CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$ FOR $n \leq 24$

by

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APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science

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This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Master of Science, Thesis-Mathematics
June, 2012

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ABSTRACT

Let $\mathbf{C}_n =< a : a^n = 1 >$ be a cyclic group of order n, $\mathbb{Z}\mathbf{C_n}$ its integral group ring and $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n})$ be the the group of normalized units of $\mathbb{Z}\mathbf{C_n}$. Symmetric elements of $\mathbb{Z}\mathbf{C_n}$, which are fixed under natural involution *, is denoted by $\mathbb{Z}\mathbf{C_n^+}$. We denote the unit group of $\mathbb{Z}\mathbf{C_n^+}$ by $\mathcal{U}(\mathbb{Z}\mathbf{C_n^+})$ and its normalized units by $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$. $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$ is torsion free part of $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n})$. $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n})$ is trivial for $n = 1, 2, 3, 4$ and 6. For $n = 5, 7, 8, 9, 10$ and 12, $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n})$ have been characterized. In this study characterization of $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$ is completed up to $n = 24$. In calculations we have used Maple and PARI softwares.

Keywords: fundamental units, symmetric units, characterization, generator, rank, cyclic group.

$n \leq 24$ İÇİN $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$ NIN BELİRLENİŞİ

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ÖZ

$\mathbf{C}_n =< a : a^n = 1 >$ bir devirli grup olsun. $\mathbb{Z}\mathbf{C}_\mathbf{n}$ onun integral grup halkası ve $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n})$ $\mathbb{Z}\mathbf{C}_\mathbf{n}$ nin normalleşmiş birimsellerinin grubu olsun. Doğal involüsyon altında sabit olan $\mathbb{Z}\mathbf{C}_\mathbf{n}$ nin simetrik elemanları $\mathbb{Z}\mathbf{C}_\mathbf{n}^+$ ile gösterilir. $\mathbb{Z}\mathbf{C}_\mathbf{n}^+$ nin birimsel grubu $\mathcal{U}(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$ ile ve normalleşmiş birimseller grubu $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$ ile gösterilir. $n = 1, 2, 3, 4$ ve 6 için $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n})$ aşikardır . $n = 5, 7, 8, 9, 10$ ve 12 için $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n})$ daha önce belirlenmiştir. Bu çalışmada $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$ nin belirlenişi $n = 24$'e kadar tamamlandı. Hesaplamalarda Maple ve PARI programları kullanıldı.

Anahtar Kelimeler: temel birimseller, simetrik birimseller, belirleme, üreteç, rank, devirli grup.

DEDICATION

To my parents Penbe and Recep TÜFEKÇİ & my brothers Fatih and Faruk TÜFEKÇİ

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CHAPTER 1

INTRODUCTION

Let $\mathbf{C}_n =< a : a^n = 1 >$ be a cyclic group of order n, $\mathbb{Z}\mathbf{C}_n$ its integral group ring and $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n)$ be the the group of normalized units of $\mathbb{Z}\mathbf{C}_n$. Now let us consider the following subring of symmetric elements of $\mathbb{Z}\mathbf{C}_n$,

$$\mathbb{Z}\mathbf{C}_n^+ = \left\{\gamma = \sum_{i=0}^{n-1} \gamma_i a^i : \gamma_i = \gamma_{n-i}\right\}.$$

We denote the unit group of $\mathbb{Z}\mathbf{C}_n^+$ by $\mathcal{U}(\mathbb{Z}\mathbf{C}_n^+)$ and its normalized units by

$$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+) = \left\{\gamma \in \mathcal{U}(\mathbb{Z}\mathbf{C}_n^+) : \sum_{i=0}^{n-1} \gamma_i = 1\right\}.$$

Let us modify Higman's result[1] for a cyclic group of order n instead of a finite abelian group:

Corollary 1.0.1. *$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+) = F$, where F is a free abelian group with rank*

$$\rho = \tfrac{1}{2}\varphi(n) - 1.$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n)$ is trivial if $n = 1, 2, 3, 4$ or 6. Torsion free part of $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n)$ has only one generator if $n = 5, 8, 10$ or 12 and it was characterized for $n = 5, 8$ and 12 in $[2-3]$ as follows :

$$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_5) = \mathbf{C}_5 \times \left\langle -1 + a + a^4 \right\rangle \tag{1.1}$$

$$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_8) = \mathbf{C}_8 \times \left\langle -1 - (a + a^{-1}) + (a^3 + a^{-3}) + 2a^4 \right\rangle$$

$$\begin{aligned} \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{12}) &= \mathbf{C}_{12} \\ &\times\ 3 + 2(a + a^{-1}) + (a^2 + a^{-2}) - (a^4 + a^{-4})\ - 2(a^5 + a^{-5}) - 2a^6 \end{aligned}$$

Another description for $n = 8$ is given by Sehgal[1] using fiber product diagram. On the other hand, ρ is 2 if $n = 7,\ 9,\ 14$ or 18. For $n = 7$, F was characterized by Karpilovsky[4] and for $n = 7$ and 9, it was characterized by Aleev[5]. A different characterization for $n = 7$ and 9 was also given by Köklüce and Kelebek[6]

$$\mathcal{U}_1(\mathbb{Z}\mathbf{C_7}) = \mathbf{C_7} \times \langle -1 + (a + a^{-1}) \rangle \times \langle -1 + 2(a + a^{-1}) - (a^2 + a^{-2}) \rangle \tag{1.2}$$

$$\begin{aligned} \mathcal{U}_1(\mathbb{Z}\mathbf{C_9}) = \mathbf{C_9} &\times \langle -1 + (a + a^{-1}) - (a^4 + a^{-4}) \rangle \\ &\times \langle -1 + (a + a^{-1}) - (a^2 + a^{-2}) + (a^3 + a^{-3}) - 2(a^4 + a^{-4}) \rangle \end{aligned} \tag{1.3}$$

In this study, we have completed the characterization of normalized units of $\mathbb{Z}\mathbf{C}_\mathbf{n}^+$ for $n \leq 24$. In the calculations we have benefited from Maple and PARI software programmes. The following table lists the ranks of cyclic groups corresponding to orders $n \leq 24$:

Table−1

Rank ρ	0	1	2	3	4	5	7	8	10
Order n	1,2,3,4,6	5,8,10,12	7,9,14,18	15,16,20,24	11,22	13,21	17	19	23

CHAPTER 2

PRELIMINARIES

2.1. DEFİNİTİONS AND SOME BASIC FACTS

We shall give some required definitions and basic facts about group G. They are taken from graduate text. (Hungerford, 1974)

Definition 2.1.1. *A nonempty subset H of a group G is* ***a subgroup*** *of G, if it is closed under the operation of G and H, with the restriction of the operation of G, is itself a group. It is denoted by $H \leq G$.*

Lemma 2.1.1. *A nonempty subset H of a group G is a subgroup of G if and only if for any elements $x,\ y \in H$ we have that*

$$x^{-1}y \in H.$$

Definition 2.1.2. *If there exists an element a in G such that $G =< a >$, then we say that G is* ***cyclic group*** *and that a is a generator of G.*

$$G =< a >= \{a^n : n \in \mathbb{Z}\}$$

Definition 2.1.3. *Let G and H be groups. A function $f\ : G \to H$* is a **homomorphism** provided

$$f(ab) = f(a)f(b) \text{ for all } a, b \in G$$

If f is surjective, f is called an **epimorphism**. If f is bijective, f is called an **isomorphism**. In this case G and H are said to be **isomorphic** and it is denoted by

$G \cong H$.

Definition 2.1.4. *Let H be a subgroup of a group G. Given an element $a \in G$,* the subsets of the form

$$aH = \{ah : h \in H\},$$

$$Ha = \{ha : h \in H\}$$

are called the **left** and **right cosets** of the subgroup H, determined by a.

Definition 2.1.5. A subgroup N of a group G which satisfies for all $a \in G$, $aNa^{-1} = N$ is said to be **normal** in G (or a normal subgroup of G); we write $N \lhd G \; if \; N$ is normal in G.

Definition 2.1.6. If N is a normal subgroup of group G and G/N is the set of all (left) cosets of N in G, then G/N which is a group of order $[G : N]$, is called the quotient group of G by N.

Definition 2.1.7. Let G be an abelian group. An element of G is called a torsion element if it is of finite order and the subgroup

$$T(G) = \{g \in G : |g| < \infty\}$$

is called **torsion subgroup** of G. If $G = T(G)$, then we say that G is **torsion** group. If $T(G) = \{1\}$, then G is said to be **torsion-free** group.

Now, we give some definitions and properties about units of group ring RG.

Definition 2.1.8. Let R be a commutative ring with identity and G be a group. **The group ring** RG of the group G over the ring R is defined by

$$RG = \left\{ \alpha = \sum_{finite} \alpha_g g : \alpha_g \in R, g \in G \right\}$$

Since $1_R.e_G = 1_{RG}$, we can define **the group of units of** RG by

$$\mathcal{U}(RG) = \{u \in RG : u.v = v.u = 1_{RG}, \exists v \in RG\}.$$

Definition 2.1.9. The ring homomorphism $\varepsilon : RG \longrightarrow R$, defined by

$$\varepsilon(\sum_{g \in G} \alpha_g g) = \sum_{g \in G} \alpha_g$$

is called the **augmentation map** of RG.

As an important consequence of the fact that ε is a ring homomorphism, we have;

$$\varepsilon(u) \in \mathcal{U}(R), \forall u \in \mathcal{U}(RG).$$

The subgroup of units of augmentation 1 in $\mathcal{U}(RG)$ is called **normalized units** of RG and it is denoted by $\mathcal{U}_1(RG)$.

Definition 2.1.10. The anti-automorphism of a group G defined by

$$\begin{array}{rccc} * : & G & \longrightarrow & G \\ & g & \longmapsto & g^* = g^{-1} \end{array}$$

is called **involution**, and can be extended linearly over R as;

$$\begin{array}{rccc} * : & RG & \longrightarrow & RG \\ & \sum \alpha_g g & \longmapsto & \sum \alpha_g g^{-1} \end{array}$$

For any $\alpha,\ \beta \in RG$, we have following basic properties for the natural involution;

1. $(\alpha + \beta)^* = \alpha^* + \beta^*$
2. $(\alpha\beta)^* = \beta^*\alpha^*$
3. $(\alpha^*)^* = \alpha$

If $G = \mathbf{C_n} =< a : a^n = 1 >$ is a cyclic group of order n. The group $\mathbf{C_n}$ has a natural involution $*$ which is defined as;

$$\begin{array}{rccc} * : & \mathbf{C_n} & \longrightarrow & \mathbf{C_n} \\ & a & \longmapsto & a^{-1}. \end{array}$$

If we extend $*$ linearly over $\mathbb{Z}$, we obtain;

$$\begin{array}{rccc} * : & \mathbb{Z}\mathbf{C_n} & \longrightarrow & \mathbb{Z}\mathbf{C_n} \\ & \sum\limits_{i=0}^{n-1} \alpha_i a^i & \longmapsto & \sum\limits_{i=0}^{n-1} \alpha_i a^{-i}. \end{array}$$

Definition 2.1.11. The elements of the group ring RG which are fixed under the involution are called **symmetric elements.** The set of symmetric elements of RG, denoted by RG^+, is expressed as follows ;

$$RG^+ = \{\alpha \in RG : \alpha^* = \alpha\}.$$

and the set of symmetric elements of $\mathbb{Z}\mathbf{C_n}$, denoted by $\mathbb{Z}\mathbf{C_n^+}$ can be expressed as follows:

$$\mathbb{Z}\mathbf{C_n^+} = \{\gamma \in \mathbb{Z}\mathbf{C_n} : \gamma_i = \gamma_{n-i}, 1 < i < n, |a| = n\}.$$

Definition 2.1.12. The elements of the group of units of a group ring which are fixed under the involution are called **symmetric units**. The set of all symmetric units of $\mathcal{U}(RG)$, is denoted by $\mathcal{U}(RG^+)$, and expressed as follows;

$$\mathcal{U}(RG^+) = \{u \in \mathcal{U}(RG) : u^* = u\}$$

or equivalently,

$$\mathcal{U}(RG^+) = \mathcal{U}(RG) \cap RG^+.$$

The group of symmetric units of integral group ring, is denoted by $\mathcal{U}(\mathbb{Z}\mathbf{C}_{\mathbf{n}}^{+})$ and defined by

$$\mathcal{U}(\mathbb{Z}\mathbf{C}_{\mathbf{n}}^{+}) = \{\gamma \in \mathcal{U}(\mathbb{Z}\mathbf{C}_{\mathbf{n}}) : \gamma^* = \gamma\}$$

and its normalized units are defined by

$$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{n}}^{+}) = \{\gamma \in \mathcal{U}(\mathbb{Z}\mathbf{C}_{\mathbf{n}}) : \sum_{i=0}^{n-1} \gamma_i = 1\}.$$

Definition 2.1.13. Let g be an element of order n in a group G. A **Bass cyclic** unit is an element of the group ring $\mathbb{Z}\mathbf{G}$ of the form:

$$\mu_i = (1 + g + \ldots + g^{i-1})^{\phi(n)} + \frac{1 - i^{\phi(n)}}{n}\hat{g},$$

where i is an integer such that $1 < i < n-1$ and $(i, n) = 1$.

Proposition 2.1.1. *Let g be an element of finite order in a group G. Then, the element*

$$\mu_i = (1 + g + \ldots + g^{i-1})^{\phi(n)} + \frac{1 - i^{\phi(n)}}{n}\hat{g},$$

where i is an integer such that $1 < i < n-1$ and $(i, n) = 1$,is, in fact, invertible, and its inverse is

$$\mu_i^{-1} = (1 + g^i + \ldots + g^{i(k-1)})^{\phi(n)} + \frac{1 - k^{\phi(n)}}{n}\hat{g},$$

where k is an integer such that $ik \equiv 1(\text{mod } n)$.

Proposition 2.1.2. *Let g be an element of finite order n in a group G and let l be integer such that $1 < l < n-1$ and $(l, n) = 1$. Then, the Bass cyclic unit*

$$\mu_l = (1 + g + \ldots + g^{l-1})^{\phi(n)} + \frac{1 - l^{\phi(n)}}{n}\hat{g}$$

is of infinite order.

Theorem 2.1.3. *Let* $\mathbf{A}$ *be a finite abelian group. Then,* $\mathcal{U}(\mathbb{Z}\mathbf{A}) = \pm\mathbf{A} \times F$, *where* F *is a free abelian group of finite rank.*

Here the rank of F is

$$\rho(F) = \frac{1}{2}(n + 1 + n_2 - 2l),$$

where n is the order of $\mathbf{A}$, n_2 is the number of cyclic subgroups of order 2 in $\mathbf{A}$ and l is the number of all cyclic subgroups of $\mathbf{A}$.

Now, we give a result of Dirichlet's unit theorem.

Corollary 2.1.4. *Let* $n > 2$ *and* ω *is* n^{th} *root of unity. Then*

$$\mathcal{U}(\mathbb{Z}[\omega]) = \pm < \omega > \times F$$

where F *is a free abelian group of rank* $\rho = \frac{1}{2}\varphi(n) - 1$.

It is a consequence of Corollary(2.1.4) that there exist $\rho = \frac{1}{2}\varphi(n) - 1$ units $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\rho$ of $\mathbb{Z}[\omega]$ such that any unit u of $\mathbb{Z}[\omega]$ may be uniquely expressed in the form

$$u = \delta\varepsilon_1^{n_1}\varepsilon_2^{n_2}\ldots\varepsilon_\rho^{n_\rho}$$

with $n_i \in \mathbb{Z}$ and $\delta \in \pm < \omega >$, the torsion subgroup of $\mathcal{U} = \mathcal{U}(\mathbb{Z}[\omega])$. The set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\rho\}$ is called **fundamental system of units** of $\mathcal{U}$.

Theorem 2.1.5. (*Bass and Milnor*) *Let* $\mathbf{A}$ *be a finite abelian group. Then the index* $(\mathcal{U}(\mathbb{Z}\mathbf{A}) : \langle\mathcal{U}(\mathbb{Z}\mathbf{C})\rangle_c)$, *where* $\mathbf{C}$ *runs over all cyclic subgroups of* $\mathbf{A}$, *is finite.*

Theorem 2.1.6. (*Bass*)*Let* $\mathbf{C}$ *be a finite cyclic group. Then the Bass cyclic units of* $\mathbb{Z}\mathbf{C}$ *generate a subgroup of finite index in* $\mathcal{U}(\mathbb{Z}\mathbf{C})$.

Putting these two results together, we get

Theorem 2.1.7. *Let* $\mathbf{A}$ *be a finite abelian group. Then* $\mathcal{B}$, *the group generated by all Bass cyclic units of* $\mathbb{Z}\mathbf{A}$, *is of finite index in* $\mathcal{U}(\mathbb{Z}\mathbf{A})$.

Theorem 2.1.8. *Let* $\mathbf{G}$ *be a finite abelian group. Then* $\mathcal{U}(\mathbb{Z}\mathbf{G})$ *is a finitely generated group.*

Lemma 2.1.2. Any $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$ can be written as

$$\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i \mathcal{C}_i$$

where $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let us denote $\mathcal{C}_i = a^i + a^{-i}$ and $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$. If $n = 2k+1$ then $\gamma = \gamma_0 + \sum_{i=1}^{k} \gamma_i \mathcal{C}_i$. If $n = 2k$ then $\gamma = \gamma_0 + \gamma_k' a^k + \sum_{i=0}^{k-1} \gamma_i \mathcal{C}_i$. By considering their augmentations, we have

$$\varepsilon(\gamma) = \begin{cases} \gamma_0 + 2\sum_{i=1}^{k} \gamma_i & , n = 2k+1 \\ \gamma_0 + \gamma_k' + 2\sum_{i=1}^{k-1} \gamma_i & , n = 2k \end{cases}$$

by modulo 2, we obtain

$$\varepsilon(\gamma) \equiv 1 (\operatorname{mod} 2) = \begin{cases} \gamma_0 \equiv 1 (\operatorname{mod} 2) & , n = 2k+1 \\ \gamma_0 + \gamma_k' \equiv 1 (\operatorname{mod} 2) & , n = 2k \end{cases}$$

By choosing γ_0 as an odd integer in both cases we obtain $\gamma_k' = 2\gamma_k$, for some $\gamma_k \in \mathbb{Z}$

$$\gamma_k' a^k = 2\gamma_k a^k = \gamma_k (a^k + a^{-k}) = \gamma_k \mathcal{C}_k.$$

So, $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_\mathbf{n}^+)$ can be written as

$$\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i \mathcal{C}_i$$

in both cases. □

Proposition 2.1.9. *Let H be a subgroup of a finite abelian group G. We can define a group epimorphism:*

$$\begin{array}{llll} \varphi: & G & \to & G/H \\ & g & \mapsto & gH \end{array}$$

If we extend φ linearly over $\mathbb{Z}$, then we get a natural ring homomorphism as follows:

$$\begin{array}{llll} \overline{\varphi}: & \mathbb{Z}G & \to & \mathbb{Z}(G/H) \\ & \sum \gamma_g g & \mapsto & \sum \gamma_g (gH) \end{array}$$

If $G/H \cong \mathbf{C}_2,\ \mathbf{C}_3,\ \mathbf{C}_4$ or $\mathbf{C}_6$ then for any torsion free unit $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{G})$, $\overline{\varphi}(\gamma) = H$.

Remark 2.1.1. If $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+)$ and $n = 2k$ then $a^k\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+)$. The coefficient of identity of either γ or $a^k\gamma$ is odd. $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+)$ can be chosen as a generator if its coefficient is odd.

Proposition 2.1.10. *Let n be odd integer then $\mathbf{C}_{2n} =< a : a^{2n} = 1 >$ and $H =< a^2 >$. Then $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+)$ and $\mathcal{U}_1(\mathbb{Z}\mathbf{H}^+)$ have the same rank.*

Proof. We consider the group epimorphism;

$$\begin{array}{llll} f: & \mathbf{C}_{2n} & \to & H \\ & a^i & \mapsto & a^{2i}. \end{array}$$

If we extend group epimorphism linearly over $\mathbb{Z}$, we obtain the following ring epimorphism:

$$\begin{array}{llll} \overline{f}: & \mathbb{Z}\mathbf{C}_{2n} & \to & \mathbb{Z}H \\ & \sum_{i=0}^{2n-1} \gamma_i a^i & \mapsto & \sum_{i=0}^{2n-1} \gamma_i a^{2i}. \end{array}$$

If we restrict $\overline{f}$ to multiplicative torsion-free group we have

$$\overline{f}: \mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+) \to \mathcal{U}_1(\mathbb{Z}\mathbf{H}^+).$$

Since

$$\begin{aligned}\rho_{2n} &= \tfrac{1}{2}\varphi(2n) - 1\\ &= \tfrac{1}{2}\varphi(2)\varphi(n) - 1\\ &= \tfrac{1}{2}\varphi(n) - 1\\ &= \rho_n,\end{aligned}$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+)$ and $\mathcal{U}_1(\mathbb{Z}\mathbf{H}^+)$ have the same rank. □

Remark 2.1.2. Let n be odd integer. If $\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i \mathcal{C}_i$ is a generator of $\mathcal{U}_1(\mathbb{Z}\mathbf{H}^+) \subset \mathbb{Z}\mathbf{C}_{2n}$, then by proposition (2.1.10) $\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i a^{2i}$ is a generator of $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_n^+)$.

By applying remark (2.1.2) to (1.1), (1.2) and (1.3) the unit groups $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{10}^+)$, $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{14}^+)$ and $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{18}^+)$ can be determined as follows:

$$\begin{aligned}\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{10}) &= \mathbf{C}_{10} \times \left\langle -1 + a^2 + a^8 \right\rangle\\ \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{14}}) &= \mathbf{C}_{\mathbf{14}} \times \left\langle -1 + (a^2 + a^{-2}) \right\rangle \times \left\langle -1 + 2(a^2 + a^{-2}) - (a^4 + a^{-4}) \right\rangle\\ \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{18}}) &= \mathbf{C}_{\mathbf{18}} \times \left\langle -1 + (a^2 + a^{-2}) - (a^8 + a^{-8}) \right\rangle \times\\ &\qquad \left\langle -1 + (a^2 + a^{-2}) - (a^4 + a^{-4}) + (a^6 + a^{-6}) - 2(a^8 + a^{-8}) \right\rangle.\end{aligned}$$

CHAPTER 3

CONSTRUCTION OF FUNDAMENTAL UNITS

Let us denote $\omega = e^{\frac{2\pi i}{n}}$ and $\alpha = \omega + \omega^{-1}$ then we can write

$$\omega^2 + \omega^{-2} = (\omega + \omega^{-1})^2 - 2 = \alpha^2 - 2.$$

Similarly, we can express $\omega^k + \omega^{-k}$ as a linear combination of $\alpha^i,\ \alpha^{i-1},\ \ldots,\ \alpha^2,\ \alpha,\ 1$ by subsituting α into the expression $1 + \omega + \omega^{-1} + \ldots + \omega^{\frac{n-1}{2}} + \omega^{-\frac{n-1}{2}}$ we get a polynomial of degree $\frac{n-1}{2}$. By factorizing this polynomial, the largest degree of this factor is the irreducible polynomial which admits α as a root.

Example 3.0.1. *Let us find minimal polynomial for* $n = 11$. *Let* $\omega = e^{\frac{2\pi i}{11}}$ *and* $\alpha = \omega + \omega^{-1}$. *Then*

$$\begin{aligned}
\omega + \omega^{-1} &= \alpha \\
\omega^2 + \omega^{-2} &= \alpha^2 - 2 \\
\omega^3 + \omega^{-3} &= \alpha^3 - 3\alpha \\
\omega^4 + \omega^{-4} &= \alpha^4 - 4\alpha^2 + 2 \\
\omega^5 + \omega^{-5} &= \alpha^5 - 5\alpha^3 + 5\alpha
\end{aligned}$$

$$\begin{aligned}
0 &= 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10} \\
&= 1 + \omega + \omega^{-1} + \omega^2 + \omega^{-2} + \omega^3 + \omega^{-3} + \omega^4 + \omega^{-4} + \omega^5 + \omega^{-5} \\
&= \alpha^5 + \alpha^4 - 4\alpha^3 - 3\alpha^2 + 3\alpha + 1
\end{aligned}$$

Since $\alpha^5 + \alpha^4 - 4\alpha^3 - 3\alpha^2 + 3\alpha + 1$ *is irreducible, minimal polynomial is*

$$min_Q(\alpha, x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$$

Example 3.0.2. *Let us compute minimal polynomial for* $n = 15$*. Similarly, let* $\omega = e^{\frac{2\pi i}{15}}$ *and* $\alpha = \omega + \omega^{-1}$

$$\begin{aligned}
\omega + \omega^{-1} &= \alpha \\
\omega^2 + \omega^{-2} &= \alpha^2 - 2 \\
\omega^3 + \omega^{-3} &= \alpha^3 - 3\alpha \\
\omega^4 + \omega^{-4} &= \alpha^4 - 4\alpha^2 + 2 \\
\omega^5 + \omega^{-5} &= \alpha^5 - 5\alpha^3 + 5\alpha \\
\omega^6 + \omega^{-6} &= \alpha^6 - 6\alpha^4 + 9\alpha^2 - 2 \\
\omega^7 + \omega^{-7} &= \alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha
\end{aligned}$$

$$\begin{aligned}
0 &= 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10} \\
&\quad + \omega^{11} + \omega^{12} + \omega^{13} + \omega^{14} \\
&= 1 + \omega + \omega^{-1} + \omega^2 + \omega^{-2} + \omega^3 + \omega^{-3} + \omega^4 + \omega^{-4} + \omega^5 + \omega^{-5} \\
&\quad + \omega^6 + \omega^{-6} + \omega^7 + \omega^{-7} \\
&= \alpha^7 + \alpha^6 - 6\alpha^5 - 5\alpha^4 + 10\alpha^3 + 6\alpha^2 - 4\alpha - 1
\end{aligned}$$

$\alpha^7 + \alpha^6 - 6\alpha^5 - 5\alpha^4 + 10\alpha^3 + 6\alpha^2 - 4\alpha - 1$ *is reducible. Thus we should factorize it. Then we obtain*

$$\alpha^7 + \alpha^6 - 6\alpha^5 - 5\alpha^4 + 10\alpha^3 + 6\alpha^2 - 4\alpha - 1 = (\alpha^4 - \alpha^3 - 4\alpha^2 + 4\alpha + 1)(\alpha^2 + \alpha - 1)(\alpha + 1).$$

Then minimal polynomial is

$$min_Q(\alpha, x) = x^4 - x^3 - 4x^2 + 4x + 1.$$

We have used PARI software to speed computations up. The following table gives us minimal polynomials corresponding to orders n :

order n	$min_Q(\alpha, x)$
11	$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$
13	$x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1$
15	$x^4 - x^3 - 4x^2 + 4x + 1$
16	$x^4 - 4x^2 + 2$
17	$x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1$
19	$x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1$
20	$x^4 - 5x^2 + 5$
21	$x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1$
23	$x^{11}+x^{10}-10x^9-9x^8+36x^7+28x^6-56x^5-35x^4+35x^3 + 15x^2 - 6x - 1$
24	$x^4 - 4x^2 + 1$

If we denote $\omega = e^{\frac{2\pi i}{n}}$ and $\alpha = \omega + \omega^{-1}$ then the image of $\mathbb{Z}\mathbf{C_n}$ under the ring homomorphism

$$\begin{array}{cccc} \psi : & \mathbb{Z}\mathbf{C_n} & \longrightarrow & \mathbb{Z}[\omega] \\ & \sum \gamma_i a^i & \longmapsto & \sum \gamma_i \omega^i \end{array}$$

gives us the ring of integers $\mathbb{Z}[\alpha]$.

To determine the normalized units $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$ of $\mathbb{Z}\mathbf{C_n^+}$ we need to find the fundamental units of $\mathbb{Z}[\alpha]$. So we have a problem. Finding fundamental units of $\mathbb{Z}[\alpha]$ is extremely a difficult problem. In fact, with the exception of some particular values of n, no specific system of fundamental units is known. The main difficulty seems to be the fact that effective calculations in $\mathcal{U}(\mathbb{Z}\mathbf{C_n})$ are intimately connected with those in $\mathcal{U}(\mathbb{Z}[\alpha])$.

In this study, we have computed fundamental units of $\mathbb{Z}[\alpha]$ by using PARI software. The following table lists the fundamental units of $\mathbb{Z}[\alpha]$ for $n \leq 24$:

Table−3

Rank ρ	Order n	fundamental units of $\mathbb{Z}[\alpha]$
4	11	$\varepsilon_1 = \alpha,\ \varepsilon_2 = \alpha+1, \varepsilon_3 = \alpha^2-2$ $\varepsilon_4 = \alpha^4+\alpha^3-3\alpha^2+3\alpha$
$min_Q(\alpha,x) = x^5+x^4-4x^3-3x^2+3x+1$		
5	13	$\varepsilon_1 = \alpha+1, \varepsilon_2 = \alpha^2-2, \varepsilon_3 = \alpha^3-3\alpha$ $\varepsilon_4 = \alpha^4-4\alpha^2+2$ $\varepsilon_5 = \alpha^4+\alpha^3-3\alpha^2-3\alpha$
$min_Q(\alpha,x) = x^6+x^5-5x^4-4x^3+6x^2+3x-1$		
3	15	$\varepsilon_1 = \alpha-1, \varepsilon_2 = \alpha^2-3$ $\varepsilon_3 = \alpha^3-3\alpha$
$min_Q(\alpha,x) = x^4-x^3-4x^2+4x+1$		
3	16	$\varepsilon_1 = \alpha-1, \varepsilon_2 = \alpha^2-1$ $\varepsilon_3 = \alpha^2+\alpha-1$
$min_Q(\alpha,x) = x^4-4x^2+2$		
7	17	$\varepsilon_1 = \alpha^3-3\alpha$ $\varepsilon_2 = \alpha^3+\alpha^2-2\alpha-1$ $\varepsilon_3 = \alpha^4-4\alpha^2+3, \varepsilon_4 = \alpha^5-4\alpha^3+3\alpha$ $\varepsilon_5 = \alpha^5-5\alpha^3+5\alpha$ $\varepsilon_6 = \alpha^6-6\alpha^4+9\alpha^2-2$ $\varepsilon_7 = \alpha^7+\alpha^6-6\alpha^5-5\alpha^4+10\alpha^3+6\alpha^2-4\alpha-2$
$min_Q(\alpha,x) = x^8+x^7-7x^6-6x^5+15x^4+10x^3-10x^2-4x+1$		
8	19	$\varepsilon_1 = \alpha^3-3\alpha, \varepsilon_2 = \alpha^4-4\alpha^2+3$ $\varepsilon_3 = \alpha^5-4\alpha^3+3\alpha$ $\varepsilon_4 = \alpha^6-6\alpha^4+9\alpha^2-2$ $\varepsilon_5 = \alpha^6+\alpha^5-5\alpha^4-4\alpha^3+6\alpha^2+3\alpha-1$ $\varepsilon_6 = \alpha^7-7\alpha^5+\alpha^4+14\alpha^3-4\alpha^2-7\alpha+2$ $\varepsilon_7 = \alpha^7+\alpha^6-6\alpha^5-6\alpha^4+9\alpha^3+9\alpha^2-\alpha-1$ $\varepsilon_8 = \alpha^7+\alpha^6-6\alpha^5-5\alpha^4+10\alpha^3+5\alpha^2-5\alpha$
$min_Q(\alpha,x) = x^9+x^8-8x^7-7x^6+21x^5+15x^4-20x^3-10x^2+5x+1$		

3	20	$\varepsilon_1 = \alpha^2 - 2, \varepsilon_2 = \alpha^2 + \alpha - 2$ $\varepsilon_3 = \alpha^3 - \alpha^2 - 3\alpha + 3$
$min_Q(\alpha, x) = x^4 - 5x^2 + 5$		
5	21	$\varepsilon_1 = \alpha - 1, \varepsilon_2 = \alpha^3 - 3\alpha$ $\varepsilon_3 = \alpha^4 - 4\alpha^2 + 1$ $\varepsilon_4 = \alpha^5 - 5\alpha^3 + \alpha^2 + 5\alpha - 2$ $\varepsilon_5 = \alpha^5 - 5\alpha^3 + 5\alpha - 1$
$min_Q(\alpha, x) = x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1$		
10	23	$\varepsilon_1 = \alpha^3 - 3\alpha, \varepsilon_2 = \alpha^3 - 3\alpha + 1$ $\varepsilon_3 = \alpha^5 - 5\alpha^3 + 5\alpha$ $\varepsilon_4 = \alpha^5 + \alpha^4 - 4\alpha^3 - 3\alpha^2 + 3\alpha + 1$ $\varepsilon_5 = \alpha^6 - 6\alpha^4 + 9\alpha^2 - 1$ $\varepsilon_6 = \alpha^6 - 6\alpha^4 + \alpha^3 + 9\alpha^2 - 3\alpha - 1$ $\varepsilon_7 = \alpha^8 - 8\alpha^6 + 21\alpha^4 - 20\alpha^2 + 5$ $\varepsilon_8 = \alpha^9 - 8\alpha^7 + 21\alpha^5 - 21\alpha^3 + \alpha^2 + 7\alpha - 1$ $\varepsilon_9 = \alpha^{10} - 9\alpha^8 + 28\alpha^6 - 35\alpha^4 + 15\alpha^2 - 1$ $\varepsilon_{10} = \alpha^{10} + \alpha^9 - 9\alpha^8 - 9\alpha^7 + + 27\alpha^6 + 27\alpha^5 - 29\alpha^4 - 29\alpha^3 + 6\alpha^2 + 6\alpha$
$min_Q(\alpha, x) = x^{11} + x^{10} - 10x^9 - 9x^8 + 36x^7 + 28x^6 - 56x^5 - 35x^4 + 35x^3 + 15x^2 - 6x - 1$		
3	24	$\varepsilon_1 = \alpha,\ \varepsilon_2 = \alpha^3 - 3\alpha + 1$ $\varepsilon_3 = \alpha^3 + \alpha^2 - 3\alpha - 2$
$min_Q(\alpha, x) = x^4 - 4x^2 + 1$		

CHAPTER 4

CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$

Now, we can start to give characterization of $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$. Let n be a prime and γ be generator of $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$. Then we know that $\gamma_0 + 2\gamma_1 + \ldots + 2\gamma_k = 1$ since γ is normalized unit. If we take ω as n^{th} root of unity and $\alpha = \omega + \omega^{-1}$, we can define a ring homomorphism

$$\begin{array}{rlcl} \psi : & \mathbb{Z}\mathbf{C}_n & \longrightarrow & \mathbb{Z}[\omega] \\ & \sum \gamma_i a^i & \longmapsto & \sum \gamma_i \omega^i \end{array} .$$

After that we have computed $\psi(\gamma)$. Since $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$, $\psi(\gamma)$ is element of $\mathcal{U}(\mathbb{Z}[\alpha])$. That is, $\psi(\gamma)$ can be written as a product of fundamental units of $\mathcal{U}(\mathbb{Z}[\alpha])$ or their inverses. Then we can obtain γ to take pre-image of $\psi(\gamma)$. $\mathcal{U}(\mathbb{Z}[\alpha])$ is denoted by $\mathcal{U}$ and product of fundamental units of $\mathcal{U}(\mathbb{Z}[\alpha])$ or their inverses is denoted by $\mathcal{V}$. Theorem(2.1.6) says that $\mathcal{U}/\mathcal{V}$ is finite. When we can not express $\mathcal{U}/\mathcal{V}$ we have found a group $\mathcal{W}$, where $\mathcal{W} \cong \mathcal{V}$. After that we have checked whether $\mathcal{U}/\mathcal{W}$ is finite or not. If n is not prime we take subgroups of prime order into consideration.

4.1. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C_{11}^+})$

Theorem 4.1.1. *The normalized units of $\mathbb{Z}\mathbf{C_{11}^+} \subset \mathbb{Z}\mathbf{C_{11}}$ are generated by the set*

$$\{-1 + \mathcal{C}_1, -1 + \mathcal{C}_2, -1 + \mathcal{C}_3, -1 + \mathcal{C}_4\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C_{11}^+}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C_{11}})$ be a generator of the group of torsion-free units.

Since

$$\rho = \frac{1}{2}\varphi(11) - 1 = \frac{1}{2}10 - 1 = 4$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{11}}^{+})$ has only 4 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \gamma_1(a + a^{-1}) + \gamma_2(a^2 + a^{-2}) + \gamma_3(a^3 + a^{-3}) \\ &+ \gamma_4(a^4 + a^{-4}) + \gamma_5(a^5 + a^{-5}) \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5\end{aligned}$$

and

$$\gamma_0 + 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 = 1 \tag{4.1.1}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{11}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 \tag{4.1.2}$$

Let us consider the following ring homomorphism:

$$\begin{array}{rccc}\psi : & \mathbb{Z}\mathbf{C}_{11} & \longrightarrow & \mathbb{Z}[\omega] \\ & \sum\gamma_i a^i & \longmapsto & \sum\gamma_i\omega^i\end{array}$$

with equations (4.1.1) and (4.1.2), the image of the unit is

$$\begin{aligned}\psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) \\ &+ \gamma_4(\omega^4 + \omega^{-4}) + \gamma_5(\omega^5 + \omega^{-5}) \\ &= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) \\ &+ \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha)\end{aligned}$$

$$= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha)$$
$$+ \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(-\alpha^4 - \alpha^3 + 3\alpha^2 + 2\alpha)$$
$$= 1 - 2\gamma_1 - 4\gamma_2 - 2\gamma_3 - 3\gamma_5 + (\gamma_1 - 3\gamma_3 + 2\gamma_5)\alpha$$
$$+ (\gamma_2 - 4\gamma_4 + 3\gamma_5)\alpha^2 + (\gamma_3 - \gamma_5)\alpha^3 + (\gamma_4 - \gamma_5)\alpha^4$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \left\{ \sum_{i=0}^{4} a_i\alpha^i : \alpha^5 + \alpha^4 - 4\alpha^3 - 3\alpha^2 + 3\alpha + 1 = 0 \right\}$, by Table-3,

$$\mathcal{U}(\mathbb{Z}[\alpha]) =< \alpha, \alpha + 1, \alpha^2 - 2, \alpha^4 + \alpha^3 - 3\alpha^2 - 3\alpha >$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{11}^{+})$ can be obtained from products of the fundamental units or their inverses which are computed by Maple. Let us say $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 >$ and $\mathcal{V} =< v_1, v_2, v_3, v_4 >$ then

$$v_1 = -\varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1} = \psi(-1 + \mathcal{C}_1)$$
$$v_2 = \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_4 = \psi(-1 + \mathcal{C}_2)$$
$$v_3 = \varepsilon_1\varepsilon_4 = \psi(-1 + \mathcal{C}_3)$$
$$v_4 = -\varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-2} = \psi(-1 + \mathcal{C}_4)$$

We should find a group $\mathcal{W}$ where $\mathcal{W} \cong \mathcal{V},$ since we can not express $\mathcal{U}/\mathcal{V}$. We have found a group $\mathcal{W} =< \varepsilon_1^5, \varepsilon_2^5, \varepsilon_3^5, \varepsilon_4^5 >$ such that

$$\varepsilon_1^5 = -v_1^2v_2^{-1}v_3^3v_4$$
$$\varepsilon_2^5 = v_1^{-4}v_2^{-3}v_3^{-1}v_4^{-2}$$
$$\varepsilon_3^5 = -v_1^{-3}v_2^{-1}v_3^{-2}v_4^{-4}$$
$$\varepsilon_4^5 = -v_1^{-2}v_2v_3^2v_4^{-1}$$

by using Maple. Then

$$\mathcal{W} \cong \mathcal{V} \Longrightarrow \mathcal{U}/\mathcal{V} \cong \mathcal{U}/\mathcal{W} \text{ and } \mathcal{U}/\mathcal{W} \cong \mathbf{C}_5 \times \mathbf{C}_5 \times \mathbf{C}_5 \times \mathbf{C}_5$$

$|\mathcal{U}/\mathcal{W}| = 625$ implies $|\mathcal{U}/\mathcal{V}| = 625$ and we have obtained

$$\begin{aligned}
\psi(-1+\mathcal{C}_1) &= -\varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1} &\Longrightarrow \quad \psi^{-1}(v_1) &= \gamma_1 = -1+\mathcal{C}_1\\
\psi(-1+\mathcal{C}_2) &= \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_4 &\Longrightarrow \quad \psi^{-1}(v_2) &= \gamma_2 = -1+\mathcal{C}_2\\
\psi(-1+\mathcal{C}_3) &= \varepsilon_1\varepsilon_4 &\Longrightarrow \quad \psi^{-1}(v_3) &= \gamma_3 = -1+\mathcal{C}_3\\
\psi(-1+\mathcal{C}_4) &= -\varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-2} &\Longrightarrow \quad \psi^{-1}(v_4) &= \gamma_4 = -1+\mathcal{C}_4
\end{aligned}$$

□

4.2. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{13}^+)$

Theorem 4.2.1. *The normalized units of* $\mathbb{Z}\mathbf{C}_{13}^+ \subset \mathbb{Z}\mathbf{C}_{13}$ *are generated by the set*

$$\{-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_3, -1+\mathcal{C}_5, 1+\mathcal{C}_3-\mathcal{C}_5\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{13}^+) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{13})$ be a generator of the group of torsion-free units. Since

$$\rho = \frac{1}{2}\varphi(13) - 1 = \frac{1}{2}12 - 1 = 5$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{13}^+)$ has only 5 generators. Then, we have

$$\begin{aligned}
\gamma &= \gamma_0 + \gamma_1(a+a^{-1}) + \gamma_2(a^2+a^{-2}) + \gamma_3(a^3+a^{-3})\\
&\quad + \gamma_4(a^4+a^{-4}) + \gamma_5(a^5+a^{-5}) + \gamma_6(a^6+a^{-6})\\
&= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6
\end{aligned}$$

and

$$\gamma_0 + 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 = 1 \tag{4.2.1}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{13}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1 \tag{4.2.2}$$

Let us consider the following ring homomorphism:

$$\begin{array}{rccc} \psi : & \mathbb{Z}\mathbf{C}_{13} & \longrightarrow & \mathbb{Z}[\omega] \\ & \sum \gamma_i a^i & \longmapsto & \sum \gamma_i \omega^i \end{array}$$

with equations (4.2.1) and (4.2.2), the image of the unit is

$$\begin{aligned} \psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) \\ &+ \gamma_4(\omega^4 + \omega^{-4}) + \gamma_5(\omega^5 + \omega^{-5}) + \gamma_6(\omega^6 + \omega^{-6}) \\ &= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) \\ &+ \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) + \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) \\ &= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) \\ &+ \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) + \gamma_6(-\alpha^5 - \alpha^4 + 4\alpha^3 + 3\alpha^2 - 3\alpha - 1) \\ &= 1 - 2\gamma_1 - 4\gamma_2 - 2\gamma_3 - 2\gamma_5 - 3\gamma_6 + (\gamma_1 - 3\gamma_3 + 5\gamma_5 - 3\gamma_6)\alpha \\ &+ (\gamma_2 - 4\gamma_4 + 3\gamma_6)\alpha^2 + (\gamma_3 - 5\gamma_5 + 4\gamma_6)\alpha^3 + (\gamma_4 - \gamma_6)\alpha^4 + (\gamma_5 - \gamma_6)\alpha^5 \end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \left\{ \sum_{i=0}^{5} a_i\alpha^i : \alpha^6 + \alpha^5 - 5\alpha^4 - 4\alpha^3 + 6\alpha^2 + 3\alpha - 1 = 0 \right\}$, by Table-3,

$$\mathcal{U}(\mathbb{Z}[\alpha]) =< \alpha + 1, \alpha^2 - 2, \alpha^3 - 2\alpha, \alpha^4 - 4\alpha^2 + 2, \alpha^4 + \alpha^3 - 3\alpha^2 - 3\alpha >$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{13}^+)$ can be obtained from products of the fundamental units or their inverses which are computed by Maple. Let us say $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 >$ and

$\mathcal{V} =< v_1, v_2, v_3, v_4, v_5 >$ then

$$\begin{aligned} v_1 &= -\varepsilon_1^{-1}\varepsilon_3\varepsilon_4\varepsilon_5^{-1} && = \psi(-1+\mathcal{C}_1) \\ v_2 &= \varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-1}\varepsilon_5 && = \psi(-1+\mathcal{C}_2) \\ v_3 &= -\varepsilon_2\varepsilon_5 && = \psi(-1+\mathcal{C}_3) \\ v_4 &= \varepsilon_1\varepsilon_3^{-1}\varepsilon_4^{-2} && = \psi(1+\mathcal{C}_3-\mathcal{C}_5) \\ v_5 &= \varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1} && = \psi(-1+\mathcal{C}_5) \end{aligned}$$

Since we can not express $\mathcal{U}/\mathcal{V}$, we have found a group $\mathcal{W} =< \varepsilon_1^3, \varepsilon_2^6, \varepsilon_3^3, \varepsilon_4^6, \varepsilon_5^6 >$ such that

$$\begin{aligned} \varepsilon_1^3 &= v_1^{-1}v_2^{-2}v_3v_5^{-1} \\ \varepsilon_2^6 &= -v_1^4v_2^{-1}v_3^5v_4^3v_5^{-2} \\ \varepsilon_3^3 &= -v_1v_2^{-1}v_3^2v_5 \\ \varepsilon_4^6 &= -v_1^{-2}v_2^{-1}v_3^{-1}v_4^{-3}v_5^{-2} \\ \varepsilon_5^6 &= -v_1^{-4}v_2v_3v_4^{-3}v_5^2. \end{aligned}$$

Thus,

$$\mathcal{U}/\mathcal{W} \cong \mathbf{C}_3 \times \mathbf{C}_3 \times \mathbf{C}_6 \times \mathbf{C}_6 \times \mathbf{C}_6.$$

Then $\mid \mathcal{U}/\mathcal{V} \mid = 1944$ since $\mid \mathcal{U}/\mathcal{W} \mid = 1944$. We have obtained

$$\begin{aligned} \psi(-1+\mathcal{C}_1) &= -\varepsilon_1^{-1}\varepsilon_3\varepsilon_4\varepsilon_5^{-1} && \Longrightarrow \psi^{-1}(v_1) = \gamma_1 = -1+\mathcal{C}_1 \\ \psi(-1+\mathcal{C}_2) &= \varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-1}\varepsilon_5 && \Longrightarrow \psi^{-1}(v_2) = \gamma_2 = -1+\mathcal{C}_2 \\ \psi(-1+\mathcal{C}_3) &= -\varepsilon_2\varepsilon_5 && \Longrightarrow \psi^{-1}(v_3) = \gamma_3 = -1+\mathcal{C}_3 \\ \psi(-1+\mathcal{C}_5) &= \varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1} && \Longrightarrow \psi^{-1}(v_4) = \gamma_4 = -1+\mathcal{C}_5 \\ \psi(1+\mathcal{C}_3-\mathcal{C}_5) &= \varepsilon_1\varepsilon_3^{-1}\varepsilon_4^{-2} && \Longrightarrow \psi^{-1}(v_5) = \gamma_5 = 1+\mathcal{C}_3-\mathcal{C}_5 \end{aligned}$$

□

4.3. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}^+_{15})$

Theorem 4.3.1. *The normalized units of* $\mathbb{Z}\mathbf{C}^+_{15} \subset \mathbb{Z}\mathbf{C}_{15}$ *are generated by the set*

$$\{-1+\mathcal{C}_3, -1+\mathcal{C}_1-\mathcal{C}_2+\mathcal{C}_3-\mathcal{C}_4+\mathcal{C}_5, -1+\mathcal{C}_2-\mathcal{C}_4+\mathcal{C}_5+\mathcal{C}_6-\mathcal{C}_7\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}^+_{15}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{15})$ be a generator of the group of torsion-free units. Since

$$\rho = \frac{1}{2}\varphi(15) - 1 = \frac{1}{2}8 - 1 = 3$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}^+_{15})$ has only 3 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \sum_{i=1}^{7}\gamma_i\mathcal{C}_i \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 + +\gamma_7\mathcal{C}_7\end{aligned}$$

Let us consider the subgroups $\mathbf{H}_1 =< a^3 >$ and $\mathbf{H}_2 =< a^5 >$ of prime orders. Since $\mathbf{C_{15}}/\mathbf{H}_1 \cong \mathbf{C_3}$ we have a group epimorphism

$$\begin{aligned}\varphi_1 : \mathbf{C_{15}} &\longrightarrow \mathbf{C_{15}}/\mathbf{H}_1 \\ a^i &\longmapsto a^i\mathbf{H}_1\end{aligned}$$

If we linearly extend φ_1 over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned}\overline{\varphi_1} : \mathbb{Z}\mathbf{C_{15}} &\longrightarrow \mathbb{Z}(\mathbf{C_{15}}/\mathbf{H}_1) \\ \sum\gamma_i a^i &\longmapsto \sum\gamma_i(a^i\mathbf{H}_1)\end{aligned}$$

$$\begin{aligned}\overline{\varphi_1}(\gamma) &= \gamma_0\mathbf{H}_1 + \gamma_1(a\mathbf{H}_1 + a^2\mathbf{H}_1) + \gamma_2(a^2\mathbf{H}_1 + a\mathbf{H}_1) + \gamma_3(\mathbf{H}_1 + \mathbf{H}_1)\\ &+ \gamma_4(a\mathbf{H}_1 + a^2\mathbf{H}_1) + \gamma_5(a^2\mathbf{H}_1 + a\mathbf{H}_1) + \gamma_6(\mathbf{H}_1 + \mathbf{H}_1) + \gamma_7(a\mathbf{H}_1 + a^2\mathbf{H}_1)\\ &= (\gamma_0 + 2\gamma_3 + 2\gamma_6)\mathbf{H}_1 + (\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7)(a\mathbf{H}_1 + a^2\mathbf{H}_1)\end{aligned}$$

and similarly since $\mathbf{C_{15}}/\ \mathbf{H}_2 \cong \mathbf{C_5}$ we have group epimorphism

$$\begin{aligned}\varphi_2 : \mathbf{C}_{15} &\longrightarrow \mathbf{C_{15}}/\mathbf{H}_2\\ a^i &\longmapsto a^i\mathbf{H}_2\end{aligned}$$

If we linearly extend φ_2 over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned}\overline{\varphi_2} : \mathbb{Z}\mathbf{C}_{15} &\longrightarrow \mathbb{Z}(\mathbf{C_{15}}/\mathbf{H}_2)\\ \sum\gamma_i a^i &\longmapsto \sum\gamma_i(a^i\mathbf{H}_2)\end{aligned}$$

$$\begin{aligned}\overline{\varphi_2}(\gamma) &= \gamma_0\mathbf{H}_2 + \gamma_1(a\mathbf{H}_2 + a^4\mathbf{H}_2) + \gamma_2(a^2\mathbf{H}_2 + a^3\mathbf{H}_2)\\ &+ \gamma_3(a^3\mathbf{H}_2 + a^2\mathbf{H}_2) + \gamma_4(a^4\mathbf{H}_2 + a\mathbf{H}_2) + \gamma_5(\mathbf{H}_2 + \mathbf{H}_2)\\ &+ \gamma_6(a\mathbf{H}_2 + a^4\mathbf{H}_2) + \gamma_7(a^2\mathbf{H}_2 + a^3\mathbf{H}_2)\\ &= (\gamma_0 + 2\gamma_5)\mathbf{H}_2 + (\gamma_1 + \gamma_4 + \gamma_6)(a\mathbf{H}_2 + a^4\mathbf{H}_2)\\ &+ (\gamma_2 + \gamma_3 + \gamma_7)(a^2\mathbf{H}_2 + a^3\mathbf{H}_2)\end{aligned}$$

$\overline{\varphi_1}(\gamma) = \mathbf{H}_1, a\mathbf{H}_1$ or $a^2\mathbf{H}_1$. Then we can find $\overline{\varphi_1}(\gamma) = \mathbf{H}_1$ and we obtain that

$$\begin{aligned}\gamma_0 + 2\gamma_3 + 2\gamma_6 &= 1\\ \gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 &= 0\end{aligned} \tag{4.3.1}$$

and $\overline{\varphi_2}(\gamma) = \mathbf{H}_2, a\mathbf{H}_2, a^2\mathbf{H}_2, a^3\mathbf{H}_2$ or $a^4\mathbf{H}_2$. Then we can find $\overline{\varphi_2}(\gamma) = \mathbf{H}_2$ and we obtain that

$$
\begin{aligned}
\gamma_0 + 2\gamma_5 &= 1 \\
\gamma_1 + \gamma_4 + \gamma_6 &= 0 \\
\gamma_2 + \gamma_3 + \gamma_7 &= 0
\end{aligned}
\tag{4.3.2}
$$

Let us subsitute $\gamma_1 = p,\ \gamma_2 = q,\ \gamma_4 = r,\ \gamma_7 = s$ in equation(4.3.1) and equation(4.3.2). We obtain,

$$
\begin{aligned}
\gamma_0 &= 1 + 2(p + q + r + s) \\
\gamma_3 &= -q - s \\
\gamma_5 &= -p - q - r - s \\
\gamma_6 &= -p - r
\end{aligned}
\tag{4.3.3}
$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{15}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$
min_Q(\alpha, x) = x^4 - x^3 - 4x^2 + 4x + 1 \tag{4.3.4}
$$

Let us consider the following ring homomorphism:

$$
\begin{array}{rccc}
\psi : & \mathbb{Z}\mathbf{C_{15}} & \longrightarrow & \mathbb{Z}[\omega] \\
 & \sum \gamma_i a^i & \longmapsto & \sum \gamma_i \omega^i
\end{array}
$$

with equations (4.3.3) and (4.3.4), the image of the unit is

$$
\begin{aligned}
\psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) \\
&\quad + \gamma_4(\omega^4 + \omega^{-4}) + \gamma_5(\omega^5 + \omega^{-5}) \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) \\
&\quad + \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha)
\end{aligned}
$$

$$= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2)$$
$$+ \gamma_5(-\alpha^4 - \alpha^3 + 3\alpha^2 + 2\alpha)$$
$$= (1 - 2\gamma_1 - 4\gamma_2 - 2\gamma_3 - 3\gamma_5) + (\gamma_1 - 3\gamma_3 + 2\gamma_5)\alpha$$
$$+ (\gamma_2 - 4\gamma_4 + 3\gamma_5)\alpha^2 + (\gamma_3 - \gamma_5)\alpha^3 + (\gamma_4 - \gamma_5)\alpha^4$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \left\{\sum_{i=0}^{3} a_i\alpha^i : \alpha^4 - \alpha^3 - 4\alpha^2 + 4\alpha + 1 = 0\right\}$, by Table-3

$$\mathcal{U}(\mathbb{Z}[\alpha]) = < \alpha - 1, \alpha^2 - 3, \alpha^3 - 3\alpha > .$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{15}}^{+})$ can be obtained single product of the fundamental units or their inverses. We have used Maple to obtain that

$$\psi(\gamma) = \varepsilon_1^2 \quad \Longrightarrow \gamma_1 = -1 + \mathcal{C}_1 - \mathcal{C}_2 + \mathcal{C}_3 - \mathcal{C}_4 + \mathcal{C}_5$$
$$\psi(\gamma) = \varepsilon_2^2 \quad \Longrightarrow \gamma_2 = -1 + \mathcal{C}_2 - \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_6 - \mathcal{C}_7.$$

Since $\mathbf{H}_1 = < a^3 >$ is a cyclic group of order 5, by equation(1.1), its unit group of integral group ring

$$\mathcal{U}_1(\mathbb{Z}\mathbf{H}_1) = \mathbf{H}_1 \times < -1 + (a^3 + a^{-3}) >,$$

so the third unit is $-1 + \mathcal{C}_3$. □

4.4. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{16}}^{+})$

Theorem 4.4.1. *The normalized units of* $\mathbb{Z}\mathbf{C}_{\mathbf{16}}^{+} \subset \mathbb{Z}\mathbf{C}_{\mathbf{16}}$ *are generated by the set*

$$\{-1 - \mathcal{C}_2 + \mathcal{C}_6 + \mathcal{C}_8, 1 - \mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_5 + \mathcal{C}_6, 1 - \mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_6 + \mathcal{C}_7\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{16}}^{+}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{16}})$ be a generator of the group of torsion-free units.

Since

$$\rho = \frac{1}{2}\varphi(16) - 1 = \frac{1}{2}8 - 1 = 3$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{16}}^{+})$ has only 3 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \sum_{i=1}^{8}\gamma_i\mathcal{C}_i \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 + \gamma_7\mathcal{C}_7 + \gamma_8 a^8\end{aligned}$$

Let us consider the subgroup $\mathbf{H} =< a^8 >$ of prime orders. Since $\mathbf{H} \cong \mathbf{C_2}$ we have a group epimorphism

$$\begin{aligned}\varphi : \mathbf{C_{16}} &\longrightarrow \mathbf{C_{16}}/\mathbf{H} \\ a^i &\longmapsto a^i\mathbf{H}\end{aligned}$$

If we linearly extend φ over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned}\overline{\varphi} : \mathbb{Z}\mathbf{C_{16}} &\longrightarrow \mathbb{Z}(\mathbf{C_{16}}/\mathbf{H}) \\ \sum\gamma_i a^i &\longmapsto \sum\gamma_i(a^i\mathbf{H})\end{aligned}$$

$$\begin{aligned}\overline{\varphi}(\gamma) = {} & \gamma_0\mathbf{H} + \gamma_1(a\mathbf{H} + a^7\mathbf{H}) + \gamma_2(a^2\mathbf{H} + a^6\mathbf{H}) + \gamma_3(a^3\mathbf{H} + a^5\mathbf{H}) + \gamma_4(a^4\mathbf{H} + a^4\mathbf{H}) \\ & + \gamma_5(a^5\mathbf{H} + a^3\mathbf{H}) + \gamma_6(a^6\mathbf{H} + a^2\mathbf{H}) + \gamma_7(a^7\mathbf{H} + a\mathbf{H}) + \gamma_8\mathbf{H} \\ & = (\gamma_0 + \gamma_8)\mathbf{H} + (\gamma_1 + \gamma_7)(a\mathbf{H} + a^7\mathbf{H}) + (\gamma_2 + \gamma_6)(a^2\mathbf{H} + a^6\mathbf{H}) \\ & + (\gamma_3 + \gamma_5)(a^3\mathbf{H} + a^5\mathbf{H}) + 2\gamma_4(a^4\mathbf{H})\end{aligned}$$

$\overline{\varphi}(\gamma) = \mathbf{H}, a\mathbf{H}, a^2\mathbf{H}, a^3\mathbf{H}, a^4\mathbf{H}, a^5\mathbf{H}, a^6\mathbf{H}$ or $a^7\mathbf{H}$. Then we can find $\overline{\varphi}(\gamma) = \mathbf{H}$ and we obtain that

$$\begin{aligned}\gamma_0 + \gamma_8 &= 1 \\ \gamma_i + \gamma_{8-i} &= 0 \text{..........} i = 1, 2, 3, 4 \end{aligned} \tag{4.4.1}$$

Let us subsitute $\gamma_0 = p,\ \gamma_1 = q,\ \gamma_2 = r,\ \gamma_3 = s$ in equation(4.4.1). We obtain,

$$\gamma_4 = 0, \gamma_5 = -s, \gamma_6 = -r, \gamma_7 = -q, \gamma_8 = 1 - p \tag{4.4.2}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{16}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^4 - 4x^2 + 2 \tag{4.4.3}$$

Let us consider the following ring homomorphism:

$$\begin{aligned} \psi : \mathbb{Z}\mathbf{C_{16}} &\longrightarrow \mathbb{Z}\left[\omega\right] \\ \sum \gamma_i a^i &\longmapsto \sum \gamma_i \omega^i \end{aligned}$$

with equations (4.4.2) and (4.4.3), the image of the unit is

$$\begin{aligned}
\psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) + \gamma_4(\omega^4 + \omega^{-4}) \\
&+ \gamma_5(\omega^5 + \omega^{-5}) + \gamma_6(\omega^6 + \omega^{-6}) + \gamma_7(\omega^7 + \omega^{-7}) + \gamma_8\omega^8 \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) \\
&+ \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) + \gamma_7(\alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha) - \gamma_8 \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_5(-\alpha^3 + 3\alpha) \\
&+ \gamma_6(-\alpha^2 + 2) + \gamma_7(-\alpha) - \gamma_8 \\
&= (\gamma_0 - 2\gamma_2 + 2\gamma_6 - \gamma_8) + (\gamma_1 - 3\gamma_3 + 3\gamma_5 - \gamma_7)\alpha \\
&+ (\gamma_2 - \gamma_6)\alpha^2 + (\gamma_3 - \gamma_5)\alpha^3 \\
&= (-1 + 2p - 4r) + (2q - 6s)\alpha + (2r)\alpha^2 + (2s)\alpha^3
\end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \left\{ \sum_{i=0}^{4} a_i\alpha^i : \alpha^4 - 4\alpha^2 + 2 = 0 \right\}$, by Table-3

$$\mathcal{U}(\mathbb{Z}[\alpha]) =< \alpha - 1, \alpha^2 - 1, \alpha^2 + \alpha - 1 > .$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}^+_{\mathbf{16}})$ can be obtained in the following product of the fundamental units or their inverses.

$$\varepsilon_2^2, \varepsilon_2^{-2}, \varepsilon_1^2\varepsilon_2, \varepsilon_1^{-2}\varepsilon_2^{-1}, \varepsilon_1^{-2}\varepsilon_2, \varepsilon_1^2\varepsilon_2^{-1}, \varepsilon_2\varepsilon_3^2, \varepsilon_2\varepsilon_3^{-2}, \varepsilon_2^{-1}\varepsilon_3^2$$

Let us say $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_1, \varepsilon_2, \varepsilon_3 >$ and $\mathcal{V} =< v_1, v_2, v_3 >$.

$$v_1 = \varepsilon_2^2$$
$$v_2 = \varepsilon_1^{-2}\varepsilon_2$$
$$v_3 = \varepsilon_2\varepsilon_3^{-2}$$

Then we have found a group $\mathcal{W} =< \varepsilon_1^4, \varepsilon_2^2, \varepsilon_3^4 >$ where $\mathcal{W} \approxeq \mathcal{V}$ such that

$$\varepsilon_1^4 = -v_1^{-1}v_2^2$$
$$\varepsilon_2^2 = v_1$$
$$\varepsilon_3^4 = v_1v_3^{-2}$$

We cannot express $\mathcal{U}/\mathcal{V}$ but

$$\mathcal{U}/\mathcal{W} \approxeq \mathbf{C}_2 \times \mathbf{C}_4 \times \mathbf{C}_4.$$

That is, $|\,\mathcal{U}/\mathcal{V}\,| = 32$ since $|\,\mathcal{U}/\mathcal{W}\,| = 32$. We have obtained generators of $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{16}}^{+})$ by means of Maple:

$$\psi(\gamma) = \varepsilon_2^2 \quad \Longrightarrow \gamma_1 = -1 - \mathcal{C}_2 + \mathcal{C}_6 + 2a^8 = -1 - \mathcal{C}_2 + \mathcal{C}_6 + \mathcal{C}_8$$
$$\psi(\gamma) = \varepsilon_1^{-2}\varepsilon_2 \quad \Longrightarrow \gamma_2 = 1 - \mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_5 + \mathcal{C}_6$$
$$\psi(\gamma) = \varepsilon_2\varepsilon_3^{-2} \quad \Longrightarrow \gamma_3 = 1 - \mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_6 + \mathcal{C}_7$$

□

4.5. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{17}}^{+})$

Theorem 4.5.1. *The normalized units of* $\mathbb{Z}\mathbf{C}_{\mathbf{17}}^{+} \subset \mathbb{Z}\mathbf{C}_{\mathbf{17}}$ *are generated by the set*

$$\{-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_4, -1+\mathcal{C}_5, -1+\mathcal{C}_6, -1+\mathcal{C}_7, 1+\mathcal{C}_8-\mathcal{C}_3\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{17}}^{+}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{17}})$ be a generator of the group of torsion-free units. Since

$$\rho = \frac{1}{2}\varphi(17) - 1 = \frac{1}{2}16 - 1 = 7$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{17}}^{+})$ has only 7 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \gamma_1(a+a^{-1}) + \gamma_2(a^2+a^{-2}) + \gamma_3(a^3+a^{-3}) \\ &+ \gamma_4(a^4+a^{-4}) + \gamma_5(a^5+a^{-5}) + \gamma_6(a^6+a^{-6}) + \gamma_7(a^7+a^{-7}) + \gamma_8(a^8+a^{-8}) \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 + \gamma_7\mathcal{C}_7 + \gamma_8\mathcal{C}_8\end{aligned}$$

and

$$\gamma_0 + 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + 2\gamma_7 + 2\gamma_8 = 1 \tag{4.5.1}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{17}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1 \tag{4.5.2}$$

Let us consider the following ring homomorphism:

$$\begin{aligned}\psi : \mathbb{Z}\mathbf{C}_{17} &\longrightarrow \mathbb{Z}[\omega] \\ \textstyle\sum \gamma_i a^i &\longmapsto \textstyle\sum \gamma_i \omega^i\end{aligned}$$

with equations (4.5.1) and (4.5.2), the image of the unit is

$$\begin{aligned}
\psi(\gamma) &= \gamma_0 + \gamma_1(\omega+\omega^{-1}) + \gamma_2(\omega^2+\omega^{-2}) + \gamma_3(\omega^3+\omega^{-3}) \\
&+ \gamma_4(\omega^4+\omega^{-4}) + \gamma_5(\omega^5+\omega^{-5}) + \gamma_6(\omega^6+\omega^{-6}) + \gamma_7(\omega^7+\omega^{-7}) + \gamma_8(\omega^8+\omega^{-8}) \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2-2) + \gamma_3(\alpha^3-3\alpha) \\
&+ \gamma_4(\alpha^4-4\alpha^2+2) + \gamma_5(\alpha^5-5\alpha^3+5\alpha) + \gamma_6(\alpha^6-6\alpha^4+9\alpha^2-2) \\
&+ \gamma_7(\alpha^7-7\alpha^5+14\alpha^3-7\alpha) + \gamma_8(\alpha^8-8\alpha^6+20\alpha^4-16\alpha^2+2) \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2-2) + \gamma_3(\alpha^3-3\alpha) \\
&+ \gamma_4(\alpha^4-4\alpha^2+2) + \gamma_5(\alpha^5-5\alpha^3+5\alpha) + \gamma_6(\alpha^6-6\alpha^4+9\alpha^2-2) \\
&+ \gamma_7(\alpha^7-7\alpha^5+14\alpha^3-7\alpha) + \gamma_8(-\alpha^7-\alpha^6+6\alpha^5+5\alpha^4-10\alpha^3-6\alpha^2+4\alpha-1) \\
&= 1-2\gamma_1-4\gamma_2-2\gamma_3-2\gamma_5-4\gamma_6-2\gamma_7-\gamma_8+(\gamma_1-3\gamma_3+5\gamma_5-7\gamma_7+4\gamma_8)\alpha \\
&+ (\gamma_2-4\gamma_4+9\gamma_6-6\gamma_8)\alpha^2 + (\gamma_3-5\gamma_5+14\gamma_7-10\gamma_8)\alpha^3 + (\gamma_4-6\gamma_6+5\gamma_8)\alpha^4 \\
&+ (\gamma_5-7\gamma_7+6\gamma_8)\alpha^5 + (\gamma_6-\gamma_8)\alpha^6 + (\gamma_7-\gamma_8)\alpha^7
\end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha])$, by Table-3,

$$\begin{aligned}
\mathcal{U}(\mathbb{Z}[\alpha]) =< \alpha^3-3\alpha, \alpha^3+\alpha^2-2\alpha-1, \alpha^4-4\alpha^2+3, \alpha^5-4\alpha^3+3\alpha, \alpha^5-5\alpha^3+5\alpha, \\
\alpha^6-6\alpha^4+9\alpha^2-2, \alpha^7+\alpha^6-6\alpha^5-5\alpha^4+10\alpha^3+6\alpha^2-4\alpha-2 >
\end{aligned}$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{17}^{+})$ can be obtained in the following product of the fundamental units or their inverses which are computed by Maple. Let $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_{5,}\varepsilon_6, \varepsilon_7 >$ and $\mathcal{V} =< v_1, v_2, v_3, v_4, v_5, v_6, v_7 >$.

$$\begin{aligned}
v_1 &= -\varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_4\varepsilon_5^{-1} && = \psi(-1+\mathcal{C}_1) \\
v_2 &= \varepsilon_1\varepsilon_3\varepsilon_4^{-1} && = \psi(-1+\mathcal{C}_2) \\
v_3 &= \varepsilon_2\varepsilon_3\varepsilon_5^{2} && = \psi(1+\mathcal{C}_8-\mathcal{C}_3) \\
v_4 &= -\varepsilon_3^{-1}\varepsilon_7 && = \psi(-1+\mathcal{C}_4) \\
v_5 &= \varepsilon_3^{-1}\varepsilon_4\varepsilon_6^{-1} && = \psi(-1+\mathcal{C}_5) \\
v_6 &= \varepsilon_1^{-1}\varepsilon_2\varepsilon_4^{-1}\varepsilon_6^{-1} && = \psi(-1+\mathcal{C}_6) \\
v_7 &= -\varepsilon_3\varepsilon_5^{-1}\varepsilon_6\varepsilon_7^{-1} && = \psi(-1+\mathcal{C}_7)
\end{aligned}$$

Then we can find a group $\mathcal{W} =< \varepsilon_1^4, \varepsilon_2^8, \varepsilon_3^8, \varepsilon_4^8, \varepsilon_5^8, \varepsilon_6^4, \varepsilon_7^8 >$ where $\mathcal{W} \cong \mathcal{V}$ such that

$$\begin{aligned}
\varepsilon_1^4 &= -v_1^{-1}v_2^2v_4v_5^2v_6^{-1}v_7\\
\varepsilon_2^8 &= -v_1^{-1}v_2^2v_3^4v_4^9v_5^6v_6^3v_7^9\\
\varepsilon_3^8 &= -v_1^5v_2^6v_3^4v_4^3v_5^2v_6v_7^3\\
\varepsilon_4^8 &= -v_1^3v_2^2v_3^4v_4^5v_5^6v_6^{-1}v_7^5\\
\varepsilon_5^8 &= v_1^{-2}v_2^{-4}v_4^{-6}v_5^{-4}v_6^{-2}v_7^{-6}\\
\varepsilon_6^4 &= -v_1^{-1}v_2^{-2}v_4v_5^{-2}v_6^{-1}v_7\\
\varepsilon_7^8 &= -v_1^5v_2^6v_3^4v_4^{11}v_5^2v_6v_7^3.
\end{aligned}$$

So we can express

$$\mathcal{U}/\mathcal{W} \cong \mathbf{C}_4 \times \mathbf{C}_4 \times \mathbf{C}_8 \times \mathbf{C}_8 \times \mathbf{C}_8 \times \mathbf{C}_8 \times \mathbf{C}_8$$

That is, $\mid \mathcal{U}/\mathcal{V} \mid = 524288$ since $\mid \mathcal{U}/\mathcal{W} \mid = 524288$. Thus we have obtained

$$\begin{aligned}
\psi(-1+\mathcal{C}_2) &= \varepsilon_1\varepsilon_3\varepsilon_4^{-1} && \Longrightarrow \gamma_2 = -1+\mathcal{C}_2\\
\psi(1+\mathcal{C}_8-\mathcal{C}_3) &= \varepsilon_2\varepsilon_3\varepsilon_5^2 && \Longrightarrow \gamma_3 = 1+\mathcal{C}_8-\mathcal{C}_3\\
\psi(-1+\mathcal{C}_4) &= -\varepsilon_3^{-1}\varepsilon_7 && \Longrightarrow \gamma_4 = -1+\mathcal{C}_4\\
\psi(-1+\mathcal{C}_5) &= \varepsilon_3^{-1}\varepsilon_4\varepsilon_6^{-1} && \Longrightarrow \gamma_5 = -1+\mathcal{C}_5\\
\psi(-1+\mathcal{C}_6) &= \varepsilon_1^{-1}\varepsilon_2\varepsilon_4^{-1}\varepsilon_6^{-1} && \Longrightarrow \gamma_6 = -1+\mathcal{C}_6\\
\psi(-1+\mathcal{C}_7) &= -\varepsilon_3\varepsilon_5^{-1}\varepsilon_6\varepsilon_7^{-1} && \Longrightarrow \gamma_7 = -1+\mathcal{C}_7
\end{aligned}$$

□

4.6. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{19}^{+})$

Theorem 4.6.1. *The normalized units of* $\mathbb{Z}\mathbf{C}_{19}^{+} \subset \mathbb{Z}\mathbf{C}_{19}$ *are generated by the set*

$$\{-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_3, -1+\mathcal{C}_4, -1+\mathcal{C}_5, -1+\mathcal{C}_6, -1+\mathcal{C}_7, -1+\mathcal{C}_8\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{19}}^{+}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{19}})$ be a generator of the group of torsion-free units.Since

$$\rho = \frac{1}{2}\varphi(19) - 1 = \frac{1}{2}18 - 1 = 8$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{19}}^{+})$ has only 8 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \gamma_1(a + a^{-1}) + \gamma_2(a^2 + a^{-2}) + \gamma_3(a^3 + a^{-3}) + \gamma_4(a^4 + a^{-4}) \\ &+ \gamma_5(a^5 + a^{-5}) + \gamma_6(a^6 + a^{-6}) + \gamma_7(a^7 + a^{-7}) + \gamma_8(a^8 + a^{-8}) + \gamma_9(a^9 + a^{-9}) \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 + \gamma_7\mathcal{C}_7 + \gamma_8\mathcal{C}_8 + \gamma_9\mathcal{C}_9\end{aligned}$$

and

$$\gamma_0 + 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + 2\gamma_7 + 2\gamma_8 + 2\gamma_9 = 1 \tag{4.6.1}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{19}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1 \tag{4.6.2}$$

Let us consider the following ring homomorphism:

$$\begin{aligned}\psi : \mathbb{Z}\mathbf{C}_{19} &\longrightarrow \mathbb{Z}[\omega] \\ \sum \gamma_i a^i &\longmapsto \sum \gamma_i \omega^i\end{aligned}$$

with equations (4.6.1) and (4.6.2), the image of the unit is

$$\begin{aligned}
\psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) + \gamma_4(\omega^4 + \omega^{-4}) \\
&+ \gamma_5(\omega^5 + \omega^{-5}) + \gamma_6(\omega^7 + \omega^{-7}) + \gamma_8(\omega^8 + \omega^{-8}) + \gamma_9(\omega^9 + \omega^{-9}) \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) \\
&+ \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) + \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) + \gamma_7(\alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha) \\
&+ \gamma_8(\alpha^8 - 8\alpha^6 + 20\alpha^4 - 16\alpha^2 + 2) + \gamma_9(\alpha^9 - 9\alpha^7 + 27\alpha^5 - 30\alpha^3 + 9\alpha) \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) \\
&+ \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) + \gamma_7(\alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha) \\
&+ \gamma_8(\alpha^8 - 8\alpha^6 + 20\alpha^4 - 16\alpha^2 + 2) \\
&+ \gamma_9(-\alpha^8 - \alpha^7 + 7\alpha^6 + 6\alpha^5 - 15\alpha^4 - 10\alpha^3 + 10\alpha^2 + 4\alpha - 1) \\
&= 1 - 2\gamma_1 - 4\gamma_2 - 2\gamma_3 - 2\gamma_5 - 4\gamma_6 - 2\gamma_7 - 3\gamma_9 + (\gamma_1 - 3\gamma_3 + 5\gamma_5 - 7\gamma_7 + 4\gamma_9)\alpha \\
&+ (\gamma_2 - 4\gamma_4 + 9\gamma_6 - 16\gamma_8 + 10\gamma_9)\alpha^2 + (\gamma_3 - 5\gamma_5 + 14\gamma_7 - 10\gamma_9)\alpha^3 \\
&+ (\gamma_4 - 6\gamma_6 + 20\gamma_8 - 15\gamma_9)\alpha^4 + (\gamma_5 - 7\gamma_7 + 6\gamma_9)\alpha^5 \\
&+ (\gamma_6 - 8\gamma_8 + 7\gamma_9)\alpha^6 + (\gamma_7 - \gamma_9)\alpha^7 + (\gamma_8 - \gamma_9)\alpha^8
\end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha])$, by Table-3,

$$\begin{aligned}
&\mathcal{U}(\mathbb{Z}[\alpha]) =< \alpha^3 - 3\alpha, \alpha^4 - 4\alpha^2 + 3, \alpha^5 - 4\alpha^3 + 3\alpha, \alpha^6 - 6\alpha^4 + 9\alpha^2 - 2, \\
&\alpha^6 + \alpha^5 - 5\alpha^4 - 4\alpha^3 + 6\alpha^2 + 3\alpha - 1, \alpha^7 - 7\alpha^5 + \alpha^4 + 14\alpha^3 - 4\alpha^2 - 7\alpha + 2, \\
&\alpha^7 + \alpha^6 - 6\alpha^5 - 6\alpha^4 + 9\alpha^3 + 9\alpha^2 - \alpha - 1, \alpha^7 + \alpha^6 - 6\alpha^5 - 5\alpha^4 + 10\alpha^3 + 5\alpha^2 - 5\alpha \;>
\end{aligned}$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{19}}^+)$ can be obtained from product of the fundamental units or their inverses which are computed by Maple. Let $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8 >$ and

$\mathcal{V} =< v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 >$.

$$
\begin{aligned}
v_1 &= \varepsilon_1^{-1}\varepsilon_4^{-1}\varepsilon_5\varepsilon_8^{-1} &&= \psi(-1+\mathcal{C}_1)\\
v_2 &= \varepsilon_1\varepsilon_5^{-1}\varepsilon_6 &&= \psi(-1+\mathcal{C}_2)\\
v_3 &= \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1}\varepsilon_8 &&= \psi(-1+\mathcal{C}_3)\\
v_4 &= -\varepsilon_1\varepsilon_2\varepsilon_6^{-1}\varepsilon_7 &&= \psi(-1+\mathcal{C}_4)\\
v_5 &= -\varepsilon_1^{-1}\varepsilon_3\varepsilon_6\varepsilon_7^{-1} &&= \psi(-1+\mathcal{C}_5)\\
v_6 &= -\varepsilon_4\varepsilon_5^{-1} &&= \psi(-1+\mathcal{C}_6)\\
v_7 &= \varepsilon_2^{-1}\varepsilon_5\varepsilon_6^{-1} &&= \psi(-1+\mathcal{C}_7)\\
v_8 &= -\varepsilon_3^{-1}\varepsilon_7^{-1}\varepsilon_8^{-1} &&= \psi(-1+\mathcal{C}_8)\\
v_9 &= \varepsilon_1\varepsilon_2\varepsilon_3^{-1}\varepsilon_4\varepsilon_7\varepsilon_8 &&= \psi(-1+\mathcal{C}_9)
\end{aligned}
$$

Then we can find a group $\mathcal{W} =< \varepsilon_1^9, \varepsilon_2^9, \varepsilon_3^9, \varepsilon_4^9, \varepsilon_5^9, \varepsilon_6^9, \varepsilon_7^9, \varepsilon_8^9 >$ such that

$$
\begin{aligned}
\varepsilon_1^9 &= -v_1^{-2}v_2^5v_3^{-1}v_4^3v_5^2v_6^{-3}v_7^4v_8\\
\varepsilon_2^9 &= -v_1^{-2}v_2^{-4}v_3^{-1}v_4^3v_5^2v_6^{-3}v_7^{-5}v_8\\
\varepsilon_3^9 &= -v_1^2v_2^4v_3v_4^6v_5^7v_6^3v_7^5v_8^{-1}\\
\varepsilon_4^9 &= -v_1^{-1}v_2^{-2}v_3^{-5}v_4^{-3}v_5v_6^3v_7^2v_8^{-4}\\
\varepsilon_5^9 &= v_1^{-1}v_2^{-2}v_3^{-5}v_4^{-3}v_5v_6^{-6}v_7^2v_8^{-4}\\
\varepsilon_6^9 &= -v_1v_2^2v_3^{-4}v_4^{-6}v_5^{-1}v_6^{-3}v_7^{-2}v_8^{-5}\\
\varepsilon_7^9 &= v_1^5v_2v_3^{-2}v_4^{-3}v_5^{-5}v_6^3v_7^{-1}v_8^{-7}\\
\varepsilon_8^9 &= v_1^{-7}v_2^{-5}v_3v_4^{-3}v_5^{-2}v_6^{-6}v_7^{-4}v_8^{-1}
\end{aligned}
$$

Then we have

$$
\mathcal{U}/\mathcal{W} \approxeq \mathbf{C}_9 \times \mathbf{C}_9 \times \mathbf{C}_9 \times \mathbf{C}_9 \times \mathbf{C}_9 \times \mathbf{C}_9 \times \mathbf{C}_9 \times \mathbf{C}_9
$$

Then $|\,\mathcal{U}/\mathcal{V}\,| = 43046721$ since $|\,\mathcal{U}/\mathcal{W}\,| = 9^8 = 43046721$.Thus we have obtained

$$
\begin{aligned}
\psi(-1+\mathcal{C}_1) &= \varepsilon_1^{-1}\varepsilon_4^{-1}\varepsilon_5\varepsilon_8^{-1} && \Longrightarrow \gamma_1 = -1+\mathcal{C}_1\\
\psi(-1+\mathcal{C}_2) &= \varepsilon_1\varepsilon_5^{-1}\varepsilon_6 && \Longrightarrow \gamma_2 = -1+\mathcal{C}_2\\
\psi(-1+\mathcal{C}_3) &= \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1}\varepsilon_8 && \Longrightarrow \gamma_3 = -1+\mathcal{C}_3\\
\psi(-1+\mathcal{C}_4) &= -\varepsilon_1\varepsilon_2\varepsilon_6^{-1}\varepsilon_7 && \Longrightarrow \gamma_4 = -1+\mathcal{C}_4\\
\psi(-1+\mathcal{C}_5) &= -\varepsilon_1^{-1}\varepsilon_3\varepsilon_6\varepsilon_7^{-1} && \Longrightarrow \gamma_5 = -1+\mathcal{C}_5\\
\psi(-1+\mathcal{C}_6) &= -\varepsilon_4\varepsilon_5^{-1} && \Longrightarrow \gamma_6 = -1+\mathcal{C}_6\\
\psi(-1+\mathcal{C}_7) &= \varepsilon_2^{-1}\varepsilon_5\varepsilon_6^{-1} && \Longrightarrow \gamma_7 = -1+\mathcal{C}_7\\
\psi(-1+\mathcal{C}_8) &= -\varepsilon_3^{-1}\varepsilon_7^{-1}\varepsilon_8^{-1} && \Longrightarrow \gamma_8 = -1+\mathcal{C}_8
\end{aligned}
$$

□

4.7. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{20}^{+})$

Theorem 4.7.1. *The normalized units of* $\mathbb{Z}\mathbf{C}_{20}^{+} \subset \mathbb{Z}\mathbf{C}_{20}$ *are generated by the set*

$$
\begin{aligned}
&\{-1-\mathcal{C}_1-\mathcal{C}_2-\mathcal{C}_3+\mathcal{C}_7+\mathcal{C}_8+\mathcal{C}_9+2\mathcal{C}_{10},\\
&-1+\mathcal{C}_1-\mathcal{C}_3+\mathcal{C}_4-\mathcal{C}_6+\mathcal{C}_7-\mathcal{C}_9+2\mathcal{C}_{10}, -1+\mathcal{C}_4\}
\end{aligned}
$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{20}^{+}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{20})$ be a generator of the group of torsion-free units.Since

$$\rho = \frac{1}{2}\varphi(20) - 1 = \frac{1}{2}8 - 1 = 3$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{20}^{+})$ has only 3 generators. Then, we have

$$
\begin{aligned}
\gamma &= \gamma_0 + \sum_{i=1}^{10}\gamma_i\mathcal{C}_i\\
&= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 + \gamma_7\mathcal{C}_7 + \gamma_8\mathcal{C}_8 + \gamma_9\mathcal{C}_9 + \gamma_{10}\mathcal{C}_{10}
\end{aligned}
$$

Let us consider the subgroups $\mathbf{H}_1 =< a^{10} >$ and $\mathbf{H}_2 =< a^4 >$ of prime orders. Since

$\mathbf{C_{20}}/\ \mathbf{H}_1 \cong \mathbf{C_{10}}$ we have a group epimorphism

$$\begin{aligned} \varphi_1 : \mathbf{C_{20}} &\longrightarrow \mathbf{C_{20}}/\mathbf{H}_1 \\ a^i &\longmapsto a^i\mathbf{H}_1 \end{aligned}$$

If we linearly extend φ_1 over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned} \overline{\varphi_1} : \mathbb{Z}\mathbf{C_{20}} &\longrightarrow \mathbb{Z}(\mathbf{C_{20}}/\mathbf{H}_1) \\ \textstyle\sum \gamma_i a^i &\longmapsto \textstyle\sum \gamma_i (a^i\mathbf{H}_1) \end{aligned}$$

$$\begin{aligned} \overline{\varphi_1}(\gamma) &= \gamma_0\mathbf{H}_1 + \gamma_1(a\mathbf{H}_1 + a^9\mathbf{H}_1) + \gamma_2(a^2\mathbf{H}_1 + a^8\mathbf{H}_1) + \gamma_3(a^3\mathbf{H}_1 + a^7\mathbf{H}_1) \\ &+ \gamma_4(a^4\mathbf{H}_1 + a^6\mathbf{H}_1) + \gamma_5(a^5\mathbf{H}_1 + a^5\mathbf{H}_1) + \gamma_6(a^6\mathbf{H}_1 + a^4\mathbf{H}_1) \\ &+ \gamma_7(a^7\mathbf{H}_1 + a^3\mathbf{H}_1) + \gamma_8(a^8\mathbf{H}_1 + a^2\mathbf{H}_1) + \gamma_9(a^9\mathbf{H}_1 + a\mathbf{H}_1) + \gamma_{10}(\mathbf{H}_1) \\ &= (\gamma_0 + \gamma_{10})\mathbf{H} + (\gamma_1 + \gamma_9)(a\mathbf{H}_1 + a^9\mathbf{H}_1) + (\gamma_2 + \gamma_8)(a^2\mathbf{H}_1 + a^8\mathbf{H}_1) \\ &+ (\gamma_3 + \gamma_7)(a^3\mathbf{H}_1 + a^7\mathbf{H}_1) + (\gamma_4 + \gamma_6)(a^4\mathbf{H}_1 + a^6\mathbf{H}_1) + (2\gamma_5)(a^5\mathbf{H}_1) \end{aligned}$$

and similarly since $\mathbf{C_{20}}/\ \mathbf{H}_2 \cong \mathbf{C_4}$ we have group epimorphism

$$\begin{aligned} \varphi_2 : \mathbf{C_{20}} &\longrightarrow \mathbf{C_{20}}/\mathbf{H}_2 \\ a^i &\longmapsto a^i\mathbf{H}_2 \end{aligned}$$

If we linearly extend φ_2 over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned} \overline{\varphi_2} : \mathbb{Z}\mathbf{C_{20}} &\longrightarrow \mathbb{Z}(\mathbf{C_{20}}/\mathbf{H}_2) \\ \textstyle\sum \gamma_i a^i &\longmapsto \textstyle\sum \gamma_i (a^i\mathbf{H}_2) \end{aligned}$$

$$\begin{aligned}\overline{\varphi_2}(\gamma) &= \gamma_0\mathbf{H}_2 + \gamma_1(a\mathbf{H}_2 + a^3\mathbf{H}_2) + \gamma_2(a^2\mathbf{H}_2 + a^2\mathbf{H}_2) + \gamma_3(a^3\mathbf{H}_2 + a\mathbf{H}_2)\\ &+ \gamma_4(\mathbf{H}_2 + \mathbf{H}_2) + \gamma_5(a\mathbf{H}_2 + a^3\mathbf{H}_2) + \gamma_6(a^2\mathbf{H}_2 + a^2\mathbf{H}_2) + \gamma_7(a^3\mathbf{H}_2 + a\mathbf{H}_2)\\ &+ \gamma_8(\mathbf{H}_2 + \mathbf{H}_2) + \gamma_9(a\mathbf{H}_2 + a^3\mathbf{H}_2) + \gamma_{10}(a^2\mathbf{H}_2 + a^2\mathbf{H}_2)\\ &= (\gamma_0 + 2\gamma_4 + 2\gamma_8)\mathbf{H}_2 + (\gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 + \gamma_9)(a\mathbf{H}_2 + a^3\mathbf{H}_2)\\ &+ (2\gamma_2 + 2\gamma_6 + 2\gamma_{10})(a^2\mathbf{H}_2)\end{aligned}$$

$\overline{\varphi_1}(\gamma) = \mathbf{H}_1, a\mathbf{H}_1, a^2\mathbf{H}_1, a^3\mathbf{H}_1, a^4\mathbf{H}_1, a^5\mathbf{H}_1, a^6\mathbf{H}_1, a^7\mathbf{H}_1, a^8\mathbf{H}_1$ or $a^9\mathbf{H}_1$. Then we can find $\overline{\varphi_1}(\gamma) = \mathbf{H}_1$ and we obtain that

$$\begin{aligned}\gamma_0 + \gamma_{10} &= 1\\ \gamma_i + \gamma_{10-i} &= 0, \quad i = 1,2,3,4,5\end{aligned} \tag{4.7.1}$$

and $\overline{\varphi_2}(\gamma) = \mathbf{H}_2, a\mathbf{H}_2, a^2\mathbf{H}_2$ or $a^3\mathbf{H}_2$. Then we can find $\overline{\varphi_2}(\gamma) = \mathbf{H}_2$ and we obtain that

$$\begin{aligned}\gamma_0 + 2\gamma_4 + 2\gamma_8 &= 1\\ \gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 + \gamma_9 &= 0\\ 2\gamma_2 + 2\gamma_6 + \gamma_{10} &= 0\end{aligned} \tag{4.7.2}$$

Let us subsitute $\gamma_1 = p,\ \gamma_2 = q,\ \gamma_3 = r,\ \gamma_4 = s$ in equation(4.7.1) and equation(4.7.2). We obtain,

$$\begin{aligned}&\gamma_0 = 1 - 2s + 2q, \gamma_6 = -s, \gamma_7 = -r, , \gamma_5 = 0\\ &\gamma_8 = -q, \gamma_9 = -p, \gamma_{10} = 2s - 2q\end{aligned} \tag{4.7.3}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{20}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^4 - 5x^2 + 5 \tag{4.7.4}$$

Let us consider the following ring homomorphism:

$$\psi : \mathbb{Z}\mathbf{C_{20}} \longrightarrow \mathbb{Z}[\omega]$$
$$\sum \gamma_i a^i \longmapsto \sum \gamma_i \omega^i$$

with equations (4.7.3) and (4.7.4), the image of the unit is

$$\begin{aligned}
\psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) \\
&+ \gamma_4(\omega^4 + \omega^{-4}) + \gamma_5(\omega^5 + \omega^{-5}) + \gamma_6(\omega^6 + \omega^{-6}) \\
&+ \gamma_7(\omega^7 + \omega^{-7}) + \gamma_8(\omega^8 + \omega^{-8}) + \gamma_9(\omega^9 + \omega^{-9}) + \gamma_{10}(\omega^{10}) \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) \\
&+ \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) + \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) \\
&+ \gamma_7(\alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha) + \gamma_8(\alpha^8 - 8\alpha^6 + 20\alpha^4 - 16\alpha^2 + 2) \\
&+ \gamma_9(\alpha^9 - 9\alpha^7 + 27\alpha^5 + 30\alpha^3 + 9\alpha) - \gamma_{10} \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^2 - 3) + \gamma_5.0 \\
&+ \gamma_6(-\alpha^2 + 3) + \gamma_7(-\alpha^3 + 3\alpha) + \gamma_8(-\alpha^2 + 2) + \gamma_9(-\alpha) - \gamma_{10} \\
&= (\gamma_0 - 2\gamma_2 - 3\gamma_4 + 3\gamma_6 + 2\gamma_8 - \gamma_{10}) + (\gamma_1 - 3\gamma_3 + 3\gamma_7 - \gamma_9)\alpha \\
&+ (\gamma_2 + \gamma_4 - \gamma_6 - \gamma_8)\alpha^2 + (\gamma_3 - \gamma_7)\alpha^3 \\
&= 1 - 10s + (2p - 6r)\alpha + (2q - 2s)\alpha^2 + (2r)\alpha^3
\end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \left\{ \sum_{i=0}^{3} a_i\alpha^i : \alpha^4 - 5\alpha^2 + 5 = 0 \right\}$, by Table-3

$$\mathcal{U}(\mathbb{Z}[\alpha]) = < \alpha^2 - 2, \alpha^2 + \alpha - 2, \alpha^3 - \alpha^2 - 3\alpha + 3 > .$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{20}}^{+})$ can be obtained from the following products of the fundamental units or their inverses;

$$\varepsilon_2^2,\ \varepsilon_3^2, \varepsilon_2^4, \varepsilon_2^2\varepsilon_3^2, \varepsilon_3^4, \varepsilon_2^2\varepsilon_3^{-2}, \varepsilon_2^{-2}\varepsilon_3^{-2}, \varepsilon_2^{-4}, \varepsilon_3^{-4}.$$

We have used Maple to obtain that

$$\begin{aligned} v_1 = \varepsilon_2^2 &\qquad = \psi(-1 - \mathcal{C}_1 - \mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_7 + \mathcal{C}_8 + \mathcal{C}_9 + 2\mathcal{C}_{10}) \\ v_2 = \varepsilon_3^2 &\qquad = \psi(-1 + \mathcal{C}_1 - \mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_6 + \mathcal{C}_7 - \mathcal{C}_9 + 2\mathcal{C}_{10}) \end{aligned}$$

Let us say $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_2, \varepsilon_3 >$ and $\mathcal{V} =< v_1, v_2 >$. Then we have

$$\mathcal{U}/\mathcal{V} \approxeq \mathcal{C}_2 \times \mathcal{C}_2$$

and $\mid \mathcal{U}/\mathcal{V} \mid = 4$. Hence we have obtained

$$\begin{aligned} \psi(\gamma) = \varepsilon_2^2 &\qquad \Longrightarrow \gamma_1 = -1 - \mathcal{C}_1 - \mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_7 + \mathcal{C}_8 + \mathcal{C}_9 + 2\mathcal{C}_{10} \\ \psi(\gamma) = \varepsilon_3^2 &\qquad \Longrightarrow \gamma_2 = -1 + \mathcal{C}_1 - \mathcal{C}_3 + \mathcal{C}_4 - \mathcal{C}_6 + \mathcal{C}_7 - \mathcal{C}_9 + 2\mathcal{C}_{10} \end{aligned}$$

In the other side, since $\mathbf{H}_1 =< a^4 >$ is a cyclic group of order 5, by equation(1.1), its unit group of integral group ring

$$\mathcal{U}_1(\mathbb{Z}\mathbf{H}_1) = \mathbf{H}_1 \times < -1 + (a^4 + a^{-4}) >,$$

so the third unit is $-1 + \mathcal{C}_4$. □

4.8. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{21}^+)$

Theorem 4.8.1. *The normalized units of $\mathbb{Z}\mathbf{C}_{21}^+ \subset \mathbb{Z}\mathbf{C}_{21}$ are generated by the set*

$$\begin{gathered} \{1 - \mathcal{C}_1 + \mathcal{C}_3 - \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_8 - \mathcal{C}_9, 1 + \mathcal{C}_1 - \mathcal{C}_4 - \mathcal{C}_5 - \mathcal{C}_6 + \mathcal{C}_9 + \mathcal{C}_{10}, \\ 1 - \mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_5 + \mathcal{C}_6 - \mathcal{C}_8 + \mathcal{C}_{10}, -1 + \mathcal{C}_1 - \mathcal{C}_2 + \mathcal{C}_7 - \mathcal{C}_8 + \mathcal{C}_9, \\ 3 - \mathcal{C}_1 + \mathcal{C}_3 - 2\mathcal{C}_4 + \mathcal{C}_5 - \mathcal{C}_6 - \mathcal{C}_7 + 2\mathcal{C}_8 - \mathcal{C}_9 + \mathcal{C}_{10}\} \end{gathered}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{21}^+) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{21})$ be a generator of the group of torsion-free units.

Since

$$\rho = \frac{1}{2}\varphi(21) - 1 = \frac{1}{2}12 - 1 = 5$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{21}}^{+})$ has 5 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \sum_{i=1}^{10}\gamma_i\mathcal{C}_i \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 + \gamma_7\mathcal{C}_7 + \gamma_8\mathcal{C}_8 + \gamma_9\mathcal{C}_9 + \gamma_{10}\mathcal{C}_{10}\end{aligned}$$

Let us consider the subgroups $\mathbf{H}_1 =< a^3 >$ and $\mathbf{H}_2 =< a^7 >$ of prime orders. Since $\mathbf{C}_{21}/\mathbf{H}_1 \cong \mathbf{C_7}$ we have a group epimorphism

$$\begin{aligned}\varphi_1 : \mathbf{C_{21}} &\longrightarrow \mathbf{C_{21}}/\mathbf{H}_1 \\ a^i &\longmapsto a^i\mathbf{H}_1\end{aligned}$$

If we linearly extend φ_1 over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned}\overline{\varphi_1} : \mathbb{Z}\mathbf{C_{21}} &\longrightarrow \mathbb{Z}(\mathbf{C_{21}}/\mathbf{H}_1) \\ \sum\gamma_i a^i &\longmapsto \sum\gamma_i(a^i\mathbf{H}_1)\end{aligned}$$

$$\begin{aligned}\overline{\varphi_1}(\gamma) = \gamma_0\mathbf{H}_1 &+ \gamma_1(a\mathbf{H}_1 + a^2\mathbf{H}_1) + \gamma_2(a^2\mathbf{H}_1 + a\mathbf{H}_1) + \gamma_3(\mathbf{H}_1 + \mathbf{H}_1) \\ &+ \gamma_4(a\mathbf{H}_1 + a^2\mathbf{H}_1) + \gamma_5(a^2\mathbf{H}_1 + a\mathbf{H}_1) + \gamma_6(\mathbf{H}_1 + \mathbf{H}_1) \\ &+ \gamma_7(a\mathbf{H}_1 + a^2\mathbf{H}_1) + \gamma_8(a^2\mathbf{H}_1 + a\mathbf{H}_1) + \gamma_9(\mathbf{H}_1 + \mathbf{H}_1) + \gamma_{10}(a\mathbf{H}_1 + a^2\mathbf{H}_1) \\ &= (\gamma_0 + 2\gamma_3 + 2\gamma_6 + 2\gamma_9)\mathbf{H}_1 + (\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 + \gamma_8 + \gamma_{10})(a\mathbf{H}_1 + a^2\mathbf{H}_1)\end{aligned}$$

and similarly since $\mathbf{C}_{21}/\ \mathbf{H}_2 \cong \mathbf{C_3}$ we have group epimorphism

$$\begin{aligned}\varphi_2 : \mathbf{C_{21}} &\longrightarrow \mathbf{C_{21}}/\mathbf{H}_2 \\ a^i &\longmapsto a^i\mathbf{H}_2\end{aligned}$$

If we linearly extend φ_2 over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned}\overline{\varphi_2} : \mathbb{Z}\mathbf{C_{21}} &\longrightarrow \mathbb{Z}(\mathbf{C_{21}}/\mathbf{H}_2)\\ \textstyle\sum\gamma_i a^i &\longmapsto \textstyle\sum\gamma_i(a^i\mathbf{H}_2)\end{aligned}$$

$$\begin{aligned}\overline{\varphi_2}(\gamma) &= \gamma_0\mathbf{H}_2 + \gamma_1(a\mathbf{H}_2 + a^6\mathbf{H}_2) + \gamma_2(a^2\mathbf{H}_2 + a^5\mathbf{H}_2) + \gamma_3(a^3\mathbf{H}_2 + a^4\mathbf{H}_2)\\ &+ \gamma_4(a^4\mathbf{H}_2 + a^3\mathbf{H}_2) + \gamma_5(a^5\mathbf{H}_2 + a^2\mathbf{H}_2) + \gamma_6(a^6\mathbf{H}_2 + a\mathbf{H}_2) + \gamma_7(\mathbf{H}_2 + \mathbf{H}_2)\\ &+ \gamma_8(a\mathbf{H}_2 + a^6\mathbf{H}_2) + \gamma_9(a^2\mathbf{H}_2 + a^5\mathbf{H}_2) + \gamma_{10}(a^3\mathbf{H}_2 + a^4\mathbf{H}_2)\\ &= (\gamma_0 + 2\gamma_7)\mathbf{H}_2 + (\gamma_1 + \gamma_6 + \gamma_8)(a\mathbf{H}_2 + a^6\mathbf{H}_2)\\ &+ (\gamma_2 + \gamma_5 + \gamma_9)(a^2\mathbf{H}_2 + a^5\mathbf{H}_2) + (\gamma_3 + \gamma_4 + \gamma_{10})(a^3\mathbf{H}_2 + a^4\mathbf{H}_2)\end{aligned}$$

$\overline{\varphi_1}(\gamma) = \mathbf{H}_1, a\mathbf{H}_1$ or $a^2\mathbf{H}_1$. Then we can find $\overline{\varphi_1}(\gamma) = \mathbf{H}_1$ and we obtain that

$$\begin{aligned}\gamma_0 + 2\gamma_3 + 2\gamma_6 + 2\gamma_9 &= 1\\ \gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 + \gamma_8 + \gamma_{10} &= 0\end{aligned} \tag{4.8.1}$$

and $\overline{\varphi_2}(\gamma) = \mathbf{H}_2, a\mathbf{H}_2, a^2\mathbf{H}_2, a^3\mathbf{H}_2, , a^4\mathbf{H}_2, a^5\mathbf{H}_2$ or $a^6\mathbf{H}_2$. Then we can find $\overline{\varphi_2}(\gamma) = \mathbf{H}_2$ and we obtain that

$$\begin{aligned}\gamma_0 + 2\gamma_7 &= 1\\ \gamma_1 + \gamma_6 + \gamma_8 &= 0\\ \gamma_2 + \gamma_5 + \gamma_9 &= 0\\ \gamma_3 + \gamma_4 + \gamma_{10} &= 0\end{aligned} \tag{4.8.2}$$

Let us subsitute $\gamma_1 = p,\ \gamma_2 = q,\ \gamma_4 = r,\ \gamma_5 = s,\ \gamma_8 = t,\ \gamma_{10} = v$ in equation(4.8.1) and equation(4.8.2). We obtain,

$$
\begin{aligned}
\gamma_0 &= 1 + 2(p + q + r + s + t + v) \\
\gamma_3 &= -r - v \\
\gamma_6 &= -p - t \\
\gamma_7 &= -p - q - r - s - t - v \\
\gamma_9 &= -q - s
\end{aligned} \tag{4.8.3}
$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{21}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1 \tag{4.8.4}$$

Let us consider the following ring homomorphism:

$$
\begin{aligned}
\psi : \mathbb{Z}\mathbf{C_{21}} &\longrightarrow \mathbb{Z}[\omega] \\
\sum \gamma_i a^i &\longmapsto \sum \gamma_i \omega^i
\end{aligned}
$$

with equations (4.8.3) and (4.8.4), the image of the unit is

$$
\begin{aligned}
\psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) \\
&+ \gamma_4(\omega^4 + \omega^{-4}) + \gamma_5(\omega^5 + \omega^{-5}) + \gamma_6(\omega^6 + \omega^{-6}) \\
&+ \gamma_7(\omega^7 + \omega^{-7}) + \gamma_8(\omega^8 + \omega^{-8}) + \gamma_9(\omega^9 + \omega^{-9}) + \gamma_{10}(\omega^{10} + \omega^{-10}) \\
&= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) \\
&+ \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) + \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) + \gamma_7(\alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha) \\
&+ \gamma_8(\alpha^8 - 8\alpha^6 + 20\alpha^4 - 16\alpha^2 + 2) + \gamma_9(\alpha^9 - 9\alpha^7 + 27\alpha^5 + 30\alpha^3 + 9\alpha) \\
&+ \gamma_{10}(\alpha^{10} - 10\alpha^8 + 35\alpha^6 - 50\alpha^4 + 25\alpha^2 - 2)
\end{aligned}
$$

$$= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) + \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha)$$
$$+ \gamma_6(\alpha^5 - 6\alpha^3 + \alpha^2 + 8\alpha - 3) - \gamma_7 + \gamma_8(-\alpha^5 + 6\alpha^3 - \alpha^2 - 9\alpha + 3)$$
$$+ \gamma_9(-\alpha^5 + 5\alpha^3 - \alpha^2 - 5\alpha + 2) + \gamma_{10}(-\alpha^4 - \alpha^3 + 4\alpha^2 + 3\alpha - 2)$$
$$= (\gamma_0 - 2\gamma_2 + 2\gamma_4 - 3\gamma_6 - \gamma_7 + 3\gamma_8 + 2\gamma_9 - \gamma_{10})$$
$$+ (\gamma_1 - 3\gamma_3 + 5\gamma_5 + 8\gamma_6 - 9\gamma_8 - 5\gamma_9 + 3\gamma_{10})\alpha$$
$$+ (\gamma_2 - 4\gamma_4 + \gamma_6 - \gamma_8 - \gamma_9 + 4\gamma_{10})\alpha^2 + (\gamma_3 - 5\gamma_5 - 6\gamma_6 + 6\gamma_8 + 5\gamma_9 - \gamma_{10})\alpha^3$$
$$+ (\gamma_4 - \gamma_{10})\alpha^4 + (\gamma_5 + \gamma_6 - \gamma_8 - \gamma_9)\alpha^5$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \left\{ \sum_{i=0}^{10} a_i \alpha^i : \alpha^6 - \alpha^5 - 6\alpha^4 + 6\alpha^3 + 8\alpha^2 - 8\alpha + 1 = 0 \right\}$, by Table-3

$$\mathcal{U}(\mathbb{Z}[\alpha]) =< \alpha - 1, \alpha^3 - 3\alpha, \alpha^4 - 4\alpha^2 + 1, \alpha^5 - 5\alpha^3 + \alpha^2 + 5\alpha - 2, \alpha^5 - 5\alpha^3 + 5\alpha - 1 > .$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{21}}^+)$ can be obtained product of the fundamental units or their inverses;

$$\varepsilon_2\varepsilon_3^2, \varepsilon_1^2\varepsilon_2^{-1}\varepsilon_4^2,\ \varepsilon_1\varepsilon_2^3\varepsilon_4^{-1}, \varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_5^2,\ \varepsilon_4^2\varepsilon_5, \varepsilon_1\varepsilon_3^{-1}\varepsilon_5^{-3}, \varepsilon_1^{-2}\varepsilon_2\varepsilon_5, \varepsilon_1^2\varepsilon_3^2\ \varepsilon_5^{-1}$$

Let us say $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{4,}\varepsilon_5 >$ and $\mathcal{V} =< v_1, v_2, v_3, v_4, v_5 >$ such that

$$v_1 = \varepsilon_2\varepsilon_3^2$$
$$v_2 = -\varepsilon_4^2\varepsilon_5$$
$$v_3 = -\varepsilon_1^{-2}\varepsilon_2\varepsilon_5$$
$$v_4 = \varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_5^2$$
$$v_5 = \varepsilon_1\varepsilon_2^3\varepsilon_4^{-1}$$

Then we can find a group $\mathcal{W} =< \varepsilon_1^{78}, \varepsilon_2^{78}, \varepsilon_3^{78}, \varepsilon_4^{78}, \varepsilon_5^{78} >$ where $\mathcal{W} \cong \mathcal{V}$ such that

$$\begin{aligned}
\varepsilon_1^{78} &= v_1^{-5} v_2^{7} v_3^{-27} v_4^{10} v_5^{14} \\
\varepsilon_2^{78} &= v_1^{4} v_2^{10} v_3^{6} v_4^{-8} v_5^{20} \\
\varepsilon_3^{78} &= v_1^{37} v_2^{-5} v_3^{-3} v_4^{4} v_5^{-10} \\
\varepsilon_4^{78} &= v_1^{7} v_2^{37} v_3^{-9} v_4^{-14} v_5^{-4} \\
\varepsilon_5^{78} &= v_1^{-14} v_2^{4} v_3^{18} v_4^{28} v_5^{8}.
\end{aligned}$$

Then we express

$$\mathcal{U}/\mathcal{W} \cong \mathbf{C}_{78} \times \mathbf{C}_{78} \times \mathbf{C}_{78} \times \mathbf{C}_{78} \times \mathbf{C}_{78}$$

That is, $\mid \mathcal{U}/\mathcal{V} \mid = 78^5$ since $\mid \mathcal{U}/\mathcal{W} \mid = 78^5$. We have used Maple to obtain that

$$\begin{aligned}
\psi(\gamma) &= \varepsilon_2 \varepsilon_3^2 & &\Longrightarrow \gamma_1 = 1 - \mathcal{C}_1 + \mathcal{C}_3 - \mathcal{C}_4 + \mathcal{C}_5 + \mathcal{C}_8 - \mathcal{C}_9 \\
\psi(\gamma) &= \varepsilon_4^2 \varepsilon_5 & &\Longrightarrow \gamma_2 = 1 + \mathcal{C}_1 - \mathcal{C}_4 - \mathcal{C}_5 - \mathcal{C}_6 + \mathcal{C}_9 + \mathcal{C}_{10} \\
\psi(\gamma) &= \varepsilon_1^{-2} \varepsilon_2 \varepsilon_5 & &\Longrightarrow \gamma_3 = 1 - \mathcal{C}_2 - \mathcal{C}_3 + \mathcal{C}_5 + \mathcal{C}_6 - \mathcal{C}_8 + \mathcal{C}_{10} \\
\psi(\gamma) &= \varepsilon_1 \varepsilon_2^{-1} \varepsilon_3 \varepsilon_5^2 & &\Longrightarrow \gamma_4 = 3 - \mathcal{C}_1 + \mathcal{C}_3 - 2\mathcal{C}_4 + \mathcal{C}_5 - \mathcal{C}_6 - \mathcal{C}_7 + 2\mathcal{C}_8 - \mathcal{C}_9 + \mathcal{C}_{10} \\
\psi(\gamma) &= \varepsilon_1 \varepsilon_2^3 \varepsilon_4^{-1} & &\Longrightarrow \gamma_5 = -1 + \mathcal{C}_1 - \mathcal{C}_2 + \mathcal{C}_7 - \mathcal{C}_8 + \mathcal{C}_9
\end{aligned}$$

□

4.9. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{22}^{+})$

Theorem 4.9.1. *The normalized units of $\mathbb{Z}\mathbf{C}_{22}^{+} \subset \mathbb{Z}\mathbf{C}_{22}$ are generated by the set*

$$\{-1 + \mathcal{C}_2, -1 + \mathcal{C}_4, -1 + \mathcal{C}_6, -1 + \mathcal{C}_8\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. If we subsitute a^2 for a we obtain desired result by considering Theorem(4.1.1) with Remark(2.1.2).

$$\{-1+a^2+a^{20}, -1+a^4+a^{18}, -1+a^6+a^{16}, -1+a^8+a^{14}\}$$

□

4.10. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{23}}^{+})$

Theorem 4.10.1. *The normalized units of* $\mathbb{Z}\mathbf{C}_{\mathbf{23}}^{+} \subset \mathbb{Z}\mathbf{C}_{\mathbf{23}}$ *are generated by the set*

$$\{-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_3, -1+\mathcal{C}_4, -1+\mathcal{C}_5, -1+\mathcal{C}_6, -1+\mathcal{C}_7, -1+\mathcal{C}_8, -1+\mathcal{C}_9, -1+\mathcal{C}_{10}\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{23}}^{+}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{23}})$ be a generator of the group of torsion-free units. Since

$$\rho = \frac{1}{2}\varphi(23) - 1 = \frac{1}{2}22 - 1 = 10$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{23}}^{+})$ has 10 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \gamma_1(a+a^{-1}) + \gamma_2(a^2+a^{-2}) + \gamma_3(a^3+a^{-3}) + \gamma_4(a^4+a^{-4}) \\ &+ \gamma_5(a^5+a^{-5}) + \gamma_6(a^6+a^{-6}) + \gamma_7(a^7+a^{-7}) + \gamma_8(a^8+a^{-8}) \\ &+ \gamma_9(a^9+a^{-9}) + \gamma_{10}(a^{10}+a^{-10}) + \gamma_{11}(a^{11}+a^{-11}) \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 \\ &+ \gamma_7\mathcal{C}_7 + \gamma_8\mathcal{C}_8 + \gamma_9\mathcal{C}_9 + \gamma_{10}\mathcal{C}_{10} + \gamma_{11}\mathcal{C}_{11}\end{aligned}$$

and

$$\gamma_0 + 2\gamma_1 + 2\gamma_2 + 2\gamma_3 + 2\gamma_4 + 2\gamma_5 + 2\gamma_6 + 2\gamma_7 + 2\gamma_8 + 2\gamma_9 + 2\gamma_{10} + 2\gamma_{11} = 1 \quad (4.10.1)$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{23}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha,x) = x^{11}+x^{10}-10x^9-9x^8+36x^7+28x^6-56x^5-35x^4+35x^3+15x^2-6x-1 \quad (4.10.2)$$

Let us consider the following ring homomorphism:

$$\begin{array}{rccc} \psi: & \mathbb{Z}\mathbf{C}_{23} & \longrightarrow & \mathbb{Z}[\omega] \\ & \sum \gamma_i a^i & \longmapsto & \sum \gamma_i \omega^i \end{array}$$

with equations (4.10.1) and (4.10.2), the image of the unit is

$$\begin{aligned} \psi(\gamma) &= \gamma_0 + \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) \\ &+ \gamma_4(\omega^4 + \omega^{-4}) + \gamma_5(\omega^5 + \omega^{-5}) + \gamma_6(\omega^7 + \omega^{-7}) + \gamma_8(\omega^8 + \omega^{-8}) \\ &+ \gamma_9(\omega^9 + \omega^{-9}) + \gamma_{10}(\omega^{10} + \omega^{-10}) + \gamma_{11}(\omega^{11} + \omega^{-11}) \\ &= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) \\ &+ \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) + \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) + \gamma_7(\alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha) \\ &+ \gamma_8(\alpha^8 - 8\alpha^6 + 20\alpha^4 - 16\alpha^2 + 2) + \gamma_9(\alpha^9 - 9\alpha^7 + 27\alpha^5 - 30\alpha^3 + 9\alpha) \\ &+ \gamma_{10}(\alpha^{10} - 10\alpha^8 + 35\alpha^6 - 50\alpha^4 + 25\alpha^2 - 2) \\ &+ \gamma_{11}(\alpha^{11} - 11\alpha^9 + 44\alpha^7 - 77\alpha^5 + 55\alpha^3 - 11\alpha) \\ &= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) \\ &+ \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) + \gamma_6(\alpha^6 - 6\alpha^4 + 9\alpha^2 - 2) + \gamma_7(\alpha^7 - 7\alpha^5 + 14\alpha^3 - 7\alpha) \\ &+ \gamma_8(\alpha^8 - 8\alpha^6 + 20\alpha^4 - 16\alpha^2 + 2) \\ &+ \gamma_9(-\alpha^8 - \alpha^7 + 7\alpha^6 + 6\alpha^5 - 15\alpha^4 - 10\alpha^3 + 10\alpha^2 + 4\alpha - 1) \\ &+ \gamma_{10}(\alpha^{10} - 10\alpha^8 + 35\alpha^6 - 50\alpha^4 + 25\alpha^2 - 2) \\ &+ \gamma_{11}(-\alpha^{10} - \alpha^9 + 9\alpha^8 + 8\alpha^7 - 28\alpha^6 - 21\alpha^5 + 35\alpha^4 + 20\alpha^3 - 15\alpha^2 - 5\alpha + 1) \end{aligned}$$

$$= 1 - 2\gamma_1 - 4\gamma_2 - 2\gamma_3 - 2\gamma_5 - 4\gamma_6 - 2\gamma_7 - 2\gamma_9 - 4\gamma_{10} - \gamma_{11}$$
$$+ (\gamma_1 - 3\gamma_3 + 5\gamma_5 - 7\gamma_7 + 9\gamma_9 - 5\gamma_{11})\alpha$$
$$+ (\gamma_2 - 4\gamma_4 + 9\gamma_6 - 16\gamma_8 + 25\gamma_{10} - 15\gamma_{11})\alpha^2$$
$$+ (\gamma_3 - 5\gamma_5 + 14\gamma_7 - 30\gamma_9 + 20\gamma_{11})\alpha^3$$
$$+ (\gamma_4 - 6\gamma_6 + 20\gamma_8 - 50\gamma_{10} + 35\gamma_{11})\alpha^4$$
$$+ (\gamma_5 - 7\gamma_7 + 27\gamma_9 - 21\gamma_{11})\alpha^5 + (\gamma_6 - 8\gamma_8 + 35\gamma_{10} - 28\gamma_{11})\alpha^6$$
$$+ (\gamma_7 - 9\gamma_9 + 8\gamma_{11})\alpha^7 + (\gamma_8 - 10\gamma_{10} + 9\gamma_{11})\alpha^8$$
$$+ (\gamma_9 - \gamma_{11})\alpha^9 + (\gamma_{10} - \gamma_{11})\alpha^{10}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha])$, by Table-3,

$$\begin{aligned}\mathcal{U}(\mathbb{Z}[\alpha]) =< &\alpha^3 - 3\alpha, \alpha^3 - 3\alpha + 1, \alpha^5 - 5\alpha^3 + 5\alpha, \alpha^5 + \alpha^4 - 4\alpha^3 - 3\alpha^2 + 3\alpha + 1,\\ &\alpha^6 - 6\alpha^4 + 9\alpha^2 - 1, \alpha^6 - 6\alpha^4 + \alpha^3 + 9\alpha^2 - 3\alpha - 1,\\ &\alpha^8 - 8\alpha^6 + 21\alpha^4 - 20\alpha^2 + 5, \alpha^9 - 8\alpha^7 + 21\alpha^5 - 21\alpha^3 + \alpha^2 + 7\alpha - 1,\\ &\alpha^{10} - 9\alpha^8 + 28\alpha^6 - 35\alpha^4 + 15\alpha^2 - 1,\\ &\alpha^{10} + \alpha^9 - 9\alpha^8 - 9\alpha^7 + 27\alpha^6 + 27\alpha^5 - 29\alpha^4 - 29\alpha^3 + 6\alpha^2 + 6\alpha \;>\end{aligned}$$

$\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}^+_{\mathbf{23}})$ can be obtained from product of the fundamental units or their inverses. Let $\mathcal{U}(\mathbb{Z}[\alpha]) = \mathcal{U} =< \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10} >$ and
$\mathcal{V} =< v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10} >$.

$$\begin{aligned}
v_1 &= -\varepsilon_1^{-1}\varepsilon_4 && = \psi(-1 + \mathcal{C}_1)\\
v_2 &= -\varepsilon_1^{-1}\varepsilon_2^2\varepsilon_3^{-1}\varepsilon_4^{-1}\varepsilon_5 && = \psi(-1 + \mathcal{C}_2)\\
v_3 &= \varepsilon_9\varepsilon_{10}^{-1} && = \psi(-1 + \mathcal{C}_3)\\
v_4 &= -\varepsilon_3^{-1}\varepsilon_5^{-1}\varepsilon_6^{-1} && = \psi(-1 + \mathcal{C}_4)\\
v_5 &= -\varepsilon_1^{-2}\varepsilon_6^{-1}\varepsilon_7^{-1}\varepsilon_8\varepsilon_9 && = \psi(-1 + \mathcal{C}_5)\\
v_6 &= \varepsilon_1\varepsilon_2^{-1}\varepsilon_8^{-1}\varepsilon_9^{-1} && = \psi(-1 + \mathcal{C}_6)\\
v_7 &= -\varepsilon_2^{-1}\varepsilon_3\varepsilon_4\varepsilon_5^{-1}\varepsilon_7^{-1}\varepsilon_{10}^{-1} && = \psi(-1 + \mathcal{C}_7)\\
v_8 &= \varepsilon_1\varepsilon_4^{-1}\varepsilon_6\varepsilon_7 && = \psi(-1 + \mathcal{C}_8)\\
v_9 &= \varepsilon_1\varepsilon_5\varepsilon_6\varepsilon_9^{-1}\varepsilon_{10} && = \psi(-1 + \mathcal{C}_9)\\
v_{10} &= \varepsilon_1\varepsilon_7\varepsilon_8^{-1}\varepsilon_{10} && = \psi(-1 + \mathcal{C}_{10})
\end{aligned}$$

Then we can find a group $\mathcal{W} =< \varepsilon_1^{11}, \varepsilon_2^{11}, \varepsilon_3^{11}, \varepsilon_4^{11}, \varepsilon_5^{11}, \varepsilon_6^{11}, \varepsilon_7^{11}, \varepsilon_8^{11}, \varepsilon_9^{11}, \varepsilon_{10}^{11} >$ which is isomorphic to $\mathcal{V}$, such that

$$\begin{aligned}
\varepsilon_1^{11} &= v_1^{-7}v_2^{-3}v_3^3v_4v_5^{-5}v_6^{-4}v_7^{-2}v_8^{-6}v_9^2v_{10}^{-1} \\
\varepsilon_2^{11} &= v_1^{-4}v_2^3v_3^{-3}v_4^{-1}v_5^{-6}v_6^{-7}v_7^2v_8^{-5}v_9^{-2}v_{10} \\
\varepsilon_3^{11} &= -v_1^{-7}v_2^{-3}v_3^{-8}v_4^{-10}v_5^{-5}v_6^{-4}v_7^{-2}v_8^{-6}v_9^{-9}v_{10}^{-1} \\
\varepsilon_4^{11} &= -v_1^4v_2^{-3}v_3^3v_4v_5^{-5}v_6^{-4}v_7^{-2}v_8^{-6}v_9^2v_{10}^{-1} \\
\varepsilon_5^{11} &= -v_1^{-2}v_2^{-4}v_3^4v_4^{-6}v_5^{-3}v_6^2v_7^{-10}v_8^{-8}v_9^{-1}v_{10}^{-5} \\
\varepsilon_6^{11} &= -v_1^9v_2^7v_3^4v_4^5v_5^8v_6^2v_7^{12}v_8^{14}v_9^{10}v_{10}^6 \\
\varepsilon_7^{11} &= v_1^2v_2^{-7}v_3^{-4}v_4^{-5}v_5^{-8}v_6^{-2}v_7^{12}v_8^{-3}v_9^{-10}v_{10}^{-6} \\
\varepsilon_8^{11} &= v_1^{-4}v_2^{-8}v_3^{-3}v_4^{-1}v_5^{-6}v_6^{-7}v_7^{-9}v_8^{-5}v_9^{-2}v_{10}^{-10} \\
\varepsilon_9^{11} &= v_1v_2^2v_3^9v_4^3v_5^7v_6^{-1}v_7^5v_8^4v_9^6v_{10}^8 \\
\varepsilon_{10}^{11} &= v_1v_2^2v_3^{-2}v_4^3v_5^7v_6^{-1}v_7^5v_8^4v_9^6v_{10}^8
\end{aligned}$$

Then we express

$$\mathcal{U}/\mathcal{W} \approxeq \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11} \times \mathbf{C}_{11}$$

That is, $\mid \mathcal{U}/\mathcal{V} \mid = 10^{11}$ since $\mid \mathcal{U}/\mathcal{W} \mid = 10^{11}$. Thus we have obtained

$$\begin{aligned}
\psi(-1+\mathcal{C}_1) &= v_1^{-7}v_2^{-3}v_3^3v_4v_5^{-5}v_6^{-4}v_7^{-2}v_8^{-6}v_9^2v_{10}^{-1} && \Longrightarrow \gamma_1 = -1+\mathcal{C}_1 \\
\psi(-1+\mathcal{C}_2) &= v_1^{-4}v_2^3v_3^{-3}v_4^{-1}v_5^{-6}v_6^{-7}v_7^2v_8^{-5}v_9^{-2}v_{10} && \Longrightarrow \gamma_2 = -1+\mathcal{C}_2 \\
\psi(-1+\mathcal{C}_3) &= v_1^{-7}v_2^{-3}v_3^{-8}v_4^{-10}v_5^{-5}v_6^{-4}v_7^{-2}v_8^{-6}v_9^{-9}v_{10}^{-1} && \Longrightarrow \gamma_3 = -1+\mathcal{C}_3 \\
\psi(-1+\mathcal{C}_4) &= v_1^4v_2^{-3}v_3^3v_4v_5^{-5}v_6^{-4}v_7^{-2}v_8^{-6}v_9^2v_{10}^{-1} && \Longrightarrow \gamma_4 = -1+\mathcal{C}_4 \\
\psi(-1+\mathcal{C}_5) &= v_1^{-2}v_2^{-4}v_3^4v_4^{-6}v_5^{-3}v_6^2v_7^{-10}v_8^{-8}v_9^{-1}v_{10}^{-5} && \Longrightarrow \gamma_5 = -1+\mathcal{C}_5 \\
\psi(-1+\mathcal{C}_6) &= v_1^9v_2^7v_3^4v_4^5v_5^8v_6^2v_7^{12}v_8^{14}v_9^{10}v_{10}^6 && \Longrightarrow \gamma_6 = -1+\mathcal{C}_6 \\
\psi(-1+\mathcal{C}_7) &= v_1^2v_2^{-7}v_3^{-4}v_4^{-5}v_5^{-8}v_6^{-2}v_7^{12}v_8^{-3}v_9^{-10}v_{10}^{-6} && \Longrightarrow \gamma_7 = -1+\mathcal{C}_7 \\
\psi(-1+\mathcal{C}_8) &= v_1^{-4}v_2^{-8}v_3^{-3}v_4^{-1}v_5^{-6}v_6^{-7}v_7^{-9}v_8^{-5}v_9^{-2}v_{10}^{-10} && \Longrightarrow \gamma_8 = -1+\mathcal{C}_8 \\
\psi(-1+\mathcal{C}_9) &= v_1v_2^2v_3^9v_4^3v_5^7v_6^{-1}v_7^5v_8^4v_9^6v_{10}^8 && \Longrightarrow \gamma_9 = -1+\mathcal{C}_9 \\
\psi(-1+\mathcal{C}_{10}) &= v_1v_2^2v_3^{-2}v_4^3v_5^7v_6^{-1}v_7^5v_8^4v_9^6v_{10}^8 && \Longrightarrow \gamma_{10} = -1+\mathcal{C}_{10}
\end{aligned}$$

□

4.11. CHARACTERIZATION OF $\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{24}}^{+})$

Theorem 4.11.1. *The normalized units of* $\mathbb{Z}\mathbf{C}_{\mathbf{24}}^{+} \subset \mathbb{Z}\mathbf{C}_{\mathbf{24}}$ *are generated by the set*

$$\{-5-2\mathcal{C}_1-4\mathcal{C}_3-3\mathcal{C}_4+2\mathcal{C}_5-2\mathcal{C}_7+3\mathcal{C}_8+4\mathcal{C}_9+2\mathcal{C}_{10}+3\mathcal{C}_{12},$$
$$3+2\mathcal{C}_2+\mathcal{C}_4-\mathcal{C}_8-2\mathcal{C}_{10}-\mathcal{C}_{12},-1+\mathcal{C}_1-\mathcal{C}_4+\mathcal{C}_5-\mathcal{C}_7+\mathcal{C}_8-\mathcal{C}_{11}+\mathcal{C}_{12}\}$$

for which $\mathcal{C}_i = a^i + a^{-i}$.

Proof. Let $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{24}}^{+}) \subset \mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{24}})$ be a generator of the group of torsion-free units. Since

$$\rho = \frac{1}{2}\varphi(24) - 1 = \frac{1}{2}8 - 1 = 3$$

$\mathcal{U}_1(\mathbb{Z}\mathbf{C}_{\mathbf{24}}^{+})$ has 3 generators. Then, we have

$$\begin{aligned}\gamma &= \gamma_0 + \sum_{i=1}^{12}\gamma_i\mathcal{C}_i \\ &= \gamma_0 + \gamma_1\mathcal{C}_1 + \gamma_2\mathcal{C}_2 + \gamma_3\mathcal{C}_3 + \gamma_4\mathcal{C}_4 + \gamma_5\mathcal{C}_5 + \gamma_6\mathcal{C}_6 \\ &+ \gamma_7\mathcal{C}_7 + \gamma_8\mathcal{C}_8 + \gamma_9\mathcal{C}_9 + \gamma_{10}\mathcal{C}_{10} + \gamma_{11}\mathcal{C}_{11} + \gamma_{12}a^{12}\end{aligned}$$

Let us consider the subgroups $\mathbf{H} =< a^{12} >$ and $\mathbf{K} =< a^{8} >$ of prime orders. Since $\mathbf{H} \cong \mathbf{C_2}$ and $\mathbf{K} \cong \mathbf{C_3}$ we have a group epimorphism

$$\begin{aligned}\varphi_1 : \mathbf{C_{24}} &\longrightarrow \mathbf{C_{24}}/\mathbf{H} \\ a^i &\longmapsto a^i\mathbf{H}\end{aligned}$$

If we linearly extend φ over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned}\overline{\varphi}_1 : \mathbb{Z}\mathbf{C_{24}} &\longrightarrow \mathbb{Z}(\mathbf{C_{24}}/\mathbf{H}) \\ \sum\gamma_i a^i &\longmapsto \sum\gamma_i(a^i\mathbf{H})\end{aligned}$$

$$\begin{aligned}\overline{\varphi_1}(\gamma) &= \gamma_0\mathbf{H} + \gamma_1(a\mathbf{H} + a^{11}\mathbf{H}) + \gamma_2(a^2\mathbf{H} + a^{10}\mathbf{H}) + \gamma_3(a^3\mathbf{H} + a^9\mathbf{H}) \\ &+ \gamma_4(a^4\mathbf{H} + a^8\mathbf{H}) + \gamma_5(a^5\mathbf{H} + a^7\mathbf{H}) + \gamma_6(a^6\mathbf{H} + a^6\mathbf{H}) \\ &+ \gamma_7(a^7\mathbf{H} + a^5\mathbf{H}) + \gamma_8(a^8\mathbf{H}+a^4\mathbf{H}) \\ &+ \gamma_9(a^9\mathbf{H} + a^3\mathbf{H}) + \gamma_{10}(a^{10}\mathbf{H} + a^2\mathbf{H}) + \gamma_{11}(a^{11}\mathbf{H}+a\mathbf{H})+\gamma_{12}\mathbf{H} \\ &= (\gamma_0 + \gamma_{12})\mathbf{H} + (\gamma_1 + \gamma_{11})(a\mathbf{H} + a^{11}\mathbf{H}) + (\gamma_2 + \gamma_{10})(a^2\mathbf{H} + a^{10}\mathbf{H}) \\ &+ (\gamma_3 + \gamma_9)(a^3\mathbf{H} + a^9\mathbf{H}) + (\gamma_4 + \gamma_8)(a^4\mathbf{H} + a^8\mathbf{H}) \\ &+ (\gamma_5 + \gamma_7)(a^5\mathbf{H} + a^7\mathbf{H}) + 2\gamma_6(a^6\mathbf{H})\end{aligned}$$

and similarly

$$\begin{aligned}\varphi_2 : \mathbf{C_{24}} &\longrightarrow \mathbf{C_{24}}/\mathbf{K} \\ a^i &\longmapsto a^i\mathbf{K}\end{aligned}$$

If we linearly extend φ_2 over $\mathbb{Z}$ then we get a natural ring homomorphism as follows

$$\begin{aligned}\overline{\varphi}_2 : \mathbb{Z}\mathbf{C_{24}} &\longrightarrow \mathbb{Z}(\mathbf{C_{24}}/\mathbf{K}) \\ \textstyle\sum\gamma_i a^i &\longmapsto \textstyle\sum\gamma_i(a^i\mathbf{K})\end{aligned}$$

$$\begin{aligned}\overline{\varphi_2}(\gamma) &= \gamma_0\mathbf{K} + \gamma_1(a\mathbf{K} + a^7\mathbf{K}) + \gamma_2(a^2\mathbf{K} + a^6\mathbf{K}) + \gamma_3(a^3\mathbf{K} + a^5\mathbf{K}) + \gamma_4(a^4\mathbf{K} + a^4\mathbf{K}) \\ &+ \gamma_5(a^5\mathbf{K} + a^3\mathbf{K}) + \gamma_6(a^6\mathbf{K} + a^2\mathbf{K}) + \gamma_7(a^7\mathbf{K} + a\mathbf{K}) + \gamma_8\mathbf{K} \\ &+ \gamma_9(a\mathbf{K} + a^7\mathbf{K}) + \gamma_{10}(a^2\mathbf{K} + a^6\mathbf{K}) + \gamma_{11}(a^3\mathbf{K}+a^5\mathbf{K})+\gamma_{12}(a^4\mathbf{K}) \\ &= (\gamma_0 + \gamma_8)\mathbf{K} + (\gamma_1 + \gamma_7 + \gamma_9)(a\mathbf{K} + a^7\mathbf{K}) + (\gamma_2 + \gamma_6 + \gamma_{10})(a^2\mathbf{K} + a^6\mathbf{K}) \\ &+ (\gamma_3 + \gamma_5 + \gamma_{11})(a^3\mathbf{K} + a^5\mathbf{K})+(\mathbf{2}\gamma_4 + \gamma_{12})(a^4\mathbf{K})\end{aligned}$$

We can find $\overline{\varphi_1}(\gamma) = \mathbf{H}$ and we obtain that

$$\begin{aligned}&\gamma_0 + \gamma_{12} = 1 \\ &\gamma_i + \gamma_{12-i} = 0..........i = 1, 2, 3, 4, 5, 6 \qquad (4.11.1)\end{aligned}$$

and similarly for $\overline{\varphi_2}(\gamma) = \mathbf{K}$,we obtain

$$\begin{aligned}
\gamma_0 + 2\gamma_8 &= 1 \\
\gamma_1 + \gamma_7 + \gamma_9 &= 0 \\
\gamma_2 + \gamma_6 + \gamma_{10} &= 0 \\
\gamma_3 + \gamma_5 + \gamma_{11} &= 0 \qquad (4.11.2)
\end{aligned}$$

Let us subsitute $\gamma_2 = p,\ \gamma_3 = q,\ \gamma_4 = r,\ \gamma_5 = s$ in equation(4.11.1) and equation(4.11.2). We obtain,

$$\begin{aligned}
&\gamma_0 = 1 + 2r, \gamma_1 = s + q, \gamma_6 = 0, \gamma_7 = -s, \gamma_8 = -r, \\
&\gamma_9 = -q, \gamma_{10} = -p, \gamma_{11} = -q - s, \gamma_{12} = -2r \qquad (4.11.3)
\end{aligned}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{24}}$ and $\alpha = \omega + \omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$, by Table-2, as follows;

$$min_Q(\alpha, x) = x^4 - 4x^2 + 1 \qquad (4.11.4)$$

Let us consider the following ring homomorphism:

$$\begin{aligned}
\psi : \mathbb{Z}\mathbf{C_{24}} &\longrightarrow \mathbb{Z}[\omega] \\
\sum \gamma_i a^i &\longmapsto \sum \gamma_i \omega^i
\end{aligned}$$

with equations (4.11.3) and (4.11.4), the image of the unit is

$$\begin{aligned}
\psi(\gamma) = \gamma_0 &+ \gamma_1(\omega + \omega^{-1}) + \gamma_2(\omega^2 + \omega^{-2}) + \gamma_3(\omega^3 + \omega^{-3}) + \gamma_4(\omega^4 + \omega^{-4}) \\
&+ \gamma_5(\omega^5 + \omega^{-5}) + \gamma_6(\omega^6 + \omega^{-6}) + \gamma_7(\omega^7 + \omega^{-7}) + \gamma_8(\omega^8 + \omega^{-8}) \\
&+ \gamma_9(\omega^9 + \omega^{-9}) + \gamma_{10}(\omega^{10} + \omega^{-10}) + \gamma_{11}(\omega^{11} + \omega^{-11}) + \gamma_{12}\omega^{12}
\end{aligned}$$

$$
\begin{aligned}
&= \gamma_0 + \gamma_1(\frac{\sqrt{6}+\sqrt{2}}{2}) + \gamma_2(\sqrt{3}) + \gamma_3(\sqrt{2}) + \gamma_4 + \gamma_6.0 \\
&+ \gamma_7(\frac{-\sqrt{6}+\sqrt{2}}{2}) - \gamma_8 + \gamma_9(-\sqrt{2}) + \gamma_{10}(-\sqrt{3}) + \gamma_{11}(\frac{-\sqrt{6}-\sqrt{2}}{2}) - \gamma_{12} \\
&= \gamma_0 + (\gamma_1 - \gamma_{11})(\frac{\sqrt{6}+\sqrt{2}}{2}) + (\gamma_2 - \gamma_{10})(\sqrt{3}) \\
&+ (\gamma_3 - \gamma_9)(\sqrt{2}) + (\gamma_5 - \gamma_7)(\frac{\sqrt{6}-\sqrt{2}}{2}) \\
&= (1+6r) + 3q\sqrt{2} + 2p\sqrt{3} + (q+2s)\sqrt{6}
\end{aligned}
$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{2}, \sqrt{3}]) = \mathcal{U}(\mathbb{Z}[\sqrt{2}]) \times \mathcal{U}(\mathbb{Z}[\sqrt{3}]) \times \mathcal{U}(\mathbb{Z}[\sqrt{6}])$

$i)\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{2}])$

$$
\begin{aligned}
&\Longrightarrow\ p = 0, q = -2s \\
&\Longrightarrow\ \psi(\gamma) = (1+6r) - 6s\sqrt{2} = \pm(1 \pm \sqrt{2})^k (k \in \mathbb{Z}) \\
&\Longrightarrow\ (1+6r) - 6s\sqrt{2} = \pm(17 \pm 12\sqrt{2}), (k = 4) \\
&\Longrightarrow\ \gamma = -5 - 2\mathcal{C}_1 - 4\mathcal{C}_3 - 3\mathcal{C}_4 + 2\mathcal{C}_5 - 2\mathcal{C}_7 + 3\mathcal{C}_8 + 4\mathcal{C}_9 + 2\mathcal{C}_{10} + 6a^{12}
\end{aligned}
$$

$ii)\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{3}])$

$$
\begin{aligned}
&\Longrightarrow\ q = s = 0 \\
&\Longrightarrow\ \psi(\gamma) = (1+6r) + 2p\sqrt{3} = \pm(2 \pm \sqrt{3})^k, (k \in \mathbb{Z}) \\
&\Longrightarrow\ (1+6r) + 2p\sqrt{3} = \pm(7 \pm 4\sqrt{3}), (k = 2) \\
&\Longrightarrow\ \gamma = 3 + 2\mathcal{C}_2 + \mathcal{C}_4 - \mathcal{C}_8 - 2\mathcal{C}_{10} - 2a^{12}
\end{aligned}
$$

$iii)\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{6}])$

$$
\begin{aligned}
&\Longrightarrow\ q = p = 0 \\
&\Longrightarrow\ \psi(\gamma) = (1+6r) + 2s\sqrt{6} = \pm(5 \pm 2\sqrt{6})^k, (k \in \mathbb{Z}) \\
&\Longrightarrow\ (1+6r) + 2s\sqrt{6} = \pm 5 \pm 2\sqrt{6}, (k = 1) \\
&\Longrightarrow\ \gamma = -1 + \mathcal{C}_1 - \mathcal{C}_4 + \mathcal{C}_5 - \mathcal{C}_7 + \mathcal{C}_8 - \mathcal{C}_{11} + 2a^{12}
\end{aligned}
$$

By considering Lemma (2.1.2) the generators can be written as follows, respectively:

$$-5-2\mathcal{C}_1-4\mathcal{C}_3-3\mathcal{C}_4+2\mathcal{C}_5-2\mathcal{C}_7+3\mathcal{C}_8+4\mathcal{C}_9+2\mathcal{C}_{10}+3\mathcal{C}_{12},$$
$$3+2\mathcal{C}_2+\mathcal{C}_4-\mathcal{C}_8-2\mathcal{C}_{10}-\mathcal{C}_{12},$$
$$-1+\mathcal{C}_1-\mathcal{C}_4+\mathcal{C}_5-\mathcal{C}_7+\mathcal{C}_8-\mathcal{C}_{11}+\mathcal{C}_{12}.$$

□

CHAPTER 5

CONCLUSION

If $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$, then we have shown that

$$\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i \mathcal{C}_i, \quad \text{where} \quad \mathcal{C}_i = a^i + a^{-i}.$$

First of all, we have obtained that $\gamma_0 + 2\gamma_1 + \ldots + 2\gamma_{k-1} + 2\gamma_k = 1$ since γ is a normalized unit. In the second step, we have denoted ω as nth root of unity and we have computed the minimal polynomial. If n is not prime, we have considered the subgroups of prime orders and we have defined ring epimorphism:

$$\bar{\varphi} : \mathbb{Z}\mathbf{C_n} \rightarrow \mathbb{Z}(\mathbf{C_n}/\mathbf{H}).$$

So, we have gotten the new equation(s) to take $\bar{\varphi}(\gamma) = H$ into consideration.

Finally, by regarding the following ring homomorphism:

$$\psi : \mathbb{Z}\mathbf{C_n} \rightarrow \mathbb{Z}[\omega]$$

we have obtained $\psi(\gamma)$ to use equations which are stated at first and second steps.

Since $\gamma \in \mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$, then $\psi(\gamma)$ is the element of $\mathcal{U}(\mathbb{Z}[\omega]) =< u_1, u_2, \ldots, u_\rho >$,where ρ is rank. Namely, $\psi(\gamma)$ can be written as product of fundamental units and inverses. We have gotten γ to take pre-image of product of fundamental units and inverses by means of Maple.

Computation of fundamental units is extremely difficult especially for great n's.We have involved this problem by using PARI software. In some cases, we have waited for two days to results obtain from Maple. Unit group of integral group ring of cyclic group was characterized for $n = 5, 7, 8, 9$ and 12 before.With this study we have completed characterization of $\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$ for $n \leq 24$.

$\mathcal{U}_1(\mathbb{Z}\mathbf{C_n^+})$	ρ	generators
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{10}^+})$	1	$-1+\mathcal{C}_2$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{11}^+})$	4	$-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_3, -1+\mathcal{C}_4$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{13}^+})$	5	$-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_3, -1+\mathcal{C}_5, 1+\mathcal{C}_3-\mathcal{C}_5$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{14}^+})$	2	$-1+\mathcal{C}_2, -1+2\mathcal{C}_2-\mathcal{C}_4$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{15}^+})$	3	$-1+\mathcal{C}_3, -1+\mathcal{C}_1-\mathcal{C}_2+\mathcal{C}_3-\mathcal{C}_4+\mathcal{C}_5,$ $-1+\mathcal{C}_2-\mathcal{C}_4+\mathcal{C}_5+\mathcal{C}_6-\mathcal{C}_7$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{16}^+})$	3	$-1-\mathcal{C}_2+\mathcal{C}_6+\mathcal{C}_8, 1-\mathcal{C}_2-\mathcal{C}_3+\mathcal{C}_5+\mathcal{C}_6,$ $1-\mathcal{C}_1+\mathcal{C}_2-\mathcal{C}_6+\mathcal{C}_7$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{17}^+})$	7	$-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_4, -1+\mathcal{C}_5, -1+\mathcal{C}_6, -1+\mathcal{C}_7, 1+\mathcal{C}_8-\mathcal{C}_3$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{18}^+})$	2	$-1+\mathcal{C}_2-\mathcal{C}_8, -1+\mathcal{C}_2+\mathcal{C}_4+\mathcal{C}_6-2\mathcal{C}_8$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{19}^+})$	8	$-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_3, -1+\mathcal{C}_4,$ $-1+\mathcal{C}_5, -1+\mathcal{C}_6, -1+\mathcal{C}_7, -1+\mathcal{C}_8$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{20}^+})$	3	$-1-\mathcal{C}_1-\mathcal{C}_2-\mathcal{C}_3+\mathcal{C}_7+\mathcal{C}_8+\mathcal{C}_9+2\mathcal{C}_{10},$ $-1+\mathcal{C}_1-\mathcal{C}_3+\mathcal{C}_4-\mathcal{C}_6+\mathcal{C}_7-\mathcal{C}_9+2\mathcal{C}_{10}, -1+\mathcal{C}_4$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{21}^+})$	5	$1-\mathcal{C}_1+\mathcal{C}_3-\mathcal{C}_4+\mathcal{C}_5+\mathcal{C}_8-\mathcal{C}_9,$ $1+\mathcal{C}_1-\mathcal{C}_4-\mathcal{C}_5-\mathcal{C}_6+\mathcal{C}_9+\mathcal{C}_{10},$ $1-\mathcal{C}_2-\mathcal{C}_3+\mathcal{C}_5+\mathcal{C}_6-\mathcal{C}_8+\mathcal{C}_{10},$ $3-\mathcal{C}_1+\mathcal{C}_3-2\mathcal{C}_4+\mathcal{C}_5-\mathcal{C}_6-\mathcal{C}_7+2\mathcal{C}_8-\mathcal{C}_9+\mathcal{C}_{10},$ $-1+\mathcal{C}_1-\mathcal{C}_2+\mathcal{C}_7-\mathcal{C}_8+\mathcal{C}_9$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{22}^+})$	4	$-1+\mathcal{C}_2, -1+\mathcal{C}_4, -1+\mathcal{C}_6, -1+\mathcal{C}_8$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{23}^+})$	10	$-1+\mathcal{C}_1, -1+\mathcal{C}_2, -1+\mathcal{C}_3, -1+\mathcal{C}_4, -1+\mathcal{C}_5,$ $-1+\mathcal{C}_6, -1+\mathcal{C}_7, -1+\mathcal{C}_8, -1+\mathcal{C}_9, -1+\mathcal{C}_{10}$
$\mathcal{U}_1(\mathbb{Z}\mathbf{C_{24}^+})$	3	$-5-2\mathcal{C}_1-4\mathcal{C}_3-3\mathcal{C}_4+2\mathcal{C}_5-2\mathcal{C}_7+3\mathcal{C}_8+4\mathcal{C}_9+2\mathcal{C}_{10}+3\mathcal{C}_{12},$ $3+2\mathrm{C}_2+\mathrm{C}_4-\mathcal{C}_8-2\mathcal{C}_{10}-\mathcal{C}_{12},$ $-1+\mathcal{C}_1-\mathcal{C}_4+\mathcal{C}_5-\mathcal{C}_7+\mathcal{C}_8-\mathcal{C}_{11}+\mathcal{C}_{12}$

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