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Inverse Conformable Sturm-Liouville Problem

Auwalu SA'IDU

PhD Thesis

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INVERSE CONFORMABLE STURM-LIOUVILLE PROBLEM

Author

Auwalu SA'IDU

Supervisor

Prof. Dr. Hikmet KEMALOĞLU

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Author:	Auwalu SA'IDU
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This thesis, which was prepared according to the thesis writing rules of the Graduate School of Natural and Applied Sciences, Fırat University, was evaluated by the committee members who have signed the following signatures and accepted unanimously as a result of the defense exam made open to the academic audience.

		Signature
Supervisor:	Prof. Dr. Hikmet KEMALOĞLU Firat Üniversity, Faculty of Science	Approved
Chair:	Prof. Dr. Hasan BULUT Firat Üniversity, Faculty of Science	Approved
Member:	Prof. Dr. Niyazi BULUT Firat Üniversity, Faculty of Science	Approved
Member:	Prof. Dr. Ali AKGÜL Siirt Üniversity, Faculty of Science	Approved
Member:	Assoc. Prof. Sertaç GÖKTAŞ Mersin Üniversity, Faculty of Science	Approved

This thesis was approved at the meeting of the Administrative Board of the Graduate School on .... / .... / 2025

Prof. Dr. Burhan ERGEN
Director of the Graduate School

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I hereby declare that I wrote this Doctoral Thesis titled "Inverse Conformable Sturm-Liouville Problem" in consistent with the thesis writing guide of the Graduate School of Natural and Applied Sciences, Firat University. I also declare that all information in it is correct, that I acted according to scientific ethics in producing and presenting the findings, cited all the references I used, and expressed all institutions or organizations or persons who supported the thesis financially. I have never used the data and the information I provide here in order to get a degree in any way.

03.02.2025

Auwalu SA'IDU

Preface

Inverse Strum-Liouville Problem is one of the current field of researches due to its vital influences. Also, currently studies on this problem are extending to the concept of non-integer value derivative which will significantly add to its importance as it comprised the derivative of fractional order. As the conformable derivative approach is new, our research focused on analyzing and extending the direct and the inverse nodal Strum-Liouville Problem (with / without delay) to the non-integer value derivative phenomena by using this new derivative approach. Our target is successfully achieved as it can been seen presented in this write up.

I would like to acknowledge that, I am grateful to the Almighty Allah for establishing me in life to the completion of my PhD programme (thesis) may his continuous blessings be upon his Noble Prophet, Muhammmad SAW and to those that follow him in obedience to the last hour. I would like to express my special thanks to my supervisor Prof.Dr. Hikmet KEMALOĞLU whom without his guidance and persistent help this work would not have been possible and also to all my lecturers in the department. I am indebted to my parents and family and all my dearest and nearest for their continuous supports in prayers and good wishes.

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Özet

Uyumlu Türevli Ters Sturm-Liouville Problemi

Auwalu SA'IDU

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Bu çalışmada, Sturm-Liouville probleminden (SLP) ile ilgili olup; uyumlu türev yardımıyla hem direkt hem de teras problemin çözümü incelenmiştir.

İlk olarak gerek mühendislikte gerek ise diğer alanlarda önemli bir yere sahip olan SLP probleminin temel özellikleri verilmiş, türevin sıfır ve bir değerleri arasındaki kesirli değerleri kullanılarak dönüşüm operatörünün varlığını ispatını verdik. Burada Hiperbolik tip kısmi türevli bir diferensiyel denklem ve bunula ilgili sınır şartları şeklinde elde edilen problem için bir Fredholm integral denklemi elde ettik. Literatürde ters nodal problem olarak bilinen problem için farklı sınır şartlarında özdeğerler, özfonksiyonlar ve nodal parametreler için asimptotik formüller verdik. Ve bunlar yardımıyla potansiyel fonksiyon için formüller verdik.

Daha sonra bazı ek koşullar altında potansiyel fonksiyon için bazı teklik teoremlerini ispat edip özel durumda bu sonuçlardan yola çıkarak Ambarzumyan teoremine ulaştık. Son olarak gecikmeli SLP için ters nodal problemin çözümünü verip problemin dengeli olduğunu gösterdik. Genel olarak elde ettiğimiz sonuçların $\beta = 1$ için klasik sonuçlar ile aynı olduğunu gösterdik.

Anahtar Kelimeler: Sturm Liouville Denklemi, Sturm Liouville problemi, Uyumlu türev, Düğüm Noktaları, Düğüm Uzunlukları, Potansiyel Fonksiyonu, Ters Problem, Gecikme Sabiti.

Abstract

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This research work talks about Sturm-Liouville Problem (SLP); the direct and the inverse nodal forms of the problem (with / without delay) using a newly established non-integer value derivative called the *conformable derivative.* At first, looking in to the vitality of the SLP in the models of the real life problems addressing scientific, engineering, economic, etc, issues and the fact that most of these problems are better described using non-integer value derivative, we considered the conformable derivative method and proved the transformation operator for the SLP using this new differentiation method on a classical Sturm-Liouville Operator. There, we obtained a Hyperbolic partial differential equation and some suitable conditions for nucleus function $N(x,\tau)$, then we finally obtained a Fredholm integral equation and the proof is validated by taking $\beta = 1$ which returns the original/classical results. In addition, we gave various basic definitions, properties, lemmas and theorems used in this thesis, and they are mostly under this new derivative.

Next, we considered the general conformable SLP and then established the spectral properties; the eigenvalues, the eigenfunctions, the nodal points, the nodal lengths, for some special boundary conditions and after that we calculated the potential functions and proved its uniqueness, in the direct case, using these spectral parameters as the direct and inverse of the conformable SLP. We considered and proved the Ambarzumyan's theorem for the conformable derivative SLP with additional condition.

Lastly, we extended the conformable SLP and its inverse problems to the constant delay problem which is also new in the literature. There we similarly obtained the spectral properties for the problem in two different set of boundary conditions and we also expressed some important results like the proof for the specification of the spectrum in one case and proving the stability of the solution in the other case.

New results, as an extension of the classical SLP (with and without delay) were established in this work in order to extend the concepts of the SLP to non-integer derivative phenomena, and the corresponding results in the classical cases of the problems discussed in this research documents can be obtained at $\beta = 1$ and also the delay concept can be relax by taking $\nu = 0$.

Keywords: Sturm Liouville Equation, Sturm Liouville problem, Conformable derivative, Nodal Points, Nodal Lengths, Potential Function, Inverse Problem, Delay Constant.

Symbols and Abbreviations

Symbols

$\langle\rangle$	:	Inner product
$L^p_\beta[a,b]$	:	Square fractional integrable functions space on $[a,b]$
μ_n	:	n. Eigenvalue
L_β	:	Sturm-Liouville operator
$\frac{d^\beta y}{dx^\beta}$	:	Fractional differential operator
${}_{RL}D^\beta_a$	:	Riemann-Liouville fractional differential operator
${}_{C}D^\beta_a$	:	Caputo fractional differential operator
D^β	:	Conformable differential operator
${}_{M}D^{\alpha,\beta}_a$	:	M-Derivative fractional differential operator
$N(x,x)$	:	Nucleus function
$L^2_\beta[a,b]$	:	Conformable Space of Real-valued squarely integrable functions
$C_\beta[a,b]$	:	Conformable Space of continuous, real or complex-valued functions
$q(x)$	:	Potential function for Sturm-Liouville operator
O	:	Limited values
o	:	Infinitesimal values
H,h	:	Impedance constants at boundary conditions
$y_n(x,\mu_n)$	:	y_n eigenfunction corresponding to μ_n eigenvalue
$x^{(n)}_j$	:	j nodal point corresponding to n
$l^{(n)}_j$	:	j nodal length corresponding to n

Abbreviations

SLP	:	Strum-Liouville Problem
SLT	:	Strum-Liouville Theory

1. Introduction

Sturm-Liouville theory begins with Sturm's original work between 1829 and 1836. Later in 1837, Sturm and Liouville published their very important work in a short note in the journal "Journal de Mathematique", they considered the boundary value problem for the differential equation below,

$$-y'' + q(x)y = \mu y \quad for \quad 0 \le x \le 1, \tag{1. 1}$$

with the boundary conditions that

$$y(0)\cos\lambda + y'(0)\sin\lambda = 0, \quad and \;\; y(1)\cos\alpha + y'(1)\sin\alpha = 0,$$

where $q(x)$ is continuous on $(0,1)$ and μ is the spectral parameter. From the solution of the boundary value problem, the parameter μ is called eigenvalue and the nontrivial solutions for (1. 1) is called eigenfunctions of the problem. The set of all eigenvalues of this problem is called the spectrum of the boundary value problem [1]. Sturm-Liouville problems (SLPs) constitute a fundamental pillar of mathematical analysis, particularly in the context of differential equations and their applications in physics and engineering. These issues are significant in light of their inherent occurrence in the separation of variables for partial differential equations, including the heat equation, wave equation, and Schrödinger equation. SLPs are of the form of eigenvalue problems, wherein the eigenvalues μ correspond to specific solutions, designated as eigenfunctions $y(x)$. The solutions of SLPs exhibit a number of noteworthy characteristics. Orthogonality is defined and the set of eigenfunctions is complete. The eigenfunctions constitute a complete set within the pertinent function space, thereby enabling expansions of other functions in terms of these eigenfunctions (e.g. Fourier series). In the case of a discrete spectrum, the spectrum of μ is ordered and bounded below under regular conditions. These properties render the Sturm-Liouville Theory (SLT) a powerful tool in mathematical physics, signal processing, and numerical methods.

Besides the studies on the SLPs there aroused the studies analyzing the concepts of the inverse SLPs. This extensions of the SLT delve into reconstructing the underlying differential operator from spectral data, which has applications in fields like geophysics, quantum mechanics, and medical imaging. The first results of inverse SLP may be considered as Ambarzumyan's theorem [2]. It says that if the spectrum of SLP with Neumann boundary conditions is $\{n^2\}$, $n \ge 0$, then the potential function $q(x) = 0$. The solution of this type of classical SLPs are possible in many ways and many results have been obtained by many researchers as in these references and there in [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 18, 34, 35, 36].

Fractional calculus is a fascinating branch of mathematics that generalises classical calculus by extending the concept of derivatives and integrals to non-integer (fractional) orders. In contrast to traditional calculus, which deals with integer-order differentiation and integration, fractional calculus allows for operations of arbitrary order giving the chance of taking the β-order derivative of a function, for $0 < \beta \leq 1$, this concept was first studied in 1695 when Leibniz asked about the derivative of order $\frac{1}{2}$ in a letter to L'Hospital. The fundamental idea of fractional calculus can be traced back to the generalisation of repeated integration and differentiation. For instance, the n-th derivative of a function can be expressed using factorials. This concept was extended to fractional orders using the Gamma function. A new differentiation method called "*Conformable Derivative*" was recently developed by Khalil et al. [37] and it satisfies the majority properties of differential calculus that were not satisfied by the earlier fractional derivative approaches, this shows that conformable derivative has some advantages. This conformable derivative, introduced to address limitations of earlier fractional derivatives, exhibits a number of properties similar to those of classical derivatives, thereby making it a useful tool for formulating fractional differential equations. Many researchers have studied, defended, improved, and successfully used the Conformable Derivative approach in many different fields, our work [38] gives our successful analysis on Conformable Sturm-Liouville operator, more details on these can be found in the references [39, 40, 41, 37, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54], it has been found to be an effective derivative approach as such in this research work we considered this new differentiation method and addressed the spectral problem on SLT. The models of the phenomena that can be describe by the SLPs are best described by fractional calculus as it involved the derivative of fractional order. Afterwards, researches on inverse and inverse nodal SLP are extended to the concept of fractional derivatives and it is now a current research field as detailed in the references [55, 56, 57, 5, 7, 58, 59, 60]. The authors in [15] introduced a formula for the potential function in inverse nodal problem, in our works [59] and [61] we treated the direct and inverse nodal problem of the SLP with eigenparameter in the boundary condition using conformable derivative approach and proved the uniqueness of the potential function, we also provided more details about it in this write up. Also, there are some results on fractional SLPs in these references and there in [62, 56, 63, 64, 57, 65, 58, 38, 66, 67, 68, 69] and many of them used the conformable derivative.

To further our studies, we extended it to the concept of direct and inverse nodal conformable SLP with delay which has the form (the classical)

$$-y'' + q(x)y(x-\nu) = \mu y \quad for \quad 0 \leq x \leq 1,$$

with any appropriate boundary conditions.

In the inverse nodal SLP with delay the goal is to determine the coefficients of the Sturm-Liouville operator and the potential function $q(x)$ or the delay parameter ν based on the given spectral parameters (or nodal information). The study of SLP with delay has applications in various fields, such as: Partial differential equations with delay terms, Quantum mechanics and quantum field theory, Control theory and optimization, Biological and engineering systems with time delays etc, that is why there is an increasing interest in the spectral analysis of such problems, more details on these can be found in [70, 71, 72, 73, 74, 75, 76]. Some researchers constructed new approaches for the solutions of the classical forms of this concept. The SLP with a large constant delay, $\nu \in \left[\frac{\pi}{2}, \pi\right)$, was taken into consideration by the authors in [77]. They developed an efficient algorithm to solve the problem, and the authors in [78] also examined the nonlinear inverse problem and determined the properties of their spectral characteristics. The case with a zero starting function, where the potential $q(x)$ is assumed to be null on the appropriate subinterval, is the focus of much inverse SLP research. The authors of [79], however, disregarded this presumption in favor of a continuously matching initial function, which causes the equation to gain a new term with a frozen argument. They successfully addressed the issue and demonstrated its spectral properties. For further insights into this topic, please refer to [70, 73, 74]. In a note worthy contribution, the authors of [70] presented an intriguing result pertaining to inverse SLP with a constant delay under a non-self-adjoint operator with a mixed boundary condition. They elucidated the spectral properties of the eigenvalues obtained.

Studies and results on non-integer value derivative inverse SLP with delay are scarce, due to that, in our research we established the direct and inverse nodal conformable SLP with delay as it can be seen in our works [75, 80]. We obtained the result under two sets of boundary conditions; the mixed and Neumann-Dirichlet boundary conditions, using conformable derivative approach, we expressed the corresponding spectral properties and we used the Lipschitz stability approach and demonstrated the stability of the solutions. The corresponding classical results can be retrieved at $\beta = 1$ and the delay concept can be reversed at $\nu = 0$. The problem of the delay by a constant, which involves changing the problem by the degree of the considered constant value, is contained in this supplement to the literature on the idea of the inverse SLP. In addition to the given information and the reviews about this research work we would like to give some details about the most important terms which are the basic and bases of many of the concerned preliminaries. Notably, all of the definitions, theorems, and lemmas that were used, together with their classical and conformable derivative proofs, came from the references mentioned, particularly [39, 40, 44, 45, 56, 64, 70].

1.1 Sturm-Liouville Operator

In this section, we discussed about the concept of operator which is highly considered due to its vitality in almost all parts of mathematics, this leads us to investigate and present the possibility of providing transformation operator for the SLP using the conformable derivative in this work. We also gave some main theorems and properties of SLP under this new differentiation approach with their proofs.

1.1.1 Transformation Operator

To define the transformation operator let us consider a topological linear space E such that $E_1, E_2 \subset E$,(closed subspaces) and A and B be two linear (may be continuous) operators.

Definition 1.1.1. *Suppose X is a linear invertible operator defined in the entire space E and acting from E_1 to E_2. If X meets the following requirements, it can be referred to as a transformation operator for a pair of operators A and B:*
1. The operator and its inverse, X and X^{-1}, are continuous in space E.
2. The form of the operator equation is

$$AX = XB \text{ or } A = XBX^{-1}.$$

Definition 1.1.2. *[81] Let X be a linear space. A norm on X is real-valued function $\|.\|$ on X defined by $\|.\| : X \to \mathbb{R}$ satisfying the following conditions*

(N1) $\|x\| \geq 0,\ x \geq 0,\ x \in X$
(N2) $\|x\| = 0, \Leftrightarrow x = 0$
(N3) $\|\beta x\| = |\alpha| \|x\|\ , \forall x \in X$ *and for all scalars* α .
(N4) $\|x+y\| \ \leq \|x\| + \|y\|, \forall x, y \in X$ *Then* $(X,\ \|.\|)$ *is called Normed space.*

Definition 1.1.3. *[81] Let X and Y be two normed linear space over the same field of scalars. A transformation defined by*

$$T : X \to Y$$

is said to be linear if ***a)*** $T(x+y) = T(x) + T(y)$.
b) $T(\alpha x) \ = \alpha T(x)$ *for all* $x, y \in X$ *and for all scalars* α.

A linear transformation $T : X \to Y$ is said to be continuous at $x_0 \in X$ if $x_n \to x_0$ then $T(x_n) \to T(x_0)$
T is said to be continuous if T is continuous at each point of X. Note that T is continuous if and only $x_n \to\ x \implies T(x_n) \to\ T(x)$.

Definition 1.1.4. *A linear transformation T from a normed space X to another normed space Y is said to be a bounded if there exists a real number $c > 0$, such that $\|T(x)\| \leq c\|x\|$ for all $x \in X$.*

Definition 1.1.5. *[81] Let X be a complex linear space. An inner-product on X is a function $\langle\rangle :\ X \times X \to \mathbb{C}$ which satisfies the following condition:*

1) $\langle x, x\rangle \ \geq\ 0,\ \langle x, x\rangle \ = 0 \iff x = 0.$
2) $\langle x, y+z\rangle \ =\ \langle x, y\rangle \ +\ \langle y, z\rangle$.
3) $\langle x, y\rangle \ =\ \overline{\langle y, x\rangle}$ *, the complex conjugate of* $\langle x, y\rangle$.
4) $\langle \alpha x, y\rangle \ =\ \alpha\ \langle x, y\rangle$.

Here $x, y, z \in X$ and α is a real number. An inner-product space is a linear space with an inner-product on it [37]. We have a relation between inner product and norm as $\|x\| = \sqrt{\langle x, x \rangle}$ A complete inner product space is said to be Hilbert space.

Lemma 1.1.1. *The space $L^p_\beta(0,a)$ associated with the norm function*

$$\|f\|_{p,\beta} = \left(\int_0^a |f(x)|^p d_\beta x \right)^{\frac{1}{p}},$$

in a Banach space.

Morever if $p = 2$, then $L^2_\beta(0,a)$ associated with the inner product $\langle f, g \rangle = \int_0^a f.g d_\beta x)$ in a Hilbert space.

1.1.2 Fractional Differential Operator

Differential operator, symbolized as D, is a function which mapped functions into their derivatives. The same is extended to the concept of fractional derivatives and all the appropriate fractional differentiation approaches can be considered as Fractional Differential Operators. We can then say, a fractional operator is a function which mapped fractional differentiable functions into their fractional derivatives. The fractional differential operators discussed here are the Riemann-Liouville, Caputo and conformable simbolized as ${}_{RL}D^\beta_a$, ${}_{C}D^\beta_a$ and D^β and each is defined and characterized based on some properties. It is based on such properties that an operator is said to be valid, good or inappropriate. The shortcomings of the operators discussed here were given below and it can be deduced that another operator can be derived as an improvement of the other.

1.1.3 Eigenvalue and Eigenfucntion

The concepts of Eigenvalue and Eigenfucntion, as explained in detail in [82], formed the basis in the SLP, as such we briefly discussed these concepts as follows.

Definition 1.1.6. *An eigenvalue of a square matrix $A = (\beta_{jk})$ is a μ such that*

$$Ax = \mu x \tag{1. 2}$$

has a solution $x \neq 0$.

A is known as the eigenvector that corresponds to that eigenvalue μ.

Definition 1.1.7. *Let H be a Hilbert space and $T : H \to H$ be a bounded linear operator on H into itself. Then the unique operator T^* defined by*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle, \forall x, y \in H$$

is called the adjoint of T.

Definition 1.1.8. *A bounded linear operator T on a Hilbert space H is said to be self-adjoint if $T^* = T$. If T is self-adjoint, then*

$$\langle Tx,y \rangle = \langle x,Ty \rangle, \forall x,y \in H.$$

1.2 Big O-notation

Big O notation is a method for characterizing functions based on their growth rates. Different functions that have the same asymptotic growth rate can be described using the same O notation. A description of a function in big O notation often provides an upper bound on the function's development rate. The little o notation, like the big O notation, is used to define several types of asymptotic growth rate constraints. These two notations typically exhibit similar characteristics [83].

1.2.1 Properties of big O notation

The following properties apply to the big O notation, that is for the functions $g(s)$, $f(s)$ and $h(s)$:

1. Reflexivity: For any function $g(s)$, $g(s) = O(g(s))$.
2. Transitivity: Given that $g(s) = O(f(s))$, and $f(s) = O(h(s))$ then $g(s) = O(h(s))$.
3. The fixed factor: If $g(s) = O(f(s))$, then $kg(s) = O(f(s))$ for every constant $k > 0$.
4. Sum Rule: $g(s) + h(s) = O(f(s))$ if $g(s) = O(f(s))$ and $h(s) = O(f(s))$.
5. Product Rule: $g(s) * h(s) = O(f(s) * k(s))$ if $g(s) = O(f(s))$ and $h(s) = O(k(s))$.
6. The Composition Rule: $f(g(s)) = O(h(s))$ if $f(s) = O(g(s))$ and $g(s) = O(h(s))$.
7. A few further properties included; for $k \neq 0$, $O(k + h(s)) = O(h(s))$, $O(kh(s)) = O(h(s))$ and $g(s) = O(g(s), f(s) = O(f(s)), then O(g(s) + f(s)) = O(max\{g(s), f(s)\})$.

1.3 Sturm-Liouville Problem

Boundary-value issues known as SLPs naturally occur when specific partial differential equation problems are solved using the separation of variables method, which is widely used in literature.

The SLP can be solved in two ways: Finding the spectral data after the potential function is provided which is the direct problem. In the second, one is asked to determine the potential function given all spectral parameters, such as eigenvalues, eigenfunctions, etc. These information and the definitions, theorem and the proofs can be found in the references [82, 84, 85].

Theorem 1.3.1. *SLP has the general form in the classical case as;*

$$(p(x)y')' + q(x)y = -\mu w(x)y \; for \;\; a < x < b \tag{1. 3}$$

with the two sets of boundary conditions that;

I the first boundary conditions

$$y(a)\cos\lambda + y'(a)\sin\lambda = 0, \;\; and \;\; y(b)\cos\alpha + y'(b)\sin\alpha = 0, \tag{1. 4}$$

II the second boundary conditions

$$y(a) = y(b), \;\; and \;\; y'(a) = y'(b),$$

where the functions $p(x), p(x)', q(x)$ and $w(x)$ (weight function) are continuous on the interval (a,b). $p(x) > 0$ and $w(x) > 0$ on $[a,b]$ and μ is a spectral parameter.

By introducing L as the Sturm-Liouville operator to (1. 3) we can represent the SLP as

$$L(y) = -\mu w(x)y,$$

where

$$L(y) = (p(x)y')' + q(x)y.$$

The first set of the boundary conditions of the SLP can be reduced to

$$y'(0) - hy(0) = 0, \;\; and \;\; y\prime(\pi) + Hy(\pi) = 0,$$

by taking the impedance constant $-h = \frac{\cos\beta}{\sin\beta}$ and $H = \frac{\cos\alpha}{\sin\alpha}$ and changing the interval to $(0,\pi)$.

Now, if we consider $Q(x,\mu)$ as the solution of (1. 3) and $Q(0,\mu) = 1$ and $Q\prime(0,\mu) = h$ then $Q(x,\mu)$ can be expressed as

$$Q(x,\mu) = \cos(\sqrt{\mu}x) + \frac{h}{\sqrt{\mu}}\sin\sqrt{\mu}x + \frac{1}{\sqrt{\mu}}\int_0^x \sin(\sqrt{\mu}(x-t))q(t)Q(t,\mu)dt$$

while the condition $Q^*(0,\mu) = 0$ and $Q^*\prime(0,\mu) = 1$ gives the solution of (1. 3) as $Q^*(x,\mu)$ and it can be expressed as

$$Q^*(x,\mu) = \frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}x + \frac{1}{\sqrt{\mu}}\int_0^x \sin(\sqrt{\mu}(x-t))q(t)Q^*(t,\mu)dt$$

1.4 Types of the SLP

There are three types of SLP (1. 3) which are as follows;

1. If $p(x) > 0$ and $w(x) > 0$ and the interval is finite $[a,b]$ then the SLP is said to be regular.

2. If $p(x)$ or $w(x)$ is zero for some x in the interval or the domain is $[0,\infty)$ then the problem is singular.

3. if $p(x) > 0, \;\; w(x) > 0$ and the functions $p(x)$, $q(x)$ and $w(x)$ are continuous functions on $[a,b]$ along with the following Boundary Conditions: $y(a) = y(b)$ and $y(a) = y(b)$ then the SLP is called periodic.

It should be noted that the most common types of SLP are regular and periodic as such most of the studies are on them.

1.5 Reducing Second Order Differential Equations to SLP

It should be remarked that the more general second order differential equations

$$-\frac{d}{dx}\{(p(x)y')\} + l(x)y = \mu w(x)y, \tag{1. 5}$$

with the functions $p(x)$ and $w(x)$ as stated in the section above, with set of the condition (1. 4) can be reduced to SLP. That is, let the function $p(x)$ be a continuous first order differentiable function and the function $p(x)w(x)$ be continuous second order differentiable function, then (1. 5) can be rewritten as

$$-p(x)y'' - p(x)'y' + l(x)y - \mu w(x)y = 0, \tag{1. 6}$$

and dividing both sides of (1. 6) by $p(x)$ and multiplying through by the integrating factor $I(x)$ we have

$$-I(x)y'' - I(x)\frac{p(x)'}{p(x)}y' + I(x)\frac{l(x)}{p(x)}y - I(x)\mu\frac{w(x)}{p(x)}y = 0,$$

that is

$$-[I(x)y']' + I(x)\frac{l(x)}{p(x)}y - I(x)\mu\frac{w(x)}{p(x)}y = 0, \tag{1. 7}$$

where $I(x) = \exp\int\frac{p(x)'}{p(x)}dx$. Clearly, (1. 7) take the form that

$$-[I(x)y']' + \frac{l(x)}{p(x)}I(x)y = w_o(x)\mu y,$$

which is a form of SLP with $q = \frac{l(x)}{p(x)}I(x)$ and the weight function as $w_o(x) = \frac{w(x)}{p(x)}I(x)$.

Example 1.5.1. *To give an example of this case, let us transform the differential equations below to the SLP form.*

$$x^2y'' + xy' + \mu xy = 0, \quad and \quad \exp(2x)y'' + \exp(2x)'y' + (x+\mu)y = 0,$$

The first equation can be just $y'' + \frac{1}{x}y' + \mu\frac{1}{x}y = 0$ *and it can be reduce by taking the integrating factor* $I(x) = \exp\int\frac{p'(x)}{p(x)}dx = \exp\int\frac{1}{x}dx = x$*, then by multiplying the equation with the integrating factor we have* $xy'' + y' + \mu y = 0$ *that is*

$$[xy']' + \mu y = 0,$$

which is a SLP with $q(x) = 0$ *and* $w_o(x) = -1$*.*
The second equation can also be $y'' + y' + x\exp(-2x)y + \mu\exp(-2x)y = 0$*, and it can also be reduce by taking the integrating factor* $I(x) = \exp\int\frac{p'(x)}{p(x)}dx = \exp\int dx = \exp(x)$*, multiplying the equation with the integrating factor we have* $x\exp(x)y'' + \exp(x)y' + x\exp(-x)y + \mu\exp(-x)y = 0$ *that is*

$$[\exp(x)y']' + x\exp(-x)y + \mu\exp(-x)y = 0,$$

which is a SLP with $q(x) = x\exp(-x)$ *and* $w_o(x) = -\exp(-x)$*.*

The special second order differential equations like Euler-Cauchy, Bessel, Legendre, Chebyshev, has SLP form as

a. Bessel Equation: $(xy')' + (x\mu^2 - \frac{m^2}{x})y = 0$,

b. Chebyshev Equation: $(\sqrt{1-x^2}y')' + \frac{n^2}{\sqrt{1-x^2}}y = 0$,

c. Legendre Equation: $((1-x^2)y')' + \mu y = 0$,

1.6 Properties of SLP

1. The eigenvalues of a SLP are all real and nonnegative.

2. The eigenvalues of a SLP can be arranged to form a strictly increasing infinite sequence; that is, $0 \leq \mu_1 < \mu_2 < \mu_3 < \ldots$ Furthermore, $\mu_n \to \infty$. as $n \to \infty$.

3. For each eigenvalue of a SLP, there exists one and only one linearly independent eigenfunction.

4. The set of eigenfunctions $\{e_1(x), e_2(x), ...\}$ of the SLP (1. 3) corresponding to distinct eigenvalues are orthogonal with respect to the weight function $w(x)$. That is

$$\int_a^b w(x)e_1(x)e_2(x)dx = 0.$$

5. The eigenvalues of the SLP (1. 3) are simple.

Definition 1.6.1. *Let $f(x)$ and $g(x)$ be two differentiable functions, then their Wronskian function is defined by*

$$W(f,g) = fg' - gf'.$$

2. Conformable Derivative/Operator

In this section, we give the description of the Conformable differentiation method and what it is about and its related issues because it is the main method used in working out this research.

2.1 Fractional Calculus

Calculus is the mathematics concerning motion or growth (change in state) of an object or situation. It was invented in 16-17th centuries in order to solve mainly the mechanical problems of the time. Such real life problems are dynamic in nature and for them to be properly addressed there is need for "Fractional Calculus " that is the integrals and derivatives of real or complex order. The letter received by Leibniz from L'Hospital in 1965 requiring information on the derivatives of $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}$ was the first mathematical motivation toward investigations and researches in addressing such situations. Mathematicians like Leibniz, L'Hospital, Euler, Lacroin, Abel, Liouville, Reimann, and many others made significant contributions in evaluating such derivative [86]. Fourier, in 1822 gave the first definition for fractical order derivative as

$$\frac{d^{\beta}y}{dx^{\beta}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mu^{\beta}d_{\mu}\int_{-\infty}^{\infty}f(\tau)\cos(\mu x - \tau\mu + \beta\frac{\pi}{2})d\tau$$

After 1832 Liouville gave his definition which is based on the formula for differentiating an exponential function as;

$$_{.}D^{\beta}f(x) = \Sigma_{k=0}^{\infty}c_k a_k^{\beta}e^{a_k x},$$

for any complex β, from which he established his "Liouville's first formula " for the differentiation of a power function and fractional integration and defined as;

$$_{L}D^{-\beta}f(x) = \frac{1}{(-1)^{\beta}\Gamma(\beta)}\int_{0}^{\infty}\varphi(x+\tau)\tau^{\beta-1}d\tau,\ -\infty < x < \infty,\ Re\,\beta$$

Later in 1847, Riemann expressed his definition as;

$$_{R}D^{\beta}f(x) = \frac{1}{\Gamma(\beta)}\int_{0}^{\infty}\frac{\varphi(\tau)}{(x-\tau)^{1-\beta}}d\tau,\ x > 0.$$

Studies on fractional calculus reached a high level in 1880 due to its versatile applications especially in engineering and sciences.

Many researchers gave many different definitions for fractional derivatives among which M-Derivative, Riemann-Liouville, Caputo and conformable derivative are the most popular ones. This research work concentrated more on the last of the above three but discussed the remaining three in brief.

2.1.1 M-derivative

To say about the M-derivative we need the Mittag-Leffler function as it is defined below.

Definition 2.1.1. *The one-parameter Mittag-Leffler function is defined as*

$$E_\beta(z) = \Sigma_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+1)},$$

The two-parameter Mittag-Leffler function is defined as

$$E_{\beta\alpha}(z) = \Sigma_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+\alpha)},$$

where $\beta > 0, \alpha > 0, z \in C$.

Definition 2.1.2. *Let* $f : [a,\infty] \to R$, $\alpha \in (0,1]$ *and* $\gamma > 0$. *Then M-derivative defined as*

$${}_M D_a^{\alpha,\gamma} f(x) = \lim_{h\to 0} \frac{f(x_i E_\gamma(h\,(x-a)^{-\alpha}) - f(x)}{h}$$

where ${}_i E_\gamma(.)$ *is truncated Mittag-Leffler function defined in* (2.1.1). *If* $a = 0$ *this derivative can be indicated as* ${}_M D^{\alpha,\gamma}$.

Note that if ${}_M D^{\alpha,\gamma}$ exists, we have ${}_M D_a^{\alpha,\gamma} f(x) = \frac{(x-a)^{1-\alpha}}{\Gamma(\gamma+1)} f'(x)$ and if $a = 0$, ${}_M D^{\alpha,\gamma} f(x) = \frac{x^{1-\alpha}}{\Gamma(\gamma+1)} f'(x)$ for Γ Gamma function.

Definition 2.1.3. *The left M-integral is defined by letting the function* f *be defined in the interval* $(a,x]$, *where* $a \geq 0, x \geq a$ *and* $0 < \alpha \leq 1$ *as*

$${}_M I_a^{\alpha,\gamma} f(x) = \int_a^x f(\tau) d_\alpha \tau = \Gamma(\gamma+1) \int_a^x (\tau-a)^{\alpha-1} f(\tau) d\tau.$$

2.1.2 Riemann-Liouville Derivative

Riemann-Liouville approach in trying to define the fractional derivative is widely accepted although it has some short coming. Below is the definition.

Definition 2.1.4. *Let* β *be any point in the interval* $[n-1,n)$, *then the* β *derivative of a function* f *is*

$${}_{RL} D_a^{\beta}(f(\tau)) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{d\tau^n} \int_a^\tau \frac{f(x)}{(\tau-x)^{\beta-n+1}} dx$$

which is the same as $\dfrac{d^n}{d\tau^n} f(\tau)$ *when* $\beta = n \in \mathbb{N}$

2.1.3 Caputo Derivative

The Caputo approach is similar to that of Riemann-Liouville only that in this case the ordinary derivative is to be taken before the integral. It is defined as;

Definition 2.1.5. *If $\beta \in [n-1, n)$, then the β derivative of a function f is*

$$ {}_{C}D_{a}^{\beta}(f(\tau)) = \frac{1}{\Gamma(n-\beta)} \int_{a}^{\tau} \frac{f^{n}(x)}{(\tau-x)^{\beta-n+1}} dx $$

and also ${}_{C}D_{a}^{\beta}(f(\tau)) = \frac{d^{n}}{d\tau^{n}} f(\tau)$ *when* $\beta = n \in \mathbb{N}$

These two popular definitions have their shortcomings due to which researching for a better and more suitable definitions continues. These shortcomings included the following facts which can be found in [37, 87, 88];

1. The Riemann-Liouville derivative does not satisfy the rule for the derivative of a constant, that is ${}_{RL}D_{a}^{\beta}(k) \neq 0$ for any constant k but it is satisfied by the Caputo derivative.

2. These two derivative approaches do not satisfy the product rule of two functions, that is ${}_{RL}D_{a}^{\beta}(fg) \neq f.{}_{RL}D_{a}^{\beta}(g) + g.{}_{RL}D_{a}^{\beta}(f)$ and ${}_{C}D_{a}^{\beta}(fg) \neq f.{}_{C}D_{a}^{\beta}(g) + g.{}_{C}D_{a}^{\beta}(f)$.

3. These two derivative approaches do not satisfy the quotient rule of two functions, that is ${}_{RL}D_{a}^{\beta}(\frac{f}{g}) \neq \frac{g.{}_{RL}D_{a}^{\beta}(f)+f.{}_{RL}D_{a}^{\beta}(g)}{g^{2}}$ for $g \neq 0$ and ${}_{C}D_{a}^{\beta}(\frac{f}{g}) \neq \frac{g.{}_{C}D_{a}^{\beta}(f)+f.{}_{C}D_{a}^{\beta}(g)}{g^{2}}$ for $g \neq 0$.

4. These two derivative approaches do not satisfy the chain rule, that is ${}_{RL}D_{a}^{\beta}(f \circ g) \neq_{RL} D_{a}^{\beta}(f(g)).{}_{RL}D_{a}^{\beta}(g)$ and ${}_{C}D_{a}^{\beta}(f \circ g) \neq_{C} D_{a}^{\beta}(f(g)).{}_{C}D_{a}^{\beta}(g)$.

5. In the Caputo definition the function f most be differentiable.

6. Although both the two definitions discussed are linear but they do not coincide with one another. The relation between them is defined by the following theorem.

   Theorem 2.1.1. *Riemann-Liouville fractional derivative is related to the Caputo fractional derivative by*

   $$ {}_{RL}D_{a}^{\beta}(f(\tau)) =_{C} D_{a}^{\beta}(f(\tau)) + \Sigma_{k=0}^{n-1} \frac{\tau^{k-\beta}}{\Gamma(k+1-\beta)} f^{k}(0). $$

   under the condition that $\beta \in [n-1, n)$, for $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$.

7. These two derivative approaches do not satisfy the basic analysis theorems like Rolle's theorem and the mean value theorem.

8. Although both the two definitions discussed have good physical meanings, they are not easy to evaluate in most cases.

2.2 Conformable Derivative

The fundamental ideas of the "conformable derivative" were developed by Khalil et al. [37] and discussed further by Abdeljawad [39]. Khalil examined the shortcomings of the most popular fractional differentiation techniques and established a new, straightforward, and easily evaluable definition of the fractional derivatives. The most fundamental characteristics of differentiation are satisfied by this new definition, which appears to be a logical progression of the conventional derivative. Below is the definition and some details with regard to this new definition.

Definition 2.2.1. *Let consider the function $w:[0,\infty)\to\mathbb{R}$, then, the β^{th} order conformable derivative of $w(x)$ is given as*

$$D^{\beta}w(x)=\lim_{h\to 0}\frac{w(x+hx^{1-\beta})-w(x)}{h},$$

for all $x>0$, $\beta\in(0,1]$.

It can be deduced from the definition 2.2.1 that if $w(x)$ is β-differentiable in some interval $(0,a)$, $a>0$, and $\lim_{h\to 0^+}w^{(\beta)}(h)$ exists, then $w^{(\beta)}(0)=\lim_{h\to 0^+}w^{(\beta)}(h)$. Also if w is differentiable (ordinary), then $D^{\beta}w(x)=x^{1-\beta}w\prime(x)$.
The conformable derivative defined for $\beta\in(0,1]$ is also defined for $\beta\in(n-1,n)$ for $n\in\mathbb{N}$. That is,

Definition 2.2.2. *Let w be an n-differentiable at x, where $x>0$ and $\beta\in(n-1,n)$, then the conformable derivative w of order β is defined as,*

$$D^{\beta}w(x)=\lim_{h\to 0}\frac{w^{\lceil\beta\rceil-1}(x+hx^{\lceil\beta\rceil-\beta})-w^{\lceil\beta\rceil-1}(x)}{h}$$

Where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.
It can also be deduced that from the definition (2.2.2) that $D^{\beta}w(x)=x^{\lceil\beta\rceil-\beta}w^{\lceil\beta\rceil}(x)$

As the concept of continuity is highly important in calculus, it is also extended to fractional calculus and the continuity at a point is defined below;

Theorem 2.2.1. *If a function $w:[0,\infty)\longrightarrow\mathbb{R}$ is β-differentiable at $x_0>0$ for $\beta\in(0,1]$ then w is continuous at $x_0>0$.*

Proof 1. This theorem is proved using the concept of limit which is also defined in fractional calculus. That is;

$$w(x_0+hx_0^{1-\beta})-w(x_0)=\frac{w(x_0+hx_0^{1-\beta})-w(x_0)}{h}.h,$$

taking the limit on both we have,

$$\lim_{h\to 0}[w(x_0+hx_0^{1-\beta})-w(x_0)]=\lim_{h\to 0}\frac{w(x_0+hx_0^{1-\beta})-w(x_0)}{h}.\lim_{h\to 0}h$$

Let $e=hx_0^{1-\beta}$ clear as $h\to 0$, $e\to 0$. Then, by the definition of *first principle* we have;

$$\lim_{e\to 0}[w(x_0+e)-w(x_0))]=f^{(\beta)}.0$$

and this implies that $\lim_{h\to 0}[w(x_0+e)] = w(x_0)$ which completes the Proof.

Theorem 2.2.2. *(Rolle's Theorem for Conformable Fractional Differentiable Functions). Let $a > 0$ and $w : [a,b] \longrightarrow \mathbb{R}$ be a given function that satisfies*
(i) w is continuous on $[a,b]$,
(ii) w is β-differentiable for some $\beta \in (0,1)$,
(iii) $w^{(\beta)}(a) = w^{(\beta)}(b) = 0$,
then, there exists $c \in (a,b)$, such that $w^{(\beta)}(c) = 0$.

Proof 2. Let w be continuous at every point of the interval $[a,b]$, β-differentiable in (a,b), and $w^{(\beta)}(a) = w^{(\beta)}(b) = 0$, then there is $c \in (a,b)$, which is a point of local extrema. Let assume, without loss of generality, that c is a point of local minimum. So,

$$w^{(\beta)}(c) = \lim_{h\to 0^+} \frac{w(c+hc^{1-\beta}) - w(c)}{h} = \lim_{h\to 0^-} \frac{w(c+hc^{1-\beta}) - w(c)}{h} \tag{2. 1}$$

Since the first limit is non-negative and the second limit is non-positive, the only possibility is $w^{(\beta)}(c) = 0$.

Theorem 2.2.3. *(Mean Value Theorem for Conformable Fractional Differentiable Functions). If $a > 0$ and $w : [a,b] \longrightarrow \mathbb{R}$ be a given function that satisfies;*
(i) w is continuous on $[a,b]$.
(ii) w is β-differentiable for some $\beta \in (0,1)$,
then, there exists $c \in (a,b)$ such that

$$w^{(\beta)}(c) = \frac{w(b) - w(a)}{\frac{b^\beta}{\beta} - \frac{a^\beta}{\beta}} \tag{2. 2}$$

Proof 3. Let us define another function

$$h(x) = w(x) + w(a) - \frac{w(b) - w(a)}{\frac{b^\beta}{\beta} - \frac{a^\beta}{\beta}} \left(\frac{x^\beta}{\beta} - \frac{a^\beta}{\beta} \right) \tag{2. 3}$$

so that the function $h(x)$ satisfies the Rolle's theorem as such there is at least one point $c \in (a,b)$ such that $h^{(\beta)}(c) = 0$. Then,

$$h^\beta(c) = w^{(\beta)}(c) - \frac{w(b) - w(a)}{\frac{b^\beta}{\beta} - \frac{a^\beta}{\beta}} = 0 \Rightarrow w^{(\beta)}(c) = \frac{w(b) - w(a)}{\frac{b^\beta}{\beta} - \frac{a^\beta}{\beta}} \tag{2. 4}$$

This completes the Proof.

2.2.1 Conformable Integrals

The conformable derivatives approach discussed included the concept of fractional integrations. The following rules where proved to be true and useful.

Definition 2.2.3. *For a function w, the conformable integral of w of order β is given as*

$$I_\beta w(x) = \int_0^x w(\tau) d_\beta \tau = \int_0^x \tau^{\beta-1} w(\tau) d\tau$$

for all $x > 0$. Where the integral is the usual Riemann integral.

Lemma 2.2.1. *Let us assume that the function w is differentiable. Then, we have for $x > a$,*

$$I_\beta^a D_x^\beta w(x) = w(x) - w(a).$$

Lemma 2.2.2. *Assume that w is differentiable. Then, for $x > a$,*

$$D_x^\beta I_\beta^a w(x) = w(x).$$

Theorem 2.2.4 (Conformable integration by parts)**.** *Let w, h be two β-differentiable functions. Then*

$$\int_a^b g(x) D_x^\beta (h(x))(x) d_\beta x = wh|_a^b - \int_a^b h(x) D_x^\beta (w(x)) d_\beta x.$$

Lemma 2.2.3. *The $\beta-$Leibniz integral rule is given in the form*

$$\begin{aligned} D_x^\beta \left[\int_{a(x)}^{b(x)} w(x,\tau) d_\beta \tau \right] &= D_x^\beta b(x) w(x, b(x)) b(x)^{\beta-1} - D_x^\beta a(x) w(x, a(x)) a(x)^{\beta-1} \\ &+ \int_a^b D_x^\beta (w(x,\tau)) d_\beta \tau. \end{aligned} \tag{2. 5}$$

for $a(x) \leq \tau \leq b(x)$ while $a(x)$ and $b(x)$ are both $\beta-$differentiable for $x_0 \leq x \leq x_1$.

Lemma 2.2.4. *Let f be β-Riemann integrable on $[a,b]$. Then,*

$$\lim_{\mu \to \pm\infty} \int_a^b f(x) \cos\left(\frac{\mu x^\beta}{\beta}\right) d_\beta x = 0$$

$$\lim_{\mu \to \pm\infty} \int_a^b f(x) \sin\left(\frac{\mu x^\beta}{\beta}\right) d_\beta x = 0$$

$$\lim_{\mu \to \pm\infty} \int_a^b f(x) e^{i\left(\frac{\mu x^\beta}{\beta}\right)} d_\beta x = 0$$

These definitions and Theorems were given by some authors [37, 39, 40, 56].

2.2.2 Properties of Conformable Derivative

The conformable Derivative has the following properties as stated in the theorem below.

Theorem 2.2.5. *Let f and w be β-differential at x, $x > 0$. Then,*
i) $D_x^\beta (af + bw) = a D_x^\beta f + b D_x^\beta w, \forall a, b \in R$

ii) $D_x^\beta(c) = 0$, (c *is a constant*)
iii) $D_x^\beta(fw) = wD_x^\beta f + fD_x^\beta w, \forall a, b \in R$
iv) $D_x^\beta(\frac{f}{w}) = \frac{w.D_x^\beta(f) - f.D_x^\beta w}{w^2}$
v) $D_x^\beta w = x^{1-\beta}.D_x w$, *when* w *is differentiable.*

The Proofs of these properties can easily be seen in [37, 87].

2.2.3 Some Examples on Conformable derivative

The examples below demonstrated the properties of the conformable derivatives and in each case, when $\beta = 1$ the normal derivatives are obtained. Now for $a, b, k \in \mathbb{R}$ let us find;
i) $D_x^\beta(\frac{b}{\beta}x^\beta + \sin\frac{1}{\beta}x^\beta)$,
ii) $D_x^\beta(x^a e^{bx} - 2k)$,
iii) $D_x^\beta\left(\frac{e^{\frac{1}{\beta}x^\beta}}{\tan(ax)}\right)$.
Clearly, all the functions in all the three problems above are differentiable them, by the fifth property the solutions are;
i) $D_x^\beta(\frac{b}{\beta}x^\beta + \sin\frac{1}{\beta}x^\beta) = D_x^\beta(\frac{b}{\beta}x^\beta) + D_x^\beta(\sin\frac{1}{\beta}x^\beta) = b + \cos\frac{1}{\beta}x^\beta$
ii) $D_x^\beta(x^a e^{bx} - 2k) = x^a D_x^\beta(e^{bx}) + e^{bx} D_x^\beta(x^a) - D_x^\beta(2k) = x^{1-\beta}e^{bx}[bx^a + ax^{(a-1)}]$
iii) $D_x^\beta\left(\frac{e^{\frac{1}{\beta}x^\beta}}{\tan(ax)}\right) = \frac{\tan(ax)D_x^\beta(e^{\frac{1}{\beta}x^\beta}) - e^{\frac{1}{\beta}x^\beta}D_x^\beta(\tan(ax))}{(\tan(ax))^2} = \frac{e^{\frac{1}{\beta}x^\beta}[(\tan(ax)) - ax^{1-\beta}\sec^2(ax)]}{(\tan(ax))^2}$ For the Integrals, let us demonstrate some of the above definitions by solving the following examples;
i) $I_\beta^a(x^k)$, $k \in \mathbb{N}$
ii) $I_{\frac{1}{2}}^a(\sqrt{x}\sin bx)$, $a, b \in \mathbb{R}$
iii) $I_\beta^a(x^{1-\beta}e^{cx})$, $a, c \in \mathbb{R}$
Now, using the definition 2.2.3 the solutions are;
i) $I_\beta^a(x^k) = \int_a^x \frac{\tau^k}{\tau^{1-\beta}}d\tau = \frac{x^{k+\beta}}{k+\beta}$, $for\ k \neq -\beta$
ii) $I_{\frac{1}{2}}^a(\sqrt{x}\sin bx) = \int_0^x \frac{\sqrt{\tau}\sin b\tau}{\tau^{1-\beta}}d\tau = \frac{1}{b}[1 - \cos bx]$
iii) $I_\beta^a(x^{1-\beta}e^{cx}) = \int_a^x \frac{\tau^{1-\beta}e^{c\tau}}{\tau^{1-\beta}}d\tau = e^{cx} - e^{ac}$
We can easily note that in all the three cases the original integral is obtained when $\beta = 1$.

2.2.4 Development on conformable derivatives

By giving definitions and performing various operations on the left and right conformable derivatives and conformable integrals of higher orders, Abdeljawad in [39] explains the evolution of the conformable calculus. Additionally, he used the conformable derivative approach to develop and explore its versions of the chain rule, exponential functions, Taylor power series expansions, Gronwall's inequality, integration by parts, Laplace transforms, and linear differential systems. Below is a brief on the work of Abdeljawad.

Definition 2.2.4. *The left conformable derivative of the function* $w : [0, \infty) \to \mathbb{R}$ *is given* *as*

$$^{a}D^{\beta}w(x) = \lim_{h\to 0}\frac{w(x+h(x-a)^{1-\beta})-w(x)}{h},$$

for all $x>0$, $\beta\in(0,1]$ *and* $a\in\mathbb{R}$ *and if* w *is* β*-differentiable in the interval* $(0,a)$, $a>0$, *then* $^{a}D^{\beta}w(a) = \lim_{h\to a^{+}}[^{a}D^{\beta}w(x)(h)]$. *The calculation is simply by* $^{a}D^{\beta}w(x) = (x-a)^{1-\beta}w\prime(x)$ *provided* w *is differentiable.*
Similarly, the right conformable derivative of the same function under the same conditions is given by $b\in\mathbb{R}$.

$$_{b}D^{\beta}w(x) = \lim_{h\to 0}\frac{w(x+h(b-x)^{1-\beta})-w(x)}{h},$$

The calculation here is also simple and it is by $_{b}D^{\beta}w(x) = -(b-x)^{1-\beta}w\prime(x)$ *provided* g *is differentiable.*

The higher order case is generalized as;

Definition 2.2.5. *The left conformable derivative of the function* $w:[0,\infty)\longrightarrow\mathbb{R}$ *of order* $\alpha\in(n-1,n), n\in\mathbb{N}$ *and* $\beta=\alpha-(n-1)$ *is defined by*

$$^{a}D^{\alpha}w(x) =^{a} D^{\beta}(w(x))^{(n-1)}$$

Similarly, the left conformable derivative of the same function under the same condition is defined as

$$_{b}D^{\alpha}w(x) = (-1)^{n}\cdot{}_{b}D^{\beta}(w(x))^{(n-1)}$$

Let us now give the corresponding conformable integral definition as follows;

Definition 2.2.6. *The left conformable integral of order* $\alpha\in(n-1,n), n\in\mathbb{N}$ *and* $\beta=\alpha-(n-1)$ *starting at* a *is defined by*

$$\boldsymbol{I}^{a}_{\alpha}(f(\tau)) = I^{a}_{n}((\tau-a)^{\beta-1}f) = \frac{1}{(n-1)!}\int_{a}^{\tau}(\tau-x)^{n-1}(x-a)^{\beta-1}f(x)dx$$

Similarly, the right conformable integral of the same order and under the same condition is given by

$$^{b}\boldsymbol{I}_{\alpha}(f(\tau)) =^{b} I_{n}((\tau-a)^{\beta-1}f) = \frac{(-1)^{n-1}}{(n-1)!}\int_{\tau}^{b}(\tau-x)^{n-1}(b-x)^{\beta-1}f(x)dx$$

2.2.5 Shortcomings of conformable derivatives

Although the conformable derivatives is an improvement on the two famous fractional derivatives approaches, it has its shortcomings. The author [37] remarked that the conformable derivative is local in nature and questioned whether its physical meaning

can easily be interpreted than the previous fractional derivatives or not.
Despite the fact that, still, there is no criteria for defining fractional derivatives Ortigueira [45] proved that the two most famous fractional derivatives approaches together with the Grunwald-Letnikov's approach share the following properties that every fractional differential operator may be expected to satisfy. These properties are;

1. Linearity
2. The identity is the zero-order operator
3. The integer-order operators give the ordinary derivative
4. Index law and commutativity
5. the generalized Leibniz rule for the derivative of a product, that is

$$D^{\beta}(f(x)g(x)) = \sum_{n=0}^{\infty} \binom{\beta}{n} D^{n}(f(x))D^{\beta-n}(g(x))$$

The author asserted that the conformable derivative does not satisfy the properties (ii),(iii) and (v).
The author [89] analyzed a general fractional model of viscoelasticity using the conformable, Riemann-Liouville and Caputo fractional derivatives and the result obtained using the first approach is inconsistent with those obtained using the latter two approaches. The latter results excellently fit with the experimental studies. But due to the fact that still there is no standard rule classifying whether a derivative is fractional or not and the facts that this conformable derivative is effective and applicable it has been employed in various fields of researches like in control theory of dynamical systems, Newton mechanics, quantum mechanics, variational calculus, arbitrary time scale problems, anomalous diffusion, diffusive transport, stochastic process and many others, with many many physical interpretations have been provided [56], it is really a development in the literature of non-integer value derivatives.

2.3 Conformable Transformation Sturm-Liouville Operator

This section presents a transformation operator for the conformable SLP which differs from the traditional Sturm-Liouville operator. Validation of the Proof is achieved by setting $\beta = 1$, which returns the classical problem. The transformation operator represents a methodology for solving the inverse Sturm-Liouville operator. In order to establish a relationship between two distinct SLPs with different potentials, an integral equation was initially employed. The potential function and the nucleus function, designated as $q(x)$ and $N(x, \tau)$ respectively, are related by a pivotal formula. A number of studies have been carried out on this topic, as referenced in the following sources [90, 91, 92, 93, 84, 38]. In order to achieve equivalent conclusions for the conformable Sturm-Liouville issue, we used the following relationship.

2.3.1 The Classical Form of the Problem

Let us consider two Sturm-Liouville operators $A = -\dfrac{d^2}{dx^2} + q(x)$ and $B = -\dfrac{d^2}{dx^2} + r(x)$, with $q(x)$ and $r(x)$ squared integrable functions and known as potentials in the theory. We considered the case where E_1 and E_2 are considered as subspaces of a topological linear space, E such that h_1 and h_2 are finite, for $h_1 \in E_1$ and $h_2 \in E_2$. Also, A and B are two differential operators from E to E under the conditions that $w\prime(0) = h_1 m(0)$ and $m\prime(0) = h_2 m(0)$. These conditions based the foundation of ordinary transformation operator for the problem which is expressed as in the theorem below [94].

Theorem 2.3.1. *Defining the transformation operator between A and B as*

$$Xm(x) = m(x) + \int_0^x N(x,\tau)m(\tau)d\tau. \tag{2. 6}$$

Then the kernel in operator (2. 6) *is a solution to the Hyperbolic equation,*

$$\frac{\partial^2}{\partial x^2}N(x,\tau) - q(x)N(x,\tau) = \frac{\partial^2}{\partial(\tau^)2}N(x,\tau) - r(x)N(x,\tau), \tag{2. 7}$$

and satisfies the conditions,

$$N(x,x) = h_2 - h_1 + \frac{1}{2}\int_0^x [q(s) - r(s)]ds, \tag{2. 8}$$

$$\left[\frac{\partial N(x,\tau)}{\partial t} - h_1 N(x,\tau)\right]\bigg|_{\tau=0} = 0. \tag{2. 9}$$

Inversely, if a the nucleus function $N(x,\tau)$ is a solution of the problem (2. 7) - (2. 9)*, then the operator X defined by* (2. 6) *is a transformation operator for A and B.*

2.3.2 The proposed conformable transformation operator

Similarly, as in the original case, we consider E as a space which consisted of all real valued functions $m(x)$ for $x \geq 0$. Let the two fractional Sturm-liouville operators be expressed as

$$A = -D_x^\beta D_x^\beta + q(x)\, and\, B = -D_x^\beta D_x^\beta + r(x),$$

here $q(x)$ and $r(x)$, $(0 \leq x < \infty)$ are continuous functions. Let E_1 and E_2 be subspaces of functions in E satisfying the boundary condition;

$$D_x^\beta m(0) = h_1 m(0)$$

and

$$D_x^\beta m(0) = h_2 m(0),$$

here h_1 and h_2 are any constants, then based on these, our proposed fractional operator is expressed by the theorem below.

Theorem 2.3.2. *The transformation operator, $X = X_{A,B}$ from E_1 to E_2 is defined*

$$Xm(x) = m(x) + \int_0^x N(x,\tau)m(\tau)d_\beta\tau \tag{2. 10}$$

the Kernel in operator (2. 10) *is a solution to the fractional differential equation;*

$$D_x^\beta(D_x^\beta N(x,\tau)) - q(x)N(x,\tau) = D_\tau^\beta(D_\tau^\beta N(x,\tau)) - r(x)N(x,\tau) \tag{2. 11}$$

and satisfies conditions,

$$N(x,x) = h_2 - h_1 + \int_0^x \frac{[q(s) - r(s)]}{1 + s^{\beta-1}} d_\beta s \tag{2. 12}$$

$$[D_\tau^\beta N(x,\tau) - h_1 N(x,\tau)]\Big|_{\tau=0} = 0. \tag{2. 13}$$

Inversely, the operator X which is given in (2. 10) *is a conformable transformation operator for A, B, on condition that $N(x,\tau)$ is a solution of* (2. 11) - (2. 13).

This is the statement of the proposed assertion and by using the facts in theorems, definitions, and lemmas above, the Proof is provided as follows.

Proof 4. Differentiating equation (2. 10) fractionally with respect to x gives

$$\begin{aligned} D_x^\beta(Xm(x)) &= D_x^\beta m(x) + D_x^\beta[\int_0^x N(x,\tau)m(\tau)d_\beta\tau] \\ &= x^{1-\beta}m'(x) + D_x^\beta(x).N(x,x)m(x).x^{\beta-1} \\ &\quad - D_x^\beta(0).N(x,0)m(0).(0)^{\beta-1} + \int_0^x D_x^\beta N(x,\tau)m(\tau)d_\beta\tau, \end{aligned} \tag{2. 14}$$

that is,

$$D_x^\beta(Xm(x)) = x^{1-\beta}m'(x) + N(x,x)m(x) + \int_0^x D_x^\beta N(x,\tau)m(\tau)d_\beta\tau. \tag{2. 15}$$

Since, $(Xm(x)) \in E_2$ and $m(x) \in E_1$, then taking as $x = 0$ in (2. 15), we obtained,

$$\begin{aligned} D_x^\beta(Xm(x))|_{x=0} = h_2(Xm(x))|_{x=0} = h_2(m(0)) = D_x^\beta m(0) + N(0,0)m(0) \\ = h_1 m(0) + N(0,0)m(0) = [h_1 + N(0,0)]m(0), \end{aligned} \tag{2. 16}$$

which implies that $N(0,0) = h_2 - h_1$. By differentiating equation (2. 15) we get,

$$\begin{aligned} D_x^\beta[D_x^\beta(Xm(x))] &= D_x^\beta[D_x^\beta m(x)] + [D_x^\beta N(x,x)]m(x) + N(x,x)[D_x^\beta m(x)] \\ &+ D_x^\beta(x)[D_x^\beta N(x,x)m(x)x^{\beta-1}](x)^{\beta-1} - D_x^\beta(0)[D_x^\beta N(x,0)m(0)(0)^{\beta-1}]x^{\beta-1} \\ &+ \int_0^x D_x^\beta[D_x^\beta N(x,\tau)m(\tau)]d_\beta\tau \\ &= D_x^\beta[D_x^\beta m(x)] + [D_x^\beta N(x,x)]m(x) + N(x,x)[D_x^\beta m(x)] \\ &+ D_x^\beta N(x,x)m(x)x^{\beta-1} + \int_0^x D_x^\beta[D_x^\beta N(x,\tau)m(\tau)]d_\beta\tau, \end{aligned} \tag{2. 17}$$

therefore we have

$$\begin{aligned} A(Xm(x)) &= -D_x^\beta D_x^\beta(Xm(x)) + q(x)(Xm(x)) \\ &= -D_x^\beta D_x^\beta m(x) - [D_x^\beta N(x,x)]m(x) - N(x,x)[D_x^\beta m(x)] - D_x^\beta N(x,x)m(x)x^{\beta-1} \\ &- \int_0^x [D_x^\beta D_x^\beta N(x,\tau)]m(\tau)d_\beta\tau + q(x)[(m(x)) + \int_0^x N(x,\tau)m(\tau)d_\beta\tau] \\ &= -D_x^\beta D_x^\beta m(x) - [D_x^\beta N(x,x)]m(x) - N(x,x)[D_x^\beta m(x)] - D_x^\beta N(x,x)m(x)x^{\beta-1} \\ &+ q(x)(m(x)) - \int_0^x [D_x^\beta D_x^\beta N(x,\tau) - q(x)N(x,\tau)]m(\tau)d_\beta\tau. \end{aligned} \tag{2. 18}$$

$Bm(x)$ under the operator X is expressed as

$$Bm(x) = [-D_x^\beta D_x^\beta + r(x)]m(x) = -D_x^\beta D_x^\beta m(x) + r(x)m(x),$$

then

$$\begin{aligned} X[Bm(x)] &= (Bm(x)) + \int_0^x N(x,\tau)[Bm(\tau)]d_\beta\tau \\ &= -D_x^\beta D_x^\beta m(x) + r(x)m(x) + \int_0^x N(x,\tau)[-D_\tau^\beta D_\tau^\beta m(\tau) + r(\tau)]m(\tau)]d_\beta\tau, \end{aligned}$$

but

$$\begin{aligned} \int_0^x N(x,\tau)D_\tau^\beta[D_\tau^\beta m(\tau)]d_\beta\tau &= N(x,\tau)D_\tau^\beta m(\tau)|_0^x - \int_0^x [D_\tau^\beta m(\tau)]D_\tau^\beta N(x,\tau)d_\beta\tau \\ &= N(x,x)D_\tau^\beta m(x) - N(x,0)D_\tau^\beta m(0) \\ &- \left[D_\tau^\beta N(x,\tau)m(\tau)|_0^x - \int_0^x m(\tau)D_\tau^\beta D_\tau^\beta N(x,\tau)d_\beta\tau\right] \\ &= N(x,x)D_\tau^\beta m(x) - N(x,0)D_\tau^\beta m(0) - D_\tau^\beta N(x,x)m(x) \\ &+ D_\tau^\beta N(x,0)m(0) + \int_0^x D_\tau^\beta D_\tau^\beta N(x,\tau)m(\tau)d_\beta\tau, \end{aligned}$$

as such $X[Bm(x)]$ become

$$\begin{aligned} X[Bm(x)] &= -D_x^\beta D_x^\beta m(x) + r(x)m(x) + [D_\tau^\beta N(x,x)]m(x) - N(x,x)D_\tau^\beta m(x) + N(x,0)D_\tau^\beta m(0) \\ &- [D_\tau^\beta N(x,0)]m(0) - \int_0^x D_\tau^\beta D_\tau^\beta[N(x,\tau) - r(x)N(x,\tau)]m(\tau)d_\beta\tau. \end{aligned} \tag{2. 19}$$

Since, $A(Xm) = X(Bm)$, then equating (2. 18) and (2. 19) gives

$$D_x^\beta D_x^\beta N(x,\tau) - q(x)N(x,\tau) = D_\tau^\beta D_\tau^\beta N(x,\tau) - r(x)N(x,\tau),$$

and also, the conditions from the same relation can be derived as

$$0 = N(x,0)D_\tau^\beta m(0) - [D_\tau^\beta N(x,0)]m(0) = h_1 m(0)N(x,0) - [D_\tau^\beta N(x,0)]m(0),$$

which implies that

$$[h_1 N(x,0) - D_\tau^\beta N(x,0)]m(0) \Rightarrow D_\tau^\beta N(x,\tau) - h_1 N(x,0)]|_{\tau=0} = 0\; sincem(0) \neq 0,$$

then we have the following condition

$$[D_\tau^\beta N(x,\tau) - h_1 N(x,\tau)]|_{\tau=0} = 0,$$

and the other condition is derived as follows.

$$-[D_x^\beta N(x,x)] - D_x^\beta N(x,x)x^{\beta-1} + q(x) = [D_\tau^\beta N(x,x)] + r(x),$$

that is

$$-[D_x^\beta N(x,x)][1 + x^{\beta-1}]q(x) = [D_\tau^\beta N(x,x)] + r(x) \Rightarrow$$

$$q - r = [D_x^\beta N(x,x)][1 + x^{\beta-1}] + D_\tau^\beta N(x,x),$$

that is at $\tau = x \Rightarrow$

$$D_x^\beta N(x,x)[1 + x^{\beta-1}] = q(x) - r(x),$$

which is equivalent to

$$N(x,x) = \int_0^x \frac{[q(s) - r(s)]}{1 + s^{\beta-1}} d_\beta s + N(0,0),\; for\, N(0,0) = h_2 - h_1.$$

This gives the Proof of the proposed assertion and it is clear that the original problem is obtainable by taking $\beta = 1$. Briefly, the Proof is that, if the transformation operator X is in the form (2. 10), then $N(x,\tau)$ is a solution of Hyperbolic problems (2. 11)to (2. 13).Conversely this assertion is also true. To complete the remainder of Proof, the solvability of problem (2. 11) to (2. 13) is shown below;
First assume that the functions $q(x)$ and $r(x)$ are differentiable. Let the variables $\xi = x + \tau$ and $\eta = x - \tau$.Then consider the function below,

$$Q(\xi,\eta) = N\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right), \tag{2. 20}$$

equations (2. 11) to (2. 13) in terms of ξ and η can be expressed as

$$D_\xi^\beta(D_\eta^\beta Q(\xi,\eta)) = \frac{1}{4}\left[q\left(\frac{\xi+\eta}{2}\right) - r\left(\frac{\xi+\eta}{2}\right)\right]Q, \tag{2. 21}$$

and satisfies the conditions,

$$Q(\xi,0) = h_2 - h_1 + \int_0^{\frac{\xi}{2}} \frac{[q(s)-r(s)]}{1+s^{\beta-1}} d_\beta s \tag{2. 22}$$

and

$$[D_\xi^\beta Q\tau^{1-\beta} - D_\eta^\beta Q\tau^{1-\beta} - h_1 Q]\Big|_{\xi=\eta} = 0. \tag{2. 23}$$

Integrating equation (2. 21) according to variable η from 0 to η gives

$$\int_0^\eta D_\xi^\beta[(D_\eta^\beta Q(\xi,\eta))]d_\beta\eta = \frac{1}{4}\int_0^\eta \left[q\left(\frac{\xi+\eta}{2}\right) - r\left(\frac{\xi+\eta}{2}\right)\right]Q d_\beta\eta,$$

that is

$$\int_0^\eta D_\eta^\beta[(D_\xi^\beta Q(\xi,\mu))]d_\beta\mu = \frac{1}{4}\int_0^\eta \left[q\left(\frac{\xi+\mu}{2}\right) - r\left(\frac{\xi+\mu}{2}\right)\right]Q d_\beta\mu$$

so that

$$D_\xi^\beta Q - D_\xi^\beta Q\Big|_{\eta=0} = \frac{1}{4}\int_0^\eta \left[q\left(\frac{\xi+\mu}{2}\right) - r\left(\frac{\xi+\mu}{2}\right)\right]Q(\xi,\mu)d_\beta\mu. \tag{2. 24}$$

It follows from condition (2. 22) that

$$\begin{aligned}
D_\xi^\beta Q\Big|_{\eta=0} &= D_\xi^\beta Q(\xi,0) = D_\xi^\beta\left[h_2 - h_1 + \int_0^{\frac{\xi}{2}} \frac{[q(s)-r(s)]}{1+s^{\beta-1}} d_\beta s\right] \\
&= D_\xi^\beta\left[\int_0^{\frac{\xi}{2}} \frac{[q(s)-r(s)]}{1+s^{\beta-1}} d_\beta s\right] \\
&= D_\xi^\beta(\frac{\xi}{2})\frac{\left[q\left(\frac{\xi}{2}\right) - r\left(\frac{\xi}{2}\right)\right]}{1+(\frac{\xi}{2})^{\beta-1}}\left(\frac{\xi}{2}\right)^{\beta-1} - D_\xi^\beta(0)\frac{[q(0)-r(0)]}{(1+(0)^{\beta-1})}(0)^{\beta-1} \\
&+ \int_0^{\frac{\xi}{2}} D_\xi^\beta \frac{[q(s)-r(s)]}{(1+(s)^{\beta-1})} d_\beta s,
\end{aligned}$$

that is

$$D_\xi^\beta Q\Big|_{\eta=0} = \frac{\left[q\left(\frac{\xi}{2}\right) - r\left(\frac{\xi}{2}\right)\right]}{2(1+(\frac{\xi}{2})^{\beta-1})}\left(\frac{\xi}{2}\right)^{\beta-1},$$

therefore (2. 24) becomes

$$\begin{aligned} D_\xi^\beta Q &= \frac{1}{4}\int_0^\eta \left[q\left(\frac{\xi+\mu}{2}\right)-r\left(\frac{\xi+\mu}{2}\right)\right]Q(\xi,\mu)d_\beta\mu + D_\xi^\beta Q\Big|_{\eta=0} \\ &= \frac{1}{4}\int_0^\eta \left[q\left(\frac{\xi+\mu}{2}\right)-r\left(\frac{\xi+\mu}{2}\right)\right]Qd_\beta\mu + \frac{\left[q\left(\frac{\xi}{2}\right)-r\left(\frac{\xi}{2}\right)\right]}{2(1+(\frac{\xi}{2})^{\beta-1})}\left(\frac{\xi}{2}\right)^{\beta-1}. \end{aligned} \tag{2. 25}$$

Integrating equation (2. 25) with respect to variable ξ from η to ξ gives

$$\begin{aligned} \int_\eta^\xi D_\xi^\beta Q d_\beta\xi &= \int_\eta^\xi D_\xi^\beta Q d_\beta\alpha = Q(\xi,\eta) \\ &= \frac{1}{4}\int_\eta^\xi d_\beta\alpha\left(\int_0^\eta \left[q\left(\frac{\alpha+\mu}{2}\right)-r\left(\frac{\alpha+\mu}{2}\right)\right]Q(\alpha,\mu)d_\beta\mu\right) \\ &+ \int_\eta^\alpha \frac{\left[q\left(\frac{\xi}{2}\right)-r\left(\frac{\xi}{2}\right)\right]}{2(1+(\frac{\xi}{2})^{\beta-1})}\left(\frac{\xi}{2}\right)^{\beta-1} d_\beta\mu + Q(\eta,\eta), \end{aligned}$$

that is

$$\begin{aligned} Q(\xi,\eta) &= \frac{1}{4}\int_\eta^\xi d_\beta\alpha\int_0^\eta \left[q\left(\frac{\alpha+\mu}{2}\right)-r\left(\frac{\alpha+\mu}{2}\right)\right]Q(\alpha,\mu)d_\beta\mu \\ &+ \int_\eta^\xi \frac{\left[q\left(\frac{\xi}{2}\right)-r\left(\frac{\xi}{2}\right)\right]}{2(1+(\frac{\xi}{2})^{\beta-1})}\left(\frac{\xi}{2}\right)^{\beta-1} d_\beta\mu + Q(\eta,\eta). \end{aligned}$$

Now, to calculate $Q(\eta,\eta)$, it follows from (2. 23) and (2. 25) that

$$\begin{aligned} 2D_\xi^\beta Q\tau^{1-\beta}\Big|_{\xi=\eta} &= \left[D_\xi^\beta Q\tau^{1-\beta} + D_\eta^\beta Q\tau^{1-\beta} + h_1 Q\right]\Big|_{\xi=\eta} \\ &= \frac{1}{2}\int_0^\eta \left[q\left(\frac{\eta+\mu}{2}\right)-r\left(\frac{\eta+\mu}{2}\right)\right]Q(\eta,\mu)d_\beta\mu + \frac{[q\left(\frac{\eta}{2}\right)-r\left(\frac{\eta}{2}\right)]}{(1+(\frac{\eta}{2})^{\beta-1})}\left(\frac{\eta}{2}\right)^{\beta-1}, \end{aligned}$$

then the last equation gives that

$$\begin{aligned} D_\xi^\beta[e^{(h_2+h_1)\eta}Q(\eta,\eta)] &= e^{(h_2+h_1)\eta}\left[\frac{[q\left(\frac{\eta}{2}\right)-r\left(\frac{\eta}{2}\right)]}{(1+(\frac{\eta}{2})^{\beta-1})}\left(\frac{\eta}{2}\right)^{\beta-1}\right] \\ &+ e^{(h_2+h_1)\eta}\left[\frac{1}{2}\int_0^\eta \left[q\left(\frac{\eta+\mu}{2}\right)-r\left(\frac{\eta+\mu}{2}\right)\right]Q(\eta,\mu)d_\beta\mu\right]. \end{aligned}$$

Integrating this equation from 0 from η and considering the equation (2. 22) leads to,

$$\begin{aligned} Q(\eta,\eta) &= -(h_2+h_1)e^{-(h_2+h_1)\eta} \\ &+ e^{-(h_2+h_1)\eta}\int_0^\eta e^{(h_2+h_1)\alpha}\left[\frac{1}{2}\int_0^\eta \left[q\left(\frac{\alpha+\mu}{2}\right)-r\left(\frac{\alpha+\mu}{2}\right)\right]Q(\alpha,\mu)d_\beta\mu\right]d_\beta\alpha \\ &+ e^{-(h_2+h_1)\eta}\int_0^\eta e^{(h_2+h_1)\alpha}\left[\frac{[q\left(\frac{\alpha}{2}\right)-r\left(\frac{\alpha}{2}\right)]}{(1+(\frac{\alpha}{2})^{\beta-1})}\left(\frac{\alpha}{2}\right)^{\beta-1}\right]d_\beta\alpha. \end{aligned}$$

Thus equation $Q(\xi,\eta)$ must satisfy the integral equation

$$\begin{aligned}
Q(\xi,\eta) &= \frac{1}{4}\int_{\eta}^{\alpha} d_{\beta}\alpha \int_{0}^{\eta}\left[q\left(\frac{\alpha+\mu}{2}\right)-r\left(\frac{\alpha+\mu}{2}\right)\right]Q(\alpha,\mu)d_{\beta}\mu \\
&+\int_{0}^{\xi}\left[\frac{[q\left(\frac{\mu}{2}\right)-r\left(\frac{\mu}{2}\right)]}{2(1+(\frac{\mu}{2})^{\beta-1})}\left(\frac{\mu}{2}\right)^{\beta-1}\right]d_{\beta}\mu \\
&+e^{-(h_2+h_1)\eta}\left(-(h_2+h_1)+\int_{0}^{\eta}e^{(h_2+h_1)\alpha}\right. \\
&\cdot\left[\frac{1}{2}\int_{0}^{\alpha}\left[q\left(\frac{\alpha+\mu}{2}\right)-r\left(\frac{\alpha+\mu}{2}\right)\right]Q(\alpha,\mu)d_{\beta}\mu\right]d_{\beta}\alpha \\
&\left.+\int_{0}^{\eta}e^{(h_2+h_1)\alpha}\left[\frac{[q\left(\frac{\alpha}{2}\right)-r\left(\frac{\alpha}{2}\right)]}{(1+(\frac{\alpha}{2})^{\beta-1})}\left(\frac{\alpha}{2}\right)^{\beta-1}\right]d_{\beta}\alpha\right).
\end{aligned} \tag{2. 26}$$

And also, if function $Q(\xi,\eta)$ satisfies the integral equation (2. 26) with $q(x)$ and $r(x)$ each differentiable once, it can be verified easily that (2. 26) is a solution of problems (2. 20) to (2. 22).

This completes the Proof and the desired result, that is establishing the transformation operator for the conformable Sturm-liouville problem, is obtained. The equivalent of equation (2. 26) in the ordinary case is a voltera-type integral equation, also, in this case (2. 26) behave the same. The assumption that the functions $q(x)$ and $r(x)$ are each differentiable will not have an impact on the approximation because continuous functions can be approximated by smooth functions. The kernel of the transformation operator is then extrapolated to the limit function $N(x,\tau)$ produced. If $A=-D_x^{\beta}D_x^{\beta}+q(x) : and : B=-D_x^{\beta}D_x^{\beta}$ we obtain two problems as,

$$\begin{aligned}
&Af=\mu f, \\
&D_x^{\beta}f(0)=0
\end{aligned} \tag{2. 27}$$

and

$$\begin{aligned}
&Bf=\mu f, \\
&D_x^{\beta}f(0)=0
\end{aligned} \tag{2. 28}$$

and the solution of the problems (2. 27)and (2. 28) are $m(x)$ and $\cos\frac{\sqrt{\mu}}{\beta}x^{\beta}$. In this case, we can write the transformation operator as

$$X\left(\cos\frac{\sqrt{\mu}}{\beta}x^{\beta}\right)=\cos\frac{\sqrt{\mu}}{\beta}x^{\beta}+\int_{0}^{x}N(x,\tau)\cos\frac{\sqrt{\mu}}{\beta}\tau^{\beta}d_{\beta}\tau.$$

Then we can obtain a partial differential equation for $N(x,\tau)$.

3. Conformable SLP

Conformable SLPs are the classical SLPs expressed under the Conformable derivative approach. As there are two ways to solve the SLPs in the classical form so also in the conformable form that is the direct and the inverse ways.
The information, the definitions, theorem and all the proofs below can be found in the references [56, 64].

3.1 Defining the Conformable SLP

Theorem 3.1.1. *Conformable SLP has the general form as*

$$D^\beta(p(x)D^\beta y) + q(x)y = -\mu w(x)y, \tag{3. 1}$$

with the conditions that

$$y(a)\cos\lambda + D^\beta y(a)\sin\lambda = 0, \text{ and } y(b)\cos\alpha + D^\beta y(b)\sin\alpha = 0,$$

where $\beta \in (0,1], x \in (a,b), p(x), D_a^\beta p(x,)q(x)$ *and* $w(x)$ *(weight function) are continuous on* (a,b). *The fractional derivative is conformable and* $p(x) > 0$ *and* $w(x) > 0$ *on* $[a,b]$ *and* μ *is a spectral parameter.*

To demonstrate the above defined problem, to find the eigenvalues and the eigenfunctions, let us consider the following example.

Example 3.1.1. *Let us consider the conformable SLP*
$-D^\beta D^\beta y + q(x)y = \mu w(x)y$ *with the conditions that* $y(0) = y(1) = 0$. *Consider* $w(x) = 1$ *and* $q(x) = 0$.
The corresponding classical problem; $y'' + \mu y = 0$, *has the eigenvalue* $\mu_n = (n\pi)^2$ *and the eigenfunction* $\varphi_n(x) = \sin(n\pi x)$. *The author of [64] gave the eigenvalue and the eigenfunction of the fractional problem as* $\mu_n = (\beta n\pi)^2$ *and* $\varphi_n(x) = \sin(n\pi x^\beta)$ *respectively. Clearly, the two solutions are the same at* $\beta = 1$.

Theorem 3.1.2. *If* $D^\beta D^\beta y$ *is continuous on* $[a,b]$ *and* $y \in C^{2\beta}[a,b]$ *then* y *is* 2β*-continuously differentiable on* $[a,b]$.

By introducing L as the Sturm-Liouville operator to (3. 1) we can represent the problem as;

$$L(y,\beta) = -\mu w(x)y$$

where

$$L(y,\beta) = D^\beta(p(x)D^\beta y) + q(x)y$$

Definition 3.1.1. *$f(x)$ and $g(x)$ are β-orthogonal with regard to the weight function $w(x) \geq 0$ if they are two fractional differentiable functions and*

$$\int_a^b w(x)f(x)g(x)d_\beta x = 0$$

Theorem 3.1.3. *With regard to the weight function $w(x)$, the eigenfunctions of the conformable SLP* (3. 1) *that correspond to different eigenvalues are β-orthogonal.*

Theorem 3.1.4. *The eigenvalues of the conformable SLP* (3. 1) *are real.*

Definition 3.1.2. *Let $f(x)$ and $g(x)$ be two conformable differentiable functions, then their conformable Wronskian function is defined by*

$$W_\beta(f,g) = f D_a^\beta g - g D_a^\beta f$$

Theorem 3.1.5. *The eigenvalues of the conformable SLP* (3. 1) *are simple.*

3.2 Conformable SLP with Eigenparameter in the Boundary Conditions

The SLP can be solved in two ways. Once the potential function has been provided, the next step is to identify the spectral data in direct problems. In the second case, the objective is to determine the potential function given all of the spectrum parameters, such as eigenvalues, eigenfunctions, and so forth. The second case is generally regarded as more challenging and intriguing when compared to the first. It is noteworthy that authors have demonstrated that the zeros of eigenfunctions can be utilized to determine the potential function. In the literature, this is referred to as the inverse nodal problem. As such, in this section we give the solutions of the inverse Conformable SLP under a special boundary conditions- carrying the eigenparameter- and the corresponding spectral properties. The classical sort of the problem is discussed in [11]. We considered the following conformable fractional SLP, $L_\beta(q,f)$, that is

$$-D_x^\beta D_x^\beta y + q(x)y = \mu y \tag{3. 2}$$

with the conditions

$$D_x^\beta y(0) = 0, \text{ and } D_x^\beta y(\pi) + f(\mu)y(\pi) = 0 \tag{3. 3}$$

where D_x^β defines the conformable derivative of order β, $0 < \beta \leq 1$, q is a real valued continuous function and

$$f(\mu) = a_1\sqrt{\mu} + a_2\sqrt{\mu}^2 + ... + a_r\sqrt{\mu}^r, a_i \in \mathbb{R}, a_r \neq 0, r \in Z^+.$$

Let $Q_1(x,\mu)$ be solution of (3. 2) under the condition $y(0)=1$, then $Q_1(x,\mu)$ can be expressed as

$$Q_1(x,\mu)=\cos\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right)+\int_0^x N(x,\tau)\cos\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau. \tag{3. 4}$$

The kernel $N(x,\tau)$ is the solution of

$$D_x^\beta(D_x^\beta N(x,\tau))-q(x)N(x,\tau)=D_\tau^\beta(D_\tau^\beta N(x,\tau)), \tag{3. 5}$$

and $N(x,x)$ satisfies

$$N(x,x)=\int_0^x \frac{1}{1+s^{\beta-1}}q(s)d_\beta s. \tag{3. 6}$$

and also, let $Q_2(x,\mu)$ be the solution of

$$-D_x^\beta D_x^\beta y+q(x)y=\mu y \text{ under the condition that } \ y(0)=0,\ D_x^\beta y(0)=\sqrt{\mu}. \tag{3. 7}$$

Conversely, if $N(x,\tau)$ is a solution of (3. 5), then (3. 4) satisfies (3. 2) under the condition $y(0)=1$. This is known as transformation operator in the SLT.
Clearly, $Q_2(x,\mu)$ can be expressed as,

$$Q_2(x,\mu)=\sin\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right)+\int_0^x N(x,\tau)\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau.$$

Now, taking the integration in (3. 4) by parts using the conformable integration it follows that

$$\begin{aligned}Q_1(x,\mu)&=\cos\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right)+\int_0^x N(x,\tau)\left[\frac{1}{\sqrt{\mu}}D_\tau^\beta\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)\right]d_\beta\tau\\&=\cos\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right)+\frac{1}{\sqrt{\mu}}N(x,x)\sin\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right)\\&-\frac{1}{\sqrt{\mu}}\int_0^x D_\tau^\beta N(x,\tau)\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau,\end{aligned}$$

that is at $x=\pi$ we have

$$\begin{aligned}Q_1(\pi,\mu)&=\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right)+\frac{1}{\sqrt{\mu}}N(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right)\\&-\frac{1}{\sqrt{\mu}}\int_0^\pi D_\tau^\beta N(\pi,\tau)\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau.\end{aligned} \tag{3. 8}$$

By fractionally differentiating (3. 4) and using $\beta-$Leibniz rule lemma 2.2.3, we have

$$\begin{aligned}D_x^\beta Q_1(x,\mu)&=-\sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right)+N(x,x)\cos\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right)\\&+\int_0^x D_x^\beta N(x,\tau)\cos\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau,\end{aligned}$$

which gives

$$D_x^\beta Q_1(\pi,\mu) = -\sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) + N(\pi,\pi)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) + \int_0^\pi D_x^\beta N(\pi,\tau)\cos\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau. \tag{3. 9}$$

$Q_1(x,\mu)$ and $Q_2(x,\mu)$ are the fundamental solutions of (3. 2), that is, the general solution is

$$Q(x,\mu) = c_1Q_1(x,\mu) + c_2Q_2(x,\mu),\ c_1,c_2 \in \mathbb{R}.$$

but, since we considered the conditions that $D_x^\beta Q_1(x,\mu)(0) = 0$ and $D_x^\beta Q_2(x,\mu)(0) \neq 0$, it implies that $c_2 = 0$, then the solution become,

$$Q(x,\mu) = c_1Q_1(x,\mu).$$

It is well known that μ is an eigenvalue of the problem (3. 2) under (3. 3) if and only if

$$W_\beta(\mu,\tau) = D_x^\beta Q_1(\pi,\mu) + f(\mu)Q_1(\pi,\mu) = 0. \tag{3. 10}$$

Now, substituting (3. 8) and (3. 9) into (3. 10) yields

$$\begin{aligned} W_\beta(\mu) = &-\sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) + N(\pi,\pi)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) \\ &+ \int_0^\pi D_x^\beta N(\pi,\tau)\cos\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau + f(\mu)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) \\ &+ \frac{f(\mu)}{\sqrt{\mu}}N(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) - \frac{f(\mu)}{\sqrt{\mu}}\int_0^\pi D_\tau^\beta N(\pi,\tau)\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^\beta\right)d_\beta\tau = 0. \end{aligned} \tag{3. 11}$$

The zeros of the function $W_\beta(\mu)$ (characteristics equation) are its roots and these roots are the eigenvalues of problem (3. 2) represented by μ.

3.2.1 Calculating the spectral parameters

Let indicate the eigenvalues for problem (3. 2) by $\sigma_p(q)$. Now consider the lemma below.

Lemma 3.2.1. *If $r = 1$, we have*

$$\sqrt{\mu_n} = \frac{\beta n}{\pi^{\beta-1}} + \frac{\beta\arctan a_1}{\pi^\beta} + \frac{\beta}{\pi^\beta n}\int_0^\pi \frac{q(s)}{(1+s^{\beta-1})}d_\beta s + O\left(\frac{1}{n^2}\right). \tag{3. 12}$$

If $r = 2$,

$$\sqrt{\mu_n} = \frac{\beta n}{\pi^{\beta-1}} + \frac{\beta}{2\pi^{\beta-1}} + \frac{\beta}{\pi^\beta\left(n+\frac{1}{2}\right)}\left[\int_0^\pi \frac{q(s)}{(1+s^{\beta-1})}d_\beta s - \frac{1}{a_2}\right] + O\left(\frac{1}{n^2}\right). \tag{3. 13}$$

If $r \geq 3$*,*

$$\sqrt{\mu_n} = \frac{\beta n}{\pi^{\beta-1}} + \frac{\beta}{2\pi^{\beta-1}} + \frac{\beta}{\pi^{\beta}\left(n+\frac{1}{2}\right)}\left[\int_0^{\pi} \frac{q(s)}{(1+s^{\beta-1})} d_{\beta}s\right] + O\left(\frac{1}{n^2}\right). \tag{3. 14}$$

Proof 5. By considering equation (3. 11), we can see that μ is an eigenvalue of the problem (3. 2) if and only if

$$\begin{aligned} &-\sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) + N(\pi,\pi)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) \\ &+\int_0^{\pi} D_x^{\beta} N(\pi,\tau)\cos\left(\frac{\sqrt{\mu}}{\beta}\tau^{\beta}\right) d_{\beta}\tau + f(\mu)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) \\ &+\frac{f(\mu)}{\sqrt{\mu}} N(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) - \frac{f(\mu)}{\sqrt{\mu}}\int_0^{\pi} D_{\tau}^{\beta} N(\pi,\tau)\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^{\beta}\right) d_{\beta}\tau = 0 \end{aligned} \tag{3. 15}$$

Now, for $r = 1, f(\mu) = a_1\sqrt{\mu}$ and as $\mu \to \infty$ and the lemma 2.2.4, (3. 15) reduced to

$$\begin{aligned} \sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) &= N(\pi,\pi)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) + a_1\sqrt{\mu}\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) \\ &\quad + a_1 N(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) + O\left(\frac{1}{\sqrt{\mu}}\right), \end{aligned} \tag{3. 16}$$

by dividing both sides by $\sqrt{\mu}\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)$, (3. 16) can be expressed as

$$\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) = a_1 + \frac{N(\pi,\pi)}{\sqrt{\mu}} + \frac{a_1 N(\pi,\pi)\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)}{\sqrt{\mu}} + O\left(\frac{1}{\mu}\right)$$

and it can further be reduce to

$$\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) - a_1 = \frac{N(\pi,\pi)[1+a_1^2]}{\sqrt{\mu}} + O\left(\frac{1}{\mu}\right). \tag{3. 17}$$

Let us, by trigonometric identity, express the left hand side of the (3. 17) as

$$\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta} - \arctan a_1\right) = \frac{\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) - a_1}{1 + a_1\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)}.$$

By this identity, (3. 17) can be expressed as,

$$\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta} - \arctan a_1\right) = \left[1 + a_1\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)\right]^{-1}\left[\frac{N(\pi,\pi)[1+a_1^2]}{\sqrt{\mu}} + O\left(\frac{1}{\mu}\right)\right]. \tag{3. 18}$$

but as $\mu \to \infty$, we have $\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) \approx a_1$ then,

$$1 + a_1\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) = 1 + a_1^2 + O\left(\frac{1}{\sqrt{\mu}}\right),$$

as such the (3. 18) becomes,

$$\begin{aligned}\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}-\arctan a_1\right)&=\left[\frac{1}{[1+a_1^2]}+O\left(\frac{1}{\sqrt{\mu}}\right)\right]\left[\frac{N(\pi,\pi)[1+a_1^2]}{\sqrt{\mu}}+O\left(\frac{1}{\mu}\right)\right]\\&=\frac{N(\pi,\pi)}{\sqrt{\mu}}+O\left(\frac{1}{\mu}\right).\end{aligned}\tag{3. 19}$$

and

$$\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}-\arctan a_1\right)=\frac{\sqrt{\mu}}{\beta}\pi^{\beta}-\arctan a_1-n\pi,$$

by which (3. 19) gives

$$\sqrt{\mu_n}=\frac{\beta n}{\pi^{\beta-1}}+\frac{\beta\arctan a_1}{\pi^{\beta}}+\frac{\beta N(\pi,\pi)}{\pi^{\beta}\sqrt{\mu}}+O\left(\frac{1}{\mu}\right).$$

By the fact that $\frac{1}{\sqrt{\mu_n}}=\frac{1}{n}+O\left(\frac{1}{n^2}\right)$, we then write,

$$\sqrt{\mu_n}=\frac{\beta n}{\pi^{\beta-1}}+\frac{\beta\arctan a_1}{\pi^{\beta}}+\frac{\beta N(\pi,\pi)}{\pi^{\beta}n}+O\left(\frac{1}{n^2}\right),\tag{3. 20}$$

that is

$$\sqrt{\mu_n}=\frac{\beta n}{\pi^{\beta-1}}+\frac{\beta\arctan a_1}{\pi^{\beta}}+\frac{\beta}{n\pi^{\beta}(1+x^{\beta-1})}\int_0^{\pi}q(s)d_{\beta}s+O\left(\frac{1}{n^2}\right).\tag{3. 21}$$

Now, for the remaining two cases we write

$$\frac{\sqrt{\mu}}{f(\mu)}=\frac{N(\pi,\pi)\cot\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)}{f(\mu)}+\cot\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)+\frac{N(\pi,\pi)}{\sqrt{\mu}}+O\left(\frac{1}{\mu}\right),$$

which reduced to

$$\cot\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)=-\frac{N(\pi,\pi)}{\sqrt{\mu}}+\frac{\sqrt{\mu}}{f(\mu)}+O\left(\frac{1}{\mu}\right)=-\frac{N(\pi,\pi)}{\sqrt{\mu}}+\frac{1}{a_r\sqrt{\mu}^{r-1}}+O\left(\frac{1}{\mu}\right).$$

Now, when $r=2$, similarly, as in the first case, we obtain

$$\cot\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)=-\frac{N(\pi,\pi)}{\sqrt{\mu}}+\frac{1}{a_2\sqrt{\mu}}+O\left(\frac{1}{\mu}\right)=\frac{\frac{1}{a_2}-N(\pi,\pi)}{\sqrt{\mu}}+O\left(\frac{1}{\mu}\right).\tag{3. 22}$$

Then by Taylor's series expansion of cotangent inverse we obtain (3. 22) as

$$\frac{\sqrt{\mu}}{\beta}\pi^{\beta}=n\pi+\frac{\pi}{2}+\frac{\left[N(\pi,\pi)-\frac{1}{a_2}\right]}{\left(n+\frac{1}{2}\right)}+O\left(\frac{1}{n^2}\right),$$

which gives

$$
\begin{aligned}
\sqrt{\mu_n} &= \frac{\beta n}{\pi^{\beta-1}} + \frac{\beta}{2\pi^{\beta-1}} + \frac{\beta\left[N(\pi,\pi) - \frac{1}{a_2}\right]}{\left(n+\frac{1}{2}\right)\pi^{\beta}} + O\left(\frac{1}{n^2}\right) \\
&= \frac{\beta n}{\pi^{\beta-1}} + \frac{\beta}{2\pi^{\beta-1}} + \frac{\beta}{\left(n+\frac{1}{2}\right)\pi^{\beta}}\left[\int_0^{\pi} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s - \frac{1}{a_2}\right] + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

For $r \geq 3$ we get

$$
\cot\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) = -\frac{N(\pi,\pi)}{\sqrt{\mu}} + \frac{1}{a_3\mu} + O\left(\frac{1}{\mu}\right) = -\frac{N(\pi,\pi)}{\sqrt{\mu}} + O\left(\frac{1}{\mu}\right),
$$

also, similarly as in the above case we obtain

$$
\begin{aligned}
\sqrt{\mu_n} &= \frac{\beta n}{\pi^{\beta-1}} + \frac{\beta}{2\pi^{\beta-1}} + \frac{\beta N(\pi,\pi)}{\left(n+\frac{1}{2}\right)\pi^{\beta}} + O\left(\frac{1}{n^2}\right) \\
&= \frac{\beta n}{\pi^{\beta-1}} + \frac{\beta}{2\pi^{\beta-1}} + \frac{\beta}{\left(n+\frac{1}{2}\right)\pi^{\beta}}\int_0^{\pi} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

3.3 Uniqueness of the Potential function

To prove the uniqueness of the potential function in this case, let's consider a second eigenvalue SLP as follows so that we proved the uniqueness by showing that $q(x) = \tilde{q}(x)$.

$$
-D_x^{\beta} D_x^{\beta} y + \tilde{q}(x) y = \mu y. \tag{3. 23}
$$

with the conditions

$$
D_x^{\beta} y(0) = 0,\; D_x^{\beta} y(\pi) + f(\mu) y(\pi) = 0,
$$

where $\tilde{q}(x)$ and $q(x)$ have the same properties.

Theorem 3.3.1. *Assume $\sigma_p(q) = \sigma_p(\tilde{q})$, then, $\int_0^{\pi}[q(s) - \tilde{q}(s)] d_\beta s = 0$.*

Proof 6. Denote μ_n by large eigenvalue for *SLP* including $q(x)$. Since $\sigma_p(q) = \sigma_p(\tilde{q})$, it follows that $\mu_n \in \sigma_p(\tilde{q})$, as such from (3. 11), we have

$$
\begin{aligned}
&-\sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) + N(\pi,\pi)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) \\
&+ \int_0^{\pi} D_x^{\beta} N(\pi,\tau)\cos\left(\frac{\sqrt{\mu}}{\beta}\tau^{\beta}\right) d_\beta \tau + f(\mu)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) \\
&+ \frac{f(\mu)}{\sqrt{\mu}} N(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right) - \frac{f(\mu)}{\sqrt{\mu}}\int_0^{\pi} D_\tau^{\beta} N(\pi,\tau)\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^{\beta}\right) d_\beta \tau = 0,
\end{aligned} \tag{3. 24}
$$

and also

$$
\begin{aligned}
&-\sqrt{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)+\tilde{N}(\pi,\pi)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)\\
&+\int_0^{\pi} D_x^{\beta}\tilde{N}(\pi,\tau)\cos\left(\frac{\sqrt{\mu}}{\beta}\tau^{\beta}\right)d_{\beta}\tau+f(\mu)\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)\\
&+\frac{f(\mu)}{\sqrt{\mu}}\tilde{N}(\pi,\pi)\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)-\frac{f(\mu)}{\sqrt{\mu}}\int_0^{\pi} D_{\tau}^{\beta}\tilde{N}(\pi,\tau)\sin\left(\frac{\sqrt{\mu}}{\beta}\tau^{\beta}\right)d_{\beta}\tau=0.
\end{aligned}
\tag{3. 25}
$$

By subtracting (3. 25) from (3. 24), we have

$$
\begin{aligned}
&[N(\pi,\pi)-\tilde{N}(\pi,\pi)]\cos\left(\frac{\sqrt{\mu_n}}{\beta}\pi^{\beta}\right)+\int_0^{\pi} D_x^{\beta}[N(\pi,\tau)-\tilde{N}(\pi,\tau)]\cos\left(\frac{\sqrt{\mu_n}}{\beta}\tau^{\beta}\right)d_{\beta}\tau\\
&+\frac{f(\mu_n)}{\sqrt{\mu_n}}[N(\pi,\pi)-\tilde{N}(\pi,\pi)]\sin\left(\frac{\sqrt{\mu_n}}{\beta}\tau^{\beta}\right)\\
&-\frac{f(\mu_n)}{\sqrt{\mu_n}}\int_0^{x} D_x^{\beta}[N(\pi,\tau)-\tilde{N}(\pi,\tau)]\sin\left(\frac{\sqrt{\mu_n}}{\beta}\tau^{\beta}\right)d_{\beta}\tau=0
\end{aligned}
\tag{3. 26}
$$

Case(I) when $r=1$, using (3. 12) and lemma 2.2.4, equation (3. 26) can be reduced to

$$
\cos\left(\frac{\sqrt{\mu_n}}{\beta}\pi^{\beta}\right)[N(\pi,\pi)-\tilde{N}(\pi,\pi)]+\frac{f(\mu_n)}{\sqrt{\mu_n}}\sin\left(\frac{\sqrt{\mu_n}}{\beta}\tau^{\beta}\right)[N(\pi,\pi)-\tilde{N}(\pi,\pi)]+O(1)=0,
$$

that is

$$
\left[\cos\left(\frac{\sqrt{\mu_n}}{\beta}\pi^{\beta}\right)+\frac{f(\mu_n)}{\sqrt{\mu_n}}\sin\left(\frac{\sqrt{\mu_n}}{\beta}\tau^{\beta}\right)\right][N(\pi,\pi)-\tilde{N}(\pi,\pi)]+O(1)=0. \tag{3. 27}
$$

Remark 3.3.1. From (3. 20) and Pythagoras theorem, the cosine and the sine functions can be expressed as follows

$$
\begin{aligned}
\cos\left(\frac{\sqrt{\mu_n}}{\beta}\pi^{\beta}\right)&\approx\cos\left(\left(\frac{\beta n}{\pi^{\beta-1}}+\frac{\beta\arctan a_1}{\pi^{\beta}}\right)\frac{\pi^{\beta}}{\beta}\right)\\
&=\cos(n\pi)\cos(arctana_1)-\sin(n\pi)\sin(arctana_1)\\
&=(-1)^n\cos(arctana_1)=(-1)^n\cos(k)\\
&=\frac{(-1)^n}{\sqrt{1+a_1^2}}
\end{aligned}
$$

where $k=arctana_1$. Similarly

$$
\sin\left(\frac{\sqrt{\mu_n}}{\beta}\pi^{\beta}\right)\approx\frac{(-1)^n a_1}{\sqrt{1+a_1^2}}
$$

By this remark, equation (3. 27) becomes

$$
(-1)^n\sqrt{1+a_1^2}[N(\pi,\pi)-\tilde{N}(\pi,\pi)]+O(1)=0.
$$

Since, for all n, $(-1)^n\sqrt{1+a_1^2} \neq 0$, the above equation implies that

$$[N(\pi,\pi) - \tilde{N}(\pi,\pi)] = 0$$

as such by (3. 6) we have

$$\int_0^x \frac{[q(s) - \tilde{q}(s)]}{1+s^{\beta-1}} d_\beta s = 0,$$

that is

$$\int_0^x [q(s) - \tilde{q}(s)] d_\beta s = 0$$

Case(II) In this case, when $r \geq 2$, by multiplying both sides of equation (3. 26) with $\frac{\sqrt{\mu_n}}{f(\mu_n)}$, we have

$$\begin{aligned}
&[N(\pi,\pi) - \tilde{N}(\pi,\pi)]\frac{\sqrt{\mu_n}}{f(\mu_n)}\cos\left(\frac{\sqrt{\mu_n}}{\beta}\pi^\beta\right) \\
&+ \frac{\sqrt{\mu_n}}{f(\mu_n)}\int_0^\pi D_x^\beta[N(\pi,\tau) - \tilde{N}(\pi,\tau)]\cos\left(\frac{\sqrt{\mu_n}}{\beta}\tau^\beta\right) d_\beta\tau \\
&+ [N(\pi,\pi) - \tilde{N}(\pi,\pi)]\sin\left(\frac{\sqrt{\mu_n}}{\beta}\tau^\beta\right) \\
&- \int_0^x D_x^\beta[N(\pi,\tau) - \tilde{N}(\pi,\tau)]\sin\left(\frac{\sqrt{\mu_n}}{\beta}\tau^\beta\right) d_\beta\tau = 0,
\end{aligned} \tag{3. 28}$$

by the virtue of the lemma 2.2.4, equation (3. 28) becomes

$$[\frac{\sqrt{\mu_n}}{f(\mu_n)}\cos(\frac{\sqrt{\mu_n}}{\beta}\pi^\beta) + \sin(\frac{\sqrt{\mu_n}}{\beta}\tau^\beta)][N(\pi,\pi) - \tilde{N}(\pi,\pi)] + O(1) = 0.$$

Remark 3.3.2. Similarly as in remark 3.3.1, the cosine and the sine functions can be expressed as follows,

$$\begin{aligned}
\sin\left(\frac{\sqrt{\mu_n}}{\beta}\pi^\beta\right) &\approx \sin\left(\left(\frac{\beta n}{\pi^{\beta-1}} + \frac{\beta}{2\pi^{\beta-1}}\right)\frac{\pi^\beta}{\beta}\right) \\
&= \sin(n\pi)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\cos(n\pi) = (-1)^n,
\end{aligned}$$

in the same way

$$\cos\left(\frac{\sqrt{\mu_n}}{\beta}\pi^\beta\right) \approx 0,$$

then by using (3. 13),(3. 14) and the remark 3.3.2 on (3. 28), we obtain

$$(-1)^n[N(\pi,\pi) - \tilde{N}(\pi,\pi)] + O(1) = 0 \tag{3. 29}$$

then (3. 29) implies that $[N(\pi,\pi)-\tilde{N}(\pi,\pi)]=0$, as such,

$$\int_0^x [q(s)-\tilde{q}(s)]d_\beta s = 0. \tag{3. 30}$$

It's clear that the only possibility for (3. 30) to be zero is when $q(s)-\tilde{q}(s)=0$. This completes the proof.

Theorem 3.3.2. *Suppose* $q \in C[0,\pi]$*, and* $\sigma_p(q)=\sigma_p(0)$*. Then* $q(x)=0$ *on* $[0,\pi]$.

Proof 7. By the theorem (3. 13), if $\sigma_p(q)=\sigma_p(0)$, then

$$\int_0^\pi q(s)d_\beta s = 0\, for,\, 0<\beta\leq 1.$$

Let y_1 be the eigenfunction corresponding to $\mu=0\in\sigma_p(q)=\sigma_p(0)$, then

$$-D_x^\beta D_x^\beta y_1 + q(x)y_1 = 0 \tag{3. 31}$$

with the conditions

$$D_x^\beta y_1(0)=0,\, D_x^\beta y_1(\pi)=0. \tag{3. 32}$$

We claim that $y_1(0)\neq 0$ and $y_1(\pi)\neq 0$ otherwise we obtain $y_1(x)=0$ is an eigenfunction of (3. 31) which is a contradiction. Now,

$$\frac{D_x^\beta D_x^\beta y_1}{y_1} = q(x) \Rightarrow D_x^\beta\left(\frac{D_x^\beta y_1}{y_1}\right)+\left(\frac{D_x^\beta y_1}{y_1}\right)^2 = q(x), \tag{3. 33}$$

and by conformable integral, integrating (3. 33) on $[0,\pi]$ gives

$$\left.\frac{D_x^\beta y_1}{y_1}\right|_0^\pi + \int_0^\pi \left(\frac{D_x^\beta y_1}{y_1}\right)^2 d_\beta x = \int_0^\pi q(x)d_\beta x = 0,$$

that is, by the boundary conditions (3. 32),

$$\int_0^\pi \left(\frac{D_x^\beta y_1}{y_1}\right)^2 d_\beta x = \int_0^\pi \left(\frac{D_x^\beta y_1}{y_1}\right)^2 x^{\beta-1}dx = 0,$$

for $0<\beta\leq 1$ and $x\in[0,\pi]$. Eventually, we get
$y_1 = c$ (a constant) and

$$-D_x^\beta D_x^\beta c + q(x)c = 0 \Rightarrow q(x)=0.$$

This completes the proof.

3.4 The Inverse Nodal Problem

In this section, the nodal lengths and points were calculated, as well as the potential function reconstruction and evidence for each. The asymptotic form of the eigenvalues for issue(3. 2) is provided by the following lemma. Moreover, the eigenvalues are the roots of the equation $W_\beta(\mu,\tau)=0$. It is evident that this function is solely reliant on μ and β, rather than x.

Let $y(x,\mu_n)$ be the eigenfunctions that correspond to the eigenvalues, and let $\{\mu_n\}_{n\geq 0}$ be the spectrum of (3. 2) under (3. 3). The Oscillation theorem makes it clear that $y(x,\mu_n)$ has precisely n roots that lie in $(0,\pi)$. Let us define $Y_\beta=\left\{y_j^n=\frac{(x_j^n)^\beta}{\beta}, x_j^n\in X_\beta\right\}, n>0$ and $j=\overline{1,n}$. Besides, $X_\beta=\left\{x_j^n : n\in\mathbb{N}\right\}$ indicates a set that consists of the nodal set of (3. 2) under (3. 3). That is , $y(x_j^n,\mu_n)=0$. In addition, we will indicate j^{th} nodal domain of the n^{th} element with $I_{\beta j}^n=\left[y_j^n, y_{j+1}^n\right]$ and also $l_j^n=y_{j+1}^n-y_j^n$ be the nodal length of the j^{th} domain. In order to solve the inverse nodal problem, we must create another function, $j_n(y)$, which is the greatest index j such that $0\leq y_j^n\leq y$. Then, if and only if $y\in\left[y_j^n, y_{j+1}^n\right]$, $j=j_n(y)$.

The nodal points are computed using the problem's eigenvalues as stated in lemma 3.2.1.

Theorem 3.4.1. *The nodal points of the problem* (3. 2) *under* (3. 3) *provide the asymptotic form as* $n\to\infty$, *for* $r=1$

$$y_j^n=\left(j-\frac{1}{2}\right)\frac{\pi^\beta}{\beta n}+\left(j-\frac{1}{2}\right)\frac{\pi^{\beta-1}\arctan a_1}{\beta n^2}+\frac{\pi^{2\beta-2}}{\beta^2n^2}N(y_j^n,y_j^n)+O\left(\frac{1}{n^3}\right),$$

and for $r\geq 2$

$$y_j^n=\left(j-\frac{1}{2}\right)\left(1+\frac{1}{2n}\right)\frac{\pi^\beta}{\beta n}+\left(1+\frac{1}{2n}\right)^2\frac{\pi^{2\beta-2}}{\beta^2n^2}N(y_j^n,y_j^n)+O\left(\frac{1}{n^3}\right).$$

Proof 8. Considering the solution of the initial value problem (3. 2) under (3. 3)

$$Q_1(x,\mu)=\cos\left(s\frac{x^\beta}{\beta}\right)+\int_0^x N(x,\tau)\cos\left(s\frac{\tau^\beta}{\beta}\right)d_\beta\tau,$$

or

$$Q_1(x,\mu)=\cos\left(s\frac{x^\beta}{\beta}\right)+\frac{N(x,x)}{s}\sin\left(s\frac{\tau^\beta}{\beta}\right)+O\left(\frac{1}{s}\right),$$

where $s=\sqrt{\mu}$, then by definition of nodal points, it yields

$$Q_1(x,\mu)=\cos\left(s\frac{x^\beta}{\beta}\right)+\frac{N(x,x)}{s}\sin\left(s\frac{\tau^\beta}{\beta}\right)+O\left(\frac{1}{s}\right)=0,$$

which gives

$$\cot\left(s\frac{x^{\beta}}{\beta}\right)=-\frac{N(x,x)}{s}+O\left(\frac{1}{s}\right),$$

that is

$$\left(s\frac{x^{\beta}}{\beta}\right)=arccot\left[-\frac{N(x,x)}{s}+O\left(\frac{1}{s}\right)\right],$$

and using Taylor's expansion of $arccotx$ near zero we have

$$y_j^n=\frac{\left(j-\frac{1}{2}\right)\pi}{s_n}+\frac{N(y_j^n,y_j^n)}{s_n^2}+O\left(\frac{1}{s_n^2}\right) \tag{3. 34}$$

If $r=1$, then from (3. 12)

$$s_n=\frac{\beta n}{\pi^{\beta-1}}+\frac{\beta\arctan a_1}{\pi^{\beta}}+O\left(\frac{1}{n^2}\right),$$

that is

$$\begin{aligned}\frac{1}{s_n}&=\frac{1}{\frac{\beta n}{\pi^{\beta-1}}\left(1+\frac{\arctan a_1}{n\pi}+O(\frac{1}{n^2})\right)}\\&=\frac{\pi^{\beta-1}}{\beta n}+\frac{\arctan a_1}{\beta n^2}\pi^{\beta-2}+O\left(\frac{1}{n^3}\right),\end{aligned} \tag{3. 35}$$

and also

$$\frac{1}{s_n^2}=\frac{\pi^{2\beta-2}}{\beta^2 n^2}+O\left(\frac{1}{n^3}\right). \tag{3. 36}$$

Then inserting (3. 35) and (3. 36) in (3. 34) gives

$$y_j^n=\frac{\left(j-\frac{1}{2}\right)\pi^{\beta}}{n}+\frac{\arctan a_1}{n^2}\left(j-\frac{1}{2}\right)\pi^{\beta-1}+\frac{\pi^{2\beta-2}N(y_j^n,y_j^n)}{\beta n^2}+O\left(\frac{1}{n^3}\right).$$

which gives the first case.

For $r\geq 2$,

$$s_n=\frac{\beta n}{\pi^{\beta-1}}+\frac{\beta}{2\pi^{\beta-1}}+O\left(\frac{1}{n}\right),$$

that is

$$\begin{aligned}\frac{1}{s_n}&=\frac{1}{\frac{\beta n}{\pi^{\beta-1}}\left(1+\frac{1}{2n}+O\left(\frac{1}{n^2}\right)\right)}=\frac{\pi^{\beta-1}}{\beta n}+\frac{\pi^{\beta-1}}{2\beta n^2}+O\left(\frac{1}{n^3}\right)\\&=\frac{\pi^{\beta-1}}{\beta n}\left(1+\frac{1}{2n}\right)+O\left(\frac{1}{n^3}\right),\end{aligned} \tag{3. 37}$$

this implies that

$$\frac{1}{s_n^2} = \frac{\pi^{2\beta-2}}{\beta^2 n^2}\left(1+\frac{1}{2n}\right)^2 + O\left(\frac{1}{n^3}\right). \tag{3. 38}$$

then using (3. 37) and (3. 38) in (3. 34), we have

$$y_j^n = (j-\frac{1}{2})\left(1+\frac{1}{2n}\right)\frac{\pi^\beta}{\beta n} + \left(1+\frac{1}{2n}\right)^2 \frac{\pi^{2\beta-2}}{\beta^2 n^2} N(y_j^n, y_j^n) + O\left(\frac{1}{n^3}\right).$$

which completes the proofs.

To obtain the nodal lengths, by the definition of the nodal lengths that $l_j^n = y_{j+1}^n - y_j^n$, we can obtain nodal lengths for $r=1$ and $r \geq 2$, respectively as follows.
The nodal lengths of the problem (3. 2) under (3. 3) are for $r=1$

$$\begin{aligned} l_j^n &= \frac{\pi^\beta}{n\beta} + \frac{\pi^{\beta-1}\arctan a_1}{\beta n^2} \\ &+ \frac{\pi^{2\beta-2}}{\beta^2 n^2}\left[\int_0^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s - \int_0^{y_j^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O\left(\frac{1}{n^3}\right) \\ &= \frac{\pi^\beta}{n\beta} + \frac{\pi^{\beta-1}\arctan a_1}{\beta n^2} + \frac{\pi^{2\beta-2}}{\beta^2 n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O\left(\frac{1}{n^3}\right), \end{aligned} \tag{3. 39}$$

for $r \geq 2$

$$\begin{aligned} l_j^n &= \left(1+\frac{1}{2n}\right)\frac{\pi^\beta}{n\beta} + \left(1+\frac{1}{2n}\right)^2 \frac{\pi^{2\beta-2}}{\beta^2 n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O\left(\frac{1}{n^3}\right) \\ &= \left(1+\frac{1}{2n}\right)\frac{\pi^\beta}{n\beta} + \left(1+\frac{1}{2n}\right)^2 \frac{\pi^{2\beta-2}}{\beta^2 n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O(\frac{1}{n^3}). \end{aligned} \tag{3. 40}$$

3.4.1 Reconstruction of the Potential

Besides calculating the spectral parameters in two cases so far, we reconstructed the potential function q case wise, $r=1$, and $r \geq 2$, as follows.

Theorem 3.4.2. *The reconstruction of the potential of the problem* (3. 2) *under* (3. 3) *for* $q \in C[0,\pi]$ *is,*
for $r=1$*,*

$$q(x) = (1+x^{\beta-1}) \lim_{n\to\infty} s_n^2 \left(\frac{s_n l_j^n}{\pi} - 1\right),$$

and $r \geq 2$

$$q(x) = (1+x^{\beta-1}) \lim_{n\to\infty} \frac{s_n^3}{\pi}\left(1+\frac{1}{2n}\right).$$

Proof 9. By definition of nodal lengths (3. 39), we have for $r=1$

$$\left(l_j^n - \pi\left(\frac{\pi^{\beta-1}}{n\beta} + \frac{\pi^{\beta-2}\arctan a_1}{\beta n^2}\right)\right)\frac{\beta^2 n^2}{\pi^{2\beta-2}} = \int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s + O\left(\frac{1}{n^3}\right),$$

then, from the (3. 35) and (3. 36) we have

$$\left(l_j^n - \frac{\pi}{s_n}\right)s_n^2 = \int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s + O\left(\frac{1}{n^3}\right),$$

that is

$$\left(\frac{s_n l_j^n}{\pi} - 1\right) = \frac{1}{s_n \pi}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O\left(\frac{1}{n}\right).$$

By mean value theorem for the integrals there is a $z \in (y_j^n, y_{j+1}^n)$ such that

$$\left(\frac{s_n l_j^n}{\pi} - 1\right) = \frac{l_j^n}{s_n \pi}\frac{q(z)}{(1+z^{\beta-1})} + O\left(\frac{1}{n}\right).$$

From the fact that, as $n \to \infty$, $l_j^n \to \frac{\pi}{s_n}$, we obtain

$$q(x) = (1+x^{\beta-1})\lim_{n\to\infty} s_n^2\left(\frac{s_n l_j^n}{\pi} - 1\right),$$

and for the proof for $r \geq 2$, from (3. 40) we have

$$\left(1+\frac{1}{2n}\right)^2 l_j^n = \pi\left(1+\frac{1}{2n}\right)^2\frac{\pi^{\beta-1}}{n\beta} + \left(1+\frac{1}{2n}\right)^2\frac{\pi^{2\beta-2}}{\beta^2 n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O\left(\frac{1}{n^5}\right),$$

then, from the (3. 37) and (3. 38) we obtain

$$\left(1+\frac{1}{2n}\right)^2 l_j^n = \pi\left(1+\frac{1}{2n}\right)\frac{1}{s_n} + \frac{1}{s_n^2}\left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O\left(\frac{1}{n^5}\right),$$

which becomes

$$\left(1+\frac{1}{2n}\right) l_j^n \frac{s_n^3}{\pi} = \left[\int_{y_j^n}^{y_{j+1}^n} \frac{q(s)}{(1+s^{\beta-1})} d_\beta s\right] + O\left(\frac{1}{n^5}\right),$$

also, by mean value theorem as in the above case, we have

$$\left(1+\frac{1}{2n}\right) l_j^n \frac{s_n^3}{\pi} = l_j^n \frac{q(z)}{(1+z^{\beta-1})} + O\left(\frac{1}{n^5}\right),$$

and by taking limit as $n \to \infty$ we have

$$q(x) = (1+x^{\beta-1})\lim_{n\to\infty} \frac{s_n^3}{\pi}\left(1+\frac{1}{2n}\right)$$

The fractional integral of q and nodal lengths make up the following function.

$$F_n(x) = \frac{2n^2\beta^2}{\pi^{2\beta-2}} \left[\frac{n\beta l_j^n}{\pi} + \frac{l_j^n}{2n^2\beta\pi^2} \int_0^\pi q(\tau) d_\beta \tau - 1 \right].$$

Following lemmas complete proof of the theorem 3.4.2.

Lemma 3.4.1. *Suppose that the sequence $g_k \in C[0,\pi]$ converges to any function g in the space L_1^β. Then for any $\varepsilon > 0$, with $j = j_n(y)$*

$$\left\| \frac{\mu_n}{\beta\pi} \int_{y_j^n}^{y_{j+1}^n} (g_k(\tau) - g(\tau))\, d_\beta \tau \right\|_1 < \varepsilon.$$

Lemma 3.4.2. *Assume that $q \in L_1^\beta(0,\pi)$ and $r = 1$. Then, as $n \to \infty$,*

$$\left\| \frac{\mu_n}{\beta\pi} \int_{y_j^n}^{y_{j+1}^n} q(\tau) d_\beta \tau - q(y) \right\|_1 \to 0, \;\; j = j_n(y).$$

Theorem 3.4.3. *$F_n(x)$ converges to the the potential function $q(x)$ in L_1^β.*

Proof 10. This theorem shows that the limit defined in the theorem 3.4.2 exists. proof of the theorem 3.4.3 is omitted.

4. The Conformable SLP with a Constant Delay

In this part, we used the conformable derivative approach to acquire the results of the direct and inverse nodal SLP with a constant delay under two sets of different boundary conditions. We also expressed the associated spectral features and demonstrated the stability of the solution. As inferred from [70], the appropriate classical findings can be obtained at $\beta = 1$. The lemma below, which is about Big O approximation, hold in the conformable derivative case as it hold in the classical derivative case, below is the proof.

Lemma 4.0.1. *If the derivative $D_\tau^\beta q(\tau)$ exist and is bounded, and for a delay constant ν, then for $0 \leq x \leq \pi$,*

$$\frac{1}{2}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}\left(\tau^\beta + (\tau-\nu)^\beta\right)\right) d_\beta\tau = O\left(\frac{1}{s}\right),$$

and

$$\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta + (\tau-\nu)^\beta\right)\right) d_\beta\tau = O\left(\frac{1}{s}\right).$$

Proof 11. The proof of the two integrals above goes similarly, as such the proof of one of them suffices the proof of the other, let us now consider the first integrals.
Let

$$T(\tau) = \eta = \frac{1}{\beta}(\tau^\beta + (\tau-\nu)^\beta) = \frac{1}{\beta}(\tau^\beta + \Delta(\tau)),$$

where $\Delta(\tau) = (\tau-\nu)^\beta$, then

$$D_\tau^\beta(T(\tau)) = D_\tau^\beta(\eta) = (1 + \frac{1}{\beta}D_\tau^\beta\Delta(\tau)) \Rightarrow d_\beta^\beta\eta = (1 + \frac{1}{\beta}D_\tau^\beta\Delta(\tau))d_\beta\tau,$$

as such

$$d_\beta^\beta\tau = \frac{d_\beta\eta}{(1 + \frac{1}{\beta}D_\tau^\beta\Delta(\chi(\eta)))},$$

where $\chi(\eta) = \frac{\tau^\beta}{\beta} - R(\eta)$ and $R(\eta) = \frac{1}{\beta}(\eta^\beta + \Delta(\eta))$ is the inverse of the function $T(\tau)$. The hypotheses of the lemma shows that $D_\tau^\beta T(\tau) > 0$, hence $T(\tau)$ is a monotone function which is conformable differentiable on $[0,\pi]$ and this implies that $\chi(\eta)$ exist and is conformable differentiable and the derivative is bounded. Therefore

$$\int_0^x q(\tau)\sin\left(\frac{s}{\beta}\left(\tau^\beta + (\tau-\nu)^\beta\right)\right) d_\beta\tau = \int_{-\nu}^{2x-\nu} \phi(\eta)\sin s\eta d_\beta\eta, \tag{4. 1}$$

where

$$\phi(\eta) = q\left((\beta\eta - \Delta(\eta))^{\frac{1}{\beta}}\right).\frac{1}{(1+\frac{1}{\beta}D_\tau^\beta \Delta(\chi(\eta)))},$$

also, by the hypotheses of the lemma, $\phi(\eta)$ is bounded and it has a bounded derivative. Integrating (4. 1) by parts gives

$$\int_{-\nu}^{2x-\nu} \phi(\eta) \sin s\eta d_\beta \eta = O\left(\frac{1}{s}\right).$$

This completes the proof.

4.1 A conformable SLP with constant delay with mixed boundary condition

In addition to being the proof of the spectrum specification, the result of SLP with a constant delay under a conformable operator with a mixed boundary condition obtained here is novel in its literature.

We consider the SLP below with conformable derivative operator

$$-D_x^\beta D_x^\beta y + q(x)y(x-\nu) = \mu y(x),\ for\ \ x \in (0,\pi) \tag{4. 2}$$

under the condition

$$y(0) = y^{(j)}(\pi) = 0,\ \ for\ j = 0,1,$$

for $\nu \in (0,\pi)$, and $q(x) \in L(\nu,\pi)$. Taking μ as the spectral parameter and the potential function $q(x) = 0$ for $x \in [0,\nu]$.

By defining an operator

$$L_\beta y(x) = -D_x^\beta D_x^\beta y + q(x)y(x-\nu)$$

then (4. 2) can be expressed as

$$L_\beta y(x) = \mu y(x),\ \ x \in (0,\pi) \tag{4. 3}$$

We assumed that the eigenvalues of (4. 3) are indicated by $\{\mu_{n_j}\}_{n\geq 1, j=0,1}$.

Consider $N \in \mathbb{N}$ such that $\nu \in \left[\frac{\pi}{N+1}, \frac{\pi}{N}\right]$ and $Q(x,\mu)$ be a solution of (4. 3) under the conditions that

$$Q(0,\mu) = 0, \quad D_x^\beta Q(0,\mu) = 1.$$

We can then expressed $Q(x,\mu)$ as

$$Q(x,\mu) = \frac{1}{\sqrt{\mu}}\sin\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right) + \frac{1}{\sqrt{\mu}}\int_0^x \sin\left(\frac{\sqrt{\mu}}{\beta}(x^\beta - \tau^\beta)\right) q(\tau)Q((\tau-\nu),\mu)d_\beta\tau, \tag{4. 4}$$

For every x in the interval $(0,\pi)$ and $j=0,1$, it is evident that $Q^{(j)}(x,\mu)$ is whole in μ of order $\frac{1}{2}$.
By the method of successive approximations,the solution of (4. 4) is

$$Q(x,\mu)=Q_0(x,\mu)+Q_1(x,\mu)+\ldots+Q_N(x,\mu),$$

for which

$$Q_0(x,\mu)=\frac{1}{\sqrt{\mu}}\sin\left(\frac{\sqrt{\mu}}{\beta}x^\beta\right), \quad for\ x\geq 0 \tag{4. 5}$$

$$Q_k(x,\mu)=\frac{1}{\sqrt{\mu}}\int_{k\nu}^{x}\sin\left(\frac{\sqrt{\mu}}{\beta}(x^\beta-\tau^\beta)\right)q(\tau)Q_{k-1}((\tau-\nu),\mu)d_\beta\tau, \tag{4. 6}$$

for $x\geq k\nu$, and $Q_k(x,\mu)=0$ for $x\leq k\nu$.
Now, for $k\geq 1$, and from (4. 6) and by definition of the conformable derivative we have,

$$D_x^\beta Q_k(x,\mu)=\int_{k\nu}^{x}\cos\left(\frac{\sqrt{\mu}}{\beta}(x^\beta-\tau^\beta)\right)q(\tau)Q_{k-1}((\tau-\nu),\mu)d_\beta\tau \quad for\ x\geq k\nu.$$

From (4. 6) we obtained

$$\begin{aligned}Q_1(x,\mu)&=\frac{1}{\sqrt{\mu}}\int_{\nu}^{x}\sin\left(\frac{\sqrt{\mu}}{\beta}(x^\beta-\tau^\beta)\right)q(\tau)Q_0((\tau-\nu),\mu)d_\beta\tau\\&=\frac{1}{\mu}\int_{\nu}^{x}\sin\left(\frac{\sqrt{\mu}}{\beta}(x^\beta-\tau^\beta)\right).\sin\left(\frac{\sqrt{\mu}}{\beta}(\tau-\nu)^\beta\right)q(\tau)d_\beta\tau,\end{aligned} \tag{4. 7}$$

so that

$$\begin{aligned}Q_1(x,\mu)&=\frac{1}{2\mu}\cos\left(\frac{\sqrt{\mu}}{\beta}(x^\beta-\tau^\beta-(\tau-\nu)^\beta)\right)\int_{\nu}^{x}q(\tau)d_\beta\tau\\&-\frac{1}{2\mu}\int_{\nu}^{x}\cos\left(\frac{\sqrt{\mu}}{\beta}(x^\beta-\tau^\beta+(\tau-\nu)^\beta)\right)q(\tau)d_\beta\tau,\end{aligned} \tag{4. 8}$$

then, we have from (4. 8) and trigonometric identities that

$$\begin{aligned}Q_1(x,\mu)&=\frac{1}{2\mu}\int_{\nu}^{x}\Bigg[\cos\frac{\sqrt{\mu}}{\beta}x^\beta\left(\cos\left(\frac{\sqrt{\mu}}{\beta}(\tau^\beta+(\tau-\nu)^\beta)\right)\right.\\&-\left.\cos\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta-\tau^\beta)\right)\right)+\sin\frac{\sqrt{\mu}}{\beta}x^\beta\left(\sin\left(\frac{\sqrt{\mu}}{\beta}(\tau^\beta+(\tau-\nu)^\beta)\right)\right.\\&-\left.\sin\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta-\tau^\beta)\right)\right)q(\tau)d_\beta\tau,\end{aligned} \tag{4. 9}$$

that is

$$
\begin{aligned}
Q_1(x,\mu) = & \frac{1}{\mu}\sin\frac{\sqrt{\mu}}{\beta}x^\beta\left[\frac{1}{2}\int_\nu^x \sin\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta-\tau^\beta)\right)q(\tau)d_\beta\tau.\right.\\
& \left.+\frac{1}{2}\int_\nu^x \sin\left(\frac{\sqrt{\mu}}{\beta}(\tau^\beta+(\tau-\nu)^\beta)\right)\right]q(\tau)d_\beta\tau\\
& +\frac{1}{\mu}\cos\frac{\sqrt{\mu}}{\beta}x^\beta\left[\frac{1}{2}\int_\nu^x \cos\left(\frac{\sqrt{\mu}}{\beta}(\tau^\beta+(\tau-\nu)^\beta)\right).\right.\\
& \left.-\frac{1}{2}\int_\nu^x \cos\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta-\tau^\beta)\right)\right]q(\tau)d_\beta\tau.
\end{aligned}
\tag{4. 10}
$$

Let us define

$$
\begin{aligned}
G(s,\tau,\nu) &= \frac{1}{2}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau \ \ and\\
H(s,\tau,\nu) &= \frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau.
\end{aligned}
\tag{4. 11}
$$

Then by (4. 11) and the lemma (4.0.1) the equation (4. 10) becomes

$$
Q_1(x,\mu) = \frac{1}{\mu}\sin\frac{\sqrt{\mu}}{\beta}x^\beta\left[H(s,\tau,\nu)+O\left(\frac{1}{\sqrt{\mu}}\right)\right]-\frac{1}{\mu}\cos\frac{\sqrt{\mu}}{\beta}x^\beta\left[G(s,\tau,\nu)+O\left(\frac{1}{\sqrt{\mu}}\right)\right],
$$

that is

$$
Q_1(x,\mu) = \frac{1}{\mu}\sin\frac{\sqrt{\mu}}{\beta}x^\beta H(s,\tau,\nu)-\frac{1}{\mu}\cos\frac{\sqrt{\mu}}{\beta}x^\beta G(s,\tau,\nu)+O\left(\frac{1}{\mu}\right).
$$

Also, differentiating equation (4. 10) gives,

$$
\begin{aligned}
D_x^\beta Q_1(x,\mu) = & \frac{1}{\sqrt{\mu}}\cos\frac{\sqrt{\mu}}{\beta}x^\beta\left[\frac{1}{2}\int_\nu^x \sin\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta-\tau^\beta)\right)q(\tau)d_\beta\tau\right.\\
& \left.+\frac{1}{2}\int_\nu^x \sin\left(\frac{\sqrt{\mu}}{\beta}(\tau^\beta+(\tau-\nu)^\beta)\right)\right]q(\tau)d_\beta\tau\\
& -\frac{1}{\sqrt{\mu}}\sin\frac{\sqrt{\mu}}{\beta}x^\beta\left[\frac{1}{2}\int_\nu^x \cos\left(\frac{\sqrt{\mu}}{\beta}(\tau^\beta+(\tau-\nu)^\beta)\right)\right.\\
& \left.-\frac{1}{2}\int_\nu^x \cos\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta-\tau^\beta)\right)\right]q(\tau)d_\beta\tau,
\end{aligned}
\tag{4. 12}
$$

and by (4. 11) and the lemma (4.0.1) the equation (4. 12) becomes

$$
D_x^\beta Q_1(x,\mu) = \frac{1}{\sqrt{\mu}}\cos\frac{\sqrt{\mu}}{\beta}x^\beta H(s,\tau,\nu)+\frac{1}{\sqrt{\mu}}\sin\frac{\sqrt{\mu}}{\beta}x^\beta G(s,\tau,\nu)+O\left(\frac{1}{\sqrt{\mu}}\right).
$$

4.1.1 The Asymptotic Formulae

The characteristics function of $L_j(q)$ can be represented as $W_j(\mu)$, where $j=0,1$ and $W_j(\mu)=Q^{(j)}(\pi,\mu)$. The $W_j(\mu)$ is also entire in μ of order $\frac{1}{2}$ since $Q^{(j)}(\pi,\mu)$ is. For the SLP $L_j(q)$, we drive the asymptotic equations as in the following for $|\sqrt{\mu}|\to\infty$ from (4. 6),

(4. 9), and (4. 10):

$$\begin{aligned}
W_{\beta_0}(\mu) &= Q(\pi,\mu) = Q_0(\pi,\mu) + Q_1(\pi,\mu) + \ldots + Q_N(\pi,\mu) \\
&= \frac{1}{\sqrt{\mu}} \sin\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) + \frac{1}{\mu}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) H(s,\tau,\nu) \\
&- \frac{1}{\mu}\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) G(s,\tau,\nu) + o\left(\mu^{\frac{1}{2}(j-k-1)} e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^\beta\right)}\right).
\end{aligned}$$

Now, to find the eigenvalues, we set $W_{\beta_0}(\mu) = 0$ and divide its both sides by $\frac{1}{\sqrt{\mu}}\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right)$ that is,

$$\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right)[1 + H(s,\tau,\nu)] = G(s,\tau,\nu) + o\left(\mu^{\frac{1}{2}(j-k)} e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^\beta\right)}\right),$$

that is

$$\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) = \frac{1}{[1 + H(s,\tau,\nu)]} G(s,\tau,\nu) + o\left(\mu^{\frac{1}{2}(j-k)} e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^\beta\right)}\right)$$

as such, also by (4. 11) and the lemma (4.0.1) we have

$$\begin{aligned}
&\tan\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) = \frac{1}{2}\int_\nu^x \cos\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta - \tau^\beta)\right) q(\tau) d_\beta\tau \\
&+ o\left(\mu^{\frac{1}{2}(j-k-1)} e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^\beta\right)}\right),
\end{aligned}$$

then by Tailor expansion of infinite series we have

$$\begin{aligned}
\frac{\sqrt{\mu}}{\beta}\pi^\beta = &\tan^{-1}\left(\frac{1}{2}\int_\nu^x \cos\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta - \tau^\beta)\right) q(\tau) d_\beta\tau \right. \\
&\left. + o\left(\mu^{\frac{1}{2}(j-k-1)} e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^\beta\right)}\right)\right),
\end{aligned}$$

that is

$$\frac{\sqrt{\mu}}{\beta}\pi^\beta = \pi n + \frac{1}{2}\int_\nu^x \cos\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^\beta - \tau^\beta)\right) q(\tau) d_\beta\tau + o\left(\frac{1}{n}\right),$$

finally, the eigenvalues for $j = 0$ is calculated as

$$\sqrt{\mu}_0 = \beta n\pi^{1-\beta} + \frac{1}{2n\pi}\int_\nu^x \cos\left(n\pi^{1-\beta}((\tau-\nu)^\beta - \tau^\beta)\right) q(\tau) d_\beta\tau + o\left(\frac{1}{n}\right).$$

Similarly, the eigenvalues for $j = 1$ is calculated as follows,

$$\begin{aligned}
W_1(\mu) &= D_x^\beta Q(\pi,\mu) = D_x^\beta Q_0(\pi,\mu) + D_x^\beta Q_1(\pi,\mu) + \ldots + D_x^\beta Q_N(\pi,\mu) \\
&= \cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) + \frac{1}{\sqrt{\mu}}\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) G(s,\tau,\nu) + \frac{1}{\sqrt{\mu}}\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^\beta\right) H(s,\tau,\nu) \\
&+ o\left(\mu^{\frac{1}{2}(j-k-1)} e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^\beta\right)}\right)
\end{aligned}$$

Now, to find the eigenvalues, we set $W_1(\mu)=0$ and divide its both sides by $\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)$ that is,

$$\cot\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)\left(1+\frac{1}{\sqrt{\mu}}G(s,\tau,\nu)\right)=\frac{1}{\sqrt{\mu}}H(s,\tau,\nu)+o\left(\mu^{\frac{1}{2}(j-k-1)}e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^{\beta}\right)}\right),$$

as such

$$\cot\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)=\frac{1}{\left(1+\frac{1}{\sqrt{\mu}}G(s,\tau,\nu)\right)}\left(\frac{1}{\sqrt{\mu}}H(s,\tau,\nu)+o\left(\mu^{\frac{1}{2}(j-k-1)}e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^{\beta}\right)}\right)\right),$$

similarly, by Tailor expansion we have

$$\begin{aligned}\frac{\sqrt{\mu}}{\beta}\pi^{\beta}=&\cot^{-1}\left(\frac{1}{2\sqrt{\mu}}\int_{\nu}^{x}\cos\left(\frac{\sqrt{\mu}}{\beta}((\tau-\nu)^{\beta}-\tau^{\beta})\right)q(\tau)d_{\beta}\tau\right)\\&+o\left(\mu^{\frac{1}{2}(j-k+1)}e^{\left(|Im\sqrt{\mu}|(\pi-\nu)^{\beta}\right)}\right)\end{aligned}$$

and finally the eigenvalue for $j=1$ is,

$$\begin{aligned}\sqrt{\mu}_1=&\beta\left(n-\frac{1}{2}\right)\pi^{1-\beta}+\frac{1}{2\left(n-\frac{1}{2}\right)\pi}\int_{\nu}^{x}\cos\left(\left(n-\frac{1}{2}\right)\pi^{1-\beta}((\tau-\nu)^{\beta}-\tau^{\beta})\right)q(\tau)d_{\beta}\tau\\&+o\left(\frac{1}{n}\right).\end{aligned}$$

4.1.2 The Specification of the Spectrum

To validate the use of the conformal derivative operator, we obtained and proved the specification of the spectrum for the characteristic function as it follows. The spectrum corresponding to a problem for a Sturm-Liouville operator for the interval $(0,\infty)$ means the complement of the set of points in a neighborhood where the spectral function $W_j(\mu)$ is constant.

Lemma 4.1.1. *The specification of the spectrum* $\{\mu_{nj}\}_{n\geq 1}$, $j=0,1$ *uniquely determines the characteristics function* $W_j(\mu)$ *by the formulas*

$$W_{\beta_0}(\mu)=\frac{\pi^{3\beta-2}}{\beta^3}\prod_{n=1}^{\infty}\left(\frac{\mu_{n0}-\mu}{n^2}\right),$$

and

$$W_{\beta_1}(\mu)=\frac{\pi^{2\beta-2}}{\beta^2}\prod_{n=1}^{\infty}\frac{(\mu_{n1}-\mu)}{\left(n-\frac{1}{2}\right)^2}.$$

Proof 12. Being $W_j(\mu)$ entire in μ of order $\frac{1}{2}$, by [95], we get

$$W_{\beta_0}(\mu)=C\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\mu_{n0}}\right),$$

now, since

$$\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{(k\pi)^2}\right),$$

it implies that

$$\tilde{W}_0(\mu)=\frac{\sin\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)}{\sqrt{\mu}}=\frac{\pi^{\beta}}{\beta}\prod_{n=1}^{\infty}\left(1-\frac{\mu\pi^{2\beta-2}}{\beta^2 n^2}\right)=\frac{\pi^{\beta}}{\beta}\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)n^2}\right),$$

then

$$\frac{W_{\beta_0}(\mu)}{\tilde{W}_0(\mu)}=C\frac{\beta^3}{\pi^{3\beta-2}}\prod_{n=1}^{\infty}\frac{n^2}{\mu_{n0}}\prod_{n=1}^{\infty}\left(1+\frac{\mu_{n0}-\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)n^2}{\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)n^2-\mu}\right),$$

now

$$\lim_{\mu\to-\infty}\frac{W_{\beta_0}(\mu)}{\tilde{W}_0(\mu)}=1 \ \ and \ \ \lim_{\mu\to-\infty}\prod_{n=1}^{\infty}\left(1+\frac{\mu_{n0}-\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)n^2}{\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)n^2-\mu}\right)=1,$$

then

$$C=\frac{\pi^{3\beta-2}}{\beta^3}\prod_{n=1}^{\infty}\frac{\mu_{n0}}{n^2},$$

and eventually, we arrived at

$$W_{\beta_0}(\mu)=C\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\mu_{n0}}\right)=\frac{\pi^{3\beta-2}}{\beta^3}\prod_{n=1}^{\infty}\left(\frac{\mu_0-\mu}{n^2}\right).$$

This completes the proof of the first case, that is for $j=0$.
For the proof for the second case, $j=1$, that is $W_{\beta_1}(\mu)$, we have by the same Hadamard's factorization theorem that

$$W_{\beta_1}(\mu)=C\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\mu_{n_1}}\right),$$

and from the fact that

$$\cos\pi z=\prod_{n=1}^{\infty}\left(1-\frac{4z^2}{(2n-1)^2}\right),$$

it implies that

$$\widetilde{W}_{\beta_1}(\mu)=\cos\left(\frac{\sqrt{\mu}}{\beta}\pi^{\beta}\right)=\prod_{n=1}^{\infty}\left(1-\frac{\mu\pi^{2\beta-2}}{\beta^2(n-\frac{1}{2})^2}\right),$$

where $\pi z = \frac{\sqrt{\mu}}{\beta}\pi^{\beta}$ then

$$\begin{aligned}
\frac{W_{\beta_1}(\mu)}{\widetilde{W}_{\beta_1}(\mu)} &= C\prod_{n=1}^{\infty}\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)\frac{1}{\mu_{n_1}}\prod_{n=1}^{\infty}\left(\frac{(n-\frac{1}{2})^2\mu_{n_1}-(n-\frac{1}{2})^2\mu}{\left(\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)(n-\frac{1}{2})^2-\mu\right)}\right)\\
&= C\frac{\beta^2}{\pi^{2\beta-2}}\prod_{n=1}^{\infty}\frac{(n-\frac{1}{2})^2}{\mu_{n_1}}\prod_{n=1}^{\infty}\left(\frac{\mu_{n_1}-\mu}{\left(\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)(n-\frac{1}{2})^2-\mu\right)}\right)\\
&= C\frac{\beta^2}{\pi^{2\beta-2}}\prod_{n=1}^{\infty}\frac{(n-\frac{1}{2})^2}{\mu_{n_1}}\prod_{n=1}^{\infty}\left(1+\frac{\mu_{n_1}-\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)(n-\frac{1}{2})^2}{\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)(n-\frac{1}{2})^2-\mu}\right),
\end{aligned}$$

as well, by taking the limit we have
$\lim_{\mu\to-\infty}\frac{W_{\beta_1}(\mu)}{\widetilde{W}_{\beta_1}(\mu)} = 1$ and $\lim_{\mu\to-\infty}\prod_{n=1}^{\infty}\left(1+\frac{\mu_{n_1}-\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)(n-\frac{1}{2})^2}{\left(\frac{\beta^2}{\pi^{2\beta-2}}\right)(n-\frac{1}{2})^2-\mu}\right) = 1,$
then

$$C = \frac{\pi^{2\beta-2}}{\beta^2}\prod_{n=1}^{\infty}\frac{\mu_{n_1}}{(n-\frac{1}{2})^2}$$

and consequently we obtained

$$\begin{aligned}
W_{\beta_1}(\mu) &= C\prod_{n=1}^{\infty}\left(1-\frac{\mu}{\mu_{n_1}}\right)\\
&= \frac{\pi^{2\beta-2}}{\beta^2}\prod_{n=1}^{\infty}\frac{\mu_{n_1}}{(n-\frac{1}{2})^2}\left(1-\frac{\mu}{\mu_{n_1}}\right) = \frac{\pi^{2\beta-2}}{\beta^2}\prod_{n=1}^{\infty}\left(\frac{\mu_{n_1}-\mu}{(n-\frac{1}{2})^2}\right).
\end{aligned}$$

This completes the proof for both the two cases.

4.2 The inverse nodal SLP with constant delay under more general boundary conditions

We consider here a more general form of the problem by looking into the boundary conditions we used and we successfully analyzed the problem as it follows.
Let consider the SLP below with conformable derivative operator;

$$D_x^{\beta}D_x^{\beta}u + q(x)u(x-\nu) + \mu u(x) = 0,\ on\ \ x\in[0,\pi] \tag{4. 13}$$

with the boundary conditions

$$u(0)\cos\omega + D_x^{\beta}u(0)\sin\omega = 0, \tag{4. 14}$$

and

$$u(\pi)\cos\alpha + D_x^{\beta}u(\pi)\sin\alpha = 0, \tag{4. 15}$$

for $q(x)$ a continuous real-valued function in the interval $[0,\pi]$ while ν is a constant in the interval $(0,\pi)$ such that $x-\nu \geq 0$, and $\omega,\alpha \in [0,\pi]$ and for $x \in [0,\pi]$ while $\mu,(\mu = s^2)$ is a real eigenparameter.
Let $Q(x,\mu)$ be a solution of (4. 13) on $[0,\pi]$ satisfying the following initial conditions,

$$Q(0,\mu) = \sin\omega \text{ and } D_x^\beta Q(0,\mu) = -\cos\omega. \tag{4. 16}$$

The lemma below proved the above statement.

Lemma 4.2.1. *Let* $Q(x,\mu)$ *be a solution of* (4. 13) *and* $\mu > 0$*, then it holds that,*

$$\begin{aligned} Q(x,\mu) &= \sin\omega\cos\left(\frac{s}{\beta}x^\beta\right) - \frac{1}{s}\cos\omega\sin\left(\frac{s}{\beta}x^\beta\right) \\ &\quad - \frac{1}{s}\int_0^x \sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau. \end{aligned} \tag{4. 17}$$

Proof 13. To prove this lemma, we can write (4. 13) as

$$-q(x)u(x-\nu) = D_x^\beta D_x^\beta u + \mu u(x),$$

since (4. 17) is a solution of (4. 13), we can write

$$\begin{aligned} &-\int_0^x q(\tau)\sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) Q(\tau-\nu,\mu)d_\beta\tau \\ &= \int_0^x \sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) D_x^\beta D_x^\beta Q(\tau-\nu,\mu)d_\beta\tau \\ &+ s^2\int_0^x \sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) Q(\tau-\nu,\mu)d_\beta\tau. \end{aligned} \tag{4. 18}$$

Now, integrating the first part of the right -hand side of (4. 18) by part twice and using the conditions (4. 16) we obtained

$$\begin{aligned} &\int_0^x \sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) D_x^\beta D_x^\beta Q(\tau-\nu,\mu)d_\beta\tau \\ &= \cos\omega\sin\left(\frac{s}{\beta}x^\beta\right) - s\sin\omega\cos\left(\frac{s}{\beta}x^\beta\right) + sQ(x,\mu) \\ &- s^2\int_0^x \sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) Q(\tau-\nu,\mu)d_\beta\tau, \end{aligned}$$

then, (4. 18) becomes

$$\begin{aligned} &-\int_0^x \sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau \\ &= \cos\omega\sin\left(\frac{s}{\beta}x^\beta\right) - s\sin\omega\cos\left(\frac{s}{\beta}x^\beta\right) + sQ(x,\mu) \end{aligned}$$

which when rearranged gives back (4. 17). This completes the proof.

Theorem 4.2.1. *The eigenvalues of the problem* (4. 13)- (4. 14) *are simple.*

Proof 14. Let $\widetilde{\mu}$ be an eigenvalue of the problem (4. 13) under (4. 14) and $\widetilde{u}(x,\widetilde{\mu})$ be a corresponding eigenfunction.Then from (4. 14) and (4. 15) it follows that the determinant

$$W[\widetilde{u}(0,\widetilde{\mu}),Q(0,\widetilde{\mu})] = \begin{vmatrix} \widetilde{u}(0,\widetilde{\mu}) & \sin\omega \\ D_x^\beta \widetilde{u}(0,\widetilde{\mu}) & -\cos\omega \end{vmatrix} = 0$$

then the functions $\widetilde{u}(x,\widetilde{\mu})$ and $Q(x,\widetilde{\mu})$ are linearly dependent on $[0,\pi]$, hence

$$\widetilde{u}(x,\widetilde{\mu}) = kQ(x,\widetilde{\mu}) \ \ for\ k \neq 0.$$

Since $Q(x,\widetilde{\mu})$ is a solution of (4. 13) under (4. 14) on $[0,\pi]$, it follows that $Q(x,\widetilde{\mu}) = 0$ identically on $[0,\pi]$ but this contradicts (4. 16).
Thus, there can not be two linearly independent eigenfunctions in a SLP, then each eigenvalue of this problem has pairwise corresponding linearly dependent eigenfunctions.

Considering the function $Q(x,\mu)$ as the non-trivial solution of (4. 13) under (4. 14), now putting $Q(x,\mu)$ in to (4. 15), we obtained the characteristic equation as

$$\begin{aligned} G(\mu) &= Q(\pi,\mu)\cos\alpha + D_x^\beta Q(\pi,\mu)\sin\alpha \\ &= \left[\sin\omega\cos\left(\frac{s}{\beta}\pi^\beta\right) - \frac{1}{s}\cos\omega\sin\left(\frac{s}{\beta}\pi^\beta\right)\right. \\ &- \left.\frac{1}{s}\int_0^\pi \sin\left(\frac{s}{\beta}(\pi^\beta - \tau^\beta)\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau\right]\cos\alpha \\ &- \left[s\sin\omega\sin\left(\frac{s}{\beta}\pi^\beta\right) + \cos\omega\cos\left(\frac{s}{\beta}\pi^\beta\right)\right. \\ &+ \left.\int_0^\pi \cos\left(\frac{s}{\beta}(\pi^\beta - \tau^\beta)\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau\right]\sin\alpha = 0, \end{aligned} \tag{4. 19}$$

by theorem 4.2.1, the set of the eigenvalues of (4. 13) under (4. 14) coincides with the set of real roots of (4. 19) as it can be seen in the next section.
Let $q^* = \int_0^\pi |q(\tau)|d_\beta\tau$, the majority of the claims, assertions, and evidence presented below are based on [72]'s chapter III.

Lemma 4.2.2. *Let $\mu \geq 4q^{*2}$, then for the solution $Q(x,\mu)$ of (4. 17), the following in equality holds;*

$$|Q(x,\mu)| \leq max\left\{\frac{1}{q^*}\sqrt{4q^{*2}\sin^2\omega + \cos^2\omega}\ ,\nu|\sin\omega|\right\},\ x\in[0,\pi] \tag{4. 20}$$

Proof 15. Let $B_\mu = max_{[0,\pi]}|Q(x,\mu)|$, then from (4. 17) it follows that for every $\mu > 0$, one of the following inequality holds,

$$B_\mu \leq \sqrt{\sin^2\omega + \frac{1}{s^2}\cos^2\omega} + \frac{1}{s}B_\mu q^*,$$

or

$$B_\mu \leq \sqrt{\sin^2\omega + \frac{1}{s^2}\cos^2\omega} + \frac{1}{s}\nu|\sin\omega|q^*$$

in each of the two cases above if $s \geq 2q^*$, then we get (4. 20) which completes the proof.

Theorem 4.2.2. *The problem* (4. 13) *under* (4. 14) *has an infinite set of positive eigenvalues.*

Proof 16. Differentiating (4. 17) according to the conformable derivative with respect to x we have,

$$\begin{aligned} D_x^\beta Q(x,\mu) = & -s\sin\omega\sin\left(\frac{s}{\beta}x^\beta\right) - \cos\omega\cos\left(\frac{s}{\beta}x^\beta\right) \\ & - \int_0^x \cos\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau. \end{aligned} \tag{4. 21}$$

Using the equation (4. 17) and (4. 21) in (4. 19) gives

$$\begin{aligned} & \left[\sin\omega\cos\left(\frac{s}{\beta}\pi^\beta\right) - \frac{1}{s}\cos\omega\sin\left(\frac{s}{\beta}\pi^\beta\right) \right. \\ & \left. -\frac{1}{s}\int_0^\pi \sin\left(\frac{s}{\beta}(\pi^\beta - \tau^\beta)\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau\right]\cos\alpha \\ & -\left[s\sin\omega\sin\left(\frac{s}{\beta}\pi^\beta\right) + \cos\omega\cos\left(\frac{s}{\beta}\pi^\beta\right)\right. \\ & \left. +\int_0^\pi \cos\left(\frac{s}{\beta}(\pi^\beta - \tau^\beta)\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau\right]\sin\alpha = 0, \end{aligned}$$

which after some algebraic rearrangements gives

$$\begin{aligned} & \cos\left(\frac{s}{\beta}\pi^\beta\right)\left[\sin(\omega-\alpha)\right. \\ & \quad \left. + \int_0^\pi \left(\frac{\sin\left(\frac{s}{\beta}(\tau^\beta\right)\cos\alpha}{s} - \cos\left(\frac{s}{\beta}(\tau^\beta\right)\sin\alpha\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau\right] \\ & \quad - \sin\left(\frac{s}{\beta}\pi^\beta\right)\left[s\sin\omega\sin\alpha + \frac{\cos\omega\cos\alpha}{s}\right. \\ & \quad \left. + \int_0^\pi \left(\frac{\cos\left(\frac{s}{\beta}(\tau^\beta\right)\cos\alpha}{s} + \sin\left(\frac{s}{\beta}(\tau^\beta\right)\sin\alpha\right) q(\tau)Q(\tau-\nu,\mu)d_\beta\tau\right] = 0. \end{aligned} \tag{4. 22}$$

The equation (4. 22) can be treated in each of the four different cases below;

1. $\sin\omega \neq 0$ *and* $\sin\alpha \neq 0$.
2. $\sin\omega \neq 0$ *and* $\sin\alpha = 0$.
3. $\sin\omega = 0$ *and* $\sin\alpha \neq 0$.
4. $\sin\omega = 0$ *and* $\sin\alpha = 0$.

*Case*1. Let μ be sufficiently large, then by lemma 4.2.2, (4. 22) may be rewritten in the form

$$s\sin\left(\frac{s}{\beta}\pi^\beta\right) + O(1) = 0. \tag{4. 23}$$

Obviously, for large s, (4. 23) has an infinite set of roots.

*Case*2. Here the equation (4. 22) has the form

$$\begin{aligned}\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\sin(\omega)+\frac{1}{s}\int_0^{\pi}\sin\left(\frac{s}{\beta}\pi^{\beta}\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\right]\\ -\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\frac{\cos\omega}{s}+\frac{1}{s}\int_0^{\pi}\cos\left(\frac{s}{\beta}\pi^{\beta}\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\right]=0,\end{aligned} \tag{4. 24}$$

multiplying (4. 24) by s and by considering lemma 4.2.2 we have

$$s\cos\left(\frac{s}{\beta}\pi^{\beta}\right)+O(1)=0, \tag{4. 25}$$

also for large s, (4. 25) has an infinite set of roots.

*Case*3. In this case the equation (4. 22) is expressed in the form

$$\begin{aligned}&\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\Bigg[-\cos\omega\sin\alpha\\ &+\int_0^{\pi}\left(\frac{\sin\left(\frac{s}{\beta}(\tau^{\beta}\right)\cos\alpha}{s}-\cos\left(\frac{s}{\beta}(\tau^{\beta}\right)\sin\alpha\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\Bigg]\\ &-\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\Bigg[\frac{\cos\omega\cos\alpha}{s}\\ &+\int_0^{\pi}\left(\frac{\cos\left(\frac{s}{\beta}(\tau^{\beta}\right)\cos\alpha}{s}+\sin\left(\frac{s}{\beta}(\tau^{\beta}\right)\sin\alpha\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\Bigg]=0.\end{aligned} \tag{4. 26}$$

We know that $x\in[0,\pi]$ and $x-\nu\geq 0$ and then $Q(\tau-\nu,\mu)=0$ if $\tau-\nu<0$, then by virtue of (4. 17) and (4. 20) on $[0,\pi]$ we have

$$Q(\tau-\nu,\mu)=O\left(\frac{1}{s}\right), \tag{4. 27}$$

multiplying bothsides of (4. 26) by s and considering (4. 27) gives

$$s\cos\left(\frac{s}{\beta}\pi^{\beta}\right)+O(1)=0. \tag{4. 28}$$

which the same as (4. 25).

*Case*4. The equation (4. 22) in this case has the form

$$\begin{aligned}\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\int_0^{\pi}\frac{\sin\left(\frac{s}{\beta}(\tau^{\beta}\right)}{s}q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\right]\\ -\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\frac{\cos\omega}{s}+\int_0^{\pi}\frac{\cos\left(\frac{s}{\beta}(\tau^{\beta}\right)}{s}q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\right]=0,\end{aligned} \tag{4. 29}$$

here also, (4. 23) holds and taking it in to account and multiplying both sides of (4. 29) with $\frac{1}{s^2}$ we obtain

$$s\sin\left(\frac{s}{\beta}\pi^{\beta}\right)+O(1)=0 \tag{4. 30}$$

as in $case1$.
Proving the theorem in each of the four cases above completes the proof of the theorem.

4.3 The asymptotic parameters

Under this section we will give the asymptotic properties of the eigenvalues and the corresponding eigenfunctions of the problem (4. 13)- (4. 15) by the conformable derivatives as follows.
Let consider s sufficiently large for all the four cases.
Now by (4. 20)

$$Q(x,\mu)=O(1), \tag{4. 31}$$

on $[-a,\pi]$ in the first and the second cases while in the third and the fourth cases where $Q(\tau-\nu,\mu)=0$ for $x-\nu<0$ and by the equations (4. 17) and (4. 20),

$$Q(x,\mu)=O\left(\frac{1}{s}\right), \tag{4. 32}$$

on $[-a,\pi]$.
The existence and continuity of the derivatives $D_s^{\beta}Q(x,\mu)$ for $0\le x\le\pi,\ |\mu|<\infty$ follows from a theorem 4.1 in chapter I in [72].

Lemma 4.3.1. *In the first and the second cases,*

$$D_s^{\beta}Q(x,\mu)=O(1),\quad for\, x\in[-\nu,\pi] \tag{4. 33}$$

Proof 17. Differentiating (4. 17) using conformable derivatives with respect to s, and by (4. 31) we obtain,

$$D_s^{\beta}Q(x,\mu)=-\frac{1}{s}\int_0^x \sin\left(\frac{s}{\beta}(x^{\beta}-\tau^{\beta})\right)q(\tau)D_s^{\beta}Q(\tau-\nu,\mu)d_{\beta}\tau+k(x,\mu), \tag{4. 34}$$

where k is the remaining term and satisfied $|k(x,\mu)|\le k_o$.
Let $C_{\mu}=Max_{[0,\pi]}|D_s^{\beta}Q(x,\mu)|$. From (4. 34)

$$C_{\mu}\le\frac{1}{s}q^*C_{\mu}+k_o$$

If $s\ge 2q^*$, then $C_{\mu}\le 2k_o$ and this validated the asymptotic expression (4. 33).

Lemma 4.3.2. *In the third and the fourth cases,*

$$D_s^{\beta}Q(x,\mu)=O\left(\frac{1}{s}\right)\quad for\, x\in[-\nu,\pi]. \tag{4. 35}$$

Proof 18. The proof goes similar to that of the lemma 4.3.1 by considering (4. 32).

Let $n \in \mathbb{N}$, then if $|\beta n\pi^{1-\beta} - \sqrt{\mu}| < \frac{1}{4}$ or $|\beta(n+\frac{1}{2})\pi^{1-\beta} - \sqrt{\mu}| < \frac{1}{4}$, the eigenvalue μ is situated near $(\beta n\pi^{1-\beta})^2$ or $(\beta(n+\frac{1}{2})\pi^{1-\beta})^2$.

Theorem 4.3.1. *Let $n \in \mathbb{N}$. Then, there is just one eigenvalue of the problem* (4. 13)-(4. 14) *near $(\beta n\pi^{1-\beta})^2$ in the first and the second cases also for the third and the fourth cases there is exactly one eigenvalue of the problem near $(\beta(n+\frac{1}{2})\pi^{1-\beta})^2$.*

Proof 19. Consider the expressions (4. 23), (4. 25), (4. 28) and (4. 30) then it can be shown by conformable differentiation with respect to s that for large s the expressions below have bounded derivative,
First Equation: Multiplying both sides of the equation (4. 31) by $\frac{1}{\sin\alpha\sin\omega}$ and applying the equation (4. 22) we have the expression that;

$$\begin{aligned}\frac{1}{\sin\alpha\sin\omega}\Bigg\{&\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\Bigg[\frac{\cos\omega\cos\alpha}{s}\\&+\int_0^{\pi}\left(\frac{\cos\left(\frac{s}{\beta}\tau^{\beta}\right)\cos\alpha}{s}+\sin\left(\frac{s}{\beta}\tau^{\beta}\right)\sin\alpha\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\Bigg]\\&-\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\Bigg[\sin(\omega-\alpha)\\&+\int_0^{\pi}\left(\frac{\sin\left(\frac{s}{\beta}\tau^{\beta}\right)\cos\alpha}{s}-\cos\left(\frac{s}{\beta}\tau^{\beta}\right)\sin\alpha\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\Bigg]\Bigg\}.\end{aligned}$$

Second Equation: Multiplying both sides of the equation (4. 32) by $\frac{s}{\sin\omega}$ and applying the equation (4. 24) we have the expression that

$$\begin{aligned}\frac{1}{\sin\omega}\Bigg\{&\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\int_0^{\pi}\sin\left(\frac{s}{\beta}\pi^{\beta}\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\right]\\&-\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\cos\omega+\int_0^{\pi}\cos\left(\frac{s}{\beta}\pi^{\beta}\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\right]\Bigg\}.\end{aligned}$$

Third Equation: Multiplying both sides of the equation (4. 33) by $\frac{s}{\sin\alpha\cos\omega}$ and applying the equation (4. 26) we have the expression that;

$$\begin{aligned}&\frac{1}{\cos\omega\sin\alpha}\Bigg\{\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\Bigg[\cos\omega\sin\alpha\\&+\int_0^{\pi}\left(\frac{\cos\left(\frac{s}{\beta}\tau^{\beta}\right)\cos\alpha}{s}+\sin\left(\frac{s}{\beta}\tau^{\beta}\right)\sin\alpha\right)sq(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\Bigg]\\&-\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\int_0^{\pi}\left(\frac{\sin\left(\frac{s}{\beta}\tau^{\beta}\right)\cos\alpha}{s}-\cos\left(\frac{s}{\beta}\tau^{\beta}\right)\sin\alpha\right)sq(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\Bigg\}.\end{aligned}$$

Fourth Equation: Multiplying both sides of the equation (4. 35) by $\frac{s^2}{\cos\omega}$ we have,

$$s\sin\left(\frac{s}{\beta}\pi^{\beta}\right)+\frac{1}{\cos\omega}\left\{\int_0^{\pi}\left(\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\cos\left(\frac{s}{\beta}\tau^{\beta}\right)\right.\right.$$
$$\left.\left.-\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\sin\left(\frac{s}{\beta}\tau^{\beta}\right)\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau\right\},$$

and applying the equation (4. 29) and some trigonometric expression we have the expression that

$$\frac{1}{\cos\omega}\int_0^{\pi}s\sin\left(\frac{s}{\beta}(\pi^{\beta}-\tau^{\beta})\right)q(\tau)Q(\tau-\nu,\mu)d_{\beta}\tau.$$

It is clear that as $s\to\infty$ the roots of (4. 23) (which is the same as (4. 30)) are situated close to the integer numbers.
Now we will show that as $s\to\infty$, only one root of (4. 23) (or (4. 30)) lies near to each $\beta n\pi^{1-\beta}$.
Considering the function

$$C(s)=s\sin\left(\frac{s}{\beta}\pi^{\beta}\right)+O(1),$$

its derivative

$$D_s^{\beta}C(s)=\sin\left(\frac{s}{\beta}\pi^{\beta}\right)+s\frac{\pi^{\beta}}{\beta}\cos\left(\frac{s}{\beta}\pi^{\beta}\right)+O(1),$$

does not vanish for s close to n for very large n. As such, the assertion follows by Rolle's Theorem. Similarly, for the equations (4. 25) (which is the same as (4. 28)), as $s\to\infty$ their roots lies near to the numbers $\beta(n-\frac{1}{2})\pi^{1-\beta}$.
Considering the equation as the function below

$$C^*(s)=s\cos\left(\frac{s}{\beta}\pi^{\beta}\right)+O(1),$$

then its derivative is

$$D_s^{\beta}C^*(s)=\cos\left(\frac{s}{\beta}\pi^{\beta}\right)-s\frac{\pi^{\beta}}{\beta}\sin\left(\frac{s}{\beta}\pi^{\beta}\right)+O(1),$$

which also for sufficiently large n does not vanish for s close to $\beta(n-\frac{1}{2})\pi^{1-\beta}$. Similarly as in the case above, for sufficiently large n there is only one root of the equations (4. 25) (or (4. 28)) close to each number of the $\beta(n-\frac{1}{2})\pi^{1-\beta}$. Which also by Rolle's Theorem the theorem is completed.

It is now possible to obtain the asymptotic expressions for the eigenvalues of the problem (4. 13) - (4. 15) due to the expressions (4. 23), (4. 25), (4. 28) and (4. 30).
Since for sufficiently large n the eigenvalues of the problem lies near $\left(\beta n\pi^{1-\beta}\right)^2$ or $\left(\beta(n+\frac{1}{2})\pi^{1-\beta}\right)^2$, let $s_n=\beta n\pi^{1-\beta}+\gamma_n$ then by equation (4. 23) or (4. 30) we have

$\sin(\gamma_n \pi) = O\left(\frac{1}{n}\right)$ and as such, for large n, we have $(\gamma_n \pi) = O\left(\frac{1}{n}\right)$ consequently, it follows that

$$s_n = \beta n \pi^{1-\beta} + O\left(\frac{1}{n}\right), \tag{4. 36}$$

which is the eigenvalues expressions, while by equations (4. 25) and (4. 28) for sufficiently large n and letting $s_n = \beta(n+\frac{1}{2})\pi^{1-\beta} + \gamma_n$ then by equation (4. 23) or (4. 30), $\sin(\gamma_n \pi) = O\left(\frac{1}{n}\right)$ and as such, for large n, we have $(\gamma_n \pi) = O\left(\frac{1}{n}\right)$ then

$$s_n = \left(n + \frac{1}{2}\right)\beta \pi^{1-\beta} + O\left(\frac{1}{n}\right), \tag{4. 37}$$

which is the other eigenvalues expressions.
The formula (4. 36) and (4. 37) make it possible to obtain the asymptotic expressions for the eigenfunctions of the problem (4. 13) - (4. 15).
Now, by considering the case 1, then from (4. 17) and (4. 31), we have

$$Q(x,\mu) = \sin\omega \cos\left(\frac{s}{\beta} x^{\beta}\right) + O\left(\frac{1}{s}\right), \tag{4. 38}$$

by putting (4. 36) in (4. 38) we have that

$$u_n(x) = Q(x,\mu_n) = \sin\omega \cos\left(n\pi^{1-\beta} x^{\beta}\right) + O\left(\frac{1}{n}\right),$$

for the case 2, equation (4. 38) also holds and by (4. 37), we have that

$$u_n(x) = Q(x,\mu_n) = \sin\omega \cos\left(\left(n - \frac{1}{2}\right)\pi^{1-\beta} x^{\beta}\right) + O\left(\frac{1}{n}\right).$$

In case 3, from (4. 17) and (4. 32), we obtain

$$Q(x,\mu) = -\frac{\cos\omega}{s} \sin\left(\frac{s}{\beta} x^{\beta}\right) + O\left(\frac{1}{s^2}\right), \tag{4. 39}$$

by equation (4. 37) we have

$$u_n(x) = Q(x,\mu_n) = -\frac{\cos\omega}{n} \sin\left(\left(n - \frac{1}{2}\right)\pi^{1-\beta} x^{\beta}\right) + O\left(\frac{1}{n^2}\right).$$

For the last case, case 4, equation (4. 39) holds as well and by equation (4. 36) we obtain

$$u_n(x) = Q(x,\mu_n) = -\frac{\cos\omega}{n} \sin\left(n\pi^{1-\beta} x^{\beta}\right) + O\left(\frac{1}{n^2}\right).$$

These are the asymptotic representations of the eigenfunctions of the problem (4. 13) - (4. 15) for $x \in [0,\pi]$. The above obtained asymptotic expressions correspond to the ones in the similar problem without the delay, now we will obtain the more exact asymptotic formulas which depend upon the delay constant and due to the lemma (4.0.1) and some additional conditions.

Theorem 4.3.2. *If* $D_x^\beta q$ *exist and bounded in* $[0,\pi]$, *then the positive eigenvalues* $\mu_n = s_n^2$ *of the problem* (4. 13)-(4. 15) *are;*
for the case 1,

$$s_n = \beta n\pi^{1-\beta} + \frac{\cot\alpha}{n\pi} - \frac{\cot\omega}{n\pi} - \frac{1}{2n\pi}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau + O\left(\frac{1}{n^2}\right).$$

for the case 2,

$$s_n = \beta\left(n+\frac{1}{2}\right)\pi^{1-\beta} - \frac{\cot\omega}{n\pi} - \frac{1}{2n\pi}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau + O\left(\frac{1}{n^2}\right).$$

for the case 3,

$$s_n = \beta\left(n+\frac{1}{2}\right)\pi^{1-\beta} + \frac{\cot\alpha}{n\pi} - \frac{1}{2n\pi}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau + O\left(\frac{1}{n^2}\right).$$

for the case 4,

$$s_n = \beta n\pi^{1-\beta} - \frac{1}{2n\pi}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau + O\left(\frac{1}{n^2}\right).$$

Proof 20. Let assume that $D_x^\beta q$ exist and bounded in $[0,\pi]$, it is easy to see that

$$x - a \geq 0 \quad for \;\; x \in [0,\pi] \tag{4. 40}$$

for the case 1,
by (4. 38) and (4. 40) we get

$$Q(\tau - a,\mu) = \sin\omega\cos\frac{s}{\beta}(\tau-\nu)^\beta + O\left(\frac{1}{s}\right), \quad on \;\; [0,\pi] \tag{4. 41}$$

putting this in to (4. 19) we obtain

$$\begin{aligned}
&\sin\omega\cos\alpha\cos\left(\frac{s}{\beta}\pi^\beta\right) - \cos\alpha\frac{1}{s}\cos\omega\sin\left(\frac{s}{\beta}\pi^\beta\right) \\
&-\frac{1}{s}\int_0^\pi \cos\alpha\sin\left(\frac{s}{\beta}(\pi^\beta - \tau^\beta)\right) q(\tau)\left[\sin\omega\cos\frac{s}{\beta}(\tau-\nu)^\beta + O\left(\frac{1}{s}\right)\right] d_\beta\tau \\
&- s\sin\omega\sin\alpha\sin\left(\frac{s}{\beta}\pi^\beta\right) - \cos\omega\sin\alpha\cos\left(\frac{s}{\beta}\pi^\beta\right) \\
&-\int_0^\pi \sin\alpha\cos\left(\frac{s}{\beta}(\pi^\beta - \tau^\beta)\right) q(\tau)\left[\sin\omega\cos\frac{s}{\beta}(\tau-\nu)^\beta + O\left(\frac{1}{s}\right)\right] d_\beta\tau = 0,
\end{aligned}$$

that is

$$\begin{aligned}
&\sin(\omega-\alpha)\cos\left(\frac{s}{\beta}\pi^{\beta}\right)-s\sin\omega\sin\alpha\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\\
&-\int_0^{\pi}\sin\omega\sin\alpha q(\tau)\frac{1}{2}\left[\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\cos\frac{s}{\beta}\left(\tau^{\beta}+(\tau-\nu)^{\beta}\right)\right.\\
&+\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\sin\frac{s}{\beta}\left(\tau^{\beta}+(\tau-\nu)^{\beta}\right)+\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\cos\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\\
&\left.+\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\sin\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right]d_{\beta}\tau+O\left(\frac{1}{s}\right)=0,
\end{aligned}$$

that is

$$\begin{aligned}
&\sin(\omega-\alpha)\cos\left(\frac{s}{\beta}\pi^{\beta}\right)-s\sin\omega\sin\alpha\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\\
&\qquad-\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\frac{\sin\omega\sin\alpha}{2}\int_0^{\pi}\left[\cos\frac{s}{\beta}\left(\tau^{\beta}+(\tau-\nu)^{\beta}\right)\right.\right.\\
&\qquad\left.\left.+\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)\right]q(\tau)d_{\beta}\tau\right]\\
&\qquad-\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\frac{\sin\omega\sin\alpha}{2}\int_0^{\pi}\left[\sin\frac{s}{\beta}\left(\tau^{\beta}+(\tau-\nu)^{\beta}\right)\right.\right.\\
&\qquad\left.\left.+\sin\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)\right]q(\tau)d_{\beta}\tau\right]+O\left(\frac{1}{s}\right)=0,
\end{aligned}$$

that is

$$\begin{aligned}
&\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\sin(\omega-\alpha)-\frac{\sin\omega\sin\alpha}{2}\int_0^{\pi}\left[\cos\frac{s}{\beta}\left(\tau^{\beta}+(\tau-\nu)^{\beta}\right)\right.\right.\\
&\quad\left.\left.+\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)\right]q(\tau)d_{\beta}\tau\right]\\
&\quad-\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\left[s\sin\omega\sin\alpha+\frac{\sin\omega\sin\alpha}{2}\int_0^{\pi}\left[\sin\frac{s}{\beta}\left(\tau^{\beta}+(\tau-\nu)^{\beta}\right)\right.\right.\\
&\quad\left.\left.+\sin\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)\right]q(\tau)d_{\beta}\tau\right]+O\left(\frac{1}{s}\right)=0.
\end{aligned}\tag{4. 42}$$

It is clear that these functions are bounded for $x\in[0,\pi]$ and $s\in(0,\infty)$. Now, from lemma 4.0.1 and the equations (4. 42) and (4. 11) we have

$$\begin{aligned}
&\cos\left(\frac{s}{\beta}\pi^{\beta}\right)\left[\sin(\omega-\alpha)-\sin\omega\sin\alpha\left(H(s,\tau,\nu)+O\left(\frac{1}{s}\right)\right)\right]\\
&-\sin\left(\frac{s}{\beta}\pi^{\beta}\right)\left[s\sin\omega\sin\alpha-\sin\omega\sin\alpha\left(G(s,\tau,\nu)+O\left(\frac{1}{s}\right)\right)\right]=0,
\end{aligned}$$

which, by dividing its both side by $\cos\left(\frac{s}{\beta}\pi^{\beta}\right)$ and $\sin\omega\sin\alpha$ implies that

$$\tan\left(\frac{s}{\beta}\pi^{\beta}\right)[s+G(s,\tau,\nu)]=\cot\alpha-\cot\omega-H(s,\tau,\nu)+O\left(\frac{1}{s}\right),$$

that is

$$\tan\left(\frac{s}{\beta}\pi^{\beta}\right)=\frac{1}{s}[\cot\alpha-\cot\omega-H(s,\tau,\nu)]+O\left(\frac{1}{s^2}\right).\tag{4. 43}$$

Let again $s_n = \beta n\pi^{1-\beta} + \gamma_n$ then from (4. 36), for large n the equation (4. 43) gives

$$\gamma_n = \frac{\cot\alpha}{n\pi} - \frac{\cot\omega}{n\pi} - \frac{1}{2n\pi}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau + O\left(\frac{1}{n^2}\right),$$

finally, the eigenvalues takes the form

$$\begin{aligned} s_n =& \beta n\pi^{1-\beta} + \frac{\cot\alpha}{n\pi} - \frac{\cot\omega}{n\pi} - \frac{1}{2n\pi}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau \\ & + O\left(\frac{1}{n^2}\right), \end{aligned} \tag{4. 44}$$

and this completes the proof for this case.

for the case 2,

The equations (4. 24) and (4. 41) we obtain

$$\begin{aligned} &\cos\left(\frac{s}{\beta}\pi^\beta\right)\left[\sin\omega + \frac{\sin\omega}{s}\int_0^\pi q(\tau)\sin\left(\frac{s}{\beta}\tau^\beta\right)\cos\left(\frac{s}{\beta}(\tau-\nu)^\beta\right) d_\beta\tau + O\left(\frac{1}{s^2}\right)\right] \\ &- \sin\left(\frac{s}{\beta}\pi^\beta\right)\left[\frac{\cos\omega}{s} + \frac{\sin\omega}{s}\int_0^\pi q(\tau)\cos\left(\frac{s}{\beta}\tau^\beta\right)\cos\left(\frac{s}{\beta}(\tau-\nu)^\beta\right) d_\beta\tau + O\left(\frac{1}{s^2}\right)\right] = 0. \end{aligned} \tag{4. 45}$$

multiplying (4. 45) by s and applying the product rule from trigonometric identity we have

$$\begin{aligned} &\cos\left(\frac{s}{\beta}\pi^\beta\right)\left[s\sin\omega + \frac{\sin\omega}{2}\int_0^\pi q(\tau)\left(\sin\frac{s}{\beta}\left(\tau^\beta + (\tau-\nu)^\beta\right)\right.\right. \\ &\qquad \left.\left. + \sin\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau + O\left(\frac{1}{s}\right)\right] - \sin\left(\frac{s}{\beta}\pi^\beta\right)\left[\cos\omega + \frac{\sin\omega}{2}\right. \\ &\qquad \left.\int_0^\pi q(\tau)\left(\cos\frac{s}{\beta}\left(\tau^\beta + (\tau-\nu)^\beta\right) + \cos\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau + O\left(\frac{1}{s}\right)\right] = 0, \end{aligned} \tag{4. 46}$$

and by considering lemma 4.0.1 and equation (4. 11)and multiplying (4. 46) by $\frac{1}{\sin\omega}$ we have

$$\begin{aligned} &\cos\left(\frac{s}{\beta}\pi^\beta\right)\left[s + G(s,\tau,\nu) + O\left(\frac{1}{s}\right)\right] \\ &\qquad - \sin\left(\frac{s}{\beta}\pi^\beta\right)\left[\cot\omega + H(s,\tau,\nu) + O\left(\frac{1}{s}\right)\right] = 0, \end{aligned}$$

then we have

$$\cot\left(\frac{s}{\beta}\pi^\beta\right) = \frac{1}{s}\left[\cot\omega + H(s,\tau,\nu) + O\left(\frac{1}{s^2}\right)\right], \tag{4. 47}$$

again, let $S_n = \beta\left(n + \frac{1}{2}\pi^{1-\beta}\right) + \gamma_n$ and recall that $\gamma_n = O\left(\frac{1}{n}\right)$ then we can say

$$\begin{aligned} \cot\left(\beta\left(n+\frac{1}{2}\right)\pi^{1-\beta} + \gamma_n\right)\frac{\pi^\beta}{\beta} &= -\tan\gamma_n\frac{\pi^\beta}{\beta} \\ &= \frac{1}{\beta\left(n+\frac{1}{2}\right)\pi^{1-\beta}}\left[\cot\omega + H(s,\tau,\nu) + O\left(\frac{1}{n^2}\right)\right], \end{aligned}$$

thus for sufficiently large n

$$\gamma_n = -\frac{1}{\left(n+\frac{1}{2}\right)\pi}\left[\cot\omega + H(s,\tau,\nu) + O\left(\frac{1}{n^2}\right)\right],$$

and finally

$$S_n = \beta\left(n+\frac{1}{2}\right)\pi^{1-\beta} - \frac{1}{n\pi}\left[\cot\omega + H(s,\tau,\nu) + O\left(\frac{1}{n^2}\right)\right],$$

which by (4. 11) completes the proof for the second case.
The proof for the third and the fourth cases follows similarly.

Theorem 4.3.3. *If $D_x^\beta q$ exist and bounded in $[0,\pi]$, then the positive eigenfunctions $u_n(x)$ of the problem (4. 13)-(4. 15) are*
for the case 1;

$$\begin{aligned}
u_n(x) = \sin\omega\Bigg[&\cos\left(n\pi^{1-\beta}x^\beta\right)\left(1+\frac{1}{2\beta n\pi^{1-\beta}}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)\\
&-\frac{\sin\left(n\pi^{1-\beta}x^\beta\right)}{n\pi}\left(\left(\cot\alpha-\cot\omega+\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)x\right.\\
&\left.+\left(\cot\omega+\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)\pi\right)\Bigg]+O\left(\frac{1}{n^2}\right).
\end{aligned}$$

for the case 2;

$$\begin{aligned}
u_n(x) = \sin\omega\Bigg[&\cos\left((n+\frac{1}{2})\pi^{1-\beta}x^\beta\right)\left(1+\frac{1}{2\beta n\pi^{1-\beta}}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)\\
&+\frac{\sin\left((n+\frac{1}{2})\pi^{1-\beta}x^\beta\right)}{\beta n\pi^{1-\beta}}\left(\left(\cot\omega+\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)x\right.\\
&\left.+\left(\cot\omega+\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)\pi\right)\Bigg]+O\left(\frac{1}{n^2}\right).
\end{aligned}$$

for the case 3;

$$\begin{aligned}
u_n(x) = -\cos\omega\Bigg[&\frac{\sin\left((n+\frac{1}{2})\pi^{1-\beta}x^\beta\right)}{\beta(n+\frac{1}{2})\pi^{1-\beta}}\left(1+\frac{1}{2\beta n\pi^{1-\beta}}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)\\
&+\frac{\cos\left((n+\frac{1}{2})\pi^{1-\beta}x^\beta\right)}{\beta^2n^2\pi^{3-2\beta}}\left(\left(\cos\alpha-\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)x\right.\\
&\left.+\left(\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right)\pi\right)\Bigg]+O\left(\frac{1}{n^3}\right).
\end{aligned}$$

for the case 4;

$$
\begin{aligned}
u_n(x) = & -\cos\omega \left[\frac{\sin\left(n\pi^{1-\beta}x^\beta\right)}{\beta n\pi^{1-\beta}} \left(1 + \frac{1}{2\beta n\pi^{1-\beta}} \int_0^x q(\tau) \sin\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau\right)\right. \\
& + \frac{\cos\left(n\pi^{1-\beta}x^\beta\right)}{n\pi^{2-\beta}x^\beta} \left(\left(\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau\right)\pi\right. \\
& \left.\left. + \left(\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta - (\tau-\nu)^\beta\right)\right) d_\beta\tau\right) x\right)\right] + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

Proof 21. For the proof of the *case 1*, consider the solution below of the problem (4. 13)-(4. 15),

$$
\begin{aligned}
Q(x,\mu) = & \sin\omega\cos\left(\frac{s}{\beta}x^\beta\right) - \frac{1}{s}\cos\omega\sin\left(\frac{s}{\beta}x^\beta\right) \\
& - \frac{1}{s}\int_0^x q(\tau)\sin\omega\sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right)\cos\left(\frac{s}{\beta}(\tau-\nu)^\beta\right) d_\beta\tau + O\left(\frac{1}{s^2}\right),
\end{aligned} \tag{4. 48}
$$

the right hand side of the equation (4. 48) can be expressed as

$$
\begin{aligned}
& \int_0^x q(\tau)\sin\omega\sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta)\right)\cos\left(\frac{s}{\beta}(\tau-\nu)^\beta\right) d_\beta\tau \\
& = \frac{1}{2}\int_0^x q(\tau)\sin\omega\left[\sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta + (\tau-\nu)^\beta)\right) + \sin\left(\frac{s}{\beta}(x^\beta - \tau^\beta - (\tau-\nu)^\beta)\right)\right] d_\beta\tau \\
& = \frac{1}{2}\int_0^x q(\tau)\sin\omega\left[\sin\left(\frac{s}{\beta}x^\beta\right)\cos\left(\frac{s}{\beta}(\tau^\beta - (\tau-\nu)^\beta)\right)\right. \\
& - \cos\left(\frac{s}{\beta}x^\beta\right)\sin\left(\frac{s}{\beta}(\tau^\beta - (\tau-\nu)^\beta)\right) \\
& \left. + \sin\left(\frac{s}{\beta}x^\beta\right)\cos\left(\frac{s}{\beta}(\tau^\beta + (\tau-\nu)^\beta)\right) - \cos\left(\frac{s}{\beta}x^\beta\right)\sin\left(\frac{s}{\beta}(\tau^\beta + (\tau-\nu)^\beta)\right)\right] d_\beta\tau \\
& = \sin\omega\left[\sin\left(\frac{s}{\beta}x^\beta\right)\left(\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}(\tau^\beta - (\tau-\nu)^\beta)\right) d_\beta\tau\right.\right. \\
& \left. + \frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}(\tau^\beta + (\tau-\nu)^\beta)\right) d_\beta\tau\right) \\
& - \cos\left(\frac{s}{\beta}x^\beta\right)\left(\frac{1}{2}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}(\tau^\beta - (\tau-\nu)^\beta)\right) d_\beta\tau\right. \\
& \left.\left. + \frac{1}{2}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}(\tau^\beta + (\tau-\nu)^\beta)\right) d_\beta\tau\right)\right] \\
& = \sin\omega\left[\sin\left(\frac{s}{\beta}x^\beta\right)\left(H(s,\tau,\nu) + O\left(\frac{1}{s}\right)\right) - \cos\left(\frac{s}{\beta}x^\beta\right)\left(G(s,\tau,\nu) + O\left(\frac{1}{s}\right)\right)\right].
\end{aligned}
$$

Now the equation (4. 48) becomes

$$
\begin{aligned}
Q(x,\mu) &= \sin\omega\cos\left(\frac{s}{\beta}x^{\beta}\right) - \frac{1}{s}\cos\omega\sin\left(\frac{s}{\beta}x^{\beta}\right) - \frac{\sin\omega}{s}\sin\left(\frac{s}{\beta}x^{\beta}\right)H(s,\tau,\nu) \\
&+ \frac{\sin\omega}{s}\cos\left(\frac{s}{\beta}x^{\beta}\right)G(s,\tau,\nu) + O\left(\frac{1}{s^2}\right) \\
&= \sin\omega\cos\left(\frac{s}{\beta}x^{\beta}\right)\left(1+\frac{G(s,\tau,\nu)}{s}\right) \\
&- \frac{\sin\left(\frac{s}{\beta}x^{\beta}\right)}{s}(\cos\omega+\sin\omega H(s,\tau,\nu)) + O\left(\frac{1}{s^2}\right) \\
&= \sin\omega\left[\cos\left(\frac{s}{\beta}x^{\beta}\right)\left(1+\frac{G(s,\tau,\nu)}{s}\right)\right. \\
&\left.-\frac{\sin\left(\frac{s}{\beta}x^{\beta}\right)}{s}(\cot\omega+H(s,\tau,\nu))\right] + O\left(\frac{1}{s^2}\right),
\end{aligned}
\tag{4. 49}
$$

then replacing s by s_n and using (4. 44) we have

$$
\begin{aligned}
Q(x,\mu) &= \sin\omega\left[\cos\left(n\pi^{1-\beta}x^{\beta}\right)\left(1+\frac{G(s,\tau,\nu)}{\beta n\pi^{1-\beta}}\right)\right. \\
&-\frac{\beta\sin\left(n\pi^{1-\beta}x^{\beta}\right)}{n\pi^{\beta}}((\cot\alpha-\cot\omega+H(s,\tau,\nu))x \\
&\left.+(\cot\omega+H(s,\tau,\nu))\pi)\right] + O\left(\frac{1}{n^2}\right),
\end{aligned}
$$

by the definition of $G(s,\tau,\nu)$ and $H(s,\tau,\nu)$ from (4. 11) above, we have

$$
\begin{aligned}
u_n(x) &= \sin\omega\left[\cos\left(n\pi^{1-\beta}x^{\beta}\right)\left(1+\frac{1}{2\beta n\pi^{1-\beta}}\int_0^x q(\tau)\sin\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau\right)\right. \\
&-\frac{\beta\sin\left(n\pi^{1-\beta}x^{\beta}\right)}{n\pi^{\beta}}\left(\cot\alpha-\cot\omega+\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau\right)x \\
&\left.+\left(\cot\omega+\frac{1}{2}\int_0^x q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau\right)\pi\right] + O\left(\frac{1}{n^2}\right)
\end{aligned}
$$

For the proof of the *case 2*, consider the equation below,

$$
\begin{aligned}
Q(x,\mu) &= \sin\omega\left[\cos\left(\frac{s}{\beta}x^{\beta}\right)\left(1+\frac{G(s,\tau,\nu)}{s}\right)\right. \\
&\left.-\frac{\sin\left(\frac{s}{\beta}x^{\beta}\right)}{s}(\cot\omega+H(s,\tau,\nu))\right] + O\left(\frac{1}{s^2}\right).
\end{aligned}
\tag{4. 50}
$$

Now replacing s by s_n and using (4. 49) in (4. 50) we have

$$
\begin{aligned}
Q(x,\mu) &= \sin\omega\left[\cos\left(\left(n+\frac{1}{2}\right)\pi^{1-\beta}x^{\beta}\right)\left(1+\frac{1}{\beta n\pi^{1-\beta}}G(s,\tau,\nu)\right)\right. \\
&\left.+\frac{\sin\left(\left(n+\frac{1}{2}\right)\pi^{1-\beta}x^{\beta}\right)}{\beta n\pi^{1-\beta}}(\cot\omega+H(s,\tau,\nu))\right] + O\left(\frac{1}{n^2}\right),
\end{aligned}
$$

hence we have

$$Q(x,\mu)=\sin\omega\left[\cos\left(\left(n+\frac{1}{2}\right)\pi^{1-\beta}x^{\beta}\right)\left(1+\frac{1}{\beta n\pi^{1-\beta}}G(s,\tau,\nu)\right)\right.$$
$$+\frac{\sin\left(\left(n+\frac{1}{2}\right)\pi^{1-\beta}x^{\beta}\right)}{\beta n\pi^{1-\beta}}((\cot\omega+H(s,\tau,\nu))x$$
$$\left.-(\cot\omega+H(s,\tau,\nu))\pi)\right]+O\left(\frac{1}{n^{2}}\right),$$

also, by the definition of $G(s,\tau,\nu)$ and $H(s,\tau,\nu)$ from (4. 11) above, the proof for the second case is completed.

Also, the proof for the third and the fourth cases follows similarly.

The eigenvalues and matching eigenfunctions that depend on the delay constant have also been computed. In addition, we will discuss the inverse problem of this SLP in the following manner.

4.4 The Inverse Nodal Problem

In this section we give the nodal points and the nodal lengths of the the problem for each of the four cases. The proofs are similar as such we give the proof of the first two cases.

Theorem 4.4.1. *The nodal points of the problem* (4. 13)-(4. 15) *are*
for the case 1:

$$y_i^n=\frac{i\pi}{s_n}+\frac{1}{s_n^2}\left[\cot\alpha-\cot\omega-\frac{1}{2}\int_0^{y_i^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau\right]+O\left(\frac{1}{n^3}\right).$$

for the case 2:

$$y_i^n=\frac{\left(i-\frac{1}{2}\right)\pi}{s_n}+\frac{1}{s_n^2}\left[\cot\omega+\frac{1}{2}\int_0^{y_i^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau\right]+O\left(\frac{1}{n^3}\right).$$

for the case 3:

$$y_i^n=\frac{\left(i-\frac{1}{2}\right)\pi}{s_n}-\frac{1}{s_n^2}\left[\cot\alpha-\frac{1}{2}\int_0^{y_i^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau\right]+O\left(\frac{1}{n^3}\right).$$

for the case 4:

$$y_i^n=\frac{i\pi}{s_n}-\frac{1}{2s_n^2}\int_0^{y_i^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau+O\left(\frac{1}{n^3}\right).$$

Proof 22. For the proof of *the first case;* from (4. 43) and by replacing π by x the nodal point can be obtained as follows,

$$\frac{x^{\beta}}{\beta}=\frac{\pi}{s}+\frac{1}{s^2}\left[\cot\alpha-\cot\omega-\frac{1}{2}\int_0^{x}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau\right]+O\left(\frac{1}{s^3}\right),$$

$$(4.\ 51)$$

now, taking $\mu=\mu_n,\ s=s_n,\ and\ x=x_i^{(n)}$, we can expressed (4. 51) as

$$\frac{(x_i^{(n)})^\beta}{\beta}=\frac{i\pi}{s_n}+\frac{1}{s_n^2}\left[\cot\alpha-\cot\omega-\frac{1}{2}\int_0^{\frac{(x_i^n)^\beta}{\beta}}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right]+O\left(\frac{1}{n^3}\right), \tag{4. 52}$$

which is the nodal points of the problem.

Now, for convenience, let us define $Y_\beta=\left\{y_i^n:y_i^n=\frac{(x_i^n)^\beta}{\beta},x_i^n\in X\right\},n\in\mathbb{N}$ and $i=\overline{1,n}$ as the nodal set of the problem while $X=\{x_i^n\}$ is the nodal set of the corresponding classical problem, as such, we can expressed (4. 52) as

$$y_i^n=\frac{i\pi}{s_n}+\frac{1}{s_n^2}\left[\cot\alpha-\cot\omega-\frac{1}{2}\int_0^{y_i^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right]+O\left(\frac{1}{n^3}\right),$$

that is

$$y_{i+1}^n=\frac{(i+1)\pi}{s_n}+\frac{1}{s_n^2}\left[\cot\alpha-\cot\omega-\frac{1}{2}\int_0^{y_{i+1}^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right]+O\left(\frac{1}{n^3}\right).$$

For the proof of *the second case;* from (4. 47) and by replacing π by x and following the steps in the proof of the first case, the nodal point can be obtained as follows

$$y_i^n=\frac{\left(i-\frac{1}{2}\right)\pi}{s_n}+\frac{1}{s_n^2}\left[\cot\omega+\frac{1}{2}\int_0^{y_i^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right]+O\left(\frac{1}{n^3}\right),$$

such that

$$y_{i+1}^n=\frac{\left(i+\frac{1}{2}\right)\pi}{s_n}+\frac{1}{s_n^2}\left[\cot\omega+\frac{1}{2}\int_0^{y_{i+1}^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau\right]+O\left(\frac{1}{n^3}\right).$$

Now using the fact that the difference of the points (y_i^n,y_{i+1}^n) is the nodal length of the problem, as such the nodal length of the first case (which is the same as the nodal length of the fourth case) is obtain as

$$l_{\beta i}^n=\frac{\pi}{s_n}-\frac{1}{2s_n^2}\int_{y_i^n}^{y_{i+1}^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau+o\left(\frac{1}{n^3}\right), \tag{4. 53}$$

while the nodal length of the second case (which is the same as the nodal length of the third case) is obtained as

$$l_{\beta i}^n=\frac{\pi}{s_n}+\frac{1}{2s_n^2}\int_{y_i^n}^{y_{i+1}^n}q(\tau)\cos\left(\frac{s}{\beta}\left(\tau^\beta-(\tau-\nu)^\beta\right)\right)d_\beta\tau+o\left(\frac{1}{n^3}\right). \tag{4. 54}$$

4.5 Reconstruction of the Potential Function

Reconstruction of the potential function using the nodal points and the nodal lengths is one of the ways in inverse problem, as such we reconstructed the potential function in each of the four cases expressed above.

For the potential function in the first case (which is the same as the potential in the fourth case) from (4. 53) we have

$$\left(\frac{l_{\beta i}^{n}s_n}{\pi}-1\right)\frac{2\pi s_n}{1}=-\int_{y_i^n}^{y_{i+1}^n}q(\tau)\cos\left(\frac{s_n}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau+o\left(\frac{1}{n^2}\right), \tag{4. 55}$$

then, by applying the mean value theorem for integrals to (4. 55) for a fixed n, there exist a point, $z\in(y_i^n,y_{i+1}^n)$ so that

$$\int_{y_i^n}^{y_{i+1}^n}q(\tau)\cos\left(\frac{s_n}{\beta}\left(\tau^{\beta}-(\tau-\nu)^{\beta}\right)\right)d_{\beta}\tau=l_{\beta i}^{n}q(z)\cos\left(\frac{s_n}{\beta}\left(z^{\beta}-(z-\nu)^{\beta}\right)\right), \tag{4. 56}$$

equations (4. 55) and (4. 56) gives

$$q(z)=-\left(\frac{l_{\beta i}^{n}s_n}{\pi}-1\right)\frac{2\pi s_n}{l_{\beta i}^{n}\cos\left(\frac{s_n}{\beta}\left(z^{\beta}-(z-\nu)^{\beta}\right)\right)}+o\left(\frac{1}{n}\right), \tag{4. 57}$$

but from (4. 53) we have

$$\frac{l_{\beta i}^{(n)}s_n}{\pi}=1+O\left(\frac{1}{n}\right), \tag{4. 58}$$

then from (4. 58) and as $n\to\infty$, (4. 57) becomes

$$q(x)=\frac{2s_n^2}{\cos\left(\frac{s_n}{\beta}\left(x^{\beta}-(x-\nu)^{\beta}\right)\right)}\left(1-\frac{l_{\beta i}^{n}s_n}{\pi}\right), \tag{4. 59}$$

then by taking the limit as $n\to\infty$ and by (4. 44), equation (4. 59) becomes

$$q(x)=\lim_{n\to\infty}\frac{2\beta^2n^2\pi^{2-2\beta}}{\cos\left(n\pi^{1-\beta}\left(x^{\beta}-(x-\nu)^{\beta}\right)\right)}\left(1-\frac{\beta n l_{\beta i}^{n}}{\pi^{\beta}}\right). \tag{4. 60}$$

Similarly, the potential function of the the second case (which is the same as the potential in the third case) from (4. 54) is calculated as

$$q(x)=\lim_{n\to\infty}\frac{2\beta^2(n+\frac{1}{2})^2\pi^{2-2\beta}}{\cos\left((n+\frac{1}{2})\pi^{1-\beta}\left(x^{\beta}-(x-\nu)^{\beta}\right)\right)}\left(\frac{\beta(n+\frac{1}{2})l_{\beta i}^{n}}{\pi^{\beta}}-1\right), \tag{4. 61}$$

then (4. 60) and (4. 61) are the potential functions of the SLP (4. 13)- (4. 15).

4.6 The Stability of the Potential

In this section, we would like to prove the stability of the problem using Lischitz stability approach which is based on metric spaces. The concept of hemeomorphism between the defined two metric spaces is considered to show the continuity.
Let $N = \mathbb{N}\setminus\{1\}$. Let Ω be the space of all admissible sequences that is

$$\Omega = \{(q,\omega,\alpha) \in L^1_{\beta_i}(0,\pi)\times(0,\pi)\}$$

$\Sigma = The\ collection\ of\ all\ double\ sequences\ X_\beta = \{X^{(n)}_{\beta_i}\}$ for $i = 1,2,3,\dots,n-1, n \in N$ such that $0 < X^{(n)}_{\beta_1} < X^{(n)}_{\beta_2} < X^{(n)}_{\beta_3} < \dots < X^{(n)}_{\beta_{n-1}} < \pi$, for each n.
Let $X_\beta \in \Sigma$ and define $X^{(n)}_{\beta_0} = 0, X^{(n)}_{\beta_n} = \pi, L^{(n)}_{\beta_i} = X^{(n)}_{\beta_{i+1}} - X^{(n)}_{\beta_i},\ and\ I^{(n)}_{\beta_i} = (X^{(n)}_{\beta_i}, X^{(n)}_{\beta_{i+1}})$ for $i = 1,2,3,\dots,n-1$.
Let the function $i_{\beta n}(x)$ on $(0,\pi)$ be define by

$$i_{\beta n}(x) = Max\{i : x^{(n)}_i \le x\},$$

and for fixed x and n, $i = i_{\beta n}(x)$ implies $x \in [x^{(n)}_i, x^{(n)}_{i+1}]$
In what follows, let define $\gamma = \frac{1}{\cos\left(n\pi^{1-\beta}\left(x^\beta-(x-\nu)^\beta\right)\right)}$ for convenience.

Definition 4.6.1. *Let $X^{(n)}_{\beta_i}, \widetilde{X}^{(n)}_{\beta_i} \in \Sigma$ with $L^{(n)}_{\beta_i}$ and $\widetilde{L}^{(n)}_{\beta_{i+1}}$ as their respective grid lengths. Define*

$$K_{\beta_n}(X_\beta, \widetilde{X}_\beta) = \beta^2\pi^{2-2\beta}\Sigma_{i=0}^{n-1} n^2\gamma|L^{(n)}_{\beta_i} - \widetilde{L}^{(n)}_{\beta_i}|.$$

Let d_o and d_Σ be two metric on Σ defined by

$$d_o(X_\beta, \widetilde{X}_\beta) = \overline{\lim_{n\to\infty}} K_{\beta_n}(X_\beta, \widetilde{X}_\beta),$$

and

$$d_\Sigma(X_\beta, \widetilde{X}_\beta) = \overline{\lim_{n\to\infty}} \frac{K_{\beta_n}(X_\beta, \widetilde{X}_\beta)}{1 + K_{\beta_n}(X_\beta, \widetilde{X}_\beta)} = \frac{d_o(X_\beta, \widetilde{X}_\beta)}{1 + d_o(X_\beta, \widetilde{X}_\beta)}, \tag{4. 62}$$

which implies that

$$d_o(X_\beta, \widetilde{X}_\beta) = \frac{d_\Sigma(X_\beta, \widetilde{X}_\beta)}{1 - d_\Sigma(X_\beta, \widetilde{X}_\beta)}, \tag{4. 63}$$

such that

$$d_\Sigma(X_\beta, \widetilde{X}_\beta) \le d_o(X_\beta, \widetilde{X}_\beta). \tag{4. 64}$$

Then, there exist a positive constant k such that

$$\frac{2}{3k} \leq d_o(X_\beta, \widetilde{X}_\beta) \leq d_\Sigma(X_\beta, \widetilde{X}_\beta) \leq 3(k+1)d_o(X_\beta, \widetilde{X}_\beta),$$

which means $d_o(X_\beta, \widetilde{X}_\beta) = 0$ only when $d_\Sigma(X_\beta, \widetilde{X}_\beta) = 0$. If this is achieved, then this inverse nodal problem is stable. The metric $d_\Sigma(.,.)$ is a pseudometric on Σ.

Proof 23. From the definition and the absolute value properties $d_\Sigma(.,.)$ is finite and symmetric. Now the following proves the triangle inequility, that is,

$$\begin{aligned}
K_{\beta_n}(X_\beta, \widetilde{X}_\beta) &= \beta^2\pi^{2-2\beta}\Sigma_{i=0}^{n-1} n^2\gamma|L_{\beta_i}^{(n)} - \widetilde{L}_{\beta_i}^{(n)}| \\
&\leq \beta^2\pi^{2-2\beta}\Sigma_{i=0}^{n-1} n^2\gamma[|L_{\beta_i}^{(n)} - L_{\beta_j}^{(n)}| + |L_{\beta_j}^{(n)} - \widetilde{L}_{\beta_i}^{(n)}|] \\
&= \beta^2\pi^{2-2\beta}\Sigma_{i=0}^{n-1} n^2\gamma|L_{\beta_i}^{(n)} - L_{\beta_j}^{(n)}| + \beta^2\pi^{2-2\beta}\Sigma_{i=0}^{n-1} n^2\gamma|L_{\beta_j}^{(n)} - \widetilde{L}_{\beta_i}^{(n)}| \\
&= K_{\beta_n}(X_\beta, Y_\beta) + K_{\beta_n}(Y_\beta, \widetilde{X}_\beta),
\end{aligned}$$

clearly

$$\frac{K_{\beta_n}(X_\beta, \widetilde{X}_\beta)}{1+K_{\beta_n}(X_\beta, \widetilde{X}_\beta)} \leq \frac{K_{\beta_n}(X_\beta, Y_\beta)}{1+K_{\beta_n}(X_\beta, Y_\beta)} + \frac{K_{\beta_n}(Y_\beta, \widetilde{X}_\beta)}{1+K_{\beta_n}(Y_\beta, \widetilde{X}_\beta)}, \tag{4. 65}$$

then from (4. 62) and (4. 65) we have

$$d_\Sigma(X_\beta, \widetilde{X}_\beta) \leq d_\Sigma(X_\beta, Y_\beta) + d_\Sigma(Y_\beta, \widetilde{X}_\beta).$$

This completes the proof.

So, $d_\Sigma(.,.)$ defines an equivalence relation $\sim$ on $\sum$. Now $(\sum^*, d_\Sigma)$ is a metric space, where $\sum^* := \sum / \sim$.
Let $\sum_1 \subset \sum$ be the subspace of all asymtotically equivalent nodal sequences and let $\sum_1^* := \sum_1 / \sim$.

Theorem 4.6.1. *The metric spaces $(\Omega, ||\cdot||_1)$ and $(\sum^*, d_\Sigma)$ are homeomorphic to each other.*

Lemma 4.6.1. *Let $X_\beta, \widetilde{X}_\beta \in \Sigma_1$, then*

i. The length of the interval $I_{\beta i}^{(n)}$ between the points $X_{\beta i}^{(n)}$ and $\widetilde{X}_{\beta i}^{(n)}$ is $O\left(\frac{1}{n^2}\right)$.

ii. For all $x \in (0,1)$, $|i_{\beta n}(x) - \widetilde{i}_{\beta n}(x)| \leq 1$ for sufficiently large n.

Proof 24. For the proof of i. above, we have from (4. 56)

$$X_{\beta i}^{(n)} - \frac{i\pi^\beta}{\beta n} = O\left(\frac{1}{n^2}\right),$$

then

$$\left|I_{\beta i}^{(n)}\right| = \left|X_{\beta i}^{(n)} - \widetilde{X}_{\beta i}^{(n)}\right| \leq \left|X_{\beta i}^{(n)} - \frac{i\pi^{\beta}}{\beta n}\right| + \left|\frac{i\pi^{\beta}}{\beta n} - \widetilde{X}_{\beta i}^{(n)}\right|$$
$$= O\left(\frac{1}{n^2}\right) + O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n^2}\right),$$

and for the proof of ii, for a fixed $x \in (0,\pi)$ and from (4. 56) again,

$$\frac{i\pi^{\beta}}{\beta n} + O\left(\frac{1}{n^2}\right) = X_{\beta i}^{(n)} \leq x \leq X_{\beta(i+1)}^{(n)} = \frac{(i+1)\pi^{\beta}}{\beta n} + O\left(\frac{1}{n^2}\right),$$

similarly

$$\frac{\widetilde{i}\pi^{\beta}}{\beta n} + O\left(\frac{1}{n^2}\right) = \widetilde{X}_{\beta \widetilde{i}}^{(n)} \leq x \leq \widetilde{X}_{\beta(\widetilde{i}+1)}^{(n)} = \frac{(\widetilde{i}+1)\pi^{\beta}}{\beta n} + O\left(\frac{1}{n^2}\right),$$

now, for a very large n, $i+1 \geq \widetilde{i}$ and $\widetilde{i}+1 \geq i$ which implies that

$$\left|i_{\beta i}(x) - \widetilde{i}_{\beta i}(x)\right| \leq 1,$$

this completes the proof.

Theorem 4.6.2. *If $X_{\beta}, \widetilde{X}_{\beta} \in \Sigma_1$ are asymptotically nodal to q and $\widetilde{q}$ in Ω, respectively, then*

$$||q - \widetilde{q}|| \leq 2d_o\left(X_{\beta}, \widetilde{X}_{\beta}\right).$$

Proof 25. From (4. 59) and for almost every, $x \in (0,\pi)$, we have

$$q(x) - \widetilde{q}(x) = \overline{\lim}_{n\to\infty} \frac{2\gamma s_n^3}{\pi}\left[\widetilde{L}_{\beta \widetilde{i}_{\beta n}(x)}^{(n)} - L_{\beta i_{\beta n}(x)}^{(n)}\right],$$

then, by Fatou's lemma, we have

$$\begin{aligned}\int_0^{\pi} |q - \widetilde{q}|\, d_{\beta}x &\leq \underline{\lim}_{n\to\infty} 2\beta^3\pi^{2-3\beta}\int_0^{\pi} \gamma n^3 \left|L_{\beta i_{\beta n}(x)}^{(n)} - \widetilde{L}_{\beta \widetilde{i}_{\beta n}(x)}^{(n)}\right| d_{\beta}x \\ &\leq 2\beta^3\pi^{2-3\beta}\overline{\lim}_{n\to\infty}\left\{n^3\int_0^{\pi}\gamma\left|L_{\beta i_{\beta n}(x)}^{(n)} - \widetilde{L}_{\beta i_{\beta n}(x)}^{(n)}\right| d_{\beta}x \right. \\ &\left. + n^3\int_0^{\pi}\gamma\left|\widetilde{L}_{\beta i_{\beta n}(x)}^{(n)} - \widetilde{L}_{\beta \widetilde{i}_{\beta n}(x)}^{(n)}\right| d_{\beta}x\right\},\end{aligned} \tag{4. 66}$$

due to lemma 4.6.1, the second term gives

$$\begin{aligned}n^3\int_0^{\pi}\gamma\left|\widetilde{L}_{\beta i_{\beta n}(x)}^{(n)} - \widetilde{L}_{\beta \widetilde{i}_{\beta n}(x)}^{(n)}\right| d_{\beta}x &= n^3\gamma\Sigma_{i=0}^{n-1}\left|\widetilde{L}_{\beta(i+1)} - \widetilde{L}_{\beta i}\right|\left|I_{\beta i}^{n}\right| \\ &= n^3\gamma\Sigma_{i=0}^{n-1} o\left(\frac{1}{n^2}\right) O\left(\frac{1}{n^2}\right) = O(1),\end{aligned}$$

since, for $1 \le i \le n-2$, (4. 53) implies that

$$\left|L_{\beta i} - \widetilde{L}_{\beta i}\right| = \frac{\pi^{2\beta-2}}{2\beta^2 n^2}\left|\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} \frac{1}{\gamma} q(\tau) d_\beta \tau - \int_{\widetilde{x}_i^{(n)}}^{\widetilde{x}_{i+1}^{(n)}} \frac{1}{\gamma}\widetilde{q}(\tau) d_\beta \tau\right| + o\left(\frac{1}{n^3}\right)$$

Thus $\left|L_{\beta i} - \widetilde{L}_{\beta i}\right| = o\left(\frac{1}{n^2}\right)$, as such the second term in (4. 66), after taking the upper limit, vanished.
Also, by virture of (4. 66) we obtain

$$\begin{aligned} \int_0^\pi |q-\widetilde{q}|\, d_\beta x &\le 2\beta^3\pi^{2-3\beta}\overline{\lim}_{n\to\infty}\Sigma_{i=0}^{n-1} n^3 \gamma L_{\beta i}^n \left|L_{\beta i}^n - \widetilde{L}_{\beta i}^n\right| \\ &\le 2\beta^3\pi^{2-3\beta}\overline{\lim}_{n\to\infty}\Sigma_{i=0}^{n-1}\gamma\left(\frac{\pi^\beta}{\beta} n^2 + O(n)\right)\left|L_{\beta i}^n - \widetilde{L}_{\beta i}^n\right| \\ &= 2\beta^2\pi^{2-2\beta}\overline{\lim}_{n\to\infty}\Sigma_{i=0}^{n-1} n^2\gamma\left|L_{\beta i}^n - \widetilde{L}_{\beta i}^n\right| \\ &= 2d_o(X_\beta, \widetilde{X}_\beta), \end{aligned} \tag{4. 67}$$

which completes the proof.

It follows from theorem 4.6.2 that for the asymptotically nodal points X_β and $\widetilde{X}_\beta$ to q and $\widetilde{q}$ in Ω the following relation holds

$$||q-\widetilde{q}||_1 \le 2d_o(X_\beta, \widetilde{X}_\beta).$$

Theorem 4.6.3. *If X_β and $\widetilde{X}_\beta$ asymptotically nodal to q and $\widetilde{q}$ in Ω, then $d_o(X_\beta, \widetilde{X}_\beta) \le ||q-\widetilde{q}||_1$.*

Proof 26. By virtue of (4. 67) and definition 4.6.1 we have

$$\begin{aligned} K_{\beta n}(X_\beta, \widetilde{X}_\beta) &\le \beta^2\pi^{2-2\beta}\Sigma_{i=0}^{n-1} n^2\gamma\left|L_{\beta i}^n - \widetilde{L}_{\beta i}^n\right| \\ &= \frac{\gamma}{2}\Sigma_{i=0}^{n-1}\left|\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}\frac{1}{\gamma}q(\tau)d_\beta\tau - \int_{\widetilde{x}_i^{(n)}}^{\widetilde{x}_{i+1}^{(n)}}\frac{1}{\gamma}\widetilde{q}(\tau)d_\beta\tau\right| + o(1) \\ &\le \frac{\gamma}{2}\Sigma_{i=0}^{n-1}\left\{\left|\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}\frac{1}{\gamma}\left(q(\tau)-\widetilde{q}(\tau)\right)d_\beta\tau\right|\right. \\ &\left. + \left|\left(\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} - \int_{\widetilde{x}_i^{(n)}}^{\widetilde{x}_{i+1}^{(n)}}\right)\frac{1}{\gamma}\widetilde{q}(\tau)d_\beta\tau\right|\right\} + o(1) \\ &\le \frac{\gamma}{2}\Sigma_{i=0}^{n-1}\left\{\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}\frac{1}{\gamma}|q(\tau)-\widetilde{q}(\tau)|\, d_\beta\tau\right. \\ &\left. + \left(\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} - \int_{\widetilde{x}_i^{(n)}}^{\widetilde{x}_{i+1}^{(n)}}\right)\frac{1}{\gamma}|\widetilde{q}(\tau)|\, d_\beta\tau\right\} + o(1), \end{aligned} \tag{4. 68}$$

but we know that

$$\begin{aligned}\Sigma_{i=0}^{n-1}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}\frac{1}{\gamma}|q(\tau)-\widetilde{q}(\tau)|\,d_\beta\tau &= \int_{x_0^{(n)}}^{x_1^{(n)}}\frac{1}{\gamma}|q(\tau)-\widetilde{q}(\tau)|\,d_\beta\tau+\int_{x_1^{(n)}}^{x_2^{(n)}}\frac{1}{\gamma}|q(\tau)-\widetilde{q}(\tau)|\,d_\beta\tau\\ &+\cdots+\int_{x_n^{(n)}}^{x_{n-1}^{(n)}}\frac{1}{\gamma}|q(\tau)-\widetilde{q}(\tau)|\,d_\beta\tau\\ &=\int_0^{\pi}\frac{1}{\gamma}|q(\tau)-\widetilde{q}(\tau)|\,d_\beta\tau=\left\|\frac{1}{\gamma}(q-\widetilde{q})\right\|_1,\end{aligned}$$

then (4. 68) becomes

$$\begin{aligned}K_{\beta n}(X_\beta,\widetilde{X}_\beta) &\le \frac{\gamma}{2}||\frac{1}{\gamma}||\,||q-\widetilde{q}||_1+\Sigma_{i=0}^{n-1}\left(\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}-\int_{\widetilde{x}_i^{(n)}}^{\widetilde{x}_{i+1}^{(n)}}\right)\frac{1}{\gamma}|\widetilde{q}(\tau)|\,d_\beta\tau+o(1)\\ &\le ||q-\widetilde{q}||_1+\Sigma_{i=0}^{n-1}\int_{\widetilde{I}_{\beta i}^{(n)}}\frac{1}{\gamma}|\widetilde{q}(\tau)|\,d_\beta\tau,\end{aligned} \tag{4. 69}$$

where $\widetilde{I}_{\beta i}^{(n)}=I_{\beta i}^{(n)}\cup I_{\beta(i+1)}^{(n)}$ and $I_{\beta(i)}^{(n)}$ is the interval bounded by $x_{\beta i}^{(n)}$ and $\widetilde{x}_{\beta i}^{(n)}$. Let $I_\beta^{(n)}=\cup\widetilde{I}_{\beta(i)}^{(n)}$, then by virtue of lemma 4.6.1 we have

$$\left|I_\beta^n\right|=\Sigma_{i=0}^{n-1}\left|\widetilde{I}_{\beta i}^n\right|=n\left|\widetilde{I}_{\beta i}^n\right|=nO\left(\frac{1}{n^2}\right)=O\left(\frac{1}{n}\right),$$

as such $\overline{lim}_{n\to\infty}\int_{\widetilde{I}_{\beta i}^{(n)}}\frac{1}{\gamma}|\widetilde{q}(\tau)|\,d_\beta\tau=0$ and (4. 69) gives $K_{\beta n}(X_\beta,\widetilde{X}_\beta)\le||q-\widetilde{q}||_1$ which implies that

$$d_o(X_\beta,\widetilde{X}_\beta)\le||q-\widetilde{q}||_1, \tag{4. 70}$$

hence the theorem is proved.

Now, we can prove the theorem 4.6.1.

Proof 27. Considering section 4.6, theorem 4.6.1 and (4. 64) it is clear that $d_\Sigma(X_\beta,\widetilde{X}_\beta)=0$ if and only if $q=\widetilde{q}$, as such Σ_1^* is in one-one correspondence with Ω.
Let

$$d_\Sigma(X_\beta,\widetilde{X}_\beta)\le\frac{1}{2},$$

it follows from (4. 63) and section 4.6 that

$$||q-\widetilde{q}||_1\le 4d_\Sigma(X_\beta,\widetilde{X}_\beta), \tag{4. 71}$$

on the contrary, it follows from theorem 4.6.3 that if $||q-\widetilde{q}||_1$ is small, then $d_\Sigma(X_\beta,\widetilde{X}_\beta)$ is also small. This and (4. 70) completes the proof.

It can be seen from (4. 70) and (4. 71) that $d_o(X_\beta,\widetilde{X}_\beta)=0$ only when $d_\Sigma(X_\beta,\widetilde{X}_\beta)=0$. as such, the inverse nodal problem is stable.

5. Conclusion

In this research work, we established and presented the direct and inverse SLP using the new non-integer value differentiation method- Conformable derivative, where we analyzed its spectral properties; the eigenvalues, the eigenfunctions, the nodal points, the nodal lengths and the potential functions under different sets of boundary conditions as an effort to extend the concepts of the SLP and its inverse problems to the non-integer value derivative concepts as it gives a more appropriate and more approximate model of the likely physical problems.

In order to test the possibility of using this new differentiation method in SLPs we used it and established a transformation operator for the SLP on a classical Sturm-Liouville operator, there we obtained a Hyperbolic type partial differential equation and the related conditions for Nucleus function $N(x,\tau)$. Finally, this problem is reduced to a Volterra integral type equation. This transformation operator has an important place in terms of the solution of the inverse problem in the SLT, especially the relation between the potential function and the kernel in the form of $N(x,x) = \int_0^x [q(s) - r(s)]ds$. A similar relationship for the conformable SLP is presented as $N(x,x) = \int_0^x \frac{[q(s)-r(s)]}{1+s^{\beta-1}} d_\beta s$.

Subsequently, we investigated the direct and inverse nodal problems associated with a conformable derivative SLP with the spectral parameter included in the boundary conditions. The spectral parameters obtained were employed to reconstruct the potential function and the uniqueness of the potential function is demonstrated. We proved the Ambarzumyan theorem to furthermore demonstrate the new derivative approach. This study is significant for two reasons. Initially, an *nth*-degree polynomial was considered in the boundary condition, resulting in the availability of exceptional cases. Secondly, in comparison to the classical derivative, the conformable derivative demonstrated greater universality.

In the last section, we extended the SLPs to the concepts of constant delay using the conformable derivative approach, there we were able to show and expressed the possibility of solving the direct and inverse SLP with constant delay and expressed the corresponding spectral properties.

We examined the problem under two different sets of boundary conditions. In the first case, we solved the direct problem and proved that the specification of the spectrum uniquely determine the characteristics function, while in the second case, we solved both the direct and the inverse problem and proved the stability of the obtained potential function using Lipschitz stability approach.

This novel derivative approach allows for the discussion of similar problems with different boundary conditions, along with their associated spectral features and also similar or other derivative approaches can be use where possible to express similar cases of the SLP so that a full non-integer derivative phase of the direct and inverse SLP could be established and this will significantly add to the literature of this special case of the second order differential equation.

RECOMMENDATIONS

This research work established, analyzed and successfully expressed the direct and inverse SLP using a non-integer value derivative called conformable derivative which is an addition to the literature of the SLP since it is an extension to the classical SLP. This work may enhance the provision of more appropriate and more approximate models of the likely real life problem and we recommended that further researches should be carried out using this new derivative under different set of the boundary conditions or another different non-integer value differentiation approaches in order to established a complete phenomena for the SLP as it may be more vital in the near future.

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CURRICULUM VITAE

Auwalu SA’IDU

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ACADEMIC ACTIVITIES

Publications:

1. Auwalu Sa’idu and Mannir Yusuf, "Application of Fuzzy Set Theory in Obtaining Optimal Preference Ordering in Multi-person Decision Making," *International Journal of Innovative Science and Research Technology,* Vol. 5, Issue 5, 2020.

2. Auwalu Sa’idu, "Practical Analysis on the Effect of Shiruma Approach in Obtaining Optimal Preference Ordering in Multi-Person Fuzzy Decision Making," *International Journal of Innovative Research & Development,* Vol. 9, Issue 4, 2020.

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