow G$ defined by $f_2(x) = -x$, which are both continuous. If 0 is closed in G, then G is Hausdorff. If a is fixed element of G, the translation T_a defined by $T_a(x) = x + a$ is a homeomorphism of G onto G since T_a is continuous and its inverse is T_{-a}. Hence, if U is any neighborhood of 0 in G, then $U + a$ is a neighborhood of a in G, and conversely, every neighborhood of a appears in this form. Thus, the topology on G is uniquely determined by the neighborhood of 0 in G.

Lemma 4.1 *((Atiyah & Macdonald, 1969), Lemma 10.1) Let H be the intersection of all neighborhoods of* 0 *in G. Then ;*

(i) *H is subgroup.*

(ii) *H is clousure of* 0.

(iii) *G/H is Hausdorff.*

(iv) *G is Hausdorff if and only if* $H = 0$.

Proof **(i)** Let U_i denote all neighborhoods of 0 in G, and H be the intersection of all these neighborhoods. Since U_i is neighborhood of 0 in G, f_1 and f_2 are continuous on U_i, and so on H. And if U_i is a neighborhod of 0, then $-U_i = \{-x \mid x \in U_i\}$. This shows that if $x \in H$ then $-x \in H$. We also need to show that H is closed under addition. Let $x, y \in H$. Since f_1 is continuous on each U_i, ${f_1}^{-1}(U_i)$ is open in $G \times G$. Consider $\bigcup(X \times Y)$, where X and Y are open subsets of G. There exists $X_{0i} \times Y_{0i} \subseteq X \times Y$ such that $(0, 0) \in X_{0i} \times Y_{0i}$. Let $V = X_{0i} \cap Y_{0i}$, so $0 \in V$. If $(x, y) \in V \times V$, then $(x, y) \in X_{0i} \times Y_{0i}$. Moreover, $(x, y) \in {f_1}^{-1}(U_i)$ for each i . If we apply f_1 both sides, we get that $x + y \in U_i$ for all i, and hence $x + y \in H$.
(ii) If $x \in H$, then $x \in U_i$ for all i. So, $0 \in x - U_i$ for all neighborhoods of 0. Thus, $x \in \{\bar{0}\}$.
(iii) We showed in (ii) that $H = \{\bar{0}\}$, so H is closed. Since H is zero of G/H, G/H is Hausdorff.
(iv) If H is 0, then by (iii) $G/H \cong G$ is Hausdorff. Conversely, suppose G is Hausdorff and $H \neq 0$. Then there exists a non-zero element $y \in H$, so $y \in U_i$ for each i. But then for any open sets U_i, U_j , $U_i \cap U_j = y \neq 0$ contradicting with G is Hausdorff. □

Remark 4.1 *Let us assume that $0 \in G$ has a countable fundamental system of neighborhoods. Then the completion $\tilde{G}$ of G may be defined by Cauchy sequences.*

Definition 4.4 *A Cauchy sequence in G defined to be sequence (x_ν) of elements of G such that for any neighborhood of* 0*, there exist an integer $s(U)$ with the property that $x_\mu - x_\nu \in U$ for all $\mu, \nu \geq s(U)$. Two Cauchy sequences $(x_\nu), (y_\nu)$ are equivalent if $(x_\nu) - (y_\nu) \longrightarrow 0$ in G.*

The set of all equivalence classes of Cauchy sequences is denoted by $\tilde{\mathbf{G}}$. We note that $\tilde{G}$ is an abelian group under addition. For $x \in G$, the class of constant sequence (x) is an element $\phi(x)$ of $\tilde{G}$ and $\phi : G \longrightarrow \tilde{G}$ is homomorphism of abelian groups. Note that ϕ is not generally injective. In fact, we have $Ker(\phi) = \bigcap U$, where U runs through all neighborhoods of 0 in G. So, ϕ is injective if and only if $Ker(\phi) = \bigcap U = 0$, that is G is Hausdorff.

Remark 4.2 *Let us assume that* $0 \in G$ *has a fundamental system of neighborhoods consisting of subgroups. Thus, we have a sequence of subgroups ;*

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq ... \supseteq G_n \supseteq ...$$

and $U \subset G$ *is a neighborhood of* 0 *if and only if it contains some* G_n*.*

For topologies given by sequences of subgroups there is an alternative purely algebraic definition of the completion. Suppose (x_v) is Cauchy sequence in G. Then, the image of (x_v) in G/G_n, say $(\bar{x_v})$, is ultimately constant, for instance $(\bar{x_v}) = (\bar{x_1}\bar{x_2}, .., \bar{x_n}, ..., a_n, ..., a_n)$. If we pass from $n+1$ to n, it is clear that $a_{n+1} \longrightarrow a_n$ under the projection

$$\phi_{n+1} : G/G_{n+1} \longrightarrow G/G_n.$$

Thus a Cauchy sequence $x_v \in G$ defines a coherent sequence (a_n) in the sense that $\phi_{n+1}(a_{n+1}) = a_n$ for all n. Thus, $\tilde{G}$ can equally well be defined as a set of coherent sequences with the obvious group structure.

Definition 4.5 *Let us consider any sequnce of groups* $\{G_n\}$ *and homomorphisms* $\phi_{n+1} : G_{n+1} \longrightarrow G_n$*. We call this* ***inverse system*** *and the group of all coherent sequences* (a_n) *(i.e. ;* $a_n \in G_n$ *and* $\phi_{n+1}(a_{n+1}) = a_n$*) is called* ***the inverse limit*** *of the system. With this definition, we have the notation* $\tilde{G} = \xleftarrow{limG/G_n}$*. If* $G \cong \tilde{G}$*, then we shall say that G is complete.*

We are interested in the topological group $G = R$ and $G_n = I^n$, where I is an ideal of a ring R. $(R, +)$ is a topological ring with $\{I^n \mid n \geq 0\}$ as a fundamental system of neighorhoods at 0.

4.1.1. I-adic Completion of a Ring

Let I be an ideal in a ring R. We have an inverse system of quotient rings :

$$R/I \longleftarrow R/I^2 \longleftarrow R/I^3 \longleftarrow ...$$

We write $\tilde{R}$ for the inverse limit of this system and $\tilde{R}$ completion of R with respect to I (or the I-adic completion). We say that R is I-adically complete if the natural map $\phi : R \longrightarrow \tilde{R}$ is an isomorphism. This implies the following conditions:

(i) For the injectivity of ϕ, we must have $Ker(\phi) = \bigcap_{i=1}^{\infty} I^n = 0$.

(ii) For the surjectivity of ϕ, for any sequence $(a_1, a_2, ..., a_n, ...)$ such that $a_{n+1} \equiv a_n \mod I^n$ for every n, there exists an element $a \in R$ such that $a \equiv a_n \mod I^n$ for all n.

The sequence $(a_1, a_2, ..., a_n, ...)$ in (ii) above is then Cauchy sequence, in that $a_m - a_n$ is very small for large m, n. Thus, condition (ii) guarentees the Cauchy sequence $(a_1, a_2, ..., a_n, ...)$ has limit a in R. And condition (i) guarentees the uniqueness of this limit and the Hausdorff condition.

Definition 4.6 *An ideal I of a ring R is called* ***nilpotent ideal*** *if $I^n = 0$ for some positive integer n. And I is called* ***nil ideal*** *if every element in I is nilpotent.*

Remark 4.3 *((Lam, 1991), Remark 21.30) If I is a nilpotent ideal of a ring R, then R is I-adically complete.*

Proof Let I be a nilpotent ideal of R. To show that R is I-adically complete, we need to show that $\phi : R \longrightarrow \tilde{R}$ is an isomorphism. Since I is nilpotent, $I^n = 0$ for some positive integer n, and so $Ker(\phi) = \cap_{i=1}^{\infty} I^n = 0$. Therefore, ϕ is one-to-one. Now consider any sequence $(a_1, a_2, ..., a_n, ...)$ such that $a_{n+1} \equiv a_n \mod I^n$;

$$\begin{aligned} a_2 &\equiv a_1 \quad (mod \ \ I) \\ a_3 &\equiv a_2 \quad (mod \ \ I^2) \\ \ldots\ldots\ldots\ldots\ldots & \\ a_n &\equiv a_{n-1} \quad (mod \ \ I^n). \end{aligned}$$

Since $I^n = 0$, $a_n = a_{n-1}$, and we have $a_i = a_n$ for all $i \geq n - 1$, which means the Cauchy sequence $(a_1, a_2, ..., a_n, ...)$ has a limit. Therefore ϕ is surjective. This proves that R is I-adicaly complete. □

4.2. Idempotent Lifting

If I is an ideal in a ring R, we say that an idempotent $x \in R/I$ can be lifted to R if there exists an idempotent element $e \in R$ whose image under the natural map $R \longrightarrow R/I$ is

x. An arbitrary ideal I, we certainly do not expect every idempotent $x \in R/I$ to be liftable. For instance, for $R = \mathbb{Z}$, if we take $I = 6\mathbb{Z}$, then $\bar{3}$ is an idempotent in R/I, which can not be lifted to R. We shall give a sufficent condition on $I \subseteq R$ which guarentee the liftability of idempotents.

Proposition 4.1 *((Bass, 1968), Proposition 2.10) Let I be a two-sided ideal in a ring R. Suppose either that I is nil or that R is I-adically complete. Then finite sets of orthogonal idempotents can be lifted modulo I, i.e, given $a_1, a_2, ..., a_m \in R$ such that $a_i a_j = \delta_{ij} a_i$ mod I $(1 \leq i, j \leq m)$, then there exist $e_1, e_2, ..., e_m \in R$ such that $e_i \equiv a_i$ mod I and $e_i e_j = \delta_{ij} e_i$ $(1 \leq i, j \leq m)$.*

Proof Let $a \in R$. For any $n > 0$,

$$1 = (a + (1-a))^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} a^{2n-j}(1-a)^j$$

Set

$$f_n(a) = \sum_{j=0}^{2n} \binom{2n}{j} a^{2n-j}(1-a)^j = 1 - \sum_{j=n}^{2n} \binom{2n}{j} a^{2n-j}(1-a)^j.$$

Then f_n is a polynomial with integer coefficients, i.e, it lies in the ideal generated by a, and we have

$$\begin{aligned} f_n(a) &\equiv 0 \quad in \ \ mod \quad a^n R, \\ f_n(a) &\equiv 1 \quad in \ \ mod \quad (1-a)^n R. \end{aligned}$$

These implies $f_n(a)^2 \equiv f_n(a)$ mod $(a(1-a))^n R$. Since $a^n R + (1-a)^n R = R$, it follows that $(a(1-a))^n R = a^n R \cap (1-a)^n R$. Hence, we also conclude that $f_n(a) = f_{n-1}(a)$ mod $(a(1-a))^{n-1} R$. So, we have $f_1(a) = \binom{2}{0} a^2 + \binom{2}{1} a(1-a) = a^2 + 2a(1-a) = 2a - a^2 = a + a(1-a) \equiv a$ mod $a(1-a)R$. Thus, $f_n(a) \equiv a$ mod $a(1-a)R$.

Now suppose $a^2 - a = a(1-a)$ is nilpotent. Then the congruences above show that, for large n, we have $f_n(a) \equiv a$ mod $(a^2 - a)R$ and $f_n(a^2) = f_n(a)$. This shows we can lift an idempotent modulo a nil ideal J (for we are then given a with $a^2 - a \in I$). If, R is I-adically complete, then we can inductively construct $e_n \in R$ such that $e_1 = a$, ${e_n}^2 \equiv e_n$ mod I^n, and $e_{n+1} \equiv e_n$ mod I^n. This is because I/I^n is nilpotent. Now $\{e_n\}$ converges to an $e \in R$, such that $e \equiv a$ mod I, and $e^2 = e$. This proves the proposition for a single idempotent.

In general, we suppose, by induction, that $e_1, e_2, ..., e_{m-1}$ have been constructed as in the proposition. Then $e = e_1 + e_2 + ... + e_{m-1}$ is idempotent and $e \equiv a_1 + a_2 + ... + a_{m-1}$ mod I. Therefore, e and a_m are orthogonal idempotents mod I. Set $f = 1 - e$ and $b = f_m(a)$. Then $b \equiv a_m$ mod I and $eb = be = 0$. Form the sequence $f_n(b)$ as above, so that the $\{f_n(b)\}$ converges to an idempotent e_m such that $e_m \equiv b$ mod I. Since each $f_n(b)$ is a polynomial in b with zero constant term and integer coefficients, we have $ef_n(b) = f_n(b)e = 0$. Therefore, e and e_m are orthogonal. For $1 \leq i \leq m$, we have $e_i e = e_i = ee_i$, so e_m is orthogonal to these $e_i' s$. □

4.3. Relation Between Idempotent Matrices and Projective Modules

If P is a finitely generated projective R-module, we may assume (replacing P by an isomorphic module) that $P \oplus Q = R^{(n)}$ for some n, and we can consider the R-module homomorphism p from $R^{(n)}$ to itself which is the identity on P and 0 on Q. Clearly, p is idempotent, i.e. $p^2 = p$. Since any R-module homomorphism $R^{(n)} \longrightarrow R^{(n)}$ is determined by the n coordinates of the images of each of the standart basis vectors, it corresponds the multiplication by a $n \times n$ matrix. In other words, P is given by an idempotent $n \times n$ matrix p which determines P up to isomorphism. On the other hand, different idempotent matrices can give rise to the same isomorphism class of projective modules.

Lemma 4.2 *((Rosenberg, 1994), Lemma 1.2.1) If p and q are idempotent matrices of possibly different size over a ring R, the corresponding finitely generated projective R-modules are isomorphic if and only if it is possible to enlarge the size of p and q (by adding zeros in the lower-right-hand corner), so that they have the same size $N \times N$ and are conjugate under the group of invertible $N \times N$ matrices over R, $GL(N, R)$.*

Proof First, let us suppose that $u \in GL(N, R)$ and $upu^{-1} = q$. Then, right multiplication by u, induces a isomorphism from $R^{(n)}p$ to $R^{(n)}q$ since $up = qu$. So, we are done in this condition.

Now suppose p and q are idempotent matrices with sizes $n \times n$ and $m \times m$, and respectively let $R^{(n)}p \cong R^{(m)}q$. We can extend isomorphism α from $R^{(n)}p$ to $R^{(m)}q$ to an R-module homomorphism $R^{(n)}$ to $R^{(m)}$ by taking $\alpha = 0$ on the complementary module $R^{(n)}(1-p)$ and the identity on $R^{(n)}p$. Similarly, we can extend α^{-1} to an R-module homomorphism from $R^{(m)}$ to $R^{(n)}$ which is 0 on $R^{(m)}(1 - q)$ and the identity on $R^{(m)}q$. So, α is given by right multiplication by a $n \times m$ matrix a, and α^{-1} is given by right multiplication by a $m \times n$ matrix b. We also have the relations from composition of homomorphisms ;

$ab = p, ba = q, a = pa = aq, b = qb = bp$. Now take $N = n + m$, and observe that

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}^2 = \begin{pmatrix} (1-p)^2 + ab & a - pa + a - aq \\ b - bp + b - qb & ba - (1-q)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(with usual block matrix notation) and that

$$\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} . \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} . \begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

Thus $\begin{pmatrix} 1-p & a \\ b & 1-q \end{pmatrix}$ is invertible, and conjugates $p \oplus 0$ to $0 \oplus q$. The letter is of course conjugate $q \oplus 0$ by permutation matrix. □

CHAPTER 5

BCS-RINGS

A matrix with unit content corresponds to rank one projective summands as follows. Let $E = Rx_1 + Rx_2 + ... + Rx_n$ be a finitely generated submodule of $R^{(n)}$. Then E is a rank one projective summand of $R^{(n)}$ if and only if the matrix $[x_1, x_2, ..., x_n]$ has unit content.

Definition 5.1 *Let B be the submodule of $R^{(n)}$. Then B is said to be **basic** if and only if the ideal generated by the contents of all vectors in B equals R. That is, the submodule B of $R^{(n)}$ is basic if and only if $c(B) = R$. In particular, if B is basic, there exist finitely many vectors $x_1, x_2, ..., x_n$ in B such that $c(x_1) + c(x_2) + ... + c(x_n) = R$.*

Definition 5.2 *A ring R is said to be **BCS-ring** if and only if each basic submodule B of $R^{(n)}$ contains a rank one projective summand of $R^{(n)}$.*

Proposition 5.1 *((Brewer & Klingler, 1988), Proposition 1) Let R be a ring with an ideal I of R contained in the $Rad(R)$. If I is nil, or R is complete in I-adic topology, then R is BCS-ring if and only if R/I is BCS-ring.*

Proof Let $\bar{B} \subseteq (R/I)^n = \bar{R}^{(n)}$ be basic, that is $c(\bar{B}) = \sum_{i=1}^{n} r_i\bar{b}_i = 1_{\bar{R}} = 1+I$, then $\sum_{i=1}^{n} r_ib_i = 1$. So, B is a basic submodule of $R^{(n)}$. Since R is assumed to be a BCS-ring , B contains a rank one summand P of $R^{(n)}$. Then P/IP is also a rank one summand of $\bar{R}^{(n)}$. Hence, $\bar{R}$ is a BCS-ring.

Conversely, if $B \subseteq R^{(n)}$ is basic, that is $\sum_{i=1}^{n} r_ib_i = 1$, then $\bar{B} \subseteq \bar{R}^{(n)}$ is also basic since $\sum_{i=1}^{n} r_i\bar{b}_i = 1_{\bar{R}}$. Since $\bar{R}$ is assumed to be a BCS-ring, $\bar{B}$ contains a rank one projective summand $\bar{P}$ of $\bar{R}^{(n)}$ and $\bar{P}$ is the image of idempotent matrix $\bar{E}$ over $\bar{R}$. Now consider the ring homomorphism

$$\phi : M_{n\times n}(R) \longrightarrow M_{n\times n}(R/I)$$

defined by $\phi((r_{ij})) = (r_{ij} + I)$, which is clearly onto. And

$$Ker(\phi) = \{(r_{ij}) \in M_{n\times n}(R) \mid \phi((r_{ij})) = 0_{M_{n\times n}(R/I)}\} = M_{n\times n}(I).$$

So, by the first isomomorphism theorem,

$$M_{n\times n}(R)/M_{n\times n}(I) \cong M_{n\times n}(R/I).$$

If I is nil, then $M_{n\times n}(I)$ is also nil. While R is complete in I-adic topology, $M_{n\times n}(R)$ is also complete in $M_{n\times n}(I)$-adic topology. Thus, by Proposition 4.1, it is possible to lift the idempotent matrix $\bar{E}$ in $M_{n\times n}(\bar{R})$ to an idempotent matrix E in $M_{n\times n}(R)$. Since E is idempotent, its image is a projective summand of $R^{(n)}$. We claim that the image of E has rank one and that E can be chosen so that image of E is contained in B.

The first claim is easy to show. The image of E reduced modulo I is just the image of $\bar{E}$, so that the image of E has rank one modulo I. Since I is contained in each maximal ideal of R, the image of E has a constant rank. So, the image of E has rank one over R.

For the second claim, as in the proof of Proposition 4.1, we can lift $\bar{E}$ to a matrix A in $M_{n\times n}(R)$, in this case choosing the columns of A from the submodule B of $R^{(n)}$, which we can do because the columns of $\bar{E}$ are assumed to belong to the submodule $\bar{B}$ of $\bar{R}^{(n)}$. For a positive integer m, consider the polynomial

$$q_m(A) = \binom{2m}{j} A^{2m-j}(1-A)^j.$$

If I is nil, then $A(1-A)$ is nilpotent. As shown in the proof of Proposition 4.1, for large m, $q_m(A) = q_m(A)^2$. Thus, we can take $E = q_m(A)$. Since A divides $q_m(A)$, we can write $E = AF$ for some matrix F in $M_{n\times n}(R)$, which means that image of E is contained in the image of A.

On the other hand, if R is complete in I-adic topology, then as shown in the Proposition 4.1, the sequence $\{q_m(A)\}$ is a Cauchy. Moreover, if E is its limit, then E is the idempotent lifting of $\bar{E}$. But A is a factor of each $q_m(A)$, so that we can write $q_m(A) = Ar_m(A)$ for each positive integer m. Since $\{r_m(A)\}$ is subsequnce of a Cauchy sequence $\{q_m(A)\}$, it is also a Cauchy sequence. So, if the matrix F in $M_{n\times n}(R)$ is its limit, then $E = AF$. Thus, in either case, the image of E is contained in the image of A, which is assumed to be contained in B. Hence, R is a BCS-ring. □

Definition 5.3 *A ring R is called a* ***reduced ring*** *if it has no non-zero nilpotent elements. Equivalently, for $x \in R$, $x^2 = 0$ implies that $x = 0$.*

Moreover, if R is reduced, then the intersection of all prime ideals which forms an ideal of R, denoted by Nil(R), is zero. Clearly, for any ring R, $R/Nil(R)$ is reduced. And if R is reduced, then the union of the all minimal prime ideals equals the set of all zero divisors in R. Note that any integral domain is a reduced ring since nilpotent elements are zero divisors. But, the converse is not true.

Definition 5.4 ***(i)*** *The* ***radical*** *of an ideal I in a ring R, denoted by* $\sqrt{I}$*, is defined as*

$$\sqrt{I} = \{r \in R \mid r^n \in I \ \ for \ \ some \ \ positive \ \ integer \ \ n\}.$$

(ii) *If I is coincides with its own radical, then I is called* ***radical ideal****. Namely, for any* $r \in R$ *and any positive integer n if* $r^n \in I$*, then* $r \in I$.

Proposition 5.2 *((Lam, 1991), Theorem 10.7) For any ring R and any ideal I of R ,* $\sqrt{I}$ *equals the intersection of all the prime ideals containing I.*

Proposition 5.3 *An ideal I in a ring R is a radical ideal if and only if* R/I *is reduced.*

Proof First, let us suppose that I is radical ideal of R. To show that R/I is reduced, we need to show that if $(r+I)^n = 0_{R/I}$ then $r+I = 0_{R/I}$, which means $r \in I$. Let $(r+I)^n = 0_{R/I}$. Then $r^n \in I$. Since I is radical ideal, $r \in I$.

Conversely, suppose R/I is reduced and $r^n \in I$ for some $r \in R$ and a positive integer n. Then $0_{R/I} = r^n + I = (r+I)^n$. Since R/I is reduced, $r + I = 0$, and so $r \in I$. This shows that I is a radical ideal. □

A nonempyt subset S of a ring R is called a **m-system** if, for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$. For instance, any multiplicatively closed set S is m-system. But, the converse is not true.

Proposition 5.4 *((Lam, 1991), Proposition 10.5) Let* $S \subseteq R$ *be an m-system and P an ideal maximal with respect to being disjoint from* S*. Then P is a prime ideal.*

Definition 5.5 *Annihilator of an ideal I in a ring R, denoted by Ann(I), defined as*

$$Ann(I) = \{r \in R \mid ri = 0, \ \ \forall i \in I\}$$

.

Lemma 5.1 *((Brewer & Klingler, 1988), Lemma 1) Let R be a reduced ring.*

(i) *If I is finitely generated ideal of R having zero annihilator, then I is contained in no minimal prime ideal of R.*

(ii) *If J is an annihilator ideal of R, then J is a radical ideal.*

(iii) *If I is an ideal of R with an annihilator J, then $(I + J)/J$ has a zero annihilator in R/J.*

Proof **(i)** Let $x \in R$. We shall first prove that if P is a minimal prime ideal of R then $P \supseteq < x >$ or $P \supseteq Ann(x)$, but not both. Since $< x > Ann(x) =< 0 >\subseteq P$, where P is a minimal prime ideal, at least one of $< x >$ or $Ann(x)$ is contained in P. Suppose that both were contained in P, and consider the multiplicative subset T of R generated by x and $R - P$. Since R is a reduced ring, no power of x vanishes. If $x^m y^n = 0$ for positive integers m and n and some element y not in P (so $y \in T$), then $(xy)^t = 0$ for some positive integer t. But, then $xy = 0$ since R is reduced and $y \in Ann(x)$, which it does not. Because we assume that $Ann(x) \subseteq P$ and $y \notin P$, so $y \notin Ann(x)$. It follows that $< 0 > \cap T = \emptyset$.

Let Q be an ideal of R, maximal without T, then by Proposition 5.4, Q is prime. If $y \notin P$, then $y \notin Q$ since $y \in T$ by definition of T and $Q \cap T = \emptyset$. Therefore, Q is properly contained in P, contradicting the minimality of P. This proves the claim.

Now, let $I =< x_1, x_2, ..., x_n >$ and P a minimal prime ideal of R with $P \supseteq I$. Then $P \supseteq 0 = Ann(I) = \bigcap_{i=1}^{n} Ann(x_i)$. Hence, $P \supseteq Ann(x_j)$ for some j, contradicting the claim of the first part of the proof.

(ii) Let $J = Ann(I)$ for an ideal I of R and $x^n \in J$ for $x \in R$ and positive integer n. To show that J is radical, we need to show that $x \in J$. Since $x^n \in J$, there exists $y \in I$ such that $0 = x^n y = (xy)^n$. Since R is reduced, $xy = 0$, and so $x \in J$.

(iii) Suppose that $(r + J)$ annihalates $(I + J)/J$. Then $(r + J)(y + j + J) = ry + rj = 0_{(I+J)/J} = J$ implies that $ry \in J$ for all $y \in I$. It follows that $ryy = ry^2 = (ry)^2 = 0$. Since R is reduced, $ry = 0$ and $r \in J$. □

Definition 5.6 *A ring R is said to be a* ***Von Neumann Regular ring*** *if for every $a \in R$, there exists $x \in R$ such that $a = a^2 x$.*

Moreover, the Krull-dimension of Von Neumann regular ring is zero, so that every prime ideal is also maximal, and hence $Rad(R) = Nil(R)$. It is easy to see that (commutative) Von Neumann Regular rings are reduced. That gives very important property that $Rad(R) = 0$. And which is most important to us, every finitely generated ideal I of R is generated by an idempotent element.

Proposition 5.5 *((Goodearl, 1979), Theorem 1.16) For a ring R the following conditions are equivalent.*

(i) *R is Von Neumann Regular.*

(ii) *R has no non-zero nilpotent elements, and all prime ideals are maximal.*

(iii) *R_M is a field for all maximal ideals M of R.*

Remark 5.1 *If R is zero dimensional and reduced, then R is Von Neumann Regular.*

Proof Since R is zero dimensional, every prime is also maximal and $Rad(R) = Nil(R)$, and since R is reduced $Rad(R) = Nil(R) = 0$. To show that R is Von Neumann Regular, we must prove that R_M is a field for any maximal ideal M of R. Since R is zero dimensional and reduced, so is R_M, and MR_M is the only prime ideal of R_M, but then $R_M = 0$ since it is nil. Hence, R_M is a field as desired. □

Lemma 5.2 *((Brewer & Klingler, 1988), Lemma 2) Let R be a von Neumann regular ring. Given $\mu_1, \mu_2, ..., \mu_m \in R^{(n)}$, there exist elements $r_1, r_2, ..., r_{m-1} \in R$ such that*

$$c(r_1\mu_1 + r_2\mu_2 + ... + r_{m-1}\mu_{m-1} + \mu_m) = c(\mu_1) + c(\mu_2) + ... + c(\mu_m).$$

Proof Since $c(\mu_i)$ is an ideal of R for each $1 \leq i \leq m$, $c(\mu_1)+c(\mu_2)+...+c(\mu_m)$ is a finitely generated ideal of R. So, $c(\mu_1) + c(\mu_2) + ... + c(\mu_m) = eR$ for some idempotent $e \in R$ since any finitely generated ideal of von Neumann regular ring is generated by an idempotent element. Now consider $c(\mu_1) + c(\mu_2) + ... + c(\mu_m) = eR$. This means, $\sum_{j=1}^{n}(\sum_{i=1}^{m} r_{ij}\mu_{ij}) = eR$, where $r_{ij} \in R$ and μ_{ij}'s are component of each $\mu_i \in R^{(n)}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Then multiplying this equation by e, we get that,

$$\sum_{j=1}^{n}(\sum_{i=1}^{m} er_{ij}\mu_{ij}) = e^2R = eR = \sum_{j=1}^{n}(\sum_{i=1}^{m} r_{ij}\mu_{ij}).$$

So, every element of the form $er_{ij}\mu_{ij} \in eR$ can be written as $r_{ij}\mu_{ij} \in R$. Thus, we can replace eR by R and assume that $c(\mu_1) + c(\mu_2) + ... + c(\mu_m) = R$.

The proof is then induction on m. If $m = 1$, then $c(r_1\mu_1) = c(\mu_1)$. Now assume that $m > 1$. By induction on m, there exist elements $r_1, r_2, ..., r_{m-2} \in R$ such that for $\mu = r_1\mu_1 + r_2\mu_2 + ... + r_{m-2}\mu_{m-2} + \mu_{m-1}$, $c(\mu) = c(r_1\mu_1 + r_2\mu_2 + ... + r_{m-2}\mu_{m-2} + \mu_{m-1}) = c(\mu_1) + c(\mu_2) + ... + c(\mu_{m-1})$. Let $c(\mu_m) = eR$ and $c(\mu) = fR$ for idempotent elements e

and f of R. Since e and f are both in $((1-e)f+e)R$ and $c(\mu)+c(\mu_m) = eR+fR = R$ $((1-e)f+e)R = R$. Now consider the equations below.

$$e((1-e)\mu+\mu_m) = e(\mu - e\mu + \mu_m) = e\mu_m = \mu_m$$

since we are able to replace eR by R. And

$$(1-e)((1-e)\mu+\mu_m) = (1-e)\mu + (1-e)\mu_m.$$

Then, subsituting $\mu_m = e((1-e)\mu+\mu_m)$ in the second equation, we get

$$(1-e)((1-e)\mu+\mu_m) = (1-e)\mu + (1-e)e((1-e)\mu+\mu_m) = (1-e)\mu.$$

Note that $c((1-e)\mu)+c(\mu_m) = R$ since $c((1-e)\mu) = (1-e)fR$ and $c(\mu_m) = eR$. And it is clear that $c((1-e)\mu)+c(\mu_m) \subseteq c((1-e)\mu+\mu_m)$, so this implies $c((1-e)\mu+\mu_m) = R$. □

Theorem 5.1 *((Brewer & Klingler, 1988), Theorem) If R is a ring of Krull dimension one, then R is BCS-ring.*

Proof Let N be the nilradical of R. To show that R is a BCS-ring, it suffices to show R/N is BCS-ring by Proposition 5.1. We note that R/N is reduced. We claim that the reduced ring R/N is a BCS-ring. So, we can assume that R itself is both one dimensional and reduced.

Let $B \subseteq R^{(n)}$ be a basic submodule with $B = R\mu_1+R\mu_2+...+R\mu_m$, i.e., $c(B) = R$. We claim that B contains a rank one projective summand of $R^{(n)}$, and we do this by induction. Suppose that all 2×2 minors of the matrix $[\mu_1, \mu_2, ..., \mu_k] = 0$. That is,

$$\begin{pmatrix} \mu_{11} & \mu_{21} & . & . & \mu_{k1} \\ \mu_{12} & \mu_{22} & . & . & \mu_{k2} \\ . & . & . & . & . \\ . & . & . & . & . \\ \mu_{1n} & \mu_{2n} & . & . & \mu_{kn} \end{pmatrix}$$

$$< (\mu_{11}\mu_{22} - \mu_{21}\mu_{12}), (\mu_{21}\mu_{32} - \mu_{31}\mu_{22}), ..., (\mu_{(k-1)(n-1)}\mu_{kn} - \mu_{k(n-1)}\mu_{(k-1)n}) >= 0.$$

This gives us that $\{\mu_1, \mu_2, ..., \mu_k\}$ is linearly dependent. This is obviously the case when $k = 1$ since all vectors are linearly dependent, that gives only one vector. If $k = m$, then B is itself a rank one projective summand of $R^{(n)}$ since $c(B) = R$.

Otherwise, we make induction essentially on the difference $m - k$. Consider the vectors $\mu_1, \mu_2, ..., \mu_k, \mu_{k+1}$, and let I be the ideal of R generated by the 2×2 minors of the matrix $[\mu_1, \mu_2, ..., \mu_k, \mu_{k+1}]$. That is,

$$\begin{pmatrix} \mu_{11} & \mu_{21} & \cdot & \cdot & \mu_{(k+1)1} \\ \mu_{12} & \mu_{22} & \cdot & \cdot & \mu_{(k+1)2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{1n} & \mu_{2n} & \cdot & \cdot & \mu_{(k+1)n} \end{pmatrix}$$

$$I =< (\mu_{11}\mu_{22} - \mu_{21}\mu_{12}), (\mu_{21}\mu_{32} - \mu_{31}\mu_{22}), ..., (\mu_{(k+1)n}\mu_{k(n-1)} - \mu_{kn}\mu_{(k+1)n}) > .$$

If $I =< 0 >$, then $\{\mu_1, \mu_2, ..., \mu_k, \mu_{k+1}\}$ is linearly dependent, and we can continue the induction.

Suppose $I \neq< 0 >$ and $Ann(I) =< 0 >$. By Lemma 5.1, I is not contained in any minimal prime ideal of R. Therefore, $\sqrt{I}$ is the intersection of all maximal ideals containing I. Now consider the quotient ring $\bar{R} = R/\sqrt{I}$. The prime ideals of $\bar{R}$ are of the form $M/\sqrt{I}$, where $M \supseteq \sqrt{I}$. So, $\bar{R}$ has Krull diemension zero. We note that it is also reduced by Proposition 5.3. Thus, $\bar{R}$ is Von Neumann Regular by Remark 5.1. Consider the vectors $\bar{\mu_1}, \bar{\mu_2}, ..., \bar{\mu_{k+1}} \in \bar{R}^{(n)}$. By Lemma 5.2, there exist elements $r_1, r_2, ..., r_k \in R$ such that $c(\bar{r_1}\bar{\mu_1} + \bar{r_2}\bar{\mu_2} + ... + \bar{r_k}\bar{\mu_k} + \bar{\mu_{k+1}}) = c(\bar{\mu_1}) + c(\bar{\mu_2}) + ... + c(\bar{\mu_k}) + c(\bar{\mu_{k+1}})$. Set $\mu = r_1\mu_1 + r_2\mu_2 + ... + r_k\mu_k + \mu_{k+1}$. Then

$$c(\mu) + \sqrt{I} = c(\mu_1) + c(\mu_2) + ... + c(\mu_{k+1}) + \sqrt{I}. \qquad (*)$$

Let M be a maximal ideal of R with $c(\mu) \subseteq M$, and let us consider the field $R' = R/M$. Since $\mu_{k+1} = \mu - (r_1\mu_1 + r_2\mu_2 + ... + r_k\mu_k)$ and $\mu = 0 \bmod M$ with respect to $c(\mu) \subseteq M$, $\mu_{k+1} = -(r_1\mu_1 + r_2\mu_2 + ... + r_k\mu_k) \bmod M$. By the induction assumption, all 2×2 minors of the matrix $[\mu_1, \mu_2, ..., \mu_k] = 0$, that is $\{\mu_1, \mu_2, ..., \mu_k\}$ is linearly dependent. Thus, for $1 \leq i \leq k$, the vectors μ_i and μ_{k+1} are linearly dependent mod M. So, $I = 0 \bmod M$, it follows that $I \subseteq M$. Since $\sqrt{I}$ is intersection of all maximal ideals containing I, $\sqrt{I} \subseteq M$. But, then $c(\mu) + \sqrt{I} \subseteq M$ and by (*), $c(\mu_1) + c(\mu_2) + ... + c(\mu_{k+1}) \subseteq M$. So, $c(\mu_1) + c(\mu_2) + ... + c(\mu_{k+1})$ is contained in any maximal ideal of contains $c(\mu)$. This allows us to replace $\mu_1, \mu_2, ..., \mu_{k+1}$ with μ and to continue the induction on the submodule of B generated by $\mu, \mu_{k+2}, , ..., \mu_m$.

There is final case to treat, the case when $I \neq 0$ and $Ann(I) = J \neq 0$. Set $\bar{R} = R/J$, and note that by Lemma 5.1(ii), J is a radical ideal so that R/J is reduced. Consider $\bar{\mu_1}, \bar{\mu_2}, ..., \bar{\mu_{k+1}} \in \bar{R}^{(n)}$. Then $\bar{I} = (I + J)/J$ is the ideal of $\bar{R}$ generated by the 2×2 minors of the matrix $[\bar{\mu_1}, \bar{\mu_2}, ..., \bar{\mu_{k+1}}]$ since $Im(I) = (I + J)/J$ in $\bar{R}$ under the natural homomorphism. By Lemma 5.1(iii), $Ann(\bar{I}) = 0_{(R+J)/J}$. So, again by Lemma 5.1(i), $\bar{I}$ is not contained in any minimal prime ideal of $\bar{R}$. Therefore, $\sqrt{\bar{I}}$ is the intersection of all maximal ideals containing $\bar{I}$. Set $\bar{\bar{R}} = \bar{R}/\sqrt{\bar{I}}$, and note that $\bar{\bar{R}}$ is zero dimensional and reduced. So, $\bar{\bar{R}}$ is Von Neumann regular. Then again by Lemma 5.2, there exist $\bar{r_1}, \bar{r_2}, ..., \bar{r_k} \in \bar{R}$ such that $c(\bar{\bar{r}}_1\bar{\bar{\mu}}_1 + \bar{\bar{r}}_2\bar{\bar{\mu}}_2 + ... + \bar{\bar{r}}_k\bar{\bar{\mu}}_k + \bar{\bar{\mu}}_{k+1}) = c(\bar{\bar{\mu}}_1) + c(\bar{\bar{\mu}}_2) + ... + c(\bar{\bar{\mu}}_{k+1})$. Set $\bar{\mu} = \bar{r_1}\bar{\mu_1} + \bar{r_2}\bar{\mu_2} + ... + \bar{r_k}\bar{\mu_k} + \bar{\mu_{k+1}}$. Then

$$c(\bar{\mu}) + \sqrt{\bar{I}} = c(\bar{\mu_1}) + c(\bar{\mu_2}) + ... + c(\bar{\mu_{k+1}}) + \sqrt{\bar{I}}. \quad (**)$$

Let M be a maximal ideal of R containing J with $c(\bar{\mu}) \subset M/J$, set $\bar{M} = M/J$, and consider the field $R'' = \bar{R}/\bar{M}$. Then $\bar{\mu_{k+1}} = -(\bar{r_1}\bar{\mu_1} + \bar{r_2}\bar{\mu_2} + ... + \bar{r_k}\bar{\mu_k}) \bmod \bar{M}$. And by the induction assumption, $\{\mu_1, \mu_2, ..., \mu_k\}$ is linearly dependent in R, so in $\bar{R}$. Thus, for $1 \leq i \leq k$ the vectors $\bar{\mu_i}$ and $\bar{\mu_{k+1}}$ are linearly dependent mod $\bar{M}$. It follows that $\bar{I} \subseteq \bar{M}$, and hence, $\sqrt{\bar{I}} \subseteq \bar{M}$. But, then by (**), $c(\bar{\mu_1}) + c(\bar{\mu_2}) + ... + c(\bar{\mu_{k+1}}) \subseteq \bar{M}$, and hence, $c(\mu_1) + c(\mu_2) + ... + c(\mu_{k+1}) \subseteq M$. On the other hand, suppose that, $c(\bar{\mu}) \subseteq \bar{M}$ for some maximal ideal M of R not containing J. It is clear that $c(\mu) \subseteq M$ and $c(\mu) = c(R\mu)$. Since $J \subseteq R$, we also have $c(J\mu) \subseteq M$, and

$$\begin{aligned}
c(J\mu) &= c(J(r_1\mu_1 + r_2\mu_2 + ... + r_k\mu_k + \mu_{k+1})) \\
&= c(r_1J\mu_1 + r_2J\mu_2 + ... + r_kJ\mu_k + J\mu_{k+1}) \\
&\supseteq c(J\mu_1 + J\mu_2 + ... + J\mu_k + J\mu_{k+1}) \\
&\supseteq c(J\mu_1) + c(J\mu_2) + ... + c(J\mu_k) + c(J\mu_{k+1}).
\end{aligned}$$

So we get that $c(J\mu_i) = Jc(\mu_i) \subseteq M$ for each $i = 1, ..., k + 1$. Since $J \nsubseteq M$, $c(\mu_i) \subseteq M$, and hence, $c(\mu_1) + c(\mu_2) + ... + c(\mu_k) + c(\mu_{k+1}) \subseteq M$. We showed that if $c(\bar{\mu}) \subseteq \bar{M}$ for some maximal ideal M of R, containing J or not, then $c(\mu_1) + c(\mu_2) + ... + c(\mu_k) + c(\mu_{k+1}) \subseteq M$.

Now consider the submodule $R\mu + J\mu_1 + J\mu_2 + ... + J\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m$ of B

and examine its content:

$$\begin{aligned} c(R\mu + ... + R\mu_m) &= c(R(r_1\mu_1 + ... + r_k\mu_k + \mu_{k+1}) + J\mu_1 + ... + J\mu_{k+1} + ... + R\mu_m) \\ &\supseteq c(R\mu_1 + ... + R\mu_{k+1} + J\mu_1 + ... + J\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m) \\ &= c(R\mu_1 + ... + R\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m + J\mu_1 + ... + J\mu_{k+1}) \\ &\supseteq c(R\mu_1 + ... + R\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m) + c(J\mu_1 + ... + J\mu_{k+1}). \end{aligned}$$

Since $B = R\mu_1 + ... + R\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m$ is basic, $c(R\mu_1 + ... + R\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m) = R$, and hence, $c(R\mu + J\mu_1 + ... + J\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m) = R$ which gives $R\mu + J\mu_1 + ... + J\mu_{k+1} + R\mu_{k+2} + ... + R\mu_m$ is also basic. Thus, for some elements $v_1, v_2, ..., v_t \in R\mu + J\mu_1 + ... + J\mu_{k+1}$, the module $Rv_1 + Rv_1 + ... + Rv_t + R\mu_{k+2} + ... + R\mu_m$ is basic.

We claim that all 2×2 minors of the matrix $[v_1, v_2, ..., v_t]$ are zero. For this, it suffices to show that for any vectors $\sigma, \tau \in R\mu + J\mu_1 + ... + J\mu_{k+1}$, all 2×2 minors of the matrix $[\sigma, \tau]$ are zero. Write

$$\begin{aligned} \sigma &= r\mu + i_1\mu_1 + i_2\mu_2 + ... + i_{k+1}\mu_{k+1} \\ \tau &= s\mu + j_1\mu_1 + j_2\mu_2 + ... + j_{k+1}\mu_{k+1}. \end{aligned}$$

All 2×2 minors of the matrix $[\mu_p, \mu_q]$ are zero for $1 \leq p, q \leq k$ by the induction assumption. All 2×2 minors of the $[i_p\mu_p, j_{k+1}\mu_{k+1}]$ and $[i_{k+1}\mu_{k+1}, j_q\mu_q]$ are zero since i_{k+1} and j_{k+1} annihilate I. Clearly, all 2×2 minors of $[i_{k+1}\mu_{k+1}, j_{k+1}\mu_{k+1}]$ vanish. Finally, since $\mu = r_1\mu_1 + ... + r_k\mu_k + \mu_{k+1}$, all 2×2 minors involving μ are zero. Thus, all 2×2 minors of $[\sigma, \tau]$ are zero. This allows us to replace $\mu_1, \mu_2, ..., \mu_{k+1}$ with $v_1, v_2, ..., v_t$ and continue the induction on the submodule of B generated by $v_1, v_2, ..., v_t, \mu_{k+1}, ..., \mu_m$. This completes both the induction and the proof. □

Corollary 5.1 *((Brewer & Klingler, 1988), Corollary 2) If R is a ring of Krull dimension less than or equals one, then R is a BCS-ring.*

Proof When the dimension of R is zero, each basic submodule of $R^{(n)}$ contains a rank one free summand of $R^{(n)}$(see, (Vasconcelos & Weibel), Proposition 1.4). When the dimension of R is one, this is the Theorem 5.1. □

Remark 5.2 *Let R be a commutative ring of dimension one. By an argument similar to the proof of the theorem, we can show that if B is a basic submodule of a free R-module*

E, and if B is generated by two elements, then B contains rank one projective submodule of E that requires at most three elements to generate.

Proof Let $B = R\mu_1 + R\mu_2$, where $\mu_1, \mu_2 \in E$ and I be the ideal of R generated by the 2×2 minors of the matrix $[\mu_1, \mu_2]$. If $I = 0$, then μ_1 and μ_2 are linearly dependent. Hence, $R\mu_1 = R\mu_2$ is rank one submodule of E with one generator.

Now suppose $I \neq 0$. If $Ann(I) = 0$, then $\bar{R} = R/\sqrt{I}$ is zero dimensional and reduced, and so Von Neumann Regular. Thus, there exists $r_1 \in R$, such that $c(\bar{r_1}\bar{\mu_1} + \bar{\mu_2}) = c(\mu_1) + c(\bar{\mu_2})$. Set $\mu = r_1\mu_1 + \mu_2$. Then, we observe that every maximal ideal M of R containing $c(\mu)$, contains $c(\mu_1) + c(\mu_2)$. This allows us to replace μ_1, μ_2 with μ, and this gives that $R\mu$ is a rank one submodule of E with one generator.

On the other hand, if $J = Ann(I) \neq 0$, then set $\bar{R} = R/J$, and consider the elements $\bar{\mu_1}, \bar{\mu_2} \in \bar{R}^{(n)}$. Then $\bar{I} = (I + J)/J$ is the ideal generated by the 2×2 minors of the matrix. So, $Ann_{\bar{R}}(\bar{I}) = 0$. By the case above, there exists $r_1 \in R$, such that for $\mu = r_1\mu_1 + \mu_2$, if $c(\bar{\mu}) \subseteq \bar{M}$ for a maximal ideal of R, containing J or not, then $c(\mu_1) + c(\mu_2) \subseteq M$.

Now consider the submodule $R\mu + J\mu_1 + J\mu_2$. Since $B = R\mu_1 + R\mu_2$ is basic, $R\mu + R\mu_1 + R\mu_2$ is also basic. It is clear that $R\mu + J\mu_1 + J\mu_2 \subseteq R\mu + R\mu_1 + R\mu_2$, where $\mu = r_1\mu_1 + \mu_2$. Then choosing $R\mu = Rv_1$, $J\mu_1 = Rv_2$ and $J\mu_2 = Rv_3$, we get that $Rv_1 + Rv_2 + Rv_3$ is a rank one projective submodule of E with 3 generators since v_1, v_2, v_3 are linearly dependent by the theorem. □

CHAPTER 6

FUTURE RESEARCH

This chapter includes some results from a research paper, which has been prepared by B. Ay Saylam and L. Klingler, that might lead to comparisons of some "weaker" forms of isomorphism. For convinience, we list their definitions.

Definition 6.1 ***(i)*** *The R-modules G and H are said to be locally isomorphic if* $G_M \cong H_M$ *for all maximal ideals M of R.*

(ii) *The R-modules G and H are said to be stably isomorphic if* $G \oplus R \cong H \oplus R$.

(iii) *Two torsionless R-modules G and H are said to be nearly isomorphic if G and H are of the same rank and, for each non-zero ideal I of R, there exists an embedding* $f : G \longrightarrow H$ *such that the ideal* $Ann_R(Coker(f))$ *is comaximal with I.*

The authors have been able to use the following facts for comparing these weaker forms of isomorphism over domains of finite character. An integral domain *R* is of finite character if every non-zero element of *R* is contained in finitely many maximal ideals of *R*. The future plan is to use the facts, which are listed below, and to see whether the methods, which are developed in (Ay Saylam & Klingler) over domains of finite character, could be adapted to almost local-global domains.

Proposition 6.1 *((Ay Saylam & Klingler)) Let R be an almost local-global integral domain and P a finitely generated projective R-module of finite rank. Then P is isomorphic to a finite rank free module direct sum with an invertible ideal.*

Lemma 6.1 *((Ay Saylam & Klingler)) Let R be an almost local-global ring and G a torsionles R-module. If the rank*$(G) \geq 2$*, then G is isomorphic to a direct sum of a free R-module and an invertible ideal of R.*

CHAPTER 7

CONCLUSION

Let R be a commutative ring. We say that R has the GCU-property if for every reachable system (F, G) over R, the matrix G has a unimodular vector in its image. If $G : R^{(m)} \longrightarrow R^{(n)}$, then this condition is equivalent to the image of G containing a rank one free summand. We gave necessary condition for domains having 2-generator property to GCU-property holds, shown by ((Brewer & Klingler, 1987)): Let R be an integral domain with 2-generator property. If R has the GCU-property, then the Picard group of R is torsion-free.

An integral domain R has the Simultaneous Basis property if and only if, given finitely generated projective R-modules $N \subseteq M$, there exists vectors $x_1, x_2, ..., x_m$ in $KM \cong K^{(m)}$ and invertible fractional ideals $J_1, J_2, ..., J_n$ and invertible integral ideals $A_1 \supseteq A_2 \supseteq ... \supseteq A_m$ $(m \leq n)$ such that

$$\begin{aligned} M &= J_1x_1 \oplus J_2x_2 \oplus J_nx_n \oplus .. \oplus J_mx_m \\ N &= A_1J_1x_1 \oplus A_2J_2x_2 \oplus .. \oplus A_nJ_nx_m, \end{aligned}$$

and we say that R satisfies the Invariant Factor Theorem if and only if R has the Simultaneous Basis Property, and, for each pair of finitely generated projective R-modules $N \subseteq M$, it is possible to decompose M and N simultaneously (as above) with the added condition that $J_1 = J_2 = ... = J_{n-1} = R$. We showed that if R is a Prüfer domain satisfying the Invariant Factor Theorem, then R has the GCU-property if and only if the Picard group of R is torsion-free ((Brewer & Klingler, 1987)).

We say that a commutative ring R has the UCS-Property if for each matrix G of unit content, the column space of G contains a rank one projective summand of the containing free module. We showed that almost local-global rings have the UCS-property. Moreover, we observed that in Prüfer domains, the UCS-property is equivalent to the Simultaneous Basis property. Using all these observations, we gave a sufficient condition for Prüfer domains and Dedekind domains to satisfy Invariant Factor Theorem, shown by (Brewer & Klingler, 1987): Let R be a Prüfer domain such that every proper homomorphic image of R is a local-global ring. Then R satisfies the Invariant Factor Theorem, so that R

has the GCU-property if and only if the Picard group of R is torsion-free.

If I is an ideal in a ring R ,we say that an idempotent $x \in R/I$ can be lifted to R if there exists an idempotent element $e \in R$ whose image under the natural map $R \longrightarrow R/I$ is x. We showed that being I is nil or R is I-adically complete guarentees the liftability of idempotents. This condition is also a sufficent condition for R to be a BCS-ring, while so the quotient ring R/I is. We showed that one dimensional rings are BCS-rings, and that almost local-global rings are also BCS-ring since they satisfy UCS-property.

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