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STABILITY ANALYSIS AND FEEDBACK STABILIZATION OF BIMODAL PIECEWISE LINEAR SYSTEMS

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Yavuz EREN

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LIST OF SYMBOLS

$\mathbb{C}^n$	n dimensioned complex vector space
$\mathbb{R}^n$	n dimensioned real vector space
Δ	Lebesque measurable functions of time
Σ	Linear time invariant system
d	Damping constant
$im[B]$	Image space of B
inf	Infimum
$ker[C]$	Kernel space of C
k	Spring constants
m_i	Mass of cart i
x^T	Transpose of x
ω	Frequency
ϵ	Epsilon
$x^{x_0,u}$	Trajectory x with initial condition x_0 and control input u
x_f	Final value of trajectory x
$\Longrightarrow$	Implies
$\Longleftrightarrow$	If and only if

LIST OF ABBREVIATIONS

BMI	Bilinear Matrix Inequality
CQLF	Common Quadratic Lyapunov Function
LMI	Linear Matrix Inequality
LTI	Linear Time Invariant
PWL	Piecewise Linear

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ABSTRACT

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This thesis deals with the quadratic stability and feedback stabilization problems for continuous bimodal piecewise linear systems. First, we provide necessary and sufficient conditions in terms of linear matrix inequalities for quadratic stability and stabilization of this class of systems. Later, these conditions are investigated from a geometric control point of view and a set of sufficient conditions for feedback stabilization are obtained.

Moreover, we consider observer design procedure for bimodal systems and we propose a simpler procedure by reducing the required conditions on the observer design.

Finally, the result for stability analysis is extended to the bimodal systems with norm-bounded uncertainties and is proposed a corollary to guarantee the robust stability for the related systems.

Key words: Piecewise linear systems, Bimodal systems, Stability of bimodal systems, Stabilization of bimodal systems

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ÖZET

ÇİFT DURUMLU PARÇALI SÜREKLİ DOĞRUSAL SİSTEMLERİN KARARLILIK ANALİZİ ve GERİBESLEME KARARLILIĞI

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Parçalı sürekli doğrusal sistemler, ayrık ve sürekli dinamikleri bünyesinde barındıran hibrit sistemlerin temel sınıflarından biridir. Çift durumlu sistemler ise, parçalı sürekli sistemlerin en basit alt sınıfıdır. Basit yapılarının yanında, kararlılık ve kararlı kılma gibi önemli problemlerin çözümü üzerindeki çalışmalarda kolaylık sağladığından, çift durumlu sistemler, hibrit kontrol teorisinin geliştirmesinde önemlidirler.

Çift durumlu sistemlerin kararlılık analizi ve kararlı kılma problemleri, günümüzde üzerine çalışılan önemli problemlerdendir. Bu bakımdan, doğrusal zamanla değişmeyen alt sistemlerin konveks kombinasyonlarının, hangi şartlar altında eşdeğer kuadratik Lyapunov fonksiyonunu paylaşacağı bu tez çalışmasında ele alınmıştır. Orijinal ve kolay test imkanı sağlayan kararlılık şartları sürekli vektör alanına sahip çift durumlu sistemler için elde edilmiştir. Bunun yanında, geribesleme ile kararlı kılma problemi detaylı ele alınıp, geometrik kontrol yaklaşımı yardımıyla, daha az tutucu sonuçlar bulunmuştur.

Ayrıca, çift durumlu sistemler için geliştirilmiş gözleyici tasarımı ele alınıp, bu tasarım için gereken şartların azaltıldığı bir tasarımı metodu önerilmiştir.

Son olarak kararlılık analizi için bulunan sonuçlar normu sınırlandırılabilen belirsizlikler içeren çift durumlu sistemler için genişletilip, ilgili sistemlerin dayanıklı kararlılığını garanti eden koşul elde edilmiştir.

Anahtar Kelimeler: Parçalı sürekli lineer sistemler, Çift durumlu sistemler, Çift durumlu sistemlerin kararlılığı, Çift durumlu sistemlerin kararlı kılınması

CHAPTER 1

INTRODUCTION

1.1 Literature Review

Piecewise linear (PWL) systems are some of the fundamental classes of hybrid systems which incorporate both discrete and continuous dynamics. PWL systems consist of some pairs of linear time invariant dynamics and a switching surface which divides the state space into subspaces according to a criterion depending on the system dynamics. In this thesis, we consider a particular class of switched PWL systems with state-dependent switchings, namely bimodal PWL systems with a continuous vector field and we focus on to investigate linear matrix inequality (LMI) conditions for the stability analysis and feedback stabilization for related systems. However, it is well known that the efficient solution of the stabilization problem can only be developed when some basic concepts of linear control theory such as well-posedness, controllability, observability and stability are fully understood.

In this context, Imura and Schaft studied the well-posedness of PWL systems in the sense of Carathéodory [1]. They derived necessary and sufficient conditions for the well-posedness of bimodal systems with single criterion in terms of an analysis based on lexicographic inequalities and the smooth continuation property of solutions. They also proposed an algorithm to solve the conditions and gave several necessary and sufficient conditions for bimodal systems to be well-posed. Furthermore, they discussed the well-posedness problem of feedback control systems with two state feedback gains switched according to a criterion depending on the state. Wu et al. also considered the class of bimodal state-based switched systems and presented necessary and sufficient conditions for the well-posedness of bimodal system by excluding the Zeno phenomenon which is called as infinitely mode transitions in a finite time interval [2].

On the other hand, controllability and observability concepts of PWL systems are also studied extensively in literature. We can refer to Sontag's book for detailed reviews and comments [3]. In summary, it should be noted that characterization of global controllability and observability of even simple PWL system classes such as bimodal or conewise is very complicated [3]. However, some remarkable studies are performed. For instance, Camlibel et al. investigated algebraic necessary and sufficient conditions for controllability of conewise linear systems that the state space representation is partitioned into conical regions which some linear dynamics are active on each of these regions separately [4]. They utilized the controllability results for push–pull systems and stated easily verifiable conditions for controllability of bimodal systems. They also adopted geometric control theory to characterize the controllability conditions, so as to lead to solve the problems like feedback stabilization, observability and controller synthesis for the related systems. Moreover, a significant study of Juloski et al. can be referred for observability concept [5]. Juloski et al. presented observer design procedures for a class of bimodal PWL systems in both continuous and discrete time. Their design approach needs input and measured output signals only, while the information on the active mode is not required.

As a last issue, stability and performance analysis on PWL systems are investigated. Even though, the literature contains rich content for those studies, some of them become prominent with constructive results. In this regard, J. L. Willems' study can be considered as a keystone for the solution of stability problem for related systems [6]. J. L. Willems obtained stability conditions of second order system by the technique of optimal quadratic common Lyapunov function and showed that the stability results obtained for a second order system by means of the circle criterion that is the same with quadratic stability technique. J. L. Willems also proved that the results are valid in general for nonlinear time-dependent feedback systems which can be considered as a threshold matter to extend the stability and stabilization results to the other classes of PWL systems. Hassibi and Boyd emphasized the importance of Lyapunov approach for the analysis and controller synthesis of PWL systems to get less conservative results

[7]. They derived sufficient conditions for stability and performance analysis by the help of Lyapunov functions in the LMI context, that can be also turned into convex optimization problem. So that, they proposed ellipsoidal outer approximation to the operating regions to reduce the conservatism for PWL systems. In the same fashion, Johansson and Rantzer developed a uniform and computationally tractable approach for stability analysis of nonlinear systems with PWL affine dynamics [8]. They also specified that this approach is promising to be generalised in a large number of directions such as performance analysis, global linearization, controller optimization and model reduction.

As been noted above, those studies are initiated to developed a useful tool, is common quadratic Lyapunov functions (CQLF), to deal with the stability and performance analysis of switching PWL systems. To date, CQLF are among the most popular tools for the related systems, both for state-independent [9,10] and state-dependent switchings [8]. One of the main reason behind their popularity is that, in most cases, such Lyapunov functions can be efficiently computed via LMIs. As such, providing sufficient conditions for stability in terms of feasibility of a set of LMIs is highly popular in the literature of linear switching systems [11,12]. However, these conditions are rather computational in nature and often do not relate to the underlying structure of the system under study, in particular for the case of state-dependent switchings.

In adition to the feasibility problem of specifying a CQLF, complexity of stability analysis for switched PWL systems is also originated from the characterization of switching surface. Samadi and Rodrigues considered the switched surface of PWL systems and presented a unified dissipative approach for stability analysis of PWL smooth systems with continuous and discontinuous vector fields [13]. They proposed a candidate Lyapunov function such that there is no need for information about attractive sliding modes on switching surfaces. In fact, their study is inspiring to deal with closed loop stabilization problem of bimodal system, but it is restrictive to syntheses the stabilization conditions for bimodal systems due to the fact that it contains a parameter being related to the decay rate of the Lyapunov function. Camlibel et al. also considered the problem of open loop stabilization for bimodal systems with state-dependent switching [14]. They adopted geometric control approach with LMI formulation and presented algebraic necessary and sufficient conditions to characterize the stabilization. Furthermore, a full connection between stabilizability and controllability is established for piecewise linear switched systems. The main advantage of their approach is to reduce burden of computation for specifying a CQLF. Also, this study can be considered as a first step to solve the closed loop stabilization problem for bimodal system with state-depended switching.

Finally, some stability results which are based on notable ideas should be noted for completeness of the survey. For example, Ibeas and de le Sen presented a stability test for multi-modal PWL system formed by a family of simultanously triangularizable system matrices with CQLF [15]. In a different perspective, Feng presented a stability analysis method for PWL discrete-time linear systems based on a PWL smooth Lyapunov function that can be obtained by solving a set of LMIs, that is numerically feasible with commercially available software [16]. Contrary to the previous discussion, Iwatani and Hara dealt with the stability problem for PWL systems from a different perspective. Instead of using Lyapunov's theory, they investigated behavior of PWL systems directly to get stability tests which are computationally tractable [17].

1.2 Objective of the Thesis

The objectives of this thesis are the stability analysis and feedback stabilization of bimodal systems. As the previous discussion reveals, there are a variety of stability analysis for bimodal systems available in the literature. Besides, many of the stability conditions obtained are non-constructive or impose significant restrictions on the controller design for the stabilization. Thus, our first objective is to present stability conditions that provide some insight into the stability properties of the bimodal systems and can contribute the feedback stabilization process of bimodal systems with continuity assumption. It turns out that continuity assumption of the underlying vector field leads to an alternative LMI based necessary and sufficient condition for bimodal systems. In turn, this alternative condition enables us to look at the feedback stabilization problem from a geometric point of view. Indeed, the main contribution of this thesis is to provide sufficient conditions for the existence of a stabilizing static state feedback for bimodal systems. Then, we are also in pursuit of less restrictive solution for the stabilization problem by the merits of geometric control theory.

However, available result on observability for bimodal systems leads to conservatism by imposing extra restrictions on the construction of Lyapunov function. Motivating from this problem, we aim to offer a less conservative and simpler procedure to design observers for bimodal system.

1.3 Hypothesis

To our best knowledge, bimodal systems are the simpliest possible form of PWL systems, therefore the results provided for bimodal systems can be used as a stepping stone to establish hybrid control theory. However, the stability analysis and stabilization of even simple classes of PWL systems is extremely complex. In this regard, this thesis is dedicated to investigate LMI conditions on the stability and state feedback stabilization for bimodal systems. Overall, the results presented here reveal a full connection between the stability and feedback stabilization for bimodal systems which were not available prior to our study. Moreover, although they are different in nature, those conditions can be applied to the stabilization of the open loop configuration.

CHAPTER 2

STABILITY ANALYSIS OF BIMODAL SYSTEMS

This chapter presents the stability conditions for bimodal PWL systems. To this end, we first formulate the quadratic stability of bimodal systems by following J.L. Willems's conceptual framework on quadratic stability analysis of nonlinear time-dependent feedback systems [6]. Then, we derive equivalent LMI conditions on the dynamics of the related systems such that the stability of the bimodal PWL systems is guaranteed. Based upon the implications of this chapter, we will deal with feedback stabilization of bimodal PWL systems and we will present theorems to construct CQLF.

2.1 The Willems' Conjecture

The result of J.L. Willems on the quadratic stability of nonlinear time-dependent feedback systems that may be useful as a starting point to derive orginal conditions for stability analysis of bimodal systems [6]. For this purpose, we begin with the Willems' conjecture.

Let us consider the general nonlinear time-dependent feedback system as follows

$$\dot{x}(t) = Ax(t) - Bk(x,t)Cx \tag{2.1}$$

where A, B and C are $(n \times n)$, $(n \times 1)$ and $(1 \times n)$ matrices respectively, and realize the transfer function $G(s) = C(sI - A)^{-1}B$ in a minimal way. The circle criterion proves asymptotic stability in the large of the solution of (2.1) for all $k(x,t)$ such that for an arbitrarily small positive ϵ

$$\alpha + \epsilon \leq k(x,t) \leq \beta - \epsilon \tag{2.2}$$

if the function

$$F(s)=\frac{1+\alpha G(s)}{1+\beta G(s)} \tag{2.3}$$

is positive real.

Theorem 2.1 [6]: There exists a positive-definite quadratic Lyapunov function $V(x)=x^T Px$ with $P=P^T>0$ such that the derivative along the solution of (2.1)

$$W(x,t)=x^T[(A^T P+PA)-k(x,t)(PBC+C^T B^T P)] \tag{2.4}$$

is negative definite for all $k(x,t)$ satisfying (2.2) if and only if $F(s)$, as defined in (2.3), is positive real. Proof of the Theorem 2.1 can be given by means of Lemma 2.2.

Lemma 2.2 [6]: There exists a positive-definite P satisfying the matrix inequality

$$A^T P+PA+(PB-gC^T)(B^T P-gC)-\alpha\beta C^T C\leq 0 \tag{2.5}$$

with $g=(\alpha+\beta)/2$ only if for all real ω,

$$1+(\alpha+\beta)ReG(j\omega)+\alpha\beta\left|G(j\omega)\right|^2\geq 0. \tag{2.6}$$

Proof of Teorem 2.1: Let $A\in\mathbb{R}^{n\times n}$ and $B,C\in\mathbb{R}^n$ where $B\neq 0$ and $C\neq 0$. Assume that there exist a symmetric positive definite matrix P such that for each $\rho\in(\alpha,\beta)$,

$$M(\rho)=(A-\rho BC^T)^T P+P(A-\rho BC^T)=A^T P+PA-\rho(PBC^T+CB^T P) \tag{2.7}$$

is negative definite. Hence, $(PBC^T+CB^T P)\neq 0$ and there exists $x_*\in\mathbb{R}^n$ such that $x_*^T(PBC^T+CB^T P)x_*\neq 0$. Let us define $H:\mathbb{R}\to\mathbb{R}$ as $H(\rho)=x_*^T M(\rho)x_*$. Thus, there exists $\rho_*>0$ such that $H(\rho)H(-\rho)<0$ for all $\rho\geq\rho_*$. This shows that $M(\rho)$ and $M(-\rho)$ can not be both negative definite for each $\rho\geq\rho_*$.

To be more precise, we can dominate the term of $A^T P+PA$ in (2.7) by increasing ρ and we can guarantee the negative definition of (2.7). To do so, let us to define $F=x_*^T(PBC^T+CB^T P)x_*>0$ and state $M(\rho)$ in quadratic form as follows.

$$\begin{aligned}x_*^T M(\rho)x_*&=x_*^T(A^T P+PA)x_*-\rho x_*^T(PBC^T+CB^T P)x_*\\&=x_*^T(A^T P+PA)x_*-\rho F\end{aligned} \tag{2.7}$$

In this case, we summarize all the possibilities of sign definitions in the following table.

Table 2.1 Sign definition of H

F	$x_*^T M(\rho) x_*$	$x_*^T M(-\rho) x_*$
$-$	>0	<0
$+$	<0	>0

We imply that, there exist $\forall\rho \geq \rho_*$ such that $M(\rho)M(-\rho) < 0$. Let us consider the closed interval of $\rho \in [\alpha, \beta]$ and we can make negative definite $H(\rho) = x_*^T M(\rho) x_*$ by choosing the value of ρ whereas approaching the α_1. If $x^T M(\alpha_1) x = 0$ for $\alpha_1 = \rho$ then, $x^T M x = 0 \iff Mx = 0$ must be satisfied with $M = M^T$. This property can be proven easily. Such as, "if only" part of the proof is obvious. For the proof of the "if" part we can define $M = N^T N$ and straightforward calculation yields $x^T M x = \|Nx\|^2$ and this means that $Nx = 0 \implies Mx = 0$.

Now, we have following implications

$$x^T M(\rho = \alpha_1) x < 0 \implies x \in \ker C^T \tag{2.8}$$

$$x^T M(\rho = \alpha_1) x = 0 \implies x \notin \ker C^T, x \in \ker M(\alpha_1) \tag{2.9}$$

and due to semi-definetness, it is obvious that $\dim(\ker C^T) = n - 1$.

Note that, negative definetness for different ρ values can be guaranteed with $PBC^T + CB^T P \neq 0$. Therefore, there exist $x_* \in \mathbb{R}^n$ such that $x_*^T (PBC^T + CB^T P) x_* \neq 0$ is satisfied. In other words, we have a relation $H : \mathbb{R} \to \mathbb{R}$ can be defined as follows

$$H(\rho) = x_*^T M(\rho) x_*. \tag{2.10}$$

Hence, there exists a $\rho_* > 0$ such that $H(\rho)H(-\rho) < 0$ is guaranteed for all $\rho \geq \rho_*$ and we deduce that both the functions of $M(\rho)$ and $M(-\rho)$ can not be made negative definite for all $\rho \geq \rho_*$.

Now, let us to define the following parameters on the value sets

$$\alpha_1 \equiv \inf\{\kappa \mid M(\rho) < 0 \quad \text{on} \quad [\kappa, \beta]\} \tag{2.11}$$

$$\beta_1 \equiv \sup\{\kappa \mid M(\rho) < 0 \quad \text{on} \quad [\alpha, \kappa]\} \tag{2.12}$$

where $\alpha_1 \in [-\infty, \infty]$, $\beta_1 \in [-\infty, \infty]$ and $\alpha_1 \le \alpha < \beta \le \beta_1$. Hence, at least one of α_1 and β_1 is finite. For example if $H(-\rho) > 0$ for some $\rho \ge \rho_*$, then α_1 is finite. Now suppose that α_1 is finite. In this case, $M(\rho)$ is negative definite for all $\rho \in (\alpha_1, \beta]$. By continuity of $M(\rho)$, $M(\alpha_1)$ must be singular and negative semidefinite.

At this stage, we need some auxiliary results which will be considered in the following claims to complete the proof.

Claim 2.3: There exist the vectors $z_1, z_2, \ldots, z_{n-1}$ such that $\{z_1, z_2, \ldots, z_{n-1}, C\}$ spans $\mathbb{R}^n$ and for all $x \in \mathbb{R}^n$, $x^T M(\alpha_1) x = -\sum_{i=1}^{n-1} (z_i^T x)^2$.

Proof of Claim 2.3: Let $v \neq 0$ be an arbitrary vector in the kernel of C^T. Then, $v^T M(\alpha_1) v = v^T M(\rho_1) v < 0$ for each $\rho \in (\alpha_1, \beta]$. Hence, a nonzero vector y in the kernel of $M(\alpha_1)$ must not be in the kernel of C^T. Thus, there exist $z_1, z_2, \ldots, z_{n-1}$ such that $\{z_1, z_2, \ldots, z_{n-1}, C\}$ spans $\mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ and $x^T M(\alpha_1) x = -\sum_{i=1}^{n-1} (z_i^T x)^2$.

Claim 2.4: Let $B^T P = rC^T + \sum_{i=1}^{n-1} (m_i z_i^T)^2$. Then $r > 0$.

Proof of Claim 2.4: Let $y \in \operatorname{span}\{z_1, z_2, \ldots, z_{n-1}\}^\perp \neq 0$. Hence $C^T y \neq 0$ as $y \neq 0$ and the vector set of $\{z_1, z_2, \ldots, z_{n-1}, C\}$ is a basis of $\mathbb{R}^n$. Thus, it is obvious that $y^T M(\alpha_1) y = 0$, using $x^T M(\alpha_1) x = -\sum_{i=1}^{n-1} (z_i^T x)^2$. Note that,

$$\begin{aligned} y^T M(\alpha_1) y &= y^T (A^T P + PA) y - y(\alpha_1 (Cr + \sum_{i=1}^{n-1} (z_i m_i^T x)^2 C^T + C(rC^T + \sum_{i=1}^{n-1} (m_i z_i^T x)^2)) y^T \\ &= y^T (A^T P + PA) y - 2\alpha_1 r (c^T y)^2. \end{aligned}$$

Since α_1 is the infumum of $\rho \in (\alpha_1, \beta]$ with $M(\rho)$ being negative definite, $y^T M(\rho) y = y^T (A^T P + PA) y - 2\rho r (C^T y)^2 < 0$ for any $\rho \in (\alpha_1, \beta]$. This implies that $r > 0$. More clearly, since α_1 is the infumum , α_1 means the first point that negative-semi definition transforms unidentifiability. Therefore, it is clear that $y^T (A^T P + PA) y = 2\alpha_1 r (C^T y)^2$. Now, let us rearrange $y^T M(\rho) y$ as follows

$$y^T M(\rho) y = y^T (A^T P + PA) y - 2\rho r (C^T y)^2$$

$$= 2\alpha_1 r (C^T y)^2 - 2\rho r (C^T y)^2 < 0 \tag{2.13}$$

$$= 2(\alpha_1 - \rho) r (C^T y)^2 < 0.$$

Consequently, it is obvious that $(\alpha_1 - \rho) < 0$ from the last inequality and the other terms are greater than the zero. So, $r > 0$.

Claim 2.5: Let us choose $r = (\beta - \alpha_1)/2$ as a positive constant. Then $(\sum_{i=1}^{n-1} m_i z_i^T y)^2 \leq \sum_{i=1}^{n-1} (z_i^T y)^2$ for all $y \in \mathbb{R}^n$.

Proof of Claim 2.5: Let y be an arbitrary nonzero vector differently from Claim 2.4. Note that,

$$y^T M(\beta) y = y^T (A^T P + PA) y - 2\beta [r(C^T y)^2 + (C^T y) \sum_{i=1}^{n-1} m_i (z_i^T y)] < 0$$

and

$$y^T M(\alpha_1) y = y^T (A^T P + PA) y - 2\alpha_1 [r(C^T y)^2 + (C^T y) \sum_{i=1}^{n-1} m_i (z_i^T y)] = -\sum_{i=1}^{n-1} (z_i^T y).$$

Therefore

$$2(\alpha_1 - \beta)[r(C^T y)^2 + (C^T y) \sum_{i=1}^{n-1} m_i (z_i^T y)] - \sum_{i=1}^{n-1} (z_i^T y)^2 < 0. \tag{2.14}$$

Now, recall the definition of r as $r = (\beta - \alpha_1)/2$. Then, the straightforward calculation yields

$$r(C^T y)^2 + (C^T y) \sum_{i=1}^{n-1} m_i (z_i^T y) + \frac{1}{4r} \sum_{i=1}^{n-1} (z_i^T y)^2 > 0. \tag{2.15}$$

If y is in the orthogonal complement of $\operatorname{span}\{z_1, z_2, \ldots, z_{n-1}\}$, then the inequality holds trivially. Now assume that the given $y \neq 0$ is not in the orthogonal complement of $\operatorname{span}\{z_1, z_2, \ldots, z_{n-1}\}$. Fix this y and v be an arbitrary nonzero vector that is orthogonal to $\operatorname{span}\{z_1, z_2, \ldots, z_{n-1}\}$. Hence $C^T v \neq 0$, $y + v \neq 0$ and $z_i^T (y + v) = z_i^T y$ for all $i = 1, 2, \ldots, n-1$. Hence, let us to write (2.16) as follows

$$r(C^T (y+v))^2 + (C^T (y+v)) \sum_{i=1}^{n-1} m_i (z_i^T y) + \frac{1}{4r} \sum_{i=1}^{n-1} (z_i^T y)^2 > 0. \tag{2.16}$$

Since $C^T v$ can be assigned arbitrary by scaling, we must have

$$r x^2 + \sum_{i=1}^{n-1} m_i (z_i^T y) x + \frac{1}{4r} \sum_{i=1}^{n-1} (z_i^T y)^2 > 0 \tag{2.17}$$

where $\forall x \in \mathbb{R}$. Due to positive definition of polynomials we get

$$\sum_{i=1}^{n-1} m_i(z_i^T y)]^2 - \sum_{i=1}^{n-1}(z_i^T y)^2 < 0. \tag{2.18}$$

The negative definiteness of $M(\beta)$ have been proved by Claim 2.5 for all x. Then, we can use Lemma 2.2 as follows

$$A^T P + PA + (PB - \frac{\alpha+\beta}{2}C^T)(B^T P - \frac{\alpha+\beta}{2}C) - \alpha\beta C^T C \leq 0 \tag{2.19}$$

such that proves (2.6) is true. Since the existence of the Lyapunov function implies (2.1) is asymptotically stable for $\alpha < k < \beta$, and at least weakly stable at $k = \alpha$ and $k = \beta$, it is concluded that $F(s)$ is positive real. Sufficiency part of the proof can be found in [18].

2.2 Bimodal Systems

In this subsection, we introduce a particular class of PWL systems with internally induced switchings, namely bimodal systems with a continuous vector field. This vector field concatenates two conical region, contain linear dynamics separately, on the state space. Bimodal PWL systems can be given by

$$\dot{x}(t) = \begin{cases} A_1 x(t) + B_1 u(t) & \text{if} \quad y(t) \leq 0 \\ A_2 x(t) + B_2 u(t) & \text{if} \quad y(t) \geq 0 \end{cases} \tag{2.20}$$

$$y(t) = C^T x(t) + Du(t) \tag{2.21}$$

where A_1, $A_2 \in \Re^{n\times n}$, B_1, B_2, $C \in \mathbb{R}^n$ and D is a scalar. $x \in \mathbb{R}^n$ is the state $u \in \mathbb{R}^m$ is the input, and all the matrices involved are of appropriate dimensions. Let assume that the dynamics is continuous along the hyperplane $\{(x,u)|C^T x + Du = 0\}$ such that

$$C^T x + Du = 0 \rightarrow A_1 x + B_1 u = A_2 x + B_2 u.$$

This fact is equivalent to

$$ker\,[C^T \quad D] \subseteq ker\,[A_1 - A_2 \quad B_1 - B_2].$$

Throughout the thesis, we assume that the right-hand side is a continuous function both in x and u, or equivalently, there exists a vector $e \in \mathbb{R}^n$ such that

$$A_1 - A_2 = eC^T \tag{2.22}$$

and $B_1 - B_2 = eD$. In this case, the right-hand side of (2.20) is a Lipschitz continuous function. Hence, for each initial state x_0 and locally-integrable input u, there exists a unique absolutely continuous function $x^{x_0,u}$ such that (2.20) holds for almost all $t \in \mathbb{R}$ and $x^{x_0,u}(0) = x_0$. If the right-hand side of (2.20) is Lipschitz continuous in the x variable, one can show that for each initial state $x_0 \in \mathbb{R}^n$ and locally integrable input $u \in \mathbb{L}^1$, there exists a unique absolutely continuous function $x^{x_0,u}$ satisfying (2.20) almost everywhere [14]. We also say that system (2.20) is completely controllable if for any pair of states (x_0, x_f) which is related start and final time respectively, there exists a locally integrable input u such that the solution $x^{x_0,u}$ of (2.20) passes through x_f, i.e. $x^{x_0,u}(\tau) = x_f$ for some $\tau \geq 0$.

Bimodal systems can be encountered in a variety of applications sometimes artificially as approximations of nonlinear systems and sometimes naturally due to the intrinsic piecewise affine behaviour. Next, we illustrate an example of mechanical system in the Figure 2.1 for the latter case.

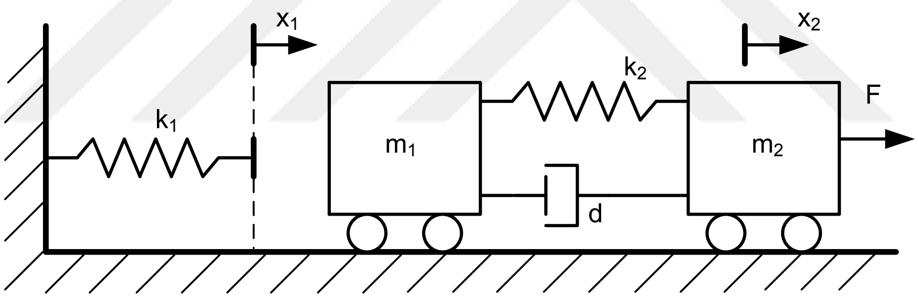

Figure 2.1 Linear mechanical system with a one-sided spring

We assume that the elements of the system behave linearly. Let x_1 and x_2 denote the displacements of the left and right carts from the tip of the leftmost spring, respectively. Let the mass of the left one cart denoted by m_1 and m_2 for the other one, the spring constants by k_1 for the leftmost one and k_2 for the other, and the damping constant by d. Then the governing differential equations can be given by

$$m_1\ddot{x}_1 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) - k' max(-x_1, 0) = 0,$$

$$m_2\ddot{x}_2 + k(x_1 - x_2) + d(\dot{x}_1 - \dot{x}_2) = F,$$

where F is the force that is applied to the right cart. By denoting the velocities of the left and right carts, respectively, by x_3 and x_4, one arrives at the following bimodal piecewise linear system:

$$\dot{x} = \begin{cases} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & \frac{-d}{m_1} & \frac{d}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2}{m_2} & \frac{-d}{m_2} & \frac{d}{m_2} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} F & \text{if } y \leq 0 \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-k_2}{m_1} & \frac{k_2}{m_1} & \frac{-d}{m_1} & \frac{d}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2}{m_2} & \frac{-d}{m_2} & \frac{d}{m_2} \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{F} & \text{if } y \geq 0 \end{cases} \quad (2.23)$$

$$y = x_1 \quad (2.24)$$

where $x = col(x_1, x_2, x_3, x_4)$. Note that, the continuity condition (2.22) is satisfied for $e = col(0, 0, \frac{-k_1}{m_1}, 0)$ as $D = 0$ and $B_1 = B_2$.

More realistic applications of bimodal systems arising from one-sided springs can be found in for instance [19-20]. These papers deal with observer design and disturbance attenuation problems, respectively, for a continuous bimodal system arising as a mathematical model of two steel beams, one supported at both ends by two leaf springs whereas the other (which is located parallel to the first one) clamped at both ends acting as a one-sided spring.

Other control systems applications in which bimodal systems arise intrinsically include for instance [21] where clutch engagement problem studied in [22].

In addition to the engineering applications, continuous bimodal systems are also encountered in various other contexts. Examples from the area of dynamical systems are included in [23-26]. In what follows, we illustrate a bimodal system arising in the study of certain partial differential equations.

The so-called Michelson system was originally studied in [26] in the context of the steady solutions of the Kuramoto–Sivashinsky equations and further studied in [27]. It can be given as a bimodal system in the form of (2.20) where

$$A_i = \begin{bmatrix} 0 & -1 & (-1)^i\lambda(1+\lambda^2) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \forall i \in \{1,2\},$$

$F = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $C^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$,

and $\lambda \in \mathbb{R}$ is aconstant. Note that the continuity assumption is satisfied with

$e^T = \begin{bmatrix} -2\lambda(1+\lambda)^2 & 0 & 0 \end{bmatrix}$.

2.3 Quadratic Stability of Bimodal Systems

In this subsection, we will formulate a theorem to test the stability of bimodal systems which is the particular case of (2.20). Consider the bimodal PWL unforced system given by

$$\dot{x}(t) = \begin{cases} A_1 x(t) & \text{if } C^T x(t) \le 0 \\ A_2 x(t) & \text{if } C^T x(t) \ge 0 \end{cases} \tag{2.25}$$

where A_1, $A_2 \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^n$. We say that the bimodal system is quadratically stable if there exists a quadratic function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) > 0$ for all $x \ne 0 \in \mathbb{R}^n$ and $dV(x(t)/dt) < 0$ for all state trajectories x of (2.25) with $x(t) \ne 0$. Equivalently, the system (2.25) is quadratically stable if and only if there exists a CQLF for the linear subsystems, that is there exists a symmetric positive definite matrix P such that

$$A_i^T P + P A_i < 0 \tag{2.26}$$

with $i \in 1, 2$.

The following theorem gives an alternative characterization for the existence of a CQLF $x \to \frac{1}{2} x^T P x$ satisfying (2.26) by exploting continuity condition which is $A_1 - A_2 = eC^T$.

Theorem 2.6: The following statements are equivalent.

1. The bimodal system (2.25) is quadratically stable.
2. There exists a symmetric positive definite matrix P such that $(A_1 - \mu e C^T)^T P + P(A_1 - \mu e C^T) < 0$ for all $\mu \in [0,1]$.

3. There exists a symmetric positive definite matrix K such that

$$\begin{bmatrix} A_1^T K + K A_1 & Ke - C \\ e^T K - C^T & -2 \end{bmatrix} < 0. \tag{2.27}$$

To prove this theorem, we need the following auxiliary result which can be derived from the proof of Theorem 1 [6].

Lemma 2.7 [6] : There exists a symmetric positive definite matrix P such that

$$(A_1 - \mu e C^T)^T P + P(A_1 - \mu e C^T) < 0 \tag{2.28}$$

for all $\mu \in (\alpha, \beta)$ only if there exists $\gamma > 0$ such that $Q = \gamma P$ satisfies

$$A_1^T Q + Q A_1 + (Qe - \frac{\alpha + \beta}{2} C)(Qe - \frac{\alpha + \beta}{2} C)^T - \alpha\beta C C^T \leq 0. \tag{2.29}$$

Proof of Teorem 2.6 :

$1 \implies 2$: If the bimodal system (2.25) quadratically stable, then there exists $P = P^T > 0$ such that

$$A_1^T P + P A_1 < 0$$

$$(A_1 - eC^T)^T P + P(A_1 - eC^T) < 0.$$

By taking convex combination for all $\mu \in [0, 1]$ we get Statement 2 of the Theorem 2.6.

$2 \implies 3$: Due to continuity there exists sufficiently small $\epsilon > 0$ such that

$$(A_1 - \mu e C^T)^T P + P(A_1 - \mu e C^T) < 0 \tag{2.30}$$

where $\mu \in [0, 1 + \epsilon]$.

Then, it follows from Lemma 2.2 in [6] that there exists $\gamma > 0$ such that $Q = \gamma P$ satisfies

$$A_1^T Q + Q A_1 + (Qe - \frac{1 + \epsilon}{2} C)(Qe - \frac{1 + \epsilon}{2} C)^T \leq 0. \tag{2.31}$$

By taking $K = \frac{2}{1+\epsilon} Q$, we obtain

$$A_1^T K + K A_1 + \frac{1 + \epsilon}{2}(Ke - C)(Ke - C)^T \leq 0. \tag{2.32}$$

Since $(Ke - C)(Ke - C)^T$ is positive semi-definite and $\epsilon > 0$, we further get

$$A_1^T K + KA_1 + \frac{1}{2}(Ke - C)(Ke - C)^T \leq 0. \tag{2.33}$$

Now, we claim that

$$A_1^T K + KA_1 + \frac{1}{2}(Ke - C)(Ke - C)^T < 0. \tag{2.34}$$

To see this, let $x \in \mathbb{C}^n$ such that

$$x^*(A_1^T K + KA_1 + \frac{1}{2}(Ke - C)(Ke - C)^T)x = 0. \tag{2.35}$$

Then, it follows from (2.32) that

$$\frac{\epsilon}{2}x^*(Ke - C)(Ke - C)^T x \leq 0.$$

Since $\epsilon > 0$, we can conclude that

$$(Ke - C)^T x = 0.$$

Now, it follows from (2.35) that

$$x^*(A_1^T K + KA_1)x = 0.$$

Since $K = \frac{2\gamma}{1+\epsilon}P$, we obtain

$$x^*(A_1^T P + PA_1)x = 0.$$

Therefore, we get $x = 0$ from (2.30). Hence, we have showed that (2.26) holds. The LMI (2.35) readily follows from (2.34) by using a Schur complement argument.

$3 \implies 1$: By taking the Schur complement with respect to "-2" of the left hand side of (2.27), we obtain

$$A_1^T K + KA_1 + \frac{1}{2}(Ke - C)(Ke - C)^T < 0.$$

Note that

$$\begin{aligned}
0 &> A_1^T K + KA_1 + \frac{1}{2}(Ke - C)(Ke - C)^T \\
&= (A_1 - eC^T)^T K + K(A_1 - eC^T) + \frac{1}{2}(Ke + C)(Ke + C)^T \\
&\geq (A_1 - eC^T)^T K + K(A_1 - eC^T).
\end{aligned}$$

Therefore, we get

$(A_1 - eC^T)^T K + K(A_1 - eC^T) < 0.$

Clearly, we also have

$A_1^T K + KA_1 < 0.$

Note that

$$
\begin{aligned}
&(A_1 - \mu eC^T)^T K + K(A_1 - \mu eC^T) \\
&\qquad = (1-\mu)(A_1^T K + KA_1) + \mu((A_1 - eC^T)^T K + K(A_1 - eC^T)) < 0
\end{aligned}
$$

for all $\mu \in [0,1]$.

Remark 2.8: Theorem 2.6 shows that one needs to solve the $(n+1)$ LMI in order to check the existence of a common Lyapunov function $A_i^T P + PA_i < 0$ given by two $(n \times n)$ LMIs. It also shows that existence of a common quadratic Lyapunov function is intimately related to a certain type of passivity of the linear system given by the quadruple $\Sigma(A_1; e; C; 1)$. More interestingly, Theorem 2.6 leads to a number of geometric suffcient conditions for feedback stabilization of bimodal systems as discussed in what follows.

CHAPTER 3

STABILIZATION OF BIMODAL SYSTEMS

This chapter deals with the stabilization problem and provides sufficient conditions for the existence of a static state feedback controller for bimodal PWL systems. After comparing the existing open loop stabilizability conditions and those presented for feedback stabilization, we provide a set of sufficient conditions for feedback stabilization in terms of the zero dynamics of one of the linear subsystems. The results of stabilization problem of related systems extensions of Lyapunov theory so as to we need to show the existence of CQLFs which guarantees stability.

3.1 Open Loop Stabilization of Bimodal Systems

We refer the following theorem that presents a full characterization of open loop stabilization problem for bimodal systems with continuous vector field [14].

Theorem 3.1 [14]: Suppose that the transfer function $D + C^T(sI - A_1)^{-1}B_1$ is not identically zero. The following statements are equivalent.

1. The Bimodal system (2.29) is stabiliazable.
2. The pair $(A_1, [B_1 \quad e])$ is stabiliazable and the implication

   $$\begin{bmatrix} v^T & \mu \end{bmatrix} \begin{bmatrix} A_i - \lambda I & B_i \\ C^T & D \end{bmatrix} = 0, 0 \leq \lambda \in \Re, v \neq 0, i = 1,2 \implies \mu_1 \mu_2 \geq 0$$

   holds.
3. The pair $(A_1, [B_1 \quad e])$ is controllable and the inequality system

$$\mu \geq 0,$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} A_i - \lambda I & B_1 \\ C^T & D \end{bmatrix} = 0,$$

$$\begin{bmatrix} z^T & \mu \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} \leq 0,$$

admits no solution $0 \neq col(z, \mu) \in \mathbb{R}^{n+1}$ and $0 \leq \lambda \in \mathbb{R}$.

Proof of Theorem 3.1: Proof of the theorem is in [14, Thm.2.3].

3.2 Quadratic Feedback Stabilization of Bimodal Systems

This subchapter aims to present novel conditions for feedback stabilization of bimodal PWL systems. It is known that the existence of a quadratic Lyapunov function is necessary and sufficient for asymptotic stability of LTI systems [28]. In consideration of this idea, we present feedback stabilization procedure for unstable bimodal systems then we investigate to find such a Lyapunov function that guarantees the stability of the forced bimodal systems. So that, we turn our attention to bimodal PWL systems with inputs of the form

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bu(t) & \text{if} \quad C^T x(t) \leq 0 \\ A_2 x(t) + Bu(t) & \text{if} \quad C^T x(t) \geq 0 \end{cases} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, and all the matrices involved are of appropriate dimensions. We assume that the right-hand side of (3.1) is continuos in both x and u, i.e. the continuity condition which is $A_1 - A_2 = eC^T$, holds. As such, for each initial state x_0 and locally-integrable input u there exists a unique absolutely continuous function $x^{x_0,u}$ such that (3.1) holds for almost all $t \in \Re$ and $x^{x_0,u}(0) = x_0$.

The problem, we address, is under what conditions there exists a state feedback of the form $u = k^T x$ which renders the closed loop bimodal system

$$\dot{x}(t) = \begin{cases} (A_1 + Bk^T) x(t) & \text{if} \quad C^T x(t) \leq 0 \\ (A_2 + Bk^T) x(t) & \text{if} \quad C^T x(t) \geq 0 \end{cases} \tag{3.2}$$

quadratically stable. In case, such a feedback exists, we say that the bimodal system (3.1) is feedback stabilizable.

In the following theorem, we state necessary and sufficient conditions for feedback stabilization of bimodal systems in terms of LMIs. Later, we will investigate geometric sufficient conditions based on this theorem. First of all, it should be noted that for a symmetric matrix M and a subspace $\mathcal{W}$of the underlying linear space, we write $M \overset{\mathcal{W}}{<} 0$ meaning that $w^T M w < 0$ for all non-zero $w \in \mathcal{W}$.

Theorem 3.2 The following statements are equivalent.

1. The Bimodal system (3.1) is feedback stabilizable.
2. There exists k and $P = P^T > 0$ such that

$$\begin{bmatrix} (A_1 + Bk^T)^T P + P(A_1 + Bk^T) & Pe - C \\ e^T P - C^T & -2 \end{bmatrix} < 0 \tag{3.3}$$

3. There exists $Q = Q^T > 0$ such that

$$\begin{bmatrix} A_1 Q + Q A_1^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} \overset{\mathcal{W}}{<} 0 \tag{3.4}$$

where $\mathcal{W} = \ker B^T \times \mathbb{R}$.

If the statement 3 holds, one can choose $k^T = -\alpha B^T Q^{-1}$ for some sufficiently large $\alpha > 0$.

Proof of Theorem 3.2 :

$1 \implies 2$: This readily follows from the application of Theorem 2.6 to bimodal system (3.2).

$2 \implies 3$: Since the LMI in (3.3) is feasible, so is

$$\begin{bmatrix} P^{-1} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} (A_1 - Bk^T)^T P + P(A_1 - Bk^T) & Pe - C \\ e^T P - C^T & -2 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & -1 \end{bmatrix} < 0. \tag{3.5}$$

where $Q = P^{-1}$. Straightforward calculations yield

$$\begin{bmatrix} A_1 Q + Q A_1^T & QC - e \\ C^T Q - e^T & -2 \end{bmatrix} + \begin{bmatrix} Bk^T Q + QkB^T & 0 \\ 0 & 0 \end{bmatrix} < 0. \tag{3.6}$$

Let $x \in \ker B^T$ and $u \in \mathbb{R}$. Then, it follows from (3.6) that

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A_1 Q + Q A_1^T & QC - e \\ C^T Q - e^T & -2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} < 0. \tag{3.7}$$

Therefore, (3.4) holds.

$3 \implies 2$: If $B=0$, then we have

$$\begin{bmatrix} A_1Q+QA_1^T & QC-e \\ C^TQ-e^T & -2 \end{bmatrix} < 0. \tag{3.8}$$

By pre- and post-multiplying this LMI by

$$\begin{bmatrix} Q^{-1} & 0 \\ 0 & -1 \end{bmatrix}$$

and defining $P=Q^{-1}$, we obtain

$$\begin{bmatrix} A_1^TP+PA_1 & Pe-C \\ e^TP-C^T & -2 \end{bmatrix} < 0 \tag{3.9}$$

which proves the claim.

If $B \neq 0$, take $k^T=-\alpha B^TQ^{-1}$. It follows from Finsler's lemma [29] that

$$\begin{bmatrix} A_1Q+QA_1^T & QC-e \\ C^TQ-e^T & -2 \end{bmatrix} + \begin{bmatrix} Bk^TQ+QkB^T & 0 \\ 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} A_1Q+QA_1^T-2\alpha BB^T & QC-e \\ C^TQ-e^T & -2 \end{bmatrix} < 0 \tag{3.10}$$

for all sufficiently large $\alpha>0$. Note that, it is enough to show that for any $\rho>0$ there exists a sufficiently large $\alpha>0$ such that

$$A_1Q+QA_1^T-2\alpha BB^T < -\rho I. \tag{3.11}$$

Note that $\mathbb{R}^n = imM \oplus imB$ where $im\ M = ker\ B^T$. Also note that

$$\begin{aligned}(Mv+\beta B)^T&\left(A_1Q+QA_1^T-2\alpha BB^T\right)(Mv+\beta B) \\ &= \begin{bmatrix} v \\ \beta \end{bmatrix}^T \begin{bmatrix} M^T(A_1Q+QA_1^T)M & M^T(A_1Q+QA_1^T)B \\ B^T(A_1Q+QA_1^T)M & B^T(A_1Q+QA_1^T)B-2\alpha\|B\|^4 \end{bmatrix} \begin{bmatrix} v \\ \beta \end{bmatrix}\end{aligned} \tag{3.12}$$

where β and v positive constants. Since $(A_1Q+QA_1^T) \overset{imM}{<} 0$, we know that $M^T(A_1Q+QA_1^T)M<0$. Then, it follows from a Schur complement argument that for any $\rho>0$ we can choose $\alpha>0$ sufficiently large enough so that

$$\begin{bmatrix} M^T(A_1Q+QA_1^T)M & M^T(A_1Q+QA_1^T)B \\ B^T(A_1Q+QA_1^T)M & B^T(A_1Q+QA_1^T)B-2\alpha\|B\|^4 \end{bmatrix} < -\rho I. \tag{3.13}$$

In view of (3.12), this implies that (3.11) holds. Therefore, we have (3.6). By pre- and post multiplying (3.10) by

$$\begin{bmatrix} Q^{-1} & 0 \\ 0 & -1 \end{bmatrix} \tag{3.14}$$

and defining

$P = Q^{-1}$,

we obtain (3.3). Note that, if we remove the restriction on the control input in Statement 3 of Theorem 3.2, then it is immediately equal to (2.27).

Remark 3.3: Although they are different in nature, conditions of Theorem 3.1 are necessary for feedback stabilization and hence should imply those of Theorem 3.2. To see this implication, note first that the (3.4) readily implies that (A_1, B) and hence $(A_1, [B\, e])$ is stabiliazable. To see the second condition of Theorem 3.1 also follows from (3.4), let $\lambda \geq 0$, $v \neq 0$ and μ_i with $i = 1, 2$ satisfy

$$\begin{bmatrix} v^T & \mu_1 \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b_i \\ c^T & d \end{bmatrix} = 0. \tag{3.15}$$

This yields that $v \in ker\ B^T$, $v^T A_i + \mu_i c^T = \lambda v^T$ and $\mu_2 = \mu_1 + v^T e$. As such, we have

$$\begin{bmatrix} v \\ \mu_1 \end{bmatrix}^T \begin{bmatrix} A_1 Q + Q A_1^T & QC - e \\ C^T Q - e^T & -2 \end{bmatrix} \begin{bmatrix} v \\ \mu_1 \end{bmatrix} = 2\lambda v^T Q v - 2\mu_1 \mu_2. \tag{3.16}$$

Since the right-hand side is negative, λ is nonnegative and Q is positive definite, one can conclude that $\mu_1 \mu_2 > 0$.

Remark 3.4: Theorem 3.2 provides necessary and sufficient conditions for feedback stabilizability in terms of certain LMIs. Next, we further investigate these LMIs with an eye towards the geometric structure of the linear subsystems of the bimodal system (3.1). To do so, we quickly introduce some notation.

A subspace $\mathcal{V} \in \mathbb{R}^n$ is called as controlled invariant if there exists $F \in \mathbb{R}^n$ such that $(A - BF^T)\mathcal{V} \in \mathcal{V}$. Let $\mathcal{V}^*(A, B, C^T)$ be the largest controlled invariant subspace that is contained in $ker\ C^T$. A subspace $\mathcal{T}$ is called as conditioned invariant, if there exists $G \in \mathbb{R}^n$ such that $(A - BG^T)\mathcal{T} \in \mathcal{T}$. Let $\mathcal{T}^*(A, B, C^T)$ be the smallest conditioned invariant subspace that contains $im\ B$.

It is well known (see e.g. [28, Prop.4]) that the transfer function $C^T(sI-A)^{-1}B$ is invertible as a rational matrix if and only if $\mathcal{V}\oplus\mathcal{T}=\mathbb{R}$ and $B\neq 0\neq C$.

The continuity condition in (2.30) has a number of useful consequences. Indeed, it can be easily verified (see e.g. [4, Prop.2.1]) that

$$\mathcal{V}^*(A_1,B,C^T)=\mathcal{V}^*(A_2,B,C^T) \tag{3.17}$$

$$\mathcal{T}^*(A_1,B,C^T)=\mathcal{T}^*(A_2,B,C^T). \tag{3.18}$$

Together with the invertibility conditions, these equalities imply that the transfer function $C^T(sI-A)^{-1}B$ is invertible if and only if so is not equal to zero. Finally, the proof of the Theorem 3.7 needs the following auxiliary result, so first of all we would like to introduce Lemma 3.5.

Lemma 3.5 For any integer $m\geq 1$, any vector $\eta\in\mathbb{R}^{m-1}$, and any positive real numbers α, β and γ there exists a symmetric matrix $\Lambda\in\mathbb{R}^{m\times m}$ such that

$$\Lambda>\alpha I \tag{3.19}$$

$$\tilde{\Lambda}+\tilde{\Lambda}^T<-\beta I \tag{3.20}$$

$$\begin{bmatrix}\tilde{\Lambda}+\tilde{\Lambda}^T & \tilde{\lambda}-\eta\\ \tilde{\lambda}^T-\eta^T & -\gamma\end{bmatrix}<0 \tag{3.21}$$

where $\tilde{\Lambda}=\Lambda_{\{2,3,\ldots,m\},\{1,2,\ldots,m-1\}}$ and $\tilde{\lambda}=\Lambda_{\{1,2,\ldots,m-1\},\{1\}}$.

Proof of Lemma 3.5: It trivially holds for $m=1$. Suppose that it holds for $m=\ell$. Take a vector $\zeta\in\mathbb{R}^{\ell}$. Let $\tilde{\zeta}=\zeta_{\{1,2,\ldots,\ell-1\}}$. Since this is a $(m-1)$-vector, there exists a symmetric positive definite matrix $\Lambda\in R^{m\times m}$ such that

$$\Lambda>\alpha I$$

$$\tilde{\Lambda}+\tilde{\Lambda}^T<-\beta I$$

$$\begin{bmatrix}\tilde{\Lambda}+\tilde{\Lambda}^T & \tilde{\lambda}-\tilde{\zeta}\\ \tilde{\lambda}^T-\tilde{\zeta}^T & -\gamma\end{bmatrix}<0. \tag{3.22}$$

Now, define $\Theta\in\mathbb{R}^{l+1\times l+1}$ as follows

$$\Theta=\left[\begin{array}{c|c}\Lambda & \begin{matrix}\tilde{\lambda}^T\\ \mu\end{matrix}\\ \hline \tilde{\lambda}\quad\mu & \rho\end{array}\right]$$

where $\tilde{\lambda} = -\Lambda_{\{2,3,\ldots,\ell\},\{\ell\}}$, μ and ρ are real numbers. Let $\tilde{\Theta} = \Theta_{\{2,3,\ldots,\ell+1\},\{1,2,\ldots,\ell\}}$ and $\tilde{\theta} = \Theta_{\{1,2,\ldots,\ell\},\{1\}}$.

It suffices to prove that μ and ρ can be chosen so that

$$\Theta > \alpha I$$

$$\tilde{\Theta} + \tilde{\Theta}^T < -\beta I$$

$$\begin{bmatrix} \tilde{\Theta} + \tilde{\Theta}^T & \tilde{\theta} - \zeta \\ \tilde{\theta}^T - \zeta^T & -\gamma \end{bmatrix} < 0.$$

Note that,

$$\tilde{\Theta} + \tilde{\Theta}^T = \begin{bmatrix} \tilde{\Lambda} + \tilde{\Lambda}^T & 0 \\ 0 & 2\mu \end{bmatrix}$$

and that

$$\begin{bmatrix} (\tilde{\Theta} + \tilde{\Theta}^T) & \tilde{\theta} - \zeta \\ \\ \tilde{\theta}^T - \zeta^T & -\gamma \end{bmatrix} = \begin{bmatrix} (\tilde{\Lambda} + \tilde{\Lambda}^T) & 0 & \tilde{\lambda} - \tilde{\zeta} \\ 0 & 2\mu & \Lambda_{1n} - \zeta_m \\ \tilde{\lambda}^T - \tilde{\zeta}^T & \Lambda_{1n} - \zeta_m & -\beta \end{bmatrix}. \tag{3.23}$$

It follows from (3.22) that (3.23) can be made negative definite by choosing μ sufficiently small. Once μ is fixed, one can choose ρ sufficiently large to satisfy $\Theta > \alpha I$ as $\Lambda > \alpha I$.

With this preparation, we are ready to provide geometric sufficient conditions for feedback stabilization

Lemma 3.6: Suppose that the transfer function $C^T(sI - A_1)^{-1}B$ is not identically zero and $\mathcal{V}^*(A_1, B, C^T) = \{0\}$. Then, the bimodal system (3.1) is feedback stabilizable.

Proof of Lemma 3.6: Assume that the transfer function of (2.25), $G(s) = p(s)/q(s) = C^T(sI - A_1)^{-1}B \neq 0$ and $\mathcal{V}^* = \{0\}$. Then, one has $C^T(sI - A)^{-1}B = p_0/(s^n + q_{n-1}s^{n-1} + \ldots + q_1 s + q_0)$ where p_0 and q_i with $i = 0, 1 \ldots n-1$ are some real numbers and we conclude that $p(s)$ is constant and $q(s) = s^n + q_{n-1}s^{n-1} + \ldots + q_1 s + q_0$. Then, one can take

Hence, after a state space transformation, we get

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -q_0 & -q_1 & -q_2 & \dots & -q_{n-1} \end{bmatrix}, \tag{3.24}$$

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & p_0 \end{bmatrix}^T; \; C = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{3.25}$$

Our goal is to show that (3.4) holds. To do so, note that, in the new coordinates, any vector $v \in \ker B^T$ is of the form $v^T = (v_1, v_2, \dots, v_{n-1}, 0)^T$. Let $\tilde{v} = (v_1, v_2, \dots, v_{n-1})$. Straightforward calculation yields that

$$\begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} AQ + QA^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \tilde{v} \\ w \end{bmatrix}^T \begin{bmatrix} \tilde{Q} + \tilde{Q}^T & \tilde{q} - \tilde{e} \\ \tilde{q}^T - \tilde{e}^T & -2 \end{bmatrix} \begin{bmatrix} \tilde{v} \\ w \end{bmatrix} \tag{3.26}$$

where $\tilde{Q} = Q_{\{2,3,\dots,n\},\{1,2,\dots,n-1\}}$, $\tilde{q} = Q_{\{1,2,\dots,n-1\},\{1\}}$, $\tilde{e} = e_{\{1,2,\dots,n-1\}}$. Then, the application of Lemma 3.5 with $m = n$, $\eta = \tilde{e}$, $\alpha = \beta = 1$ and $\gamma = 2$ yields a symmetric matrix Q such that,

$$Q = \alpha I, \begin{bmatrix} \tilde{Q} + \tilde{Q}^T & \tilde{q} - \tilde{e} \\ \tilde{q}^T - \tilde{e}^T & -2 \end{bmatrix} < 0.$$

Therefore we have

$$\begin{bmatrix} v \\ w \end{bmatrix}^T \begin{bmatrix} A_1 Q + QA_1^T & QC - e \\ C^T Q - e^T & -2 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \tilde{v} \\ w \end{bmatrix}$$

for all $v \in ker B^T$. Then, it follows from Theorem 3.1 such that (3.4) holds.

Note that, the hypothesis of the Theorem 3.6, i.e. invertibility of transfer function $C^T(sI - A)^{-1}B$ and $\mathcal{V}^* = 0$, imply that $C^T(sI - A)^{-1}B$ has no zeros. In the following theorem, we show that feedback stabilization can be achieved in case all zeros are on the left half plane, that is when the system $\Sigma(A_1, B, C^T)$ is minumum phase.

Theorem 3.7 : Suppose that the transfer function $C^T(sI - A)^{-1}B$ is not identically zero. Let $\mathcal{V}^* = \mathcal{V}^*(A_1, B, C^T)$ and F be such that $(A_1 - BF^T)\mathcal{V}^* \in \mathcal{V}^*$. Suppose that $(A - BF^T)\mid_{\mathcal{V}^*}$ is Hurwitz. Then, the bimodal system (3.1) is quadratically feedback stabilizable.

Proof of Theorem 3.7 : Before giving the proof of the theorem, we would like to introduce some notations and notions from geometric control approach. Consider the system $\Sigma(A_1, B, C, D)$. Let $\mathcal{V}^*$ and $\mathcal{T}^*$, respectively, which has been already denoted as the largest output-nulling controlled invariant and the smallest input-containing conditioned invariant subspaces of the system $\Sigma(A_1, B, C, D)$. Also let $F \in \mathcal{F}(\mathcal{V})$ which means the friend of vector F. Apply the feedback law, $u = -Fx + v$, where v is the new input. Then the system $\Sigma(A_1, B, C, D)$ becomes

$$\dot{x} = (A_1 - BF)x + Bv \tag{3.27}$$

$$y = (C - DF) + Dv \tag{3.28}$$

Obviously, controllability is invariant under this feedback. Moreover, the system $\Sigma(A_1, B, C, D)$ and $\Sigma(A_1 - BF, B, C - DF, D)$ share the same $\mathcal{V}^*$ and $\mathcal{T}^*$ due to Proposition 2.1 in Ref. [4]. Since the transfer matrix $C^T(sI - A_1)^{-1}B + D$ is invertible as rational matrix, Proposition 2.1 in Ref. [28] implies that the state space $\mathbb{R}^n$ admits the following decomposition $\mathbb{R}^n = \mathcal{V}^* \oplus \mathcal{T}^*$. Let the dimensions of the subspaces $\mathcal{V}^*$ and $\mathcal{T}^*$ be n_1 and n_2, respectively. Also let the vectors $(x_1, x_2, \ldots, x_n)$ be a basis for $\mathbb{R}^n$, such that the first n_1 vectors form a basis for $\mathcal{V}^*$ and the last n_2 for $\mathcal{T}^*$. Also let $G \in \mathcal{G}(\mathcal{T})$. One immediately gets

$$B - GD = \begin{bmatrix} 0 \\ \tilde{b}_2 \end{bmatrix} \tag{3.29}$$

$$C - DF = \begin{bmatrix} 0 & \tilde{c}_2 \end{bmatrix} \tag{3.30}$$

in the coordinates that are adapted to the earlier basis as $\mathcal{V}^* \subseteq ker\ (C - DF)$ and $(B - GD) \subseteq \mathcal{T}^*$. Here, $\tilde{b}_2$ and $\tilde{c}_2$ are are $n_2 \times m$ and $p \times n_2$ matrices, respectively.

Note that, $(A_1 - BF - GC + GDF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $(A_1 - BF - GC + GDF)\mathcal{T}^* \subseteq \mathcal{T}^*$ according to (see e.g. [4, Prop.2.1]). Therefore, the matrices $(A_1 - BF - GC + GDF)$, $(B - GD)$ and $(C - DF)$ should be of the form $\left[\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 & * \end{smallmatrix}\right]^T$ and $\left[\begin{smallmatrix} 0 & * \end{smallmatrix}\right]$ in the new coordinates where the row (column) blocks have n_1 and n_2 rows (columns), respectively. Let the matrices F and G be partitioned as

$$F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}, G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{3.31}$$

With this partitions and one gets

$$A_1 - \begin{bmatrix} g_1 D f_1 & g_1 D f_2 \\ b_2 f_1 & b_2 f_2 \end{bmatrix} - \begin{bmatrix} g_1 D f_1 & g_1 c_2 \\ g_2 D f_1 & g_2 c_2 \end{bmatrix} + \begin{bmatrix} g_1 D f_1 & g_1 D f_2 \\ g_2 D f_1 & g_2 D f_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ 0 & \tilde{A}_{22} \end{bmatrix} \Rightarrow$$

$$A_1 = \begin{bmatrix} \tilde{A}_{11} + g_1 D f_1 & g_1 c_2 \\ b_2 f_1 & \tilde{A}_{22} + b_2 f_2 + g_2 c_2 - g_2 D f_2 \end{bmatrix} \tag{3.32}$$

$$B = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix} = \begin{bmatrix} g_1 D \\ \tilde{b}_2 \end{bmatrix} \tag{3.33}$$

$$C = \begin{bmatrix} \tilde{c}_1 & \tilde{c}_2 \end{bmatrix} = \begin{bmatrix} D f_1 & \tilde{c}_2 \end{bmatrix} \tag{3.34}$$

Now, we can start the proof. Our first aim is to put the system (3.1) into a certain canonical form as considered above. By applying the feedback law $u = -F^T x + v$, where v is the new input, we get

$$\dot{x}(t) = \begin{cases} (A_1 - BF^T)x(t) & \text{if} \quad C^T x(t) \leq 0 \\ (A_2 - BF^T)x(t) & \text{if} \quad C^T x(t) \geq 0. \end{cases} \tag{3.35}$$

Clearly, this bimodal system is feedback stabilizable if and only if so is (3.1). Since the transfer function $C^T(sI - A_1)^{-1}B$ is invertible, $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$. Let the dimensions of the subspaces $\mathcal{V}^*$ and $\mathcal{T}^*$ be n_1 and n_2, respectively. Also let the vectors $\{x_1, x_2, \ldots, x_n\}$ be a basis for $\mathbb{R}^n$, such that the first n_1 vectors form a basis for $\mathcal{V}^*$ and the last n_2 for $\mathcal{T}^*$. Let G be such that $(A_1 - GC^T)\mathcal{T}^* \subseteq \mathcal{T}^*$.

In the new coordinates as $\mathcal{V}^* \subseteq \; ker\; C^T$ and $im\; B \subseteq \mathcal{T}^*$. By using the above note, we can assign the subspaces in the new coordinates such as $(A_1 - BF^T - GC^T)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $(A_1 - BF^T - GC^T)\mathcal{T}^* \subseteq \mathcal{T}^*$. Therefore, the matrix $(A_1 - BF^T - GC^T)$ can be made of the diagonal form in the new coordinates where the row (column) blocks have n_1 and n_2, rows (columns), respectively. With the above partitions, one gets

$$A_1 - BF^T = \begin{bmatrix} A_{11} & g_1 c_2 \\ 0 & A_{22} \end{bmatrix} \tag{3.36}$$

In view of Theorem 3.2, existence of a positive definite matrix Q such that

$$\begin{bmatrix} (A_1 - BF^T)Q + Q(A_1 - BF^T)^T & Qc - e \\ c^T Q - e^T & -2 \end{bmatrix} \overset{\mathcal{W}}{<} 0 \tag{3.37}$$

where $\mathcal{W} = kerB^T \times \mathbb{R}$ is enough to prove statement.

Let a symmetric matrix Q be of the form

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}. \tag{3.38}$$

Straightforward calculations yield

$$\begin{bmatrix} A_{11}Q_{11}+Q_{11}A_{11}^T+g_1c_2^TQ_{12}^T+Q_{12}c_2g_1^T & A_{11}Q_{12}+Q_{12}A_{22}^T+g_1c_2^TQ_{22} & Q_{12}c_2-e_1 \\ A_{22}Q_{12}^T+Q_{12}^TA_{11}^T+Q_{22}c_2g_1^T & A_{22}Q_{22}+Q_{22}A_{22}^T & Q_{22}c_2-e_2 \\ c_2^TQ_{12} & -2 & \end{bmatrix} \quad (3.39)$$

By left-multiplying by

$$\left[\begin{array}{cc|c} I & 0 & -g_1 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right]$$

and right-multiplying by its transpose, we get

$$\left[\begin{array}{cc|c} A_{11}Q_{11}+Q_{11}A_{11}^T+g_1e_1^T+e_1g_1^T-2g_1g_1^T & A_{11}Q_{12}+Q_{12}A_{22}^T+g_1e_2^T & Q_{12}c_2-e_1+2g_1 \\ A_{22}Q_{12}^T+Q_{12}^TA_{11}^T+e_2g_1^T & A_{22}Q_{22}+Q_{22}A_{22}^T & Q_{22}c_2-e_2 \\ \hline c_2^TQ_{12}^T-e_1^T+2g_1^T \quad c_2^TQ_{22}-e_2 & & -2 \end{array}\right]. \quad (3.40)$$

By the hypothesis, A_{11} is Hurwitz. Let a symmetric matrix Q (partitioned accordingly) be of the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}. \quad (3.41)$$

After straightforward calculations we get (3.37) as the following form

$$\begin{bmatrix} A_{11}Q_1+Q_1A_{11}^T & g_1c_2^TQ_2 & -e_1 \\ Q_2c_2g_1^T & A_{22}Q_2+Q_2A_{22}^T & Q_2c_2-e_2 \\ -e_1^T & c_2^TQ_2-e_2 & -2 \end{bmatrix}. \quad (3.42)$$

In the new coordinates, $v \in \ker B^T$ if and only if $v=(v_1,v_2)$ and $v_2 \in \ker b_2^T$. Note that, $\mathcal{V}^*(A_{22},b_2,c_2^T)=\{0\}$. Then, Lemma 3.6 implies that there exists a symmetric positive definite matrix Q_2 and positive constant β such that,

$$Q_2 > \beta I$$

$$\begin{bmatrix} (A_{22})Q_2+Q(A_{22}^T)^T & Q_2c_2-e_2 \\ c_2^TQ_2-e_2^T & -2 \end{bmatrix} \overset{\mathcal{W}}{<} 0. \quad (3.43)$$

Note that, $(A_1-BF^T)\,|_{\mathcal{V}^*}$ can be identified with Hurwitz-A_{11} due to hypothesis. Therefore for any $R=R^T<0$ one can find $Q_1=Q_1^T>0$ such that $A_{11}Q_1+Q_1A_{11}^T=R$. Then, it follows from a Schur complement argument that we can choose Q_1 such that (3.37) holds.

Remark 3.8: Lemma 3.7 and Theorem 3.8 also imply strictly positive realness (SPR). In particular, there exists k such that the system $(A_1 - Bk^T, e, C^T, 1)$ is SPR.

CHAPTER 4

OBSERVER DESIGN FOR BIMODAL SYSTEMS

In this section, we consider observer design procedure for the class of bimodal systems. For this purpose, we consider a significant study on observer design procedure for bimodal systems devised by Juloski et al. [5]. The results in [5] lead to conservatism by imposing extra restrictions on the construction of Lyapunov function. Motivating from this problem, we streamline this observer design procedure for related systems. We reduce the number of LMIs to be solved from the orginal two to one and we also show that one of the parameters required for the design can be eliminated. As a first step, we consider Luenberger observer design procedure presented by Juloski et al., in the next subsection.

4.1 Observer Design Problem for Bimodal Systems

Let us to reconsider the bimodal piecewise linear system given by

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bu(t) & \text{if } H^T x(t) \leq 0 \\ A_2 x(t) + Bu(t) & \text{if } H^T x(t) \geq 0 \end{cases} \tag{4.1}$$

$$y(t) = Cx(t) \tag{4.2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^n$ and are the state, output and the input of the system. The input $u : \Re^+ \rightarrow \Re^m$ is assumed to be an integrable function. The matrices A_1, $A_2 \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $H \in \mathbb{R}^n$. The hyperplane defined by $ker\ H^T$ separates the n-dimensional real state space into the two half-spaces. Considering class of bimodal PWL systems has identical input distribution matrix B and output matrix C for both modes. The vector field of the system is continuous over the switching plane, i.e. $A_1(x) = A_2(x)$ when $H^T(x) = 0$. It is straightforward to show that $A_2 = A_1 + GH^T$ for some vector G of appropriate dimensions. In this case equation (4.1) can be written as follows.

$$\dot{x}(t) = A_1 x + G\max(0, H^T x) + Bu \tag{4.3}$$

Juloski et al. considered the system (4.1) in [5] and proposed a continuous time bimodal observer of the following structure

$$\dot{\hat{x}}(t) = \begin{cases} A_1\hat{x} + Bu + L_1(y-\hat{y}) & \text{if} \quad H^T\hat{x} + K^T(y-\hat{y}) \leq 0 \\ A_2\hat{x} + Bu + L_2(y-\hat{y}) & \text{if} \quad H^T\hat{x} + K^T(y-\hat{y}) \geq 0 \end{cases} \tag{4.4}$$

$$\hat{y}(t) = C\hat{x}(t) \tag{4.5}$$

where $\hat{x} \in \mathbb{R}^n$ is estimated state at time t and {L_1, L_2} $\in \mathbb{R}^{n\times p}$ and $K \in \mathbb{R}^p$ are design parameters. The dynamics of the state estimation error, $e := x - \hat{x}$ can be described as follows

$$\dot{e} = \begin{cases} (A_1 - L_1C)e, & H^Tx < 0, H^T\hat{x} + K^T(y-\hat{y}) < 0 \\ (A_2 - L_2C)e + \Delta Ax, & H^Tx < 0, H^T\hat{x} + K^T(y-\hat{y}) > 0 \\ (A_1 - L_1C)e - \Delta Ax, & H^Tx > 0, H^T\hat{x} + K^T(y-\hat{y}) < 0 \\ (A_2 - L_2C)e, & H^Tx > 0, H^T\hat{x} + K^T(y-\hat{y}) > 0 \end{cases} \tag{4.6}$$

where x satisfies (4.1) and $\hat{x}$ satisfies (4.4). By substituting $\hat{x} = x - e$ in (4.5), we see that the right hand side of the state estimation error dynamics is PWL in the variable (e, x). Note that, such dynamics are not autonomous, but depend on the state of the observer. Juloski et.al. presented the following theorem to design observer for bimodal system [5].

Theorem 4.1 [5]: The state estimation error dynamics (4.6) is globally asymptotically stable for all $x : \mathbb{R}^+ \to \mathbb{R}^n$ that satisfy system (4.1) in the sense of Lyapunov approach, if there exist matrices $P > 0$, L_1, L_2, k and constants $\lambda \geqslant 0$, $\mu > 0$ such that the following set of matrix inequalities are satisfied:

$$\begin{bmatrix} (A_2 - L_2C)^TP + P(A_2 - L_2C) + \mu I_n & -PGH^T + \dfrac{\lambda}{2}(H - C^TK)H^T \\ -HG^TP + \dfrac{\lambda}{2}H(H - C^TK)^T & -\lambda HH^T \end{bmatrix} \leq 0 \tag{4.7}$$

$$\begin{bmatrix} (A_1-L_1C)^TP+P(A_1-L_1C)+\mu I_n & PGH^T+\dfrac{\lambda}{2}(H-C^TK)H^T \\ HG^TP+\dfrac{\lambda}{2}H(H-C^TK)^T & -\lambda HH^T \end{bmatrix} \leq 0 \qquad (4.8)$$

Proof of the Theorem 4.1 can be found in [5].

Note that, the parameters λ and μ appearing in the statement and the matrix inequalities are related to the S-procedure, and are not directly needed in the design of the observer.

4.2 Simplified Observer Design for Bimodal Systems

We prove that the LMIs in Theorem 4.1 imply to each other and we reduce the number of LMIs to be solved from the orginal two to one. We also show that the parameter K can be chosen zero. By this way, we offer a simpler structure for an observer (4.4-4.5). Following theorem states that only one of the inequalities in Theorem 4.1 is needed to design observer for bimodal system.

Theorem 4.2 : Assume there exist $P=P^T>0$, L_2, K and λ, $\mu\in\mathbb{R}$ with $\lambda\geq 0$, $\mu>0$ such that LMI in (4.7) is satisfied. Then LMI in (4.8) is satisfied with $L_1=L_2-GK$.

Proof: Our goal is to present one of the two inequalities (4.7) and (4.8) needs to be solvable to design observer for bimodal systems. Now, we should consider the cases $\lambda=0$ and $\lambda>0$. In the first case, (2,2)-block term of the matrix on the left-hand side of the first LMI in (4.7) is zero. Also, the (1,2) block term must be zero to satisfy the LMI. Hence, $-PGH^T=P(A_1-A_2)=0$. Since $P>0$ this imples that $A_1=A_2$ and choosing $L_1=L_2$ and $K=0$ guarantees that the LMI in (4.8) is satisfied.

Then, we deal the case $\lambda>0$ and let us rewrite the left hand sides of (4.7) and (4.8) as $T*T_1*T'$ and $T*T_2*T'$ respectively, with the matrices T, T_1 and T_2 where

$$T=\begin{bmatrix} I_n & 0 \\ 0 & H \end{bmatrix},$$

$$T_1=\begin{bmatrix} (A_2-L_2C)^TP+P(A_2-L_2C)+\mu I_n & -PG+\dfrac{\lambda}{2}(H-C^TK) \\ -G^TP+\dfrac{\lambda}{2}H(H-C^TK)^T & -\lambda \end{bmatrix},$$

$$T_2 = \begin{bmatrix} (A_1 - L_1C)^T P + P(A_1 - L_1C) + \mu I_n & PG + \dfrac{\lambda}{2}(H - C^T K) \\ G^T P + \dfrac{\lambda}{2} H(H - C^T K)^T & -\lambda \end{bmatrix}.$$

We claim that the LMIs (4.7) and (4.8) are satisfied if and only if $T_1 \leq 0$ and $T_2 \leq 0$, respectively. To prove this claim, we need the following Lemma.

Lemma 4.3: If $N^T M N \leq 0$ and N is full row rank, then $M \leq 0$.

Since $H \neq 0$ and $T \in \mathbb{R}^{2n*(n+1)}$ has full column rank, also T' full row rank. Then, it follows from Lemma 4.3 that our claim holds. By taking the Schur complement with respect to $-\lambda$ in T_1 and T_2, we obtain the following bilinear matrix inequalities(BMIs), respectively.

$$(A_2 - L_2C)^T P + P(A_2 - L_2C) + \mu I_n \\ + \frac{1}{\lambda}(-PG + \frac{\lambda}{2}(H - C^T K))(-G^T P + \frac{\lambda}{2}(H - C^T K))^T \leq 0 \quad (4.9)$$

$$(A_1 - L_1C)^T P + P(A_1 - L_1C) + \mu I_n \\ + \frac{1}{\lambda}(PG + \frac{\lambda}{2}(H - C^T K))(G^T P + \frac{\lambda}{2}(H - C^T K))^T \leq 0 \quad (4.10)$$

Note that, $A_2 = A_1 + GH^T$. Let us substitute A_2 in (4.9) and proceed the shtraightforward computations. Then, the left hand-sides of (4.9) and (4.10) are equal to each other if and only if,

$$-C^T L_2^T P - PL_2C + \frac{1}{2}PGK^T C + \frac{1}{2}C^T K G^T P = \\ -C^T L_1^T P - PL_1C - \frac{1}{2}PGK^T C - \frac{1}{2}C^T K G^T P. \quad (4.11)$$

Consequently, equality condition (4.11) occurs if and only if $L_1 = L_2 - GK$.

Due to the symmetry in the inequalities (4.7) and (4.8) make possible the alternative expression of Theorem 4.2 that is stated with following corollary.

Corollary 4.4: Assume there exist $P = P^T > 0$, L_1, K and λ, $\mu \in \mathbb{R}$ with $\lambda \geq 0$, $\mu > 0$ such that LMI in (4.8) is satisfied. Then LMI in (4.8) is satisfied with $L_2 = L_1 + GK$.

Finally, we are ready to give the main theorem of this section which makes it possible to design observer for bimodal systems as follows.

Theorem 4.5: The following statements are equivalent:

1. There exist P, λ, L_2 and K such that

$$\begin{bmatrix} (A_2 - L_2C)^T P + P(A_2 - L_2C) & PG + \frac{\lambda}{2}(H - C^T K) \\ G^T P + \frac{\lambda}{2}H(H - C^T K)^T & -\lambda \end{bmatrix} \leq 0. \tag{4.12}$$

2. There exist $P = P^T > 0$ such that

$$\begin{bmatrix} A_2^T P + PA_2 & PG + \frac{\lambda}{2}H \\ G^T P + \frac{\lambda}{2}H^T & -\lambda \end{bmatrix} \overset{\mathcal{W}}{\leq} 0 \tag{4.13}$$

where $\mathcal{W} = \ker \mathrm{C} \times \mathbb{R}$. Moreover, assume that Statement 2 holds for some P and λ. Then, there exist $\alpha > 0$ such that Statement 1 holds for P, λ, $L_2 = -\alpha P^{-1} C^T$ and $K = 0$. Hence, any of the two statements above is equivalent with

3. The linear, time invariant system $(A_2, G, -\frac{\lambda}{2}H, \frac{\lambda}{2})$ can be passified by the state feedback law with $L_2 = L_1 + GK$.

Proof of Theorem 4.5:

1$\Longrightarrow$2 : Let us rewrite first statement as follows

$$\begin{bmatrix} A_2^T P + PA_2 & PG + \frac{\lambda}{2} \\ G^T P + \frac{\lambda}{2}H^T & -\lambda \end{bmatrix} + \begin{bmatrix} -C^T L_2^T P - PL_2C & -\frac{\lambda}{2}C^T K \\ -\frac{\lambda}{2}K^T C & 0 \end{bmatrix} \leq 0$$

Let $x \in \ker C$ and $u \in \mathbb{R}$ then the second term in the left-hand side become zero on $\ker C \times \mathbb{R}$. Also one can choose $\lambda = 2$ then we get the statement 2 of the Theorem 4.5.

Note that, λ is a scaling factor for the P, so that statement 2 of the Theorem 4.5 is also feasible with λ.

2$\Longrightarrow$1 : Assume that the statement 2 of the Theorem 4.5 holds. Then, statement 2 of the Theorem 4.5 implies that

$$\begin{bmatrix} (A_2 + L_2C)^T P + P(A_2 + L_2C) & PG + \frac{\lambda}{2}H \\ G^T P + \frac{\lambda}{2}H^T & -\lambda \end{bmatrix} \leq 0.$$

Straighforward calculation yields

$$\begin{bmatrix} A_2P + PA_2 + C^TL_2^TP + PL_2C^T & PG + \dfrac{\lambda}{2}H \\ G^TP + \dfrac{\lambda}{2}H^T & -\lambda \end{bmatrix} \leq 0. \tag{4.14}$$

(4.14) can be written as follows by the help of Finsler's lemma [29]

$$\begin{bmatrix} A_2^TP + PA_2 & PG + \dfrac{\lambda}{2}H \\ G^TP + \dfrac{\lambda}{2}H^T & -\lambda \end{bmatrix} + \begin{bmatrix} -2\alpha C^TC & 0 \\ 0 & 0 \end{bmatrix} \leq 0.$$

Define $L_2 = \alpha P^{-1}C^T$ with the sufficiently large positive parameter α and we conclude that

$$\begin{bmatrix} (A_2 - L_2C)^TP + P(A_2 - L_2C) & PG + \dfrac{\lambda}{2}(H - C^TK) \\ G^TP + \dfrac{\lambda}{2}H(H - C^TK)^T & -\lambda \end{bmatrix} \leq 0. \tag{4.15}$$

Note that (4.15) is the version of Statement 1 of Theorem 4.5 with $K = 0$ as stated in (4.13) for the same theorem.

3$\Longrightarrow$1 : Statement 1 of Theorem 4.5 is equivalent to the Positive-Real Lemma for the system $(A_2 - L_2C, G, -\dfrac{\lambda}{2}H, -\lambda)$.

Remark 4.6: While choosing $K \neq 0$ in observer design structure (4.4)-(4.5) yields better transient error dynamics. On the other hand, under the assumption of the following Corollary the dynamics of the observer with $L_1 = L_2$ is continuous.

Corollary 4.7: Assume that (4.7) holds with P, λ, μ, L_2 and K. Then there exists positive $\alpha \in \mathbb{R}$ such that the (4.7) also holds with P, λ, μ, $L_2 = \alpha P^{-1}C^T$ and $K = 0$.

Proof of Corollary 4.7: Considering Theorem 4.2, Theorem 4.5, $K = 0$ and $L_1 = L_2 = L$, then the observer state matrices of $(A_1 - LC)$ and $(A_2 - LC)$ differ by the term GH^T. Hence, the dynamics of the observer for bimodal system is continuous.

CHAPTER 5

ROBUST STABILITY TEST FOR BIMODAL SYSTEMS

In a similar way of nominal stability problem of bimodal systems, robust stability analysis problem is dealt with in this section. As in the nominal stability analysis, we also need to show the existence of CQLF which guarantees robust stability. We know that the existence of a quadratic Lyapunov function is necessary and sufficient for asymptotic stability of LTI systems with parametric and system uncertainties [30]. Those uncertainties can be originated from environment such as external errors and disturbances. Also, some of them may occur in operating conditions such as nonlinearities that have not been accounted for in the modeling process. So that, it is crucial to take into account them to construct a CQLF. To that end, we consider the robust stability analysis problem for bimodal systems with norm-bounded uncertainty such that under which conditions, a number of convex combination of LTI systems share a CQLF. We present a corollary to construct CQLF to check the robust stability of bimodal systems with norm-bounded uncertainties. Then, to illustrate the effectiveness of the corollary, we present a mechanical system in bimodal configuration and simulate the switching mechanism of the system with nominal and norm-bounded uncertain parameters

5.1 Robust Stability Test for Bimodal Systems

Let us reconsider bimodal piecewise linear system (2.25) with uncertainty as follows

$$\dot{x}(t) = \begin{cases} (A_1 + \Delta A_1(t))x(t) & \text{if } C^T x(t) \leq 0 \\ (A_2 + \Delta A_2(t))x(t) & \text{if } C^T x(t) \geq 0. \end{cases} \quad (5.1)$$

Note that, we have the continuity assumption such that

$$A_1 - A_2 + \Delta A_1 - \Delta A_2 = eC^T \quad (5.2)$$

$$\Delta A_1(t) = E_1^T \Delta_1(t) D_1 \quad (5.3)$$

$$\Delta A_2(t) = E_2^T \Delta_2(t) D_2 \quad (5.4)$$

where E_1, E_2, D_1 and D_2 are known uncertainty matrices of appropriate dimensions. Note that, Δ_1 and Δ_2 are unknown Lebesque measurable functions of time which satisfy

$$\Delta_1^T \Delta_1 \leq I, \quad (5.5)$$

$$\Delta_2^T \Delta_2 \leq I. \quad (5.6)$$

The next corollary presents a robust stability condition to check the stability of (5.1) in the view of (5.5) and (5.6).

Corollary 5.1: System (5.1) with (5.3) and (5.4) is robustly stable if there exist a positive symmetric matrix M, a positive constant ϵ such that the following condition holds.

$$\begin{bmatrix} A_1^T M + M A_1 + \epsilon E_1^T E_1 & Me - C & D_1^T \\ * & -2 & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0 \quad (5.7)$$

Proof of Corollary 5.1 : Consider the inequality (2.27) given in Section 2 . Exchanging A_1 with $A_1 + \Delta A_1$ and K with $M = M^T > 0$ the condition (2.27) turns into

$$\begin{bmatrix} (A_1 + \Delta A_1)^T M + M(A_1 + \Delta A_1) & Me - C \\ (Me - C)^T & -2 \end{bmatrix} < 0 \quad (5.8)$$

and substituting (5.3) and (5.4) into (5.8) yields,

$$\begin{bmatrix} A_1^T K + D_1^T \Delta_1^T(t) E_1 M + M A_1 + M E_1^T \Delta_1(t) D_1 & Me - C \\ (Me - C)^T & -2 \end{bmatrix} < 0. \quad (5.9)$$

Note that (5.9) can be decomposed as,

$$\begin{bmatrix} A_1^T M + M A_1 & Me - C \\ (Me-C)^T & -2 \end{bmatrix} + \begin{bmatrix} D_1^T \\ 0 \end{bmatrix} \Delta_1^T(t) \begin{bmatrix} E_1 & 0 \end{bmatrix} + \begin{bmatrix} E_1^T \\ 0 \end{bmatrix} \Delta_1^T(t) \begin{bmatrix} D_1 & 0 \end{bmatrix} < 0. \qquad (5.10)$$

Defining

$$\tilde{A} = \begin{bmatrix} A_1^T M + M A_1 & Me - C \\ (Me-C)^T & -2 \end{bmatrix},$$

$$\tilde{E} = \begin{bmatrix} E_1^T \\ 0 \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} D_1 & 0 \end{bmatrix}$$

and some straightforward calculations yields,

$$\tilde{A} + \tilde{D}\Delta_1(t)\tilde{E} + \tilde{E}^T \Delta_1(t)\tilde{D}^T < 0. \qquad (5.11)$$

However, one can find a positive constant ϵ such that,

$$E_1^T \Delta_1(t)^T D_1 + D_1^T \Delta_1(t) E_1 \leq \frac{1}{\epsilon} E_1^T E_1 + \epsilon D_1^T D_1 \qquad (5.12)$$

holds [31]. Therefore in view of (5.12), if

$$\tilde{A} + \frac{1}{\epsilon}\tilde{D}\tilde{D}^T + \epsilon \tilde{E}^T \tilde{E} < 0 \qquad (5.13)$$

holds then (5.11) holds too. Then applying Schur complement to (5.13) allows to write

$$\begin{bmatrix} \tilde{A} + \epsilon \tilde{E}\tilde{E}^T & \tilde{D} \\ \tilde{D}^T & -\epsilon I \end{bmatrix} < 0 \qquad (5.14)$$

which is nothing but (5.7).

Remark 5.2: Note that, this corollary clearly states that one need only check the above LMI based condition to determine the stability of bimodal systems with norm-bounded uncertainty.

5.2 Simulation Studies

Consider the mechanical system shown in Figure 5.1. Let the mass of the cart denoted by m, damping constant by d and the spring constants by k_{s1} for the left most and k_{s2} for the other. Firstly, we assume that the system is nominal. Defining the state variables as $x_1 = \dot{x}$, $x_2 = \ddot{x}$ which represents distance and velocity of the mass respectively, allows to obtain following governing equations

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -k_{s1}/m & -d/m \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F & \text{if } C^T x \leq 0.1 \\ \begin{bmatrix} 0 & 1 \\ -(k_{s1}+k_{s21})/m & -d/m \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F & \text{if } C^T x \geq 0.1 \end{cases} \tag{5.15}$$

where F is the force applied to the cart, c is output and x is state vectors which are $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} x_1 & x_2 \end{bmatrix}$,respectively.

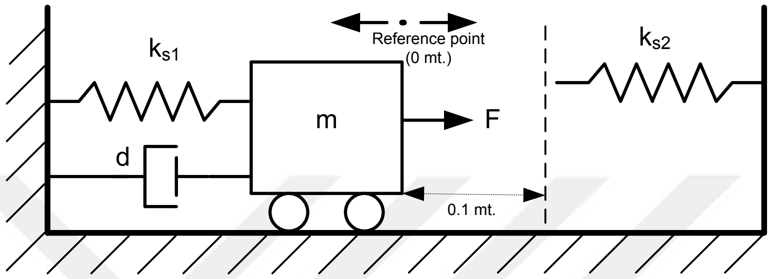

Figure 5.1 Bimodal configurated spring-mass-damper mechanical system

Firstly, we apply Theorem 2.1 to check the nominal stability of bimodal system given in (5.15). Let the system parameters be as $m = 250$ kg, $k_{s1} = 160000$ N/m, $k_{s2} = 16000$ N/m and $d = 1000$ Ns/m. In this perspective, let us first solve the LMI (2.25) to find a common symmetric definite positive matrix K. According to Theorem 2.1, we have conducted a computer simulation using LMIlab toolbox in MATLAB. After the solving the related LMI with MATLAB-LMI-lab solver we get the following result.

$$K = \begin{bmatrix} 1.2185 & 0.0016 \\ 0.0016 & 0.0019 \end{bmatrix}$$

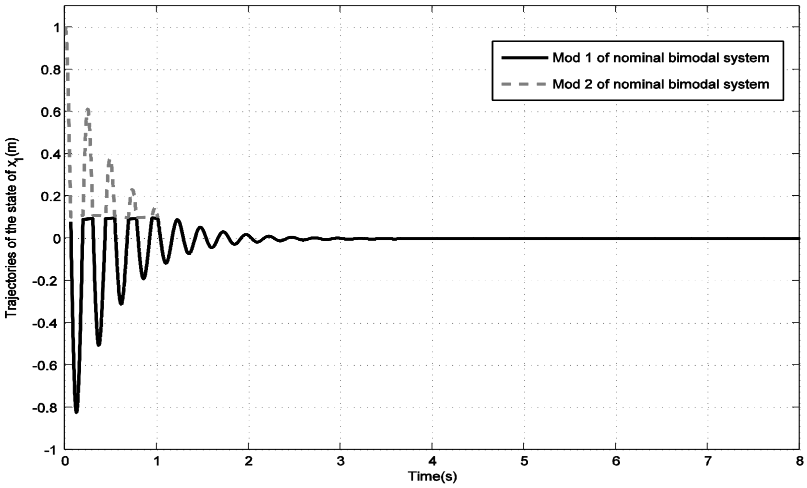

Figure 5.2 Simulation of bimodal configurated nominal spring-mass-damper mechanical system

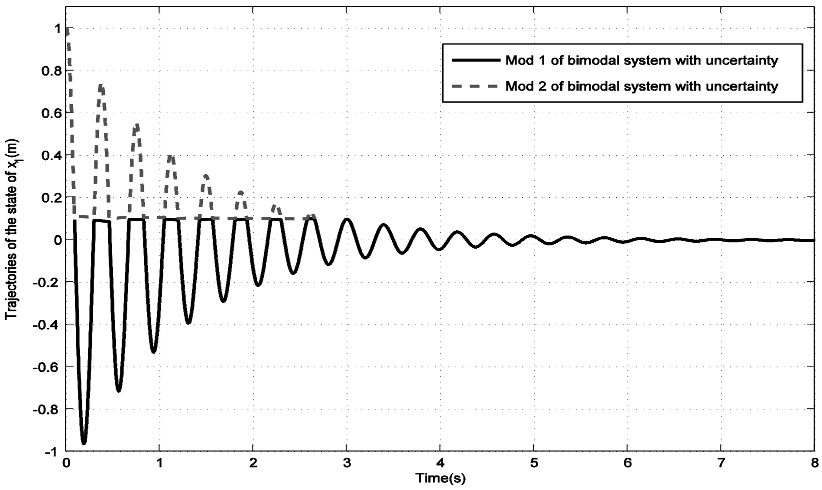

Figure 5.3 Simulation of bimodal configurated spring-mass-damper mechanical system with constant uncertainty

Now, we reformulate the system in Figure 5.1 with uncertainty under the assumption of $||\Delta|| \leqslant I$. Then, the governing equations can be given as follows.

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{cases} \left(\begin{bmatrix} 0 & 1 \\ -k_{s1}/m & -C/m \end{bmatrix} + [\Delta A_1(t)] \right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F & \text{if } C^T x_1 \leq 0.1 \\ \left(\begin{bmatrix} 0 & 1 \\ -(k_{s1}+k_{s2})/m & -c/m \end{bmatrix} + [\Delta A_2(t)] \right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} F & \text{if } C^T x_1 \geq 0.1 \end{cases} \tag{5.16}$$

It is assumed that the spring parameter k and damper parameter d have $\%60$ admissible uncertainties in their exact values. Hence, we can assign the vectors E_1 and D_1 as $\begin{bmatrix}0 & 1\end{bmatrix}^T$ and $\begin{bmatrix}(0.6 * k_{s1}/m & 0.6 * d/m)\end{bmatrix}$, respectively. System parameters are same as the previous example and solving the LMI (5.7) to find a positive real symmetric matrix M via Matlab-LMIlab solver we get the following result with $\epsilon = 1.3989$

$$M = \begin{bmatrix} 42.2501 & 2.1272 \\ 2.1272 & 0.7813 \end{bmatrix}. \tag{5.17}$$

Finally, we present elementary simulation studies to visualize the switching mechanism of bimodal configurated spring-mass-damper mechanical system. First of all, we render the trajectory changing of exact knowledge case on the Figure 5.2. Then, we deal with the uncertain case which is allowed $\%60$ change in k and d parameters, on the Figure 5.3 and time varying uncertainty taken $0.6 * k_{s1} * sin(2\pi t)$ on Figure 5.4. It should be noted that decreasing on damper coefficient affects to the system at most. Hence, one can state that constant type uncertainty have more negative impact on the stability of the system than the sinusoidal type uncertainty with respect to time. Besides, it could be underlined that sinusoidal type uncertainty affects to attenuation path more than the the other type of uncertainties.

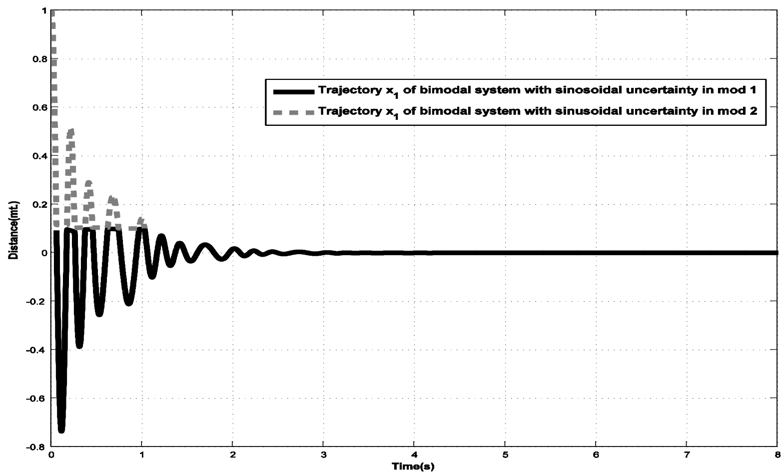

Figure 5.4 Simulation of bimodal configurated spring-mass-damper mechanical system with sinusoidal uncertainty

CHAPTER 6

RESULTS AND DISCUSSIONS

In this thesis, we investigated the quadratic stability of bimodal PWL systems with continuous vector field. After establishing necessary and sufficient conditions for quadratic stability and feedback stabilization in terms of LMIs, we provide sufficient conditions for feedback stabilization in terms of the zero dynamics for one of the two linear subsystems. Also, we discuss the relations between the existing open loop stabilizability conditions and those for the feedback stabilization. Based on the approach and the results of this thesis, several further research possibilities arise. One of the immediate issue is relaxing the continuity assumption on considering system class. Also, conditions obtained for the stability can be extended into a large class of bimodal PWL systems with relay, saturation, dead-zone or time delay.

On the other hand, stability analysis and controller design for multi-modal PWL systems is also challenging problem. Despite the considerable contribution of numerous studies, a full algebraic analogue of stability analysis and stabilization of multi-modal case has not been reported yet. In this context, our treatment has the potential to bridge the gap between stability theory and controller design of related systems due to the fact that it employs tools from geometric control theory.

REFERENCES

[1] Imura, J., and Schaft, A., (2000). "Characterization of Well- Posedness of Piecewise-Linear Systems", IEEE Transactions on Automatic Control, 45:1600-1619.

[2] Wu, C.Z. and Teo, K.L. and Rehbock, V. and Liu, G.G., (2009). "Existence and Uniqueness of Solutions of Piecewise Nonlinear Systems", Nonlinear Analysis, 71:6109-6115.

[3] Sontag, E.D., Mathematical Control Theory, New york, Springer, 1998.

[4] Camlibel, M.K. and Heemels, W.P.M.H.(Maurice) and Schumacher J.M.(Hans), (2008.1). "Algebraic Necessary and Sufficient Conditions for the Controllability of Conewise Linear Systems", IEEE Transaction on Automatic Control, 53:762-774.

[5] Juloski, A. L. and Heemels, W.P.M.H. and Weiland, S.,(2007). "Observer design for a class of piecewise linear systems ", International Journal of Robust and Nonlinear Control, 17:1387-1404.

[6] Willems, J.L., (1973). "The Circle Criterion and Quadratic Lyapunov Functions for Stability Analysis", IEEE Transaction on Automatic Control, 18(2):184.

[7] Hassibi, A. and Boyd, S., (1998). "Quadratic Stabilization and Control of Piecewise-Linear Systems", Proceedings of the American Control Conference, June 1998, Pennsylvania, 3659-3664.

[8] Johansson, M. and Rantzer, A., (1998). "Computation of Piecewise Quadratic Lyapunov Functions for Hybrid Systems", IEEE Transaction on Automatic Control, 43:555-559.

[9] Agrachev, A.A. and Liberzon, D., (2001). "Lie-algebraic stability criteria for switched systems", SIAM J. Control Optim., 40(1):253-269.

[10] Liberzon, D., (1999)."Stability of switched systems: a lie-algebraic condition", Systems & Control Letters, 37(3):117- 122.

[11] Camlibel, M.K. and Pang, J.S. and Shen, J., (2006). "Conewise Linear Systems: Non-zenoness and Observability", SIAM J. Control Optim., 45(5):1769–1800Xu, J. and Xie, L., (2006).

[12] Pavlov, A., Pogromsky, A., Van De Wouw, N., and Nijmeijer, H., (2007). "On Converges Proporties of Piecewise Affine Systems", International Journal of Control, 80:1233-1247.

[13] Samadi, B. and L. Rodrigues, A., (2011). "Unified Dissipativity Approach for Stability Analysis of Piecewise Smooth Systems", Automatica , 47:2735-2742.

[14] Camlibel, M.K. and Heemels, W.P.M.H.(Maurice) and Schumacher J.M.(Hans), (2008.2). "A Full Characterization of Stabilizability of Bimodal Piecewise Linear Systems with Scalar Inputs", Automatica, 44:1261-1267.

[15] Ibeas, A. and de le Sen, M., (2009). "Exponential Stability of Simultaneously Triangularizable Switched Systems with Explicit Calculation of a Common Lyapunov Function", Applied Mathematics Letters, 22:1549-1555.

[16] Feng, G., (2002). "Stability Analysis of Piecewise Discrete-TimeLinear Systems", IEEE Transactions on Automatic Control, 47(7):1600-1619.

[17] Iwatani, Y. and Hara, S., (2006). "Stability Tests and Stabilization for Piecewise Linear Systems Based on Poles and Zeros of Subsystems", Automatica, 42:1685-1695.

[18] Willems, J.L., Stability Theory of Dynamical System , New york, Wiley, 1970.

[19] Doris, A. and Juloski, A. L. and Mihajlovic, N. and Heemels, W. P. M. H. and van de Wouw, N. and Nijmeijer, H., (2008). "Observer designs for experimental non-smooth and discontinuous systems", IEEE Transactions on Control Systems Technology, 16:1323–1332.

[20] Doris, A. and van de Wouw, N. and Heemels, W. P. M. H. and Nijmeijer, H.,(2010). "A disturbance attenuation approach for continuous piecewise affine systems: control design and experiments", ASME Journal of Dynamic Systems, Measurement and Control, 132:044502-1–044502-7.

[21] van der Heijden, A. C. and Serrarens, A. F. A. and Camlibel, M. K. and Nijmeijer, H, (2007). "Hybrid optimal control of dry clutch engagement",International Journal of Control, 80:1717–1728.

[22] Vanek, B. and Bokor, J. and Balas, G. J. and Arndt, R. E. A, (2007). "Longitudinal motion control of a high-speed supercavitation vehicle. International Journal of Control", 13:159–184.

[23] Carmona, V. and Fernandez-Garcia, S. and Fernandez-Sanchez, F. and Garcia-Medina, E., and Teruel, A. E., (2012). "Reversible periodic orbits in a class of 3d continuous piecewise linear systems of differential equations", Nonlinear Analysis. Theory,Methods & Applications, 75:5866–5883.

[24] Carmona, V. and Fernandez-Garcia, S. and Freire, E., (2011). "Periodic orbits for perturbations of piecewise linear systems", Journal of Differential Equations, 250: 2244–2266.

[25] Michelson, D., (1986). "Steady solutions of the Kuramoto-Sivashinsky equation", Physica D: Nonlinear Phenomena, 19:89–111.

[26] Webster, K. N. and Elgin, J. N., (2003). "Asymptotic analysis of the Michelson system", Nonlinearity, 16:2149–2162.

[27] Carmona, V. and Fernandez-Sanchez, F. and Teruel, A., (2008). " Existence of a reversible Tpointheteroclinic cycle in a piecewise linear version of the Michelson system", SIAM Journal on Applied Dynamical Systems, 7:1032–1048.

[28] Aling, H. and Schumacher, J.M., (1984). "A nine-fold canonical decomposition for linear systems", International Journal of Control, 39:805-1984.

[29] Boyd, S. and Ghaoul, L. E. and Feron, E. and Balakrishnan, V., (1984). "Linear Matrix Inequalities in System and Control Theory", SIAM.

[30] Trentelman, H. L. and Stoorvogel A. A. and Hautus M., (2001). Control Theory for Linear Systems, Springer.

[31] Li X. and de Souza E., (1997). "Delay-dependent robust stability and stabilization of uncertain linear delay systems: A linear matrix inequality approach", IEEE Transactions on Automatic Control, 42(8): 1144-1148.

APPENDIX-A

GEOMETRİC CONTROL REVIEW

Appendix A introduces some notation and properties for geometric control theory from the the book of Trentelman et. al [30].

Let $\mathcal{X}$ be a linear space and assume that $\mathcal{V}$ and $\mathcal{W}$ subspaces of $\mathcal{X}$. In such case, $\mathcal{V} \cap \mathcal{W}$ and $\mathcal{V} + \mathcal{W}$ are also subspaces. The smallest subspace containing both $\mathcal{V}$ and $\mathcal{W}$ is donated as $\mathcal{V} + \mathcal{W}$. Besides, $\mathcal{V} \cap \mathcal{W}$ is the largest subspace contained in both $\mathcal{V}$ and $\mathcal{W}$. Also it can be said that if a subspace $\mathcal{L}$ satisfies $\mathcal{V} \subset \mathcal{L}$ and $\mathcal{W} \subset \mathcal{L}$ then this implies $\mathcal{V} + \mathcal{W} \subset \mathcal{L}$. In this context, the following formula which is called as modular rule

$$\mathcal{R} \cap (\mathcal{V} + \mathcal{W}) = \mathcal{V} + (\mathcal{R} \cap \mathcal{W}) \qquad \text{(A1)}$$

is valid for a subspace $\mathcal{R}$.

Another important notion is linear independency. Let $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_k$ be subspaces. Then the subspaces are called linearly independent if every $x \subset \mathcal{V}_1 + \mathcal{V}_2 + ... + \mathcal{V}_k$ has a unique representation of the form $x = x_1 + x_2 \dots x_k$ with $x_i \subset \mathcal{V}_i (i = 1, ..., k)$. Moreover, if $x_i \subset \mathcal{V}_i (i = 1, ..., k)$ and $x = x_1 + x_2 \dots x_k$ imply that $x_1 = x_2 ... = x_k = 0$.

Null space of a vector space is stated as the symbol "0". Then, it can be also characterized the linear independence as follows.

$$\mathcal{V}_i \cap \sum_{j \neq i} = \mathcal{V}_j = 0 (i = 1, ..., k) \qquad \text{(A2)}$$

If $\mathcal{V}_1, \mathcal{V}_2 ... \mathcal{V}_k$ are linear independent subspaces, then their sum $\mathcal{V}$ is called the direct sum of $\mathcal{V}_1, \mathcal{V}_2 ... \mathcal{V}_k$ as following notation.

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_k = \bigoplus_{i=1}^{k} \mathcal{V}_k \tag{A3}$$

Linear complementary is also a fundamental notion for linear spaces. If $\mathcal{V}$ is a subspace then there exists a subspace $\mathcal{W}$ such that $\mathcal{V} \oplus \mathcal{W} = \mathcal{X}$ which is called linear complement of $\mathcal{V}$. It can be constructed by choosing a basis $q_1, q_2 ... q_k$ of $\mathcal{V}$ and then extending it to a basis $q_1, q_2 ... q_n$ of $\mathcal{X}$. In this case, the span of $q_{k+1}, q_{k+2} ... q_n$ is a linear complement of $\mathcal{V}$. Note that, a linear complement is not unique.

Linear Maps

Let consider a linear map $A : X \to Y$. It can be defined $kernel$ and $image$ spaces of A as follows.

$$ker\ A \triangleq \{x \in X \mid Ax = 0\} \tag{A4}$$

$$im\ A \triangleq \{x \in X \mid Ax\} \tag{A5}$$

If $imA = Y$ then A is called as surjective and if $kerA = 0$, A is injective. Also, it is known that A is isomorphic if A is surjective and injective. Then, A has an inverse map and denoted by A^{-1}.

Based on the $kernel$ and $image$ spaces the followinf inclusions can be defined

$$A \mid imB := imB + AimB + ... + A^{n-1}imB \tag{A6}$$

$$kerC \mid A := kerC \cap A^{-1}kerC \cap + ... + A^{1-n}kerC \tag{A7}$$

and it is well known that

$$(A \mid imB) = (kerB^T \mid A^T)^{\perp} \tag{A8}$$

where $\mathcal{W}^{\perp}$ denotes the orthogonal space of $\mathcal{W}$.

It can be also commented that in general, if $A : X \to Y$ is a not necessarily invertible linear map and if $\mathcal{V}$ is a subspace of Y, then the inverse image of $\mathcal{V}$ is the subspace of X as defined

$$A^{-1}\mathcal{V} \triangleq \{x \in X \mid Ax \in \mathcal{V}\}. \tag{A9}$$

Let $A : X \to X$ and a subspace $\mathcal{V}$ of X, it said that $\mathcal{V}$ is $A-invariant$ if for all $x \in X$. Then $Ax \in \mathcal{V}$ holds. Moreover, this can be stated as $A\mathcal{V} \in \mathcal{V}$.

If $\mathcal{V} \in X$ then a basis $q_1, q_2...q_n$ of X for which $q_1, q_2...q_k$ is a basis of $\mathcal{V}(k = dim(\mathcal{V}))$ is called a basis of X adapted to $\mathcal{V}$. More generally, if $\mathcal{V}_1, \mathcal{V}_2...\mathcal{V}_r$ is a chain of subspace such that $\mathcal{V}_1 \subset \mathcal{V}_2... \subset \mathcal{V}_r$ then a basis $q_1, q_2, ..., q_n$ of $\mathcal{X}$ is said to be adapted to this chain if there exist $k_1, k_2, ..., k_r$ such that $q_1, q_2, ..., q_k$ is a basis of $\mathcal{V}_i$ for $(i = 1, ..., r)$. If $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_r$ are subspaces of X such that $\mathcal{X} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus ... \oplus \mathcal{V}_r$, it is said that a basis $q_1, q_2, ..., q_n$ is adapted to $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_r$ if there exist numbers $k_1, k_2...k_{r+1}$ such that $k_1 = 1$, $k_{r+1} = n+1$ and $q_{k_i}, ..., q_{k_{i+1}-1}$ is a basis of $\mathcal{V}_i$ for $i = 1, ..., r$.

It is also stated a linear map in matrix or vector representation. For example, if $B : U \to X$ is a linear map satisfying $imB \subset \mathcal{V}$, it can be chosen a basis $q_1, q_2, ..., q_n$ of X adapted to $\mathcal{V}$. Also, $p_1, p_2, ..., p_m$ can be chosen as the basis of U. Then, the matrix representation of B with respect to the basis is the following form

$$\mathrm{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{A10}$$

where $\mathrm{B} \in \mathbb{R}^{k \times m}$. This form clearly satisfies the condition of $B \subset \mathcal{V}$.

Upper or lower triangular matrix forms give advantageous for the analysis of the mechanical systems. For example, let $A : X \to X$, $\mathcal{V} \subset X$. Let also $q_1, q_2, ..., q_n$ be a basis of X adapted to $\mathcal{V}$. Then, the matrix of A with respect to this basis is

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}. \tag{A11}$$

The property $A_{21} = 0$ is a consequence of the $A-invariance$ of $\mathcal{V}$. The matrix of $A \mid \mathcal{V}$ is $A_{11} = 0$.

Systems with inputs and outputs

Let us consider the linear system with following equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{A12}$$

$$y(t) = Cx(t) + Du(t) \tag{A13}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output at time $t \in \mathbb{R}$.

(A, B, C, D) are linear maps or matrices between spaces. (x, u, y) vectors are usually defined on positive real axis and called as state, input and output vectors, respectively. Input function take values from the outside of the system and the class of admissible input functions are denoted as $\mathbf{U}$. $\mathbf{U}$ can be chosen different system classes but has the slicing property i.e., if $u_1 \in \mathbf{U}$ and $u_2 \in \mathbf{U}$, then for any $\theta > 0$, the function u_3 defined by $u_3 \triangleq u_1(t)(0 \leq t < \theta)$ and $u_3 \triangleq u_2(t)(t \geq \theta)$ is in $\mathbf{U}$. It is assumed that input function take values m-dimensioned positive reel space $\mathcal{U}$. Besides, state variable vector x have values in a n-dimensioned space $\mathcal{X}$ which is called as state space and output of the system (y) have values in a p-dimensioned space $\mathcal{Y}$.

All the system is represented with $\Sigma(A, B, C, D)$ or just with Σ. Moreover, the map D which is from input to the output, is feed-through term is irrelevant in many problems. So, the system is often represented with $\Sigma(A, B, C)$. It is known that, the solution of differential equation of $\Sigma(A, B, C)$ which is

$$x_u(t, x_0) = e^{At}x_0 + \int_{x_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau; \quad x(0) = x_0 \,. \tag{A14}$$

Output function $y_u(t, x_0)$ can be given as

$$y_u(t, x_0) = Ce^{At}x_0 + \int_{x_0}^{t} Ce^{At}Bu(\tau)d\tau. \tag{A15}$$

It can be reduced the system equations by using the Laplace transformation to the transfer function from input to output as follows

$$G(s) \triangleq C(sI - A)^{-1}B + D. \tag{A16}$$

Proposition A.1: Characterization of invertibility of $G(s)$,

(1) $G(s)$ is left-invertible as a rational matrix, iff $\mathcal{V}^* \cap \mathcal{T}^* = \{0\}$ and $\begin{bmatrix} B \\ D \end{bmatrix}$ is of full column rank.

(2) $G(s)$ is right-invertible as a rational matrix, iff $\mathcal{V}^* \bigoplus \mathcal{T}^* = \{\mathbb{R}^n\}$ and $\begin{bmatrix} C & D \end{bmatrix}$ is of full row rank.

(3) $G(s)$ is invertible as a rational matrix, iff $\mathcal{V}^* \bigoplus \mathcal{T}^* = \{\mathbb{R}^n\}$ and $\begin{bmatrix} B \\ D \end{bmatrix}$ is of full column rank and $\begin{bmatrix} C & D \end{bmatrix}$ is of full row rank.

Remark A.2: If $G(s) = C^T(sI - A)^{-1}B \neq 0$ and $\mathcal{V}^* = \{0\}$, then $\begin{bmatrix} C & D \end{bmatrix}$ is scalar and the system $G(s)$ does not has zero dynamics.

Alternatively, it can be characterized the invertibility of the transfer function as follows.

Proposition A.3([28])**:** The transfer matrix $D + C(sI - A)^{-1}B$ is invertible as a rational matrix if and only if $\mathcal{V}^* \oplus \mathcal{T}^* = \mathbb{R}^n$, $\begin{bmatrix} C & D \end{bmatrix}$ is of full row rank, and $\begin{bmatrix} B & D^T \end{bmatrix}$ is of full column rank. Moreover, the inverse is polynomial if and only if $\mathcal{V}^* \cap (A \mid imB) \subseteq (kerC \mid A)$ and $(A \mid imB) \subseteq \mathcal{V}^* + (kerC \mid A)$.

On the other hand, the linear system with feedback configuration ($\Sigma_{K,L}$) can be given by

$$\dot{x} = (A - BK - GC + GDK)x + (B - GD)v \tag{A17}$$

$$y = (C - DK)x + Dv \tag{A18}$$

This system is obtained from (A12), by applying both state feedback $u = -Kx + v$ and output injection $-Ly$.

Proposition A.4: Let $K \in \mathbb{R}^{m \times n}$ and $L \in \mathbb{R}^{n \times p}$ be given. The following statement hold:

(1) $(A \mid imB) = (A - BK \mid imB)$

(2) $(kerC \mid A) = (kerC \mid A - GC)$

(3) $\mathcal{V}^*(\Sigma_{K,L}) = \mathcal{V}^*(\Sigma)$

(4) $\tau^*(\Sigma_{K,L}) = \tau^*(\Sigma)$

The next proposition relates the invertibility of the transfer matrix to controlled and conditioned invariant subspaces.

A subspace $\mathcal{V}$ is called as output-nulling controlled invariant if for some matrix K the inclusion $(A - BK)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq ker(C - DK)$ hold. The set of such subspaces is non-empty and closed under subspace addition, the set has a maximal element which is donated as $\mathcal{V}^*(\Sigma)$. Specifically, $\mathcal{V}^*(\Sigma)$ is called as largest controlled invariant space. The notation $\mathcal{K}(\mathcal{V})$ stands for the set $\{K \mid (A - BK)\mathcal{V} \subseteq \mathcal{V}; \mathcal{V} \subseteq ker(C - DK)\}$. Also, $\mathcal{K}(A, B, C, D)$ can be written as $\mathcal{K}(\mathcal{V}^*(A, B, C, D))$. Note that, $\mathcal{V}^*$ is the limit of the subspaces and

$$\mathcal{V}^0 = \mathbb{R}^n \tag{A19}$$

$$\mathcal{V}^i = \{x \mid Ax + Bu \in \mathcal{V}^{i-1}; Cx + Du = 0 \quad \text{for some} \quad u\}. \tag{A20}$$

Dually, it can be state a subspace τ is input containing conditioned invariant if for some L the inclusions$(A - GC)\tau \subseteq \tau$ and $im(B - GD) \subseteq \tau$ hold. As the set of such subspaces is non-empty and closed under subspace intersection, it has a minimal element $\tau^*(\Sigma)$.

It is also written $\tau^*(\Sigma)$ as simply τ^* for the purpose of notational convenient. The notation $\mathfrak{L}(\tau)$ represents the set $\{G \mid (A - GC)\tau \subseteq \tau; im(B - GD) \subseteq \tau\}$. As $\mathfrak{L}(A, B, C, D)$ is represented with $\mathfrak{L}(\tau^*(A, B, C, D))$, it can be said that $(A \mid imB) \supseteq \tau^*(A, B, C, D)$.

Next proposition give general characterization of the spaces of $\mathcal{V}$ and $\mathcal{T}$.

Proposition A.5 : Following statements are equivalent:

(1) $\mathcal{V}$ is controlled invariant : iff there exists F such that $(A - BF)\mathcal{V} \subseteq \mathcal{V}$

(2) $\mathcal{V}$ is controlled invariant output nulling: if there exists F such that $(A - BF)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq ker(C - DF)$

(3) $\mathcal{V}^*$ is the largest controlled invariant nulling subspace.

(4) $\mathcal{T}$ is conditioned invariant : iff there exists G such that $(A - GC)\mathcal{T} \subseteq \mathcal{T}$

(5) $\mathcal{T}$ is conditioned invariant input containing: iff there exists G such that $(A - GC)\mathcal{T} \subseteq \mathcal{T}$ and $im(B - GD)\mathcal{T} \subseteq \mathcal{T}$

(6) $\mathcal{T}^*$ is the smallest conditioned invariant input containing

it can be used the transformation of the subspaces for complex problems. So that, let $v_1, \ldots, v_k$ and $v_{k+1}, \ldots, v_n$ be basis for $\mathcal{V}^*$ and $\mathcal{T}^*$, respectively. Note that, the following implications can be written.

$$(A - BF)\mathcal{V}^* \subseteq \mathcal{V}^* \text{ and } \mathcal{V}^* \subseteq ker(C - DF) \tag{A21}$$

$$(A - GC)\mathcal{T}^* \subseteq \mathcal{T}^* \text{ and } im(B - GD)\mathcal{T}^* \subseteq \mathcal{T}^* \tag{A22}$$

Note also that $(C - DF)$ is in the $kernel\ space$ of $\mathcal{V}^*$ and $(B - GD)$ is in the $image\ space$ of $\mathcal{T}^*$. Therefore (A23) and (A24) are similar spaces with (A21) and (A22), respectively.

$$(A-BF-GC+GDF)\mathcal{V}^* \subseteq \mathcal{V}^* \tag{A23}$$

$$(A-BF-GC+GDF)\mathcal{T}^* \subseteq \mathcal{T}^*. \tag{A24}$$

Hence, it can be written the spaces in diagonal matrix form as follows.

$$(A-BF-GC+GDF) = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \tag{A25}$$

Also one immediately gets

$$B-GD = \begin{bmatrix} 0 \\ * \end{bmatrix} \tag{A26}$$

$$C-DF = \begin{bmatrix} 0 \\ * \end{bmatrix}^T \tag{A27}$$

in the coordinates that are adapted to the earlier basis as $\mathcal{V} \subseteq \ker(C-DF)$ and $(B-GD) \subseteq \mathcal{T}$.

The continuity condition which is between two dynamics (for example, $A_1 - A_2 = eC^T$) has a number of useful consequences. Indeed, it can be verified

$$\mathcal{V}^*(A_1, B, C^T) = \mathcal{V}^*(A_2, B, C^T) \tag{A28}$$

$$\mathcal{T}^*(A_1, B, C^T) = \mathcal{T}^*(A_2, B, C^T) \tag{A29}$$

which $\Sigma(A_1, B, C^T)$ and $\Sigma(A_2, B, C^T)$ are LTI systems.

Together with the invertibility conditions, those equalities imply that the transfer function $C^T(sI - A_1)^{-1}$ is invertible if and only if so is $C^T(sI - A_2)^{-1}$.

APPENDIX-B

CONTROLLABILITY OF BIMODAL SYSTEMS

In this appendix, the solution of controllability problem for conewise linear systems is adressed. Camlibel et. al. presented the solution of the controllability problem of related systems in [4]. They derived algebraic necessary and sufficient conditions for the related problem. Firstly, they considered push-pull system for controllability problem, then they extended the solution into conewise linear systems. As the bimodal systems are an example of conewise linear systems, this solution contains the controllability problem of bimodal systems.

First of all, it should be to introduced conewise linear systems (CLS). CLS can be given as follows.

$$\dot{x} = Ax + f(u) \tag{B1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n \times n}$ and $f : \mathbb{R}^m \implies \mathbb{R}^n$ is a continuous conewise linear function. Those systems are also considered as push-pull system by the help of scalar input function u.

$$\dot{x}(t) = Ax + \begin{cases} B_1 u(t) & \text{if } u \leq 0 \\ B_2 u(t) & \text{if } u(t) \geq 0 \end{cases} \tag{B2}$$

It is well know that the system (B1) is completly controllable if for any pair of state dynamics $(x_0, x_f) \in \mathbb{R}^{n \times n}$, there exists a locally integrable input function u such that the solution $x^{x_0,u}$ of (B1) satisfies $x^{x_0,u}(T) = x_f$ for some $T > 0$. Also, the system (B1) is reachable from zero if for any state $x_f \in \mathbb{R}^n$, there exists a locally integrable input function u such that the solution $x^{x_0,u}$ of (B1) satisfies $x^{0,u}(T) = x_f$ for some $T > 0$. The theorem B1 gives the necessary and sufficient conditions for the controllability of push-pull systems [4].

Theorem B.1 : The following statements are equivalent.

1) The system (B1) is completely controllable.

2) The system (B1) is completely controllable with C^∞ inputs.

3) The system (B1) is reachable from zero.

4) The system (B1) is reachable from zero with C^∞ inputs.

5) The implication

$$z^T exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \; u \in \mathbb{R}^m \implies z = 0 \tag{B3}$$

holds.

6) The pair $(A[M^1 M^2 ... M^r])$ is completely controllable with respect to $\mathcal{Y}_1 \times \mathcal{Y}_2 ... \times \mathcal{Y}_r$.

Proof of Theorem B.1: The implications ($2 \implies 1$), ($1 \implies 3$) and ($3 \implies 4$) are clear.

($3 \implies 5$): Suppose that Statement (3) holds. Let $z \in \mathbb{R}^n$ be such that

$$z^T exp(At) f(u) \geq 0 \tag{B4}$$

for all $t \geq 0$ and $u \in \mathbb{R}^m$. Then, the solution x of (B1) with zero initial condition the following inequaility can be written.

$$z^T x(T) = z^T \int_0^T exp(A(T-s)) f(u(s)) ds \geq . \tag{B5}$$

$x(T)$ may take any arbitrary value by choosing a suitable function so z must be zero.

($5 \implies 6$): Suppose that statement 5 of the theorem B1 holds. Then, one can need to use next theorem [4].

Theorem B.2: Consider the linear time invariant system with input as follows with a solid cone U as the restraint set.

$$\dot{x} = Ax + B(u) \tag{B6}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$

Then, (B6) is completely controllable with respect to U if and only if the following conditions hold.

1) The pair (A, B) is controllable.

2) The implication

$\lambda \in \mathbb{R}$, $z \in \mathbb{R}$, $z^T A = \lambda z^T$, $B^T z \in U^* \implies z = 0$

holds. By the help of Theorem B.2 following two implications are needed to show.

1) the pair $(A[M^1 M^2 ... M^r])$ is controllable.

Proof : Let $s' \in \mathbb{C}$ and $v \in \mathbb{C}^n$ be such that

$$v^*[s'I - A \quad M^1 M^2 ... M^r] = 0. \tag{B7}$$

(B6) means that

$$s'v^* = v^* A \tag{B8}$$

$$v^* M^i = 0 \tag{B9}$$

for all $i = 1, 2, ..., r$. Let σ and ω be, respectively, the real and imaginary parts of s'. Also let v_1 and v_2 be, respectively the real and imaginary parts of v. Then, (B8) and (B9) can be written in terms of σ, ω, v_1 and v_2 as follows

$$\begin{bmatrix} v_1^T \\ v_1^T \end{bmatrix} A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \begin{bmatrix} v_1^T \\ v_1^T \end{bmatrix} \tag{B10}$$

$$v_1^T M^i = v_2^T M^i = 0 \tag{B11}$$

for all $i = 1, 2, ..., r$. Also, (B10) results in

$$\begin{bmatrix} v_1^T \\ v_1^T \end{bmatrix} exp(At) = exp\left(\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t \right) \begin{bmatrix} v_1^T \\ v_1^T \end{bmatrix} \tag{B12}$$

(B11) and (B12) imply that $v_j^T exp(At) M^i = 0$ for all t, i and $j \in \{1, 2\}$. In view of statement 5 of Theorem B.1 v_1 and v_2 must be zero. Consequently, the pair $(A[M^1 M^2 ... M^r])$ is controllable.

2) The implication $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$,

$$z^T A = \lambda z^T, (M_i)^T z \in \mathcal{Y}_i^* \text{ for all } i = 1, 2, ..., r \implies z = 0 \tag{B13}$$

holds.

Proof: Let $z \in \mathbb{Z}$ and $\lambda \in \mathbb{R}$ be such that

$$z^T A = \lambda z^T \tag{B14}$$

$$(M_i)^T z \in \mathcal{Y}_i^* \tag{B15}$$

for all $i = 1, 2, ..., r$. It is known that, $z^T M^i v$ is nonnegative for any $v \in \mathcal{Y}_i$. .thus, one gets $z^T f(v) \geq 0$ for all v. Note that, $z^T exp(At) = exp(\lambda t) z^T$ due to (B14). Then, $z^T exp(At) f(v) \geq 0$ for all $v \in \mathbb{R}^m$. In view of statement 5 of Theorem 4.1 this implies that $z = 0$. Consequently Statement 6 of Theorem B.1 follows from above implications and Theorem B.2.

($5 \implies 4$): This implication follows from the following lemma.

Lemma B.3: Consider the system (B1) and suppose that the implication

$$z^T exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \text{ and } u \in \mathbb{R}^m \implies z = 0 \tag{B16}$$

holds. Then, there exist a positive real number T and an integer l such that for a given state x_f, one can always find vectors $\eta^{i,j} \in \mathcal{Y}_i$ for $i = 1, 2, ..., r$ and $j = 0, 1, 2, ..., l-1$ such that the state x_f can be reached from the zero state in time T by the application of the input

$$\bar{u}(t) = \eta^{i,j} \Theta^{\Delta l}(t - (jr + i - 1)\Delta l) \tag{B17}$$

for $(jr + i - 1)\Delta l \leq t \leq (jr + i)\Delta l$ where $\Delta l = T/(lr)$ and $\Theta^{\Delta} : \mathbb{R} \to \mathbb{R}$ is a nonnegative valued $\mathbb{C}^{\infty}$ function with $supp(\Theta^{\Delta}) \subseteq (\Delta/4, 3\Delta/4)$ and $\int_0^{\Delta} \Theta^{\Delta}(t) = 1$.

Proof: First of all, it can be shown that, if (B16) holds then there exists a positive real number T such that the implication,

$$z^T exp(At) f(u) \geq 0 \text{ for all } t \in [0, T] \text{ and } u \in \mathbb{R}^m \implies z = 0 \tag{B18}$$

holds. To see this, suppose that the (B18) does not hold for any T. Therefore, for all T, there exists $0 \neq z_T \in \mathbb{R}^n$ such that

$$z^T exp(At) f(u) \geq 0 \text{ for all } t \in [0, T] \text{ and } u \in \mathbb{R}^m \implies z = 0 \tag{B19}$$

Without the loss of generality, it can be assumed that $||z_T|| = 1$. Then the sequence $\{z_T\}_{T \in \mathbb{N}}$ admits a convergent subsequences due to the well-known Bolzano-

Weierstrass theorem. Let z_∞ denote its limit. Note that, $||z_\infty|| = 1$. It can be claimed that

$$z_\infty^T exp(At)f(u) \geq 0 \quad \text{(B20)}$$

for all $t \geq 0$ and $u \in \mathbb{R}^m$. To show this, suppose that $z_\infty^T exp(At')f(u') < 0$. Then for sufficiently large T', one has $z_T^T exp(At')f(u') < 0$ and $t' < T'$. However, this can not happen due to (B19). In view of (B16), (B20) yields $||z_\infty|| = 0$. Hence, by contradiction, there exists a positive real number T such that the implication (B18) holds.

Now, consider the input function in (B17). Note that

$$f(\bar{u}(t)) = M^i \bar{u}(t) \text{ if } (jr + i - 1)\Delta l \leq t \leq (jr + i)\Delta l.$$

The solution of (B1) corresponding to $x(0) = 0$ and $u = \bar{u}$ is given by

$$x(T) = \int_0^T exp[A(T - s)]f(\bar{u}(s))ds \quad \text{(B21)}$$

Straightforward calculation yields that

$$x(T) = \Lambda(\Delta_l) \sum_{j=1}^{l-1} \sum_{i=1}^{r} exp[A(T - (jr + i - 1)\Delta_l)]M^i \eta^{i,j} \quad \text{(B22)}$$

where $\Lambda(\Delta) = \int_0^\Delta exp(-As)\Theta^\Delta ds$. Then it is enough to show that there exists an integer l such that the previous equation is solvable in $\eta^{i,j} \in \mathcal{Y}_i$ for all $i = 1, 2, ..., r$ and $j = 0, 1, ..., l - 1$ for any $x(T) \in R^n$. For this reason, it is needed to generalized Farkas' lemma.

Lemma B.4: Let $H \in \mathbb{R}^{P \times N}$, $q \in \mathbb{R}^P$ and a closed convex cone $C \subseteq \mathbb{R}^N$ be given. Suppose that HC is closed. Then, either the primal system

$$Hv = q, \ v \in C$$

has a solution $v \in \mathbb{R}^P$, but never both.

A consequence of this lemma is that, if the implication

$$w^T Hv \geq 0 \text{ for all } v \in C \implies w = 0 \quad \text{(B23)}$$

holds, then the primal system has a solution for all q. It is assumed that the (B22) is the primal system. Note that, $\Lambda(\Delta_l)$ is nonsingular for all sufficiently large l, as it converges to the identity matrix as l tends to infinity. As $\mathcal{Y}_i$ is polyhedral cone $\Lambda(\Delta_l)exp(A\tau)M^i\mathcal{Y}_i$ must be polyhedral. Hence, closed for all sufficiently large l and for all τ. Therefore, in view of (B23), in order to show that for an integer l, (B22) has a solution for arbitrary $x(T)$. Indeed, the relation

$$z^T\Lambda(\Delta_l)\sum_{j=1}^{l-1}\sum_{i=1}^{r}exp[A(T-(jr+i-1)\Delta_l)]M^i\eta^{i,j}\geq 0 \tag{B24}$$

for all $\eta^{i,j}\in\mathcal{Y}_i$, $i=1,2,...,r$ and $j=0,1,...,l-1$ can be only satisfied by $z=0$. To see this, suppose that for each integer l, there exists $z_l\neq 0$ such that

$$z_l^T\Lambda(\Delta_l)\sum_{j=1}^{l-1}\sum_{i=1}^{r}exp[A(T-(jr+i-1)\Delta_l)]M^i\eta^{i,j}\geq 0 \tag{B25}$$

for all $\eta^{i,j}\in\mathcal{Y}_i$, $i=1,2,...,r$ and $j=0,1,...,l-1$. Also, $||z_l||$can be taken as 1. In view of the Bolzano-Weierstrass theorem, it can be assumed, without the loss of generality, that the sequence $\{z_l\}$ converges, say to z_∞, as l tends to infinity. Now, fix i and $t\in[0,T]$. It can be verified that there exists a subsequence $\{l_k\}\subset\mathbb{N}$ such that the inequality $(jl_kr+i-1)\Delta l_k\leq T-t\leq(jl_kr+i)\Delta l_k$ holds for some $jl_k\in\{1,2,...,l_k\}$. Note that Θ^Δ converges to a Dirac impulse as Δ tends to zero. Hence, $\Lambda(\Delta_l)$ converges to the identity matrix as l tends to infinity. Let $l=l_k$ and $j=j_{l_k}$ in (B25). By taking the limit, one gets

$$z_\infty^T exp(At)M^i\eta\geq 0 \tag{B26}$$

for all $t\in[0,T]$, $\eta\in\mathcal{Y}_i$ and $i=1,2,...,r$. Consequently, one has the following inequality

$$z_\infty^T exp(At)f(u)\geq 0 \tag{B27}$$

for all $t\in[0,T]$ and $u\in\mathcal{U}$. Therefore, z_∞ must be zero due to (B18). This is contradiction.

($6 \implies 5$) Suppose that 6 holds. It follows from theorem B.2 such that

a) the pair $(A[M^1M^2...M^r])$ is controllable,

b) the implication $\lambda\in\mathbb{R}$, $z\in\mathbb{R}^n$,

$z^TA=\lambda z^T,(M^i)^Tz\in\mathcal{Y}_i^*$ for all $i=1,2,...,r \implies z=0$

holds. Now, the following lemma shoul be applied.

Lemma B.5: Let $G \in \mathbb{R}^{N \times N}$ and $H \in \mathbb{R}^{N \times M}$ be given. Also let $\mathcal{W} \subseteq R^M$ be such that its convex hull nonempty interior in R^M. Suppose that th pair (G, H) is controllable and the implication

$$\lambda \in \mathbb{R},\ z \in \mathbb{R}^n,\ z^T G = \lambda z^T,\ H^T z \in \mathcal{W}^* \implies z = 0 \tag{B28}$$

holds. Then, also the implication

$$z^T exp(Gt) H v \geq 0 \text{ for all } t \geq 0 \text{ and } v \in \mathcal{W} \implies z = 0 \tag{B29}$$

holds. Take $G = A$, $H = [M^1 M^2 ... M^r]$ and $\mathcal{W} = \mathcal{Y}_1 \times \mathcal{Y}_2 ... \mathcal{Y}_r$. It follows from (a) and (b) that the hypothesis of the above lemma is satisfied. Hence, the implication

$$z^T exp(A)[M^1\ M^2\ ...\ M^r] v \geq 0 \text{ for all } t \geq 0 \text{ and } v \in \mathcal{W} = \mathcal{Y}_1 \times \mathcal{Y}_2 ... \mathcal{Y}_r \implies z = 0$$

holds. Then, the implication

$$z^T exp(At) f(u) \geq 0 \text{ for all } t \geq 0 \text{ and } u \in \mathcal{U} \implies z = 0 \tag{B30}$$

holds.

($6 \implies 1$) Note that if the statement 6 holds for the system (B1), does it for the time reversed version of the system (B1). Then, the statement 4 holds for (B1). This means controllability.

($4 \implies 2$) If the statement 4 holds, then one steer any initial state first to zero then, to any final state in view of Lemma B.3.

Lemma B.6: Consider the (B1) such that $p = m$ and the transfer matrix $D + C(sI - A)^{-1}$ is invertible as a rational matrix. Then, the following statements are equivalent.

1) The CLS (B1) is completely controllable

2) The push-pull system

$$\dot{x}_1 = A_{11} x_1 + g(y) \tag{B31}$$

is completely controllable.

Proof of Lemma B.6: For the proof of the lemma, some auxiliary results are needed. First of all, following lemma which guarantees the existence of smooth functions lying in a given polyhedral cone, is needed.

Lemma B.7: Let $\mathcal{Y} \subseteq \mathbb{R}^p$ be a polyhedral cone and y be a $\mathbb{C}^\infty$ function, such that $y(t) \in \mathcal{Y}$ for all $t \in [0, \epsilon]$, where $0 < \epsilon < 1$. Then, there exists a $\mathbb{C}^\infty$ function, $\bar{y}$ such that:

a) $\bar{y}(t) = y(t)$, for all $t \in [0, \epsilon]$;

b) $\bar{y}^{(k)}(1) = 0$, for all $k = 0, 1, ...$; and

c) $\bar{y}(t) \in \mathcal{Y}$, for all $t \in [0, 1]$.

Proof: Let only prove the case $p = 1$ and $\mathcal{Y} = \mathbb{R}_+$. The rest of the proof is generalization to the higher dimensional case. Let $\bar{\bar{y}}(t)$ be a $\mathbb{C}^\infty$ function, such that $\bar{\bar{y}}(t) = 1$ for $t \leq \epsilon/4$, $\bar{\bar{y}}(t) > 1$ for $\epsilon/4 \leq t \leq 3\epsilon/4$ and $\bar{\bar{y}}(t) = 0$ for $3\epsilon/4 \leq t$. Such a function can be derived from so-called bump function in [24, Lemma 1.2.3] by integration and scaling. So, the product of y and $\bar{\bar{y}}(t)$ proves the claim. The other auxiliary results are related with the existence of the solutions of conewise linear systems(CLS) with certain proporties[11, Lemmas 2.4 and 3.3].

Proposition 3.5: Consider the CLS (1) with $u = 0$. Then, for each initial state x_0, there exists an index set i and a positive number ϵ such that $y(t) \in \mathcal{Y}_i$ for all $t \in [0, \epsilon]$.

Now we can turn the proof. Obviously, statement 1 implies to statement 2. For the rest, it is enough to show that the system (A11) is controllable, if statement 2 holds.

Note that,

$\mathcal{V}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \{0\}$ and $\mathcal{T}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = \mathbb{R}^{n_2}$ for all $i = 1, 2, ..., r$ due to (A23-24) and Proposition 2.1, transfer function $D + C_2(sI - A_{22} + M_2^i C_2)^{-1}(B_2 + M_2^i D)$ has a polynomial inverse for all $i = 1, 2, ..., r$.

Take any $x_{10}, x_{1f} \in \mathbb{R}^{n_1}$ and $x_{20}, x_{2f} \in \mathbb{R}^{n_2}$. Consider the system (A11) and apply $v = 0$. According to Proposition 3.5 in [4] it can be found an index i_0 and an arbitrarily small positive number ϵ such that $y(t) \in \mathcal{Y}_{i0}$ for all $t \in [0, \epsilon]$. By applying Lemma B.7, one can get a $\mathbb{C}^\infty$ function y_{in} such that:

a) $y_{in} = y(t)$, for all $t \in [0, \epsilon]$;

b) $y_{in}^{(k)}(1) = 0$, for all $k = 0, 1, ...$; and

c) $y_{in}(t) \in \mathcal{Y}_{i0}$, for all $t \in [0, 1]$.

Then by applying Proposition 2.4 and 2.5 in [4] to the system $\Sigma(A_{22} + M_2^{i0}C_2, B_2 + M_2^{i0}D, C_2, D)$, one can find an input v_{in} such that the output y of (A11) is identically y_{in} and the state x_2 satisfies $x_2(0) = x_{20}$. Note that the input v_{in} should be zero on the interval $[0, \epsilon]$ by the construction of y_{in} and invertibility. Moreover, $x_{21} = 0$ due to (b) and Proposition 2.5 in [4]. Therefore, the input v_{in} steers the state $col(x_{10}, x_{20}))$ to $col(x'_{10}, 0))$ where $x'_{10} := x_1(1)$ [4]. Also, it is can come up with an input v_{out}, such that it steers a state $col(x'_{1f}, 0))$ to $col(x_{1f}, x_{2f}))$. Now, it is needed to show that the state $col(x'_{10}, 0))$ can be steered to $col(x'_{1f}, 0))$. To see this, Theorem B.1 have to be applied. This theorem gives a positive number $T > 0$ and a $\mathbb{C}^{\infty}$ function $y = y_{mid}$, such that the solution x_1 of (A11) satisfies $x_1(0) = x'_{10}$ and $x_1(T) = x'_{1f}$. According to Lemma B.3, function y_{mid} can be chosen such that $y_{mid}^{(j)}(0) = y_{mid}^{(j)}(T) = 0$ for all $j = 0, 1, ...$ Moreover, one can find a finite number of points, say $0 = t_0 < t_1 < ... < t_Q = T$, such that $y_{mid}(t) \in \mathcal{Y}_{i_q}$ whenever $t \in [t_q, t_{q+1}]$.

Since the transfer function $D + C_2(sI - A_{22} + M_2^i C_2)^{-1}(B_2 + M_2^i D)$ has a polynomial inverse for all $i = 1, 2, ..., r$, repeated application of Proposition 2.4 in [4] to the system $\Sigma(A_{22} + M_2^{i0}C_2, B_2 + M_2^{i0}D, C_2, D)$ yields an input v_{mid} and a state trajectory x_2 are satisfied for $y = y_{mid}$. Also, $x_2(0) = x_2(T) = 0$ due to Proposition 2.5 in [4]. Consequently, the concatenation of v_{in}, v_{mid} and v_{out} steers the state $col(x_{10}, x_{20})$ to the state $col(x_{1f}, x_{2f})$.

By combining the previous lemma with Theorem B.1, main result of controllability problem is as follows.

Theorem B.8: Consider the (B1) such that $p = m$ and the transfer matrix $D + C(sI - A)^{-1}$ is invertible as a rational matrix. Then, the CLS is completely controllable if and only if:

1) the relation

$$\sum_{i=1}^{r}\langle A+M^iC \mid im(B+M^iD)\rangle = \mathbb{R}^n \tag{B32}$$

is satisfied. And,

2) the implication

$$\begin{bmatrix} z^T & w_i^T \end{bmatrix}\begin{bmatrix} A+M^iC-\lambda I & B+M^iD \\ C & D \end{bmatrix} = 0 \tag{B33}$$

where $\lambda \in \mathbb{R}$, $z \in \mathbb{R}^n$, $w_i \in \mathbb{R}^m$, $w_i \in \mathcal{Y}_i^*$ for all $i = 1, 2, ..., r$, then $z = 0$ holds.

Proof of Theorem B.8: If it is considered Lemma1 in [31] and Theorem B.1, it is enough to show that the conrollability of the pair

$$(A_{11}, [L_1 + M_1^1 \;\; L_1 + M_1^2 \; ... \; L_1 + M_1^r]) \tag{B34}$$

with respect to $\mathcal{Y}_1 \times \mathcal{Y}_2 ... \times \mathcal{Y}_r$ is equivalent to the conditions presented in Theorem B.8. Note that the former is equivalent to the following conditions:

a) the pair $(A_{11}, [L_1 + M_1^1 \;\; L_1 + M_1^2 \; ... \; L_1 + M_1^r])$ is controllable and

b) the implication

$$z^T A_{11} = \lambda z^T, \lambda \in \mathcal{R}], (L_1 + M_1^i)^T z \in \mathcal{Y}_i^* \text{ for all } i \implies z = 0$$

holds.

It is aimed to prove the equivalance of (a) to 1 and of (b) to 2.

7) $a \Leftrightarrow 1$

Note that ,

$\langle A + M_iC \mid im(B + M_iD)\rangle = \langle (A - BK) + M_i(C - DK) \mid im(B + M_iD)\rangle$

for any K due to Proposition 2.1 in [10]. Take $K \in \mathcal{K}(\mathcal{V}^*)$. Note that the condition in 1of Theorem B.8 is invariant under state space transformations. Therefore, one can take

$$(A - BK) + M_i(C - DK) = \begin{bmatrix} A_{11} & (L_1 + M_1^i)C_2 \\ 0 & A_22 + M_2^iC_2 \end{bmatrix}, \tag{B35}$$

$$B + M_iD = \begin{bmatrix} (L_1 + M_1^i)D \\ B_2 + M_2^iD \end{bmatrix}. \tag{B36}$$

Let $\mathcal{R}_i$ denote $\langle (A-BK)+M_i(C-DK) \mid im(B+M_iD)\rangle$. Note that $\mathcal{R}_i$ is an input-containing conditioned invariant subspace of the system $\Sigma(A,B,C,D)$. Hence, T^*, the smallest of the input-containing conditioned invariant subspaces must be contained in $\mathcal{R}_i$. In the present case, following inclusions hold

$$im \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix} \subseteq \mathcal{R}_i. \tag{B37}$$

Now, it is needed the following lemma.

Lemma B.9: Let $\mathcal{O}$, $\mathcal{P}$ and $\mathcal{Q}$ be vector spaces such that $\mathcal{O}=\mathcal{P}\oplus\mathcal{Q}$. Also let $\pi_{\mathcal{P}}(\pi_{\mathcal{Q}})$: $\mathcal{O}\rightarrow\mathcal{Q}$ satisfy the following properties:

a)$\mathcal{P}$ is $\mathcal{F}$-invariant,

b) $\pi_{\mathcal{P}}F\pi_{\mathcal{P}}=\tilde{F}$,

c)$\mathcal{Q}\subseteq\langle F \mid imG\rangle$.

Then, $\langle \tilde{F} \mid (\pi_{\mathcal{P}}F\pi_{\mathcal{Q}})\langle + im(\pi_{\mathcal{P}}G) \subseteq \langle F \mid imG\rangle$.

Proof: Note that

$$\tilde{F}\langle F \mid im\rangle = \pi_{\mathcal{P}}F\pi_{\mathcal{P}}\langle F \mid imG\rangle \tag{B38}$$

$$=\pi_{\mathcal{P}}F(\mathcal{P}\cap\langle F \mid imG\rangle) \tag{B39}$$

$$\subseteq \pi_{\mathcal{P}}(\mathcal{P}\cap\langle F \mid imG\rangle) \tag{B40}$$

$$\subseteq (\mathcal{P}\cap\langle F \mid imG\rangle) \subseteq \langle F \mid imG\rangle. \tag{B41}$$

This shows that the subspace $\langle F \mid imG\rangle$ is $\tilde{F}$-invariant. Note also that

$$im\,\pi_{\mathcal{P}}F\pi_{\mathcal{Q}}=\pi_{\mathcal{P}}F\mathcal{Q}\subseteq\pi_{\mathcal{P}}F\langle F \mid imG\rangle\subseteq\pi_{\mathcal{P}}\langle F \mid imG\rangle\subseteq\langle F \mid imG\rangle$$

and

$$im\,\pi_{\mathcal{P}}G\subseteq im\,G\subseteq\langle F \mid imG\rangle. \tag{B42}$$

Last two inclusions show that the subspace $\langle F \mid imG\rangle$ contains $im\,(\pi_{\mathcal{P}}F\pi_{\mathcal{Q}})+im\,(\pi_{\mathcal{P}})G$. Since $\langle \tilde{F} \mid (im\,\pi_{\mathcal{P}}F\pi_{\mathcal{Q}})\rangle + im\,(\pi_{\mathcal{P}})G$ is the smallest $\tilde{F}$-invariant subspace that contains $im\,(\pi_{\mathcal{P}}F\pi_{\mathcal{Q}})+im\,(\pi_{\mathcal{P}})G$, the inclusion

$\langle \tilde{F} \mid (\pi_{\mathcal{P}}F\pi_{\mathcal{Q}})\rangle + im(\pi_{\mathcal{P}}G) \subseteq \langle F \mid imG\rangle$

holds.

Now, Let take

$$\mathcal{O} = \mathbb{R}^n, \mathcal{P} = im\begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}, \mathcal{O} = im\begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix}, \mathcal{S} = \mathbb{R}^m \quad \text{(B43)}$$

$$F^i = (A - BK) + M^i(C - DK) \quad G^i = B + M^i D \quad \text{(B44)}$$

$$\tilde{F} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that

$\pi_{\mathcal{P}} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}$ and $\pi_{\mathcal{Q}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}$.

Then, one has

$$\pi_{\mathcal{P}} F^i \pi_{\mathcal{P}} = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{(B45)}$$

$$\pi_{\mathcal{P}} F^i \pi_{\mathcal{Q}} = \begin{bmatrix} 0 & (L_1 + M_1^i)C_2 \\ 0 & 0 \end{bmatrix}, \quad \text{(B46)}$$

$$\pi_{\mathcal{P}} G^i = \begin{bmatrix} (L_1 + M_1^i)D \\ 0 \end{bmatrix}. \quad \text{(B47)}$$

Note that, the first hypothesis of Lemma B.9 satisfies due to (B35) and (B36). It follows from (B43) and (B44) that the second one is also satisfied. Finally, the third follows from (B37). Then, Lemma B.9 results in

$$\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid im\begin{bmatrix} (L_1 + M_1^i)C_2 & 0 \\ (L_1 + M_1^i)D & 0 \end{bmatrix} \rangle \subseteq \mathcal{R}_i. \quad \text{(B48)}$$

By the invertibility hypothesis, the matrix $\begin{bmatrix} C_2 & D \end{bmatrix}$ must be of full row rank. Then, the previous inclusion can be written as

$$\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid im\begin{bmatrix} (L_1 + M_1^i) \\ 0 \end{bmatrix} \rangle \subseteq \mathcal{R}_i\,. \quad \text{(B49)}$$

Summing both sides over i, one gets

$$\sum_{i=1}^{r} \langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid im\begin{bmatrix} (L_1 + M_1^i) \\ 0 \end{bmatrix} \rangle \subseteq \sum_{i=1}^{r} \mathcal{R}_i\,. \quad \text{(B50)}$$

This implies that

$$\langle \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} \mid im \begin{bmatrix} L_1 + M_1^1 & L_1 + M_1^2 & \dots & L_1 + M_1^r \\ 0 & 0 & \dots & 0 \end{bmatrix} \rangle \subseteq \sum_{i=1}^{r} \mathcal{R}_i \, . \quad \text{(B51)}$$

Together with (B37), the previous inclusion implies that the implication "$a \Leftrightarrow 1$" holds. For the reverse, suppose that 1 holds but (a) does not. Then, there exists a nonzero vector z and $\lambda \in \mathbb{C}$ such that

$$z^*[\lambda I A_{11} \; L_1 + M_1^1 \; L_1 + M_1^2 \; ...L_1 + M_1^r] = 0. \quad \text{(B52)}$$

It can be verified that the real part of z, say w, belongs to $\mathcal{R}_i^{\perp}$ for all i. Thus, w belongs to $\bigcap_{i=1}^{r} \mathcal{R}_i^{\perp} = (\sum_{i=1}^{r} \mathcal{R}_i)^{\perp}$ .This contradicts 1.

8) $b \Leftrightarrow 2$: Statement 2 is invariant under state space transformations. So it is enough to prove the statement for the system (A11). Let $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^{n_1}$, $z \in \mathbb{R}^{n_2}$ and $w \in \mathbb{R}^{m}$ be such that the following product is equal to zero:

$$\begin{bmatrix} v \\ z \\ w \end{bmatrix}^T = \begin{bmatrix} A_{11} - \lambda I & (L_1 + M_1^i)C_2 & (L_1 + M_1^i)D \\ 0 & A_{22} + M_2^i C_2 - \lambda I & B_2 + (L_1 + M_2^i)D \\ 0 & C_2 & D \end{bmatrix}. \quad \text{(B53)}$$

This would result in

$$v^T A_{11} = \lambda v^T \begin{bmatrix} z \\ w_i + (L_1 + M_1^i)v \end{bmatrix}^T \begin{bmatrix} A_{22} + M_2^i C_2 - \lambda I & B_2 + (L_1 + M_2^i)D \\ C_2 & D \end{bmatrix} = 0.$$

Note that $\mathcal{V}^*(A_{22} + M_2^i C_2, B_2 + M_2^i D, C_2, D) = 0$ for all i. Then, it follows from Proposition 2.3 in [4] that $z = 0$ and $w_i^T = v^T(L_1 + M_1^i)$. This implies that (b) is equivalent to 2.

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PUBLISHMENTS

Papers

1. Y. Eren, J. Shen and M.K. Camlibel, Quadratic stability and stabilization of bimodal piecewise linear systems, Automatica, 2014.

2. I.B. Kucukdemiral, L. Ucun, Y. Eren, H. Gorgun and G. Cansever, Ellipsoid Based L_2 Controller Design for LPV Systems with Saturating Actuators, Turkish Journal of Electrical Engineering and Computer Science, 2010.

3. Y. Eren, O. Erdinc, H. Gorgun, M. Uzunoglu, B. Vural, Fuzzy Logic Based Supervisory Controller for a FC/UC Hybrid Vehicular Power System, Int. Journal of Hydrogen Energy, 2009.

Conference Papers

1. O.Erdinc, Y.Eren, H.Gorgun, M.Uzunoglu, Dynamic Modeling and Intelligent Control of a Hybrid PV/FC/Battery Power System for Stand Alone Applications, 5th International Ege Energy Symposium and Exhibition, 2011.

2. I.B. Kucukdemiral, L. Ucun, Y. Eren, H. Gorgun, A. Delibasi and G. Cansever, L2 Controller Design for LPV Systems with Saturating , IFAC Mechatronics Conference, 2010.