STOCHASTIC ANALYSIS WITH APPLICATIONS TO QUANTUM MECHANICS

by

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality as a thesis for the degree of Master of Science.

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ABSTRACT

STOCHASTIC ANALYSIS

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In this thesis I describe a kind theory of stochastic analysis in Hilbert space, quantum states, and Jaynes - Cumming Model under random phase telegraph noise. I apply the stochastic process in quantum systems and explore entanglement properties of the atoms-fields interaction under random phase telegraph noise.

Keywords: Entanglement, Jaynes-Cumming Model, Random Phase Telegraph Noise.

ÖZET

KUANTUM MEKANİĞİNDEKİ UYGULAMALARI İLE

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Bu tezde, Hilbert uzayda stokastik analizin, kuantum durumlarının ve Jaynes - Cumming Modelin rasgele faz telegraf gürültü altında bir çeşit teorisini tanımlıyorum. Stokastik süreci kuantum sistemlerine uyguluyorum ve rasgele faz telegraf gürültü altında atom-alan etkileşimlerinin dolaşıklık özelliklerini araştırıyorum.

Anahtar Kelimeler: Dolaşıklık, Jaynes-Cumming Model, Rasgele Faz Telegraf Gürültü.

To My Family

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CHAPTER 1

INTRODUCTION

In the mathematically formulation of quantum mechanics, developed by Paul Dirac[1] and John von Neumann[2], the possible states of a quantum mechanical system are represented by unit vectors (called ”state vectors”) residing in a complex separable Hilbert space (variously called the ”state space” or the ”associated Hilbert space” of the system) well defined up to a complex number of norm 1 (the phase factor). In other words, the possible states are points in the projectivization of a Hilbert space, usually called the complex projective space. The exact nature of this Hilbert space is dependent on the system; for example, the state space for position and momentum states is the space of square-integrable functions, while the state space for the spin of a single proton is just the product of two complex planes. Each observable is represented by a maximally-Hermitian (precisely: by a self-adjoint) linear operator acting on the state space. Each eigenstate of an observable corresponds to an eigenvector of the operator, and the associated eigenvalue corresponds to the value of the observable in that eigenstate. If the operator's spectrum is discrete, the observable can only attain those discrete eigenvalues.

The time evolution of a quantum state is described by the Schrödinger equation, in which the Hamiltonian, the operator corresponding to the total energy of the system, generates time evolution.

The inner product between two state vectors is a complex number known as a probability amplitude. During a measurement, the probability that a system collapses from a given initial state to a particular eigenstate is given by the square of the absolute value of the probability amplitudes between the initial and final states. The possible results of a measurement are the eigenvalues of the operator - which explains the choice of Hermitian operators, for which all the eigenvalues are real. We can find the probability distribution of an observable in a given state by computing the spectral decomposition of the corresponding operator. Heisenberg's uncertainty principle is represented by the statement that the operators corresponding to certain observables do not commute.

The Schrödinger equation acts on the entire probability amplitude, not merely its absolute value. Whereas the absolute value of the probability amplitude encodes information about probabilities, its phase encodes information about the interference between quantum states. This gives rise to the wave-like behavior of quantum states.

It turns out that analytic solutions of Schrödinger's equation are only available for a small number of model Hamiltonians, of which the quantum harmonic oscillator, the particle in a box, the hydrogen molecular ion and the hydrogen atom are the most important representatives. Even the helium atom, which contains just one more electron than hydrogen, defies all attempts at a fully analytic treatment. There exist several techniques for generating approximate solutions. For instance, in the method known as perturbation theory one uses the analytic results for a simple quantum mechanical model to generate results for a more complicated model related to the simple model by, for example, the addition of a weak potential energy. Another method is the "semi-classical equation of motion" approach, which applies to systems for which quantum mechanics produces weak deviations from classical behavior. The deviations can be calculated based on the classical motion. This approach is important for the field of quantum chaos.

An alternative formulation of quantum mechanics is Feynman's path integral formulation, in which a quantum-mechanical amplitude is considered as a sum over histories between initial and final states; this is the quantum- mechanical counterpart of action principles in classical mechanics.

Quantum entanglement, also called the quantum non-local connection, is a possible property of a quantum mechanical state of a system of two or more objects in which the quantum states of the constituting objects are linked together so that one object can no longer be adequately described without full mention of its counterpart - even if the individual objects are spatially separated in a spacelike manner. This interconnection leads to non-classical correlations between observable physical properties of remote systems, often referred to as nonlocal correlations.

For example, quantum mechanics holds that observables such as spin are indeterminate until such time as some physical intervention is made to measure the spin of the object in question. In the singlet state of two spins it is equally likely that any given particle will be observed to be spin-up as that it will be spin-down. Measuring any number of particles will result in an unpredictable series of measures that will tend more and more closely to half up and half down. However, if this experiment is done with entangled particles the results are quite different. When two members of an entangled pair are measured, one will always be spin-up and the other will be spin-down. The distance between the two particles is irrelevant.

Theories involving 'hidden variables' have been proposed in order to explain this result; these hidden variables account for the spin of each particle, and are determined when the entangled pair is created. It may appear then that the hidden variables must be in communication no matter how far apart the particles are, that the hidden variable describing one particle must be able to change instantly when the other is measured. If the hidden variables stop interacting when they are far apart, the statistics of multiple measurements must obey an inequality (called Bell's inequality), which is, however, violated - both by quantum mechanical theory and in experiments.

When pairs of particles are generated by the decay of other particles, naturally or through induced collision, these pairs may be termed "entangled", in that such pairs often necessarily have linked and opposite qualities, i.e. of spin or charge. The assumption that measurement in effect "creates" the state of the measured quality goes back to the arguments of, among others: Schrödinger, and Einstein, Podolsky, and Rosen(EPR paradox[3]) concerning Heisenberg's uncertainty principle and its relation to observation (Copenhagen interpretation[4]). The analysis of entangled particles by means of Bell's theorem, can lead to an impression of non-locality (that is, that there exists a connection between the members of such a pair that defies both classical and relativistic concepts of space and time). This is reasonable if it is assumed that each particle departs the scene of the pair's creation in an ambiguous state (as per a possible interpretation of Heisenberg). In such a case, for a given measurement either outcome remains a possibility; only measurement itself would precipitate a distinct value. On the other hand, if each particle departs the scene of its "entangled creation" with properties that would unambiguously determine the value of the quality to be subsequently measured, then a postulated instantaneous transmission of information across space and time would not be required to account for the result. The Bohm interpretation postulates that a guide wave exists connecting what are perceived as individual particles such that the supposed hidden variables are actually the particles themselves existing as functions of that wave.

Observation of wavefunction collapse can lead to the impression that measurements performed on one system instantaneously influence other systems entangled with the measured system, even when far apart. Yet another interpretation of this phenomenon is that quantum entanglement does not necessarily enable the transmission of classical information faster than the speed of light because a classical information channel is required to complete the process.

The Jaynes-Cummings model (JCM) is a theoretical model in quantum optics. It describes the system of a two-level atom interacting with a quantized mode of an optical cavity, with or without the presence of light. The JCM is of great interest in atomic physics and quantum optics both experimentally and theoretically.

This model was originally proposed in 1963 by Edwin Jaynes and Fred Cummings in order to study the relationship between the quantum theory of radiation and the semi-classical theory in describing the phenomenon of spontaneous emission[5].

In the earlier semi-classical theory of field-atom interaction, only the atom is quantized and the field is treated as a definite function of time rather than as an operator. The semi-classical theory can explain many phenomena that are observed in modern optics, for example the existence of Rabi cycles in atomic excitation probabilities for radiation fields with sharply defined energy (narrow bandwidth). The JCM serves to find out how quantization of the radiation field affects the predictions for the evolution of the state of a two-level system in comparison with semi-classical theory of light-atom interaction. It was later discovered that the revival of the atomic population inversion after its collapse is a direct consequence of discreteness of field states (photons [6, 7]). This is a pure quantum effect that can be described by the JCM but not with the semi-classical theory.

Twenty four years later, a beautiful demonstration of quantum collapse and revival was observed in a one-atom maser by Rempe, Walther, and Klein [8]. Before that time research groups were able to build experimental setups capable of enhancing the coupling of an atom with a single field mode, simultaneously suppressing other modes. Experimentally, the quality factor of the cavity must be high enough to consider the dynamics of the system as equivalent to the dynamics of a single mode field. With the advent of one-atom masers it was possible to study the interaction of a single atom (usually a Rydberg atom) with a single resonant mode of the electromagnetic field in a cavity from an experimental point of view [9, 10] and study different aspects of the JCM.

In order to more precisely describe the interaction between an atom and a laser field, the model is generalized in different ways. Some of the generalizations are applying initial conditions[11], consideration of dissipation and damping in the model [12, 13], consideration of multilevel atoms and multiple atoms [14], and multi-mode description of the field [15].

It was also discovered that during the quiescent intervals of collapsed Rabi oscillations the atom and field exist in a macroscopic superposition state (a Schrödinger cat). This discovery offers the opportunity to use the JCM to elucidate the basic properties of quantum correlation (entanglement) [16]. In another work the JCM is employed to model transfer of quantum information [17].

Another important generalization of the JCM is the incorporation of the noise into the model as given by Joshi *et.al.*[18]. Kayhan *et.al.*[19, 20, 21, 22]. used this generalized model to calculate the entanglement dynamics in the atom-field interaction.

In this thesis we first review the works in Ref [20, 21, 22] and go further to calculate the entanglement dynamics in the two atoms-two fields system. We find that there is no three partite entanglement of the time evolution system.

We should note that all calculations in the thesis are verified in *MATHEMATICA*.

The organization of the thesis is as follows.

Chapter 2 contains a general information about stochastic analysis and quantum states.

In Chapter 3 we review one atom-one field interaction under random phase telegraph noise.

In Chapter 4 we review entanglement of one atom-two fields interactions under random phase telegraph noise.

In Chapter 5 we investigate two atoms-two fields interaction under random phase telegraph noise as a new work.

In Chapter of conclusion, we deduce that in atoms-fields interactions the effects of random phase telegraph noise is different.

CHAPTER 2

BASIC CONCEPTS

2.1 Stochastic Analysis

In this chapter, we present the mathematical theory of stochastic analysis[23, 24, 25, 26, 27, 28].

A stochastic process, or sometimes random process, is the counterpart to a deterministic process (or deterministic system) in probability theory. Instead of dealing with only one possible 'reality' of how the process might evolve under time (as is the case, for example, for solutions of an ordinary differential equation), in a stochastic or random process there is some indeterminacy in its future evolution described by probability distributions. This means that even if the initial condition (or starting point) is known, there are many possibilities the process might go to, but some paths are more probable and others less.

2.1.1 Probability Spaces and Random Variables

In this section we recall the basic vocabulary and results of probability theory. A probability space associated with a random experiment is a triple $(\Omega; \mathtt{F}; \mathtt{P})$ where:

1. Ω is the set of all possible outcomes of the random experiment, and it is called the sample space.

2. $\mathtt{F}$ is a family of subsets of Ω which has the structure of a σ-field:

   - $\theta \epsilon \mathtt{F}$

   - If $\mathtt{A} \in \mathtt{F}$,then its complement $\mathtt{A}^c$ also belong to $\mathtt{F}$

   - $\mathtt{A}_1, \mathtt{A}_2, ... \in \mathtt{F} \Rightarrow \bigcup_{i=1}^{\infty} \mathtt{A}_i \in \mathtt{F}$

3. P is a function with associates $\mathtt{P(A)}$ to each set $\mathtt{A} \in \mathtt{F}$ $\left(that\ is, \mathtt{A}_i \cap \mathtt{A}_j = \emptyset \quad if \quad i \neq j\right)$,

$$\mathrm{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathtt{P}\left(\mathtt{A}_i\right)$$

The elements of the $\sigma-$field $\mathtt{F}$ are called events and the mapping P is called a probability measure. In this way we have the following interpretation of this model:

$$\mathtt{P}\left(\mathtt{A}\right) ='' \ probability\ that\ the\ event\ \mathtt{F}\ occurs''$$

The set $\emptyset$ is called the empty event and it has probability zero. Indeed, the additivity property (3) implies

$$\mathtt{P}\left(\emptyset\right) + \mathtt{P}\left(\emptyset\right) + \dots\dots\dots = \mathtt{P}\left(\emptyset\right).$$

The set Ω is also called the certain set and by property (3) it has probability one. Usually, there will be other events $\mathtt{A} \subset \Omega$ such that $\mathtt{P}\left(\mathtt{A}\right) = 1$, then its customary to say that the statement is true almost surely, or that the statement holds for almost all $\omega \in \Omega$.

The axioms 1, 2 and 3 lead to the following basic rules of the probability calculus:

$$\begin{aligned} \mathtt{P}\left(\mathtt{A} \cup \mathtt{B}\right) &= \mathtt{P}\left(\mathtt{A}\right) + \mathtt{P}\left(\mathtt{B}\right) \quad if \quad \mathtt{A} \cap \mathtt{B} = \emptyset \\ \mathtt{P}\left(\mathtt{B}\right) &= 1 - \mathtt{P}\left(\mathtt{A}\right) \\ \mathtt{A} &\subset \mathtt{B} \Longrightarrow \mathtt{P}\left(\mathtt{A}\right) \leq \mathtt{P}\left(\mathtt{B}\right) \end{aligned}$$

2.1.2 Probability Distributions of Random Variables

We have two types of random variable *discrete* and *continuous*. All random variables (discrete or continuous) have a cumulative distrubition function, which is the characterizetion of the probability distribution $F_x(x) : \mathbb{R} \longrightarrow [0, 1]$ defined by

$$F_x(x) = P(X \leq x) = P_x((-\infty, x]) \tag{2.1}$$

which satisfies

- F_x is non-decreasing,
- Right continuous and

$$\lim_{x \to -\infty} F_x(x) = 0$$

$$\lim_{x \to +\infty} F_x(x) = 1$$

2.1.3 Discrete probability distribution

A probability distribution called *discrete* if its cumulative distribution function only increases in jumps. More precisely, a probability distribution is discrete if there is a finite or countable set whose probability is 1. It is also sometimes called the *probability function* or the *probability mass function*

$$P(X = x_i) = p(x_i) \tag{2.2}$$

It satisfies the following conditions:

- $0 \leq p(x_i) \leq 1$
- $\sum p(x_i) = 1$

2.1.4 Continuous distribution function

A probability distribution is called *continuous* if its cumulative distribution function is *continuous*, which means that it belongs to a random variable X for which $P[X = x] = 0$ for all x in $\mathbb{R}$.

If we have a absolutely continuous distributions we can characterized this distributions by a probability density function:

$$F_x(x) = P[X \leq x] = \int_{-\infty}^{x} f_x(t)dt \tag{2.3}$$

and if, in addition, the density is continuous, then $F'_x(x) = f_x(x)$.

2.1.5 Expected Value

The *expected value* of a random variable indicates its *average* or *central value*. It is useful summary value (a number) of variable's distribution.
Stating the expected variables with the same expected value can have very different distributions. There are other useful descriptive measures which affect the shape of the distribution, for example variance.
The expected value of a random variable X is symbolised by E(X) or μ.
If X is a discrete random variable whit possible values $x_1, x_2, x_3, ..., x_n$ and $p(x_i)$ denotes $P(X = x_i)$, then the expected value of X is defined by:

$$\mu = E(X) = \sum x_i p(x_i) \tag{2.4}$$

where the elements are summed over all values of the random variable X.

If X is a continuous random variable whit probability density function f(x), then the expected value of X is defined by

$$\mu = E(X) = \int x f(x)dx \tag{2.5}$$

2.1.6 n-Dimensional Random Variable

We say that $X = (X_1, ..., X_n)$ is an n dimensional random vector if its components are random variables. This equivalent to say that X is a random variable whit values in $\mathbb{R}^n$.

The mathematical expectation of an n-dimensional random vector X is, by definition, the vector

$$E(X) = (E(X_1), ..., E(X_n)) \tag{2.6}$$

As in the case of real-valued random variables we introduce the distribution of an n-dimensional random vector X as the probability measure defined on the Borel σ-field of $\mathbb{R}^n$ by

$$P_x(B) = P(X^{-1}(B)) = P(X \in B) \tag{2.7}$$

We will say that a random vector X has a probability density f_x if $f_x(x)$ is nonnegative function on $\mathbb{R}^n$, measurable with respect to the Borel σ-field and such that

$$P\{a_i < X_i < b_i, i = 1, ..., n\} = \int_{a_n}^{b_n} ... \int_{a_1}^{b_1} f_x(x_1, ..., x_n) dx_1...dx_n, \tag{2.8}$$

for all $a_i < b_i$. Notice that

$$\int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} f_x(x_1, ..., x_n) dx_1...dx_n = 1. \tag{2.9}$$

2.1.7 Independent Random Variable

Two events $A, B \in \mathcal{F}$ are said independent provided

$$P(A \cap B) = P(A)P(B). \tag{2.10}$$

Given an arbitrary collection of events $\{A_i, i \in I\}$, we say that the events of collection are independent provided

$$P(A_{i_1} \cap ... \cap A_{i_k}) = P(A_{i_1})...P(A_{i_k}) \tag{2.11}$$

for every finite subset of indexes $\{i_1, ..., i_k\} \subset I$.

Suppose that X, Y are two independent random variables with finite expectation.

Then the product XY has also finite expectation and

$$E(XY) = E(X)E(Y) \tag{2.12}$$

More generally , if $X_1, ..., X_n$ are independent random variables,

$$E[g_1(X_1)...g_n(X_n)] = E[g_1(X_1)]...E[g_n(X_n)], \tag{2.13}$$

where g_i are measurable functions such that $E[|g_i(X_i)|] < \infty$.

2.1.8 Conditional Probability

The conditional probability of an event A given another event B such that $P(B) > 0$ is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{2.14}$$

We see that A and B are independent if and only if $P(A \mid B) = P(A)$. The conditional probability of the event A modified by the additional information that the event B has occurred.

2.1.9 Stochastic Process

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process is a collection

$$\{X_t \mid t \in T\} \tag{2.15}$$

of random variables X_t defined on $(\Omega, \mathcal{F}, P)$ where T is a set, called index set of the process $\{X_t \mid t \in T\}$. T is usually (but not always) a subset of $\mathbb{R}$. X is sometimes known as a random function.

Given any t the possible values of X_t are called the states of the process at t. The set of all states, for all t of a stochastic process is called its state space.

If T is discrete, then the stochastic process is a *discrete-time process*. If T is an interval of $\mathbb{R}$ then $\{X_t \mid t \in T\}$ is a *continuous-time process*. If T can be linearly ordered, then t is also known as the time.

A stochastic process X with state space S can be thought of either of following three ways.

- As a collection of random variables, X_t for each t in the index set T.
- As a function in two variables $t \in T$ and $\omega \in \Omega$

$$X : T \times \Omega \longrightarrow S, (t, \omega) \longmapsto X_t(\omega). \tag{2.16}$$

  The process is said to be measurable, or jointly measurable if it is $\mathcal{B}(T) \otimes \mathcal{F} / \mathcal{B}(S)$ measurable. Here, $\mathcal{B}(T)$ and $\mathcal{B}(S)$ are the Borel σ algebras on T and s respectively.

- In terms of sample paths. Each $\omega \in \Omega$ maps to a function

$$T \longrightarrow S, t \longmapsto X_t(\omega) \tag{2.17}$$

In the following sections we will analysis some of the most important stochastic processes which are Poisson process and Markov process.

2.2 Poisson Process

A Poisson process[29, 30] is the stochastic in which events occur continuously and independently of one another. Examples that are well-modeled as Poisson processes include the telephone calls arriving a switchboard, page view requests to a website, and rainfall.

Let N(t) denote the number of events having occurred before time t. Such a stochastic process $\{N(t), t \geq 0\}$ is called a *counting process*. It takes values on non-negative integers and is non-decreasing in time t. The increment $N(t) - N(s), s < t$, counts the number of events having occurred during the time interval $(s, t]$. It is usually assumed that $N(0) = 0$.

A Poisson process is a continuous-time *counting process* $\{N(t), t \geq t\}$ that possesses the following properties:

- N(0)=0
- *Independent increments* (the number of occurrence counted in disjoint intervals are independent from each other)
- *Stationary increments* (the probability distribution of the number of occurrences counted any time interval only dependents on the length of the interval)
- arrivals don't occur simultaneously

Consequence of the definition include

- The probability distribution of $N(t)$ is a *Poisson distribution*.
  The Poisson distribution is a discrete probability distribution that expresses the probability of a number of events occurring in a fixed period of time if these events occur with a known average rate and independently of the time since the last event.
- The probability distribution of the waiting time until next occurred is an exponential distribution.
  The exponential distributions are a class of continuous probability distributions. They describe the times between events in a Poisson process, i.e. a process in which events occur continuously and independently at a constant average rate.

2.2.1 Homogeneous Poisson Process

A homogeneous Poisson process has a constant rate parameter function $\lambda(t) = \lambda$, such that the number of events in time interval $(t, t + \tau]$ follows a Poisson distribution with parameter $\lambda\tau$. This relation gives us

$$P[(N(t+\tau) - N(t)) = k] = \frac{e^{-\lambda\tau}(\lambda\tau)^k}{k!} \qquad k = 0, 1, 2, ..., \tag{2.18}$$

where $N(t+\tau) - N(t)$ is the number of events in time interval $(t, t+\tau]$, the rate λ is the expected value of "events" or "arrivals" that occur per unit time.

2.2.2 Non-Homogeneous Poisson Process

In general, the rate parameter may change over time; such a process is called a *non-homogeneous Poisson process* or *inhomogeneous Poisson process*. In this case, the generalized rate function is given as $\lambda(t)$. Now the expected number of events between time a and b is

$$\lambda_{a,b} = \int_a^b \lambda(t)dt. \tag{2.19}$$

Thus,the number of arrivals in the time interval $(a, b]$, given as $N(b) - N(a)$, follows a Poisson distribution with associated parameter $\lambda_{a,b}$

$$P[(N(b) - N(a)) = k] = \frac{e^{-\lambda_{a,b}}(\lambda_{a,b})^k}{k!} \qquad k = 0, 1, ... \tag{2.20}$$

A homogeneous Poisson process may be viewed as a special case when $\lambda(t) = \lambda$, a constant rate.

2.2.3 Properties

In its most general form, the only two conditions for a stochastic process to be a Poisson process are:

- *Orderliness*: which roughly means

$$\lim_{\Delta t \longrightarrow 0} P[N(t+\Delta t) - N(t) > 1 \mid N(t+\Delta t) - N(t) \geq 1] = 0 \tag{2.21}$$

which implies that arrivals don't occur simultaneously.

- *Memorylessness*: the number of arrivals occuring in any bounded interval of time t is independent of the number of arrivals occuring before time t.

These seemingly unrestrictive conditions actually impose a great deal of structure in the Poisson process. In particular, they imply that the time between consecutive events (called interarrival times) are independent random variables. For the homogeneous Poisson process, these inter-arrival times are exponentially distributed with parameter λ(mean $1/\lambda$); Also, the memorylessness property shows that the number of events in one time interval is independent from the number of events in an interval that is disjoint from the first interval. This latter property is known as the *independent increments* property of the Poisson process.

2.3 Markov Process

A Markov process ([31] - [37]) is a mathematical model for the random evolution of a *memoryless system*, that is, one for which the likelihood of a given future state, at any given moment, depends only on its present state, and not on any states.
In common description, a stochastic process with the *Markov property*, or memorylessness, is one for which *conditional* on the present state of the system, its future and past are independent.
Often, the term *Markov chain* is used to mean a Markov process which has a discrete (finite or countable) state - space.

2.3.1 Definition

A stochastic process whose state at time t is $X(t)$, for $t > 0$, and whose history of states is given by $x(s)$ for times $s < t$ is a Markov process if

$$P\left[X(t+h) = y \mid X(s) = x(s), \forall s \leq t\right] = P\left[X(t+h) = y \mid X(t) = x(t)\right], \quad \forall h > 0. \tag{2.22}$$

That is, the probability of its having state y at time $t + h$, conditioned on having the particular state $x(t)$ at time t, is equal to the conditional of its having that same state y but conditioned on its value for all previous times before t. This captures the idea that its future state is independent of its past states.

Markov processes are typically termed *(time-) homogeneous* if

$$P\left[X(t+h) = y \mid X(t) = x\right] = P\left[X(h) = y \mid X(0) = x\right], \quad \forall t, h > 0, \tag{2.23}$$

and otherwise are termed *(time-)inhomogeneous (or(time-)nonhomogeneous)*. Homogeneous Markov process, usually being simpler than inhomogeneous ones, form the most important class of Markov processes.

2.3.2 Markovian Representations

In some case, apparently non-Markovian processes may still have Markovian representations, constructed by expanding the concept of the 'current' and 'future' states. For example, let X be a non-Markovian process. Then define a process, such that each state of Y represents a time-interval of states of X, i.e. mathematically

$$Y(t) = \{X(s) : s \in [a(t), b(t)]\}. \tag{2.24}$$

If Y has the Markov property, then it is a *Markovian representation* of X. In this case, X is also called *a second-order Markov process*. *Higher-order Markov processes* are defined analogously.

2.3.3 Markov Chain

A Markov chain is a stochastic process with the Markov property. We talk about it above subsection. At each step the system may change its state from the current state to another state (or remain in the same state) according to a probability distribution. The changes of state are called transitions, and the probabilities associated with various state-changes are called transition probabilities.

Formal Definition

A Markov chain is a sequence of random variables $X_1, X_2, X_3, \ldots$ with the Markov property, namely that, given the present state, the future and past states are independent. Formally,

$$P(X_{n+1} = x \mid X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_{n+1} = x \mid X_n = x_n). \tag{2.25}$$

The possible values of X_i form a countable set S called the state space of the chain.

Properties

The probability of going from i to sate j in n steps is

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \tag{2.26}$$

and the single-step transition is

$$p_{ij} = P(X_1 = j \mid X_0 = i). \tag{2.27}$$

For a time-homogeneous Markov chain:

$$p_{ij}^{(n)} = P(X_{n+1} = j \mid X_k = i) \tag{2.28}$$

and

$$p_{ij} = P(X_{k+1} = j \mid X_k = i) \tag{2.29}$$

so, the $n-$step transition satisfies the Chapman-Kolmogorov equation, that for any k such that $0 < k < n$,

$$p_{ij}^{(n)} = \sum_{r \in S} p_{ir}^{(k)} p_{rj}^{(n-k)} \tag{2.30}$$

where S is the state space of the Markov chain. The marginal distribution $P(X_n = x)$ is the distribution over states at time n. The initial distribution is $P(X_0 = x)$. The evolution of the process through one time step is described by

$$P(X_n = j) = \sum_{r \in S} p_{rj} P(X_{n-1} = r) = \sum_{r \in S} p_{rj}^{(n)} P(X_0 = r). \tag{2.31}$$

in which the (n) is an *index*, not an exponent.

2.4 Random Phase Telegraph noise

Random switching between discrete levels is known as random phase telegraph noise[38, 39, 40]. The telegraph model of a random process $x(t)$ (we will use in following chapters $x(t) = \phi(t)$) rests on the following assumptions.

- The quantity $x(t)$ remains constant except during infinitesimally brief jumps, when it changes to a new constant value.
- The process $x(t)$ is stationary in time and the jumps occur at random.
- The jump process is Markovian: the value of $x(t)$ immediately after a jump depends, at most, only upon the value immediately preceding the jump, not upon any prior history.

In short, the telegraph constitutes a stationary Markov chain.

It is well known that the dynamics of such Markov chain is completely described by two basic functions $c(\alpha; t)$ and $f(\alpha \mid \beta; t)$ governing all possible transitions among the different states α at time t will change its state in the interval $t, t + dt$. The function $f(\alpha \mid \beta; t)$ is the conditional probability that in the interval $t, t + dt$ this change takes the telegraph from state β, given that the transition *ends* in the state α. From the definition of the conditional probability $f(\alpha \mid \beta; t)$ it is clear that for every fixed state α and time t we have

$$\sum_{\beta} f(\alpha \mid \beta; t) = 1. \tag{2.32}$$

We adopt a common simplification and specialize to the case of generalized Poisson processes. The random phase telegraph noise is a Poisson process ; the probability that its state changes n times in an interval of length Δt is given by

$$P_n = e^{-\widetilde{n}}(\widetilde{n})^n/n!, \tag{2.33}$$

where the mean number $\widetilde{n}$ is related to Δt through the dwell time T (i.e., the mean time between interruptions) for the telegraph:

$$\widetilde{n} = \Delta t/T. \tag{2.34}$$

These are special Markov chains for which all $c(\alpha; t)$ are equal to the same constant value, and the functions $f(\alpha \mid \beta; t)$ are time independent

$$c(\alpha; t) = \frac{1}{T}, \tag{2.35}$$

$$f(\alpha \mid \beta; t) = f(\alpha \mid \beta). \tag{2.36}$$

These definitions imply that the number of transitions within a finite interval Δt has a Poisson-like distribution with a mean value equal to $\Delta t/T$. The probability of remaining in a given telegraph state decreases with time as $\exp(- \mid t \mid /T)$.

2.5 Qubit

The simplest nontrivial quantum mechanical system is a two-state system which can be described by a vector in two-dimensional complex Hilbert space. Such a system is called a *qubit*[41]. The favorite qubit of many physicists is the spin of a spin-1/2 particle, for instance an electron. We will use spins of such particles to illustrate entanglement.

If we measure the z-component of the spin, the result is either up or down. This measurement corresponds to the observable operator which we call Z. After the measurement the state of the particle is an eigenstate of the observable. We denote these states by $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$, for spin parallel to and antiparallel to the z-axis. They are mutually orthogonal, and we take them as the unit basis vectors of the spin Hilbert space. In addition to the eigenstate, the particle may be in any superposition of the eigenstates,

$$|\psi\rangle = \alpha \mid \uparrow_z\rangle + \beta \mid \downarrow_z\rangle \tag{2.37}$$

such that $|\alpha|^2 + |\beta|^2 = 1$. When the spin along the z-axis is measured, we obtain +1 whit a probability $|\alpha|^2$ and -1 with probability $|\beta|^2$ (where the spin is in units of $\hbar/2$). After the measurement the state is collapses into the corresponding eigenstate.

As the overall phase of a quantum state does not have any physical significance, only the phase difference between α and β is important, and for convenience we often choose α to be real. In order to also include the property $|\alpha|^2 + |\beta|^2 = 1$, we can write the spin state (2.37) of the particle as

$$|\psi\rangle = \cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle. \tag{2.38}$$

This gives a nice and intuitive picture of the state space of the spin-1/2 particle, and thereby any qubit. All possible states are covered when the parameters are restricted to $0 \leq \phi \leq 2\pi$ and $0 \leq \phi \leq \pi$. Thus, any state corresponds to a point on the unit sphere, whit the azimuthal angle given by ϕ and the polar angle given by θ. Such a sphere is called a Bloch sphere, and is very useful tool for visualising states of a qubit. Note that for $\theta = \pi$, the phase of (2.38) changes as ϕ changes, but the state is still the same. This is reflected on the Bloch sphere, as the azimuthal angle is irrelevant at the poles.

We may choose another set of orthonormal basis vectors. For instance

$$| \uparrow_x\rangle = \frac{1}{\sqrt{2}}(| \uparrow_z\rangle + | \downarrow_z\rangle) \qquad | \downarrow_x\rangle = \frac{1}{\sqrt{2}}(| \uparrow_z\rangle - | \downarrow_z\rangle). \tag{2.39}$$

These are the eigenvectors of the X-operator, and they are the resulting states if we measure the spin along the x-axis. Likewise, we may measure the spin along any axis, and find a corresponding set of eigenvectors. If we denote the basis vectors

$$| \uparrow_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad | \uparrow_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.40}$$

then the operators for the spin along the x, y and z axes are nothing more than the Pauli matrices

$$X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.41}$$

For the spin along any other unit vector, say $a\vec{x} + b\vec{y} + c\vec{z}$ where $a^2 + b^2 + c^2 = 1$, the spin operator is $aX + bY + cZ$. All operators of this form have eigenvalues ± 1 corresponding to spin parallel and antiparallel to the vector.

As already mentioned, other system are equally good qubits as a spin-1/2 particle. Among the more popular ones are the polarisation of a photon and the energy levels of a two-level system. They can be represented by a Bloch sphere just as well as the spin of an electron can. The advantage of thinking about a spin-1/2 particle is that the vectors on the Bloch sphere corresponds to actual spin directions in real space.Thus, a qubit in a state close to the y-axis on the Bloch sphere, has, when the qubit is a spin-1/2 particle, a spin pointing in a direction close to the y-axis.

In general the particular implementation of the qubit is not important for our purpose, so we simplify denote the basis vectors $|0\rangle$ and $|1\rangle$, which in the case of a spin-1/2 particle are typically $| \uparrow_\mathbf{z}\rangle$ and $| \downarrow_\mathbf{z}\rangle$, respectively. The superpositions of these states corresponding to the spin along the x-axis, are often denoted $|+\rangle$ and $|-\rangle$, according to the sign in the superposition.

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2.42}$$

The name qubit comes from quantum bit and it plays the same role in quantum information theory as the bit does in classical information theory. The bit is a system which can be in one of two states, usually denoted 0 and 1. Thus, each bit is capable of storing enough information to answer exactly one yes/no question. As we have just seen, the qubit is a much richer structure. In addition to being capable of being in the states $|0\rangle$ and $|1\rangle$, just as a bit, it can be in a continuum of superpositions between those states, given by two parameters. Still, only a single bit of information can be read from a single qubit. When the spin of a spin-1/2 particle is measured along an axis the only possible results are parallel and antiparallel to the chosen axis. After the measurement the state is changed, and from the result of the measurement we know what it is. No more information about the original state can be retrieved.

2.6 Entangled State

From the simplest quantum systems whatsoever, we now turn to look at the simplest composite quantum systems[48, 49, 50]. A composite system is a system which consists of two or more parts, and the simplest one is a system consisting of two qubits. We call the two systems A and B. The composite state of the two systems in then simply the tensor product (or direct product) of the two states.

$$|\Psi_{prod}\rangle = |\psi\rangle_A \otimes |\phi\rangle_B \tag{2.43}$$

Such a state is called a product state, but product states are not the only physically realisable states. If we let the two systems interact with each other, any superposition of product states is realisable. Hence, a general composite state can be written as

$$|\Psi\rangle = \sum_{ij} \alpha_{ij} |\psi_i\rangle_A \otimes |\phi_j\rangle_B \tag{2.44}$$

where $\sum |\alpha_{ij}|^2 = 1$ and the sets $\{|\psi_i\rangle\}$ and $\{|\phi_j\rangle\}$ are orthonormal basis for the two subsystems. Any composite state that is not a product state is called an *entangled state*.

A composite quantum state consisting of two parts only, is called a *bipartite state*, as opposed to *multipartite states* which consists of more than two parts.

For a single qubit we could always change to another basis where the outcome of a Z measurement would be given. For a spin-1/2 particle this means that the spin is always pointing in some direction, even though the state will show up as a superposition in a basis where the state is not one of the basis states. If the particle is entangled with another particle, though, the direction of the spin of that particle alone is not well defined.

2.7 Pure and Mixed States

2.7.1 Pure states

In quantum mechanics, states of knowledge cannot be complete, meaning that an observer cannot know the result of all possible measurements perfectly well. However, some states of knowledge are still better than others and there remains a notion of maximal knowledge. Those quantum states of maximal knowledge are the pure states[51]:

$$\rho = |\psi\rangle\langle\psi|$$

where the wavevector is an element of the associated Hilbert space $\psi \in \mathcal{H}$. The states of non-maximal knowledge are referred to as *mixed states*.

Some simple relations for pure states are $\rho^2 = \rho$ and $Tr(\rho^2) = Tr(\rho) = 1$

When we are given a bipartite state the first question of interest is whether it is entangled or not. This is the question of separability. A separable state is a state which is not entangled, so each of the subsystems can be given a description on its own. When it comes to pure states, a state is separable if and only if it is a product state. As indicated in 2.6, a pure state is a product state if it can be written as

$$|\psi_{prod}\rangle = |\psi\rangle_A \otimes |\phi\rangle_B \tag{2.45}$$

for any pure state $|\psi\rangle_A$ and $|\phi\rangle_B$. If the state is not a product state, it is an entangled state.

In practice, though, one needs to find the right basis vectors to be able to write a product state as in (2.45). A simpler criterion is desired. The Schmidt decomposition[47] is helpful in this respect. The number of terms in the Schmidt decomposition is called the Schmidt number , sometimes also Schmidt rank. That is, the number n in

$$|\psi\rangle = \sum_{k=1}^{n} \lambda_k |a'_k\rangle \otimes |b'_k\rangle. \tag{2.46}$$

The Schmidt number is at most $min(d_A, d_B)$, where $d_A(d_B)$ is the dimension of subsystem A (B). But when some of the Schmidt coefficients are zero, those terms may be skipped giving a lower Schmidt number. Remember that the Schmidt coefficients are simply the square root of the eigenvalues of any of the reduced density matrices. Hence, the Schmidt number is simply the number of nonzero eigenvalues of ρ_A or ρ_B.

What the Schmidt number tells us physically is basically how many degrees of freedom that are entangled. And as a product state can be written as (2.45) it is obvious that its Schmidt number is 1. For pure states this is a necessary and sufficient criterion for separability.

2.7.2 Mixed States

A pure quantum state is a state which can be described by a single ket vector, other states are *convex combinations*[1] of pure states which are called *mixed states*.

A mixed state cannot be described as a ket vector. Instead, it is described by its associated density matrix (or density operator), usually denoted ρ. Note that density matrices can describe both mixed and pure states, treating them on the same footing. A simple criterion for checking whether a density matrix is describing a pure or mixed state is that the trace of ρ^2 is equal to 1 if the state is pure, and less than 1 if the state is mixed. Another, equivalent, criterion is that the von Neumann entropy is 0 for a pure state, and strictly positive for a mixed state.

The structure of mixed state bipartite entanglement is richer than its pure state equivalent. Also, the more difficult task of finding out whether a given mixed state is separable or not gives rise to classes of states which can be identified by other criteria.

2.8 Density Operators and Mixed States

We want to be able to describe the state of a subsystem that is entangled with another subsystem to which we do not have access. State vectors cannot be used for this purpose, and we need another representation of quantum states, namely *density operators*. This is not only useful for describing "locally" a subsystem which is part of an entangled system for which we know the state. The formalism is also necessary for describing quantum mechanical experiments - where noise is inevitable - and doing quantum statistical mechanics. In the laboratory it is impossible to isolate the quantum systems under study completely. The systems become entangled with the environment through unwanted, but unavoidable, interactions. Nevertheless, it is necessary to describe the system without taking the environment into account.

In the density operator formalism, we describe quantum states by operators on the system's Hilbert space instead of unit vectors on it. For any quantum state vector $|\psi\rangle$,

[1]A convex combination of elements $\mathrm{x}_1, \mathrm{x}_2, ..., \mathrm{x}_\mathrm{N}$ is an element which can be written $\sum_i \mathrm{p}_i \mathrm{x}_i$ for $\mathrm{p}_i \geq 0$ and $\sum_i \mathrm{p}_i = 1$. A convex combination of two points is a point on the straight line connecting them.

the corresponding density operator is the projection operator $|\psi\rangle\langle\psi|$. As linear operators may be - and often is - represented by matrices, density operators are also called *density matrices*.

So far we have only represented the state in a new way, using density operators instead of vectors. But what if we do not know exactly in what state our system is? Imagine, for example, that we have a machine in our lab which outputs particles in state $|\psi\rangle$ (think of it as some component of a spin). But once in a while events beyond our control and knowledge (a spike in the electric grid, the turning on of an electromagnet in a neighbouring lab, etc.) make the machine malfunction slightly producing states which are close to, but not exactly $|\psi\rangle$. Let us for simplicity say that the state $|\psi'\rangle$ is always created when the machine malfunctions, and that the probability for that to happen to any given particle is p. Given this uncertainty, the state cannot be described by a state vector. Still, in terms of density matrices it is described as

$$\rho = (1 - \mathrm{p})\,|\psi\rangle\,\langle\psi| \; + \; \mathrm{p}\,|\psi'\rangle\,\langle\psi'|. \tag{2.47}$$

The density operator does not tell us from what pure states the state was prepared. For example, we can have a state $|0\rangle$ or $|1\rangle$, each with probability 1/2. The density operator is then $\frac{1}{2}(\,|0\rangle\langle 0| + |1\rangle\langle 1|\,) = \frac{1}{2}\mathbb{I}$. The same density operator can be made by mixing the states $|+\rangle = \frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle\,)$ and $|-\rangle = \frac{1}{\sqrt{2}}(\,|0\rangle - |1\rangle\,)$ in equal amounts, or the three states $|0\rangle$, $|+\rangle$ and $\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ in amounts $\mathrm{p}_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$, $\mathrm{p}_2 = \frac{\sqrt{3}}{3+\sqrt{3}}$ and $\mathrm{p}_3 = 1 - \frac{\sqrt{3}}{3}$, respectively. We call these different collections of pure states with corresponding probabilities, $\{\mathrm{p}_i, |\psi_i\rangle\}$, *ensembles*. Sometimes it is also called a *realisation* and we say that an ensemble *realises* a mixed state. Hence, when we describe a state from an ensemble by its density operator, we discard the information about which ensemble the mixed state was made from. The density operator still describes the mixed state as well as can be done, as states from different ensembles having the same density operator are experimentally indistinguishable.

This indistinguishability makes the density operator a perhaps more intuitive representation of a state than the state vector. For example, a state vector has an arbitrary overall phase factor, so two vectors differing by a phase factor represent the same state. Still, when the vector is put together with its dual vector to form a pure state density operator, the phase factor vanishes because of the complex conjugation. Hence, all state vectors representing the same state are represented by the same density operator.

To complete our discussion about density operators, we should take note of some of their properties. For a more thorough discussion, see e.g. [69].Density operators are (i) positive[2], meaning that for any vector $|v\rangle$, $\langle v|\rho|v\rangle \geq 0$, or equivalently that it is Hermitian with nonnegative eigenvalues. Also, (ii) the trace of a density operator is unity, $tr(\rho) = 1$. It turns out that any operator satisfying these two criteria can be realised by a pure state ensemble. Hence, we may take these criteria as the defining properties of a density operator.

We will now see how the density operators are used to describe individual parts of an entangled system. In fact, we will see that even when the composite system is in a pure state, if the subsystems are entangled, the individual subsystems are in mixed states.

Formally, ignoring some degrees of freedom in a system (like the ones corresponding to a particle out of reach) is done by *tracing out* the relevant degrees of freedom from the density operator. This is also called taking the partial trace with respect to the degrees of freedom that are to be traced out. Any bipartite state can be written as

$$\rho = \sum_{ijkl} c_{ijkl}|a_i\rangle\langle a_j| \otimes |b_k\rangle\langle b_l|$$

and the partial trace with respect to system B of a bipartite system is defined as

$$\begin{aligned} tr_B(\rho) &\equiv \sum_{ijkl} c_{ijkl}|a_i\rangle\langle a_j| \otimes tr(|b_k\rangle\langle b_l|) \\ &= \sum_{ijkl} c_{ijkl}\langle b_l|b_k\rangle\, |a_i\rangle\langle a_j| \\ &= \sum_{ij} c_{ij}|a_i\rangle\langle a_j| \end{aligned} \tag{2.48}$$

[2]Positive operators here refer to positive semidefinite operators. Positive definite operators, on the other hand, satisfy$\langle v|\rho|v\rangle > 0$ for any $|v\rangle$ and have only positive eigenvalues.

with $C_{ij} = \sum_{kl} c_{ij}\langle b_l|b_k\rangle$. The density operator obtained by tracing out one part of the system is called the reduced density operator.

We can perform the partial trace in an example. The density matrix is

$$\rho = |\alpha|^2|00\rangle\langle 00| + \alpha\beta^* |00\rangle\langle 11| + \alpha^*\beta |11\rangle\langle 00| + |\beta|^2|11\rangle\langle 11|, \tag{2.49}$$

so the reduced density matrix becomes

$$\begin{aligned}
\rho_A &= tr_B(\rho) \\
&= |\alpha|^2\langle 0|0\rangle|0\rangle\langle 0| + \alpha\beta^* \langle 0|1\rangle|0\rangle\langle 1| + \alpha^*\beta \langle 1|0\rangle|1\rangle\langle 0| + |\beta|^2\langle 1|1\rangle|1\rangle\langle 1| \\
&= |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|.
\end{aligned} \tag{2.50}$$

which is just what we expected.

In section 2.6 we saw what we meant by an entangled state as long as the states were pure. But this leaves the question of when a mixed state is entangled.

We say that a mixed state is *separable* if and only if it can be realised as a mixture (convex combination) of locally prepared states [3]. That is, it can be written as[4]

$$\sum_i p_i \rho_i^{(A)} \otimes \rho_i^{(B)} \tag{2.51}$$

where p_i forms a probability distribution; $p_i \geq 0$ and $\sum_i p_i = 1$. This also means [46] that the state can be written as a mixture of pure product states,

$$\sum_{kl} p_{kl} |\psi_k^{(A)}\rangle\langle\psi_k^{(A)}| \otimes |\psi_l^{(B)}\rangle\langle\psi_l^{(B)}|. \tag{2.52}$$

Hence, a mixture of entangled states need not be entangled, but a mixture of separable states is always separable. Mixture is a process which destroys entanglement. This is because by discarding information about which of a number of entangled states the

[3] This definition is due to Werner [44] who called them "classically correlated" states.

[4] In fact it was considered sufficient that such a sum approximated the state arbitrarily well. It was shown in [46], though, that in the finite dimensional case any separable state could be expressed in d^2 terms, where d is the dimension of the Hilbert space of the composite system.

system is in, it can no longer be distinguished from a mixture of separable states.

2.9 Separable States

Recall from section 2.8 that a bipartite state is called separable if it can be realised as a mixture of product states,

$$\sum_i \mathrm{p}_i \rho_i^{(A)} \otimes \rho_i^{(B)} \tag{2.53}$$

with $\mathrm{p}_i \geq 0$ and $\Sigma_i \mathrm{p}_i = 1$. Given a density matrix in some basis the task of finding local density matrices to satisfy (2.53) can be enormous. Even for the simplest system which may be entangled, a pair of qubits, a density operator would typically look like

$$\begin{aligned} \rho \;\; = \;\; & \frac{3}{24}|00\rangle\langle 00| + \frac{\sqrt{2}}{24} i|00\rangle\langle 01| + \frac{\sqrt{2}}{12} i|00\rangle\langle 11| \\ & -\frac{\sqrt{2}}{24} i|01\rangle\langle 00| + \frac{1}{4}|01\rangle\langle 01| - \frac{\sqrt{2}}{12} i||01\rangle\langle 10| \\ & +\frac{\sqrt{2}}{12} i|10\rangle\langle 01| + \frac{5}{24}|10\rangle\langle 10| - \frac{\sqrt{2}}{24} i|10\rangle\langle 11| \\ & -\frac{\sqrt{2}}{12} i|11\rangle\langle 00| + \frac{\sqrt{2}}{24} i|11\rangle\langle 10| + \frac{5}{12}|11\rangle\langle 11| \end{aligned} \tag{2.54}$$

or in matrix representation in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis,

$$\rho = \begin{pmatrix} \frac{3}{24} & \frac{\sqrt{2}}{24} i & 0 & \frac{\sqrt{2}}{12} i \\ -\frac{\sqrt{2}}{24} i & \frac{1}{4} & -\frac{\sqrt{2}}{12} i & 0 \\ 0 & \frac{\sqrt{2}}{12} i & \frac{5}{24} & -\frac{\sqrt{2}}{24} i \\ -\frac{\sqrt{2}}{12} i & 0 & \frac{\sqrt{2}}{24} i & \frac{5}{12} \end{pmatrix} \tag{2.55}$$

It is not easy to see that this can be written as (2.53). Still, it is constructed as a mixture of two product states. It can be written as (2.53). Still, it is constructed as a mixture of two product states. It can be written as

$$\begin{aligned}\rho &= \mathrm{p}_1\rho_1^{(A)}\otimes\rho_1^{(B)}+\mathrm{p}_2\rho_2^{(A)}\otimes\rho_2^{(B)}\\ &= \frac{1}{2}\left(\frac{1}{2}|1\rangle\langle 1|+\frac{1}{2}|-\rangle\langle -|\right)\otimes|\psi^+_{\frac{2}{3}}\rangle\langle\psi^+_{\frac{2}{3}}|+\frac{1}{2}\left(\frac{1}{2}\mathbb{I}/2+\frac{1}{2}|+\rangle\langle +|\right)\otimes|\psi^-_{\frac{2}{3}}\rangle\langle\psi^-_{\frac{2}{3}}| \quad (2.56)\end{aligned}$$

where

$$|\psi^{+}_{\frac{2}{3}}\rangle = \frac{1}{\sqrt{3}}|0\rangle \pm \sqrt{\frac{2}{3}}i|1\rangle$$

It could of course be realised by a variety of other mixtures as well.

This example shows that we could well use an operational criterion for separability, like the Schmidt number for pure states. In general, no easily computable necessary and sufficient criterion is known. Still, we have a necessary condition which is easy to compute. This is known as the PPT[5]-criterion or Peres-Horodecki criterion [45, 46]. It states that if the state is separable, then the partial transpose of the density operator with respect to one subsystem is positive. A transpose of an operator needs to be taken with respect to a basis, and the resulting operator depends on the basis. Actually, if A^T is the transpose of the operator with respect to one basis, any operator of the form $U\,A^T\,U^\dagger$ for unitary U satisfying $U = U^T$ (in the same basis) is a transpose of the operator with respect to another basis. However, since the different bases are connected by a unitary transformation, the eigenvalues are independent of the basis. Therefore, any orthonormal basis will do for taking the transposition. More formally, if we choose an orthonormal product basis $\{|v_i v_j\rangle\} \equiv \{|v_i\rangle\otimes|v_j\rangle\}$ for the state ρ, the partial transpose ρ^{T_B} is defined by its matrix elements

$$\rho^{T_B}_{m\mu,n\nu} = \langle v_m v_\mu|\rho^{T_B}|v_n v_\nu\rangle = \rho_{m\nu,n\mu}. \qquad (2.57)$$

For systems of dimension 2×2 and 2×3 the PPT-criterion is also sufficient for the state to be separable [46]. For larger systems, though, there exist entangled PPT states

[5]Positive Partial Transpose

[55]. The PPT-criterion is still so useful that the class of PPT states is an important class of states, strictly larger than the class of separable states. Anything that can be proved for all PPT states, is automatically true for separable states.

2.10 Von Neumann Entropy

Before treating the general case of mixed states we will consider the simpler case of entanglement measures of pure states. The treatment is facilitated by the fact that a pure state contains no classical correlations between the subsystems, so any correlation present must be of quantum nature. As mentioned in section 2.3, even if an entangled state is pure the states of the subsystems - described locally by the reduced density operator - are mixed. For pure states, then, the amount of "mixedness" turn out to be a good measure of entanglement.

Our measure for how mixed a quantum state ñ is, will be the von Neumann entropy, defined as

$$S(\rho) = -\,tr\,(\rho \log \rho) \tag{2.58}$$

where we take the logarithm base-2 as is the custom in information theory. It is most easily calculated from the nonzero eigenvalues λ_i of ρ as

$$S(\rho) = -\sum_i \lambda_i \log \lambda_i \tag{2.59}$$

What the von Neumann entropy essentially describes, is the uncertainty in a quantum state. It is zero for pure states, and smaller for a mixture of two nonorthogonal states than a similar mixture of two orthogonal states.

2.11 Entanglement of Formation

The entanglement of formation is a straightforward generalisation of the entropy of entanglement to mixed states. We define the entanglement of formation for a pure state as the entropy of entanglement. For a given ensemble of pure states $\mathcal{E} = \{p_i, |\psi_i\rangle\}$ the entanglement of formation is the average of the entropy of entanglement for the states in the ensemble

$$E_f(\mathcal{E}) \equiv \sum_i p_i E_E(|\psi_i\rangle)\,. \tag{2.60}$$

A mixed state can be realised by a multitude of pure state ensembles, with different entanglement of formation. As any of those ensembles realises the mixed state, the natural definition for the entanglement of formation for a mixed state is the entanglement of formation for the "most economic" ensemble. That is, the entanglement of formation for a mixed state is defined as

$$E_f(\rho) \equiv \inf_{\mathcal{E}} \sum_i p_i E_E(|\psi_i\rangle)\,. \tag{2.61}$$

where the infimum is taken over all ensembles $\mathcal{E} = \{p_i, |\psi_i\rangle\}$ that realises the state ρ.

2.11.1 Concurrence

A concept related to the entanglement of formation is the *concurrence* [72, 73]. It is defined for a system of two qubits. For a general state ρ of two qubits, let $\widetilde{\rho}$ be the spin-flipped state

$$\widetilde{\rho} \equiv (Y \otimes Y)\rho^* (Y \otimes Y) \tag{2.62}$$

where the Y is the Pauli Y operator and ρ^* is the complex conjugate of ρ, both taken in the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Let the Hermitian matrix R be defined as

$$R \equiv \sqrt{\sqrt{\rho}\,\widetilde{\rho}\,\sqrt{\rho}} \tag{2.63}$$

with eigenvalues in decreasing order $\{\lambda_i\}$. The concurrence is then defined as

$$C(\rho) \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{2.64}$$

The concurrence can be used as an entanglement measure for two qubits. The great advantage is that it is easily computable. But more important is that it is directly related to the entanglement of formation, providing an explicit formula for the entanglement of formation in the case of two qubits. Let the function $\mathcal{E}$ be given by

$$\mathcal{E}(C) \equiv h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right) \tag{2.65}$$

where h is the binary entropy function

$$h(x) \equiv -x \log x - (1 - x) \log(1 - x). \tag{2.66}$$

Then the entanglement of formation is simply given by [72, 73]

$$\mathrm{E}_f(\rho) = \mathcal{E}\left(C(\rho)\right). \tag{2.67}$$

There have been various attempts to generalise the concept of concurrence. Uhlmann [75] considered general conjugations instead of the special (2.62). This was further generalised to concurrence vectors by Auderaert et al [76]. The previously mentioned convex roof extended negativity coincides with the concurrence in the case of two qubits [66], and thus provides another generalisation. Some of those generalisations - along with other aspects of concurrence and entanglement of formation - are reviewed in [77]

2.11.2 Multipartite Systems

In this section, we study more than bipartite systems. We expressed the entanglement of bipartite systems and measurement in the previous sections. Now we focus multipartite systems[74].

The questions about entanglement literally acquire new dimensions when more that two subsystems are considered. Between two parties all relations are between those two parties. Once we increase the number of parties to three, we have three pair relations in the system. Still, those three pair relations do not describe all correlations in the system (neither classical nor quantum). Consider for instance the pure tripartite entangled state called the GHZ[6] state

$$|GHZ\rangle \equiv \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{2.68}$$

If we trace out any one of the systems, we get the density matrix shared by the two remaining parties

$$\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \tag{2.69}$$

which is separable. Thus, the state contains no bipartite entanglement between the systems, but the tripartite entanglement is maximal.

For separability, some notions can be taken directly over from bipartite systems, though. A state is separable if and only if it can be written as a convex combination of product states;

$$\sum_i \mathrm{p}_i \rho_i^{(A)} \otimes \rho_i^{(B)} \otimes \ldots \otimes \rho_i^{(N)} \tag{2.70}$$

whit $\mathrm{p}_i \geq 0$ and $\sum_i \mathrm{p}_i = 1$.

The classification of states does not end with this. An entangled multipartite state may or may not still be entangled after one of the subsystems are ignored (i.e. traced out). A multipartite state that loses its entanglement when any of the subsystems is traced out, is called *multiseparable*.

The PPT (Positive Partial Transpose) criterion for separability trivially extends to the multipartite case. If a multipartite state is separable, then transposing any number of the subsystems must also give a valid (i.e. positive) density operator.

The meaning of entanglement measures in the multipartite setting is more vague than in the bipartite. In the bipartite setting it quantifies in some way how much that

[6]Daniel M. Greenberger, Michael A. Horne and Anton Zeilinger

can be achieved by the two parties together. In the multipartite case there are more possibilities.

For discriminating the genuine three-partite entangled states from other entangled states or separable states, we search the state preparation fidelity to adopt and study if the three-partite entangled states can be generated in the system[20]. For a three-qubit state, the state preparation fidelity, F, is defined as

$$F(\rho) = \langle GHZ|\rho|GHZ\rangle, \tag{2.71}$$

where $|GHZ\rangle$ is the three-partite GHZ state. A sufficient condition to determine the genuine three-partite entanglement is whether $F(\rho) > 1/2$ or not. If $F(\rho) > 1/2$, the system is three-partite entanglement , otherwise it is not entanglement.

CHAPTER 3

REVIEW OF ONE ATOM-ONE FIELD INTERACTION UNDER RANDOM PHASE TELEGRAPH NOISE

In this chapter, we restudy one atom-one field interaction under random phase telegraph noise. For this purpose, we use *generalized Jaynes-Cummings model with random telegraph noise* by Joshi and *et.al.*[18, 78]. The Jaynes-Cummings model of a two -level atom interacting with a single quantized mode of radiation field is generalized to include the effects of stochastic fluctuations in the atom-field coupling. The fluctuations are modelled with random telegraph process by Joshi ana *et.al.*Using Markovian property of the telegraph noise, we rewrite in these two cases an equation for the density operator averaged over the distribution of fluctuations.

3.1 Generalized Jaynes-Cumming Model with Random Telegraph Noise

In this section, we review the basic formulation of the stochastic Jaynes-Cumming Model (JCM [79]) by Joshi and *et.al.* After introducing the physical model and the basic properties of the phase telegraph noise, we go on solutions. Exact equations are rewritten for the relevant density matrix averaged over the fluctuates in phase.

3.1.1 Formulation

We consider the interaction of a single two-level atom with a single mode of radiation field. The atom is characterized by spin-1/2 angular-momentum operators $S_{\pm}, S_z$, while the field described by the annihilation and the creation operators a and $a^{\dagger}$, respectively. In the usual rotating-wave approximation, the Hamiltonian of the system takes the form

$$H = \omega_0 S_z + \omega_0 a^\dagger a + [g^*(t) S_+ a + g(t) S_- a^\dagger], \tag{3.1}$$

where the coupling g(t) between the atom and the field is assumed to be time dependent. In this section we will study an alternative model that represents noise by means of discrete jump processes. A simple example of such a jump process is two-state random telegraph.

In order to illustrate such a model we know that

$$g(t) = g_0 e^{-i\phi(t)}, \tag{3.2}$$

where the nonstochastic amplitude g_0 is a positive real quantity while the $\phi(t)$ is treated as a stochastic variable. In the following, we have, telegraph noise where $\phi(t)$ randomly jumps between two possible values(states) a and -a, the correlation function

$$\begin{aligned} \langle g(t) g(t') \rangle &= g_0^2 \langle \exp\{i[\phi(t) - \phi(t')]\} \rangle \\ &= g_0^2 [\cos^2 a + \sin^2 a \exp(-2|t - t'|/T)], \end{aligned} \tag{3.3}$$

where the T is the mean dwell time of the telegraph (The jumps are separated by an average time interval called the mean dwell time.)

3.1.2 Solution

We now proceed to study in the presence of telegrapher noise. For this purpose, we rewrite the relevant equation of motion from the basic Liouville-von Neumann equation for the density operator. It is convenient to use of the interaction representation and the evolution equation as

$$i\frac{\partial \rho}{\partial t} = [H_1, \rho] = H_1 \rho - \rho H_1, \tag{3.4}$$

where H_1 denotes the interaction Hamiltonian

$$H_1 = g_0 \left\{ \exp[i\phi(t)] S_+ a + \exp[-i\phi(t)] S_- a^\dagger \right\}. \tag{3.5}$$

We now introduce the states

$$\begin{aligned} |\Psi_m^+\rangle &= |n, e\rangle, \\ |\Psi_m^-\rangle &= |n+1, g\rangle, \end{aligned} \tag{3.6}$$

and

$$\rho_{mn}^{\pm,\mp}(t) = \langle \Psi_m^\pm | \rho(t) | \Psi_n^\mp \rangle \tag{3.7}$$

From the above equation we can define

$$\rho_{mn}^{++}(t) = \langle ne | \rho(t) | ne \rangle \tag{3.8}$$

$$\rho_{mn}^{+-}(t) = \langle ne | \rho(t) | n+1g \rangle \tag{3.9}$$

$$\rho_{mn}^{-+}(t) = \langle n+1g | \rho(t) | ne \rangle \tag{3.10}$$

$$\rho_{mn}^{--}(t) = \langle n+1g | \rho(t) | n+1g \rangle \tag{3.11}$$

Using equation (3.4) we have that

$$i\frac{d}{dt}\rho_{mn}^{++}(t) = H_1 \rho_{mn}^{++}(t) - \rho_{mn}^{++}(t) H_1 \tag{3.12}$$

$$i\frac{d}{dt}\rho_{mn}^{+-}(t) = H_1 \rho_{mn}^{+-}(t) - \rho_{mn}^{+-}(t) H_1 \tag{3.13}$$

$$i\frac{d}{dt}\rho_{mn}^{-+}(t) = H_1 \rho_{mn}^{-+}(t) - \rho_{mn}^{-+}(t) H_1 \tag{3.14}$$

$$i\frac{d}{dt}\rho_{mn}^{++}(t) = H_1 \rho_{mn}^{++}(t) - \rho_{mn}^{++}(t) H_1 \tag{3.15}$$

Now we obtain from equations that (3.12), (3.13), (3.14), (3.15)

$$\frac{d}{dt}\rho_{mn}^{++} = i\left[\alpha_n e^{-i\phi(t)}\rho_{mn}^{+-}(t) - \alpha_m e^{+i\phi(t)}\rho_{mn}^{-+}(t)\right] \quad (3.16)$$
$$\frac{d}{dt}\rho_{mn}^{+-} = i\left[\alpha_n e^{+i\phi(t)}\rho_{mn}^{++}(t) - \alpha_m e^{+i\phi(t)}\rho_{mn}^{--}(t)\right] \quad (3.17)$$
$$\frac{d}{dt}\rho_{mn}^{-+} = i\left[\alpha_n e^{-i\phi(t)}\rho_{mn}^{--}(t) - \alpha_m e^{-i\phi(t)}\rho_{mn}^{++}(t)\right] \quad (3.18)$$
$$\frac{d}{dt}\rho_{mn}^{--} = i\left[\alpha_n e^{+i\phi(t)}\rho_{mn}^{-+}(t) - \alpha_m e^{-i\phi(t)}\rho_{mn}^{+-}(t)\right] \quad (3.19)$$

in which

$$\alpha_m = g_0\sqrt{m+1} \quad (3.20)$$

These equations have to be solved taking into account the stochastic nature of the phase variable $\phi(t)$. For this purpose, it is convenient to first solve equations (3.17) and (3.18) formally and insert to equations (3.16) and (3.19). Then we first make the convenient solution, for this purpose we integrate $\frac{d}{dt'}\rho_{mn}^{+-}(t')$ and $\frac{d}{dt'}\rho_{mn}^{-+}(t')$ from 0 to t.

$$\int_0^t \frac{d}{dt'}\rho_{mn}^{+-}(t')dt' = \rho_{mn}^{+-}(t) - \rho_{mn}^{+-}(0) = \rho_{mn}^{+-}(t) \quad (3.21)$$
$$\int_0^t \frac{d}{dt'}\rho_{mn}^{-+}(t')dt' = \rho_{mn}^{-+}(t) - \rho_{mn}^{-+}(0) = \rho_{mn}^{-+}(t) \quad (3.22)$$

We assumed that $\rho_{mn}^{\pm,\mp}(0) = 0$ and also $t' < t$. Now we insert the convenient equations to the equations (3.16) and (3.19).

$$\begin{aligned}
\frac{d}{dt}\rho_{mn}^{++}(t) &= i\left[\alpha_n e^{-i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{+-}(t')dt' - \alpha_m e^{+i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{-+}(t')dt'\right] \\
&= i\alpha_n e^{-i\phi(t)}\int_0^t dt'\, i\left[\alpha_n e^{+i\phi(t')}\rho_{mn}^{++}(t') - \alpha_m e^{+i\phi(t')}\rho_{mn}^{--}(t')\right] \\
&\quad -i\alpha_m e^{+i\phi(t)}\int_0^t dt'\, i\left[\alpha_n e^{-i\phi(t')}\rho_{mn}^{--}(t') - \alpha_m e^{-i\phi(t')}\rho_{mn}^{++}(t')\right] \\
&= -\int_0^t dt'\Big\{\alpha_n e^{-i[\phi(t)-\phi(t')]}\left[\alpha_n\rho_{mn}^{++}(t') - \alpha_m\rho_{mn}^{--}(t')\right] \\
&\quad +\alpha_m e^{i[\phi(t)-\phi(t')]}\left[\alpha_m\rho_{mn}^{++}(t') - \alpha_n\rho_{mn}^{--}(t')\right]\Big\} \quad (3.23)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{d}{dt}\rho_{mn}^{--}(t) &= i\left[\alpha_n e^{+i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{-+}(t')dt' - \alpha_m e^{-i\phi(t)}\int_0^t \frac{d}{dt'}\rho_{mn}^{+-}(t')dt'\right] \\
&= i\alpha_n e^{+i\phi(t)}\int_0^t dt' i\left[\alpha_n e^{-i\phi(t')}\rho_{mn}^{--}(t') - \alpha_m e^{-i\phi(t')}\rho_{mn}^{++}(t')\right] \\
&\quad -i\alpha_m e^{-i\phi(t)}\int_0^t dt' i\left[\alpha_n e^{+i\phi(t')}\rho_{mn}^{++}(t') - \alpha_m e^{+i\phi(t')}\rho_{mn}^{--}(t')\right] \\
&= \int_0^t dt'\Big\{\alpha_n e^{-i[\phi(t)-\phi(t')]}\left[\alpha_n\rho_{mn}^{++}(t') - \alpha_m\rho_{mn}^{--}(t')\right] \\
&\quad +\alpha_m e^{i[\phi(t)-\phi(t')]}\left[\alpha_m\rho_{mn}^{++}(t') - \alpha_n\rho_{mn}^{--}(t')\right]\Big\} \qquad (3.24)
\end{aligned}$$

In describing equations (3.16) and (3.19) in the above form the idea behind it, is to make use of Markovian property of the telegrapher noise. According to this, since $t' < t$, the part involving the exponential factor in equations (3.23) and (3.24) gets decoupled from the other part, which depends on t'. This allows us to take the average over the stochastic distribution of the telegrapher noise. Using the symbol $\langle\ \rangle$ to denote such an average and writing

$$c(t-t') = \langle e^{\pm i[\phi(t)-\phi(t')]}\rangle, \qquad (3.25)$$

Now we take the average of equations (3.23) and (3.24), then we will use equation (3.25) and obtain;

$$\begin{aligned}
\frac{d}{dt}\langle\rho_{mn}^{++}(t)\rangle &= -\int_0^t dt'\Big\{\alpha_n\langle e^{-i[\phi(t)-\phi(t')]}\rangle\left[\alpha_n\langle\rho_{mn}^{++}(t')\rangle - \alpha_m\langle\rho_{mn}^{--}(t')\rangle\right] \\
&\quad +\alpha_m\langle e^{i[\phi(t)-\phi(t')]}\rangle\left[\alpha_m\langle\rho_{mn}^{++}(t')\rangle - \alpha_n\langle\rho_{mn}^{--}(t')\rangle\right]\Big\} \\
&= -\int_0^t dt' c(t-t')\Big\{(\alpha_n^2+\alpha_m^2)\langle\rho_{mn}^{++}(t')\rangle - 2\alpha_n\alpha_m\langle\rho_{mn}^{--}(t')\rangle\Big\}, \qquad (3.26) \\
\frac{d}{dt}\langle\rho_{mn}^{--}(t)\rangle &= \int_0^t dt'\Big\{\alpha_m\langle e^{-i[\phi(t)-\phi(t')]}\rangle\left[\alpha_n\langle\rho_{mn}^{++}(t')\rangle - \alpha_m\langle\rho_{mn}^{--}(t')\rangle\right] \\
&\quad +\alpha_n\langle e^{i[\phi(t)-\phi(t')]}\rangle\left[\alpha_m\langle\rho_{mn}^{++}(t')\rangle - \alpha_n\langle\rho_{mn}^{--}(t')\rangle\right]\Big\} \\
&= \int_0^t dt' c(t-t')\Big\{2\alpha_n\alpha_m\langle\rho_{mn}^{++}(t')\rangle - (\alpha_n^2+\alpha_m^2)\langle\rho_{mn}^{--}(t')\rangle\Big\} \qquad (3.27)
\end{aligned}$$

For convenience we may introduce two quantities

$$\mathbf{F}^{\pm}_{mn}(t) = \frac{1}{2}\{\langle\rho^{++}_{mn}(t)\rangle \pm \langle\rho^{--}_{mn}(t)\rangle\} \tag{3.28}$$

and if we make addition or substraction between equations (3.26) and (3.27) we can rewrite them in a new form.Firstly we make addition

$$\begin{aligned}
\frac{d}{dt}\{\langle\rho^{++}_{mn}(t)\rangle + \langle\rho^{--}_{mn}(t)\rangle\} &= \int_0^t dt' c(t-t')\left\{-(\alpha_m-\alpha_n)^2\langle\rho^{++}_{mn}(t')\rangle - (\alpha_m-\alpha_n)^2\langle\rho^{--}_{mn}(t')\rangle\right\} \\
2\frac{d}{dt}\mathbf{F}^{+}_{mn}(t) &= \int_0^t dt' c(t-t')\left[-(\alpha_m-\alpha_n)^2\right]\{\langle\rho^{++}_{mn}(t')\rangle + \langle\rho^{--}_{mn}(t')\rangle\} \\
\frac{d}{dt}\mathbf{F}^{+}_{mn}(t) &= \int_0^t dt' c(t-t')\left[-(\alpha_m-\alpha_n)^2\right]\frac{1}{2}\{\langle\rho^{++}_{mn}(t')\rangle + \langle\rho^{--}_{mn}(t')\rangle\} \\
\frac{d}{dt}\mathbf{F}^{+}_{mn}(t) &= -(\Omega^{-}_{mn})^2\int_0^t dt' c(t-t')\mathbf{F}^{+}_{mn}(t') \qquad (3.29)
\end{aligned}$$

Now we make substraction

$$\begin{aligned}
\frac{d}{dt}\{\langle\rho^{++}_{mn}(t)\rangle - \langle\rho^{--}_{mn}(t)\rangle\} &= \int_0^t dt' c(t-t')\left\{(\alpha_m+\alpha_n)^2\langle\rho^{++}_{mn}(t')\rangle - (\alpha_m+\alpha_n)^2\langle\rho^{--}_{mn}(t')\rangle\right\} \\
2\frac{d}{dt}\mathbf{F}^{-}_{mn}(t) &= \int_0^t dt' c(t-t')(\alpha_m+\alpha_n)^2\{\langle\rho^{++}_{mn}(t')\rangle - \langle\rho^{--}_{mn}(t')\rangle\} \\
\frac{d}{dt}\mathbf{F}^{-}_{mn}(t) &= \int_0^t dt' c(t-t')(\alpha_m+\alpha_n)^2\frac{1}{2}\{\langle\rho^{++}_{mn}(t')\rangle - \langle\rho^{--}_{mn}(t')\rangle\} \\
\frac{d}{dt}\mathbf{F}^{-}_{mn}(t) &= -(\Omega^{+}_{mn})^2\int_0^t dt' c(t-t')\mathbf{F}^{-}_{mn}(t') \qquad (3.30)
\end{aligned}$$

in which,
$(\Omega^{\pm}_{mn})^2 = (\alpha_m \pm \alpha_n)^2 = g_o^2(\sqrt{m+1} \pm \sqrt{n+1})^2$.
Now we may solve equations (3.29) and (3.30) using Laplace transform techniques. The form of the correlation function c(t) depends on the telegrapher process considered.

The correlation function $C(t-t')$ for the phase telegraph is given by equation (3.3)

$$C(t-t') = \cos^2 a + \sin^2 a e^{-2|t-t'|/T}. \tag{3.31}$$

Inserting equation (3.31) in equations (3.29) and (3.30) and taking the Laplace transform with respect to t, we obtain $\widetilde{\mathbf{F}}^{\pm}_{mn}(z)$. Firstly we obtain $\widetilde{\mathbf{F}}^{+}_{mn}(z)$;

$$\begin{aligned}
\frac{d}{dt}\mathbf{F}^{+}_{mn}(t) &= -(\Omega^{-}_{mn})^2 \int_0^t dt' c(t-t')\mathbf{F}^{+}_{mn}(t') \\
\mathcal{L}\left\{\frac{d}{dt}\mathbf{F}^{+}_{mn}(t)\right\} &= -(\Omega^{-}_{mn})^2 \mathcal{L}\left\{\int_0^t dt' c(t-t')\mathbf{F}^{+}_{mn}(t')\right\} \\
z\,\widetilde{\mathbf{F}}^{+}_{mn}(z) - \mathbf{F}^{+}_{mn}(0) &= -(\Omega^{-}_{mn})^2 \mathcal{L}\{c(t)\}\,\widetilde{\mathbf{F}}^{+}_{mn}(z) \\
\widetilde{\mathbf{F}}^{+}_{mn}(z) &= \frac{\mathbf{F}^{+}_{mn}(0)}{z + (\Omega^{-}_{mn})^2 \mathcal{L}\{c(t)\}} \qquad (3.32)
\end{aligned}$$

Since $c(t) = \cos^2 a + \sin^2 a e^{-\frac{2t}{T}}$ we have

$$\mathcal{L}\{c(t)\} = \frac{\cos^2 a}{z} + \frac{\sin^2 a}{2/T + z} \qquad (3.33)$$

If we insert equation (3.33) in equation (3.32) and simplify it, we obtain

$$\widetilde{\mathbf{F}}^{+}_{mn}(z) = \frac{z(z+2/T)\mathbf{F}^{+}_{mn}(0)}{z^3 + 2z^2/T + (\Omega^{-}_{mn})^2 2\cos^2 a/T + z(\Omega^{-}_{mn})^2} \qquad (3.34)$$

In a similar procedure we can obtain that $\widetilde{\mathbf{F}}^{-}_{mn}(z)$; that is

$$\widetilde{\mathbf{F}}^{-}_{mn}(z) = \frac{z(z+2/T)\mathbf{F}^{-}_{mn}(0)}{z^3 + 2z^2/T + (\Omega^{+}_{mn})^2 2\cos^2 a/T + z(\Omega^{+}_{mn})^2} \qquad (3.35)$$

where $\widetilde{\mathbf{F}}^{\pm}_{mn}(z)$ is the Laplace transform of $\mathbf{F}^{\pm}_{mn}(t)$. When m=n

$$(\Omega^{-}_{mn})^2 = (\alpha_m - \alpha_n)^2 = (\alpha_n - \alpha_n)^2 = 0$$

Thus,we obtain from equation (3.34)

$$\mathbf{F}^{+}_{nn}(t) = \mathbf{F}^{+}_{nn}(0) \qquad (3.36)$$

On the other hand, when $m \neq n$, from Inverse Laplace transform of the equations (3.34) and (3.35) we get the general solution

$$\mathbf{F}^{\pm}_{mn}(t) = \mathbf{F}^{\pm}_{mn}(0)\left\{\sum_{j=1}^{3}\frac{\lambda_j(\lambda_j + 2/T)}{\prod_{k\neq j}(\lambda_j - \lambda_k)}\exp(\lambda_j t)\right\} \tag{3.37}$$

where λ_j are the roots of the cubic equation in (3.34) and (3.35)

$$\lambda^3 + (2/T)\lambda^2 + (\Omega^{\pm}_{mn})^2\lambda + (2/T)(\Omega^{\pm}_{mn})^2\cos^2 a = 0 \tag{3.38}$$

Now we extend calculation thinking random phase telegraph jumps from $\pm a$ to infinity as formulated by Joshi and *et.al*. Let $\phi(t)$ follow the distribution.

$$dQ(\tau) = \exp(-\frac{\tau}{T})\frac{d\tau}{T} \tag{3.39}$$

The above distribution specifies the probability of duration of each such jump interval. We consider only that case in which $\phi(t)$s in the neighbouring intervals are not correlated. So, at any instant, the probability of finding a given ϕ remains the same and equal to $dQ(\tau)$, and there is no limitation on the form of this distribution. In other words, $\phi(t)$ is undergoing random continuous change of Markov type. To determine the dynamical evolution of system we need to know the unitary transformation $U(t, t')$ such that

$$\rho(t) = U(\phi, t, t')\rho(t')U^{-1}(\phi, t, t') \tag{3.40}$$

At the end of each (ith) interval we find the density matrix $\rho(t)$ which is the initial condition for the next matrix; so, if in the interval (0,t) there are k changes in ϕ,then

$$\begin{aligned}\rho(t, t_k) \;&=\; \mathbf{U}\left(\phi(t_k); t, t_k\right)...\mathbf{U}\left(\phi(t_1); t_2, t_1\right)\mathbf{U}(\phi(0); t_1, 0)\,\rho(0)\\ &\quad\times \mathbf{U}^{-1}\left(\phi(0); t_1, 0\right)\mathbf{U}^{-1}\left(\phi(t_1); t_2, t_1\right)...\mathbf{U}^{-1}\left(\phi(t_k); t, t_k\right).\end{aligned} \tag{3.41}$$

The expression in equation (3.41) is of multiplicative nature and hence it is quite easy to average it over. The probability (in the interval (0,t)) that k changes of ϕ have actually occurred at successive instants $t_1, t_2, ..., t_k$ and a certain sequence $\phi_1, \phi_2, ..., \phi_k$ (where $\phi_i = \phi(t_i)$) was realized between them is obviously equal to

$$\mathrm{d}Q\,(t_1, t_2, ..., t_k; \phi_1, \phi_2, ..., \phi_k, t) = \exp\left(-\frac{t}{T}\right)\prod_{i=1}^{k}\left(\frac{\mathrm{d}t_i}{T}\right)\prod_{i=0}^{k}\mathrm{d}Q(\phi_i). \quad (3.42)$$

The average density operator can be thus written as

$$\begin{aligned}\widetilde{\rho}(t)\exp\left(\frac{t}{T}\right) &= \sum_{k=0}^{\infty}\frac{1}{T^k}\int_0^t \mathrm{d}t_k \int_0^{t_k}\mathrm{d}t_{k-1}... \\ &\times \int_0^{t_2}\mathrm{d}t_1 \int \mathrm{d}Q(\phi_k)\int \mathrm{d}Q(\phi_{k-1})...\int \mathrm{d}Q(\phi_0)\rho(t, t_k). \quad (3.43)\end{aligned}$$

Note that the term with $k = 0$ (when ϕ does not change at all in the interval (0,t)) will not contain integrals with respect to the time and is equal simply to

$$\int \mathrm{d}Q(\phi_0)\rho(t, 0).$$

By using the recurrence relation (3.39), we can multiply both sides of equation (3.43) from the left and the right by $\mathbf{U}^{-1}(\phi, \tau, t)$ and $\mathbf{U}(\phi, \tau, t)$ respectively and also by $\mathrm{d}Q(\phi)\mathrm{d}t/T$, then integrate with respect to time from 0 to τ and eliminate the entire series using equation (3.43). After some simplifications it is easy to show that

$$\begin{aligned}\rho(\tau)\exp\left(\frac{\tau}{T}\right) &= \int U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0)dQ(\phi) \\ &+\frac{1}{T}\int \exp\left(\frac{t}{T}\right)\int U(\phi;\tau,t)\rho(t)U^{-1}(\phi;\tau,t)dQ(\phi)dt, \quad (3.44)\end{aligned}$$

where we define the horizontal bar sign over ρ for convenience.

Next, we define

$$\begin{aligned}\mathbf{G}_{lk}^{im} &= \int \mathbf{U}_{ik}(\phi;\tau,t)\mathbf{U}_{lm}^{-1}(\phi;\tau,t)\mathrm{d}Q(\phi), \\ \mathbf{G}_{lk}^{im} &= \mathbf{G}_{kl}^{mi}, \quad (3.45)\end{aligned}$$

so we get

$$\begin{aligned}\rho_{im}(\tau) &= \exp\left(-\frac{\tau}{T}\right)\mathrm{Tr}\left[\mathbf{G}^{im}(\tau,o)\rho(0)\right] \\ &+\frac{1}{T}\int_0^\tau \mathrm{d}t\exp\left(-\frac{(\tau-t)}{T}\right)\mathrm{Tr}\left[\mathbf{G}^{im}(\tau,t)\rho(t)\right]. \end{aligned} \tag{3.46}$$

This is the statistical average over the random variable $\phi(t)$. In order to determine dynamical evolution of the system one has to determine **G**. The problem is now simplified because we have to deal whit an interval in which ϕ (or alter natively g) is constant and the change in ρ is perfectly regular. Thus, by knowing **G**, we can find the average variation in the system, unbounded with respect to time.

Now we will firstly calculate the H_{int} matrix for the basis $\{|ne\rangle, |n+1g\rangle\}$

$$\begin{aligned} H_{11} &= \langle ne|\, H_{int}\, |ne\rangle \\ H_{12} &= \langle ne|\, H_{int}|\, n+1g\rangle \\ H_{21} &= \langle n+1g|\, H_{int}\, |ne\rangle \\ H_{22} &= \langle n+1g|\, H_{int}\, |n+1g\rangle \end{aligned} \tag{3.47}$$

from the above equation H_{int} matrix

$$H_{int} = \begin{pmatrix} 0 & g_o e^{-i\phi}\sqrt{n+1} \\ g_o e^{+i\phi}\sqrt{n+1} & 0 \end{pmatrix} \tag{3.48}$$

The unitary matrix

$$\mathbf{U} = \exp(-iH_{int}t) = \begin{pmatrix} \cos(\theta_n t) & -i\exp(-i\phi)\sin(\theta_n t) \\ -i\exp(+i\phi)\sin(\theta_n t) & \cos(\theta_n t) \end{pmatrix} \tag{3.49}$$

and the inverse of the unitary matrix

$$\mathbf{U}^{-1} = \begin{pmatrix} \cos(\theta_n t) & i\exp(-i\phi)\sin(\theta_n t) \\ i\exp(+i\phi)\sin(\theta_n t) & \cos(\theta_n t) \end{pmatrix} \tag{3.50}$$

in which $\theta_n = g_0(n+1)^{1/2}$. The elements to be averaged in the calculations of $\mathbf{G}$ are those containing the factors $\exp(\pm i\phi)$. Since the phases are equally probable, that is

$$\mathrm{d}Q = \frac{\mathrm{d}\phi}{2\pi}, \tag{3.51}$$

from equation (3.45)

$$\begin{aligned}
\mathbf{G}^{11}_{11} &= \int_0^{2\pi} \mathbf{U}_{11}(\phi;\tau,t).\mathbf{U}^{-1}_{11}(\phi;\tau,t)\mathrm{d}Q(\phi) \\
&= \int_0^{2\pi} \cos[\theta_n(\tau-t)].\cos[\theta_n(\tau-t)]\frac{\mathrm{d}\phi}{2\pi} \\
&= \cos^2[\theta_n(\tau-t)] \\
\mathbf{G}^{11}_{12} &= \int_0^{2\pi} \mathbf{U}_{12}(\phi;\tau,t).\mathbf{U}^{-1}_{11}(\phi;\tau,t)\mathrm{d}Q(\phi) \\
&= \int_0^{2\pi} -ie^{-i\phi}\sin[\theta_n(\tau-t)]\cos[\theta_n(\tau-t)]\frac{\mathrm{d}\phi}{2\pi} \\
&= 0 \qquad\qquad (3.52) \\
\mathbf{G}^{11}_{21} &= \mathbf{G}^{11}_{12} \\
\mathbf{G}^{11}_{22} &= \int_0^{2\pi} \mathbf{U}_{12}(\phi;\tau,t).\mathbf{U}^{-1}_{21}(\phi;\tau,t)\mathrm{d}Q(\phi) \\
&= \int_0^{2\pi} (-i)e^{-i\phi}\sin[\theta_n(\tau-t)](+i)e^{i\phi}\sin[\theta_n(\tau-t)]\frac{\mathrm{d}\phi}{2\pi} \\
&= \sin^2[\theta_n(\tau-t)]
\end{aligned}$$

We have thus

$$\mathbf{G}^{11} = \begin{pmatrix} \cos^2[\theta_n(\tau-t)] & 0 \\ 0 & \sin^2[\theta_n(\tau-t)] \end{pmatrix} \tag{3.53}$$

Similarly from (3.45)

$$
\begin{aligned}
\mathbf{G}_{11}^{22} &= \int_o^{2\pi} \mathbf{U}_{12}(\phi;\tau,t)\mathbf{U}_{21}^{-1}(\phi;\tau,t)\mathrm{d}Q(\phi) \\
&= \sin^2[\theta_n(\tau-t)] \\
\mathbf{G}_{12}^{22} &= \int_o^{2\pi} \mathbf{U}_{22}(\phi;\tau,t)\mathbf{U}_{12}^{-1}(\phi;\tau,t)\mathrm{d}Q(\phi) \\
&= 0 \qquad (3.54) \\
\mathbf{G}_{12}^{22} &= \mathbf{G}_{21}^{22} \\
\mathbf{G}_{22}^{22} &= \int_o^{2\pi} \mathbf{U}_{22}(\phi;\tau,t)\mathbf{U}_{22}^{-1}(\phi;\tau,t)\mathrm{d}Q(\phi) \\
&= \cos^2[\theta_n(\tau-t)]
\end{aligned}
$$

We have thus

$$
\mathbf{G}^{22} = \begin{pmatrix} \sin^2[\theta_n(\tau-t)] & 0 \\ 0 & \cos^2[\theta_n(\tau-t)] \end{pmatrix} \qquad (3.55)
$$

We need not calculate all elements of the density matrix using (3.45). For two-level atom we have $\rho_{11} = 1 - \rho_{22}$ and $\rho_{12} = \rho_{21}^*$; so it is perfectly sufficient to know only ρ_{11} and ρ_{12} to determine the complete dynamics of the system.

From equation (3.46)

$$
\begin{aligned}
\rho_{11}(\tau) &= \exp\left(-\frac{\tau}{T}\right)\mathrm{Tr}\left[\mathbf{G}^{11}(\tau,o)\rho(0)\right] \\
&\quad + \frac{1}{T}\int_0^{\tau} \mathrm{d}t \exp\left(-\frac{(\tau-t)}{T}\right)\mathrm{Tr}\left[\mathbf{G}^{11}(\tau,t)\rho(t)\right]. \qquad (3.56)
\end{aligned}
$$

We get

$$
\rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} \qquad (3.57)
$$

and

$$\mathbf{G}^{11}(\tau, o) = \begin{pmatrix} \cos^2[\theta_n \tau] & 0 \\ 0 & \sin^2[\theta_n \tau] \end{pmatrix}$$

Then we multiply

$$\mathbf{G}^{11}(\tau, 0).\rho(0) = \begin{pmatrix} \cos^2[\theta_n \tau]\rho_{11}(0) & \cos^2[\theta_n \tau]\rho_{12}(0) \\ \sin^2[\theta_n \tau]\rho_{21}(0) & \sin^2[\theta_n \tau]\rho_{22}(0) \end{pmatrix}$$

Then we obtain that

$$\begin{aligned} \mathrm{Tr}\left[\mathbf{G}^{11}(\tau, o)\rho(0)\right] &= \rho_{11}(0)\cos^2[\theta_n \tau] + \rho_{22}(0)\sin^2[\theta_n \tau] \\ &= \rho_{11}(0) + [\rho_{22}(0) - \rho_{11}(0)]\sin^2[\theta_n \tau] \end{aligned} \tag{3.58}$$

We get

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix} \tag{3.59}$$

Using the equations (3.53) and (3.59)

$$\mathbf{G}^{11}(\tau, t).\rho(t) = \begin{pmatrix} \cos^2[\theta_n(\tau - t)]\rho_{11}(t) & \cos^2[\theta_n(\tau - t)]\rho_{12}(t) \\ \sin^2[\theta_n(\tau - t)]\rho_{21}(t) & \sin^2[\theta_n(\tau - t)]\rho_{22}(t) \end{pmatrix}$$

$$\begin{aligned} \mathrm{Tr}\left[\mathbf{G}^{11}(\tau, t).\rho(t)\right] &= \rho_{11}(t)\cos^2[\theta_n(\tau - t)] + \rho_{22}(t)\sin^2[\theta_n(\tau - t)] \\ &= \rho_{11}(t) + [\rho_{22}(t) - \rho_{11}(t)]\sin^2[\theta_n(\tau - t)] \end{aligned} \tag{3.60}$$

Inserting equations (3.58) and (3.60) in equation (3.56) we obtain that

$$\begin{aligned} \rho_{11}(\tau)\exp\left(+\frac{\tau}{T}\right) &= \rho_{11}(0) + [\rho_{22}(0) - \rho_{11}(0)]\sin^2[\theta_n \tau] \\ &\quad + \frac{1}{T}\int_0^\tau \exp\left(+\frac{t}{T}\right)\left\{\rho_{11}(t) + [\rho_{22}(t) - \rho_{11}(t)]\sin^2[\theta_n(\tau - t)]\right\} \end{aligned} \tag{3.61}$$

Similarly

$$\rho_{22}(\tau)\exp\left(+\frac{\tau}{T}\right) = \rho_{22}(0) + [\rho_{11}(0) - \rho_{22}(0)]\sin^2[\theta_n\tau] + \frac{1}{T}\int_0^\tau \exp\left(+\frac{t}{T}\right)\Big\{\rho_{22}(t) + [\rho_{11}(t) - \rho_{22}(t)]\sin^2[\theta_n(\tau - t)]\Big\} \tag{3.62}$$

To quantify of experimental interest called inversion is defined as $W_n = \rho_{11} - \rho_{22}$ and can be evaluated using equations (3.61) and (3.62).

Firstly we substract equations (3.61) and (3.62):

$$(\rho_{11}(\tau) - \rho_{22}(\tau))\exp\left(\frac{\tau}{T}\right) = \Big\{[\rho_{11}(0) - \rho_{22}(0)] + 2[\rho_{22}(0) - \rho_{11}(0)]\sin^2[Q_n\tau]\Big\} + \frac{1}{T}\int_0^\tau \exp\left(\frac{t}{T}\right)\Big\{[\rho_{11}(t) - \rho_{22}(t)] + 2[\rho_{22}(t) - \rho_{11}(t)]\sin^2[\theta_n(\tau - t)]\Big\} \tag{3.63}$$

Since

$$W_n = \rho_{11} - \rho_{22}\,, \qquad W_n(0) = \rho_{11}(0) - \rho_{22}(0) \tag{3.64}$$

Substituting (3.64) in equation (3.63) we obtain

$$W_n\exp\left(+\frac{\tau}{T}\right) = W_n(0)[1 - 2\sin^2(\theta_n\tau)] + \frac{1}{T}\int_0^\tau \exp\left(+\frac{t}{T}\right)\Big\{1 - 2\sin^2[\theta_n(\tau - t)]\Big\}W_n(t)\mathrm{d}t. \tag{3.65}$$

This equation describes the relaxation of the population and can be easily solved by the known procedure of differentiation. Differentiating (3.65) three times and carrying out necessary simplifications we have

$$\frac{dW_n}{d\tau}\exp\left(\frac{\tau}{T}\right)+\frac{1}{T}W_n\exp\left(\frac{\tau}{T}\right)= W_n(0)[-(2\theta_n)\sin(2\theta_n\tau)]$$
$$+\frac{1}{T}\int_0^\tau \exp\left(\frac{t}{T}\right)[-(2\theta_n)\sin(2\theta_n(\tau-t))]W_n(t)\mathrm{d}t+\exp\left(\frac{\tau}{T}\right)W_n(\tau)\frac{1}{T} \tag{3.66}$$

$$\frac{d^2W_n}{d\tau^2}\exp\left(\frac{\tau}{T}\right)+\frac{dW_n}{d\tau}\exp\left(\frac{\tau}{T}\right)\frac{2}{T}+\frac{1}{T^2}W_n\exp\left(\frac{\tau}{T}\right)= W_n(0)[-4\theta_n^2]\cos(2\theta_n\tau)$$
$$+\frac{1}{T}\int_0^\tau \exp\left(\frac{t}{T}\right)[-(4\theta_n^2)\cos(2\theta_n(\tau-t))]W_n(t)\mathrm{d}t+\exp\left(\frac{\tau}{T}\right)W_n(\tau)\frac{1}{T^2}+\exp\left(\frac{\tau}{T}\right)\frac{dW_n}{d\tau}\frac{1}{T}$$

$$\left(\frac{d^2W_n}{d\tau^2}+\frac{dW_n}{d\tau}\frac{1}{T}\right)\exp\left(\frac{\tau}{T}\right)=(-4\theta_n^2)W_n\exp\left(\frac{\tau}{T}\right) \tag{3.67}$$

$$\left(\frac{d^3W_n}{d\tau^3}+\frac{d^2W_n}{d\tau^2}\frac{1}{T}\right)\exp\left(\frac{\tau}{T}\right)+\left(\frac{d^2W_n}{d\tau^2}\frac{1}{T}+\frac{dW_n}{d\tau}\frac{1}{T^2}\right)\exp\left(\frac{\tau}{T}\right)$$
$$=\frac{dW_n}{d\tau}(-4\theta_n^2)\exp\left(\frac{\tau}{T}\right)+(-4\theta_n^2)\frac{1}{T}W_n\exp\left(\frac{\tau}{T}\right) \tag{3.68}$$

Equations (3.66),(3.67) and (3.68) gives us

$$\frac{d^3W_n}{d\tau^3}+\frac{2}{T}\frac{d^2W_n}{d\tau^2}+\left(\frac{1}{T^2}+(2\theta_n)^2\right)\frac{dW_n}{d\tau}+\frac{(2\theta_n)^2}{T}W_n=0 \tag{3.69}$$

where the required solution should satisfy the following initial conditions:

$$\frac{dW_n}{d\tau}=0 \qquad\qquad \frac{d^2W_n}{d\tau^2}=-(2\theta_n)^2W_n(0). \tag{3.70}$$

We use of Laplace transform to solve equation (3.69) together with the initial conditions mentioned in equation (3.70),

$$\mathcal{L}\left\{\frac{d^3W_n}{d\tau^3}+\frac{2}{T}\frac{d^2W_n}{d\tau^2}+\left(\frac{1}{T^2}+(2\theta_n)^2\right)\frac{dW_n}{d\tau}+\frac{(2\theta_n)^2}{T}W_n,\tau,s\right\}=0$$

We have

$$\widetilde{W}_n(s) = \frac{s^2 + \frac{2s}{T} + \frac{1}{T^2}}{s^3 + \frac{2s^2}{T} + \frac{s}{T^2} + (2\theta_n)^2 s + \frac{(2\theta_n)^2}{T}} W_n(0) \tag{3.71}$$

By Inverselaplace transform of equation 3.71 we get

$$\begin{aligned} W_n(t) &= \Big(\cos[2g_0(n+1)^{1/2} \mathrm{Z}_n t] \\ &\quad + \frac{1}{2g_o T(n+1)^{1/2}\mathrm{Z}_n} \sin[2g_0(n+1)^{1/2}\mathrm{Z}_n t] \Big) \exp\left(-\frac{t}{T}\right), \end{aligned} \tag{3.72}$$

in which

$$\mathrm{Z}_n = \left(\frac{1}{16g_0^2 T^2 (n+1)} - 1 \right)^{1/2} \tag{3.73}$$

Now, we begin a new calculation in which we use the master equation (3.44) to find the time evolution density matrix. During the process we use Laplace transform techniques. Since the master equation (3.44) equal to:

$$\begin{aligned} \rho(\tau) \exp\left(\frac{\tau}{T}\right) &= \int U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0)dQ(\phi) \\ &\quad + \frac{1}{T}\int \exp(\frac{t}{T}) \int U(\phi;\tau,t)\rho(t)U^{-1}(\phi;\tau,t)dQ(\phi)dt, \end{aligned} \tag{3.74}$$

By defining

$$\rho(\tau) = \begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) \end{pmatrix}, \; \rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix}, \; \rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) \\ \rho_{21}(t) & \rho_{22}(t) \end{pmatrix} \tag{3.75}$$

and from equations (3.49) and (3.50)

$$\mathbf{U}(\phi;\tau,0) = \begin{pmatrix} \cos(\theta_n\tau) & -i\exp(-i\phi)\sin(\theta_n\tau) \\ -i\exp(+i\phi)\sin(\theta_n\tau) & \cos(\theta_n\tau) \end{pmatrix}, \tag{3.76}$$

$$\mathbf{U}^{-1}(\phi;\tau,0) = \begin{pmatrix} \cos(\theta_n\tau) & i\exp(-i\phi)\sin(\theta_n\tau) \\ i\exp(+i\phi)\sin(\theta_n\tau) & \cos(\theta_n\tau) \end{pmatrix} \tag{3.77}$$

Similarly,

$$\mathbf{U}(\phi;\tau,t) = \begin{pmatrix} \cos[\theta_n(\tau-t)] & -i\exp(-i\phi)\sin[\theta_n(\tau-t)] \\ -i\exp(+i\phi)\sin[\theta_n(\tau-t)] & \cos[\theta_n(\tau-t)] \end{pmatrix}, \tag{3.78}$$

$$\mathbf{U}^{-1}(\phi;\tau,t) = \begin{pmatrix} \cos[\theta_n(\tau-t)] & i\exp(-i\phi)\sin[\theta_n(\tau-t)] \\ i\exp(+i\phi)\sin[\theta_n(\tau-t)] & \cos[\theta_n(\tau-t)] \end{pmatrix} \tag{3.79}$$

We begin calculation getting the integrations on the right and side of the master equation

$$\int_0^{2\pi} \mathbf{U}(\phi;\tau,0)\rho(0)\mathbf{U}^{-1}(\phi;\tau,0)\frac{d\phi}{2\pi} = \begin{pmatrix} \cos^2[\theta_n\tau]\rho_{11}(0) & \cos^2[\theta_n\tau]\rho_{12}(0) \\ +\sin^2[\theta_n\tau]\rho_{22}(0) & \\ & \sin^2[\theta_n\tau]\rho_{11}(0) \\ \cos^2[\theta_n\tau]\rho_{21}(0) & +\cos^2[\theta_n\tau]\rho_{22}(0) \end{pmatrix} \tag{3.80}$$

$$\int_0^{2\pi} \mathbf{U}(\phi;\tau,t)\rho(t)\mathbf{U}^{-1}(\phi;\tau,t)\frac{d\phi}{2\pi} = \begin{pmatrix} \cos^2[\theta_n(\tau-t)]\rho_{11}(t) & \cos^2[\theta_n(\tau-t)]\rho_{12}(t) \\ +\sin^2[\theta_n(\tau-t)]\rho_{22}(t) & \\ & \sin^2[\theta_n(\tau-t)]\rho_{11}(t) \\ \cos^2[\theta_n(\tau-t)]\rho_{21}(t) & +\cos^2[\theta_n(\tau-t)]\rho_{22}(t) \end{pmatrix} \tag{3.81}$$

Then we divide both side of the equation $\exp\left(\frac{\tau}{T}\right)$. After division we get Laplace transform with respect to τ;

$$\mathcal{L}\left\{\begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) \end{pmatrix}\right\} = \mathcal{L}\left\{\exp\left(-\frac{\tau}{T}\right)\begin{pmatrix} \cos^2[\theta_n\tau]\rho_{11}(0) & \cos^2[\theta_n\tau]\rho_{12}(0) \\ +\sin^2[\theta_n\tau]\rho_{22}(0) & \\ & \sin^2[\theta_n\tau]\rho_{11}(0) \\ \cos^2[\theta_n\tau]\rho_{21}(0) & +\cos^2[\theta_n\tau]\rho_{22}(0) \end{pmatrix}\right\}$$

$$+\mathcal{L}\left\{\frac{1}{T}\int dt\exp\left(-\frac{\tau-t}{T}\right)\begin{pmatrix} \cos^2[\theta_n(\tau-t)]\rho_{11}(t) & \cos^2[\theta_n(\tau-t)]\rho_{12}(t) \\ +\sin^2[\theta_n(\tau-t)]\rho_{22}(t) & \\ & \sin^2[\theta_n(\tau-t)]\rho_{11}(t) \\ \cos^2[\theta_n(\tau-t)]\rho_{21}(t) & +\cos^2[\theta_n(\tau-t)]\rho_{22}(t) \end{pmatrix}\right\}$$

(3.82)

Thus we have

$$\left\{\{\rho_{11}(s),\rho_{12}(s)\},\{\rho_{21}(s),\rho_{22}(s)\}\right\} =$$

$$\left\{\{\frac{T((1+Ts)^2+2T^2\theta_n^2)\rho_{11}(0)+2T^3\theta_n^2\rho_{22}(0)}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2},\frac{T((1+Ts)^2+2T^2\theta_n^2)\rho_{12}(0)}{(1+Ts)^3+4\tau_o^2(1+Ts)\theta_n^2}\},\right.$$

$$\left.\{\frac{T((1+Ts)^2+2T^2\theta_n^2)\rho_{12}(0)}{(1+Ts)^3+4\tau_o^2(1+Ts)\theta_n^2},\frac{2T^3\theta_n^2\rho_{11}(0)+((1+Ts)^2+2T^2\theta_n^2)\rho_{22}(0)}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2}\}\right\}$$

$$+\frac{1}{T}\left\{\{\frac{T((1+Ts)^2+2T^2\theta_n^2)\rho_{11}(s)+2T^3\theta_n^2\rho_{22}(s)}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2},\frac{T((1+Ts)^2+2T^2\theta_n^2)\rho_{12}(s)}{(1+Ts)^3+4\tau_o^2(1+Ts)\theta_n^2}\},\right.$$

$$\left.\{\frac{T((1+Ts)^2+2T^2\theta_n^2)\rho_{12}(s)}{(1+Ts)^3+4\tau_o^2(1+Ts)\theta_n^2},\frac{2T^3\theta_n^2\rho_{11}(s)+((1+Ts)^2+2T^2\theta_n^2)\rho_{22}(s)}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2}\}\right\}$$

(3.83)

After addition, we get

$$\left\{\{\rho_{11}(s),\rho_{12}(s)\},\{\rho_{21}(s),\rho_{22}(s)\}\right\} =$$

$$\left\{\{\frac{((1+Ts)^2+2T^2\theta_n^2)\rho_{11}(s)+T(\rho_{11}(0)+T(s(2+Ts)\rho_{11}(0)+2\theta_n^2(\rho_{22}(s+T(\rho_{11}(0+\rho_{22}(0)))))}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2},\right.$$

$$\frac{((1+Ts)^2+2T^2\theta_n^2)(\rho_{12}(s)+T\rho_{12}(0))}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2}\},\{\frac{((1+Ts)^2+2T^2\theta_n^2)(\rho_{12}(s)+T\rho_{12}(0))}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2},$$

$$\left.\frac{2T^2\theta_n^2\rho_{11}(s)+((1+Ts)^2+2T^2\theta_n^2)\rho_{22}(s)+2T^3\theta_n^2\rho_{11}(0)+\tau((1+Ts)^2+2T^2\theta_n^2)\rho_{22}(0)}{(1+Ts)^3+4T^2(1+Ts)\theta_n^2}\}\right\} \tag{3.84}$$

When we solve the equation (3.84) above, we get ;

$$\rho_{11}(s) = \frac{(s+Ts^2+2T\theta_n^2)\rho_{11}(0)+2T\theta_n^2\rho_{22}(0)}{s(s+\tau s^2+4T\theta_n^2)} \tag{3.85}$$

$$\rho_{12}(s) = \frac{((1+Ts)^2+2T^2\theta_n^2)\rho_{12}(0)}{s(1+Ts)^2+2T(1+2Ts)\theta_n^2} \tag{3.86}$$

$$\rho_{21}(s) = \frac{((1+Ts)^2+2T^2\theta_n^2)\rho_{21}(0)}{s(1+Ts)^2+2T(1+2Ts)\theta_n^2} \tag{3.87}$$

$$\rho_{22}(s) = \frac{2T\theta_n^2\rho_{11}(0)+(s+Ts^2+2T\theta_n^2)\rho_{22}(0)}{s(s+\tau s^2+4T\theta_n^2)} \tag{3.88}$$

Now, we get the Inverselaplace transform of equations (3.85) and (3.88). We do not need the (3.86) and (3.87).

$$\rho_{11}(t)=\frac{1}{2}\left(\rho_{11}(0)+e^{-\frac{t}{2T}}\left(\cosh[2g_0(n+1)^{1/2}z_nt]+\frac{sinh[2g_0(n+1)^{1/2}Z_nt]}{2g_0T(n+1)^{1/2}Z_n}\right)\Big(\rho_{11}(0)-\rho_{22}(0)\Big)+\rho_{22}(0)\right)$$

$$\rho_{22}(t)=\frac{1}{2}\left(\rho_{11}(0)-e^{-\frac{t}{2T}}\left(\cosh[2g_0(n+1)^{1/2}z_nt]+\frac{sinh[2g_0(n+1)^{1/2}Z_nt]}{2g_0T(n+1)^{1/2}Z_n}\right)\Big(\rho_{11}(0)-\rho_{22}(0)\Big)+\rho_{22}(0)\right)$$

(3.89)

in which

$$Z_n = \left(\frac{1}{16g_0^2 T^2(n+1)} - 1 \right)^{1/2} \tag{3.90}$$

We know from equation (3.70) $W_n = \rho_{11} - \rho_{22}$. So if we use the(3.89) we have

$$\begin{aligned} W_n(t) &= \Bigg(\cos[2g_0(n+1)^{1/2} Z_n t] \\ &\quad + \frac{1}{2g_o T(n+1)^{1/2} Z_n} \sin[2g_0(n+1)^{1/2} Z_n t] \Bigg) \exp\left(-\frac{t}{T}\right) W_n(0) \end{aligned} \tag{3.91}$$

The exact solution in equation (3.72) is completely the same as equation (3.91).

CHAPTER 4

REVIEW OF ENTANGLEMENT OF ONE ATOM-TWO FIELDS INTERACTION UNDER RANDOM PHASE TELEGRAPH NOISE

In this chapter, we review one atom-two fields interaction under random phase telegraph noise and re-investigate the results [20]. The illustrated model is the JCM with random telegraph process. We review entanglement in the interaction system and its subsystems. We find one atom-one field subsystem is not entangled, but field-field subsystem is entangled under random phase telegraph noise as given [20].

4.1 Formulation and Exact Solution

A two-level atom described by spin-1/2 operators $(S_{\pm}, S_z)$ interacts with a single mode of two identical optical cavity fields described by the annihilation (a,b) and the creation $(a^{\dagger}, b^{\dagger})$ operators. The configuration of the system resembles the system of Plenio and Huelga [81] but no thermal driving is involved and the radiation fields are in resonance with the atomic transition frequency ω_o. The Hamiltonian of the system with rotating wave approximation under the assumed conditions is ($\hbar = 1$)

$$
\begin{aligned}
H &= \omega_0 S_z + \omega_0 a^{\dagger} a + \omega_0 b^{\dagger} b \\
&\quad + (g_1^*(t) S_+ a + g_1(t) S_- a^{\dagger}) + (g_2^*(t) S_+ b + g_2(t) S_- b^{\dagger}) \qquad (4.1)
\end{aligned}
$$

in which the interaction part is

$$
H_{int}(t) = g^*(t) S_+ a + g(t) S_- a^{\dagger} + g^*(t) S_+ b + g(t) S_- b^{\dagger} \qquad (4.2)
$$

where g(t) is the time-dependent coupling coefficient between the atom and the fields, which is considered to be completely stochastic:

$$g(t) = g_0 e^{i\phi(t)} \tag{4.3}$$

g_o is a real number. In calculations we set $g_0 = 1$. $\phi(t)$ is jump manner fluctuating stochastic variable between arbitrary phase occur according to a Poisson process. The jumps are separated by an average time interval called the mean dwell time, which is the mean inter-arrival time of the Poisson process. The mean dwell time is a predominant factor that determines the strength of the noise. $\phi(t)$ for neighboring intervals are not correlated. Consequently, the probability of finding $\phi(t)$ is the same-constant for any instant of time. Thus, $\phi(t)$ can go through continuous random change of the Markov type, which allows one to take the average over the stochastic fluctuations.

We initiated the state of the system as the cavity fields in the vacuum state $|00\rangle$ and the atom in the excited state $|e\rangle$. Hence the initial density matrix of the system becomes

$$\rho(0) = |00\rangle\langle 00| \otimes |e\rangle\langle e|. \tag{4.4}$$

Then the time-dependent interaction density matrix can be written

$$\begin{aligned} \rho(\tau)\exp\left(\frac{\tau}{T}\right) &= \int U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0)dQ(\phi) \\ &+\frac{1}{T}\int \exp(\frac{t}{T})\int U(\phi;\tau,t)\rho(t)U^{-1}(\phi;\tau,t)dQ(\phi)dt, \end{aligned} \tag{4.5}$$

where T is the mean dwell time and $dQ(\phi) = \frac{d\phi}{2\pi}$.

Firstly our purpose is to obtain the solution of density matrix. We will use of the Laplace transformation techniques during the solution process.

Now we obtain H_{int} matrix with basis $\{|00e\rangle, |10e\rangle, |01e\rangle, |11e\rangle, |00g\rangle, |10g\rangle, |01g\rangle, |11g\rangle\}$

$$\begin{array}{llll}
H_{11} = \langle 00e|H_{int}|00e\rangle & H_{12} = \langle 00e|H_{int}|10e\rangle & .... & H_{18} = \langle 00e|H_{int}|11g\rangle \\
H_{21} = \langle 10e|H_{int}|00e\rangle & H_{22} = \langle 10e|H_{int}|10e\rangle & .... & H_{28} = \langle 10e|H_{int}|11g\rangle \\
\vdots & \vdots & .... & \vdots \\
\vdots & \vdots & .... & \vdots \\
H_{81} = \langle 11g|H_{int}|00e\rangle & H_{82} = \langle 11g|H_{int}|10e\rangle & .... & H_{88} = \langle 11g|H_{int}|11g\rangle
\end{array} \tag{4.6}$$

From the above equation (4.6), we obtained the H_{int} matrix:

$$H_{int} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e^{-i\phi} & e^{-i\phi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-i\phi} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ e^{i\phi} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{i\phi} & e^{i\phi} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

We know that the unitary matrix equal to

$$\mathbf{U} = \exp(-iH_{int}\tau).$$

So if we use the equation (4.7), we have

$$\mathbf{U} = \Big\{\{\cos[\sqrt{2}\tau], 0, 0, 0, 0, -\frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}, -\frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}, 0\},$$
$$\{0, \cos^2[\frac{\tau}{\sqrt{2}}], -\sin^2[\frac{\tau}{\sqrt{2}}], 0, 0, 0, 0, -\frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}\},$$
$$\{0, -\sin^2[\frac{\tau}{\sqrt{2}}], \cos^2[\frac{\tau}{\sqrt{2}}], 0, 0, 0, 0, -\frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}\},$$
$$\{0, 0, 0, 1, 0, 0, 0, 0\},$$
$$\{0, 0, 0, 0, 1, 0, 0, 0\},$$
$$\{\frac{\sin[\sqrt{2}\tau](-i\cos[\phi] + \sin[\phi])}{\sqrt{2}}, 0, 0, 0, 0, \cos^2[\frac{\tau}{\sqrt{2}}], -\sin^2[\frac{\tau}{\sqrt{2}}], 0\},$$
$$\{\frac{\sin[\sqrt{2}\tau](-i\cos[\phi] + \sin[\phi])}{\sqrt{2}}, 0, 0, 0, 0, -\sin^2[\frac{\tau}{\sqrt{2}}], \cos^2[\frac{\tau}{\sqrt{2}}], 0\},$$
$$\{0, \frac{\sin[\sqrt{2}\tau](-i\cos[\phi] + \sin[\phi])}{\sqrt{2}}, \frac{\sin[\sqrt{2}\tau](-i\cos[\phi] + \sin[\phi])}{\sqrt{2}}, 0, 0, 0, 0, \cos[\sqrt{2}\tau]\}\Big\} \tag{4.8}$$

The inverse of unitary matrix

$$\mathbf{U}^{-1} = \Big\{\{\cos[\sqrt{2}\tau], 0, 0, 0, 0, \frac{\sin[\sqrt{2}\tau](i\cos[\phi] + \sin[\phi])}{\sqrt{2}}, \frac{\sin[\sqrt{2}\tau](i\cos[\phi] + \sin[\phi])}{\sqrt{2}}, 0\},$$
$$\{0, \cos^2[\frac{\tau}{\sqrt{2}}], -\sin^2[\frac{\tau}{\sqrt{2}}], 0, 0, 0, 0, \frac{\sin[\sqrt{2}\tau](i\cos[\phi] + \sin[\phi])}{\sqrt{2}}\},$$
$$\{0, -\sin^2[\frac{\tau}{\sqrt{2}}], \cos^2[\frac{\tau}{\sqrt{2}}], 0, 0, 0, 0, \frac{\sin[\sqrt{2}\tau](i\cos[\phi] + \sin[\phi])}{\sqrt{2}}\},$$
$$\{0, 0, 0, 1, 0, 0, 0, 0\},$$
$$\{0, 0, 0, 0, 1, 0, 0, 0\},$$
$$\{\frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}, 0, 0, 0, 0, \cos^2[\frac{\tau}{\sqrt{2}}], -\sin^2[\frac{\tau}{\sqrt{2}}], 0\},$$
$$\{\frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}, 0, 0, 0, 0, -\sin^2[\frac{\tau}{\sqrt{2}}], \cos^2[\frac{\tau}{\sqrt{2}}], 0\},$$
$$\{0, \frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}, \frac{ie^{i\phi}\sin[\sqrt{2}\tau]}{\sqrt{2}}, 0, 0, 0, 0, \cos[\sqrt{2}\tau]\}\Big\} \tag{4.9}$$

By defining

$$\rho(0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.10)$$

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t) & \rho_{15}(t) & \rho_{16}(t) & \rho_{17}(t) & \rho_{18}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t) & \rho_{25}(t) & \rho_{26}(t) & \rho_{27}(t) & \rho_{28}(t) \\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) & \rho_{34}(t) & \rho_{35}(t) & \rho_{36}(t) & \rho_{37}(t) & \rho_{38}(t) \\ \rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t) & \rho_{44}(t) & \rho_{45}(t) & \rho_{46}(t) & \rho_{47}(t) & \rho_{48}(t) \\ \rho_{51}(t) & \rho_{52}(t) & \rho_{53}(t) & \rho_{54}(t) & \rho_{55}(t) & \rho_{56}(t) & \rho_{57}(t) & \rho_{58}(t) \\ \rho_{61}(t) & \rho_{62}(t) & \rho_{63}(t) & \rho_{64}(t) & \rho_{65}(t) & \rho_{66}(t) & \rho_{67}(t) & \rho_{68}(t) \\ \rho_{71}(t) & \rho_{72}(t) & \rho_{73}(t) & \rho_{74}(t) & \rho_{75}(t) & \rho_{76}(t) & \rho_{77}(t) & \rho_{78}(t) \\ \rho_{81}(t) & \rho_{82}(t) & \rho_{83}(t) & \rho_{84}(t) & \rho_{85}(t) & \rho_{86}(t) & \rho_{87}(t) & \rho_{88}(t) \end{pmatrix} \quad (4.11)$$

and

$$\rho(\tau) = \begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) & \rho_{13}(\tau) & \rho_{14}(\tau) & \rho_{15}(\tau) & \rho_{16}(\tau) & \rho_{17}(\tau) & \rho_{18}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) & \rho_{23}(\tau) & \rho_{24}(\tau) & \rho_{25}(\tau) & \rho_{26}(\tau) & \rho_{27}(\tau) & \rho_{28}(\tau) \\ \rho_{31}(\tau) & \rho_{32}(\tau) & \rho_{33}(\tau) & \rho_{34}(\tau) & \rho_{35}(\tau) & \rho_{36}(\tau) & \rho_{37}(\tau) & \rho_{38}(\tau) \\ \rho_{41}(\tau) & \rho_{42}(\tau) & \rho_{43}(\tau) & \rho_{44}(\tau) & \rho_{45}(\tau) & \rho_{46}(\tau) & \rho_{47}(\tau) & \rho_{48}(\tau) \\ \rho_{51}(\tau) & \rho_{52}(\tau) & \rho_{53}(\tau) & \rho_{54}(\tau) & \rho_{55}(\tau) & \rho_{56}(\tau) & \rho_{57}(\tau) & \rho_{58}(\tau) \\ \rho_{61}(\tau) & \rho_{62}(\tau) & \rho_{63}(\tau) & \rho_{64}(\tau) & \rho_{65}(\tau) & \rho_{66}(\tau) & \rho_{67}(\tau) & \rho_{68}(\tau) \\ \rho_{71}(\tau) & \rho_{72}(\tau) & \rho_{73}(\tau) & \rho_{74}(\tau) & \rho_{75}(\tau) & \rho_{76}(\tau) & \rho_{77}(\tau) & \rho_{78}(\tau) \\ \rho_{81}(\tau) & \rho_{82}(\tau) & \rho_{83}(\tau) & \rho_{84}(\tau) & \rho_{85}(\tau) & \rho_{86}(\tau) & \rho_{87}(\tau) & \rho_{88}(\tau) \end{pmatrix} \quad (4.12)$$

We apply the appropriate form of equations (4.8) and (4.9) with the conditions in equations (4.10), (4.11) and (4.12) to the master equation (5.5). Firstly we make the integrating parts respect to the ϕ in the interval $[0, 2\pi]$. Then we divide the both sides of the equation (we obtained) $e^{\left(+\frac{\tau}{T}\right)}$. After division we use of the Laplace transformation respect to the $\{\tau, s\}$. Then we solve the equation and obtain

$$\rho_{11}(s), \rho_{12}(s), ..., \rho_{18}(s), ..., \rho_{81}(s), ..., \rho_{88}(s).$$

Then we use the Inverselaplace transform to obtain the density matrix. We simplify the solution then we have the nonzero elements :

$$\rho_{11}(t) = \frac{1}{4}\left(2 + 2e^{-\frac{t}{2T}}\left(\cosh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right] + \frac{\sinh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right]}{\sqrt{1-32T^2}}\right)\right) \tag{4.13}$$

$$\rho_{66}(t) = \frac{1}{4}\left(1 - e^{-\frac{t}{2T}}\cosh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right] - \frac{e^{-\frac{t}{2T}}\sinh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right]}{\sqrt{1-32T^2}}\right) \tag{4.14}$$

$$\rho_{76}(t) = \frac{1}{4}\left(1 - e^{-\frac{t}{2T}}\cosh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right] - \frac{e^{-\frac{t}{2T}}\sinh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right]}{\sqrt{1-32T^2}}\right) \tag{4.15}$$

$$\rho_{77}(t) = \frac{1}{4}\left(1 - e^{-\frac{t}{2T}}\cosh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right] - \frac{e^{-\frac{t}{2T}}\sinh\left[\frac{\sqrt{1-32T^2}\,t}{2T}\right]}{\sqrt{1-32T^2}}\right) \tag{4.16}$$

The other elements of the density matrix are all zero.
The solution is

$$\begin{aligned} \rho(t) &= \rho_{11}(t)|00e\rangle\langle 00e| + \rho_{66}(t)|10g\rangle\langle 10g| + \rho_{67}(t)|10g\rangle\langle 01g| \\ &\quad + \rho_{76}(t)|01g\rangle\langle 10g| + \rho_{77}(t)|01g\rangle\langle 01g| \end{aligned} \tag{4.17}$$

4.2 Results

We use (4.17) for investigate the effects of the phase telegraph noise on the entanglement properties of the system.

Now we can obtain a subsystem which is containing field 1 and the atom by tracing out of the degree of freedom of the second field , using the equation (4.17). That is, the reduced density matrix $\rho^{af,1}(t)$;

$$\begin{aligned}\rho^{af,1}(t) &= \langle 0|\rho(t)|0\rangle + \langle 1|\rho(t)|1\rangle \\ &= \rho_{11}(t)|0e\rangle\langle 0e| + \rho_{66}(t)|1g\rangle\langle 1g| + \rho_{77}(t)|0g\rangle\langle 0g|\end{aligned} \tag{4.18}$$

and repeating it for tracing out the degree of freedom of the first field to obtain the reduced density matrix for field 2 and the atom subsystem $\rho^{af,2}(t)$ is

$$\begin{aligned}\rho^{af,2}(t) &= \langle 0|\rho(t)|0\rangle + \langle 1|\rho(t)|1\rangle \\ &= \rho_{11}(t)|0e\rangle\langle 0e| + \rho_{66}(t)|0g\rangle\langle 0g| + \rho_{77}(t)|1g\rangle\langle 1g|\end{aligned} \tag{4.19}$$

From equations (4.18) and (4.19), we can say that $\rho^{af,1}(t)$ and $\rho^{af,2}(t)$ gives the same density matrix.

$$\rho^{af,1}(t) = \rho^{af,2}(t) \tag{4.20}$$

Now, by tracing out the degree of the freedom of the atom,we obtain the reduced density matrix $\rho^{ff}(t)$, for the subsystem containing two fields which is

$$\begin{aligned}\rho^{ff}(t) &= \langle e|\rho(t)|e\rangle + \langle g|\rho(t)|g\rangle \\ &= \rho_{11}(t)|00\rangle\langle 00| + \rho_{66}(t)|10\rangle\langle 10| + \rho_{67}(t)|10\rangle\langle 01| \\ &\quad +\rho_{76}(t)|01\rangle\langle 10| + \rho_{77}(t)|01\rangle\langle 01|\end{aligned} \tag{4.21}$$

For these systems, the degree of the entanglement can be quantified by the Wootters's concurrence C(t) which is defined as

$$C(t) = \max\{0, \lambda_1(t) - \lambda_2(t) - \lambda_3(t) - \lambda_4(t)\} \tag{4.22}$$

where $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t) \geq \lambda_4(t)$ are the square roots of the eigenvalues of the matrix R(t) which is

$$R(t) = \rho(t)(\sigma_y \otimes \sigma_y)\rho^*(t)(\sigma_y \otimes \sigma_y) \tag{4.23}$$

The Pauli spin matrix [82] σ_y is

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.24}$$

And the tensor product of $(\sigma_y \otimes \sigma_y)$

$$(\sigma_y \otimes \sigma_y) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.25}$$

Now we will calculate the concurrence C(t) of these subsystems for to check entanglement properties.

Firstly we will make calculations for the field 1 and atom subsystem.From the above equation (4.18) the matrix form of the reduced density matrix $\rho^{af,1}(t)$, with the basis $\{|oe\rangle, |1e\rangle, |0g\rangle, |1g\rangle\}$

$$\rho^{af,1}(t) = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{77}(t) & 0 \\ 0 & 0 & 0 & \rho_{66}(t) \end{pmatrix} \tag{4.26}$$

Then we obtain $\left(\rho^{af,1}\right)^*(t)$. After these we can obtain $R^{af,1}(t)$. From equation (4.23)

$$R^{af,1}(t) = \rho^{af,1}(t)(\sigma_y \otimes \sigma_y)\left(\rho^{af,1}\right)^*(t)(\sigma_y \otimes \sigma_y) \tag{4.27}$$

Using the equations (4.26) and (4.25) we can obtain the $R^{af,1}(t)$. Then we need the eigenvalues of $R^{af,1}(t)$. In calculations we find two of the eigenvalues zero and the two of them nonzero. Then we get the square roots of the eigenvalues and then we order them from greater one to smaller one. From the definitions of concurrence in equation (4.22) we calculate the C(t) and we find it zero ($C(t) = 0$). Thus we can say that the subsystem which is containing one atom-one field, does not entangle.

Since $\rho^{af,1}(t) = \rho^{af,2}(t)$, we do not need a new calculation for $\rho^{af,2}(t)$. We will have same result again.

Lastly, we calculate the concurrence of field-field subsystem. From the above equation (4.21) the matrix form of the reduced density matrix $\rho^{ff}(t)$, with the basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$

$$\rho^{ff}(t) = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & 0 \\ 0 & \rho_{66}(t) & \rho_{67}(t) & 0 \\ 0 & \rho_{76}(t) & \rho_{77}(t) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.28}$$

We use the same procedure to calculate concurrence with field-field subsystem. We obtain

$$R^{ff}(t) = \rho^{ff}(t)(\sigma_y \otimes \sigma_y)\left(\rho^{ff}\right)^*(t)(\sigma_y \otimes \sigma_y) \tag{4.29}$$

Then we find the eigenvalues of $R^{ff}(t)$. We find the three of eigenvalues zero and the one of them is non zero. After these we get square roots of eigenvalues and then order them greater one to smaller one. From the definitions of concurrence in equation (4.22) we calculate the C(t) and we plot it. And we analyze the entanglement of this subsystem.

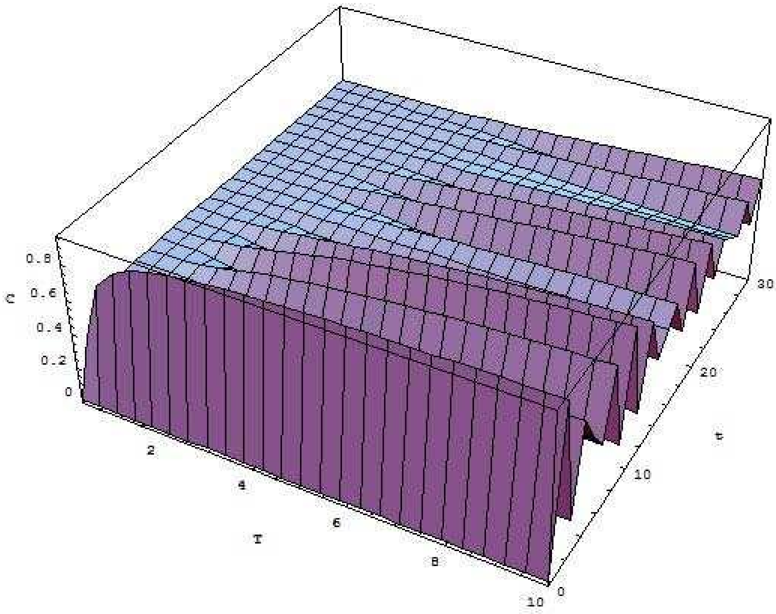

Figure 4.1: Concurrence C in the field-field interaction subsystem under RPTN with time t and mean dwell time T.

In Figure 4.1 we find the concurrence C over 0.5. We conclude that field-field subsystem is always entangled from [20]. In one atom-two field interaction system under phase telegraph noise by tracing out of the degree of the freedom of atom or field we find the subsystems and investigate their entanglement properties. Here the Wootters's concurrence can be used as a measure of entanglement. The concurrence C between the cavity fields reaches its peak value in a very short time. Then, as time passes, it oscillates and gradually decays until it takes a finite value. As the mean dwell time T increases, the effect of noise declines. But, at sufficiently small values of T, the decay of entanglement stabilizes to a finite value and remains in a stationary state. Thus, in these subsystems we find that one atom-one field interaction subsystem does not entangle under RPTN, but field-field interaction subsystem is entangled.

CHAPTER 5

THREE-PARTITE ENTANGLEMENT OF ONE FIELD-TWO ATOMS INTERACTION SYSTEM UNDER RANDOM PHASE TELEGRAPH NOISE

In this chapter we study a new work that is; two-atoms and two-fields interaction system under random phase telegraph noise. In the previous chapter we have reviewed bipartite system, now we increase the number of states. We look at three-partite entanglement subsystem and generalized the methods of the bipartite systems to multipartite system and we use fidelity as a measure of entanglement.

5.0.1 Formulation and Exact Solution

We have considered the interaction of two two-level atoms described by spin-$1/2$ operators $S_{\pm}^{(1,2)}, S_z^{(1,2)}$ with a single mode of two identical optical cavity fields described by the annihilation (a, b) and the creation $(a^{\dagger}, b^{\dagger})$ operators. The Hamiltonian of the system is

$$\begin{aligned} H &= \omega_0 S_z^1 + \omega_0 S_z^2 + \omega_0 a^{\dagger} a + \omega_0 b^{\dagger} b \\ &\quad + \Big(g^*(t) S_+^1 a + g(t) S_-^1 a^{\dagger} + g^*(t) S_+^1 b + g(t) S_-^1 b^{\dagger} \\ &\quad + g^*(t) S_+^2 a + g(t) S_-^2 a^{\dagger} + g^*(t) S_+^2 b + g(t) S_-^2 b^{\dagger} \Big) \end{aligned} \tag{5.1}$$

in which the interaction part is

$$\begin{aligned} H_{int} &= g^*(t) S_+^1 a + g(t) S_-^1 a^{\dagger} + g^*(t) S_+^1 b + g(t) S_-^1 b^{\dagger} \\ &\quad + g^*(t) S_+^2 a + g(t) S_-^2 a^{\dagger} + g^*(t) S_+^2 b + g(t) S_-^2 b^{\dagger} \end{aligned} \tag{5.2}$$

where $g(t)$ is the time-dependent coupling coefficient between the atoms and the fields. In the random phase telegraph noise interaction problems, $g(t)$ is taken to be completely stochastic, and it is defined as

$$g(t) = g_0 e^{i\phi(t)} \tag{5.3}$$

where g_0 is a positive real constant amplitude. In calculations we set $g_0 = 1$. $\phi(t)$ is a stochastic variable fluctuating between different arbitrary phases in a manner of jumps. The jumps in the phase occur according to the Poisson process and they are separated by an average time interval called the mean dwell time. $\phi(t)$ for neighboring intervals are not correlated. The probability of finding $\phi(t)$ remains the same at any instant of time. Thus, $\phi(t)$ can undergo continuous random change of Markov type which allows us to take the average over the stochastic fluctuations.

In the system we define the initial states; the cavity fields in the vacuum state $|00\rangle$, atom 1 in the excited state $|e\rangle$ and atom 2 in the ground state $|g\rangle$. Thus, the initial density matrix of the system is

$$\rho(0) = |00eg\rangle\langle 00eg| \tag{5.4}$$

Then the time-dependent interaction density matrix can be written

$$\begin{aligned}\rho(\tau)\exp\left(\frac{\tau}{T}\right) &= \int U(\phi;\tau,0)\rho(0)U^{-1}(\phi;\tau,0)dQ(\phi)\\ &+\frac{1}{T}\int \exp(\frac{t}{T})\int U(\phi;\tau,t)\rho(t)U^{-1}(\phi;\tau,t)dQ(\phi)dt,\end{aligned} \tag{5.5}$$

where T is the mean dwell time and $dQ(\phi) = \frac{d\phi}{2\pi}$. In this integral equation, the statistical average is taken over the stochastic variable $\phi(t)$. Hence, $\rho(t)$ is the noise averaged time evolution density matrix of the system.

Firstly our purpose is to obtain the solution of time evolution density matrix. We will use of the Laplace transformation techniques during the solution process.

We obtain the H_{int} interaction matrix in the sub-Hilbert space spanned by the basis

$\{|00eg\rangle, |00ge\rangle, |10gg\rangle, |01gg\rangle\}$

$$
\begin{array}{llll}
H_{11} = \langle 00eg|H_{int}|00eg\rangle & H_{12} = \langle 00eg|H_{int}|00ge\rangle & .... & H_{14} = \langle 00eg|H_{int}|01gg\rangle \\
H_{21} = \langle 00ge|H_{int}|00eg\rangle & H_{22} = \langle 00ge|H_{int}|00ge\rangle & .... & H_{24} = \langle 00ge|H_{int}|01gg\rangle \\
\vdots & \vdots & .... & \vdots \\
H_{41} = \langle 01gg|H_{int}|00eg\rangle & H_{42} = \langle 01gg|H_{int}|00ge\rangle & .... & H_{44} = \langle 01gg|H_{int}|01gg\rangle
\end{array}
\tag{5.6}
$$

From equation (5.6) we obtain the H_{int} matrix ;

$$
H_{int} = \begin{pmatrix} 0 & 0 & g_0 e^{-i\phi} & g_0 e^{-i\phi} \\ 0 & 0 & g_0 e^{-i\phi} & g_0 e^{-i\phi} \\ g_0 e^{i\phi} & g_0 e^{i\phi} & 0 & 0 \\ g_0 e^{i\phi} & g_0 e^{i\phi} & 0 & 0 \end{pmatrix} \tag{5.7}
$$

The unitary matrix,

$$
U = \exp(-iH_{int}\tau) \tag{5.8}
$$

so, we can obtain the unitary matrix U by using the equation (5.7), then we have

$$
U = \begin{pmatrix} \cos^2[g_0\tau] & -\sin^2[g_0\tau] & -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] \\ -\sin^2[g_0\tau] & \cos^2[g_0\tau] & -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] \\ -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & \cos^2[g_0\tau] & -\sin^2[g_0\tau] \\ -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & -\frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & -\sin^2[g_0\tau] & \cos^2[g_0\tau] \end{pmatrix} \tag{5.9}
$$

From (5.9) the inverse of unitary matrix:

$$U^{-1} = \begin{pmatrix} \cos^2[g_0\tau] & -\sin^2[g_0\tau] & \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] \\ -\sin^2[g_0\tau] & \cos^2[g_0\tau] & \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] \\ \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & \cos^2[g_0\tau] & -\sin^2[g_0\tau] \\ \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & \frac{1}{2}ie^{-i\phi}\sin[2g_0\tau] & -\sin^2[g_0\tau] & \cos^2[g_0\tau] \end{pmatrix} \tag{5.10}$$

We rewrite the initial density matrix from equation (5.4) in the matrix form

$$\rho(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.11}$$

and define

$$\rho(t) = \begin{pmatrix} \rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t) \\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) & \rho_{34}(t) \\ \rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t) & \rho_{44}(t) \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} \rho_{11}(\tau) & \rho_{12}(\tau) & \rho_{13}(\tau) & \rho_{14}(\tau) \\ \rho_{21}(\tau) & \rho_{22}(\tau) & \rho_{23}(\tau) & \rho_{24}(\tau) \\ \rho_{31}(\tau) & \rho_{32}(\tau) & \rho_{33}(\tau) & \rho_{34}(\tau) \\ \rho_{41}(\tau) & \rho_{42}(\tau) & \rho_{43}(\tau) & \rho_{44}(\tau) \end{pmatrix} \tag{5.12}$$

We apply the appropriate form of equations (5.9) and (5.10) with the conditions in equations (5.11) and (5.12) to the master equation (5.5). Firstly we make the integrating parts respect to the ϕ in the interval $[0, 2\pi]$. Then we divide the both sides of the equation (we obtained) $e^{\left(+\frac{\tau}{T}\right)}$. After division we use of the Laplace transformation respect to the $\{\tau, s\}$. Then we solve the equation and obtain

$$\rho_{11}(s), \rho_{12}(s), ..., \rho_{14}(s), ..., \rho_{41}(s), ..., \rho_{44}(s) \tag{5.13}$$

After these calculation we use the Inverselaplace transform to obtain time evolution density matrix. We simplify the calculation and then we have nonzero elements

$$\rho_{11}(t) = \frac{1}{8}\Big(3 + e^{-t/2T}\Big(\cosh[\nu t/2T] + 4\cosh[\omega t/2T] + \frac{\sinh[\nu t/2T]}{\nu} + \frac{4\sinh[\omega t/2\tau]}{\omega}\Big)\Big) \tag{5.14}$$

$$\rho_{12}(t) = \frac{1}{8}\Big(-1 + e^{-t/2T}\Big(\cosh[\nu t/2T] + \frac{\sinh[\nu t/2T]}{\nu}\Big)\Big) \tag{5.15}$$

$$\rho_{21}(t) = \frac{1}{8}\Big(-1 + e^{-t/2T}\Big(\cosh[\nu t/2T] + \frac{\sinh[\nu t/2T]}{\nu}\Big)\Big) \tag{5.16}$$

$$\rho_{22}(t) = \frac{1}{8}\Big(3 + e^{-t/2T}\Big(\cosh[\nu t/2T] - 4\cosh[\omega t/2T] + \frac{\sinh[\nu t/2T]}{\nu} - \frac{4\sinh[\omega t/2\tau]}{\omega}\Big)\Big) \tag{5.17}$$

$$\rho_{33}(t) = \frac{1}{8}\Big(1 + e^{-t/2T}\Big(-\cosh[\nu t/2T] - \frac{\sinh[\nu t/2T]}{\nu}\Big)\Big) \tag{5.18}$$

$$\rho_{34}(t) = \frac{1}{8}\Big(1 + e^{-t/2T}\Big(-\cosh[\nu t/2T] - \frac{\sinh[\nu t/2T]}{\nu}\Big)\Big) \tag{5.19}$$

$$\rho_{44}(t) = \frac{1}{8}\Big(1 + e^{-t/2T}\Big(-\cosh[\nu t/2T] - \frac{\sinh[\nu t/2T]}{\nu}\Big)\Big) \tag{5.20}$$

in which $\nu = \sqrt{1 - 64T^2 g_0^2}$ and $\omega = \sqrt{1 - 16T^2 g_0^2}$. The other elements of the time evolution density matrix are all zero. From above equations the time evolution density matrix is

$$\begin{aligned}\rho(t) = {}& \rho_{11}(t)|00eg\rangle\langle 00eg| + \rho_{12}(t)|00eg\rangle\langle 00ge| + \rho_{22}(t)|00ge\rangle\langle 00ge|\rho_{33}(t)|10gg\rangle\langle 10gg| \\ & +\rho_{34}(t)|10gg\rangle\langle 01gg| + \rho_{34}(t)|10gg\rangle\langle 01gg| + \rho_{43}(t)|01gg\rangle\langle 10gg| \\ & +\rho_{44}(t)|01gg\rangle\langle 01gg|\end{aligned} \tag{5.21}$$

5.0.2 *Results*

We can use equation (5.21) for investigate the effects of the random phase telegraph noise on the entanglement of the system.

We can obtain subsystems by tracing out of the degree of the freedom of one of the fields and atoms.

Now, we obtain a subsystem which is containing field 1 and two atoms by tracing out the degree of the freedom of the second field, using equation (5.21). That is, the reduced density matrix $\rho^{aaf,1}$;

$$\begin{aligned}
\rho^{aaf,1}(t) &= \langle 0|\rho(t)|0\rangle + \langle 1|\rho(t)|1\rangle \\
&= \rho_{11}(t)|0eg\rangle\langle 0eg| + \rho_{12}(t)|0eg\rangle\langle 0ge| + \rho_{21}(t)|0ge\rangle\langle 0eg| \\
&\quad +\rho_{22}(t)|0ge\rangle\langle 0ge| + \rho_{33}(t)|1gg\rangle\langle 1gg| + \rho_{44}(t)|0gg\rangle\langle 0gg| \qquad (5.22)
\end{aligned}$$

and repeating it for tracing out the degree of the freedom of the first field to obtain the reduced density matrix for field 2 and the atoms subsystem $\rho^{aaf,2}(t)$;

$$\begin{aligned}
\rho^{aaf,2}(t) &= \langle 0|\rho(t)|0\rangle + \langle 1|\rho(t)|1\rangle \\
&= \rho_{11}(t)|0eg\rangle\langle 0eg| + \rho_{12}(t)|0eg\rangle\langle 0ge| + \rho_{21}(t)|0ge\rangle\langle 0eg| \\
&\quad +\rho_{22}(t)|0ge\rangle\langle 0ge| + \rho_{33}(t)|0gg\rangle\langle 0gg| + \rho_{44}(t)|1gg\rangle\langle 1gg| \qquad (5.23)
\end{aligned}$$

From equations (5.22) and (5.23), we say that $\rho^{aaf,1}(t)$ and $\rho^{aaf,2}(t)$ give the same reduced density matrix.

$$\rho^{aaf,1}(t) = \rho^{aaf,2}(t) \qquad (5.24)$$

So, it is sufficient to investigate one of the subsystems for the entanglement properties(they have the same results).

For these systems we investigate whether the three-partite entanglement can appear or not. For discriminating the genuine three-partite entangled states from other entangled states or separable states, we searched the state preparation fidelity to adopt and study if the three partite entangled states can be generated in the system.For a three - partite state, the state preparation fidelity, F, is defined as

$$F(\rho) = \langle GHZ|\,\rho\,|GHZ\rangle, \qquad (5.25)$$

where $|GHZ\rangle$ is the three-partite GHZ state. A sufficient condition to determine the genuine three-partite entanglement is whether $F(\rho) > 1/2$ or not. We considered the state density matrix in equation (5.21) and choose the GHZ state to be $|GHZ\rangle$ [20]

$$\begin{aligned}
|GHZ\rangle \;\; &= \;\; \frac{1}{4}\Big[(|0\rangle - i|1\rangle) \otimes (|g\rangle + |e\rangle) \otimes (|g\rangle - |e\rangle) \\
&\quad\; - (|0\rangle + i|1\rangle) \otimes (|g\rangle - |e\rangle) \otimes (|g\rangle + |e\rangle)\Big]
\end{aligned} \tag{5.26}$$

We rewrite the equation (5.26) and we obtain that

$$|GHZ\rangle = \frac{1}{2}\left[-|0ge\rangle + |0eg\rangle - i|1gg\rangle + i|1ee\rangle\right] \tag{5.27}$$

Thus,

$$\langle GHZ| = \frac{1}{2}[-\langle 0ge| + \langle 0eg| + i\langle 1gg| - i\langle 1ee|] \tag{5.28}$$

Now we calculate the fidelity $F(\rho)$ by using equations (5.27), (5.28) and (5.22) in equation (5.25), we obtain that

$$\begin{aligned}
F(t) \;\; &= \;\; \langle GHZ|\rho^{aaf,1}(t)|GHZ\rangle \\
&= \;\; \frac{1}{4}\left[\rho_{11}(t) - \rho_{12}(t) - \rho_{21}(t) + \rho_{22}(t) + \rho_{33}(t)\right] \\
&= \;\; e^{-(1+\nu)t/2T}\left(1 - \nu + 18e^{(1+\nu)t/2T}\nu - e^{\nu t/T}(1+\nu)\right)\Big/64\nu
\end{aligned} \tag{5.29}$$

in which $\nu = \sqrt{1 - 64T^2 g_0^2}$.

After these calculations we plot the $F(t)$ and analyze entanglement possibilities.

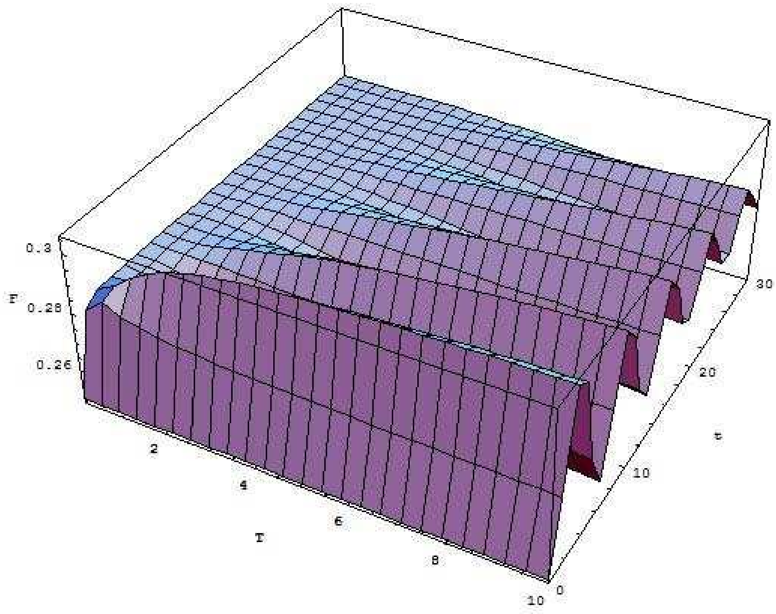

Figure 5.1: Fidelity F of one field-two atom interaction subsystem under RPTN with the time t and mean dwell time T.

In Figure 5.1 we plot the fidelity F as a function of t and the mean dwell time T. As a measure of the entanglement between three-partite state F reaches its maximum value in a very short time and gets values under 0.28. Here we observe that its maximum value of F is almost 0.28. As the value of mean dwell time T increases, the strength and the lifetime of phases of possibilities of entanglement increases. But the system does not entangle. It is clear that $F(t)$ is not over $1/2$ in time evolution of the system. From this view, we point out that one field and two atom interaction of three-partite state under random phase telegraph noise does not entangle.

CHAPTER 6

CONCLUSION

In this thesis, we investigated the interactions of atoms and fields under random phase telegraph noise in a stochastic structure. We used generalized Jaynes-Cumming model under random phase telegraph noise by Joshi *et.al*. We have studied the entanglement properties of these interactions in different systems and subsystems, like atom-field, field-field and atom-atom-field interactions systems.

We firstly gave a general information about stochastic analysis and quantum states in chapter 2. We searched special cases Poisson process, Markov Process and random phase telegraph noise. For quantum states, we gave information about quantum qubits, entangled states, pure and mixed states, density operators. After these we analyzed entanglement of formation.

Then, in chapter 3 we reviewed one atom-one field interaction under random phase telegraph noise. For this purpose, we have studied generalized Jaynes-Cumming model with random phase telegraph noise by Joshi and *et.al*. We searched analytically the equations of Jaynes-Cumming model. It is emphasizedi the importance of the Poisson process and Markovian property in the random phase telegraph noise.

All then, in chapter 4 we reviewed entanglement of one atom-two fields interactions under random phase telegraph noise by [20]. In calculation procedure, we used Laplace transformation techniques and also for random phase telegraph noise Poisson process and Markovian property is important too. It is convenient to used Wootters's concurrence as a measure of entanglement in the bipartite subsystems. We reviewed that atom - field interaction subsystem does not entangle but field-field interaction subsystem entangled under random phase telegraph noise.

Finally, in chapter 5 we investigated two atoms-two fields interaction under random phase telegraph noise as a new work. In this interaction system we used another method of entanglement measure called fidelity for three-partite subsystems. We analyzed one field-two atoms subsystems entanglement property. We concluded that it does not entangle under random phase telegraph noise.

In result, we can deduce that in atoms-fields interactions the effects of random phase telegraph noise is different. In atom-field interaction system we found that it does not entangle but field-field interaction system entangled as indicated in chapter 4. For three-partite states, in one field-two atoms interaction system we found that it does not entangle in chapter 5. From this view, if we investigate more multi-partite interaction systems under random phase telegraph noise than, may be, we can generalize the effects of noise for entanglement property of multi-partite state systems.

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