

DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

**COVER-THIN MODULES OVER PATH ALGEBRAS
OF QUIVERS**

by
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July, 2024

İZMİR

COVER-THIN MODULES OVER PATH ALGEBRAS OF QUIVERS

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Graduate School of Natural And Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of
Science in Mathematics**

**by
İrem YILDIZ**

July, 2024

İZMİR

M.Sc. THESIS EXAMINATION RESULT FORM

We have read the thesis entitled “**COVER-THIN MODULES OVER PATH ALGEBRAS OF QUIVERS**” completed by **İREM YILDIZ** under supervision of **ASSOC. PROF. DR. ENGIN MERMUT** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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COVER-THIN MODULES OVER PATH ALGEBRAS OF QUIVERS

ABSTRACT

This thesis investigates the representation theory of quivers, with a particular focus on cover-thin representations for the Kronecker-3 quiver. The Kronecker-3 quiver consists of two vertices connected by three parallel arrows. The main objective is to determine the number of isomorphism classes of the cover-thin representations in a given dimension. Thin representations are defined as those in which the vector space corresponding to each vertex has dimension at most 1. Using covering theory, it has been demonstrated that a class of finite subquivers of the universal covering quiver of Kronecker-3, up to shifting, corresponds to an isomorphism class of an indecomposable cover-thin representation of the Kronecker-3 quiver. To compute the number of these subquivers, the properties of a subquiver with n vertices are analyzed, and a group action is used for accurate counting. Subsequently, using SageMath to manage group automorphisms and combinatorial calculations, the number of indecomposable cover-thin representations on Kronecker-3 for a given dimension were determined. The intuitive findings obtained from the combinatorial calculations conducted with SageMath have shown that the number of these isomorphism classes coincide with the sequence of super ballot numbers, and this relationship is proven.

Keywords: Quiver representations, Path algebras, Cover-thin modules, Thin representations, Super ballot numbers, Universal covering quiver.

KUİVERLERİN YOL CEBİRLERİ ÜZERİNDEKİ ÖRTÜ-İNCE MODÜLLER

ÖZ

Bu tez, özellikle Kronecker-3 kuiveri için örtü-ince temsiller üzerine odaklanarak, kuiverlerin temsil teorisini araştırmaktadır. Kronecker-3 kuiveri, üç paralel ok ile birbirine bağlanmış iki köşeden oluşmaktadır. Ana hedef, verilen bir boyutta bu örtü-ince temsillerin izomorfizm sınıflarının sayısını belirlemektir. İnce temsiller, her bir köşeye karşılık gelen vektör uzayının en fazla bir boyuta sahip olduğu temsiller olarak tanımlanır. Örtü teorisi kullanılarak, Kronecker-3 kuiverine ait evrensel örtü kuiverinin kaydırmaya göre bir sonlu alt-kuiverlerinin sınıfı, Kronecker-3 kuiveri için bir örtü-ince ayrıştırılamaz temsili izomorfizm sınıfına karşılık geldiği gösterilmektedir. Altkuiverlerin sayısını hesaplamak için, n köşeli bir altquiverin hangi özelliklere sahip olduğu incelenip, doğru sayma yapabilmek için grup etkisi kullanılmaktadır. Ardından grup otomorfizmaları ve kombinatorik hesaplamalarla başa çıkabilen SageMath kullanılarak Kronecker-3 üzerindeki verilen bir boyuttaki ayrıştırılamaz örtü-ince temsillerin sayıları hesaplanmaktadır. SageMath kullanılarak gerçekleştirilen kombinatorik hesaplamalar sonucunda elde edilen sezgisel bulgular, bu izomorfizm sınıflarının sayısının süper ballot sayıları dizisiyle örtüşmektedir ve bu ilişki kanıtlanmaktadır.

Anahtar kelimeler: Kuiver temsilleri, Örtü-ince modüller, İnce kuiver temsiller, Süper ballot sayıları, Evrensel örtü quiveri.

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LIST OF SYMBOLS

K_n	: Kronecker- n quiver
$\tilde{K}_n$	: Universal covering quiver of the Kronecker- n quiver
$\mathbb{K}$	: An algebraically closed field
T_n	: n -regular (undirected) Tree
$\text{ct}(n)$	: The number of isomorphism classes of cover-thin indecomposable representations of the Kronecker-3 quiver K_3 with dimension n
Q	: A quiver (Q_0, Q_1, s, t)
$\tilde{Q}$	: The universal covering quiver of a quiver Q
Q_0	: Vertex set of a quiver Q
Q_1	: Arrow/edge set of a quiver Q
$s(e)$	: Source of an arrow/edge e
$t(e)$	: Target of an arrow/edge e
$c(e)$	: Color of an arrow/edge e
Q^{-1}	: Double quiver constructed from a quiver Q
σ^{-1}	: Formal inverse of an arrow σ
$\bar{Q}$	: Underlying graph of a quiver Q
$\{a, b\}$	: An undirected edge with endpoints a and b
$\sigma_{(i,w)}$	: An arrow of a universal covering quiver $\tilde{Q}$ of some quiver Q where i is a vertex of Q and w a walk of Q with $t(w) = i$, σ is an arrow of Q with source i
$W(Q)$	: The set of walks of a quiver Q
A_∞	: Line quiver with infinitely many vertices
A_n	: Line quiver with n vertices
π	: Covering morphism $\pi : \tilde{Q} \rightarrow Q$ defined from the universal covering quiver $\tilde{Q}$ of a quiver Q to the quiver Q
Π	: The push-down functor $\Pi : \text{rep}_k \tilde{Q} \rightarrow \text{rep}_k Q$ constructed by covering quiver morphism π
$Q(M)$	: The support quiver of a representation M of a quiver Q
$[M_\sigma]$	: The matrix representing the linear map M_σ in some basis

- E_{ij} : The elementary matrix where the only nonzero entry is the ij -th entry which is equal to 1
- $d(a, b)$: Distance between vertices a and b
- $\deg(v)$: Degree of a vertex v
- $e(v)$: eccentricity of a vertex v
- B_n : n -th entry of the sequence of the super ballot numbers



CHAPTER ONE

INTRODUCTION

Kronecker-3 represents one of the fundamental linear algebra problems known as the 3-Kronecker problem. Let (A, B, C) and (A', B', C') be two 3-tuples of matrices of the same size. We define an equivalence relation between (A, B, C) and (A', B', C') , denoted by $\sim$, if there exist invertible matrices U and T such that $A' = UAT^{-1}$, $B' = UBT^{-1}$, and $C' = UCT^{-1}$. The direct sum is defined as follows.

$$(A, B, C) \oplus (D, E, F) = (A \oplus D, B \oplus E, C \oplus F).$$

Currently, the challenge lies in enumerating all the indecomposables. In the context of the Kronecker- n quivers, the definitions are naturally extended to n -tuples. For a detailed analysis of the Kronecker-2 problem using matrices and purely linear algebraic methods, the reader is referred to Barot (2015).

The primary objective of this thesis is not solely to catalog all indecomposable modules over the path algebra of the Kronecker-3 quiver for a given $\mathbb{K}$ -dimension d , up to isomorphism. Instead, it focuses on identifying and enumerating a more manageable subset of these modules — the cover-thin indecomposable modules — for the same dimension d . The definition and characterization of cover-thin modules will be developed step by step in the following sections.

First and foremost, it should be stated that the problem being worked on for this thesis was suggested by the esteemed mathematician Professor Claus Michael Ringel, who came for a workshop in Turkey in 2014. The primary reference for cover-thin modules in this thesis is Claus Michael Ringel's paper, "Indecomposable Representations of the Kronecker Quivers"; see Ringel (2013). For covering theory, the paper I utilized for the definition and main properties of the universal covering quiver is "Tree Modules" from Thorsten Weist (Weist, 2012). Additionally, for further information about the theory and related topics, de la Peña (2022) is referred to.

In Chapter 2 Preliminaries, the first section contains the necessary information about quivers. Since quivers are directed graphs, and both graph theory and representation theory of quivers are needed in the main chapter of the thesis, this section includes a few definitions not standard in the quiver representation theory. These definitions were added to ensure clarity in the mathematical writing, as we did not take them from the main textbooks Assem et al. (2006) and Schiffler (2014).

In the second section of Chapter 2, the main definitions necessary for the representation theory of quivers, along with a few examples, are provided. Almost all the information in this section can be referenced from Schiffler's book Schiffler (2014). Besides these two main topics, we also utilize important theories from group theory, graph theory, and category theory in the main part of the thesis. However, since only basic information is used, these details and their references are given as needed.

The third chapter, titled 'Cover-thin Representations', is the main chapter of this thesis. After the Kronecker-3 quiver is defined and the notation is fixed, the necessary definitions from covering theory and the universal covering quiver, along with examples, are introduced to start this chapter. The universal covering quiver of the Kronecker-3 quiver is denoted by $\tilde{K}_3$. The structure of this quiver is examined, and, due to the properties of the covering morphism, the edges of this quiver are colored with three colors. The preservation of edge colors under the covering morphism is highlighted as an important feature.

Before providing the definition of cover-thin representations, whose isomorphism classes we will count, we define the necessary push-down functor. Subsequently, we state that this functor maps indecomposable objects to indecomposable objects. Following this result, the primary focus becomes the property of being cover-thin.

A lift of a cover-thin representation is thin; therefore, we provide several propositions detailing the properties of thin indecomposable representations in the universal covering quiver. Proposition 3.1.16 emphasizes the significant relationship between a thin indecomposable representation and its support quiver. This implies that each subquiver

of a finite quiver provides the isomorphism classes of one of the thin indecomposable representations over that quiver.

We then move on to the critical Theorem 3.1.17 for this thesis problem. This theorem states that instead of counting the isomorphism classes of n -dimensional cover-thin indecomposable representations of K_3 , we can count the n -vertex subquivers of the universal covering quiver $\tilde{K}_3$ of K_3 up to shifting. Thanks to this theorem, we will no longer perform counting on the abstract objects, which are representations, but on the more concrete and physical objects, which are graphs.

The undirected and uncolored form of this universal cover representation is, in fact, the infinite tree T_3 , where each vertex has a degree of exactly 3. In Remark 3.2.1, we take a finite subquiver T of T_3 , assign coloring and directions, and transform T into a subquiver of the universal cover representation. We gather all possible coloring and direction assignments into a set denoted by $\mathcal{S}$. Subsequently, to ensure that each subquiver is counted exactly once, we define a group action on this set. This involves identifying quivers in $\mathcal{S}$ that are essentially the same subquiver of the universal cover quiver, and then counting the orbits under this group action.

Of course, for each n -vertex T , we obtain a different result. To count the isomorphism classes of all cover-thin representations, we need to define the set $\mathcal{S}$ and the group action for each n -vertex $T \subseteq T_3$ and then count the orbits. By summing these numbers for each non-isomorphic T , we will obtain the count of n -dimensional cover-thin modules.

However, this is not an easy task. Since we do not yet have a simple formula for this process, we utilize SageMath, which can handle algebraic objects and graphs.

Let us denote the number of isomorphism classes of n -dimensional indecomposable cover-thins by $ct(n)$. Using SageMath, we examined the sequence of $ct(n)$ for n from 2 to 15. You can see this sequence in the second output of the SageMath codes. We noticed that this finite sequence exactly matches the sequence of super ballot numbers for n from 2 to 15.

With this insight, we took a further step and proved in the last two theorems that the recursive formula for super ballot numbers actually counts the number of n -vertex subquivers of $\tilde{K}_3$, up to shifting. Consequently, the n -th super ballot number gives the count of isomorphism classes of n -dimensional cover-thin indecomposable representations.

In the final chapter, we discuss the results we obtained, as well as some intuitive ideas that could build upon these results and further advance this research.



CHAPTER TWO

PRELIMINARIES

2.1 Quivers

Definition 2.1.1. A **graph** consists of two sets:

- V , the set of points called vertices.
- E , the set of lines called edges, which connect pairs of vertices.

Each edge links two vertices. If there is exactly one edge connecting two vertices a and b , it is denoted by $\{a, b\}$. If multiple edges connect the same two vertices, these are denoted by $\{a, b\}_1, \{a, b\}_2$, and so on, to distinguish between them. This indexed notation is used only when necessary to specify multiple connections between the same pair of vertices; otherwise, the simpler notation is used.

When we say graph we always mean undirected graph in this thesis.

Definition 2.1.2. A **quiver** Q is a quadruple (Q_0, Q_1, s, t) , where Q_0 is the set of vertices, Q_1 is the set of edges and $s, t : Q_1 \rightarrow Q_0$ are functions that specify the direction of the edges; specifically, s maps an arrow to its starting point and t maps an arrow to its terminal point. Here, an arrow refers to a directed edge.

Definition 2.1.3. The **in-degree** of a vertex $i \in Q_0$ of Q is the cardinality of the set $\{\alpha \in Q_1 \mid t(\alpha) = i\}$. The **out-degree** of a vertex $i \in Q_0$ of Q is the cardinality of the set $\{\alpha \in Q_1 \mid s(\alpha) = i\}$. The **degree** of a vertex is the sum of in-degree and out-degree of the given vertex. Given two vertices in a quiver, we say that they are **adjacent** if one is the starting and the other is the ending point of an arrow in the quiver. For given two vertices in a graph, we say two vertices are adjacent if both vertices are endpoints of the same edge.

Two edges in a graph are said to be **incident** if they share a common vertex. Formally, given a graph $G = (V, E)$ where V is the set of vertices and E is the set of edges, two

edges $e_1 = \{u, v\}$ and $e_2 = \{v, w\}$ are incident if there exists a vertex $v \in V$ such that v is an endpoint of both e_1 and e_2 . In other words, two edges are incident if they are connected to each other through a shared vertex. For a quiver, i.e, directed graph, two edges are incident if they are incident in the underlying graph of that quiver.

Definition 2.1.4. A quiver is **finite** if the sets of vertices and arrows are finite. A quiver is called **locally finite** if each vertex has finitely many vertices that are adjacent to it.

Note that a quiver can be locally finite and not finite. A basic example would be n -regular trees, n regular trees defined as graphs where they do not have any cycle and each vertex has degree n , so by defining any orientation on these n -regular trees we will get a locally finite quiver since every vertex is adjacent to finitely many vertices.

Remark 2.1.5. Let us give a notation for a line graph that we will denote by A_∞ . It is a graph where each vertex has exactly degree 2 and does not contain any cycle, so it goes to both side looks like an infinite chain.

Definition 2.1.6. Given a directed graph $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices and Q_1 is the set of directed edges with functions s and t determining the source and target of each edge respectively, the **underlying graph** of Q is the undirected graph obtained by replacing every directed edge $\alpha : s(\alpha) \rightarrow t(\alpha)$ for each $\alpha \in Q_1$ with an undirected edge $\bar{\alpha} : \{s(\alpha), t(\alpha)\}$. This process removes the orientation from each edge, forming an undirected graph.

Definition 2.1.7. A path in Q is a sequence $p = (i_1 | \alpha_1 \alpha_2 \alpha_3 \dots \alpha_n | i_{n+1})$ such that $\alpha_j : i_j \rightarrow i_{j+1} \in Q_1$ where we define the source of the path p to be $s(p) = s(\alpha_1) = i_1$, and the target of the path p to be $t(p) = t(\alpha_n) = i_{n+1}$ and for every $i \in \{1, 2, 3, \dots, n-1\}$, we have $s(\alpha_{i+1}) = t(\alpha_i)$.

Definition 2.1.8. The **length** of a path p in a quiver Q is defined as the number of arrows it contains.

The **distance** between two vertices a and b in a quiver Q is denoted by $d(a, b)$ and is defined as the minimum length among all paths from a to b .

Definition 2.1.9. Given two paths p and q in a quiver Q , where $p = (i_1 | \alpha_1 \alpha_2 \dots \alpha_m | i_{m+1})$ and $q = (j_1 | \beta_1 \beta_2 \dots \beta_n | j_{n+1})$, the **multiplication of paths** p and q , denoted by pq , is

defined if and only if $t(p) = s(q)$ (i.e., $i_{m+1} = j_1$). The result of this multiplication is a new path from $s(p)$ to $t(q)$ formed by joining the sequences of arrows in p and q , denoted as:

$$pq = (i_1 | \alpha_1 \alpha_2 \dots \alpha_m \beta_1 \beta_2 \dots \beta_n | j_{n+1})$$

This new path starts at $s(p)$ which is i_1 , passes through the sequence of arrows $\alpha_1 \alpha_2 \dots \alpha_m$ followed by $\beta_1 \beta_2 \dots \beta_n$, and ends at $t(q)$, which is j_{n+1} .

Definition 2.1.10. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. We define the **double quiver** Q^{-1} of Q as follows:

- The vertex set of Q^{-1} remains the same as that of Q , denoted by $Q_0^{-1} = Q_0$.
- The arrow set Q_1^{-1} includes every arrow α from Q_1 and a corresponding formal inverse arrow α^{-1} for each α , where $\alpha^{-1} : t(\alpha) \rightarrow s(\alpha)$. Thus, the set of arrows for Q^{-1} is given by:

$$Q_1^{-1} = \{\alpha^{-1} : t(\alpha) \rightarrow s(\alpha) \mid \alpha : s(\alpha) \rightarrow t(\alpha) \in Q_1\} \cup Q_1.$$

This configuration doubles the arrow set of Q , comprising both the original arrows and their inverses.

Definition 2.1.11 (Connected quiver). We say a quiver Q is connected if for every pair $(i, j) \in Q_0^2$ there exist a path in the double quiver Q^{-1} .

Some may consider that, the paths in the double quiver of Q , as the undirected paths of the underlying graph of Q .

Definition 2.1.12. (Bongartz & Gabriel, 8182) Let $Q = (Q_0, Q_1, s, t)$ be a quiver, and let $Q^{-1} = (Q_0, Q_1^{-1}, s, t)$ be its double quiver. A **walk** in quiver Q is defined as a path in the double quiver Q^{-1} . A walk ω in Q is denoted by $\omega = p_1 p_2 \dots p_n$, where each p_i is an arrow in Q_1^{-1} for $i \in \{1, 2, \dots, n\}$. As walks are paths in the double quiver, path notation is applicable.

An equivalence relation on walks is defined as follows: for any arrow $\alpha \in Q_1^{-1}$, the walk containing a sequence $(i \mid \alpha \alpha^{-1} \mid i)$ is equivalent to $(i \parallel i)$, representing

simplification i.e, $(i||i) \sim (i|\alpha\alpha^{-1}|i)$. Thus, a walk $(i | \beta\alpha\alpha^{-1}\gamma | j)$ is equivalent to $(i | \beta\gamma | j)$, which can be seen by decomposing $(i | \beta\alpha\alpha^{-1}\gamma | j)$ into $(i | \beta | k)(k | \alpha\alpha^{-1} | k)(k | \gamma | j)$, where k is the target of β and the source of α , and the walk $(k | \alpha\alpha^{-1} | k)$ simplifies to $(k || k)$. Therefore, we have

$$(i | \beta\alpha\alpha^{-1}\gamma | j) \sim (i | \beta | k)(k || k)(k | \gamma | j) \sim (i | \beta\gamma | j).$$

Paths and walks are considered up to this equivalence. The set of all such walks within a quiver Q is denoted by $W(Q)$.

Definition 2.1.13 (Subquiver). A quiver $C = (C_0, C_1, s_c, t_c)$ is a **subquiver** of $Q = (Q_0, Q_1, s, t)$ if $C_0 \subseteq Q_0$, $C_1 \subseteq Q_1$ and the functions s_c, s_1 are the restrictions of the functions s, t respectively.

Definition 2.1.14 (Connected component). A subquiver Q' of Q is a **connected component** of Q if it is connected and if Q' is the greatest connected subquiver of Q that contains Q' .

Definition 2.1.15 (Morphism of quivers). Let $Q = (V, E, s, t)$ and $Q' = (V', E', s, t)$ be quivers. A **morphism** $m = (m_v, m_e) : Q \rightarrow Q'$ of quivers is a pair of functions $m_v : V \rightarrow V'$ and $m_e : E \rightarrow E'$ making the following diagrams (2.1) commutative

$$\begin{array}{ccc} E & \xrightarrow{m_e} & E' \\ s \downarrow & & \downarrow s \\ V & \xrightarrow{m_v} & V' \end{array} \quad \begin{array}{ccc} E & \xrightarrow{m_e} & E' \\ t \downarrow & & \downarrow t \\ V & \xrightarrow{m_v} & V' \end{array} \quad (2.1)$$

Definition 2.1.16 (Cycle). A **cycle in a graph** is a sequence of distinct vertices $v_1, v_2, \dots, v_k$ (with $k \geq 2$) such that there is an edge between v_i and v_{i+1} for $i = 1, 2, \dots, k-1$, and an edge between v_k and v_1 . This sequence of vertices forms a closed loop, where each vertex is connected to exactly two others, and the graph traversal returns to the starting vertex without retracing any edge or vertex.

Definition 2.1.17. A **cycle in a quiver** Q is defined as a walk $\omega = \alpha_1\alpha_2 \dots \alpha_k$ in Q , where:

- $k \geq 2$ is the minimum number of arrows among all walks equivalent to ω under the defined equivalence relation for walks.
- The source of the first arrow and the target of the last arrow in ω are the same, i.e., $s(\alpha_1) = t(\alpha_k)$, which ensures that ω starts and ends at the same vertex.

If $k = 1$, then ω is defined as a **loop**, which is a special case of a cycle where the walk consists of a single arrow that starts and ends at the same vertex.

Definition 2.1.18 (Directed cycle). A **directed cycle** in a quiver is a path of length $k \geq 2$ that starts and ends in the same vertex.

Definition 2.1.19 (Tree graph). A connected graph is called a **tree** if it does not consist cycles. When we refer to a quiver as a **tree**, it means the underlying undirected graph is a tree.

2.2 Representation of Quivers

Definition 2.2.1 (Representation of a Quiver). A **representation** M of a quiver Q is a tuple $(M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ defined on the vertex set Q_0 and the arrow set Q_1 of the quiver Q . For every vertex $i \in Q_0$, M_i is a vector space over $\mathbb{K}$, and for every arrow $\alpha : i \rightarrow j$, $M_\alpha : M_i \rightarrow M_j$ is a $\mathbb{K}$ -linear map.

A representation is called **zero representation** if for every vertex the corresponding vector space is zero and for every arrow the corresponding linear map is the zero map. An **element** of a representation $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is a sequence $(m_i)_{i \in Q_0}$ such that for every arrow $\alpha : j \rightarrow l$ of Q , $M_\alpha(m_j) = m_l$.

Definition 2.2.2 (Direct sum). Let $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $N = (N_i, N_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be two representations of a quiver Q . The **direct sum** of two representations defined as

$$M \oplus N = (M_i \oplus N_i, M_\alpha \oplus N_\alpha)$$

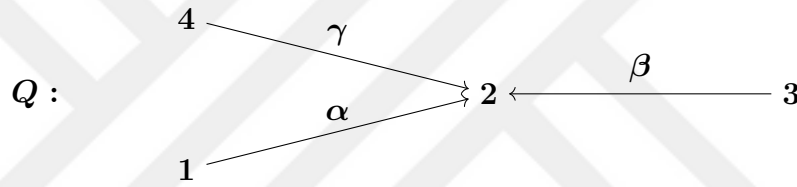
The matrix representative of the linear map $M_\alpha \oplus N_\alpha$ is

$$\begin{bmatrix} [M_\alpha] & 0 \\ 0 & [N_\alpha] \end{bmatrix}$$

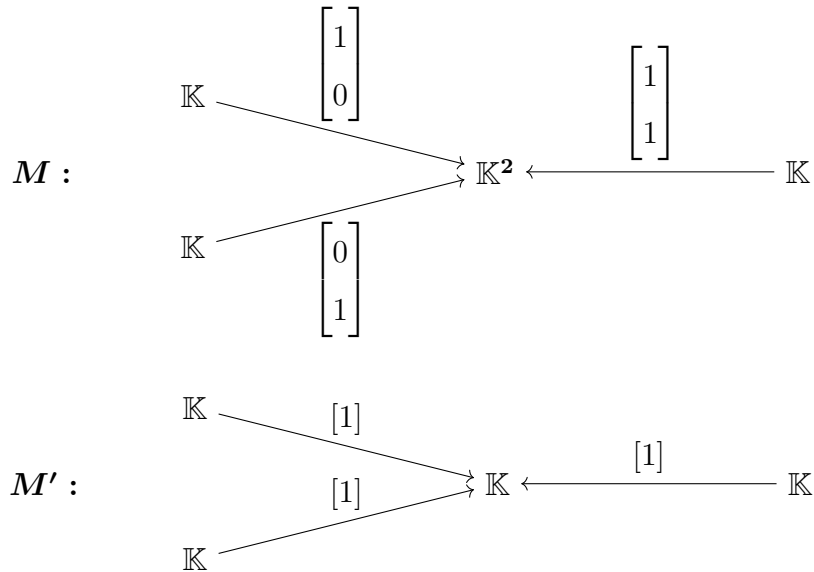
where $[M_\alpha], [N_\alpha]$ are the matrix representatives of the linear maps M_α and N_α respectively.

Let us look into an example for direct sum of representations of a given quiver Q . For this example we refer to (Schiffler, 2014).

Example 2.2.3. Let Q be the following quiver.



Take the two following representations M, M' be representations of Q .



Then the direct sum $M \oplus M'$ of representations M and M' of the quiver Q is the following representation.

$$\begin{array}{ccccc}
 & & \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} & & \\
 & \mathbb{K}^2 & \searrow & & \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\
 M \oplus M' : & & & \mathbb{K}^3 & \longleftarrow \mathbb{K}^2 \\
 & \mathbb{K}^2 & \nearrow & & \\
 & & \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & &
 \end{array}$$

Definition 2.2.4 (Indecomposable representation). We call a nonzero representation M indecomposable whenever $M \cong N \oplus L$ then $N = 0$ or $L = 0$.

Definition 2.2.5 (Morphism of Representations). Let $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $N = (N_i, N_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be two representations of a quiver Q . A morphism of representations $f : M \rightarrow N$ is a collection $(f_i)_{i \in Q_0}$ of linear maps $f_i : M_i \rightarrow N_i$ such that for each arrow $\alpha : i \rightarrow j$ in Q_1 the following diagram commutes.

$$\begin{array}{ccc}
 M_i & \xrightarrow{M_\alpha} & M_j \\
 f_i \downarrow & & \downarrow f_j \\
 N_i & \xrightarrow{N_\alpha} & N_j
 \end{array}$$

That is $f_j \circ M_\alpha = N_\alpha \circ f_i$.

In addition, $f = (f_i)_{i \in Q_0}$ is called a monomorphism if each f_i is injective, an endomorphism if each f_i is surjective, an isomorphism if each f_i is bijective.

Let us look into an example of direct summands of a given representation of a quiver Q . For this example we refer to (Schiffler, 2014).

Example 2.2.6. Let Q be the following quiver.

$$Q : \quad 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

and M be the following representation of Q .

$$M : \quad \mathbb{K}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}} \mathbb{K}^3 \xleftarrow{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \mathbb{K}$$

We can observe that $M \cong N \oplus L$ where N and L are the following representations.

$$N : \quad \mathbb{K} \xrightarrow{[1]} \mathbb{K} \xleftarrow{[1]} \mathbb{K}$$

$$L : \quad \mathbb{K} \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \mathbb{K}^2 \xleftarrow{[0]} 0$$

Both N and L are nonzero representations, so M is not an indecomposable representation.

Definition 2.2.7 (Dimension of a representation). For a given representation $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$, dimension of M is denoted by $\dim M$ and defined as a collection $(\dim M_i)_{i \in Q_0}$.

Total dimension of a representation $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is defined as $\sum_{i \in Q_0} \dim M_i$.

Definition 2.2.8 (Thin representation). A representation M of a quiver Q is called thin, if the vector space M_i has dimension less than or equal to 1 for every vertex i of Q .

Definition 2.2.9 (Subrepresentation). For given two representations of Q ,

$$M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1} \quad \text{and} \quad N = (N_i, N_\alpha)_{i \in Q_0, \alpha \in Q_1},$$

N is called a subrepresentation of M if for every $i \in Q_0$, we have $N_i \subseteq M_i$ and for every arrow $\alpha : i \rightarrow j$, we have the restriction map $N_\alpha = M_\alpha|_{N_i}$.

As a result, we need to have $M_\alpha(N_i) \subseteq N_j$ for the map N_α to be well-defined.

Definition 2.2.10 (Support quiver). Let M be a representation of a quiver Q . Then the support quiver $Q(M) = (Q(M)_0, Q(M)_1, s, t)$ of the representation M of Q is defined as follows.

$$Q(M)_0 = \{x \in Q_0 \mid M_x \neq 0\}$$

$$Q(M)_1 = \{\alpha \in Q_1 \mid M_\alpha \neq 0\}$$

Orientations of the arrows remain same, equivalently, for every arrow $\alpha \in Q(M)_1$, we have $\alpha : s(\alpha) \rightarrow t(\alpha)$ where $s(\alpha)$ and $t(\alpha)$ are the source and target of α respectively, in the quiver Q .

CHAPTER THREE

COVER-THIN REPRESENTATIONS

In this chapter, we assume that $\mathbb{K}$ is an algebraically closed field, and all quivers are assumed not to contain directed cycles. For notations and definitions not explicitly stated here we refer to the book Assem et al. (2006), and we follow similar notations throughout.

The path algebra $\mathbb{K}Q$ of a quiver Q is the algebra spanned by all paths in Q . The multiplication operation in $\mathbb{K}Q$ is defined by extending linearly the previously defined path multiplication in 2.1.9. In the path algebra $\mathbb{K}Q$, the identity element is represented by the sum of all paths of length zero, denoted by e_i , where i runs over the vertex set Q_0 . Each e_i corresponds to a vertex i and acts as left identity element for paths starting at i , and as right identity element for paths ending at i . It is important to note that a path algebra might not have an identity element. This situation occurs, for example, in any path algebra constructed from a quiver with infinitely many vertices. In such cases, the sum of the infinitely many paths of length zero, denoted by e_i , where i runs over the vertex set, is not well-defined.

A category consists of a family of objects and morphisms between these objects. The category of representations over an arbitrary quiver Q , denoted by $\text{rep}_{\mathbb{K}}Q$, has as objects all the representations with finite total dimension, which is the sum of the dimensions of the vector spaces appearing in each vertex of the quiver Q . The category of finite dimensional representations, denoted by $\text{Rep}_{\mathbb{K}}Q$, has as objects all the finite dimensional representations, meaning the representations with finite dimensional vector spaces in every vertex of the quiver Q . Define $\text{mod-}\mathbb{K}Q$ as the category of finitely generated modules over $\mathbb{K}Q$. If Q is a finite quiver without directed cycles, then it was noticed that there exists an equivalence between $\text{mod-}\mathbb{K}Q$ and $\text{Rep}_{\mathbb{K}}Q$. For further details, and the proof of the equivalence, we refer to (Schiffler, 2014, 5.4) and (Assem et al., 2006, Theorem 3.1.6).

Through this established equivalence, we aim to derive results using the representation theory of quivers, which will subsequently be applied to the category of finitely generated modules over the path algebra we have investigated.

We will assume that all of our representations belong to the category $\text{rep}_{\mathbb{K}} Q$.

3.1 Cover-thin Representations of Kronecker-3 Quiver

We start by introducing the Kronecker-3 quiver.

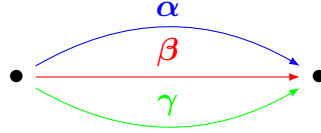


Figure 3.1 Kronecker-3 quiver K_3

From now on we will fix the names of the vertices and arrows of the Kronecker-3 quiver to be more clear. The source vertex is 1, the sink vertex is 2 and arrows from 1 to 2 are called α, β, γ .

We first define the cover of a quiver (Franzen, 2022).

Definition 3.1.1 (Cover of a quiver). A cover of a quiver Q is a pair consisting of a quiver C and a morphism $c = (c_0, c_1) : C \rightarrow Q$ of quivers, which is surjective on both vertices and arrows, and that satisfies the below condition. That is, the maps $c_0 : C_0 \rightarrow Q_0$ and $c_1 : C_1 \rightarrow Q_1$ are surjective, and the following condition is satisfied: For any $i \in Q_0$ and $k \in C_0$ where $c_0(k) = i$, the map $c_1 : C_1 \rightarrow Q_1$ induces bijections:

$$\{\beta \in C_1 \mid s(\beta) = k\} \xrightarrow{c_1} \{\alpha \in Q_1 \mid s(\alpha) = i\}$$

and

$$\{\beta \in C_1 \mid t(\beta) = k\} \xrightarrow{c_1} \{\alpha \in Q_1 \mid t(\alpha) = i\}.$$

This is the general definition for a cover of a quiver; however, we will use a well-constructed cover called the universal covering quiver, defined as follows (Weist, 2012).

Definition 3.1.2. The universal covering quiver $\tilde{Q}$ of Q is given by the vertex set

$$\tilde{Q}_0 = \{(i, \omega) \mid i \in Q_0, \omega \in W(Q), t(\omega) = i\},$$

where $W(Q)$ is the set of all walks in Q , and the arrow set is

$$\tilde{Q}_1 = \{ \alpha_{(i,\omega)} : (i,\omega) \rightarrow (j,\omega\alpha) \mid (\alpha : i \rightarrow j \in Q_1), \omega \in W(Q), t(\omega) = i \}.$$

The universal covering quiver $\tilde{Q}$ of Q is also a cover of Q with the following morphism

$$\pi = (\pi_0, \pi_1) : \tilde{Q} \longrightarrow Q$$

defined as follows. For a vertex $(i,\omega) \in \tilde{Q}_0$, $\pi_0(i,\omega) = i$ and, for an arrow $\alpha_{(i,\omega)} \in \tilde{Q}_1$, $\pi_1(\alpha_{(i,\omega)}) = \alpha$.

Example 3.1.3. Let Q be the Kronecker-2 quiver.

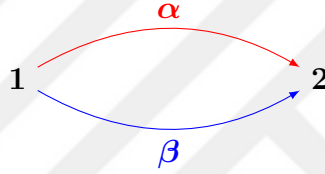


Figure 3.2 Kronecker-2 quiver K_2

The set of walks of the Kronecker-2 quiver is the following quiver consisting of two connected components:

$$W(Q) = \{ e_1, e_2, \alpha, \beta, \alpha^{-1}, \beta^{-1}\alpha\beta^{-1}, \beta\alpha^{-1}, \alpha\beta^{-1}\alpha, \beta\alpha^{-1}\beta, \dots \} \quad (3.1)$$

The set of vertices of the universal covering quiver of Kronecker-2 quiver is as follows:

$$\tilde{Q}_0 = \left\{ \begin{array}{l} (1, e_1), (2, e_2), (2, \alpha), (2, \beta), (1, \alpha^{-1}), (1, \beta^{-1}), \\ (1, \alpha\beta^{-1}), (1, \beta\alpha^{-1}), (2, \alpha^{-1}\beta), (2, \beta^{-1}\alpha), \dots \end{array} \right\} \quad (3.2)$$

Then, the universal covering quiver $\tilde{K}_2$ of K_2 is the following quiver.

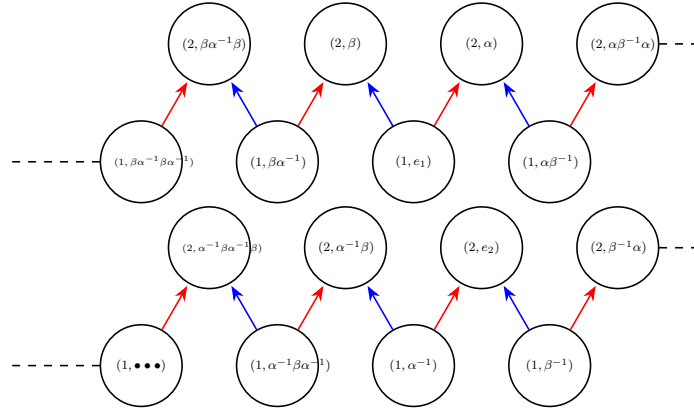


Figure 3.3 The universal covering quiver $\tilde{K}_2$ of the Kronecker-2 quiver K_2

In this example, all red arrows correspond to the arrow α , and blue arrows correspond to β . Consider the arrow $\alpha_{(1, e_1)} : (1, e_1) \rightarrow (2, \alpha)$. It is apparent from this illustration that all arrows in $\tilde{Q}$ can be uniquely determined by their source vertex and the corresponding arrow in the quiver Q .

Furthermore, it is important to note that the two components cannot be interconnected. This is because any vertex (i, w) , where w is a path of length n , cannot be connected to a vertex (j, w') in the other component, where w' has length $n + 1$ or $n - 1$. Specifically, a vertex with a walk of an odd number of length in the second entry has a target of 2 in the first component of $\tilde{K}_2$, whereas in the second component of $\tilde{K}_2$, the target is 1. Consequently, for (i, w) , if $i = 1$, then in the other component, the paths with length $n + 1$ and $n - 1$ will correspond to $(1, \cdot)$, and the Kronecker-2 does not include any arrows from 1 to itself, and similarly, if $i = 2$, then in the other component, the paths with length $n + 1$ and $n - 1$ will correspond to $(2, \cdot)$, and the Kronecker-2 quiver does not include any arrows from 2 to itself.

Since a component is a copy of the other, we will assume for simplicity assume that the universal cover of Kronecker-2 is the following quiver.

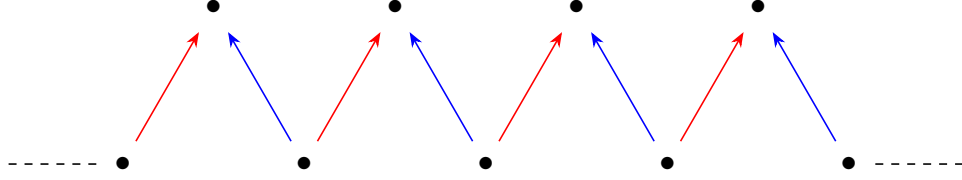


Figure 3.4 The universal covering quiver $\tilde{K}_2$

Example 3.1.4. For the Kronecker-3 quiver Q with source vertex called 1, target vertex called 2 and arrows α, β, γ , we have the following set of walks.

$$W(Q) = \left\{ \begin{array}{l} e_1, e_2, \alpha, \beta, \gamma, \alpha^{-1}, \beta^{-1}, \gamma^{-1}, \alpha\beta^{-1}, \alpha\gamma^{-1}, \beta\alpha^{-1}, \beta\gamma^{-1}, \\ \gamma\alpha^{-1}, \gamma\beta^{-1}, \alpha^{-1}\beta, \alpha^{-1}\gamma, \beta^{-1}\alpha, \beta^{-1}\gamma, \gamma^{-1}\alpha, \gamma^{-1}\beta, \dots \end{array} \right\}. \quad (3.3)$$

The set of vertices of the universal covering quiver of the Kronecker-3 quiver is given as follows:

$$\tilde{Q}_0 = \left\{ \begin{array}{l} (1, e_1), (2, e_2), (2, \alpha), (2, \beta), (2, \gamma), (1, \alpha^{-1}), (1, \beta^{-1}), (1, \gamma^{-1}), \\ (1, \alpha\beta^{-1}), (1, \alpha\gamma^{-1}), (1, \beta\gamma^{-1}), (1, \beta\alpha^{-1}), (2, \alpha^{-1}\beta), (2, \beta^{-1}\alpha), \dots \end{array} \right\} \quad (3.4)$$

The Figure 3.5 corresponds to the universal covering quiver $\tilde{K}_3$ of the Kronecker-3 quiver.

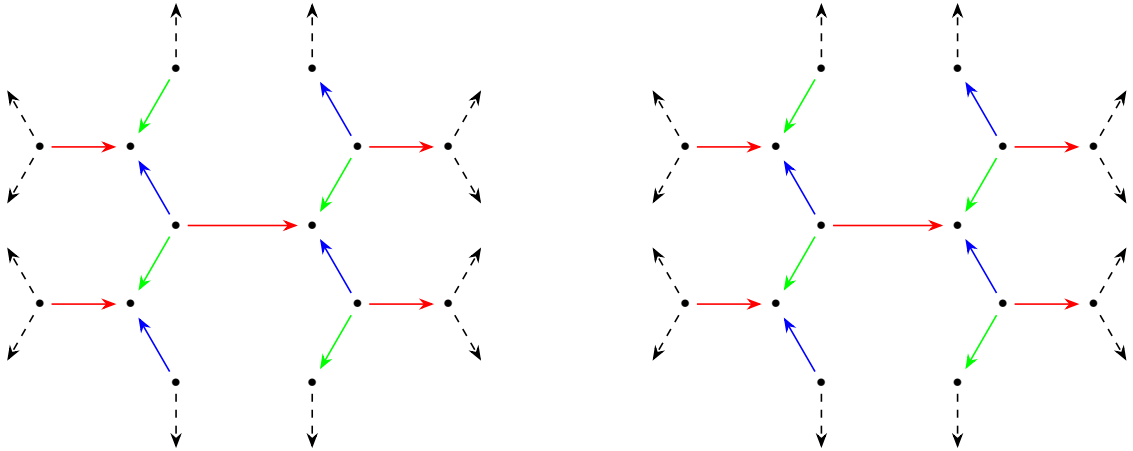


Figure 3.5 The universal covering quiver $\tilde{K}_3$ of the quiver K_3

Let w be a walk of K_3 where $t(w) = 1$, so $(1, w)$ is a vertex in $\tilde{K}_3$ and for the arrows α, β, γ of K_3 , there are arrows in $\tilde{K}_3$ defined as follows:

$$\alpha_{(1,w)} : (1, w) \rightarrow (2, w\alpha)$$

$$\beta_{(1,w)} : (1, w) \rightarrow (2, w\beta)$$

$$\gamma_{(1,w)} : (1, w) \rightarrow (2, w\gamma)$$

Similarly, for a walk w of K_3 with $t(w) = 2$, $(2, w)$ is a vertex in $\tilde{K}_3$ and for an arrow α, β, γ of K_3 , there are arrows in $\tilde{K}_3$ defined as follows:

$$\alpha_{(2,w\alpha^{-1})} : (2, w\alpha^{-1}) \rightarrow (1, w)$$

$$\beta_{(2,w\beta^{-1})} : (2, w\beta^{-1}) \rightarrow (1, w)$$

$$\gamma_{(2,w\gamma^{-1})} : (2, w\gamma^{-1}) \rightarrow (1, w)$$

This gives us that for every walk w of K_3 , there is a vertex of $\tilde{K}_3$ assigned as a source or a sink $((1, w)$ or $(2, w))$, and each vertex of $\tilde{K}_3$ has degree 3. To be more specific, each vertex has either an in-degree of 3 and an out-degree of 0, or an out-degree of 3 and an in-degree of 0, but not both.

For a vertex v of $\tilde{K}_3$, there are three arrows adjacent to it. These three arrows have the property that two incident arrows cannot have the same image under the covering morphism π .

The universal covering quiver of the Kronecker-3 quiver includes two isomorphic components. One copy is constructed starting from the vertex $(1, e_1)$, and the other from the vertex $(2, e_2)$. These are the only components of the universal covering quiver of the Kronecker-3 quiver.

By gathering all of the properties of $\tilde{K}_3$ we described above, we are going to define an edge-coloring on $\tilde{K}_3$ as follows.

Definition 3.1.5. We will define an edge-coloring on the universal covering quiver $\tilde{K}_3 = (\tilde{Q}_0, \tilde{Q}_1, s, t)$ of $K_3 = (Q_0, Q_1, s, t)$. Define a coloring function $c : \tilde{Q}_1 \rightarrow \{\alpha, \beta, \gamma\}$ where for two distinct incident edges $e_1, e_2 \in \tilde{Q}_1$, $c(e_1) \neq c(e_2)$. We shall follow the

definition of $\tilde{K}_3$ as in Example 3.1.4. For every arrow $\sigma \in Q_1 = \{\alpha, \beta, \gamma\}$ and $\sigma_{(i,\omega)} \in \tilde{Q}_1$, define $c(\sigma_{(i,\omega)}) = \sigma$ where $\sigma \in \{\alpha, \beta, \gamma\} = Q_1$. We also can take α as red, β as blue and γ as green. So that we can actually draw the edges of the universal covering quiver $\tilde{K}_3$ with three colors.

Lemma 3.1.6. *Take a walk $p = \sigma_1\sigma_2\sigma_3 \cdots \sigma_n$ of length n . The coloring sequence of p is $c(p) = c(\sigma_1)c(\sigma_2)c(\sigma_3) \cdots c(\sigma_n)$ and $s(p)$ is the source of the given walk p . Every walk in $\tilde{K}_3$ can be identified by its coloring sequence with its source.*

Proof. Starting with $s(p)$, each color will give rise to a unique edge that starts with vertex $s(p)$, because two incident edges cannot have the same color. $\square$

We will define the shifting automorphism of $\tilde{K}_3$ as following.

Definition 3.1.7. Let $\tilde{K}_3 = (\tilde{Q}_0, \tilde{Q}_1, s, t, c)$. For a given walk $p \in W(\tilde{K}_3)$, the shifting of $\tilde{K}_3$ given by the walk p , is defined as follows:

- For every vertex u in $\tilde{Q}_0$, take the unique walk q from $v = s(p)$ to u .
- Define the shifting automorphism $f : \tilde{K}_3 \rightarrow \tilde{K}_3$ where $f(u)$ is given by the target of the unique path given by the coloring sequence $c(q^{-1})c(p)c(q)$ where the source of the path is the vertex u of $\tilde{K}_3$.
- Since all vertices are shifted, ensure that the arrows also carry accordingly. Additionally, the directions and colors of the arrows will remain the same under the shifting automorphism of $\tilde{K}_3$.
- Extend the walk p to an infinite line quiver C of the form A_∞ . Let $d = d(f(v), v)$ represent the distance, then the defined shifting is generally along the line C with a length d .

See the following article for more details (Möller, 1991). The origin of the information is the source (Tits, 1970, Proposition 3.2), we refer to the first one because of the language issue.

Remark 3.1.8. Take a subquiver $Q' = (Q'_0, Q'_1, s, t, c)$ of $\tilde{K}_3 = (\tilde{Q}_0, \tilde{Q}_1, s, t, c)$. Let f be the shifting automorphism of $\tilde{K}_3$ given by the walk p as in the Definition 3.1.7. Then

$f(Q') = (fQ'_0, fQ'_1, s, t, c)$ is also a subquiver of $\tilde{K}_3$ defined as follows:

$$\begin{aligned} fQ'_0 &= \{f(v) \in \tilde{Q}_0 \mid v \in Q'_0\} \\ fQ'_1 &= \{f(\sigma) \in \tilde{Q}_1 \mid \sigma \in Q'_1\} \\ s(f(\sigma)) &= f(s(\sigma)) \\ t(f(\sigma)) &= f(t(\sigma)) \\ c(f(\sigma)) &= c(\sigma). \end{aligned}$$

Source, target and coloring of any arrow of Q' is preserved under f by its definition.

Let us define the functor Π , referred to as the **push-down functor** (Weist, 2012). This functor plays a crucial role in the context of the representation theory we are exploring.

Definition 3.1.9 (Push down functor).

$$\begin{aligned} \Pi : \text{rep}_{\mathbb{K}} \tilde{Q} &\longrightarrow \text{rep}_{\mathbb{K}} Q \\ \tilde{M} &\longmapsto \Pi(\tilde{M}) = M \end{aligned} \tag{3.5}$$

For a given representation $\tilde{M}$, $\Pi(\tilde{M}) = M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$ where

$$M_i = \bigoplus_{\substack{\omega \in W(Q), \\ t(\omega)=i}} \tilde{M}_{(i,\omega)} \tag{3.6}$$

and for $\alpha : i \rightarrow j \in Q_1$, $M_\alpha : M_i \rightarrow M_j$ is defined by

$$M_\alpha|_{M_{(i,\omega)}} = \tilde{M}_{\alpha(i,\omega)} \tag{3.7}$$

For an element $m \in M_i$, $M_\alpha(m)$ is given by

$$M_\alpha(m) = \sum_{\substack{\omega \in W(Q), \\ t(\omega)=i}} \tilde{M}_{\alpha(i,\omega)}(m_{(i,\omega)}) \tag{3.8}$$

where $m_{(i,\omega)} \in \tilde{M}_{(i,\omega)}$ and

$$m = \sum_{\substack{\omega \in W(Q), \\ t(\omega)=i}} m_{(i,\omega)}.$$

For explained theory of the push-down functor we refer to, (de la Peña, 2022).

Definition 3.1.10 (Cover-thin representation). (Ringel, 2013) We call a representation M over Q a cover-thin representation if there exist a thin representation $\tilde{M}$ over $\tilde{Q}$ such that $\Pi(\tilde{M}) = M$ where Π is the push-down functor.

Let $F : \text{mod-}\mathbb{K}\tilde{Q} \longrightarrow \text{rep}_{\mathbb{K}} Q$ be the functor that gives the categorical equivalence mentioned above (Assem et al., 2006). We say a module M over the path algebra $\mathbb{K}Q$ is a cover-thin module if the representation $F(M)$ of Q is a cover-thin. The following theorem is important to be able to count indecomposable representations and to use the push-down functor in order to count some subfamily of indecomposable representations. We refer for more details to (Weist, 2012, Theorem 2.4) and Green (1981).

Theorem 3.1.11. (Weist, 2012) Let Q be a quiver, $\tilde{M}$ be a representation of $\tilde{Q}$ and $\Pi(\tilde{M}) = M$ where $\Pi : \text{rep}_{\mathbb{K}}(\tilde{Q}) \longrightarrow \text{rep}_{\mathbb{K}}(Q)$ is the push-down functor. Then M is indecomposable if and only if $\tilde{M}$ is indecomposable.

Now, the following propositions discuss the relationship between the number of subtrees of $\tilde{Q}$ and the number of thin representations. We are trying to reduce the problem into counting on trees with the following results.

For completeness, we shall give the proof of the next four propositions using the ideas from the workshop notes of Claus Michael Ringel, see (Ringel, 2014).

Proposition 3.1.12. If M is an indecomposable representation of a quiver Q , then the support quiver $Q(M)$ of M is connected.

Proof. We will prove it by contrapositive. Let us assume $Q(M)$ (defined in 2.2.10) is not connected. Let Q' be one of the connected components and let Q'' be the union of the remaining connected components. Then Q' and Q'' are nonempty and the disjoint

union $Q' \sqcup Q''$ is the support quiver $Q(M)$. Now we will define representations $A = (A_i, A_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $B = (B_i, B_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q where

$$A_i = \begin{cases} M_i & \text{if } i \in Q'_0 \\ 0 & \text{otherwise} \end{cases}, \quad A_\alpha = \begin{cases} M_\alpha & \text{if } \alpha \in Q'_1 \\ 0 & \text{otherwise} \end{cases} \quad (3.9)$$

and

$$B_i = \begin{cases} M_i & \text{if } i \in Q''_0 \\ 0 & \text{otherwise} \end{cases}, \quad B_\alpha = \begin{cases} M_\alpha & \text{if } \alpha \in Q''_1 \\ 0 & \text{otherwise} \end{cases} \quad (3.10)$$

Since the intersection $Q'_0 \cap Q''_0$ is empty, for every vertex i of Q , we have either $A_i = 0$ or $B_i = 0$, so that we have $A_i \cap B_i = 0$ and thus $A \cap B = 0$. Now we check if $A + B = M$. $A_i + B_i = M_i$ since for every $i \in Q_0$ we choose either $A_i = 0$ and $B_i = M_i$ or $A_i = M_i$ and $B_i = 0$. Now we need to check the arrows. For each arrow $\alpha : i \rightarrow j$ both i and j belong to the same component by definition. So for $\alpha \in Q'_1$, we have $A_\alpha = M_\alpha$ and $B_\alpha = 0$ and for $\alpha \in Q''_1$, we have $B_\alpha = M_\alpha$ and $A_\alpha = 0$. Therefore $A \oplus B = M$ with A and B are both nonzero since Q' and Q'' are nonempty quivers. $\square$

As a result of this proposition, if there is an indecomposable representation of $\tilde{Q}$, then it is on a connected component of $\tilde{Q}$ because the support quiver of an indecomposable representation has to be connected.

Remark 3.1.13. Let M be an indecomposable thin representation of $\tilde{K}_3$. Then since $Q(M)$ has to be connected, M can be considered as a representation of one of the connected components of $\tilde{K}_3$. And since both of the connected components of $\tilde{K}_3$ are isomorphic by a relabelling, we can assume $\tilde{K}_3$ as one of the connected components of it.

We will assume throughout this chapter that universal covering quiver of Kronecker-3 quiver has one connected component because of what we explained in the remark above.

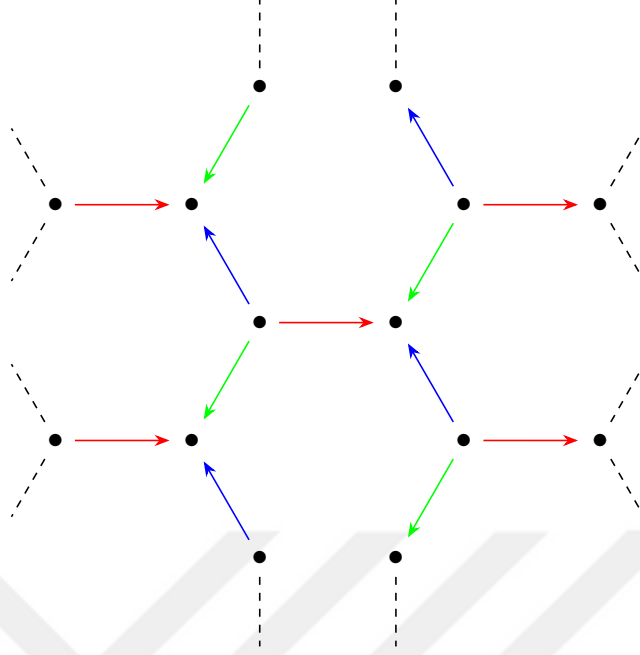


Figure 3.6 The universal covering quiver $\tilde{K}_3$

Proposition 3.1.14. Let M and M' be representations of a quiver Q which is a tree. If $M \cong M'$, then $Q(M) = Q(M')$.

Proof. Let $M = (M_i, M_\alpha)_{i \in Q_0, \alpha \in Q_1}$, $M' = (M'_i, M'_\alpha)_{i \in Q_0, \alpha \in Q_1}$ be representations of $Q = (Q_0, Q_1, s, t)$ such that $M \cong M'$ and let $Q(M) = (V, E, s|_E, t|_E)$ be the support quiver of M , $Q(M') = (V', E', s|_{E'}, t|_{E'})$ be the support quiver of M' . For every $i \in V$, we have $M_i \neq 0$, and by the isomorphism $M_i \cong M'_i$, we have $M'_i \neq 0$ implies that $i \in V'$. In this case, we have $V \subseteq V'$. Similarly, $V' \subseteq V$. Let $\phi = (\phi_i)_{i \in Q_0} : M \rightarrow M'$ be an isomorphism. For every arrow $\alpha : i \rightarrow j$ in $Q(M)$, we have $M_\alpha \neq 0$, so the following diagram commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{M_\alpha} & M_j \\ \phi_i \downarrow & & \downarrow \phi_j \\ M'_i & \xrightarrow{M'_\alpha} & M'_j \end{array}$$

Then the equality $M'_\alpha \circ \phi_i = \phi_j \circ M_\alpha$ holds. Since M_α is a nonzero linear map so is M'_α . Hence α is an arrow in the support quiver of M' . Thus $E \subseteq E'$. Similarly, $E' \subseteq E$. $\square$

The next proposition states that to understand what a thin indecomposable representation is, it is sufficient to look at its support quiver.

Proposition 3.1.15. Let Q be a quiver and a tree, and M be a thin representation of Q . Then M is indecomposable if and only if $Q(M)$ is connected.

Proof. It has been already shown in the Proposition 3.1.12 that if M is an indecomposable, then the support quiver of M is connected. So we will show the converse of the proposition only. Now assume that for a thin representation M , we have that the support quiver $Q(M)$ is connected. If M is not indecomposable, then there exist nonzero representations A, B such that $M = A \oplus B$. Because of the direct sum, we have $Q(M) = Q(A) \cup Q(B)$. To be clear, union does not mean union of quivers, it means both union on vertex and arrow sets. For every $i \in Q_0$, we have either $A_i = 0$ and $B_i \cong \mathbb{K}$ or $A_i \cong \mathbb{K}$ and $B_i = 0$. The first case; $A_i = 0$ and $B_i \cong \mathbb{K}$, implies that i is a vertex in $Q(B)$ not in $Q(A)$. The second case; $A_i \cong \mathbb{K}$ and $B_i = 0$, means that i is a vertex in $Q(A)$ and not a vertex in $Q(B)$. Shortly, $Q(A)_0 \cap Q(B)_0 = \emptyset$. Let $a \in Q(A)_0$ and $b \in Q(B)_0$ be arbitrary vertices. Since $Q(M)$ is connected there is an undirected path p between vertices a and b . Let us write $p = (a|p_1p_2p_3 \cdots p_t|b)$. From 1 to t take the first i such that source and target of p_i are not in the same set from the following sets of vertices: $Q(A)_0, Q(B)_0$. Without loss of generality choose p_i to be an arrow with source in $Q(A)$ and target in $Q(B)$ and call it $\alpha : i \rightarrow j$. By the direct sum $A \oplus B = M$, we have:

$$(A \oplus B)_\alpha = M_\alpha : M_i \longrightarrow M_j$$

In fact, we have the operator:

$$\begin{aligned} M_\alpha : A_i \oplus B_i &\longrightarrow A_j \oplus B_j \\ a + b &\longmapsto A_\alpha(a) + B_\alpha(b), \end{aligned}$$

where $a \in A_i, b \in B_i$. Note that, $A_\alpha : A_i \rightarrow A_j$ is the zero map since j is not in $Q(A)_0$, which means that $A_j = 0$; and $B_\alpha : B_i \rightarrow B_j$ is the zero map since i is not in $Q(B)_0$, which means $B_i = 0$. Collecting this, we have $M_\alpha = 0$, which is contradiction. So there is no such path to connect a vertex in $Q(A)$ to a vertex in $Q(B)$. That contradicts with the assumption $Q(M)$ to be connected. So M has to be indecomposable. $\square$

The following proposition tells us that, for a finite tree quiver Q , every connected subquiver of Q gives rise to the isomorphism class of a thin representation of Q .

Proposition 3.1.16. Let Q be a quiver that is a tree and M, M' be indecomposable thin representations of Q . Then $M \cong M'$ if and only if $Q(M) = Q(M')$.

Proof. The if part of the theorem is true for arbitrary representations M, M' of Q . So we will only show the only if part. Assume that M, N are indecomposable thin representations of a finite tree quiver Q such that $Q(M) = Q(N) = G$. We will prove by induction on n , where n is the number of vertices of G , which is finite because $\dim(M)$ is finite. If G has only one vertex, call it i , then both N_i and M_i are one dimensional vector spaces, and so isomorphic to the field $\mathbb{K}$. Thus $M \cong S_i \cong N$ where S_i is a simple representation where the only nonzero vector space is in the vertex i which is one dimensional. Let us assume that G has at least 2 vertices and is obtained from a quiver Q' by attaching an arm of the form A_2 . Define representations $N' = N|_{Q'}$ and $M' = M|_{Q'}$. Q' has $n - 1$ vertices. Thus, by induction, we have $N' \cong M'$. Let $\phi' = (\phi_i)_{i \in Q'_0} : M' \rightarrow N'$ be an isomorphism. Let us say that the arrow is connected to Q' at the vertex v and so the arrow α is either $v \rightarrow u$ or $u \rightarrow v$. Then we define a morphism $\phi = (\phi_i)_{i \in G} : M \rightarrow N$. We only need to define $\phi_u : M_u \rightarrow N_u$. According to the Definition 2.2.5, of morphism between representations, we have the following commutative diagram for the case; α is an arrow from $u \rightarrow v$.

$$\begin{array}{ccc} M_u & \xrightarrow{\phi_u} & N_u \\ M_\alpha \downarrow & & \downarrow N_\alpha \\ M_v & \xrightarrow{\phi_v} & N_v \end{array}$$

We get the equality $\phi_v(M_\alpha(m)) = N_\alpha(\phi_u(m))$ for every $m \in M_u$. So we choose $\phi_u = N_\alpha^{-1} \circ \phi_v \circ M_\alpha$. For the case, if α is an arrow $v \rightarrow u$, then we have the following commutative diagram.

$$\begin{array}{ccc} M_v & \xrightarrow{\phi_v} & N_v \\ M_\alpha \downarrow & & \downarrow N_\alpha \\ M_u & \xrightarrow{\phi_u} & N_u \end{array}$$

We get the equality $\phi_u(M_\alpha(m')) = N_\alpha(\phi_v(m'))$ for every $m' \in M_v$. This means we need to choose $\phi_u = N_\alpha \circ \phi_v \circ M_\alpha^{-1}$.

Note that in general M_α and N_α are not invertible. Since M and N are thin representations and α belong to the support quiver of both of them, then M_α and N_α are nonzero linear maps between one dimensional vector spaces. This means that they are isomorphisms. $\square$

This means that for any subquiver Q' of a finite tree quiver Q , there exists a thin indecomposable representation with its support quiver Q' . A simple construction can be as follows: Place the field $\mathbb{K}$ at every vertex of the given quiver and the identity morphism for every arrow. The representations obtained in this way are called normal form, following from the notes of Claus Michael Ringel, see (Ringel, 2014).

The proposition above seems to allow us to count cover-thin indecomposable representations. The idea is to count support quivers that corresponds to thin indecomposable representations so that we can count cover-thin representations by using the push-down functor. However, another question we must ask is: Can an indecomposable cover-thin representation M of Kronecker-3 be the image of multiple non-isomorphic thin indecomposable representations of $\tilde{K}_3$ under the push-down functor? The next result will show the answer.

Theorem 3.1.17. *Let $\tilde{Q} = \tilde{K}_3$ be the universal covering quiver of the Kronecker-3 quiver $Q = K_3$ and M and N be cover-thin indecomposable representations of Q with total dimension n , where $\tilde{M}, \tilde{N}$ are thin indecomposable representations of $\tilde{Q}$ with total dimension n such that $\Pi(\tilde{M}) = M$ and $\Pi(\tilde{N}) = N$ where Π is the push-down functor. Then the following statements are equivalent.*

- (i) $M \cong N$ as representations of K_3 .
- (ii) *There exist a shift automorphism f of $\tilde{Q}$ such that $f(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$, where $\tilde{Q}(\tilde{M})$ and $\tilde{Q}(\tilde{N})$ are the support quivers of $\tilde{M}$ and $\tilde{N}$ respectively.*

Proof. Let M and N be cover-thin representations of Q with total dimension n , where $\tilde{M}, \tilde{N}$ are thin representations of $\tilde{Q}$ with total dimension n such that $\Pi(\tilde{M}) = M$ and $\Pi(\tilde{N}) = N$ where $\tilde{M}$ and $\tilde{N}$ are of the normal form (i.e. each vertex in the support quiver, has one dimensional vector space $\mathbb{K}$ and each nonzero linear map in the representation is

the identity map on $\mathbb{K}$.) and Π is the push-down functor. Then we will show that $M \cong N$ if and only if there exist a shift automorphism f of $\tilde{Q}$ such that $f(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$.

Now first assume that there exist a shift automorphism f of $\tilde{Q}$ such that $f(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$. Let $\tilde{Q}(\tilde{M}) = Q^M = (Q_0^M, Q_1^M, s, t)$. Since Q^M has bipartite orientation (i.e. each vertex is either a sink (out-degree is 0) or a source (in-degree is zero)), there are sets S^M, T^M such that $Q_0^M = S^M \sqcup T^M$ where S^M is the set of vertices of Q^M that are sources and T^M is the set of vertices of Q^M that are sinks. Let

$$S^M = \{(1, a_1), (1, a_2), (1, a_3), \dots, (1, a_k)\}, \quad T^M = \{(2, b_1), (2, b_2), (2, b_3), \dots, (2, b_t)\}$$

where $a_1, a_2, \dots, a_k$ are some walks of K_3 with target vertex 1 and $b_1, b_2, \dots, b_t$ are walks of K_3 with target vertex 2. Since for every vertex $(1, a_i) \in S^M$, we have $\tilde{M}_{(1, a_i)} = \mathbb{K}$, we get

$$M_1 = \bigoplus_{\substack{\omega \in W(Q), \\ t(\omega)=1}} \tilde{M}_{(1, w)} = \bigoplus_{(1, a_i) \in S^M} \tilde{M}_{(1, a_i)} = \bigoplus_{(1, a_i) \in S^M} \mathbb{K} = \mathbb{K}^k$$

and

$$M_2 = \bigoplus_{\substack{\omega \in W(Q), \\ t(\omega)=2}} \tilde{M}_{(2, w)} = \bigoplus_{(2, b_i) \in T^M} \tilde{M}_{(2, b_i)} = \bigoplus_{(2, b_i) \in T^M} \mathbb{K} = \mathbb{K}^t.$$

For every arrow σ of K_3 (i.e., $\sigma \in \{\alpha, \beta, \gamma\}$), denote the matrix form of the linear map M_σ by $[M_\sigma]$. Remember from Definition 3.8:

$$M_\alpha(m) = \sum_{\substack{\omega \in W(Q), \\ t(\omega)=i}} \tilde{M}_{\alpha(i, \omega)}(m_{(i, \omega)})$$

Since we know that for any arrow $\sigma_{(1, a_j)} : (1, a_j) \rightarrow (2, b_i)$, we have $\tilde{M}_{\sigma_{(1, a_j)}} : \tilde{M}_{(1, a_j)} \rightarrow \tilde{M}_{(2, b_i)}$. Thus the matrix form of the linear map $\tilde{M}_{\sigma_{(1, a_j)}}$ with the extended form $\tilde{M}_{\sigma_{(1, a_j)}} : \mathbb{K}^k \rightarrow \mathbb{K}^t$ is E_{ij} . So for any $\sigma \in \{\alpha, \beta, \gamma\}$, define

$$[M_\sigma] = \sum_{(\sigma_{(1, a_j)} : (1, a_j) \rightarrow (2, b_i)) \in Q_1^M} E_{ij},$$

where E_{ij} is the $t \times k$ elementary matrix.

Similarly for $\tilde{Q}(\tilde{N}) = Q^N = (Q_0^N, Q_1^N, s, t)$ we know that the shifting automorphism f of $\tilde{Q} = \tilde{K}_3$ carries Q^M to Q^N and respects the orientation of arrows. Thus, we have $Q_0^N = S^N \sqcup T^N$ where $S^N = \{f((1, a_i)) | (1, a_i) \in S^M\}$ and $T^N = \{f((2, b_i)) | (2, b_i) \in T^M\}$. Then, in a similar way, we will get the following results.

$$N_1 = \bigoplus_{\substack{\omega \in W(Q), \\ t(\omega)=1}} \tilde{N}_{(1,w)} = \bigoplus_{f(1,a_i) \in S^N} \tilde{N}_{f(1,a_i)} = \bigoplus_{(1,a_i) \in S^M} \mathbb{K} = \mathbb{K}^k$$

$$N_2 = \bigoplus_{\substack{\omega \in W(Q), \\ t(\omega)=2}} \tilde{N}_{(2,w)} = \bigoplus_{f(2,b_i) \in S^N} \tilde{N}_{f(2,b_i)} = \bigoplus_{(2,b_i) \in T^M} \mathbb{K} = \mathbb{K}^t.$$

And then since the shifting automorphism f preserves adjacency and coloring, there is an arrow $(\sigma_{(1,a_j)} : a_j \rightarrow b_i) \in Q_1^M$ if and only if there is an arrow $(f(\sigma_{(1,a_j)}) : f(a_j) \rightarrow f(b_i)) \in Q_1^N$. So $M_\sigma = N_\sigma$ for every $\sigma \in \{\alpha, \beta, \gamma\}$. Thus $M \cong N$.

For the other implication, assume that $M \cong N$ where $\tilde{M}$ and $\tilde{N}$ are of the normal form. Then for every arrow $\sigma \in \{\alpha, \beta, \gamma\}$ of K_3 , we have $[M_\sigma] = \sum_{(i,j) \in X} E_{ij}$ where

$$X = \left\{ \begin{array}{l} (i, j) \mid (\sigma_{(1,a_j)} : (1, a_j) \rightarrow (2, b_i)) \in \tilde{Q}(\tilde{M})_1 \text{ where } (1, a_j) \text{ is in the} \\ \text{set of the sources of } \tilde{Q}(\tilde{M}), (2, b_i) \text{ is in the set of sinks of } \tilde{Q}(\tilde{M}) \end{array} \right\}$$

Observe that an arrow with color σ can not have incident edges to it with the same color σ . This means in the summation $\sum_{(i,j) \in X} E_{ij}$ if E_{ij} is in the summation of $[M_\sigma]$, E_{il} can not be in the summation of $[M_\sigma]$ for $l \neq j$ and E_{lj} can not be in the summation of $[M_\sigma]$ for $l \neq i$. Thus, the followings are equivalent.

- There exists an arrow $(\sigma_{(1,a_j)} : (1, a_j) \rightarrow (2, b_i)) \in \tilde{Q}(\tilde{M})$,
- E_{ij} is a summand of $[M_\sigma]$, i.e, $\sum_{(i,j) \in X} E_{ij}(e_j) = e_i$, where e_i is the standart basis element in $\mathbb{K}^k$ with only nonzero coordinate is i -th one,
- $M_\sigma(e_j) = e_i$.

Take $\phi = (\phi_1, \phi_2) : M \rightarrow N$ as an isomorphism between cover-thin indecomposable representations of Q where $\phi_1 : M_1 \rightarrow N_1$ and $\phi_2 : M_2 \rightarrow N_2$.

Since ϕ_1 and ϕ_2 are isomorphisms $(\phi_1(e_1), \phi_1(e_2), \dots, \phi_1(e_k))$ and $(\phi_2(e_1), \phi_2(e_2), \dots, \phi_2(e_t))$ are bases for N_1 and N_2 respectively. In addition, take the set $\{(1, a'_1), (1, a'_2), \dots, (1, a'_k)\}$ as set of sources and $\{(2, b'_1), (2, b'_2), \dots, (2, b'_t)\}$ as the set of sinks of $\tilde{Q}(\tilde{N})$. Then for every arrow $\sigma \in \{\alpha, \beta, \gamma\}$, we have $[N_\sigma] = \sum_{(i,j) \in Y} E_{ij}$ where

$$Y = \left\{ (i, j) \mid (\sigma_{(1, a'_j)} : (1, a'_j) \rightarrow (2, b'_i)) \in \tilde{Q}(\tilde{N})_1 \text{ where } (1, a'_j) \text{ is in the set of the sources of } \tilde{Q}(\tilde{N}), (2, b'_i) \text{ is in the set of sinks of } \tilde{Q}(\tilde{N}) \right\}$$

For every arrow $\sigma \in \{\alpha, \beta, \gamma\}$, we have that $\phi_2 \circ M_\sigma = N_\sigma \circ \phi_1$. Thus for $e_i \in M_1$ the followings are equivalent.

- $M_\sigma(e_i) = e_j$,
- $\phi_2(M_\sigma(e_i)) = N_\sigma(\phi_1(e_i))$ so that $\phi_2(e_j) = N_\sigma(\phi_1(e_i))$,
- E_{ij} is a summand of $[N_\sigma] = \sum_{(i,j) \in Y} E_{ij}$, i.e., $\sum_{(i,j) \in Y} E_{ij}(\phi_1(e_i)) = \phi_2(e_j)$,
- There exists an arrow $(\sigma_{(1, a'_j)} : (1, a'_j) \rightarrow (2, b'_i)) \in \tilde{Q}(\tilde{N})_1$.

Thus by using equivalent expressions above, we get that there exists an arrow $(\sigma_{(1, a'_j)} : (1, a'_j) \rightarrow (2, b'_i)) \in \tilde{Q}(\tilde{N})_1$ if and only if there exists an arrow $(\sigma_{(1, a_j)} : (1, a_j) \rightarrow (2, b_i)) \in \tilde{Q}(\tilde{M})$. So let us define a morphism $f : \tilde{Q}(\tilde{M}) \rightarrow \tilde{Q}(\tilde{N})$ between support quivers of M and N as follows.

- For every source vertex $(1, a_j) \in Q_0^M$, $f(1, a_j) = (1, a'_j)$.
- For every sink vertex $(2, b_i) \in Q_0^M$, $f(2, b_i) = (2, b'_i)$.
- For every arrow $(\sigma_{(1, a_j)} : (1, a_j) \rightarrow (2, b_i)) \in Q_1^M$,

$$f(\sigma_{(1, a_j)}) = \sigma_{(1, a'_j)} : (1, a'_j) \rightarrow (2, b'_i).$$

It is clear by definition that the colors and adjacency are preserved. In addition, source vertices match with sources and sinks match with sinks so the orientation is also preserved under f . It follows that f gives a well-defined isomorphism between $\tilde{Q}(\tilde{M})$ and $\tilde{Q}(\tilde{N})$ that preserves edge-coloring.

Now we will show that f gives rise to a shifting automorphism $\tilde{f}$ of $\tilde{Q}$ where $\tilde{f}(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$.

- (1) $\tilde{Q}(\tilde{M}) \cong \tilde{Q}(\tilde{N})$ such that the isomorphism respects edge-coloring.
- (2) There exists an automorphism $\tilde{f}$ of $\tilde{Q}$ that respects edge-coloring of $\tilde{Q}$ and $\tilde{f}(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$.
- (3) There exists a shift automorphism $\tilde{f}$ of $\tilde{Q}$ such that $\tilde{f}(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$.

Now we will show that (1) implies (2) and (2) implies (3).

(1 $\Rightarrow$ 2) : Let $\tilde{Q}(\tilde{M}) \cong \tilde{Q}(\tilde{N})$ that respects edge-coloring. Then consider the underlying graphs (i.e, graphs without considering directions and colors of edges), Q_m and Q_n of $\tilde{Q}(\tilde{M})$ and $\tilde{Q}(\tilde{N})$ respectively. Then Q_m and Q_n are connected subgraphs of T_3 , and $Q_m \cong Q_n$ where T_3 is the regular tree of degree 3 (i.e, the tree which every vertex has degree 3), T_3 is the underlying graph of $\tilde{Q}$. Then by Tits (1970) and Möller (1991), there exists an automorphism g of T_3 which carries Q_m to Q_n . Then define $\tilde{f} : \tilde{Q} \rightarrow \tilde{Q}$ as follows:

For every vertex i , we have $\tilde{f}(i) = g(i)$. For every edge e , we have $\tilde{f}(e) = g(e)$ such that $\tilde{f}(s(e)) = s(\tilde{f}(e))$, $\tilde{f}(t(e)) = t(\tilde{f}(e))$ and $c(\tilde{f}(e)) = c(e)$. In this case $\tilde{f}$ is an isomorphism of $\tilde{Q}$ which respects coloring and carries $\tilde{Q}(\tilde{M})$ to $\tilde{Q}(\tilde{N})$.

(2 $\Rightarrow$ 3) : An automorphism of $\tilde{Q}$ can be seen as an automorphism of T_3 , since T_3 is the underlying graph of $\tilde{Q}$. By Tits (1970) and Möller (1991) an automorphism g of T_3 satisfies one and only one of the following:

- (i) A vertex is fixed by g .
- (ii) For an edge e with endpoints x and y , $g(e) = e$ where $g(x) = y$ and $g(y) = x$.
- (iii) There is a chain L of the form A_∞ that is subgraph of T_3 such that $f(L) = L$ and no vertex of T_3 is fixed under g . In this case we call g a shifting, mentioned as translation in (Möller, 1991).

Let $d = \inf_{v \in V_{T_3}} (d(v, g(v)))$, where V_{T_3} is the vertex set of T_3 . In the case of shifting, the chain L is unique and g induces on L a shifting of distance d and $d(\alpha, g(\alpha)) = d$ if and only if v is a vertex in the chain L . (Möller, 1991).

Then $\tilde{f}$ will satisfy one of (1), (2) and (3) above. If a vertex is fixed by $\tilde{f}$ then one edge will be carried to another where they are incident to each other by a vertex that is fixed under $\tilde{f}$. Since two incident edges cannot have same color, $\tilde{f}$ cannot fix a vertex.

If $\tilde{f}$ fixes an edge but change its vertices as described in (2), then since $\tilde{f}$ transposes the endpoints of an edge it will also change the direction of that edge, which shows us that $\tilde{f}$ can not preserve orientation. Only option that is left for $\tilde{f}$, is $\tilde{f}$ to be a shifting automorphism.

Until now we showed that $M \cong N$ if and only if there exists a shift automorphism $\tilde{f}$ of $\tilde{Q}$ such that $\tilde{f}(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$ where $\tilde{M}$ and $\tilde{N}$ are of the normal form.

For representations M' and N' of the quiver Q , we will finally show that $M' \cong N'$ if and only if there exists a shift automorphism f of $\tilde{Q}$ such that $f(\tilde{Q}(\tilde{M}')) = \tilde{Q}(\tilde{N}')$ without assuming that $\tilde{M}'$ and $\tilde{N}'$ are of the normal form.

Let $\tilde{M}$ be an indecomposable thin representation of the normal form of $\tilde{M}'$, so $\tilde{M} \cong \tilde{M}'$ and similarly let $\tilde{N}$ be an indecomposable thin representation of the normal form of $\tilde{N}'$, so $\tilde{N} \cong \tilde{N}'$.

$\tilde{M} \cong \tilde{M}'$ implies that $M = \Pi(\tilde{M}) \cong \Pi(\tilde{M}') = M'$ and similarly $\tilde{N} \cong \tilde{N}'$ implies that $N = \Pi(\tilde{N}) \cong \Pi(\tilde{N}') = N'$.

Assume that there exists a shift automorphism $\tilde{f}$ of $\tilde{Q}$ such that $\tilde{f}(\tilde{Q}(\tilde{M}')) = \tilde{Q}(\tilde{N}')$. For the representations $\tilde{M}$ and $\tilde{N}$ that are of the regular form corresponding to the support quivers $\tilde{Q}(\tilde{M})$, $\tilde{Q}(\tilde{N})$ respectively, we have that $M = \Pi(\tilde{M}) \cong \Pi(\tilde{N}) = N$. Thus we get, $M' \cong M \cong N \cong N'$ so $M' \cong N'$.

Now we will prove the other side of the implication. Assume that $\Pi(\tilde{M}) = M \cong N = \Pi(\tilde{N})$. Then there exists an automorphism $\tilde{f}$ of $\tilde{Q}$ such that $\tilde{f}(\tilde{Q}(\tilde{M})) = \tilde{Q}(\tilde{N})$ where $\tilde{M}$ and $\tilde{N}$ are of the normal form. But for any thin representation $\tilde{M}'$ and $\tilde{N}'$ such that $\tilde{M} \cong \tilde{M}'$ and $\tilde{N} \cong \tilde{N}'$ the corresponding support quivers are equal, i.e, $\tilde{Q}(\tilde{M}) = \tilde{Q}(\tilde{M}')$ and $\tilde{Q}(\tilde{N}) = \tilde{Q}(\tilde{N}')$. This gives us the desired proof, since we have $\tilde{f}(\tilde{Q}(\tilde{M}')) = \tilde{f}(\tilde{Q}(\tilde{M})) =$

$$\tilde{Q}(\tilde{N}) = \tilde{Q}(\tilde{N}').$$

□

Corollary 3.1.18. *Let $\tilde{Q} = \tilde{K}_3$ be the universal cover of the Kronecker-3 quiver $Q = K_3$. Define $\text{ct}(d)$ as the number of isomorphism classes of cover-thin indecomposable representations over Q with total dimension d . Then $\text{ct}(d)$ is equal to the number of subtrees of $\tilde{Q}$, up to shifting, with d vertices.*

3.2 Counting the Cover-thin Representations of K_3

We will start by constructing a subquiver of the universal covering quiver $\tilde{K}_3$ of the Kronecker-3 quiver K_3 .

Remark 3.2.1. Let $T = (V, E)$ be an undirected connected tree with $n \geq 2$ vertices with the following property: for every vertex $v \in V$, $\deg(v) \leq 3$, i.e, T is a subtree of T_3 , which is the tree with each vertex having degree 3. Define a set $\mathcal{S}$ for an undirected tree $T = (V, E)$ whose elements have the following properties;

- (i) Every element is a tree of the form (V, E, s, t, c) where V is the vertex set of T , E is the edge set of T . $s, t : E \rightarrow V$ are functions that specify a direction for each edge, s is the source function and t is the target function; s and t can only give as result one of the endpoints of a given edge. $c : E \rightarrow \{\alpha, \beta, \gamma\}$ is the coloring function that specifies the color of each edge as α, β, γ .
- (ii) T is the underlying graph of each element.
- (iii) The orientation is bipartite, i.e, each vertex is either a sink or a source.
- (iv) For any distinct edges $e_1, e_2 \in E$, if e_1 and e_2 are incident to each other then $c(e_1) \neq c(e_2)$.

Observe that the set $\mathcal{S}$ contains only all the subquivers of $\tilde{K}_3$ whose underlying graph is T .

Theorem 3.2.2. *For a tree T and the set $\mathcal{S}$ defined as in the remark above, the cardinality of $\mathcal{S}$ is $|\mathcal{S}| = 3 \cdot 2^{e-t}$ where e is the number of edges of T and t is the number of vertices with degree 3.*

Proof. Take a finite tree $T = (V, E)$ with properties given in the Remark 3.2.1. Then we will count how many different colored and directed version we can construct from T considering the properties in the remark above. Take an arbitrary edge $e = u, v \in E$. The direction can be chosen in 2 ways: $u \rightarrow v$ or $v \rightarrow u$. After choosing the direction of the first edge take the set of edges that are incident to e . Then, for edges that $s(e)$ is the common vertex, choose the vertex $s(e)$ as source since the orientation has to be bipartite and the other endpoint has to be target. For edges that $t(e)$ is the common vertex, again because of bipartite orientation now $t(e)$ is sink so choose the vertex $t(e)$ as their target, and the other endpoint is the source. The tree is connected so we apply this method until all edges become directed. This shows that direction of an arbitrary edge in the tree determines the bipartite orientation. So we have two ways to construct bipartite orientation for T . Now, we will count how many ways to color the edges with three colors. Since T is a finite tree, there are at least two leaves. Take an arbitrary leaf $v \in V$, then take the only edge e that one of the endpoints is v . This edge can be colored in three different ways. Then check the degree of the other endpoint of it, say u . If degree of u is one then T has only one edge and two vertices, so we are done, there are three ways to color. If degree of u is two then there is only one edge that is incident to e . There are two ways to choose color for the second edge since incident edges can not have same color. If degree of u is three then there are two more edges that are incident to e . Say e_1 and e_2 . The order of coloring is not important so take e_1 , there are 2 ways to color e_1 , and then there is only one color left for e_2 . So for edge e , we have multiple 3, then for edge e_1 multiple is 2 and for e_2 the multiple is 1. Continue with this method until all the tree is colored. As a result the first vertex we choosed has multiple 3 but then since tree is connected the other edges have multiple 2 or 1. If the degree of vertex is two then the next edge has multiple 2 but if the degree of an edge is 3 then the next two vertex has multiple 2 and 1. For counting the power of 2 first count the number of edges, say e , then for every vertex of degree three we choose one edge with the multiplicity as 1 so take the number of vertices of degree three out, that is $e - t$. and then take the first edge that we choose arbitrarily: $e - t - 1$. We have $3 \cdot 2^{e-t-1}$. We multiply by two, for colors and orientations, so we get: $3 \cdot 2^{e-t}$. $\square$

Definition 3.2.3. Let T be a tree and $\mathcal{S}$ be the set defined as in the remark above, and

let G be the group of automorphisms of T . Define the group action of G on the set $\mathcal{S}$ as follows:

$$\begin{aligned} G \times \mathcal{S} &\longrightarrow \mathcal{S} \\ (g, S) &\longmapsto g \cdot S, \end{aligned}$$

where $g \cdot S = (V', E', s', t', c')$, $V' = \{gv \mid v \in V\}$, $E' = \{ge = \{gu, gv\} \mid \{u, v\} = e \in E\}$. The source, target, and color functions are defined as $s' : E' \rightarrow V'$ where $s'(ge) = g(s(e))$, $t' : E' \rightarrow V'$ where $t'(ge) = g(t(e))$, and $c' : E' \rightarrow \{\alpha, \beta, \gamma\}$ where $c'(ge) = c(e)$ for every $e \in E$.

Every group action is associated to an equivalence relation; in this case, it is defined by saying that $S \sim S'$ if and only if there exists $g \in G$ such that $g \cdot S = S'$. The equivalence class of $S \in \mathcal{S}$ is called the orbit of S , defined as $G_S = \{g \cdot S \mid g \in G\}$. We denote the set of trees fixed by g by $\mathcal{S}^g = \{S \in \mathcal{S} \mid g \cdot S = S\}$. Note that each orbit consists of trees that are isomorphic to each other as directed colored trees. Because the action of the automorphism group G on the set $\mathcal{S}$ is defined to be the isomorphism as directed colored trees that respects coloring and orientation. Thus the number of orbits will be the number of isomorphism classes of 3-edge-colored bipartite trees with underlying graph T . For counting the number of orbits, we will use Burnside's Lemma, which states the number of orbits with the following formula:

$$\frac{1}{|G|} \cdot \sum_{g \in G} |\mathcal{S}^g| \quad (3.11)$$

The next result is about counting the set of trees fixed by g which is $\mathcal{S}^g$.

Proposition 3.2.4. Let the set $\mathcal{S}$ and the group G be defined as in the Remark 3.2.1. For a directed colored tree S in the set $\mathcal{S}$ and g in G , $g \cdot S = S$ implies that $g = 1$.

In order to prove the proposition we first will give some results.

Lemma 3.2.5. Let the set $\mathcal{S}$ and the group G be defined as in the Remark 3.2.1. For a given tree $S = (V, E, s, t, c)$ in the set $\mathcal{S}$ and g in G , if $g \neq 1$, then there exists a vertex $v \in V$ such that $g \cdot v \neq v$.

Proof. Suppose, for the sake of contradiction, that $g \neq 1$ and $g \cdot v = v$ for every vertex $v \in V$. Consider an edge $e = \{u, v\}$. Then, we have $ge = \{gu, gv\} = \{u, v\} = e$, so $ge = e$ for every edge $e \in E$.

For the source, target, and color of an edge $e = \{u, v\}$, we find:

$$s'(g \cdot e) = g \cdot s(e) = s(e),$$

$$t'(g \cdot e) = g \cdot t(e) = t(e),$$

$$c'(g \cdot e) = c(e),$$

for all $e \in E$. This contradicts the assumption that $g \neq 1$, as it implies that there must exist at least one vertex $v \in V$ such that $g \cdot v \neq v$. $\square$

Lemma 3.2.6. *Let the set $\mathcal{S}$ and the group G be defined as in the Remark 3.2.1 and let $S = (V, E, s, t, c)$ be a directed colored tree in the set $\mathcal{S}$. Define $V_0 = \{v \in V \mid g \cdot v = v\} \subseteq V$. Let $v_1 \in V \setminus V_0$, $v_0 \in V_0$ for a given walk $p = (v_1 = w_0 | \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_n | w_{n+1} = v_0)$ where $s(\alpha_i) = w_i$ and $t(\alpha_i) = w_{i+1}$, for $i = 0, 1, \dots, n$. Then there are adjacent vertices $w_j, w_l \in V$ such that $w_j \in V_0$ and $w_l \notin V_0$.*

Proof. Let $T = (V, E)$ be the underlying graph of the tree S . Since T is a tree, there is a unique (undirected) path between v_0 and v_1 . So $p = (v_1 = w_0 | \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_n | w_{n+1} = v_0)$ is that path. First check whether $w_1 \in V_0$. If so take $j = 0$ and $l = 1$, if not check whether $w_2 \in V_0$, if so take $j = 1$ and $l = 2$. If not, check whether $w_3 \in V_0$, so on. So take the first $i \in 0, 1, 2, \dots, n + 1$ that satisfy $w_i \in V_0$. Then choose $j = i - 1$ and $l = i$. $\square$

For a finite tree, the **diameter** D of the tree is defined as the maximum distance between any two vertices within the tree. This maximum distance, D , is achieved specifically between two vertices u and w , such that $D = d(u, w)$. The paths that have length diameter D of the tree are called **diameter paths**. We define the **eccentricity** $e(v)$ of a vertex v as the maximum distance from v to any other vertex in the tree, expressed as $e(v) = \max_{u \in V} d(u, v)$, where V represents the set of all vertices in the tree. This measure allows us to identify the farthest point from a given vertex.

The **radius** r of the tree is determined as the minimum eccentricity among all vertices, formulated as $r = \min_{v \in V} e(v)$. A vertex is termed a **center** if its eccentricity equals the radius of the tree. Consequently, the elements of the set $C = \{v \in V \mid e(v) = r\}$ are considered centers of the tree. In the subsequent result, we will demonstrate that any automorphism of the tree either fixes a center or an edge between two centers.

The following lemma is not giving any new information but we give a proof from our perspective using similar ideas that are already included in the literature.

Lemma 3.2.7. *Every finite graph that is a tree possesses at least one vertex or one edge that remains unchanged under any automorphism of it.*

Proof. Let f be an automorphism of a finite graph T that is a tree. Then by the definition, for any edge e with endpoints u and v , there exists an edge $f(e)$ with endpoints $f(u)$ and $f(v)$, so it preserves adjacency between vertices.

Let $p = (v_1|e_1e_2e_3 \cdots e_{k-1}|v_k)$ be a path in the tree, where $e_i = \{v_i, v_{i+1}\}$ for every $i \in \{1, 2, 3, \dots, k-1\}$.

Because the automorphism f preserves adjacency, if there exists an edge e_i between v_i and v_{i+1} , there also must be an edge $f(e_i)$ between $f(v_i)$ and $f(v_{i+1})$ for all $i \in \{1, 2, \dots, k-1\}$. Thus f maps the path p to another path

$$p' = (f(v_1)|f(e_1)f(e_2) \cdots f(e_{k-1})|f(v_k)).$$

Since every path goes to a path with the same length under the automorphism f , the distance between two vertices, i.e length of the unique path between two vertices must be preserved under the automorphism f . Thus, we have $d(u, v) = d(f(u), f(v))$ for any vertices v, u .

Now, we will show that any center lies on each diameter path. Suppose for the contrary that there exists a center vertex c and a diameter path from x_1 to x_n such that c is not on this particular diameter path.

Observe that the center(s) c_d of the diameter path (i.e, the center of the diameter path

as a subgraph of T) have eccentricity $e(c_d) = k$ in the tree T , assuming that the diameter is $2k - 1$ or $2k$. To show that, remember the eccentricity of a vertex c_d is the maximum of all distances from c_d to other vertices. For sake of contradiction, assuming $e(c_d) \neq k$. Then, we have $e(c_d) > k$, since between c_d and one of the endpoints of the diameter path has distance k and eccentricity defined as maximum distance between c_d and any other vertices in the graph. One of the endpoints of the diameter path say x_1 has distance k with the vertex c_d if $e(c_d) > k$, then there is a vertex v with $d(c_d, v) > k$. Take the path from x_1 to v , going through the path x_1 to c_d and then c_d to v , which is longer than $2k$. This contradicts with the diameter path being $2k - 1$ or $2k$.

Take the unique path p from c to x_1 . Then we can separate p into two paths, $p = p_1p_2$ such that when ordering the vertices in the path from c to x_1 take the first vertex that lie on the diameter path call it x_i with $t(p_1) = s(p_2) = x_i$.

Take also the unique path p' from c to x_n . That path $p' = p'_1p'_2$ can also be separated into two paths such that when ordering the vertices in the path from c to x_n the first vertex that lies on the diameter path is again x_i with $t(p_1) = s(p'_2) = x_i$. Note that if such different vertices lies on the diameter path, then there would be cycles in the tree.

$$\ell(p) + \ell(p') = \ell(p_1) + \ell(p_2) + \ell(p_1) + \ell(p'_2) = 2 \cdot \ell(p_1) + D > D \quad (3.12)$$

where $\ell(p)$ is the length of the path p . the above equality means that one of the paths p or p' is longer than k which implies the eccentricity of center c is bigger than k , i.e, $e(c) > e(c_d)$, contradicting with the assumption that the center has the minimum eccentricity.

If a diameter path P has odd number of vertices then we will take the vertex that cuts the diameter in half. If the diameter path P has even number of vertices this means it has odd number of edges so we will take the endpoints of the edge such that when we remove that edge from the diameter path P , the rest two paths have the same length. And all the other vertices that are on the diameter path have larger eccentricity since they are closer to one of the endpoints. This also shows that a tree has a center or two centers that are adjacent.

Now we are going to show that a center maps to a center under f . First for any vertex v , since the distance is preserved under f and the set of vertices $V = \{f(u) \in V \mid u \in V\}$, we have the equality:

$$e(v) = \max_{u \in V} d(u, v) = \max_{u \in V} d(f(u), f(v)) = \max_{u \in V} d(u, f(v)) \quad (3.13)$$

Then for the radius, we have

$$r = \min_{v \in V} \max_{u \in V} d(u, v) = \min_{v \in V} \max_{u \in V} d(f(u), f(v)) \quad (3.14)$$

The following equation shows that a center maps to a center under f :

$$\begin{aligned} e(c) &= \max_{u \in V} d(u, c) \\ &= \max_{u \in V} d(f(u), f(c)) \\ &= \max_{u \in V} d(u, f(c)) = e(f(c)). \end{aligned}$$

Last equality occurs because $\{f(u) \mid u \in V\} = V$. For the last part of the proof, we will show that the center or the edge between centers must be preserved under the automorphism f . If there is only one center in the tree, then it is trivial; the center is fixed. If there are two centers in the tree, then the centers c_1 and c_2 are adjacent. Therefore, there is an edge between them, call it e_c . There are two cases for the images of the centers under f . In the first case, $f(c_1) = c_1$ and $f(c_2) = c_2$. In the second case, $f(c_1) = c_2$ and $f(c_2) = c_1$. Clearly, in both cases, the edge e_c is fixed. $\square$

Proposition 3.2.8. Let the set $\mathcal{S}$ and the group G be defined as in the remark 3.2.1. For a directed colored tree S in set $\mathcal{S}$ and $g \in G$, $g \cdot S = S$ implies that $g = 1$.

Proof. Suppose for the contrary that there exists $g \in G$ such that $g \neq 1$ and $g \cdot S = S$, where $S = (V, E, s, t, c)$. Let $V_0 = \{v \in V \mid g \cdot v = v\} \subseteq V$.

A tree has either one center or two centers. If there is one center, it must be fixed under every automorphism in G ; if there are two centers, they are adjacent, so the edge between

the two centers must be fixed under any automorphism. Thus, an edge or a vertex is fixed under g . If a vertex v is fixed, then $v \in V_0$, so V_0 is nonempty. If an edge is fixed, say edge $e = \{u, v\}$, then $g \cdot e = e$ implies $g \cdot u = v$ and $g \cdot v = u$, or $g \cdot u = u$ and $g \cdot v = v$.

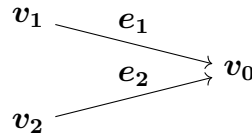
For the first case, we check the directions. If $e : u \rightarrow v$, then we have the following commutative diagram:

$$\begin{array}{ccc} e & \xrightarrow{g} & g \cdot e \\ s \downarrow & & \downarrow s' \\ u & \xrightarrow{g} & A \end{array}$$

By the commutative diagram, $A = s'(g \cdot e) = u$ and $A = g \cdot u = v$, which is a contradiction, as we cannot have both.

Therefore, the second case must be correct, and that means u and v are fixed so that V_0 is a nonempty set. Now, since $g \neq 1$, by Lemma 3.2.5, there exists a vertex $v_1 \in V$ such that v_1 is not fixed under g , which means that $v_1 \in V \setminus V_0$, and there exists a vertex $v_0 \in V_0$ since V_0 is nonempty. In every connected tree, there exists a unique path between any two vertices. Therefore, there is a unique path, not a directed path since T is not a directed tree, $p = (v_1 = w_0 | \alpha_0 \alpha_1 \alpha_2 \cdots \alpha_n | w_{n+1} = v_0)$ where $s(\alpha_i) = w_i$ and $t(\alpha_i) = w_{i+1}$.

By Lemma 3.2.6, we can assume that v_1 and v_0 are endpoints of an edge $e_1 : v_1 \rightarrow v_0$ or $e_1 : v_0 \rightarrow v_1$. Suppose $e_1 : v_1 \rightarrow v_0$. Then let $g \cdot v_1 = v_2 \in V$ and $g \cdot e_1 = e_2$. Since $v_1 \notin V_0$, we have $g \cdot v_1 = v_2 \neq v_1$ and $g \cdot e_1 = e_2 \neq e_1$. Since $v_0 = g \cdot v_0$ and $g \cdot (t(e_1)) = t'(g \cdot e_1)$, we know that $g \cdot (t(e_1)) = g \cdot v_0 = v_0$ and $t'(g \cdot e_1) = t(e_2)$. So we must have $t(e_2) = v_0$. Thus, we have a subgraph of S as follows:



Since $g e_1 = e_2$, we have $c(e_1) = c(g \cdot e_1) = c(e_2)$. Because edges e_1 and e_2 are

incident and cannot have the same color, this leads to a contradiction. We must conclude that $g = 1$. $\square$

Corollary 3.2.9. *The number of orbits of the group action defined in 3.2.3 is given by $\frac{|\mathcal{S}|}{|G|}$.*

Proof. The result follows from the Burnside's Lemma and Proposition 3.2.8. For $g = 1$, $|S^g| = |\mathcal{S}|$ and for $g \neq 1$, $|S^g| = 0$. Thus the number of orbits is equal to

$$\frac{1}{|G|} \cdot \sum_{g \in G} |S^g| = \frac{|\mathcal{S}|}{|G|} \quad (3.15)$$

$\square$

Remark 3.2.10. Every element of the set $\mathcal{S}$ defined on a tree T with n vertices is a subquiver of $\tilde{Q} = \tilde{K}_3$ (the universal covering quiver of the Kronecker-3 quiver $Q = K_3$) with n vertices. Since every automorphism of a directed colored tree can be seen as an automorphism of the underlying graph of that tree, the number of orbits of the action defined in 3.2.3 counts the number of subquivers of $\tilde{Q}$, up to isomorphism which respects coloring, with a fixed underlying graph T . Then for every nonisomorphic subtree T of T_3 with n vertices, count the number of orbits defined in 3.2.3 and call it S_T . The number of subquivers of $\tilde{Q}$, up to isomorphism that respects coloring, with n vertices is the following.

$$\sum_{T \in \mathcal{T}} S_T \quad (3.16)$$

where $\mathcal{T}$ is the set of isomorphism classes of trees with n vertices and each tree in $\mathcal{T}$ is a subtree of the 3-regular tree T_3 , i.e, the tree which every vertex has degree 3. The summation will give the number of isomorphism classes of cover-thin indecomposable representations of Kronecker-3 quiver $K - 3$, with dimension $n \geq 2$, by Theorem 3.1.17.

For the computation above, we use SageMath. There are the inputs and outputs for computing in the case $n = 7$. The second output shows a few elements of the sequence where the i -th element is the number of isomorphism classes of cover-thin indecomposable representations of K_3 with dimension i . The automorphism group of each undirected tree is listed in the outputs, e represents the identity automorphism in those lists.

```

[14]: from sage.graphs.trees import TreeIterator #use the library
      ↪ for trees.
n=7# Specify the number of vertices
#define total_tree_count to be zero each time we change the
      ↪ tree t we will add 1 to this number so it will count the
      ↪ number of isoclasses of undirected trees with n vertices
total_tree_count= 0
#define total_cover_thin_count to be zero and
      ↪ #for every tree we will calculate the number of orbits and
      ↪ then add to total_cover_thin_count
      ↪ #In the end total_cover_thin_count will give the number of
      ↪ isoclasses of cover-thin indecomposables with dimension n.
total_cover_thin_count= 0
#TreeIterator(n) includes all isoclasses of undirected trees
      ↪ with n vertices. So take a tree with n vertices from
      ↪ TreeIterator(n)

for count, t in enumerate(TreeIterator(n), start=1):
    # Check if all vertices have a degree of 3 or less
    if all(t.degree(v) <= 3 for v in t.vertices()):
        # count the number of trees in TreeIterator(n)
        total_tree_count += 1

        # Plot the graph t
        t.plot(title=f"Tree {total_tree_count} with {n}
      ↪ vertices and degree <= 3").show()

        # Count vertices with degree 3 in t
        vertices_degree3 = sum(1 for v in t.vertices() if t.
      ↪ degree(v) == 3)

        # Get automorphism group of t and its order
        symmetries = t.automorphism_group()
        order_of_aut_group= symmetries.order()
        #put all automorphism group elements in a set
        automorphisms = set(symmetries)

        # Count the number of edges
        edge = t.size()

        # Calculate the number of orbits for the current graph
      ↪ t
        number_of_orbits= (3 * 2 ** (edge - vertices_degree3))/
      ↪ order_of_aut_group

        # Sum up number of orbits

```

```

total_cover_thin_count += number_of_orbits

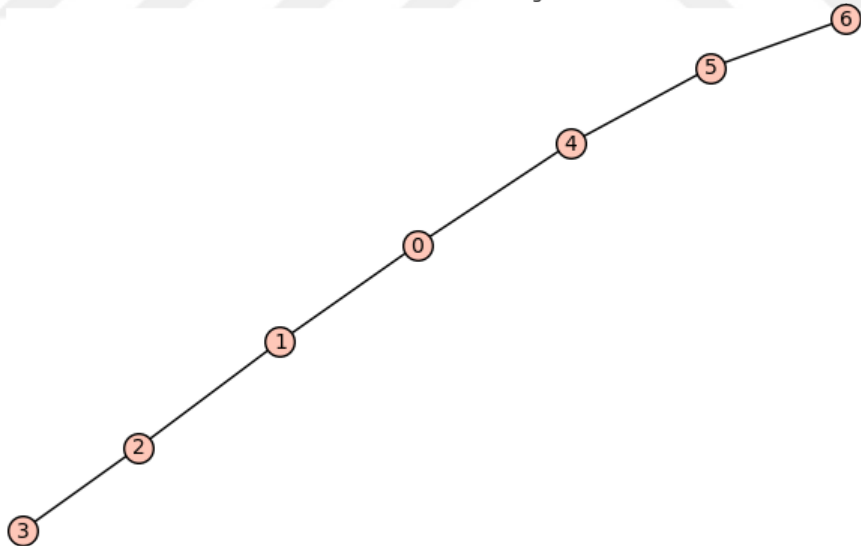
#Print results for the current graph t
print(f"Number of vertices with degree 3 in Tree_
→{total_tree_count}: {vertices_degree3}")
print(f"group automorphism of the Tree_
→{total_tree_count} : {symmetries}")
print(f"set of elements in the Group automorphism of_
→the tree {total_tree_count} : {automorphisms}")
print(f"Number of automorphisms for Tree_
→{total_tree_count} : {order_of_aut_group}")
print(f"Number of edges for Tree {total_tree_count}:_
→{e}")
print(f"the number of orbits for Tree_
→{total_tree_count}: {number_of_orbits}")

_
→print("-----")

# Print the total number of trees and total sum of x values
print(f"Total number of trees with {n } vertices :_
→{total_tree_count}")
print(f"Total number of cover thin with dim {n} :_
→{total_cover_thin_count}")

```

Tree 1 with 7 vertices and degree ≤ 3



Number of vertices with degree 3 in Tree 1: 0
group automorphism of the Tree 1 : Permutation Group with_
→generators
[(1,4)(2,5)(3,6)]

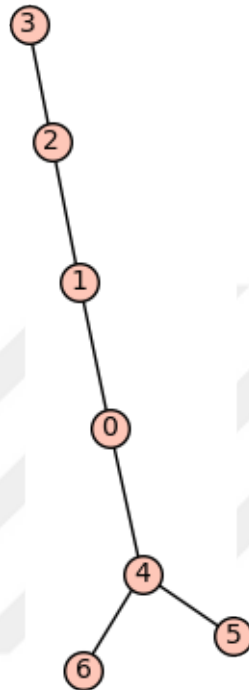
set of elements in the Group automorphism of the tree 1 : $\{(), \begin{smallmatrix} \rightarrow \\ \rightarrow \end{smallmatrix} (1,4)(2,5)(3,6)\}$

Number of automorphisms for Tree 1 : 2

Number of edges for Tree 1: e

the number of orbits for Tree 1: 96

Tree 2 with 7 vertices and degree ≤ 3



Number of vertices with degree 3 in Tree 2: 1

group automorphism of the Tree 2 : Permutation Group with $\begin{smallmatrix} \rightarrow \\ \rightarrow \end{smallmatrix}$

$\rightarrow$ generators $[(5,6)]$

set of elements in the Group automorphism of the tree 2 : $\{(), \begin{smallmatrix} \rightarrow \\ \rightarrow \end{smallmatrix} (5,6)\}$

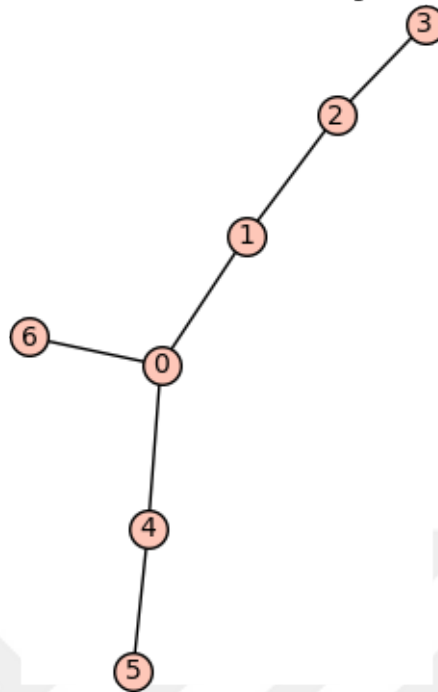
$\rightarrow(5,6)\}$

Number of automorphisms for Tree 2 : 2

Number of edges for Tree 2: e

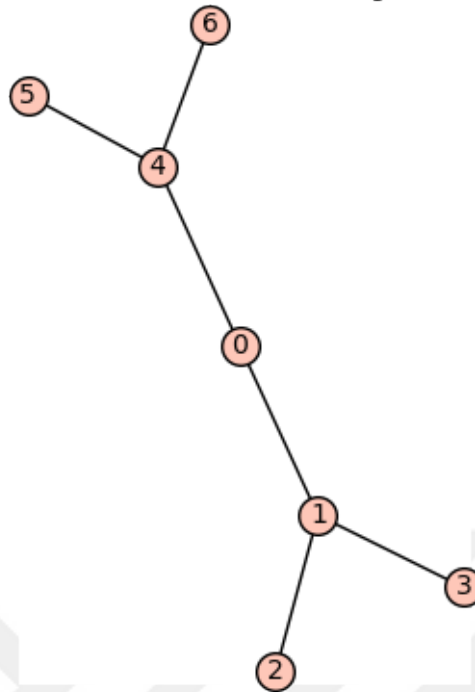
the number of orbits for Tree 2: 48

Tree 3 with 7 vertices and degree ≤ 3



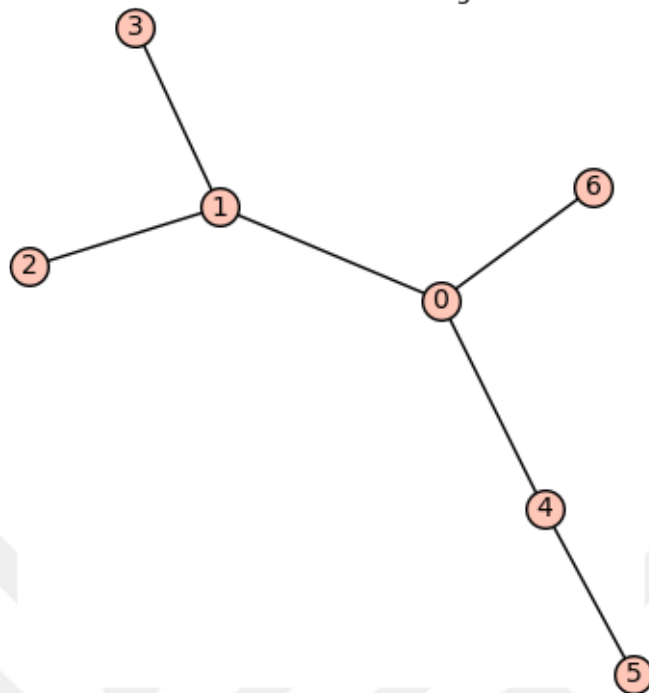
Number of vertices with degree 3 in Tree 3: 1
 group automorphism of the Tree 3 : Permutation Group with
 $\hookrightarrow$ generators $[()]$
 set of elements in the Group automorphism of the tree 3 : $\{()\}$
 Number of automorphisms for Tree 3 : 1
 Number of edges for Tree 3: e
 the number of orbits for Tree 3: 96

Tree 4 with 7 vertices and degree ≤ 3



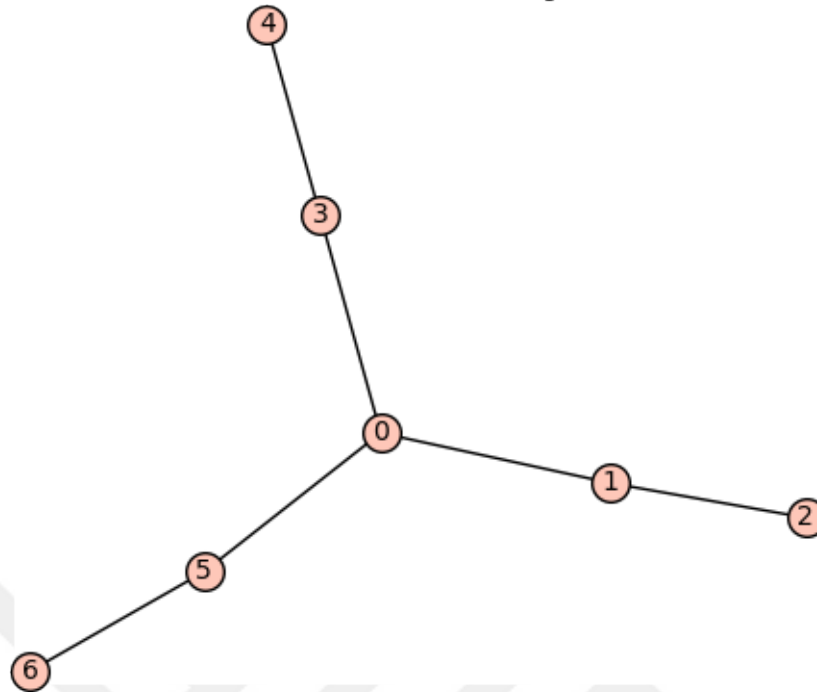
Number of vertices with degree 3 in Tree 4: 2
 group automorphism of the Tree 4 : Permutation Group with
 ↪generators [(5,6),
 (2,3), (1,4)(2,5)(3,6)]
 set of elements in the Group automorphism of the tree 4 :
 ↪{(1,4)(2,5)(3,6),
 (2,3), (2,3)(5,6), (1,4)(2,5,3,6), (1,4)(2,6,3,5), (5,6),
 ↪(1,4)(2,6)(3,5), ()}
 Number of automorphisms for Tree 4 : 8
 Number of edges for Tree 4: e
 the number of orbits for Tree 4: 6

Tree 5 with 7 vertices and degree ≤ 3



Number of vertices with degree 3 in Tree 5: 2
 group automorphism of the Tree 5 : Permutation Group with
 $\hookrightarrow$ generators $[(2,3)]$
 set of elements in the Group automorphism of the tree 5 : $\{(), \hookrightarrow (2,3)\}$
 Number of automorphisms for Tree 5 : 2
 Number of edges for Tree 5: e
 the number of orbits for Tree 5: 24

Tree 6 with 7 vertices and degree ≤ 3



Number of vertices with degree 3 in Tree 6: 1
 group automorphism of the Tree 6 : Permutation Group with
 ↳ generators
 $[(3,5)(4,6), (1,3)(2,4)]$
 set of elements in the Group automorphism of the tree 6 :
 ↳ $\{(1,5,3)(2,6,4),$
 $(1,3)(2,4), (1,3,5)(2,4,6), (3,5)(4,6), (), (1,5)(2,6)\}$
 Number of automorphisms for Tree 6 : 6
 Number of edges for Tree 6: e
 the number of orbits for Tree 6: 16

 Total number of trees with 7 vertices : 6
 Total number of cover thin with dim 7 : 286

```

[15]: from sage.graphs.trees import TreeIterator #use the library
      ↪ for trees.
      #n is the nnumber of vertices for a tree, here we take n from
      ↪ 2 to 17 to list cover-thin indecomposables of total
      ↪ dimension n=1,2,3,...17.
      #range(x,y) gives integers from x to y-1.
      for n in range(2,16):
          #define total_cover_thin_count to be zero and
          #for every tree we will calculate the number of orbits and
          ↪ then add to total_cover_thin_count
          #In the end total_cover_thin_count will give the number of
          ↪ isoclasses of cover-thin indecomposables with dimension n.

          total_cover_thin_count= 0
      #TreeIterator(n) includes all isoclasses of undirected trees
      ↪ with n vertices. So take a tree with n vertices from
      ↪ TreeIterator(n)
          for count, t in enumerate(TreeIterator(n), start=1):
              # Check if all vertices have a degree of 3 or less
              if all(t.degree(v) <= 3 for v in t.vertices()):

                  # Count vertices with degree 3
                  vertices_degree3 = sum(1 for v in t.vertices() if
      ↪ t.degree(v) == 3)

                  # Get automorphism group and its order:

                  #this is the automorphism group of t
                  symmetries = t.automorphism_group()

                  #this is the order of the automorphism group
                  order_of_aut_group= symmetries.order()

                  # Count the number of edges
                  edge = t.size()

                  # Calculate number of orbits for the current graph t
      ↪ and group action defined on t
                  number_of_orbits= (3 * 2 ** (edge -
      ↪ vertices_degree3))/order_of_aut_group

                  # Sum up x values
                  total_cover_thin_count += number_of_orbits

          print(f"for n ={n} : {total_cover_thin_count}")

```

```
for n =2 : 3
for n =3 : 6
for n =4 : 14
for n =5 : 36
for n =6 : 99
for n =7 : 286
for n =8 : 858
for n =9 : 2652
for n =10 : 8398
for n =11 : 27132
for n =12 : 89148
for n =13 : 297160
for n =14 : 1002915
for n =15 : 3421710
```

[]:

3.3 Cover-thin indecomposable representations of K_3 and the Super Ballot Numbers

We realized after computing some elements in the sequence that the numbers match the super ballot numbers. Then, with this inspiration, we obtained the following results.

Theorem 3.3.1. *Call H_n the set of edge-colored subquivers of $\tilde{K}_3$ with n vertices, up to shifting. Call P_n the set of edge-colored subquivers of $\tilde{K}_3$, up to shifting, with n vertices of degree two and no vertex of degree one. There is a bijection between H_n and P_n . Thus, $|H_n| = |P_n|$.*

Proof. Define a function $f : H_n \rightarrow P_n$ as follows. Take a tree S from the set H_n and attach arms of type A_2 to every vertex of S until the degree of all vertices become three. The additional arrows have the appropriate direction that will be coherent with the bipartite orientation of the tree S . The colors of the edges will be one of the missing colors from $\{\alpha, \beta, \gamma\}$. Let $f(S)$ be the subquiver in P_n obtained in this way.

Let S_1 and S_2 be trees in H_n . Then $f(S_1) \cong f(S_2)$ implies $S_1 \cong S_2$. This is because the vertices of S_i become vertices of degree three in $f(S_i)$, and since vertices of degree three should match with vertices of the same degree under isomorphism, we have $S_1 \cong S_2$. Thus, f is well-defined.

For the inverse of f , define $g : P_n \rightarrow H_n$. For a tree T in P_n , delete all the leaves (a leaf is a vertex of degree one) and edges whose endpoints are leaves. The function g is well-defined since when we delete all the leaves from a tree, the result cannot be different trees. It is clear that f and g are inverses of each other, so f and g are bijections.

□

Definition 3.3.2. The sequence called super ballot numbers is defined with the following formula (Gessel, 1992) for all integer $n \geq 0$:

$$B_n = \frac{6 \cdot (2n)!}{n! \cdot (n+2)!} \quad (3.17)$$

A recurrence relation of the super ballot numbers is also given in the following: (Gessel

& Xin, 2005):

$$B_{p+2} = \sum_{k \in \mathbb{N}} 2^{p-2k} \binom{p}{2k} B_k \quad (3.18)$$

where $p \in \mathbb{N}$ the initial values given as $B_0 = 3$ and $B_1 = 2$. We assume that the set of natural number $\mathbb{N}$ includes 0.

For more information about super ballot numbers we refer to (Gessel, 1992, Gessel & Xin, 2005).

Theorem 3.3.3. *The number of isomorphism classes of cover-thin indecomposable representations with dimension n is the n -th super ballot number $B_n = \frac{6 \cdot (2n)!}{n! \cdot (n+2)!}$.*

Proof. We will prove using the recurrence relation of the super ballot numbers given in the definition above. Let us first rewrite the recurrence relation of super ballot numbers by replacing $p + 2 = n$ and $k = t$, where t will represent the number of vertices of degree three. So we will prove the following,

$$B_n = \sum_{t \in \mathbb{N}} 2^{n-2t-2} \binom{n-2}{2t} B_t \quad (3.19)$$

Let H_n be defined as in Theorem 3.3.1. Let $h_n = |H_n|$. Then h_n is the number of isomorphism classes of cover-thin indecomposable representations of dimension n by Theorem 3.1.17. Take $h_0 = 3$ to be the initial condition.

In the case $n = 1$, a tree has only one vertex which can be either a source or a sink that will correspond to the following indecomposable representations.

$$\mathbb{K} \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} 0 \quad 0 \begin{array}{c} \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \mathbb{K}$$

Even though a tree consisting of one vertex may seem like it has only one isomorphism class, which is itself, here the source vertex and sink vertex are chosen to be a subquiver of $\tilde{K}_3$. Since a source vertex and a sink vertex cannot be shifted one to another, we count them as two different trees. So $h_1 = 2$.

For $n \geq 2$, our claim is to prove the equation (3.19) with initial values $h_0 = 3$ and $h_1 = 2$.

By Theorem 3.1.17, we will count the subquivers of $\tilde{K}_3$ with n vertices up to shifting.

Let Z_t be the set of subquivers of $\tilde{K}_3$ with n vertices and t vertices of degree three, where $n \geq 2$ and $t, n \in \mathbb{N}$.

For $t_1 \neq t_2$, where t_1 and t_2 are the numbers of vertices of degree three for the quivers Q_1 and Q_2 respectively, $Q_1 \not\cong Q_2$. Since the vertices must match with the vertices of the same degree under isomorphism, Z_{t_1} and Z_{t_2} are disjoint sets. For the cardinality of the disjoint union of the sets, we have the following equality, and since the Z_t 's are disjoint sets, the union of all will give H_n .

$$h_n = |H_n| = \left| \bigcup_{t \in \mathbb{N}} Z_t \right| = \sum_{t \in \mathbb{N}} |Z_t| \quad (3.20)$$

So we created a sum that runs over the number of vertices of degree three.

For $t = 0$ and $n \geq 2$, we have a quiver of type A_n .

For $t = 0, n \geq 2$, we have a quiver of type A_n which is of the following form.

$$x_1 \text{ --- } x_2 \text{ --- } x_3 \text{ --- } x_4 \text{ - - - - - } x_{n-1} \text{ --- } x_n$$

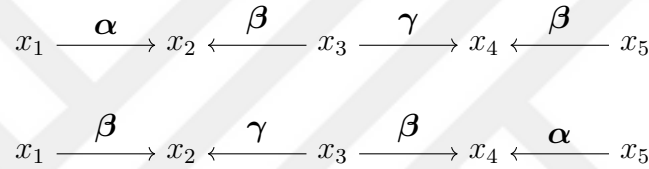
We need to fix a vertex to start, to get rid of the shiftings easily. So, we choose one of the leaves, since there are only two of them in type A_n .

Start from x_1 , and for the first edge, we can choose one of the colors from the set $\{\alpha, \beta, \gamma\}$. For the rest of the edges, there are two colors to choose from, depending on the previous edge. For example, if the first edge is α , then the second edge, which is incident to the first one, can be either β or γ . Suppose that the second edge is γ , then the third one can be α or β . Thus, we have $3 \cdot 2^{n-2}$ different ways to choose the coloring of the edges.

Now, we need to consider if there is anything we counted twice and include bipartite orientations in the counting. Let us count in two cases separately: n is an odd or an even number.

If n is an odd number, then there are two different bipartite orientations. It can start with a source vertex and end with a source vertex, or start with a sink and end with a sink.

So, we multiply by 2, and get $3 \cdot 2^{n-1}$. Then, since x_1 and x_n are both source vertices or both sink vertices, we count the coloring twice for each choice, as in the following example.

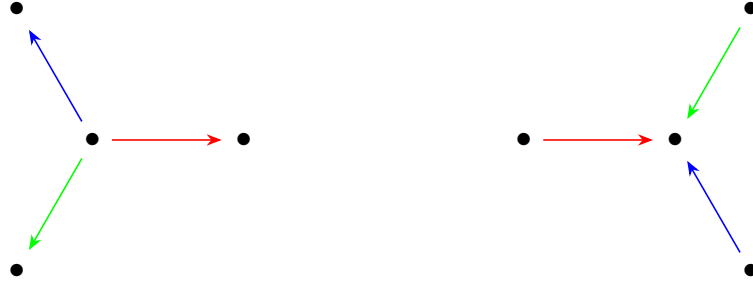


The second one is just the version where we fixed x_n at the start instead of x_1 . One may think of it as a shift from x_1 to x_n with the given coloring sequence $\alpha\beta\gamma\beta$. One may also think that it is an automorphism of the graph A_n and that the two graphs in the figure are in the same orbit under the group action we defined earlier in this section. Since we count everything twice, we divide by 2 and get $3 \cdot 2^{n-2}$.

If n is an even number, then there is only one bipartite orientation that the type A_n has, which will have one source and one sink at the endpoints. When we start from a source, it cannot be the same as starting from a sink, and since there is only one source vertex which has degree one, we are sure that in the formula $3 \cdot 2^{n-2}$ we are not counting anything twice. As a result, no matter if n is an odd or an even number, $Z_0 = 3 \cdot 2^{n-2}$.

Case $t = 1$:

We start with one vertex of degree three. That vertex can be either a source or a sink, as shown in the following figures.

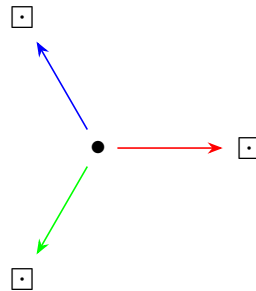


We start with the vertex of degree three. Since there are no other vertices with degree three, we fix the starting vertex, and the remaining $n - 1 - 3 = n - 4$ arrows will be placed one after another like a line quiver.

We will calculate for one of the figures, i.e., for a source vertex of degree three or for a sink vertex of degree three, and then in the end we will multiply the answer by 2.

First, let us count how many different ways we can choose and how many edges to place at the ends of the arrows with colors α, β, γ that we showed in the figure. Since we have not chosen the colors of the $n - 4$ edges yet, we may think of the $n - 4$ edges as identical objects. Additionally, placing them at the ends of the edges that correspond to the colors α, β, γ means placing them into 3 different boxes. So, we assume that there are boxes at the endpoints of the edges, which have an endpoint of degree three.

So, we have the following figure.



The number of ways to put $n - 4$ identical objects into 3 different boxes is

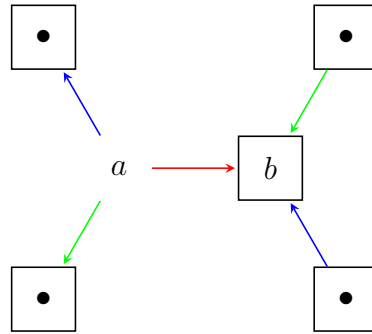
$$\binom{(n-4) + 3 - 1}{3 - 1} = \binom{n-2}{2} \quad (3.21)$$

Now we will count how many ways to color the edges of this subquiver. For the first three edges we already have colors α, β, γ . Each arrow in the boxes is already connected to some edge with color, so each edge have two options to be colored. We get 2^{n-4} . As we describe in the beginning the starting vertex could be a source or a sink so we multiply our result by $2 = h_1$ and get:

$$Z_1 = 2^{n-4} \cdot \binom{n-2}{2} \cdot h_1 \quad (3.22)$$

Case $t = 2$:

We will start with two vertices with degree three. By Theorem 3.3.1, we know that there are three ways to combine two vertices of degree three. One of them is the following:



This is the smallest quiver, in the sense that the number of vertices of the quiver is the smallest possible number with the condition that there are two vertices of degree three, i.e., no vertex has degree two.

Note that there are six vertices in the quiver, because for the first vertex of degree three we start with four vertices, and for each additional vertex of degree three we add two more vertices. So in general, there are $(4 + 2 \cdot (t - 1)) = 2 \cdot t + 2$ vertices in a quiver with t

vertices of degree three and no vertices of degree two.

We are going to start from one vertex, namely a in the figure, to color the edges and distribute the edges to the ends of the edges that start with a vertex of degree three.

First, notice that except for the starting vertex a , we can put edges to all the vertices that are left in the graph since all the vertices are the endpoints of some vertex of degree three. The reason why we do not include one vertex and we call it the starting vertex is the following: If we put some edges into the box at vertex a and the box at vertex b , then by including the arrow between them, we will have a bigger box that includes both. So instead, we can fix the vertex between them and put the box at only one of them. In this case, b is the box that we will put edges into.

Now, we observe that in general there are $(2 \cdot t + 2) - 1 = 2 \cdot t + 1$ vertices which will be considered to have a box.

There are five boxes and $n - 1 - 5 = n - 6$ edges in the case when $t = 2$. There are

$$\binom{n - 6 + 5 - 1}{5 - 1} = \binom{n - 2}{4} \quad (3.23)$$

ways to distribute $n - 6$ identical edges into five different boxes.

In general, there are $(2t + 1)$ boxes and $n - 1 - (2t + 1) = n - 2 - 2t$ edges. So, in general, for $t \in \mathbb{N}$, there are

$$\binom{(n - 2t - 2) + (2t + 1) - 1}{(2t + 1) - 1} = \binom{n - 2}{2t} \quad (3.24)$$

ways to distribute the edges into boxes.

Now we will count the number of ways to coloring the edges. For edges out of the boxes we already know how to color them, for the starting vertex a , three incident edges have colors α, β, γ , and for the rest of the edges we know that one edge will determine the colors of the two other incident edges for a vertex of degree three. So we will paint the $n - 6$ edges in a $2^{(n-6)}$ different ways, for the $t = 2$ case. In general, there are $2^{(n-2t-2)}$

ways to paint the $(n - 2t - 2)$ edges.

By using Theorem 3.3.1, and the results we explained above, for the case $t = 2$, we have the following result.

$$Z_2 = 2^{(n-6)} \cdot \binom{n-2}{2t} \cdot h_2 \quad (3.25)$$

Using our general results for a $t \in \mathbb{N}$ that we explained above, we know that there are h_t ways to combine t many vertices of degree three. And then there are $\binom{n-2}{2t}$ many ways to distribute the edges into the boxes to shape all the quiver. Lastly, to color the edges there are $2^{(n-2t-2)}$ many ways. As a result we get the following equation.

$$Z_t = 2^{(n-2t-2)} \cdot \binom{n-2}{2t} \cdot h_t \quad (3.26)$$

Using the equation (3.20), we have the following recurrence relation, which is the recurrence relation of the super ballot numbers.

$$h_n = \sum_{t \in \mathbb{N}} 2^{(n-2t-2)} \cdot \binom{n-2}{2t} \cdot h_t \quad (3.27)$$

□

CHAPTER FOUR

CONCLUSION

This thesis focused on counting the number of isomorphism classes of cover-thin indecomposable modules over the Kronecker-3 quiver. The principal achievement of this study was the identification of a novel correspondence between the number of isomorphism classes of n -dimensional cover-thin indecomposable representations and the n -th super ballot numbers.

Initially, our research focused on understanding the structure and properties of thin representations within the framework of trees, as these are foundational elements in the theory of quiver representations. A critical step was examining the relationship facilitated by the push-down functor, which preserves indecomposability. Our findings, supported by the main Theorem (3.1.17), conclusively determined that the number of isomorphism classes of cover-thin indecomposable representations is effectively quantified by the count of n -vertex subquivers within the universal covering quiver of Kronecker-3, up to shifting.

To count these subquivers efficiently, we developed a method using group actions and implemented it with SageMath. The outputs suggested a surprising match with the sequence of super ballot numbers, leading to a proof that these numbers effectively enumerate the isomorphism classes of the representations studied.

Looking ahead, investigating the number of isomorphism classes of d -dimensional cover-thin indecomposable representations of the Kronecker- n quiver is another idea to continue this research. Since the universal covering quiver of Kronecker- n is expected to have similar properties, the ideas used in the SageMath computations might be generalized to Kronecker- n for $n \geq 2$. Additionally, using covering theory and similar combinatorial methods, it is possible to research the number of cover-thin indecomposable representations of other types of quivers.

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