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QUALITATIVE BEHAVIOUR OF A CLASS OF PDE

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ABSTRACT

QUALITATIVE BEHAVIOUR OF A CLASS OF PDE

This dissertation addresses the initial and periodic boundary value problems for a fourth-order pseudo-parabolic equation given by

$$u_t - a\Delta u_t - \Delta u + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2}\nabla u), \quad (x,t) \in \Omega \times (0,T)$$

where the gradient non-linearity and the pseudo-term are as specified above. The problem is subject to the initial condition

$$u(x,0) = u_0(x), \quad u_0 \in L^2(\Omega), \quad x \in \Omega,$$

and periodic boundary conditions

$$\forall x \in \Gamma_i, \quad 0 < t < T, \quad i = 1,2,$$

$$u(x,t) = u(x + L_i e_i, t), \quad u_{x_i}(x,t) = u_{x_i}(x + L_i e_i, t),$$

where

$$\Omega = (0, L_1) \times (0, L_2), \quad \Gamma_i = \partial\Omega \cap \{x_i = 0\}.$$

$a \geq 0$, and $p > 2$. A local existence-uniqueness result for mild solutions was established for any initial data in $L^2(\Omega)$. It was also demonstrated that mild solutions are weak solutions to the problem. Additionally, the existence of blow-up solutions was proven, and the blow-up time was shown to have a lower bound.

ÖZET

BIR PDE SINIFININ NITELIKSEL DAVRANIŞI

Bu tezde, aşağıdaki gradiyenti doğrusal olmayan pseudo-parabolik denklem için sınır ve başlangıç değer problemi,

$$u_t - a\Delta u_t - \Delta u + \Delta^2 u = -\nabla \cdot (|\nabla u|^{p-2}\nabla u).$$

$$u(x,0) = u_0(x), \quad u_0 \in L^2(\Omega), \quad x \in \Omega,$$

$$\forall x \in \Gamma_i, \;\; 0 < t < T, \quad i = 1,2,$$
$$u(x,t) = u(x + L_i e_i, t), \quad u_{x_i}(x,t) = u_{x_i}(x + L_i e_i, t),$$

incelenmiştir. Burada Ω ve $\partial\Omega$

$$\Omega = (0, L_1) \times (0, L_2), \quad \Gamma_i = \partial\Omega \cap x_i = 0.$$

şeklinde tanımlanmıştır.

Verilen $a \geq 0$ ve $p > 2$, ve $L^2(\Omega)$ verilen başlangıç değerler için bu uzayda çözümün varlığı ve tekliği ispatlanmıştır. Ayrıca, bu çözümün problemin zayıf çözümü olduğu gösterilmiş ve bir patlama(blow-up) sonucu da elde edilmiştir.

DEDICATION

To my family...

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LIST OF ABBREVIATIONS

PDE	Partial Differential Equations
$\mathbb{N}$	Natural Numbers
$\mathbb{R}$	Real Numbers
$\mathbb{R}_+$	Positive real numbers
$\mathbb{R}^m$	All ordered m-length tuples made up of real numbers
C	Space of continuous functions
C^1	Space of functions whose first derivative is continuous
C^∞	The set of functions with infinite differentiability
C_c^∞	The space of infinitely differentiable functions with compact-support
$C[0,T]$	The continuous function space defined on $[0,T]$
H	Hilbert spaces
H^2	Hardy space with square norm
H^s	The space of Sobolev functions having $s-th$ order derivatives in L^2
$\dot{H}^s$	Homogeneous Sobolev space
H^2_{per}	Periodically Hilbert space
$\dot{H}^s$	Periodically homogeneous Hilbert space
$\dot{H}^s$	Homogeneous Sobolev space
I	Identity operator
L^q	Function spaces with q-norm for finite dimensional vector spaces for $q = 1, 2$ and ∞
$W^{s,q}$	Functions in a Sobolev space have derivatives up to order s that lie within the L^q space
Δ	Laplace operator
∇	Gradient operator
Ω	A region in $\mathbb{R}^m$ that is open, bounded, and connected
$\partial\Omega$	A boundary of Ω

1. INTRODUCTION

1.1. STATEMENT OF THE PROBLEM

In this thesis, the following fourth-order pseudo-parabolic equation is examined:

$$u_t - a\Delta u_t - \Delta u + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2}\nabla u), \quad (x,t) \in \Omega \times (0,T) \tag{1.1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad u_0 \in L^2(\Omega), \quad x \in \Omega, \tag{1.2}$$

and periodic boundary conditions

$$\forall x \in \Gamma_i, \;\; 0 < t < T, \quad i = 1,2, \qquad u(x,t) = u(x + L_i e_i, t), \tag{1.3}$$

$$u_{x_i}(x,t) = u_{x_i}(x + L_i e_i, t), \tag{1.4}$$

where

$$\Omega = (0, L_1) \times (0, L_2), \quad \Gamma_i = \partial\Omega \cap \{x_i = 0\}.$$

Here, $\partial\Omega$ is the boundary of Ω, $a \geq 0$, and $p > 2$. For $u_0 \not\equiv 0$, it is assumed that

$$\int_\Omega u_0(x) = 0. \tag{1.5}$$

The equation (1.1) and the condition (1.5) imply that $\frac{d}{dt}\int_\Omega u dx = 0$, indicating that periodic initial value functions with zero average yield periodic solutions with zero average.

1.2. LITERATURE SURVEY

Pseudo-parabolic equations, also known as Sobolev-type equations, form a class of partial differential equations (PDEs) that arise in various scientific and engineering disciplines. These equations play a crucial role in modeling physical phenomena involving both diffusion and wave propagation. Applications of pseudo-parabolic equations are diverse, including phenomena such as heat conduction in materials with memory, viscoelasticity, and fluid flow in porous media [1,2,3,4].

The linear Pseudo-parabolic equations are generally expressed in the form:

$$u_t - \eta \Delta u_t = k \Delta u, \tag{1.6}$$

where Δ denotes the Laplacian operator. Showalter and Ting, in [5], established connections between pseudo-parabolic and parabolic equations for a mixed boundary value problem associated with the partial differential equation (1.6). They considered the cases $\eta = 0$ and $\eta \neq 0$.

For $\eta = 0$, the equation simplifies to:

$$u_t = k \Delta u. \tag{1.7}$$

They demonstrated the uniqueness, existence, and regular behavior of the solutions. Furthermore, they found that the solution depends continuously on η, and as η approaches zero, the solution of (1.6) converges to the solution of (1.7).

In the 1960s and 1970s, significant progress was made in understanding the mathematical properties of pseudo-parabolic equations. Sergey Sobolev [6,7] made substantial contributions by analyzing their functional-analytic aspects, which led to the alternate name as Sobolev-type equations. These types of equations are also studied in [8,9]. These studies focused on understanding the existence-uniqueness of solutions to these equations.

Moreover, in this thesis, the aim is to analyze the effect of the pseudo-parabolic term $-a\Delta u_t$ and the diffusion term $-\Delta u$ in equation (1.1).

For $a = 0$, equation (1.1) turns into a special form of so called thin-film equations:

$$u_t + A_1 \Delta u + A_2 \Delta^2 u + A_3 \nabla \cdot (|\nabla u|^2 \nabla u) + A_4 \Delta |\nabla u|^2 = 0, \tag{1.8}$$

where $u(x,t)$ and $A_1 \Delta u$ indicate the terms used to represent the thickness of the film during epitaxial growth and the diffusion effects from evaporation and condensation. $A_2 \Delta^2 u$

and $A_3\nabla \cdot (|\nabla u|^2 \nabla u)$ are the capillarity-driven surface diffusion and atomic displacements, respectively, and the term $A_4\Delta|\nabla u|^2$ illustrates the movement of an atom. For a detailed description of this model, please see [10]. Thin film equations are long-standing topics of research, as shown in [10,11,12,13,14,15,16].

The study of the solutions of blow-up for the second-order nonlinear boundary value problems with initial conditions for parabolic equations has been a long-standing issue we refer [11,14,17,18,19,20,21,22,23,24,25]. For the fourth-order nonlinear parabolic equations, see the references: [11,25,26]. In [11], Feng and Xu studied the problem:

$$u_t + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) \tag{1.9}$$

with the initial condition

$$u(x,0) = u_0(x),\ u_0 \in L^2(\Omega),\ x \in \Omega,\ u \equiv 0 \tag{1.10}$$

on a two-dimensional torus. They derived an existence-uniqueness result when $2 < p < 3$. Moreover, they accomplished to contribute a result for the finite-time blow-up solutions for (1.9). In [26], thanks to the potential well theory; Zhou derived a finding for a blow-up, assuming the initial energy is positive for the initial-boundary-value problem of a form of a fourth-order thin film equation with a nonlocal source term. In [25], Philippin examined the initial-boundary value problem:

$$u_t + \Delta^2 u = k(x)|u|^{p-1}u,\ x \in \Omega,\ 0 < t < T,$$

$$u(x,t) = 0,\ \frac{\partial u}{\partial n} = 0,\ for\ x \in \partial\Omega,\ 0 < t < T,$$

$$u(x,0) = u_0(x),\ x \in \Omega,\ \Omega \in \mathbb{R}^n,\ n \geq 2, and\ k(x) > 0$$

and existence-uniqueness of the solutions. In addition, this study presented a blow-up result and provided a lower bound for the blow-up time.

On the other hand, the effect of the pseudo-parabolic term has been studied by many researchers. For example, Lakshmipriya, Gnanavel, Balachandran, and Yong-Ki addressed the following pseudo-parabolic problem in [27]:

$$w_t - \Delta w_t - \operatorname{div}\left(|\nabla w|^{p(x)-2}\nabla w\right) = |w|^{s(x)-2}w + |w|^{h-2}w \log|w|, \quad (x,t) \in \Omega \times (0,\infty),$$

with the boundary condition:

$$w(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty),$$

and the initial condition:

$$w(x,0) = w_0(x), ; x \in \bar{\Omega},$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$. Their study demonstrated about weak solutions to this equation with logarithmic nonlinearity over the interval $[0,T)$. They examined their existence and blow-up. Similar studies can be found in [28,29,30].

Polat investigated a different pseudo-parabolic equation under periodic boundary conditions, deriving a lower bound for finite time blow-up [17]. He analyzed:

$$u_t + \alpha\Delta u_t - \Delta u + \Delta^2 u = |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}u\,dx,\ x \in \Omega,\ 0 < t < T,$$

with the initial condition:

$$u(x,0) = u_0(x),$$

and

$$\frac{\partial u}{\partial n} = 0,\ \ x \in \partial\Omega,\ 0 < t < T$$

$$u(x,0) = u_0(x),\ u_0 \in \dot{H}_{per}(\Omega),\ x \in \Omega,\ u_0 \not\equiv 0$$

and periodic boundary conditions:

$$u(x,t) = u(x + L_i e_i, t),\ u_{x_i}(x,t) = u_{x_i}(x + L_i e_i, t)\ forall\ x \in \Gamma_i,\ 0 < t < T,$$

where:

$$\Omega = (0,L_1) \times \ldots \times (0,L_n) \subset \mathbb{R}^n,\ n = 2\ or\ 3, \Gamma = \partial\Omega \cap x_i = 0,$$

$\partial\Omega$ indicates the boundary of the set Ω, and $p > 1$, along with $u_0 \not\equiv 0$,

$$\int_\Omega u_0(x)\, dx = 0.$$

Polat also examines the following equation [14]:

$$u_t - u_{xx} + u_{xxxx} = |u|^{p-1}u - \frac{1}{a}\int_0^a |u|^{p-1}u, dx, \quad x \in \mathbb{R}, , t > 0,$$

with periodic boundary conditions:

$$u(x,t) = u(x+a,t), \quad \forall x \in \mathbb{R}, , t > 0,$$

and the initial condition:

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},$$

where $p > 1$, $u_0 \in \dot{H}^2_{\text{per}}(\Omega)$, $\Omega = (0,a)$, and $\int_0^a u_0(x), dx = 0$ with $u_0 \not\equiv 0$. This work investigates an equation describing thin films featuring a second-order diffusion component, a term with fourth-order derivatives, and a non-local term representing the source, all under periodic boundary conditions. Additionally, Polat demonstrated the occurrence of its finite-time blow-up for solutions.

In addition to the examples given above, numerous other studies have explored pseudo-parabolic equations, including those referenced in [5,31,32,33,34,35,36], highlighting the extensive depth of research in this area.

Apart from that, we can also say that pseudo-parabolic equations have been broadly categorized into areas such as heat conduction with memory effects, fluid flow in porous media, viscoelastic materials, biological systems, numerical methodologies, nonlinear equations, multidimensional problems, and coupled systems. Detailed discussions and examples of these studies are available in references such as [3,4,5,31,37,32,33,38].

1.3. FORMAT OF THE THESIS

This thesis consists of five chapters. Chapter 1 provides an overview of the literature of pseudo-parabolic equations, (1.1)-(1.2). It also reviews prior studies in the field and presents the formulation of the problem (1.1)-(1.2), supported by illustrative examples from existing literature.

Chapter 2 introduces the functional spaces, fundamental definitions, and lemmas necessary for establishing the existence and uniqueness properties of solutions.

In Chapter 3, the existence-uniqueness of the mild solution to (1.1)-(1.2) are established using the Banach Fixed Point Theorem and Lipschitz continuity. This chapter also proves the main theorem and demonstrates that the mild solution qualifies as a weak solution.

Chapter 4 focuses on blow-up results, proving the occurrence of blow-up and deriving both upper and lower bounds for the blow-up time of the solution.

Finally, Chapter 5 examines the distinguishing characteristics of pseudo-parabolic equations compared to parabolic equations, emphasizing their influence on the obtained results.

2. PRELIMINARIES

In this section, the necessary definitions, including the definitions of Sobolev spaces and the statements of the theorems used to prove the existence and uniqueness of the solution, are provided.

We proceed by giving the defitions of solution spaces:

$$L^2(\Omega) = \{u : \Omega \to \mathbb{R} : \int_\Omega |u(x)|^2 dx < \infty\}, \quad \text{where } \Omega = (0, L_1) \times (0, L_2).$$

is the Lebesgue space endowed with the inner product and norm

$$(u, v) = \int_0^a u(x)v(x)dx, \quad \|u\|^2 = \int_\Omega u^2 \, dx.$$

respectively.

$$H^2(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \ for \ |\alpha| \leq 2\},$$

and

$$\dot{H}^2_{per}(\Omega) := \{u \in H^2_{per}(\Omega) : \int_\Omega u \ dx = 0\},$$

which are referred to as the regular and periodic Hilbert spaces, respectively. The inner product and its norm are given as

The pair $(\dot{H}^2(\Omega), \|.\|_{\dot{H}^2(\Omega)})$ denotes Hilbert space for:

$$(u, v) := \int_\Omega \nabla u.\nabla u \ dx + \int_\Omega \Delta u \Delta v \ dx,$$

and the norm

$$\|u\|^2_{H^2(\Omega)} := \|\nabla u\|^2 + \|\Delta u\|^2.$$

In addition, $L^q(\Omega)$ is defined as the Lebesque space with the norm $\|u\|_q = \left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}}$. The Sobolev Space $W^{s,q}(\Omega)$ for $1 \leq q < \infty$ is defined as the subset of functions f in $L^q(\Omega)$

such that f and its weak derivatives up to an integer order s are in L^q. That is,

$$W^{s,q}(\Omega) = \Big\{u : u \in L^q(\Omega),\ u_x \in L^q(\Omega), ..., \frac{\partial^s u}{\partial x^s} \in L^q(\Omega), \left(\int_\Omega \left[|u|^q + |u_x|^q + ... + |\frac{\partial^s u^q}{\partial x^s}\ ]\ dx|\right]\right)^{\frac{1}{q}} < \infty\Big\}.$$

For $q \to \infty$, the norm is defined by

$$\|f\|_{k,\infty} = \max_{i=0,...,k} \|f^{(i)}\|_\infty = \max_{i=0,...,k}\left(ess \sup_t |f^{(i)}(t)|\right).$$

In the inequality above, $ess \sup$ means essential supremum. This allows us to generalize the maximum of a function in a useful way. Let $f \in L^q(\Omega)$ with $q \geq 1$ and $\hat{f}(k)$ be The Fourier transform of $f(k)$ at the frequencies $k \in \mathbb{Z}^2$ is given by

$$\hat{F}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx. \tag{2.1}$$

Moreover, when $q = 2$ and $s \in \mathbb{R}$, the Sobolev space by $H^s(\Omega)$ and the homogeneous Sobolev space by $\dot{H}^s(\Omega)$ are denoted by the norms

$$\|f\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^2} (1 + |k|^2)^s |\hat{f}(k)|^2 = \|(I - \Delta)^{s/2} f\|_{L^2}^2,$$
$$\|f\|_{\dot{H}^s}^2 = \sum_{k\in\mathbb{Z}^2} (|k|^2)^s |\hat{f}(k)|^2 = \|(-\Delta)^{s/2} f\|_{L^2}^2,$$

respectively. Here, I is the identity operator and $(-\Delta)^{s/2}$ shows Fourier multiplier with symbol $|k|^s$, $k \neq 0$. The strongly continuous semi-group of operators, which are generated by

$\mathcal{L}= (I - a\Delta)^{-1}(-\Delta + \Delta^2)$ on L^2, are defined by the inverse Fourier transform

$$e^{-t\mathcal{L}} f = F^{-1}\left(te^{-t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} f\right),$$

on Ω.

Definition 1. *A function $u : C([0,T]; L^2(\Omega))$ for $0 \leq T < \infty$ is defined as a mild solution for* *the problem (1.1)-(1.2) on* $[0,T]$ *given the initial data conditions* $u_0 \in L^2(\Omega)$*, if it satisfies*

$$\nu(u)(t) = u(t) = e^{-t\mathcal{L}}u_0 - \int_0^t e^{-(t-s)\mathcal{L}}(I - a\Delta)^{-1}(\nabla \cdot (|\nabla u|^{p-2}\nabla u))\, ds \tag{2.2}$$

for $0 \le t < T$.

Definition 2. *A function* $u \in L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; H^2(\Omega))$ *is defined as a weak solution to (1.1)-(1.2) on the interval* $[0,T]$ *with initial condition* $u_0 \in L^2(\Omega)$*, if for every test function* $\zeta \in C_c^\infty([0,T] \times \Omega)$*, the following weak formulation is satisfied:*

$$\begin{aligned}\int_\Omega u_0\zeta(0)dx + \int_\epsilon^t \int_\Omega u\frac{\partial}{\partial t}\zeta dxdt \\ = \int_\epsilon^t \int_\Omega \mathcal{L}\zeta u\, dxdt \\ - \int_\epsilon^t \int_\Omega (I - a\Delta)^{-1}|\nabla u(s)|^{p-2}\nabla u \cdot \nabla\zeta\, dxdt.\end{aligned} \tag{2.3}$$

for $0 \le t < T$.

Definition 3. *Let* X *be a Banach space. The operator* $T : X \to X$ *is called a contraction mapping if there exists a constant* α *such that and* $\|Tx - Ty\| \le \alpha\|x - y\|$ *for all* $x, y \in X$.

Definition 4. *Let* (X, d) *be a metric space. A function* $T : X \to X$ *is called a contraction mapping on* X *if a constant* β *with* $0 \le \beta < 1$ *exists* $d(T(x), T(y)) \le Kd(x, y)$ *for all* $x, y \in X$.

Theorem 1. *(Banach Fixed Point Theorem) Given that* $T : X \to X$ *is a contraction, a unique fixed point* $x*$ *satisfies* $Tx* = x*$*. Moreover, the sequence* (x_j) *defined by* $x_k = Tx_{k-1}$ *with any starting point* x_0 *converges to the* $x*$ *of* T*. (where* $j = 1, 2, 3, \ldots$*).*

Definition 5. *A function* f *defined on* $S \subset \mathbb{R}^n$ *with values in* $\mathbb{R}^m$ *is said to be Lipschitz continuous at* $x \in S$ *if there exists a constant* k *such that* $\|f(y) - f(x)\| \le k\|y - x\|$ *for all points* $y \in S$ *sufficiently close to* x.

3. LOCAL EXISTENCE AND UNIQUENESS

In this section, the existence and uniqueness of the problem (1.1)-(1.2) are established using the Banach Contraction Mapping Theorem, thereby proving the Main Theorem. Furthermore, it is demonstrated that the mild solution of the problem (1.1)-(1.2) also qualifies as a weak solution.

3.1. LOCAL EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS

Consider the following Banach spaces to attain our main results.
For $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}$,

$$E_S = \begin{cases} \sup\limits_{0\le t\le T} t^{\frac{1}{2}} \|\nabla u\|_2 < \infty, & \text{if } a \ge 1, \\ \sup\limits_{0\le t\le T} t^{\frac{1}{4}} \|\nabla u\|_2 < \infty, & \text{if } 0 \le a < 1, \end{cases}$$

and

$$\bar{E_S} = C([0,T]; L^2(\Omega) \cap E_S),$$

with the norm

$$\|u\|_{\bar{E_S}} = \begin{cases} \max\left(\sup\limits_{0\le t\le T} \|u\|_2, \sup\limits_{0\le t\le T} t^{\frac{1}{2}} \|\nabla u\|_2 \right), & \text{if } a \ge 1, \\ \max\left(\sup\limits_{0\le t\le T} \|u\|_2, \sup\limits_{0\le t\le T} t^{\frac{1}{4}} \|\nabla u\|_2 \right), & \text{if } 0 \le a < 1. \end{cases}$$

Throughout this thesis, the above spaces and the following lemmas will be used, which are crucial in proving the existence and uniqueness of the mild solution:

Lemma 1. *There exists constants $C^1 > 0$ and $C^2 > 0$ satisfying*

$$\|e^{-t\mathcal{L}}(I - a\Delta)^{-1}(\nabla \cdot f)\|_2 \le \begin{cases} C^1 t^{-\frac{1}{4}} \|f\|_1 & \text{if } a \ge 1, \\ \dfrac{C^2}{(1+a)} t^{-\frac{1}{2}} \|f\|_1 & \text{if } 0 \le a < 1. \end{cases}$$

Proof. By using Plancherel's identity and the definition of the operator $e^{-t\mathcal{L}}$, one finds

$$\begin{aligned}
&\|e^{-t\mathcal{L}}(I-a\Delta)^{-1}(\nabla\cdot f)\|_2^2\\
&\qquad = \sum_{k\in\mathbb{Z}^2}\frac{|k|^2}{(1+a|k|^2)^2}e^{-2t|k|^2\frac{1+|k|^2}{1+a|k|^2}}|\hat{f}(k)|^2\\
&\qquad \le \sup_{k\in\mathbb{Z}^2}|\hat{f}|(k)|^2\sum_{k\in\mathbb{Z}^2}\frac{|k|^2}{(1+a|k|^2)^2}e^{-2t|k|^2\frac{1+|k|^2}{1+a|k|^2}}\\
&\qquad \le \|f\|_1^2\int_{\mathbb{R}^2}\left(\frac{|x|^2}{(1+a|x|^2)^2}e^{-2t|x|^2\frac{1+|x|^2}{1+a|x|^2}}\right)dx. \qquad (3.1)
\end{aligned}$$

The above integral in polar coordinates is split into two parts:

$$\begin{aligned}
I &:= \int_{\mathbb{R}^2}\left(\frac{|x|^2}{(1+a|x|^2)^2}e^{-2t|x|^2\frac{1+|x|^2}{1+a|x|^2}}\right)dx\\
&= \int_0^{2\pi}\int_0^{\infty}\left(\frac{r^2}{(1+ar^2)^2}e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdrd\theta\\
&= 2\pi\int_0^{\infty}\left(\frac{r^2}{(1+ar^2)^2}e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdr\\
&= 2\pi\int_0^{1}\left(\frac{r^2}{(1+ar^2)^2}e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdr\\
&\qquad +2\pi\int_1^{\infty}\left(\frac{r^2}{(1+ar^2)^2}e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)rdr. \qquad (3.2)
\end{aligned}$$

In (3.2), the first integral is proper and it converges to a positive number, C_1. The second integral can be written as

$$I_2 = 2\pi\int_1^{\infty}\left(\frac{r^2 r}{(1+ar^2)^2}e^{-2tr^2\frac{1+r^2}{1+ar^2}}\right)dr.$$

Since $\dfrac{r^2(1+r^2)}{1+ar^2}\ge\dfrac{r^2}{a}$ for $a\ge 1$, and for $0<t\le 1$,

$$I_2 \le 2\pi\int_1^{\infty}\left(\frac{r}{a^2r^2}e^{-2t\frac{r^2}{a}}\right)dr = \frac{2\pi}{a^2}\int_1^{\infty}\left(\frac{\sqrt{r}\sqrt{r}}{r^2}e^{-2t\frac{r^2}{a}}\right)dr. \qquad (3.3)$$

Applying Hölder inequality to the right side of (3.3), we obtain

$$\frac{2\pi}{a^2}\int_1^{\infty}\left(\frac{\sqrt{r}\sqrt{r}}{r^2}e^{-2t\frac{r^2}{a}}\right)dr \le \frac{2\pi}{a^2}\left(\int_1^{\infty}re^{-4t\frac{r^2}{a}}dr\right)^{\frac{1}{2}}\left(\int_1^{\infty}\frac{r}{r^4}dr\right)^{\frac{1}{2}}. \qquad (3.4)$$

By computing the above integrals, the following is obtained:

$$I_2 \leq C_2 t^{-\frac{1}{2}} e^{-\frac{2t}{a}} \leq C_2 t^{-\frac{1}{2}} \quad for \quad C_2 = \frac{\pi}{2a^{\frac{3}{2}}}, \tag{3.5}$$

and

$$I \leq C_1 + C_2 t^{-\frac{1}{2}}.$$

For $C^1 = max\{C_1, C_2\}$, one gets

$$I \leq C^1(1 + t^{-\frac{1}{2}}). \tag{3.6}$$

If equation (3.6) is used in (3.1), then for $0 < t \leq 1$ one gets

$$\|e^{-t\mathcal{L}}(I - a\Delta)^{-1}(\nabla.f)\|_2^2 \leq C^1(1 + t^{-\frac{1}{2}})\|f\|_1^2 \leq C^1 t^{-\frac{1}{2}}\|f\|_1^2.$$

In addition, $0 \leq a < 1$, $\quad \frac{r^2 + r^4}{1 + ar^2} > r^2$ and $0 < t \leq 1$

$$I_2 \leq 2\pi \int_1^\infty e^{-2tr^2} \frac{r^2}{(1 + ar^2)^2} r dr.$$

Taking $s = 1 + ar^2$ in the above inequality,

$$\frac{\pi}{a^2} e^{\frac{2t}{a}} \int_{1+a}^\infty e^{-\frac{2ts}{a}} \frac{s - 1}{s^2} ds = \frac{\pi}{2t} e^{-2t} \frac{1}{(1 + a)^2} \leq \frac{C^2}{(1 + a)^2 t},$$

where $C^2 = \frac{\pi}{2}$. Hence,

$$\|e^{-t\mathcal{L}}(I - a\Delta)^{-1}(\nabla.f)\|_2^2 \leq \frac{C^2}{(1 + a)^2} t^{-1}\|f\|_1^2.$$

□

Lemma 1 can be generalized further as in the following statement:

Lemma 2. *There exists a constant $C > 0$ satisfying*

$$\|e^{-t\mathcal{L}} f\|_2^2 \leq C t^{-\frac{p-2}{2}} \|f\|_{\frac{2}{p-1}}. \tag{3.7}$$

Proof. By employing Plancherel's identity, the definition of the operator $e^{-t\mathcal{L}}$, and utilizing the Hausdorff-Young inequality in Ω, the following inequality is derived for $a \geq 1$ and $2 < p < \frac{5}{2}$:

$$\begin{aligned}
\|e^{-t\mathcal{L}}f\|_2^2 &= \sum_{k\in\mathbb{Z}^2} e^{-2t|k|^2\frac{1+|k|^2}{1+a|k|^2}}|\hat{f}(k)|^2 \leq \left(\sum_{k\in\mathbb{Z}^2} e^{\frac{-2t|k|^2}{p-2}\frac{1+|k|^2}{1+a|k|^2}}\right)^{p-2}\left(\sum_{k\in\mathbb{Z}^2}|\hat{f}(k)|^{\frac{2}{5-2p}}\right)^{5-2p} \\
&\leq C\|f\|^2_{\frac{2}{p-2}}\left(\int_{\mathbb{R}^2} e^{\frac{-2t|x|^2}{p-2}\frac{1+|x|^2}{1+a|x|^2}}dx\right)^{p-2} \leq Ct^{-(p-2)}\|f\|_{\frac{2}{p-2}}.
\end{aligned}$$

For the special case $p = \frac{5}{2}$, the term $\|e^{-t\mathcal{L}}f\|_2^2$ is directly bounded as

$$\|f\|_{l^\infty(\mathbb{Z}^2)} \cdot \sum_{k\in\mathbb{Z}^2} e^{-4t|k|^2}\frac{1+|k|^2}{1+a|k|^2}.$$

Consequently, the desired result is obtained.

We proceed to establish this lemma for the cases $0 \leq a < 1$ and $2 < p < 3$. Utilizing Plancherel's identity, the definition of the operator $e^{-t\mathcal{L}}$, and employing the Hausdorff-Young inequality in Ω the following inequality is derived for $0 \leq a < 1$ and $2 < p < 3$:

$$\begin{aligned}
\|e^{-t\mathcal{L}}f\|_2^2 &= \sum_{k\in\mathbb{Z}^2} e^{-2t|k|^2\frac{1+|k|^2}{1+a|k|^2}}|\hat{f}(k)|^2 \leq \left(\sum_{k\in\mathbb{Z}^2} e^{\frac{-2t|k|^2}{p-2}\frac{1+|k|^2}{1+a|k|^2}}\right)^{p-2}\left(\sum_{k\in\mathbb{Z}^2}|\hat{f}(k)|^{\frac{2}{3-p}}\right)^{3-p} \\
&\leq C\|f\|^2_{\frac{2}{p-2}}\left(\int_{\mathbb{R}^2} e^{\frac{-2t|x|^2}{p-2}\frac{1+|x|^2}{1+a|x|^2}}dx\right)^{p-2} \leq Ct^{-(p-2)}\|f\|_{\frac{2}{p-2}}.
\end{aligned}$$

For the specific case $p = 3$, the term $\|e^{-t\mathcal{L}}f\|_2^2$ is directly bounded as

$$\|f\|_{l^\infty(\mathbb{Z}^2)} \cdot \sum_{k\in\mathbb{Z}^2} e^{-2t|k|^2}\frac{1+|k|^2}{1+a|k|^2}.$$

Consequently, the desired result is obtained. □

Lemma 3. *For any $s \geq 0$, there exists constants $C^3 > 0$ and $C^4 > 0$ such that*

$$\|(-\Delta)^{-\frac{s}{2}}e^{-t\mathcal{L}}f\|_2 \leq \begin{cases} C^3 t^{-\frac{s}{2}}\|f\|_2 & \text{if } a \geq 1, \\ C^4 t^{-\frac{s}{4}}\|f\|_2 & \text{if } 0 \leq a < 1. \end{cases}$$

Proof. By using Plancherel's identity and the definition of the operator $e^{-t\mathcal{L}}$

for $a \geq 1$, one can write:

$$
\begin{aligned}
\|(-\Delta)^{\frac{s}{2}} e^{-t\mathcal{L}} f\|_2^2 &= \sum_{k\in\mathbb{Z}^2} |k|^{2s} e^{-2t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} |\hat{f}(k)|^2 \\
&\leq \sum_{k\in\mathbb{Z}^2} |k|^{2s} e^{-2t\frac{|k|^2}{a}} |\hat{f}(k)|^2 \\
&\leq C^3 \left(\sup_{x\in\mathbb{R}^+} x^{2s} e^{-t\frac{x^2}{a}}\right) \sum_{k\in\mathbb{Z}^2} |\hat{f}(x)|^2 \leq C^3 t^{-s} \|f\|_2^2.
\end{aligned}
$$

For $0 \leq a < 1$, one can write:

$$
\begin{aligned}
\|(-\Delta)^{\frac{s}{2}} e^{-t\mathcal{L}} f\|_2^2 &= \sum_{k\in\mathbb{Z}^2} |k|^{2s} e^{-2t|k|^2 \frac{1+|k|^2}{1+a|k|^2}} |\hat{f}(k)|^2 \\
&\leq \sum_{k\in\mathbb{Z}^2} |k|^{2s} e^{-2ta|k|^4} |\hat{f}(k)|^2 \\
&\leq C^4 \left(\sup_{x\in\mathbb{R}^+} x^{2s} e^{-tax^4}\right) \sum_{k\in\mathbb{Z}^2} |\hat{f}(x)|^2 \leq C^4 t^{-\frac{s}{2}} \|f\|_2^2.
\end{aligned}
$$

□

Lemma 4. ***(i)*** *For $2 < p < \frac{5}{2}$, $a \geq 1$, and $0 < T \leq 1$, a constant $C_1 > 0$ exists satisfying for the operator $\nu : \bar{E_S} \to \bar{E_S}$ satisfies*

$$\|\nu(u)\|_{\bar{E_S}} \leq C_1 \left(\|u_0\|_2 + T^{\frac{5-2p}{4}} \|u\|_{\bar{E_S}}^{p-1}\right). \tag{3.8}$$

(ii) *For $2 < p < 3$, $0 \leq a < 1$, and $0 < T \leq 1$, a constant $\hat{C_1} > 0$ exists satisfying the operator $\nu : \bar{E_S} \to \bar{E_S}$ satisfies*

$$\|\nu(u)\|_{\bar{E_S}} \leq \hat{C_1} \left(\|u_0\|_2 + (1+a)^{-1} T^{\frac{3-p}{4}} \|u\|_{\bar{E_S}}^{p-1}\right). \tag{3.9}$$

Proof. To prove **(i)**, it is enough to show the two assertions given below hold:

Assertion 1 If $u \in \bar{E_S}$, then $\nu(u) \in C([0,T]; L^2(\Omega))$;

Assertion 2 If $u \in \bar{E_S}$, then $\sup_{0\leq t\leq T} t^{\frac{1}{2}} \|\nabla(\nu(u))\|_2 \leq \infty$.

Throughout the remaining computations, the operator T is denoted by $(I - a\Delta)^{-1}$. In the proofs of these two inequalities, the equality $s = t\xi$ will be used when needed. To prove

Assertion 1, Lemma 1 and Lemma 3 are used with $2 < p < \frac{5}{2}$ as follows:

$$
\begin{aligned}
\|v(u)\|_2 &\leq C\left(\|u_0\|_2 + \int_0^t \|e^{-(t-s)\mathcal{L}} T\nabla\cdot(|\nabla u|^{p-2}\nabla u)\|_2\, ds\right)\\
&\leq C\left(\|u_0\|_2 + \int_0^t (t-s)^{-\frac{1}{4}} \|\,(|\nabla u|^{p-2}\nabla u)\|_1\, ds\right)\\
&\leq C\left(\|u_0\|_2 + \int_0^t (t-s)^{-\frac{1}{4}} \|\nabla u\|_2^{p-1}\, ds\right)\\
&\leq C\left(\|u_0\|_2 + \int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{(p-1)}{2}} (s^{\frac{1}{2}}\|\nabla u\|_2)^{p-1}\, ds\right)\\
&\leq C\left(\|u_0\|_2 + t^{\frac{5-2p}{4}} \int_0^1 (1-\xi)^{-\frac{1}{4}} \xi^{-\frac{(p-1)}{2}} \|u\|_{\bar{E_S}}^{p-1}\, d\xi\right)\\
&\leq C\left(\|u_0\|_2 + t^{\frac{5-2p}{4}} \|u\|_{\bar{E_S}}^{p-1}\right).
\end{aligned}
$$

Similarly, for Assertion 2,

$$
\begin{aligned}
\|\nabla v(u)(t)\|_2 &\leq \|\nabla e^{-t\mathcal{L}} u_0\|_2\\
&\quad + \int_0^t \|\nabla e^{-\frac{(t-s)}{2}\mathcal{L}}\|_{2\to 2} \|e^{-\frac{(t-s)}{2}\mathcal{L}} T(\nabla\cdot(|\nabla u|^{p-2}\nabla u))\|_2\, ds\\
&\leq Ct^{\frac{-1}{2}}\|u_0\|_2 + C\int_0^t (t-s)^{-\frac{1}{2}}(t-s)^{-\frac{1}{4}} \|\,|\nabla u|^{p-2}\nabla u)\|_1\, ds\\
&\leq Ct^{\frac{-1}{2}}\|u_0\|_2 + C\int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{(p-1)}{2}} (s^{\frac{1}{2}}\|\nabla u\|_2)^{p-1}\, ds\\
&\leq Ct^{\frac{-1}{2}}\|u_0\|_2 + Ct^{\frac{3-2p}{4}} \int_0^1 (1-\xi)^{-\frac{3}{4}} \xi^{-\frac{(p-1)}{2}} \|u\|_{\bar{E_S}}^{p-1}\, d\xi\\
&\leq C\left(t^{\frac{-1}{2}}\|u_0\|_2 + t^{\frac{3-2p}{4}} \|u\|_{\bar{E_S}}^{p-1}\right). \qquad (3.10)
\end{aligned}
$$

Multiplying the inequality (3.10) by $t^{\frac{1}{2}}$, one obtains

$$
t^{\frac{1}{2}}\|\nabla v(u)(t)\|_2 \leq C\left(\|u_0\|_2 + t^{\frac{5-2p}{4}}\|u\|_{\bar{E_S}}^{p-1}\right),
$$

for any $t \in [0, T]$. Merging these two assertions, the proof of **(i)** is completed where $v : \bar{E_S} \to \bar{E_S}$ is a bounded operator.

Similarly, to prove the second part, it is enough to show the two assertions given below hold:

Assertion 1 If $u \in \bar{E_S}$, then $v(u) \in C([0, T]; L^2(\Omega))$;

Assertion 2 If $u \in \bar{E_S}$, then $\sup_{0\leq t\leq T} t^{\frac{1}{4}}\|\nabla(v(u))\|_2 \leq \infty$.

In the proofs of these two inequalities, the equality $s = t\xi$ will be used when needed. To prove Assertion 1, Lemma 1 and Lemma 3 are used with $2 < p < 3$,

$$\begin{aligned}
\|v(u)\|_2 &\leq C\left(\|u_0\|_2 + \int_0^t \|e^{-(t-s)\mathcal{L}}T\nabla\cdot(|\nabla u|^{p-2}\nabla u)\|_2\,ds\right)\\
&\leq C\left(\|u_0\|_2 + (1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{2}}\|\,(|\nabla u|^{p-2}\nabla u)\|_1\,ds\right)\\
&\leq C\left(\|u_0\|_2 + (1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{2}}\|\nabla u\|_2^{p-1}\,ds\right)\\
&\leq C\left(\|u_0\|_2 + (1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{2}}s^{-\frac{(p-1)}{4}}(s^{\frac{1}{4}}\|\nabla u\|_2)^{p-1}\,ds\right)\\
&\leq C\left(\|u_0\|_2 + (1+a)^{-1}t^{\frac{3-p}{2}}\int_0^1 (1-\xi)^{-\frac{1}{2}}\xi^{-\frac{(p-1)}{4}}\|u\|_{\bar{E_S}}^{p-1}d\xi\right)\\
&\leq C\left(\|u_0\|_2 + (1+a)^{-1}t^{\frac{3-p}{4}}\|u\|_{\bar{E_S}}^{p-1}\right).
\end{aligned}$$

Similarly, for Assertion 2,

$$\begin{aligned}
\|\nabla v(u)(t)\|_2 &\leq \|\nabla e^{-t\mathcal{L}}u_0\|_2 + \int_0^t \|\nabla e^{-\frac{(t-s)}{2}\mathcal{L}}\|_2\|e^{-\frac{(t-s)}{2}\mathcal{L}}T(\nabla\cdot(|\nabla u|^{p-2}\nabla u))\|_2\,ds\\
&\leq Ct^{-\frac{1}{4}}\|u_0\|_2 + C(1+a)^{-1}\int_0^t (t-s)^{-\frac{1}{4}}(t-s)^{-\frac{1}{2}}\|\,|\nabla u|^{p-2}\nabla u)\|_1\,ds\\
&\leq Ct^{-\frac{1}{4}}\|u_0\|_2 + C(1+a)^{-1}\int_0^t (t-s)^{-\frac{3}{4}}s^{-\frac{(p-1)}{4}}(s^{\frac{1}{4}}\|\nabla u\|_2)^{p-1}\,ds\\
&\leq Ct^{-\frac{1}{4}}\|u_0\|_2 + C(1+a)^{-1}t^{\frac{2-p}{4}}\int_0^1 (1-\xi)^{-\frac{3}{4}}\xi^{-\frac{(p-1)}{4}}\|u\|_{\bar{E_S}}^{p-1}\,d\xi\\
&\leq C\left(t^{-\frac{1}{4}}\|u_0\|_2 + (1+a)^{-1}t^{\frac{2-p}{4}}\|u\|_{\bar{E_S}}^{p-1}\right). \qquad (3.11)
\end{aligned}$$

Multiplying the inequality (3.11) by $t^{\frac{1}{4}}$, one gets

$$t^{\frac{1}{4}}\|\nabla v(u)(t)\|_2 \leq C\left(\|u_0\|_2 + (1+a)^{-1}t^{\frac{3-p}{4}}\|u\|_{\bar{E_S}}^{p-1}\right),$$

for any $t \in [0, T]$. Merging these two assertions, the proof of **(ii)** is completed. This means $v : \bar{E_S} \to \bar{E_S}$ is a bounded operator. □

Lemma 5. ***(i)*** *For $2 < p < \frac{5}{2}$, $a \geq 1$, and $0 < T \leq 1$, there exists a constant C_2 such that the operator $v : \bar{E_S} \to \bar{E_S}$ is a Lipschitz continuous map. That is, it satisfies:*

$$\|v(u_1) - v(u_2)\|_{\bar{E_S}} \leq C_2 T^{\frac{5-2p}{4}}(\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2})\|u_1 - u_2\|_{\bar{E_S}}. \qquad (3.12)$$

(ii) *For $2 < p < 3$, $0 \leq a < 1$, and $0 < T \leq 1$, there exists a constant $\hat{C}_2$ such that the operator $\nu : \bar{E_S} \longrightarrow \bar{E_S}$ is a Lipschitz continuous map. That is, it satisfies:*

$$\|\nu(u_1) - \nu(u_2)\|_{\bar{E_S}} \leq \hat{C}_2(1+a)^{-1} T^{\frac{3-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}. \tag{3.13}$$

Proof. To prove **(i)**, it is sufficient to show the following two inequalities:

I$_1$ $\|\nu(u_1) - \nu(u_2)\|_2 \leq Ct^{\frac{5-2p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}$;

I$_2$ $\sup_{t \in [0,T]} t^{\frac{1}{2}} \|\nabla\nu(u_1) - \nabla\nu(u_2)\|_2 \leq Ct^{\frac{5-2p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}$.

In the proofs of these two inequalities, the equality $s = t\xi$ will be used when needed. To prove **I$_1$**, Lemma 1 and Lemma 3 are used with $2 < p < \frac{5}{2}$,

$$\begin{aligned}
\|\nu(u_1) - \nu(u_2)\|_2 &\leq C\int_0^t (t-s)^{-\frac{1}{4}} \|\, |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\|_1 \, ds \\
&\leq C\int_0^t (t-s)^{-\frac{1}{4}} \|\, |\nabla u_1 - \nabla u_2|(|\nabla u_1|^{p-2} + |\nabla u_2|^{p-2})\|_1 \, ds \\
&\leq C\int_0^t (t-s)^{-\frac{1}{4}} \|\nabla u_1 - \nabla u_2\|_2 (\|\, |\nabla u_1|^{p-2}\|_2 + \|\, |\nabla u_2|^{p-2}\|_2) \, ds \\
&\leq C\int_0^t (t-s)^{-\frac{1}{4}} s^{-\frac{1}{2}} \|u_1 - u_2\|_{\bar{E_S}} s^{-\frac{(p-2)}{2}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \, ds \\
&\leq Ct^{\frac{5-2p}{4}} \int_0^1 (1-\xi)^{-\frac{1}{4}} \xi^{\frac{2-p}{2}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}} \, d\xi \\
&\leq Ct^{\frac{5-2p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}.
\end{aligned}$$

Similarly, for $\mathbf{I_2}$, one can write

$$\begin{aligned}
&\|\nabla v(u_1) - \nabla v(u_2)\|_2 \\
&\leq \int_0^t \|\nabla e^{-\frac{(t-s)}{2}\mathcal{L}}\|_2 \|e^{-\frac{(t-s)}{2}\mathcal{L}} T(\nabla \cdot (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2})\nabla u_2)\|_2 ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}} (t-s)^{-\frac{1}{4}} \| \, |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\|_1 \, ds \\
&\leq C \int_0^t (t-s)^{-\frac{3}{4}} \|\nabla u_1 - \nabla u_2\|_2 (\| \, |\nabla u_1|^{p-2}\|_2 + \| \, |\nabla u_2|^{p-2}\|_2) \, ds \\
&\leq C \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} \|u_1 - u_2\|_{\bar{E_S}} s^{-\frac{(p-2)}{2}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \, ds \\
&\leq Ct^{\frac{3-2p}{4}} \int_0^1 (1-\xi)^{-\frac{3}{4}} \xi^{\frac{1-p}{2}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}} \, d\xi, \\
&\leq Ct^{\frac{3-2p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}.
\end{aligned} \tag{3.14}$$

Multiplying the inequality (3.14) by $t^{\frac{1}{2}}$, one gets

$$t^{\frac{1}{2}} \|\nabla v(u_1) - \nabla v(u_2)\|_2 \leq Ct^{\frac{5-2p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}},$$

for all $t \in [0, T]$. Merging the above two inequalities, **(i)** is shown. That is, $v : \bar{E_S} \to \bar{E_S}$ is a Lipschitz continuous map. To prove **(ii)**, in the same way as the previous inequality, it is enough to show the following inequalities hold:

$\mathbf{I_3}$ $\quad \|v(u_1) - v(u_2)\|_2 \leq C(1+a)^{-1} t^{\frac{3-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}};$

$\mathbf{I_4}$ $\quad \sup_{t \in [0,T]} t^{\frac{1}{2}} \|\nabla v(u_1) - \nabla v(u_2)\|_2 \leq C(1+a)^{-1} t^{\frac{3-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}.$

In the proofs of these two inequalities, the equality $s = t\xi$ will be used when needed. To prove $\mathbf{I_3}$, Lemma 1 and Lemma 3 are used and $2 < p < 3$ is assumed:

$$
\begin{aligned}
\|v(u_1) - v(u_2)\|_2 &\leq C^* \int_0^t (t-s)^{-\frac{1}{2}} \| \, |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\|_1 \, ds \\
&\leq C^* \int_0^t (t-s)^{-\frac{1}{2}} \| \, |\nabla u_1 - \nabla u_2|(|\nabla u_1|^{p-2} + |\nabla u_2|^{p-2})\|_1 \, ds \\
&\leq C^* \int_0^t (t-s)^{-\frac{1}{2}} \|\nabla u_1 - \nabla u_2\|_2(\, \| \, |\nabla u_1|^{p-2}\|_2 + \| \, |\nabla u_2|^{p-2}\|_2) \, ds \\
&\leq C^* \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{4}} \|u_1 - u_2\|_{\bar{E_S}} s^{-\frac{(p-2)}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \, ds \\
&\leq C^* t^{\frac{3-p}{4}} \int_0^1 (1-\xi)^{-\frac{1}{2}} \xi^{\frac{2-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}} \, d\xi \\
&\leq C(1+a)^{-1} t^{\frac{3-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}},
\end{aligned}
$$

in which $C^* = C(1+a)^{-1}$. Similarly,for $\mathbf{I_4}$, one has

$$
\begin{aligned}
&\|\nabla v(u_1) - \nabla v(u_2)\|_2 \\
&\leq \int_0^t \|\nabla e^{-\frac{(t-s)}{2}\mathcal{L}}\|_2 \|e^{-\frac{(t-s)}{2}\mathcal{L}} T(\nabla \cdot (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2))\|_2 ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2}} (t-s)^{-\frac{1}{4}} \| \, |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\|_1 \, ds \\
&\leq C^* \int_0^t (t-s)^{-\frac{3}{4}} \|\nabla u_1 - \nabla u_2\|_2(\, \| \, |\nabla u_1|^{p-2}\|_2 + \| \, |\nabla u_2|^{p-2}\|_2) \, ds \\
&\leq C^* \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{4}} \|u_1 - u_2\|_{\bar{E_S}} s^{-\frac{(p-2)}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \, ds \\
&\leq C^* t^{\frac{2-p}{4}} \int_0^1 (1-\xi)^{-\frac{3}{4}} \xi^{\frac{1-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}} \, d\xi \\
&\leq C(1+a)^{-1} t^{\frac{2-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}, \qquad (3.15)
\end{aligned}
$$

in which $C^* = C(1+a)^{-1}$. Multiplying the inequality (3.15) with $t^{\frac{1}{4}}$, one gets:

$$
t^{\frac{1}{4}} \|\nabla v(u_1) - \nabla v(u_2)\|_2 \leq C(1+a)^{-1} t^{\frac{3-p}{4}} (\|u_1\|_{\bar{E_S}}^{p-2} + \|u_2\|_{\bar{E_S}}^{p-2}) \|u_1 - u_2\|_{\bar{E_S}}
$$

for any $t \in [0, T]$. Combining these two inequalities, the proof of **(ii)** is completed. That is, $v : \bar{E_S} \to \bar{E_S}$ is a Lipschitz continuous map. □

The main result of this study is given below.

Theorem 2. ***(Main Theorem) (i)*** *Let $u_0 \in L^2(\Omega)$ such that $a \geq 1$, $\|u_0\|_2 \leq \frac{R}{C_0}$, where*

$C_0 = \max\{C_1, C_2\}$ for the positive constants C_1 and C_2 provided by Lemma 4 and Lemma 5, and $2 < p < \frac{5}{2}$. Then $0 \leq T \leq 1$ exists, based on $\|u_0\|_2$, so that (1.1)-(1.2) accepts a unique mild solution u over the interval $[0, T] \in \bar{E}_S$.

***(ii)** Let $u_0 \in L^2(\Omega)$ such that $0 \leq a < 1$, $\|u_0\|_2 \leq \frac{R}{C_0^*}$, where $C_0^* = \max\{\hat{C}_1, \hat{C}_2\}$ for the positive constants $\hat{C}_1$ and $\hat{C}_2$ provided by Lemma 4 and Lemma 5, and $2 < p < 3$. Then $0 \leq T \leq 1$ exists, based on $\|u_0\|_2$, such that (1.1)-(1.2) accepts a unique mild solution u over the interval $[0, T] \in \bar{E}_S$.*

Proof. **(i)** Let $B_R(0)$ be the closed ball with radius R, and

$$T \leq \min\{1, (2C_0 R^{p-2})^{-\frac{4}{5-2p}}\}.$$

Then, Lemma 4 suggests that, for every $u \in B_R(0)$,

$$\|v(u)\|_{\bar{E_S}} \leq R.$$

Moreover, Lemma 5 states that, for every $u_1, u_2 \in B_R(0)$ and $l_1 < 1$,

$$\|v(u_1) - (u_2)\|_{\bar{E_S}} \leq l_1 \|u_1 - u_2\|_{\bar{E_S}}.$$

By the Banach Contraction Mapping Theorem, a fixed point of v in $B_R(0)$ is unique. Thus, this fixed point u is the unique mild solution of (1.1)-(1.2) for the initial data u_0.

(ii) Let $B_R(0)$ be the closed ball with radius R, and

$$T \leq min\{1, ((1 + a)^{-1} 2C_0^* R^{p-2})^{-\frac{4}{3-p}}\}.$$

Then, Lemma 4 implies that, for every $u \in B_R(0)$,

$$\|v(u)\|_{\bar{E_S}} \leq R.$$

Moreover, Lemma 5 denotes that for every $u_1, u_2 \in B_R(0)$ and $l_2 < 1$,

$$\|v(u_1) - (u_2)\|_{\bar{E_S}} \leq l_2 \|u_1 - u_2\|_{\bar{E_S}}.$$

By the Banach Contraction Mapping Theorem, a fixed point of ν in $B_R(0)$ is unique. Thus, this fixed point u is the unique mild solution of (1.1)-(1.2) for the initial data u_0. □

3.2. WEAK SOLUTIONS

In this section, we show that the mild solution of the problem in (1.1) with the initial condition (1.2) meets the criteria for being a weak solution. To demonstrate this, we first derive the energy inequality. Next, we prove that the mild solution belongs to the functional spaces $L^2([0,T];H^2(\Omega))$ and $L^2([0,T];H^{-2}(\Omega))$, satisfying the regularity conditions required for weak solutions. Using compactly supported test functions and the properties of the semigroup $e^{-t\mathcal{L}}$, we confirm that the mild solution conforms to the weak formulation of the problem. Finally, we apply the Dominated Convergence Theorem to complete the proof.

Proposition 1. *If u is a mild solution on the interval $[0,T]$, then it is a weak solution for the initial value problem (1.1)-(1.2) over the same interval $[0,T]$. Moreover, the following energy inequality is satisfied:*

For any $t \in [0,T]$, $2 < p < \frac{5}{2}$, and $a \geq 1$,

$$\|u(t)\|_2^2 + 2\int_0^t \|\Delta u(s)\|_2^2\, ds \leq \|u_0\|_2^2 + 2\int_0^t \|\nabla u(s)\|_p^p ds + C\int_0^t \|\nabla u(s)\|_2^2\, ds. \quad (3.16)$$

Proof. We first rewrite the mild solution of the problem (1.1)-(1.2) for $0 < \epsilon \leq t \leq T$ as follows:

$$\nu(u)(t) = u(t) = e^{-t\mathcal{L}}u_0 - \int_0^t e^{-(t-s)\mathcal{L}}(I - a\Delta)^{-1}(\nabla \cdot (|\nabla u|^{p-2}\nabla u))\, ds.$$

Reformulating this solution at initial point $t = \epsilon$ we get

$$\nu(u)(t) = u(t) = e^{-(t-\epsilon)\mathcal{L}}u(\epsilon) - \int_0^t e^{-(t-s)\mathcal{L}}(I - a\Delta)^{-1}(\nabla \cdot (|\nabla u|^{p-2}\nabla u))ds \quad (3.17)$$

which holds under the conditions

$$u \in C([\epsilon,T];L^2(\Omega)),$$

with

$$u \in L^2([\epsilon, T]; H^2(\Omega)).$$

For $u(x, .) : [\epsilon, T] \to H^2(\Omega)$, we denote

$$\int_{\epsilon}^{T} \|u(t)\|_{H^2(\Omega)}^2 dt = \|u\|_{L^2([\epsilon,T];H^2(\Omega))}.$$

Now we will show that $\Delta u \in L^2([\epsilon, T] \times \Omega)$.

From the Main theorem,

$$u \in L^2([\epsilon, T]; H^2(\Omega)),$$

and

$$u_x \in L^2([\epsilon, T]; H^1(\Omega)),$$

with

$$u_{xx} \in L^2([\epsilon, T]; H^0(\Omega)) = L^2([\epsilon, T]; L^2(\Omega)).$$

As a result, it follows that

$$\Delta u \in L^2([\epsilon, T]; L^2(\Omega)).$$

Consequently, by employing the Δ for both sides of (3.17), using the decay properties of the semigroup $e^{-t\mathcal{L}}$, and noting that $u \in \bar{E}_S$ on the interval $[0, T]$, we obtain:

$$\Delta v(u)(t) = \Delta u(t) = e^{-(t-\epsilon)\mathcal{L}} \Delta u(\epsilon) - \int_0^t e^{-(t-s)\mathcal{L}} (I - a\Delta)^{-1} \Delta R(s)\, ds \qquad (3.18)$$

where $R(s) = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ and $F(\nabla u(s)) = (|\nabla u|^{p-2} \nabla u)$.

By employing the triangle inequality and the bound $\|(I-a\Delta)^{-1}\| \leq 1$,

$$\begin{aligned}
\|\Delta u(t)\|_2 &\leq C\left(\|e^{-(t-\epsilon)\mathcal{L}}\nabla\nabla u(\epsilon)\|_2 + \int_\epsilon^t \|e^{-(t-s)\mathcal{L}}(I-a\Delta)^{-1}\Delta R(s)\|_2\, ds\right) \\
&\leq C\left((t-\epsilon)^{-\frac{1}{2}}\|\nabla u(\epsilon)\|_2 + \int_\epsilon^t \|e^{-(t-s)\mathcal{L}}\Delta R(s)\|_2\, ds\right) \\
&\leq C\left((t-\epsilon)^{-\frac{1}{2}}\|u(\epsilon)\|_{\dot{H}^1} + \int_\epsilon^t \|e^{-(t-s)\mathcal{L}}\Delta(\nabla.(|\nabla u|^{p-2}\nabla u))\|_2\, ds\right) \\
&\leq C\left((t-\epsilon)^{-\frac{1}{2}}\|u(\epsilon)\|_{\dot{H}^1} + \int_\epsilon^t (t-s)^{-\frac{3}{2}}(t-s)^{-\frac{p-2}{2}}\|\nabla u\|_{L^2}^{p-1}\, ds\right) \\
&\leq C(t-\epsilon)^{-\frac{1}{2}}\epsilon^{\frac{1}{2}}\|u(\epsilon)\|_{\dot{H}^1}\epsilon^{-\frac{1}{2}} + C\int_\epsilon^t (t-s)^{-\frac{p+1}{2}}\|\nabla u\|_{L^2}^{p-1}\, ds \\
&\leq C\left((t-\epsilon)^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}}\|u\|_{\bar{E_S}} + Ct^{2-p}\|u\|_{\bar{E_S}}^{p-1}\right)
\end{aligned}$$

is obtained. From the above inequality, the right-hand side belongs to L^2 for $2 < p < \frac{5}{2}$ and $\epsilon > 0$.

We give the proofs of the claims:

Statement 1. $F(\nabla u) \in L^2([\epsilon,T]; L^{\frac{1}{p-2}}(\Omega))$.

Statement 2. $M(u) \in L^1([\epsilon,T];\Omega)$.

The first statement follows directly from the Gagliardo-Nirenberg inequality, for $\frac{2}{3} < \frac{1}{p-1} < 1$.

$$\begin{aligned}
\int_\epsilon^T \|F(\nabla u)\|_{\frac{1}{p-2}}^2\, ds = \int_\epsilon^T \|\nabla u\|_{\frac{1}{p-2}}^{2(p-1)} ds \\
&\leq C\int_\epsilon^T \left(\|\Delta u\|_2^2\|u\|_2^{2(p-2)} + \|u\|_2^{p-1}\right) ds \\
&\leq C\int_\epsilon^T \left(\|u\|_{C([\epsilon,T];L^2(\Omega))}^{2(p-2)}\|u\|_{L^2([\epsilon,T];H^2(\Omega))}^2 + L\|u\|_{C([\epsilon,T];L^2(\Omega))}^{p-1}\right) ds.
\end{aligned}$$

To establish the second statement, we first compute the following:

$$|M(u)| = |\nabla\cdot(|\nabla u|^{p-2}\nabla u) \leq |\nabla u|^{p-2}|\Delta u| + (p-2)|\nabla u|^{p-4}|(\nabla u)^T\nabla^2 u\nabla u|,$$

where u^T stands for a vector transpose $u \in \mathbb{R}^2$. Namely,

$$\begin{aligned}\int_\epsilon^T \|M(u)\|_1\, ds &\le \int_\epsilon^T \left\||\nabla u|^{p-2}|\Delta u|\right\|_1\, ds\\ &+ (p-2)\int_\epsilon^T \left\||\nabla u|^{p-4}|(\nabla u)^T\nabla^2 u\nabla u|\right\|_1\, ds\\ &= \int_\epsilon^T \left\||\nabla u|^{p-2}|\Delta u|\right\|_1 ds + (p-2)\int_\epsilon^T \left\||\nabla u|^{p-2}\frac{|(\nabla u)^T\nabla^2 u\nabla u|}{|\nabla u|^2}\right\|_1\, ds\\ &\le C\int_\epsilon^T \|\nabla u\|_{2(p-2)}^{p-2}\cdot\|\Delta u\|_2\, ds\\ &\le C\|u\|^2_{L^2([\epsilon,T];H^2(\Omega))} + C\|u\|^{2(p-2)}_{\bar{E}_S} < \infty\end{aligned}$$

is showed. Next, for $\zeta \in C^\infty([\epsilon,T]\times\Omega)$ by Fubini's theorem, along with the self-adjointness of the operator $e^{-t\mathcal{L}}$ on L^2, we derive:

$$\begin{aligned}&\int_\Omega \zeta(t)u(t)dx\\ &= \int_\Omega e^{-(t-s)\mathcal{L}}\zeta(t)u(\epsilon)dx + \int_\epsilon^t\int_\Omega e^{-(t-\epsilon)\mathcal{L}}(I-a\Delta)^{-1}\nabla\zeta(t).F(u(s))\, dxds. \qquad (3.19)\end{aligned}$$

Since $\zeta(t)$ and $\nabla\zeta(t)$ belong to the set of elements for $\mathcal{L}$ for all $t \in [\epsilon,T]$, and $\mathcal{L}$ creates an analytic semi-group, it follows that:

$$-e^{-(t-\epsilon)\mathcal{L}}\zeta(t) = \lim_{t\to 0}\frac{e^{-(t+h-\epsilon)\mathcal{L}}\zeta(t) - e^{-(t-\epsilon)\mathcal{L}}\zeta(t)}{h},$$

and

$$-e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t) = \lim_{t\to 0}\frac{e^{-(t-\epsilon)\mathcal{L}}\zeta(t+h) - e^{-(t-\epsilon)\mathcal{L}}\zeta(t)}{h}.$$

By Leibnitz's rule:

$$\frac{d}{dt}e^{-(t-\epsilon)\mathcal{L}}\zeta(t) = e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t) - \mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\zeta(t).$$

In particular, combining $e^{-(t-\epsilon)\mathcal{L}}\zeta(t)$ with any function $f(t)$ that is absolutely continuous on

$[\epsilon, T]$, with values in $L^2(\Omega)$ and a time-integrable derivative, yields:

$$\frac{d}{dt}\int_\Omega e^{-(t-\epsilon)\mathcal{L}}\zeta(t)f(t)dx$$
$$= \int_\Omega \left(-\mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\zeta(t)f'(t)\right) dx. \qquad (3.20)$$

Similarly, for ζ replaced by $\nabla\zeta$, we obtain:

$$\frac{d}{dt}\int_\Omega e^{-(t-\epsilon)\mathcal{L}}\nabla\zeta(t)f(t)dx$$
$$= \int_\Omega \left(-\mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\nabla\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\nabla\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\nabla\zeta(t)f'(t)\right) dx(3.21)$$

Recall that:

In a Banach space X that is both separable and reflexive, a function defined on X is absolutely continuous, considering time as the variable, if and only if the function admits a weak derivative almost everywhere and is integrable in the Böchner sense.

By applying equation (3.20) for $f(t) = u(\epsilon)$ and (3.21) for $f(t) = F(\nabla u(s))$, and differentiating (3.19),

$$\frac{d}{dt}\int_\Omega \zeta(t)u(t)dx$$
$$= \int_\Omega \left(e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t) - \mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\zeta(t)\right) u(\epsilon)\, dx$$
$$+ \int_\Omega\int_\epsilon^t \left(e^{-(t-s)\mathcal{L}}\frac{\partial}{\partial t}((\nabla\zeta(t)) - \mathcal{L}e^{-(t-s)\mathcal{L}}(\nabla\zeta(t))\right)(I - a\Delta)^{-1}(|\nabla u(s)|^{p-2}\nabla u(s))\, dtdx$$
$$+ \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(t)(|\nabla u(t)|^{p-2}\nabla u(t))\, dx$$

is obtained. By rearranging terms, the expression above simplifies to:

$$\frac{d}{dt}\int_\Omega \zeta(t)u(t)dx$$
$$= \int_\Omega \frac{\partial}{\partial t}\zeta(t)\left(e^{-(t-\epsilon)\mathcal{L}}u(\epsilon) - \int_\epsilon^t (e^{-(t-s)\mathcal{L}}(I - a\Delta)^{-1}\nabla.(|\nabla u|^{p-2}\nabla u)(s)\, ds\right) dx$$
$$- \int_\Omega \mathcal{L}\zeta(t)\left(e^{-(t-\epsilon)}u(\epsilon) - \int_\epsilon^t e^{-(t-s)\mathcal{L}}(I - a\Delta)^{-1}\nabla.(|\nabla u|^{p-2}\nabla u)(s)\, ds\right) dx$$
$$+ \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(t)(|\nabla u(t)|^{p-2}\nabla u(t))dx.$$

All of the terms on the right-hand side may be formulated as:

$$\int_\Omega \left(\frac{\partial}{\partial t}\zeta(t) - \mathcal{L}\zeta(t)\right) u(t)\,dx + \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(t)(|\nabla u(t)|^{p-2}\nabla u(t))\,dx.$$

By integrating this result over $[\epsilon, t]$ for $t \in [\epsilon, T]$, we arrive at:

$$\begin{aligned}\int_\Omega \zeta(t)u(t)dx - \int_\Omega \zeta(\epsilon)u(\epsilon)dx \\ = \int_\epsilon^t \int_\Omega \frac{\partial}{\partial t}\zeta(s)u(s)dxds - \int_\epsilon^t \int_\Omega \mathcal{L}\zeta(s)u(s)\,dxds \\ + \int_\epsilon^t \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(s).(|\nabla u(s)|^{p-2}\nabla u(s))\,dxds. \qquad (3.22)\end{aligned}$$

We assert that $\frac{\partial}{\partial t}u \in L^2([\epsilon, T]; H^{-2})$. To validate this, it suffices to show that the last term of (3.22) is well-defined for $\zeta \in L^2([\epsilon, T]; H^2)$ with $\frac{\partial}{\partial t}\zeta \in L^2([\epsilon, T]; H^{-2})$. Specifically:

$$\begin{aligned}&\int_\epsilon^t \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(s) \cdot (|\nabla u(s)|^{p-2}\nabla u(s))\,dxds \\ &= \int_\epsilon^t \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(s) \cdot F(\nabla u(s))\,dxds \\ &\leq \int_\epsilon^t \|\nabla\zeta(s)\|_{\frac{1}{5-2p}} \cdot \|F(\nabla u(s))\|_{\frac{1}{p-2}}\,ds \\ &\leq C\left(\int_\epsilon^t (\|\Delta\zeta\|_2^2 + \|\Delta\zeta\|_2^2)\,ds\right)^{\frac{1}{2}} \cdot \|F(\nabla u)\|_{L^2([\epsilon,T];L^{\frac{1}{p-2}}(\Omega))} \\ &\leq C\|\zeta\|_{L^2([\epsilon,T];H^2)}\|F(\nabla u)\|_{L^2([\epsilon,T];L^{\frac{1}{p-2}}(\Omega))} \\ &< \infty,\end{aligned}$$

in which the final terms proceed from Statement 1 in addition to the Gagliardo-Nirenberg inequality:

Let $u \in L^2([\epsilon, T], H^2) \cap H^1([\epsilon, T], H^{-2})$. Consequently, there exists a sequence

$$\{u_m\} \subseteq C^1([\epsilon, T]; H^2)$$

satisfying the following conditions:

- $u^m \to u$ as strongly in $C([\epsilon, T], L^2) \cap L^2([\epsilon, T], H^{-2})$ as $m \to \infty$, and

- $\frac{\partial}{\partial t}u^m \to \frac{\partial}{\partial t}u$ weakly in $L^2([\epsilon,T],H^{-2})$ as $m\to\infty$.

These properties imply that

$$\left|\int_\epsilon^t\int_\Omega(\nabla u^m-\nabla u)F(u(s))\,dxds\right|$$

$$\leq \|\nabla u^m-\nabla u\|_{L^2\left((\epsilon,T),L^{\frac{1}{5-2p}}\right)}\|F(\nabla u)\|_{L^2\left([\epsilon,T];L^{\frac{1}{p-2}}\right)}$$

$$\leq C\|\,u^m-u\|_{L^2([\epsilon,T];H^2)}\|F(\nabla u)\|_{L^2\left([\epsilon,T];L^{\frac{1}{p-2}}\right)},$$

which converges to 0 as $m\to\infty$. Using u^m as a test function (3.22), and taking the limit, $m\to\infty$, the following is obtained:

$$\int_\Omega|u(t)|^2dx-\int_\Omega|u(\epsilon)|^2dx=\int_\epsilon^t\int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds-\int_\epsilon^t\int_\Omega\mathcal{L}u^2(s)\,dxds$$
$$+\int_\epsilon^t\int_\Omega(I-a\Delta)^{-1}\nabla u(s).(|\nabla u(s)|^{p-2}\nabla u(s))\,dxds.$$

This can take the form of

$$\|u(t)\|_2^2-\|u(\epsilon)\|_2^2$$
$$\leq\int_\epsilon^t\int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds-\int_\epsilon^t\int_\Omega\mathcal{L}u^2(s)\,dxds+\int_\epsilon^t\|\nabla u(s)\|_p^pds. \tag{3.23}$$

By rearranging and considering the inequality $||a|-|b||\leq|a-b|$, the terms in the last equality (3.23),

$$\|u(t)\|_2^2-\|u(\epsilon)\|_2^2$$
$$\leq\int_\epsilon^t\int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds-\int_\epsilon^t\|\Delta u(s)\|_2^2\,ds+C\int_\epsilon^t\|\nabla u(s)\|_2^2\,ds+\int_\epsilon^t\|\nabla u(s)\|_p^pds.$$

Since, $u\in L^2([\epsilon,T];H^2)$ and $\frac{\partial}{\partial t}u\in L^2([\epsilon,T];H^{-2})$, the map $t\to\|u(t)\|_2^2$ is absolutely continuous. By applying the Fundamental Theorem of Calculus,

$$\int_\epsilon^t\int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds=\frac{1}{2}\left(\|u(t)\|_2^2-\|u(\epsilon)\|_2^2\right).$$

By substituting this result in the previous equation, the energy inequality becomes

$$\begin{aligned}&\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2 \\ &\quad\le \frac{1}{2}\left(\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2\right) - \int_\epsilon^t \|\Delta u(s)\|_2^2\, ds + C\int_\epsilon^t \|\nabla u(s)\|_2^2\, ds + \int_\epsilon^t \|\nabla u\|_p^p ds,\end{aligned}$$

and

$$\frac{1}{2}\left(\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2\right) + \int_\epsilon^t \|\Delta u(s)\|_2^2\, ds \le \int_\epsilon^t \|\nabla u(s)\|_p^p ds + C\int_\epsilon^t \|\nabla u(s)\|_2^2\, ds.$$

By multiplying by 2, we derive the following inequality:

$$\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2 + 2\int_\epsilon^t \|\Delta u(s)\|_2^2\, ds \le 2\int_\epsilon^t \|\nabla u(s)\|_p^p ds + C\int_\epsilon^t \|\nabla u(s)\|_2^2\, ds. \quad (3.24)$$

We now demonstrate that letting $\epsilon \to 0$ in this inequality yields an energy inequality valid over the interval $[0, T]$. Using the Gagliardo-Nirenberg inequality, we have the bound

$$\int_\epsilon^t \|\nabla u(s)\|_p^p\, ds \le \int_\epsilon^t \|\Delta u(s)\|_2^2\, ds + C\int_\epsilon^t \left(\|u(s)\|_2^{\frac{2}{5-2p}} + \|u(s)\|_2^p\right) ds.$$

Substituting this into (3.24) and applying the definition of $\bar{E_S}$, we obtain

$$\int_\epsilon^t \|\Delta u(s)\|_2^2 \le \|u(\epsilon)\|_2^2 - \|u(t)\|_2^2 + \int_\epsilon^t \|\nabla u(s)\|^2\, ds + C\int_\epsilon^T \left(\|u(s)\|_2^{\frac{2}{5-2p}} + \|u(s)\|_2^p\right) ds$$

Simplifying further,

$$\int_\epsilon^t \|\Delta u(s)\|_2^2 \le C\left(\|u\|_{\bar{E_S}}^p + L\|u\|_{\bar{E_S}}^{\frac{2}{5-2p}} + L\|u\|_{\bar{E_S}}^p + \|u\|_{\bar{E_S}}^2\right),$$

where the right-hand side is independent of ϵ. This result implies that $L^2([0,T); H^2)$.

Through the Dominated Convergence theorem, assuming $\epsilon \to 0$ in (3.24) gives the energy inequality obtained as follows:

$$\|u(t)\|_2^2 + 2\int_0^t \|\Delta u(s)\|_2^2\, ds \le \|u_0\|_2^2 + 2\int_0^t \|\nabla u(s)\|_p^p ds + C\int_0^t \|\nabla u(s)\|_2^2\, ds.$$

Moreover, the continuity of the function $\|u(t)\|_2$ at $t = 0$ can be verified directly. As

$u \in L^2([0,T]; H^2)$, the right-hand side of (3.24) remains uniformly bounded as $\epsilon \to 0$.

Finally, substituting $\epsilon \to 0$ into the weak formulation and using a test function $\zeta \in C_c^\infty([0,T) \times \Omega)$, we get

$$\begin{aligned}\int_\Omega u_0 \zeta(0) dx + \int_\epsilon^t \int_\Omega u \frac{\partial}{\partial t} \zeta dx dt \\ = \int_\epsilon^t \int_\Omega \mathcal{L} \zeta u \, dx dt \\ - \int_\epsilon^t \int_\Omega (I - a\Delta)^{-1} |\nabla u(s)|^{p-2} \nabla u \cdot \nabla \zeta \, dx dt. \end{aligned} \tag{3.25}$$

Thus, the mild solution satisfies the weak formulation over the interval $[0,T]$. □

Performing similar steps for cases $2 < p < 3$ and $0 \leq a < 1$, the same energy inequality can be obtained as follows:

Proposition 2. *If u is a mild solution on the interval $[0,T]$, then it is a weak solution for the initial value problem* (1.1)-(1.2) *over the same interval $[0,T]$. Moreover, the following energy inequality is satisfied:*

For any $t \in [0,T]$, $2 < p < 3$, and $0 \leq a < 1$,

$$\|u(t)\|_2^2 + 2\int_0^t \|\Delta u(s)\|_2^2 \, ds \leq \|u_0\|_2^2 + 2\int_0^t \|\nabla u(s)\|_p^p ds + C\int_0^t \|\nabla u(s)\|_2^2 \, ds. \tag{3.26}$$

Proof. We first rewrite the mild solution of the problem (1.1) for $0 < \epsilon \leq t \leq T$ as follows:

$$v(u)(t) = u(t) = e^{-t\mathcal{L}} u_0 - \int_0^t e^{-(t-s)\mathcal{L}} (I - a\Delta)^{-1} (\nabla \cdot (|\nabla u|^{p-2} \nabla u)) \, ds.$$

Reformulating this solution at initial point $t = \epsilon$ we get

$$v(u)(t) = u(t) = e^{-(t-\epsilon)\mathcal{L}} u(\epsilon) - \int_0^t e^{-(t-s)\mathcal{L}} (I - a\Delta)^{-1} (\nabla \cdot (|\nabla u|^{p-2} \nabla u)) ds \tag{3.27}$$

which holds under the conditions

$$u \in C([\epsilon, T]; L^2(\Omega)),$$

with

$$u \in L^2([\epsilon, T]; H^2(\Omega)).$$

For $u(x, .) : [\epsilon, T] \to H^2(\Omega)$, we denote

$$\int_{\epsilon}^{T} \|u(t)\|_{H^2(\Omega)}^2 dt = \|u\|_{L^2([\epsilon,T];H^2(\Omega))}.$$

Now we will show that that $\Delta u \in L^2([\epsilon, T] \times \Omega)$.

From the Main theorem,

$$u \in L^2([\epsilon, T]; H^2(\Omega))$$

and

$$u_x \in L^2([\epsilon, T]; H^1(\Omega))$$

with

$$u_{xx} \in L^2([\epsilon, T]; H^0(\Omega)) = L^2([\epsilon, T]; L^2(\Omega)).$$

As a result, it follows that

$$\Delta u \in L^2([\epsilon, T]; L^2(\Omega)).$$

Consequently, by applying the Laplacian operator the Δ of both sides of (3.27), using the decay properties of the semigroup $e^{-t\mathcal{L}}$, and noting that $u \in \bar{E}_S$ on the interval $[0, T]$, we achieve:

$$\Delta v(u)(t) = \Delta u(t) = e^{-(t-\epsilon)\mathcal{L}} \Delta u(\epsilon) - \int_0^t e^{-(t-s)\mathcal{L}} (I - a\Delta)^{-1} \Delta R(s)\, ds \tag{3.28}$$

where $R(s) = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ and $F(\nabla u(s)) = (|\nabla u|^{p-2} \nabla u)$.

Employing the triangle inequality and the bound $\|(I-a\Delta)^{-1}\| \leq 1$,

$$\begin{aligned}
\|\Delta u(t)\|_2 &\leq C\left(\|e^{-(t-\epsilon)\mathcal{L}}\nabla\nabla u(\epsilon)\|_2 + \int_\epsilon^t \|e^{-(t-s)\mathcal{L}}(I-a\Delta)^{-1}\Delta R(s)\|_2\,ds\right)\\
&\leq C\left((t-\epsilon)^{-\frac{1}{4}}\|\nabla u(\epsilon)\|_2 + \int_\epsilon^t \|e^{-(t-s)\mathcal{L}}\Delta R(s)\|_2\,ds\right)\\
&\leq C\left((t-\epsilon)^{-\frac{1}{4}}\|u(\epsilon)\|_{\dot{H}^1} + \int_\epsilon^t \|e^{-(t-s)\mathcal{L}}\Delta(\nabla.(|\nabla u|^{p-2}\nabla u))\|_2\,ds\right)\\
&\leq C\left((t-\epsilon)^{-\frac{1}{4}}\|u(\epsilon)\|_{\dot{H}^1} + \int_\epsilon^t (t-s)^{-\frac{3}{4}}(t-s)^{-\frac{p-2}{2}}\|\nabla u\|_{L^2}^{p-1}\,ds\right)\\
&\leq C(t-\epsilon)^{-\frac{1}{4}}\epsilon^{\frac{1}{4}}\|u(\epsilon)\|_{\dot{H}^1}\epsilon^{-\frac{1}{4}} + C\int_\epsilon^t (t-s)^{\frac{1-2p}{4}}\|\nabla u\|_{L^2}^{p-1}\,ds\\
&\leq C\left((t-\epsilon)^{-\frac{1}{4}}\epsilon^{-\frac{1}{4}}\|u\|_{\bar{E_S}} + Ct^{\frac{6-3p}{4}}\|u\|_{\bar{E_S}}^{p-1}\right).
\end{aligned}$$

is obtained. From the above inequality, the right-hand side belongs to L^2 for $2 < p < 3$ and $\epsilon > 0$.

The proofs of the claims are provided:

Statement 1. $F(\nabla u) \in L^2([\epsilon, T]; L^{\frac{1}{p-2}}(\Omega))$.

Statement 2. $M(u) \in L^1([\epsilon, T]; \Omega)$.

The first statement follows directly from the Gagliardo-Nirenberg inequality, for $\frac{1}{2} < \frac{1}{p-1} < 1$.

$$\begin{aligned}
\int_\epsilon^T \|F(\nabla u)\|_{\frac{1}{p-2}}^2\,ds &= \int_\epsilon^T \|\nabla u\|_{\frac{1}{p-2}}^{2(p-1)}ds\\
&\leq C\int_\epsilon^T \left(\|\Delta u\|_2^2\|u\|_2^{2(p-2)} + \|u\|_2^{p-1}\right)ds\\
&\leq C\int_\epsilon^T \left(\|u\|_{C([\epsilon,T];L^2(\Omega))}^{2(p-2)}\|u\|_{L^2([\epsilon,T];H^2(\Omega))}^2 + L\|u\|_{C([\epsilon,T];L^2(\Omega))}^{p-1}\right)ds.
\end{aligned}$$

To establish the second statement, first compute the following:

$$|M(u)| = |\nabla\cdot(|\nabla u|^{p-2}\nabla u) \leq |\nabla u|^{p-2}|\Delta u| + (p-2)|\nabla u|^{p-4}|(\nabla u)^T\nabla^2 u\nabla u|,$$

where u^T stands for a vector transpose of $u \in \mathbb{R}^2$. Namely,

$$\begin{aligned}\int_\epsilon^T \|M(u)\|_1\, ds &\le \int_\epsilon^T \left\||\nabla u|^{p-2}|\Delta u|\right\|_1\, ds \\ &\quad + (p-2)\int_\epsilon^T \left\||\nabla u|^{p-4}|(\nabla u)^T\nabla^2 u\nabla u|\right\|_1\, ds \\ &= \int_\epsilon^T \left\||\nabla u|^{p-2}|\Delta u|\right\|_1 ds + (p-2)\int_\epsilon^T \left\||\nabla u|^{p-2}\frac{|(\nabla u)^T\nabla^2 u\nabla u|}{|\nabla u|^2}\right\|_1\, ds \\ &\le C\int_\epsilon^T \|\nabla u\|_{2(p-2)}^{p-2}\cdot\|\Delta u\|_2\, ds \\ &\le C\|u\|^2_{L^2([\epsilon,T];H^2(\Omega))} + C\|u\|^{2(p-2)}_{\bar{E}_S} < \infty\end{aligned}$$

is showed. Next, for $\zeta \in C^\infty([\epsilon,T]\times\Omega)$ by Fubini's theorem, along with the self-adjointness of the operator $e^{-t\mathcal{L}}$ on L^2, we derive:

$$\begin{aligned}&\int_\Omega \zeta(t)u(t)dx \\ &= \int_\Omega e^{-(t-s)\mathcal{L}}\zeta(t)u(\epsilon)dx + \int_\epsilon^t\int_\Omega e^{-(t-\epsilon)\mathcal{L}}(I-a\Delta)^{-1}\nabla\zeta(t).F(u(s))\,dxds.\end{aligned} \tag{3.29}$$

Since $\zeta(t)$ and $\nabla\zeta(t)$ belong to the set of elements for $\mathcal{L}$ for all $t\in[\epsilon,T]$, and $\mathcal{L}$ creates an analytic semi-group, it follows that:

$$-e^{-(t-\epsilon)\mathcal{L}}\zeta(t) = \lim_{t\to 0}\frac{e^{-(t+h-\epsilon)\mathcal{L}}\zeta(t) - e^{-(t-\epsilon)\mathcal{L}}\zeta(t)}{h},$$

and

$$-e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t) = \lim_{t\to 0}\frac{e^{-(t-\epsilon)\mathcal{L}}\zeta(t+h) - e^{-(t-\epsilon)\mathcal{L}}\zeta(t)}{h}.$$

By Leibnitz's rule:

$$\frac{d}{dt}e^{-(t-\epsilon)\mathcal{L}}\zeta(t) = e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t) - \mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\zeta(t).$$

In particular, combining $e^{-(t-\epsilon)\mathcal{L}}\zeta(t)$ with any function $f(t)$ that is absolutely continuous on $[\epsilon,T]$, with values in $L^2(\Omega)$ and a time-integrable derivative, yields: In particular, combining $e^{-(t-\epsilon)\mathcal{L}}\zeta(t)$ with any function $f(t)$ that is absolutely continuous on $[\epsilon,T]$, with values in

$L^2(\Omega)$ and a time-integrable derivative, yields:

$$\frac{d}{dt}\int_\Omega e^{-(t-\epsilon)\mathcal{L}}\zeta(t)f(t)dx$$
$$= \int_\Omega \left(-\mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\zeta(t)f'(t)\right) dx. \tag{3.30}$$

Similarly, for ζ replaced by $\nabla\zeta$, we obtain:

$$\frac{d}{dt}\int_\Omega e^{-(t-\epsilon)\mathcal{L}}\nabla\zeta(t)f(t)dx$$
$$= \int_\Omega \left(-\mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\nabla\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\nabla\zeta(t)f(t) + e^{-(t-\epsilon)\mathcal{L}}\nabla\zeta(t)f'(t)\right) dx \tag{3.31}$$

Recall that:

In a Banach space X that is both separable and reflexive, a function defined on X is absolutely continuous, considering time as the variable, if and only if the function admits a weak derivative almost everywhere and is integrable in the Böchner sense.

By applying equation (3.30) for $f(t) = u(\epsilon)$ and (3.31) for $f(t) = F(\nabla u(s))$, and differentiating (3.29), we arrive the following equation:

$$\frac{d}{dt}\int_\Omega \zeta(t)u(t)dx$$
$$= \int_\Omega \left(e^{-(t-\epsilon)\mathcal{L}}\frac{\partial}{\partial t}\zeta(t) - \mathcal{L}e^{-(t-\epsilon)\mathcal{L}}\zeta(t)\right) u(\epsilon)\, dx$$
$$+ \int_\Omega\int_\epsilon^t \left(e^{-(t-s)\mathcal{L}}\frac{\partial}{\partial t}((\nabla\zeta(t)) - \mathcal{L}e^{-(t-s)\mathcal{L}}(\nabla\zeta(t))\right)(I - a\Delta)^{-1}(|\nabla u(s)|^{p-2}\nabla u(s))\, dtdx$$
$$+ \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(t)(|\nabla u(t)|^{p-2}\nabla u(t))\, dx.$$

By rearranging terms, the expression above simplies to:

$$\frac{d}{dt}\int_\Omega \zeta(t)u(t)dx$$
$$= \int_\Omega \frac{\partial}{\partial t}\zeta(t)\left(e^{-(t-\epsilon)\mathcal{L}}u(\epsilon) - \int_\epsilon^t (e^{-(t-s)\mathcal{L}}(I - a\Delta)^{-1}\nabla.(|\nabla u|^{p-2}\nabla u)(s)\, ds\right) dx$$
$$- \int_\Omega \mathcal{L}\zeta(t)\left(e^{-(t-\epsilon)}u(\epsilon) - \int_\epsilon^t e^{-(t-s)\mathcal{L}}(I - a\Delta)^{-1}\nabla.(|\nabla u|^{p-2}\nabla u)(s)\, ds\right) dx$$
$$+ \int_\Omega (I - a\Delta)^{-1}\nabla\zeta(t)(|\nabla u(t)|^{p-2}\nabla u(t))dx.$$

All of the terms on the right-hand side may be formulated as:

$$\int_{\Omega}\left(\frac{\partial}{\partial t}\zeta(t) - \mathcal{L}\zeta(t)\right)u(t)\,dx + \int_{\Omega}(I - a\Delta)^{-1}\nabla\zeta(t)(|\nabla u(t)|^{p-2}\nabla u(t))\,dx.$$

By integrating this result over $[\epsilon, t]$ for $t \in [\epsilon, T]$. We arrive at:

$$\begin{aligned}
&\int_{\Omega}\zeta(t)u(t)dx - \int_{\Omega}\zeta(\epsilon)u(\epsilon)dx \\
&\qquad = \int_{\epsilon}^{t}\int_{\Omega}\frac{\partial}{\partial t}\zeta(s)u(s)dxds - \int_{\epsilon}^{t}\int_{\Omega}\mathcal{L}\zeta(s)u(s)\,dxds \\
&\qquad + \int_{\epsilon}^{t}\int_{\Omega}(I - a\Delta)^{-1}\nabla\zeta(s).(|\nabla u(s)|^{p-2}\nabla u(s))\,dxds.
\end{aligned} \tag{3.32}$$

We assert that $\frac{\partial}{\partial t}u \in L^2([\epsilon, T]; H^{-2})$. To validate this, it suffices to show that the last term of (3.32) is well-defined for $\zeta \in L^2([\epsilon, T]; H^2)$ with $\frac{\partial}{\partial t}\zeta \in L^2([\epsilon, T]; H^{-2})$. Specifically:

$$\begin{aligned}
&\int_{\epsilon}^{t}\int_{\Omega}(I - a\Delta)^{-1}\nabla\zeta(s)\cdot(|\nabla u(s)|^{p-2}\nabla u(s))\,dxds \\
&= \int_{\epsilon}^{t}\int_{\Omega}(I - a\Delta)^{-1}\nabla\zeta(s)\cdot F(\nabla u(s))\,dxds \\
&\leq \int_{\epsilon}^{t}\|\nabla\zeta(s)\|_{\frac{1}{3-p}}\cdot\|F(\nabla u(s))\|_{\frac{1}{p-2}}\,ds \\
&\leq C\left(\int_{\epsilon}^{t}(\|\Delta\zeta\|_2^2 + \|\Delta\zeta\|_2^2)\,ds\right)^{\frac{1}{2}}\cdot\|F(\nabla u)\|_{L^2([\epsilon,T];L^{\frac{1}{p-2}}(\Omega))} \\
&\leq C\|\zeta\|_{L^2([\epsilon,T];H^2)}\|F(\nabla u)\|_{L^2([\epsilon,T];L^{\frac{1}{p-2}}(\Omega))} \\
&< \infty,
\end{aligned}$$

in which the final terms proceed from Statement 1 in addition to the Gagliardo-Nirenberg inequality:

Let $u \in L^2([\epsilon, T], H^2) \cap H^1([\epsilon, T], H^{-2})$. Consequently, there exists a sequence

$$u_m \subseteq C^1([\epsilon, T]; H^2)$$

satisfying the following conditions:

- $u^m \to u$ as strongly in $C([\epsilon, T], L^2) \cap L^2([\epsilon, T], H^{-2})$ as $m \to \infty$, and

- $\frac{\partial}{\partial t}u^m \to \frac{\partial}{\partial t}u$ weakly in $L^2([\epsilon,T],H^{-2})$ as $m \to \infty$.

These properties imply that

$$\left|\int_\epsilon^t \int_\Omega (\nabla u^m - \nabla u)F(u(s))\,dxds\right|$$
$$\leq \|\nabla u^m - \nabla u\|_{L^2\left((\epsilon,T),L^{\frac{1}{3-p}}\right)} \|F(\nabla u)\|_{L^2\left([\epsilon,T];L^{\frac{1}{p-2}}\right)}$$
$$\leq C\|\, u^m - u\|_{L^2([\epsilon,T];H^2)} \|F(\nabla u)\|_{L^2\left([\epsilon,T];L^{\frac{1}{p-2}}\right)},$$

which converges to 0 as $m \to \infty$. Using u^m as a test function (3.32), and taking the limit $m \to \infty$, the following is obtained:

$$\int_\Omega |u(t)|^2 dx - \int_\Omega |u(\epsilon)|^2 dx = \int_\epsilon^t \int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds - \int_\epsilon^t \int_\Omega \mathcal{L}u^2(s)\,dxds$$
$$+ \int_\epsilon^t \int_\Omega (I - a\Delta)^{-1}\nabla u(s).(|\nabla u(s)|^{p-2}\nabla u(s))\,dxds$$

This can take the form of

$$\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2$$
$$\leq \int_\epsilon^t \int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds - \int_\epsilon^t \int_\Omega \mathcal{L}u^2(s)\,dxds + \int_\epsilon^t \|\nabla u\|_p^p ds. \tag{3.33}$$

By rearranging and considering the inequality $||a| - |b|| \leq |a - b|$, the terms in the last equality (3.33),

$$\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2$$
$$\leq \int_\epsilon^t \int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds - \int_\epsilon^t \|\Delta u(s)\|_2^2\,ds + C\int_\epsilon^t \|\nabla u(s)\|_2^2\,ds + \int_\epsilon^t \|\nabla u(s)\|_p^p ds.$$

Since, $u \in L^2([\epsilon,T];H^2)$ and $\frac{\partial}{\partial t}u \in L^2([\epsilon,T];H^{-2})$, the map $t \to \|u(t)\|_2^2$ is absolutely continuous. By applying the Fundamental Theorem of Calculus,

$$\int_\epsilon^t \int_\Omega u(s)\frac{\partial}{\partial t}u(s)\,dxds = \frac{1}{2}\left(\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2\right).$$

By substituting this result in the previous equation, the energy inequality becomes

$$\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2 \\ \leq \frac{1}{2}\left(\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2\right) - \int_\epsilon^t \|\Delta u(s)\|_2^2\, ds + C\int_\epsilon^t \|\nabla u(s)\|_2^2\, ds + \int_\epsilon^t \|\nabla u(s)\|_p^p ds,$$

and

$$\frac{1}{2}\left(\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2\right) + \int_\epsilon^t \|\Delta u(s)\|_2^2\, ds \leq \int_\epsilon^t \|\nabla u(s)\|_p^p ds + C\int_\epsilon^t \|\nabla u(s)\|_2^2\, ds.$$

By multiplying by 2, we derive the inequality

$$\|u(t)\|_2^2 - \|u(\epsilon)\|_2^2 + 2\int_\epsilon^t \|\Delta u(s)\|_2^2\, ds \leq 2\int_\epsilon^t \|\nabla u(s)\|_p^p ds + C\int_\epsilon^t \|\nabla u(s)\|_2^2\, ds. \quad (3.34)$$

We now demonstrate that letting $\epsilon \to 0$ this inequality yields an energy inequality valid over the interval $[0, T]$. Using the Gagliardo-Nirenberg inequality, we have the bound

$$\int_\epsilon^T \|\nabla u(s)\|_p^p\, ds \leq \int_\epsilon^T \|\Delta u(s)\|_2^2\, ds + C\int_\epsilon^T \left(\|u(s)\|_2^{\frac{2(p-1)}{3-p}} + \|u(s)\|_2^p\right) ds.$$

Substituting this into (3.34) and applying de definition of $\bar{E_S}$, we obtain

$$\int_\epsilon^t \|\Delta u(s)\|_2^2 \leq \|u(\epsilon)\|_2^2 - \|u(t)\|_2^2 + \int_\epsilon^t \|\nabla u(s)\|^2\, ds + C\int_\epsilon^T \left(\|u(s)\|_2^{\frac{2(p-1)}{3-p}} + \|u(s)\|_2^p\right) ds$$

Simplifying further,

$$\int_\epsilon^t \|\Delta u(s)\|_2^2 \leq C\left(\|u\|_{\bar{E_S}}^p + L\|u\|_{\bar{E_S}}^{\frac{2(p-1)}{3-p}} + L\|u\|_{\bar{E_S}}^p + \|u\|_{\bar{E_S}}^2\right),$$

where the right-hand side is independent of ϵ. This result implies that $L^2([0, T); H^2)$.

Through the Dominated Convergence theorem, assuming $\epsilon \to 0$ in (3.34) gives the energy inequality is obtained as follows:

$$\|u(t)\|_2^2 + 2\int_0^t \|\Delta u(s)\|_2^2\, ds \leq \|u_0\|_2^2 + 2\int_0^t \|\nabla u(s)\|_p^p ds + C\int_0^t \|\nabla u(s)\|_2^2\, ds.$$

Moreover, the continuity of the function $\|u(t)\|_2$ at $t = 0$ can be verified directly. As

$u \in L^2([0,T);H^2)$, the right-hand side of (3.34) remains uniformly bounded as $\epsilon \to 0$. Additionally, for $\zeta \in C_c^\infty([0,T);H^2)$, the sequence $\left(\frac{\partial u}{\partial t}\right)_{X[\epsilon,T]}$ converges weakly to $\frac{\partial u}{\partial t}$ in $L^2([0,T);H^{-2})$.

Finally, substituting $\epsilon \to 0$ into the weak formulation and using a test function $\zeta \in C_c^\infty([0,T) \times \Omega)$, we get

$$\begin{aligned}\int_\Omega u_0\zeta(0)dx + \int_\epsilon^t \int_\Omega u\frac{\partial}{\partial t}\zeta dxdt \\ = \int_\epsilon^t \int_\Omega \mathcal{L}\zeta u\, dxdt \\ - \int_\epsilon^t \int_\Omega (I - a\Delta)^{-1}|\nabla u(s)|^{p-2}\nabla u \cdot \nabla\zeta\, dxdt. \end{aligned} \tag{3.35}$$

Thus, the mild solution satisfies the weak formulation over the interval $[0,T]$. □

4. BLOW-UP SOLUTIONS

In this chapter, the blow-up result for the solutions of (1.1)-(1.2) is presented. To achieve this goal, the following functionals are defined:

$$\begin{aligned}\Phi &= \|u\|_2^2 + a\|\nabla u\|_2^2,\\ \Psi &= \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2}\|\,\Delta u\|_2^2.\end{aligned}$$

The main theorem of this study is given as follows:

Theorem 3. *Assume that* $p > 2$, $u_0 \in H^2(\Omega)$, $u_0 \neq 0$, *and*

$$\Psi(0) = \frac{1}{p}\|\nabla u_0\|_p^p - \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{2}\|\,\Delta u_0\|_2^2.$$

In that case, for $u(x,t)$ *of (1.1)-(1.2), some* $T^* > 0$ *exists satisfying*

$$\lim_{t\to T^*}\Phi(t) = \infty$$

in which

$$T^* = \frac{\Phi^{\frac{2-p}{2}}(0)}{p(p-2)}.$$

Theorem 3 was formulated using the notions of [25].

Proof. Considering (1.1)-(1.2),

$$\begin{aligned}\Phi'(t) &= 2\left[\|\nabla u\|_p^p - \|\nabla u\|_2^2 - \|\,\Delta u\|_2^2\right] \qquad (4.1)\\ \Psi'(t) &= \|u_t\|_2^2 + a\|\nabla u_t\|_2^2.\end{aligned}$$

is obtained. By the Schwarz inequality, one has

$$\Phi(t)\Psi'(t) = \left(\|u\|_2^2 + a\|\nabla u\|_2^2)(\|u_t\|_2^2 + a\|\nabla u_t\|_2^2\right) \geq \frac{1}{4}(\Phi'(t))^2. \qquad (4.2)$$

Combining (4.1) and (4.2), one gets

$$\Phi(t)\Psi'(t) \geq \frac{1}{4}\Phi'(t)\Phi'(t) \geq \frac{p}{2}\Phi'(t)\Psi(t), \tag{4.3}$$

which turns into

$$\Phi(t)\Psi'(t) - \frac{p}{2}\Phi'(t)\Psi(t) \geq 0.$$

Now consider

$$\left(\Psi(t)\Phi(t)^{-\frac{p}{2}}\right)' = \Phi(t)^{\frac{-p-2}{2}}\left(\Psi'(t)\Phi(t) - \frac{p}{2}\Phi'(t)\Psi(t)\right) \geq 0. \tag{4.4}$$

From this one gets

$$\Psi(t)\Phi(t)^{-\frac{p}{2}} \geq \Psi(0)\Phi(0)^{-\frac{p}{2}} := M, \tag{4.5}$$

and

$$\Psi(t) \geq M\Phi(t)^{\frac{p}{2}}.$$

Using (4.1), the following is obtained:

$$\frac{\Phi'(t)\Phi(t)^{-\frac{p}{2}}}{2p} \geq M. \tag{4.6}$$

This is equivalent to

$$\frac{\left(\Phi(t)^{\frac{2-p}{2}}\right)'}{(2-p)p} \geq M,$$

and

$$\frac{1}{(2-p)p}\left((\Phi(t))^{\frac{2-p}{2}} - (\Phi(0))^{\frac{2-p}{2}}\right) \geq Mt.$$

Thus, one gets

$$(\Phi(t))^{\frac{2-p}{2}} \geq \left(\Phi^{\frac{2-p}{2}}(0) - p(p-2)t\right). \tag{4.7}$$

Taking the roots of both sides of (4.7), one gets the following:

$$\Phi(t) \geq \left(\Phi^{\frac{2-p}{2}}(0) - p(p-2)t\right)^{\frac{2}{2-p}}.$$

This expression gives the upper bound of the time interval, i.e, $t \leq T^* = \frac{\Phi^{\frac{2-p}{2}}(0)}{p(p-2)}$ for the solution blow-up. □

Now, we give the next theorem to specify a finite time on $(0, T_*)$. Moreover, the amount $\|\nabla u\|_2^2 + a\|\ \Delta u\|_2^2$ stays bounded which is based on [22,25]. Furthermore, T_* is a lower bound for t. In accordance with the Poincaré inequality, one has

$$\|u\|_2^2 + a\|\nabla u\|_2^2 \le \lambda_1(a\|\Delta u\|_2^2 + \|\ \nabla u\|_2^2),\ t \in (0, T_*),$$

where λ_1 corresponds to the first eigenvalue $-\Delta u = \lambda u$ in the context of periodic boundary conditions.

Theorem 4. *Let $u(x,t)$ represent the solution to the problem formulated by (1.1)-(1.2). Then, a number $T_* = \dfrac{\beta^{2-p}(0)}{(p-2)C^{2p-2}(\Omega)} > 0$ exists satisfying*

$$\beta(t) = \int_\Omega (|\nabla u|^2 + a|\Delta u|^2)\, dx \tag{4.8}$$

remains bounded in $(0, T_*)$.

Proof. By using Green's identity and employing (1.1)-(1.2)

$$\begin{aligned} \beta'(t) &= 2\int_\Omega \Delta u(-u_t + a\Delta u_t)\, dx \\ &= 2\int_\Omega \Delta u(-\Delta u + (-\Delta)^2 u + \nabla \cdot (|\nabla u|^{p-2}\nabla u))\, dx, \\ &= -2\|\Delta u\|^2 - 2\|\nabla\Delta u\|^2 - 2\int_\Omega \nabla\Delta u \cdot (|\nabla u|^{p-2}\nabla u))\, dx. \end{aligned} \tag{4.9}$$

By the inequality $|2|\delta u||u|^p| \le |\delta u|^2 + |u|^{2p}$ the last term above gives

$$|-2\int_\Omega |\nabla\Delta u||\nabla u|^{p-1} dx| \le \|\nabla\Delta u\|_2^2 + \|\nabla u\|_{2p-2}^{2p-2}.$$

Substituting this into (4.9) $\beta'(t) \le \|\nabla u\|_{2p-2}^{2p-2}$. Now, by using the Sobolev inequality

$$\|\nabla u\|_{2p-2}^{2p-2} \le C^{2p-2}(\Omega)(\|\ \nabla u\|_2^2 + a\|\Delta u\|_2^2)^{p-1}. \tag{4.10}$$

So, one gets

$$\beta'(t) \le C^{2p-2}(\Omega)(\|\ \nabla u\|^2 + a\|\Delta u\|^2)^{p-1} = C^{2p-2}(\Omega)\beta^{p-1}(t).$$

If this inequality is solved, the following is obtained:

$$\beta^{2-p}(t) \geq \beta^{2-p}(0) - (p-2)C^{2p-2}(\Omega)t. \tag{4.11}$$

Taking the root of both sides of (4.11), one finds

$$\beta(t) \geq \left(\beta^{2-p}(0) - (p-2)C^{2p-2}(\Omega)t\right)^{\frac{1}{2-p}}.$$

Therefore, $T_* = \dfrac{\beta^{2-p}(0)}{(p-2)C^{2p-2}(\Omega)}$ as obtained. □

5. CONCLUSIONS

Higher-order pseudo-parabolic equations are derived from parabolic equations by incorporating the term $-a\Delta u_t$. These equations are essential for modeling complex physical phenomena where conventional parabolic or hyperbolic equations fall short. Pseudo-parabolic equations are valuable in numerous scientific and engineering applications due to their capacity to account for memory effects, non-local behavior, and to maintain the smoothness of solutions (see [39,40,41,42,43]).

By setting $a = 0$ in the term $-a\Delta u_t$, these equations generalize the thin-film equation, which is relevant in various scientific fields such as biology and physics. This encompasses the spreading of low-amplitude long waves, heat conduction, the infiltration of homogeneous fluids through cracked rock, the unidirectional travel of nonlinear dispersive long wavesand many other phenomena.

In this research, it was observed that two different solution spaces were obtained, and these spaces are defined as follows:

For $a \geq 1$, the solution space is:

$$E_S = \{u : \mathbb{R}_+ \times \Omega \to \mathbb{R} | \sup_{0 \leq t \leq T} t^{\frac{1}{2}} \|\nabla u\|_2 < \infty\},$$

and for $0 \leq a < 1$, the solution space is:

$$E_S = \{u : \mathbb{R}_+ \times \Omega \to \mathbb{R} | \sup_{0 \leq t \leq T} t^{\frac{1}{4}} \|\nabla u\|_2 < \infty\}.$$

These spaces are defined, respectively.

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