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Influence du confinement sur le développement spatio-temporel de sillages

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Abstract

This thesis focuses on the stability of wake flows, especially on the influence of confinement on their stability properties. Several analyses are conducted to this end. At first, a local stability analysis under the WKBJ approach is carried out on a family of inviscid confined jet/wakes flows with or without surface tension at interfaces. Both temporal and spatio-temporal behaviors of such flows are examined and commented. We are particularly interested in the destabilizing effect due to the blockage noticed for wake flows at medium confinement. This aspect is also investigated by means of the counterpropagating Rossby waves (CRW) perspective. Furthermore, the influence of confinement on non-parallel viscous wakes is examined by means of direct numerical simulations (DNS). It turns out that when prescribing the Reynolds number to $Re = 100$ the destabilizing influence at medium confinement is confirmed either with free slip or no slip conditions at the walls. Increasing Re up to 500, more subtle destabilizing phenomena are encountered, under the form of a vacillating front. By means of large-eddy simulations (LES), we further examine the effect of confinement on turbulent wake flows for $Re = 5000$. Examining several quantities as turbulent kinetic energy, dissipation rate, one-dimensional spectrum and third order velocity structure function, we conclude that the confinement favors the development of fully 3D turbulence.



Résumé

L'object de ce travail est l'étude de la stabilité de sillages et, en particulier, de l'influence du confinement sur leurs propriétés de stabilité. Differentes analyses sont effectuées pour parvenir à ce but. Une analyse locale, sous la condition WKBJ, est effectuée en premier lieu, en utilisant comme modèle une famille de sillages non visqueux et confinés en présence ou pas d'une tension de surface aux interfaces. Le developpement temporel ainsi que spatio-temporel des tels écoulements sont examinés et commentés. En particulier, on se interesse à l'effet destabilisant d'un confinement modéré sur le developpement d'un sillage. Cet aspect est aussi étudié en utilisant la perspective des ondes Rossby contra-propagatives (CRW). L'influence du confinement sur des sillages non-parallèles et visqueux est également examinée par des simulations numériques directes (DNS). En fixant le nombre de Reynolds à $Re = 100$, l'effet destabilisant pour un confinement modéré est confirmé avec aux parois des conditions de glissement ou bien d'adhérence. En augmentant le nombre de Reynolds à $Re = 500$, des phénomènes plus complexes ont été observés, sous la forme d'un front vacillant. En outre, avec des simulations des grandes échelles (LES), on examine l'effet du confinement sur des sillages turbulents à $Re = 5000$. L'étude de plusieurs quantités, comme l'énergie cinétique turbulente, son taux de dissipation, des spectres de vitesse et la fonction de structure de troisième ordre, a mis en évidence que le confinement favorise le développement de la turbulence 3D.



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Chapter 1

Introduction



1.1 Confinement

This thesis is devoted to the analysis of the influence of confinement on spatially evolving obstacle-free wakes. In general, we wish to understand some of the differences between flows that are infinite in the crossflow direction, which we will call unconfined flows, and those that are finite in that direction, which we will call confined flows (even though they are all treated as unconfined in the streamwise direction). In practice, at least in laboratory experiments, there are no truly unconfined flows; for example, any shear flow, (i.e. boundary layers, mixing layers, jet/wakes) in a wind tunnel is usually modeled as unconfined in the crossflow direction, although it is confined by the walls of the wind tunnel itself. What seems clear, however, is that if the boundaries are far enough away from the location of the shear layer then the presence of walls should only have a small influence on the stability of the flow. However in many physical

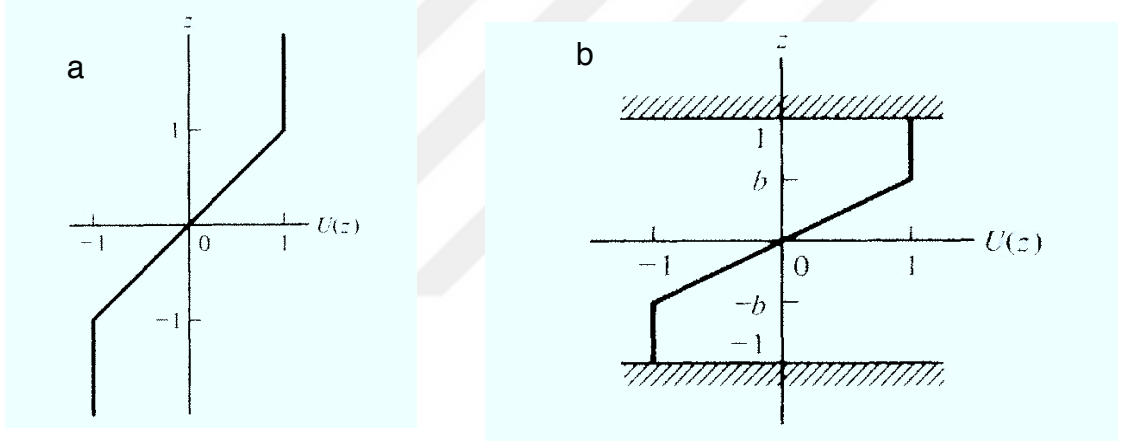


Figure 1.1: (a) Unconfined and (b) confined piecewise broken line-mixing layer studied by Rayleigh. Drazin & Reid (1981) [28].

problems this assumption is not valid and the confinement is relevant to determine the flow properties. Furthermore, it is sometimes not easy to estimate when the presence of the walls can be neglected. For example, Behr *et al.* (1995) [4] have discovered that the minimum distance at which the influence of lateral boundaries vanishes is larger than what is commonly assumed in a circular cylinder wake. Moreover, Healey (2007, 2010) [42, 44] has shown for plane and rotating disk boundary layers that the influence of confinement has to be considered also when the wall is added far-away from the shear. Therefore, it becomes fundamental to determine what occurs if walls are added to several model flows and provide a physical interpretation to the results.

Rayleigh [89] was the first in 1894 to study the effect of the confinement for a mixing layer by means of a temporal stability analysis, comparing the two models illustrated in figure 1.1. He found that the presence of confinement was stabilizing. Starting from this result, in general the addition of solid boundaries is expected to

exert a stabilizing influence on any inviscid linearly unstable waves. In the present work we report several cases where this intuition is contradicted.

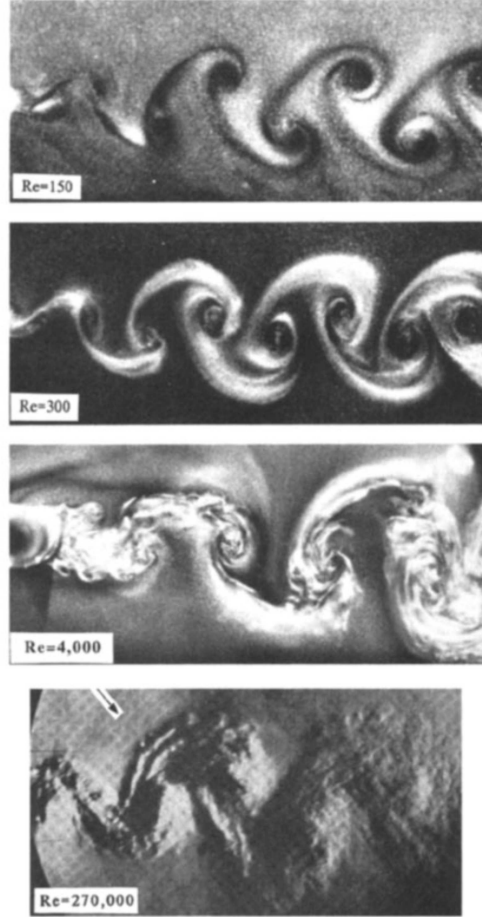


Figure 1.2: Von Kármán vortex street for different Reynolds number. Williamson (1996) [108].

1.2 The influence of confinement on wakes

In this thesis mainly wake flows are analyzed. This calls for a brief summary on the effect of confinement in such flows, in particular confined bluff-body wakes. For unconfined bluff bodies wakes, the onset of von Kármán vortex street [107] is attributed to a self-sustained oscillating mechanism, as supported by experiments conducted by Provansal *et al.* [88], see Williamson (1996) [108] for a detailed review on the subject. Figure 1.2 shows this vortex shedding phenomenon for different Reynolds number.

For cylinder wakes, the differences between confined and unconfined flows are

threefold: (a) presence of vorticity in the incoming flow, (b) confinement effect, and (c) production of vorticity at the walls due to no slip boundary conditions. In most studies, the parameter to measure the confinement is named blockage ratio β , which is defined as the ratio between the diameter of the cylinder D and the thickness of the domain H . In this thesis, we measure the blockage by means of the confinement parameter h , which for the cylinder corresponds to the dimensionless width of the free section, so that $h = (1 - \beta)/\beta$.

1.2.1 Confined 2D cylinder wakes

In first experiments on the weakly confined wake ($h > 4$) of a circular cylinder at $40 < Re < 140$ (Re being based on the cylinder diameter) Shair *et al.* [97] found that the critical Reynolds number Re_c associated to the transition between the steady and the unsteady regimes increases with the confinement. This stabilizing influence was confirmed by the experiments conducted by Coutenceau & Bouard (1977) [23] on confined cylinder wakes for $h > 5.67$.

However, the influence of confinement is more complex. More recently, Sahin & Owens (2004) [96] have increased the confinement until $h = 0.11$ discovering that for higher blockage ratio the influence of confinement is more subtle. While the critical Reynolds number Re_c increases as h is increased if $h > 1$, confinement is destabilizing in the range $0.43 < h < 1$, before recovering a stabilizing influence for $0.19 < h < 0.43$, and revealing more complex dynamics than supercritical Hopf bifurcation at these large amounts of confinement. The stabilizing effect shown by Sahin & Owens (2004) [96] for $h > 1$ was confirmed by the recent experimental study of Rehimy *et al.* (2008) [93], conducted on confined wakes ($h = 2$) past a circular cylinder, who found $Re_c = 108$ to be compared to the classical $Re_c = 47$ for unconfined cylinder wakes [108]. Concerning the Strouhal number of two-dimensional cylinder wakes, it was shown that it is enhanced by adding confinement. Both numerical [25, 15] and experimental [25, 93] studies have confirmed this tendency.

Another interesting effect due to confinement in cylinder wakes is the inversion of von Kármán street in the wake, as noticed by Zovatto & Pedrizzetti (2001) [112] and Sahin & Owens (2004) [96] and examined in details by Camarri & Giannetti (2007) [15]. More precisely, vortices are alternatively shed in the wake as in the unconfined case, i.e. if the flow is from left to right, clockwise and counterclockwise vortices are shed from the upper and lower sides of cylinder, respectively. However, at a certain distance the trajectories of the two families of vortices intersect and, further downstream, their position with respect to the symmetry line is inverted, i.e. counterclockwise and clockwise are positioned in the upper and lower half of the wake, respectively. This phenomenon is illustrated in figure 1.3. However, Camarri & Giannetti [15] have shown that the inversion is mainly due to the vorticity of the incoming flow and not to a direct influence of the blocking walls.

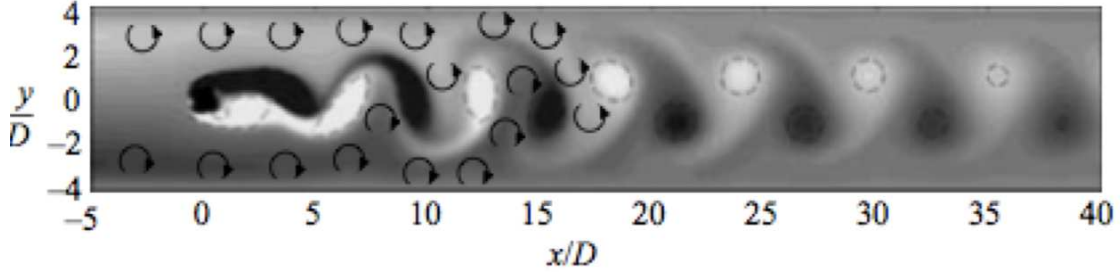


Figure 1.3: Inversion of the Von Kármán vortex street for confined square cylinder wakes. Camarri & Giannetti (2007) [15].

1.2.2 Confined 3D cylinder wakes

Concerning laminar flows, Williamson [108] has shown that the appearance of coherent structures along the spanwise direction for an unconfined 3D cylinder wakes occurs for $Re = 190$. He distinguished the structures in two different modes: mode A, which appears at $Re = 190$ and mode B, which is encountered starting from $Re = 260$. Blackburn (1994) [9] has experimentally illustrated that, including walls, this critical Reynolds number decreases with respect to the blockage ratio. This fact is confirmed by Rehimi *et al.* [93] who have noticed the appearance of 3D structures at $Re = 159$ for $h = 2$. These structures are similar to mode A noticed by Williamson, while the mode B instabilities appear at $Re = 400$. Therefore, here the influence of confinement seems to be stabilizing or destabilizing depending on the 3D-instability type.

At higher Reynolds number, i.e. $Re = 45000$, in a semi-confined flow, Bearman & Zdravkovich (1978) [3] have shown that the shedding vortex phenomenon is regularized for a certain range of confinement when a plate is approached to the cylinder. Also, confined cylinder turbulent wakes have been examined. Richter & Naudascher (1976) [94] have experimentally studied the influence of confinement on several properties of such flows for $10^4 < Re < 10^6$: Strouhal number, mean-drag coefficient and fluctuating-lift coefficient are enhanced by confinement. Concerning the latter parameter, its value for $h = 1$ is eight times larger than for the unconfined wake. Furthermore, the laminar/turbulent transition seems to be also favored when the blockage ratio is increased. More recently the destabilization due to confinement on turbulent square cylinder wakes was noticed by Kim *et al.* (2004) [59] in their large-eddy simulations, comparing $h = 4$ with $h = \infty$ at $Re = 3000$. In particular, the presence of blocking walls increases the drag force, lift fluctuations and Strouhal number.

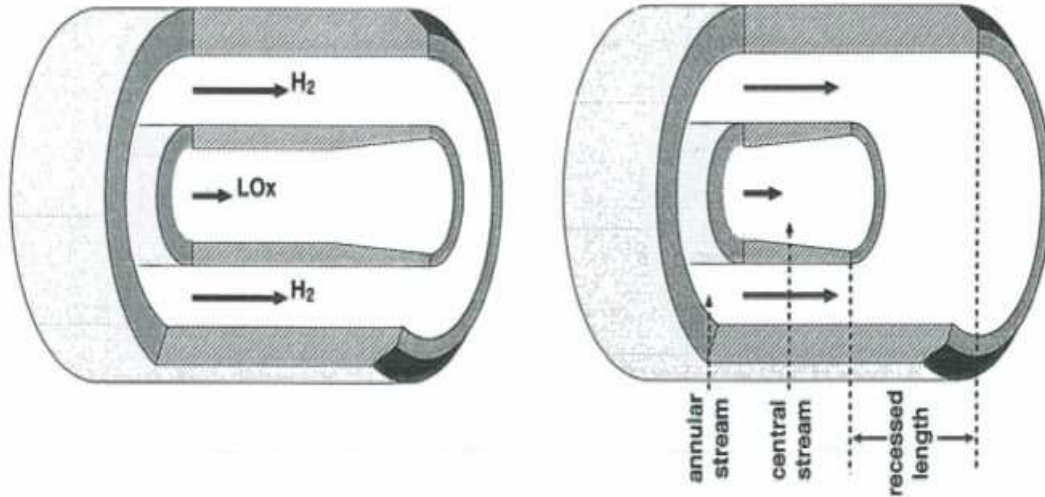


Figure 1.4: Cut-away diagram of coaxial injectors of rockets. Left: without recess. Right: with recess. Juniper's thesis (2001) [53].

1.2.3 Recessed coaxial injectors

An interesting problem connected to the influence of confinement in wake flows is the behaviour of the coaxial injector used in the rocket engines when it recessed by a duct, as illustrated in figure 1.4. In particular, Gill (1978) [35] has shown that recessing the central tube of a coaxial injector improves the mixing of the two streams. However, for two decades this destabilizing effect was not understood. More recent experiments conducted by Leroux (2001) [64] have compared a water/air injector implementing recess of several lengths, concluding that when the recess is added, the instability is promoted. In figure 1.5 some examples of these experiments are illustrated.

Earlier, Rehab *et al.* (1997) [92] have shown that a recirculation bubble is created in the middle of coaxial jets in the unrecessed injector, see figure 1.6. Moreover, this recirculation bubble has a wake-like behaviour. Successively, Juniper (2001) [53] noticed by means of experiments conducted in a cryogenic test facility that the recess enhances the spiraling or flapping instability which is a characteristic of wakes. Finally, Juniper & Candel (2003) [57] have noticed that the recess encourages the wake-like behaviour of the flow, and particularly that the destabilizing influence of the confinement is due to the growth of the wake-type instability. In order to better appreciate these findings, we have to introduce the notions of convective and absolute instability in next section 1.3.1.

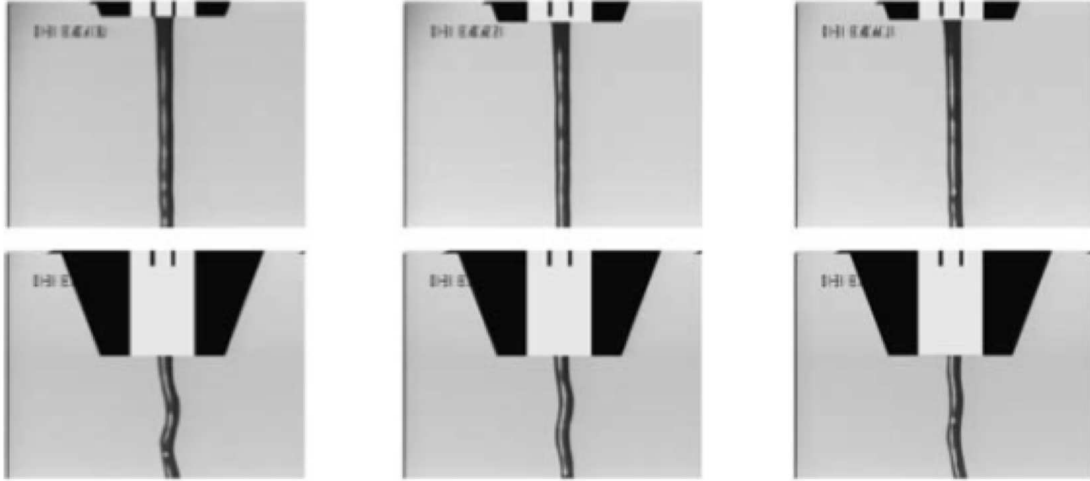


Figure 1.5: Visualization of the water jet in a laboratory-scale. Recess value: 0 mm, 1 mm, 4 mm, 5 mm, 8 mm, and 10 mm. Juniper & Candel (2003) [64].

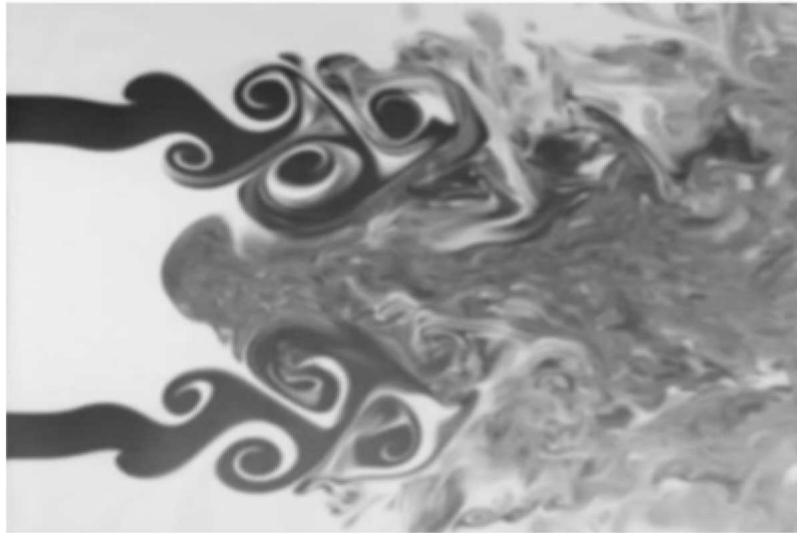


Figure 1.6: Mixing of two coaxial jets with recirculation bubble. Rehab *et al.* (1997) [92].

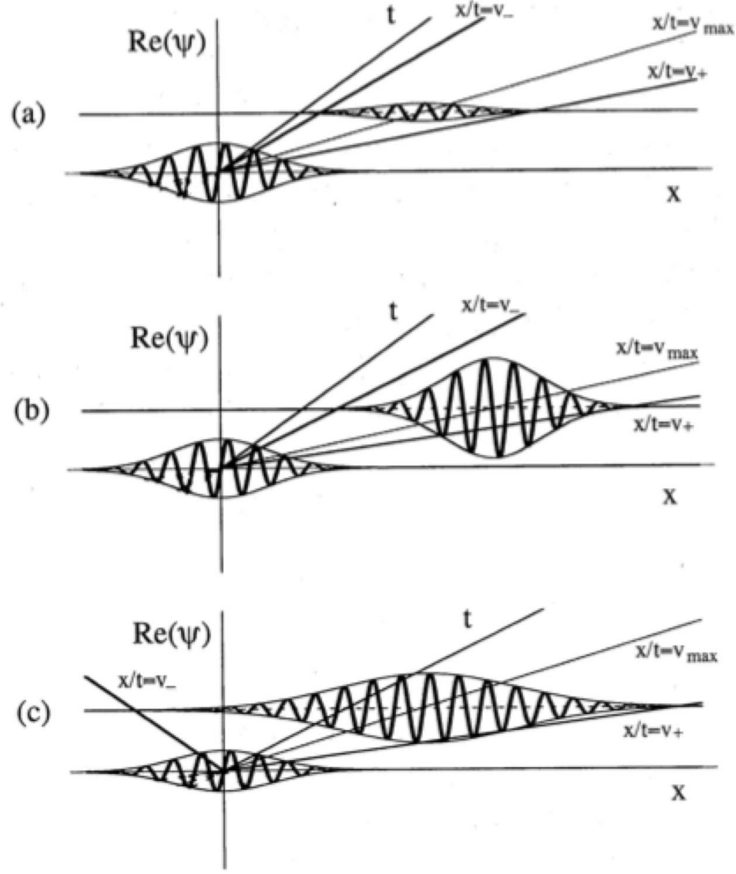


Figure 1.7: Linear response to a localized initial state of (a) stable flow, (b) convectively unstable flow, and (c) absolutely unstable flow. Huerre & Rossi (2000) [51].

1.3 From local analysis to nonlinear global modes

1.3.1 Spatio-temporal stability: convective and absolute instability

Open shear flows can be studied at first in the framework of the local stability theory, for which only the crossflow direction is an eigendirection. The perturbations at each streamwise station is examined by studying the stability of the fictitious parallel flow obtained by extending to infinity the base flow profiles prevailing at each streamwise direction. In the local approach, the flow can be stable or unstable, and in the latter case the instability is distinguished in convective or absolute [14, 6], see Huerre & Rossi (2000) [51] for a comprehensive review on the subject. Figure 1.7 illustrates the three cases in figures a, b and c, respectively.

In a convectively unstable flow, the impulse response of a perturbations is ultimately advected away from the source location. In this case the flow behaviour is like a noise amplifier, where the advection overwhelms the upstream propagation of the small-amplitude wave packet generated by initial perturbations, either natural or forced. As illustrated in figure 1.7b the wave packet is swept downstream with the flow while growing. As a final consequence of this behaviour, convectively unstable flows may amplify the external noise, but return to basic state in absence of a continuous forcing.

Conversely, if the impulse response is amplified at the source and gradually contaminates the entire space, the flow is absolutely unstable. The flow, in this situation, behaves like an hydrodynamic oscillator, where the spreading of the wave packet withstands the downstream advection. Hence, the wave packet propagates both upstream and downstream and finally grows in time at any fixed location of the reference frame, as shown in figure 1.7c. In the long-time limit, the energy is continuously extracted from the base flow within the absolute region and transferred to perturbations, yielding the self-sustained oscillations. According to the weakly non parallel analysis of Chomaz *et al.* (1991) [20], the absolutely unstable region has to reach a critical size to observe synchronized oscillations.

1.3.2 Nonlinear global modes

The advantage of the local stability analysis is the small cost in computations. However when the flow exhibits a sharp spatial pattern and frequency selection, i.e. in the case of an oscillator, additional criteria are required to unambiguously associate the local instability properties to the global dynamics. The term global highlights the fact that the oscillations encompass the entire flow field. The connection between local and global stability can be done by the weakly non parallel or Wentzel Kramers Brillouin Jeffrey (WKBJ) approach [51], where one assumes that the streamwise variations of the base flow are slow in comparison to the typical instability wavelength. Therefore, it is possible to assume that the perturbations at any streamwise station develop as if the base flow was locally parallel.

Chomaz (1992) [16] first derived a criterion for the onset of oscillations by analyzing the Ginzburg-Landau model equations in the fully nonlinear regime, giving birth to the so-called theory of the nonlinear global modes. This theory was confirmed by further studies of Couairon & Chomaz (1999) [21] and Pier *et al.* (2001) [85], predicting moreover the frequency of global oscillations. Pier & Huerre (2001) [84] have obtained the same results, analyzing more realistic boundary layer 2D-viscous equations. At the origin of the theory of nonlinear global modes, there is the idea that the flow is dominated by a stationary front acting like a wavemaker, and that its frequency and spatial structure are determined by the local linear stability properties at the upstream boundary of the absolute unstable region. Despite the fact that the slow streamwise variation assumption is not respected, many predictions of the theory of nonlinear global modes have been shown to hold in real flows, as example,

swirling jets [32, 33], hot round jets [65], inter-disk flows [106] and, of course wake flows, as detailed hereafter.

The concept of absolute/convective transition was early associated to cylinder wakes. In fact, one can say that the largest part of literature on the connection between real flows and nonlinear global modes concerns wake flows. Following Triantafyllou *et al.* (1986) [102], Monkewitz & Nguyen (1987) [74] and Monkewitz (1988) [75] demonstrated that the von Kármán vortex street could be associated to a pocket of absolute instability located in the lee of the cylinder. Furthermore Pier (2002) [83] has observed that the frequency of the von Kármán street (see figure 1.2) computed by means of direct numerical simulations matches within 10% accuracy with the theoretically predicted frequency over the range of Reynolds number $100 \leq Re \leq 180$.

The presence of a region of absolute instability was confirmed by Hanneman & Oertel (1989) [38] and Hammond & Redekopp (1997) [37] in blunt body wakes as well as in the related situation of obstacle-free wakes pioneered by Triantafyllou & Karniadakis (1990) [103] and studied successively by Pier & Huerre (2001) [84] and Chomaz (2004) [17]: instead of computing the dynamics of the whole flow around the cylinder, the cylinder region was cut away and the flow computed in a semi-infinite domain, prescribing at the inlet either the axial velocity profile measured just behind the cylinder or a prescribed wake profile.

Many variants of absolute/convective analysis of planar and axisymmetric wakes may also be mentioned to understand the importance of the local analysis in wakes instability. Wakes, including the influence of the shear layer thickness [75], Reynolds number [75], density ratio [111], compressibility [72], added swirling flow component [67] have been analyzed, to name a few. The recent addition of the confinement as parameter influencing the absolute/convective analysis of wakes is due to Juniper & Candel (2003) [57] and will be commented in section 1.4.2.

1.4 The role of the confinement at the light of the spatio-temporal analysis

1.4.1 Reflecting wall-normal modes due to confinement

Before discussing the role of the confinement on wakes, it is worthwhile mentioning the work of Healey [40, 41, 42, 43, 44], which gives an original interpretation of the influence of walls in open flows. In the unconfined rotating-disk boundary layer, he showed [40, 41] the existence of modes which propagates in the wall-normal direction, growing exponentially. Furthermore he also explained how the energy of this wall-normal growth is sustained. These modes have a character of convective instability in the crossflow direction and afterwards have been detected also in plane boundary layers [42] and homogeneous mixing layers [43]. The role of confinement was also analyzed in these cases. If the flow is unconfined this growing wall-normal disturbance eventually exits the reference frame, while if it is confined by a plate, even far from

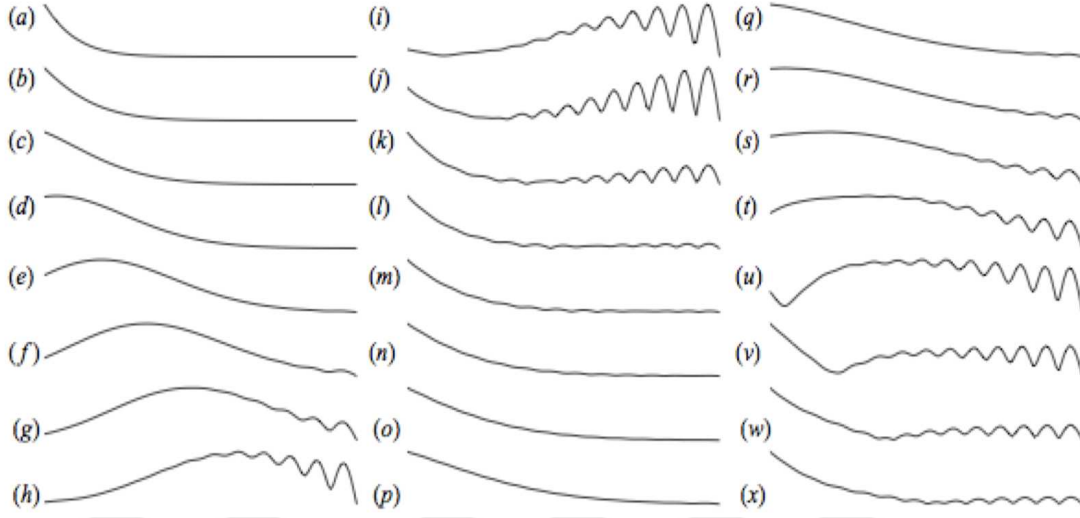


Figure 1.8: Plots of the disturbance magnitude in confined plane boundary layer as a function of normal direction at successive times. Healey (2007) [42].

the boundary layer, then eventually this disturbance reaches the plate and sets up a standing wave in the wall-normal direction. This yields an apparent enhanced absolute growth rate and favors absolute instability. This mechanism is illustrated in figure 1.8.

1.4.2 Confined wakes

Delbende & Chomaz (1998) [27] were the first to examine the effect of confinement on obstacle-free wakes by means of a spatio-temporal analysis, using periodic boundary conditions in the crossflow direction. They have found that a drastic reduction of the domain size quenches the pairing instability and that a more regular stable vortex street is then obtained. Afterwards, Juniper & Candel [57] have tried to explain, in light of the absolute/convective instability concepts, the destabilizing influence of the recess in a coaxial injector, see section 1.2.3. In particular, they have illustrated that this destabilizing influence is due to the wake-like behaviour of the central flow and moreover suggested that the confining recess increases the length of the absolute region in the wake flow, as shown in figure 1.9. In conclusion the recess encourages the transition to a wake-like global mode and so, to self-sustained oscillations.

More recently Juniper (2006) [54] has conducted a spatio-temporal analysis on a inviscid jet/wake top hat profile in order to confirm this hypothesis. He found that a medium confinement of order $h = 1$ promotes absolute instability of wakes, as depicted in figure 1.10. He found that distinct modes exist in both inner and outer streams in the limit of weak and strong confinement. Therefore, the growth in the absolute nature of the instability at medium confinement is explained by the

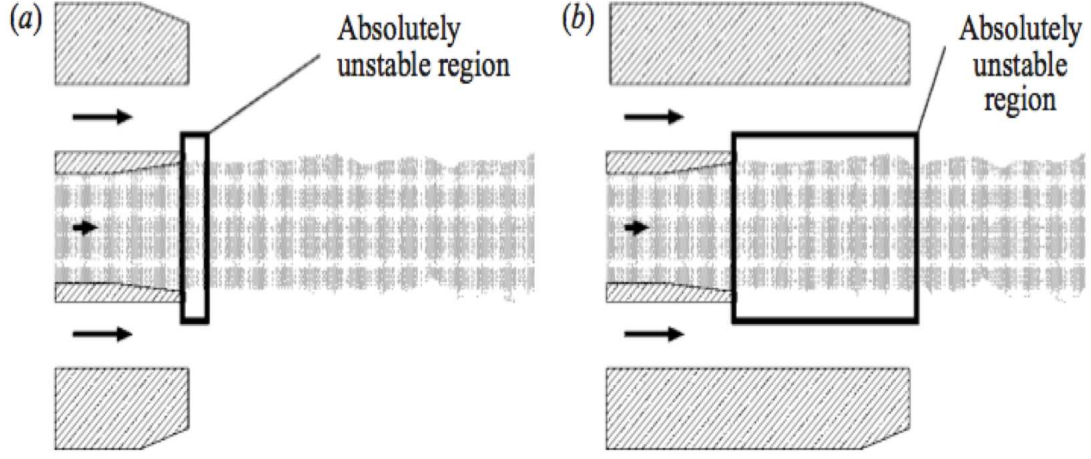


Figure 1.9: Regions of absolute instability in a coaxial injector. (a) Without recess, (b) with recess. Juniper & Candel (2003) [57].

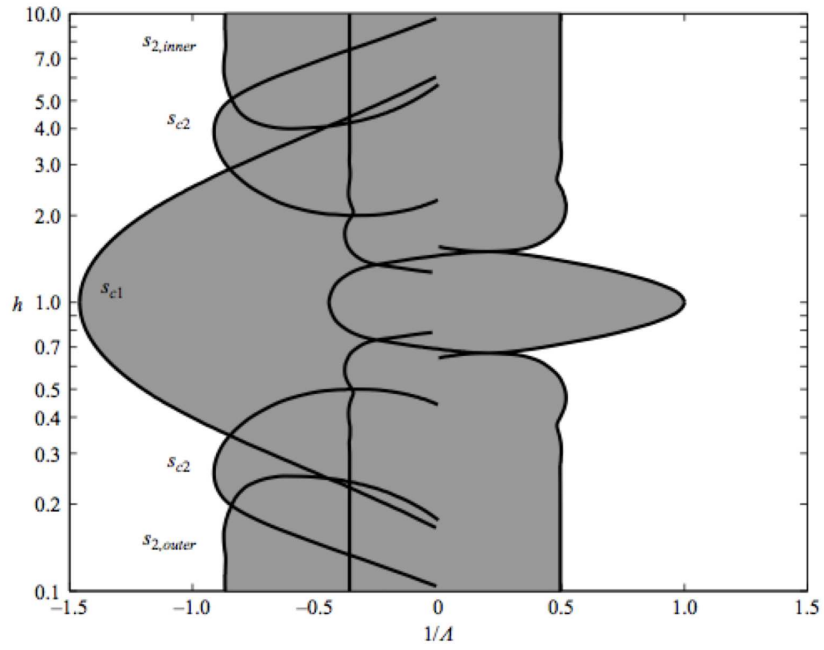


Figure 1.10: Areas of absolute instability (dark grey) as a function of shear and confinement for sinuous mode with uniform density. Wake flows are depicted for $\Lambda^{-1} < 0$, while jets for $\Lambda^{-1} > 0$. Juniper (2006) [54].

constructive interaction of these modes when the flows have a similar thickness.

Also, the influence of density ratio was examined by Juniper [54] discovering that the convective/absolute limit of figure 1.10 is moved towards stronger co-flow wakes if the inner flow is denser than the outer ones.

Analyzing round geometry Juniper (2008) [56] found that the manifold of perturbations with zero azimuthal wavenumber m is very similar to varicose manifold, while $m = 1$ is similar to the sinuous one. Hence, for round geometry the tendency towards absolute instability at moderate confinement was also found. Finally, this tendency was confirmed on several variations of parallel wake flow models including finite shear thickness [55, 91], surface tension [90] and viscous effects [91].

1.5 Outline of the thesis

In the present work, we analyze the influence of confinement on obstacle-free wakes by means of different studies: local stability analysis, direct numerical simulations and large-eddy simulations. As previously discussed that for cylinder wakes (section 1.2.1) introducing blocking walls creates three differences with the unconfined flow: (a) the presence of vorticity in the incoming flow, (b) effect of the blockage, and (c) creation of vorticity at boundaries. In obstacle-free wakes, the inlet profile has always a finite vorticity in the sheared region as for unconfined flows. Thus, point (a) is not object of this study, while the other two points will be largely treated.

In chapter 2, we analyze inviscid wakes by means of a local stability analysis when implementing free slip conditions at the walls, in order to isolate point (b), i.e. the effect of the blockage. Several confined models are constructed, implementing parameters as shear layer thickness, surface tension, and density ratio.

Chapter 3 is devoted to direct numerical simulations (DNS) using as inlet profile the top hat wake examined by Juniper (2006) [54]. Here, by means of DNS we can relax two unphysical assumptions of chapter 2: the absence of viscosity and the parallelism of the wake. Particularly, two different Reynolds numbers, i.e. $Re = 100$ and 500 , are chosen. In this manner we attempt a comparison with the nonlinear global mode theory for confined wake flows. The simulations are conducted imposing free slip or no slip conditions at lateral boundaries in order to highlight the role of blockage independently from the influence of boundary layers.

Confined turbulent wakes are then studied in chapter 4, prescribing the Reynolds number to $Re = 5000$. By means of large-eddy simulations (LES), we can study how the confinement influences the strength and the nature of the turbulence. Furthermore, the role of boundary layers are discussed also for confined turbulent wakes. Finally, conclusions are drawn in chapter 5.

For the numerical simulations carried out in this thesis, we have used the support of the IDRIS (*Institut du Développement et des Ressources en Informatique Scientifique*) as a part of the project "Etude et contrôle de phénomènes turbillonnaires" (references i2009024055 and i2010024055), as well as the Mésocentre SIGAMM of the

Observatory of Côte Azur as a part of the project "Instabilités de sillages".



Chapter 2

Local stability analysis



2.1 Introduction

The relevance and the practical importance of wakes behind bluff bodies is at the origin of a sustained effort towards the detailed understanding of their dynamics. As a archetypical example, the onset of von Kármán vortex street [107] in the wake of a bluff body is attributed to a self-sustained oscillating mechanism, as supported by experiments conducted by Provansal *et al.* [88]. These global oscillations are related to the concepts of absolute (A) and convective (C) instability, introduced by Briggs [14] and Bers [6], which describe the behaviour of the impulse response on an unstable medium. If an impulsive small amplitude perturbation generates an exponential growth at the location of its generation, the flow is so-called absolutely unstable. Conversely, if the wave packet is convected away from the impulse place and leaves the flow undisturbed, the instability is said to be convective.

Koch [60] was among the first to suggest a connection between the onset of self-sustained global oscillations in wakes and the existence of a region of absolute instability immediately downstream of the obstacle. Through an analysis of the Ginzburg-Landau equations, Couairon & Chomaz [21] and Pier *et al.* [85] have then described the structure and dynamics of nonlinear global modes triggered by the presence of a region of absolute instability in a weakly non-parallel flow, i.e. a flow, whose variations following the longitudinal direction are small with respect to the dominant wavelength of the instability. These results were then successfully applied by Pier & Huerre [84] to a synthetic wake governed by the incompressible Navier-Stokes equations.

Earlier, Monkewitz & Nguyen [74] and Monkewitz [75] have applied the concepts of absolute and convective instability to a family of parallel two dimensional wakes representative of typical local streamwise velocity distributions. The base flow family analyzed in their papers has a continuous velocity profile and allows to vary both the velocity ratio, Λ , defined as ratio of the velocity jump to the mean velocity, and the shear layer thickness δ_w through the independent control of wake depth and width. Monkewitz [75] has shown the existence of an *optimal* shear layer thickness for which the absolute character of the instability is maximum.

The confinement of the stream is another parameter of interest for wake flows. It is therefore natural to enquire about the influence of confinement onto the C/A properties of parallel wake flows. Such an approach has been followed in a series of recent papers by Juniper and co-workers [57, 54, 55, 90, 91], with the aim to elucidate the enhanced mixing in confined coaxial injectors. Different models of two-dimensional wakes were studied enabling the analysis of the combined effect of confinement and velocity ratio in addition to other parameters, like surface tension [90], non uniform density [54], viscosity [91] or a vanishing shear layer thickness [55]. It was found in all cases that a moderate confinement enhances the absolute nature of the instability.

In a very recent paper, Rees & Juniper (2010) [91] have analyzed an ad hoc family of continuous velocity profiles of finite shear layer thickness. Four specific values of the

shear layer thickness were considered and the authors showed that a finite shear layer thickness was breaking the $(h, 1/h)$ symmetry existing in the vanishing shear layer thickness limit: strongly confined flows were stabilized as the shear layer thickness was increased. However, the influence on the absolute frequency and wavenumbers was not examined. In addition, the authors have conducted a weakly non parallel analysis of a non parallel stationary solution of the Navier Stokes equations based on the local stability properties of the exact velocity profiles and not on the locally best fitting profiles in their family. Despite existing differences between spatio-temporal properties of local profiles and their best fitting approximations for round wakes, Meliga *et al.* (2009) [72] have shown the relevance of using fitted profiles, using the radial analogue of Monkewitz’s family of profiles discussed above.

Though the absolute/convective character of an instability in an open flow has been shown to play a key role in its overall behavior (see Huerre & Monkewitz (1990) [50] for a review), the relevance of temporal stability theory on the determination of the behavior of convective flows should not be underestimated. In the case of Holmboe’s tube experiment for instance, where the mean flow rate vanishes and spatial forcing can be realized experimentally, temporal stability theory is particularly relevant [87]. This confined shear layer situation, displaying a symmetric confinement, however differs from our flow geometry, and will not be further analyzed (see Criminale *et al.* (2003) [24], section 2.3.2, for a brief discussion).

More generally, when the flow is ”strongly” convectively unstable, Gaster’s transformation relating the spatial growth-rate to the temporal growth-rate applies [34], and the long term response to external noise is dominated by the most unstable temporally growing mode. Still, for many physically relevant confined immiscible jet/wake flow parameter settings, Gaster’s transformation is not valid. The temporal stability analysis remains however valid and significant: it directly yields a measure of the maximum growth rate of the impulse response wavepacket (which is also accessible by the full spatio-temporal analysis employed in Juniper (2007) [54] and Rees and Juniper (2009) [90]).

The temporal stability is also connected to the so-called counterpropagating Rossby waves (CRW) perspective. The origin of this perspective is a paper of Bretherton (1966) [13], where it was shown that the wave development in flows featuring two distinct potential-vorticity gradients can be interpreted in terms of the interaction of two distinct Rossby waves, named counter-propagating by Hoskins *et al.* (1985) [48] since their direction is opposite to the underlying potential vorticity gradient. More recently the interaction of these CRWs was used by Heifetz *et al.* (1999) [45] to describe an unconfined inviscid mixing layer. Also the Kelvin-Helmoltz instability has then been interpreted in the CRW perspective as a limiting case of the mixing layer with vanishing shear layer thickness [47]. Afterwards, Umurhan & Heifetz (2007) [104] and Bakas & Ioannou (2009) [2] have revisited instabilities associated respectively to Holmboe modes and free surface shear flows with this physical description. The only attempt to connect the CRWs and spatio-temporal instability is

due to Bishop & Heifetz (2000) [8], analyzing the baroclinic Eady's model. Our purpose is to reinterpret both the temporal and the spatio-temporal stability of confined jets/wakes as the interaction of CRWs and moreover to integrate the surface tension into this formalism.

The outline of the chapter is as follows. In section 2.2, the models are described and their dispersion relations are derived. Section 2.3 is devoted to the temporal stability analysis while its interpretation by means of the perspective of the CRWs is delineated in section 2.4. Finally, the spatio-temporal analysis and its applications are examined in section 2.5.

2.2 Models description

2.2.1 Theoretical framework and governing equations

The models analyzed in this chapter can be summarized in two general classes of jet/wakes in a confined channel. The channel has infinite extent in the longitudinal x -direction and confining walls are located at $y = \pm(h_1 + h_2)$. The flow is assumed to be inviscid and incompressible. In accordance with the inviscid nature of the flow, slip conditions are imposed as boundary conditions and a parallel flow assumption is made. Furthermore, the velocity profile has a non-zero component only in the longitudinal direction $\mathbf{U} = U(y)\mathbf{e}_x$, with constant pressure $P = \bar{P}$. The first model, illustrated in section 2.2.1.1 is constituted by three fluids having as velocity profile a piecewise broken line, presenting at the interfaces a surface tension, while in section 2.5.3.1 we describe the second one which is constituted by only one fluid with a continuous velocity profile.

2.2.1.1 Piecewise broken line profile with surface tension

The first model flow consists of three fluids sandwiched between two flat plates, as illustrated in figure 2.1. The inner flow (B) has width $2[h_1 + \delta_w \cdot (\pi - \frac{1}{2})]$, where π is a parameter $0 \leq \pi \leq 1$ and density ρ_1 , while the outer flows (A and C) have the same width $h_2 + \delta_w \cdot (\frac{1}{2} - \pi)$ and density ρ_2 . As previously mentioned, the three fluids are supposed to be inviscid and sliding on the walls. A restoring force due to surface tension σ is assumed at the interfaces, whose position is controlled by the parameter π , $y_\pi = \pm[h_1 + \delta_w(\pi - \frac{1}{2})]$.

The dimensional velocity profile of the model flow is

$$U(y) = \begin{cases} U_2 & \text{for } |y| \geq h_1 + \frac{\delta_w}{2} \\ U_2 - \frac{(U_1 - U_2)}{\delta_w}(|y| - h_1) & \text{for } h_1 - \frac{\delta_w}{2} < |y| < h_1 + \frac{\delta_w}{2} \\ U_1 & \text{for } |y| \leq h_1 - \frac{\delta_w}{2}. \end{cases} \quad (2.1)$$

Starting from incompressible 2D Euler's equations, it is possible to linearize them

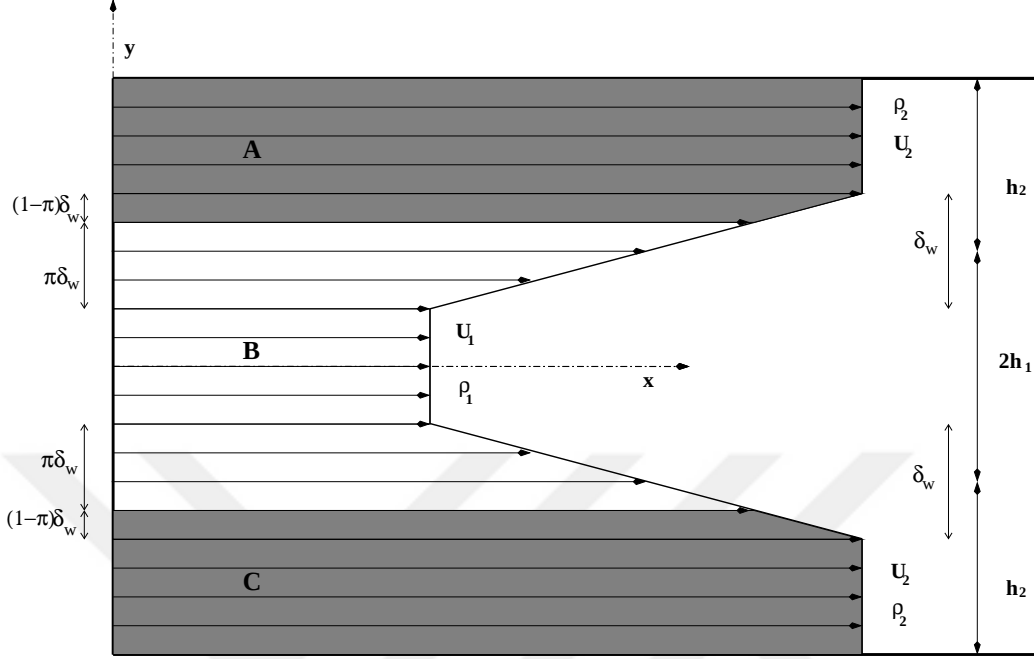


Figure 2.1: The model consists of three flows sandwiched between two flat plates including the surface tension acting at the interfaces.

about the base flow $\mathbf{U}$ assuming small perturbations on the flow.

$$ik(U - \frac{\omega}{k})\tilde{u} + \frac{dU}{dy}\tilde{v} + ik\tilde{p} = 0, \quad (2.2a)$$

$$ik(U - \frac{\omega}{k})\tilde{v} + \frac{d\tilde{p}}{dy} = 0, \quad (2.2b)$$

$$ik\tilde{u} + \frac{d\tilde{v}}{dx} = 0, \quad (2.2c)$$

where we have introduced the classic normal mode decomposition: for a generic quantity $\hat{a}$ derives from the decomposition in normal modes $a' = \tilde{a}(y)e^{i(kx - \omega t)}$. At this point, the streamfunction ψ' of the perturbation can conveniently be introduced. Suppose that $u' = \frac{\partial\psi'}{\partial y}$ and $v' = -\frac{\partial\psi'}{\partial x}$ and take also normal modes decomposition $\psi'(x, y, t) = \phi(y)e^{i(kx - \omega t)}$. Therefore, $\tilde{u} = \frac{d\phi}{dy}$ and $\tilde{v} = -ik\phi$ are obtained. On using these results into equations (2.2), the Rayleigh stability equation is obtained

$$(U - \frac{\omega}{k})(\phi'' - k^2\phi) - U''\phi = 0. \quad (2.3)$$

For more details on the derivation of eq. (2.3), see Drazin and Reid (chapter 8) [28].

Observe in fig. 2.1 that the flow could be divided in seven sub-domains, separated by $y = h_1 \pm \frac{\delta_w}{2}$, where the vorticity has a discontinuity, and by the interfaces

$y = \pm y_\pi$. The total streamfunction can be decomposed into basic and perturbation contributions:

$$\Psi_i(x, y, t) = U_i y + \phi_i(y) \cdot e^{i(kx - \omega t)} \quad (2.4)$$

with $i=1,2,3,4,5,6,7$. Since $U_i''(y) = 0$ for each sub-domain i , the solution of Rayleigh equation degenerates in an harmonic solution $\phi_i = A_i e^{ky} + B_i e^{-ky}$, yielding 14 unknowns (A_i, B_i). Since the model is symmetric with respect to the x -axis, two different perturbations can be distinguished: varicose perturbations, where the shear layers move as mirror images of each other, and sinuous perturbations, where the shear layers move parallel to each other. Any perturbation can be formed from a linear combination of a varicose and a sinuous perturbation and, taking advantage of this symmetry, the fourteen unknowns reduce to seven. Considering only one half of the model, further constraints are to be imposed: imposition of zero y -velocity at the wall, continuity of the particle displacement and of the pressure (except when the surface tension is present) on the interfaces between sub-domains. Thus, the equations of motion and the boundary conditions are reduced to a single relationship between the frequency ω , the wavenumber k and the parameters of the model ($U_j, \rho_j, h_j, \delta_w, \pi, \sigma$). The dispersion relation can only be satisfied for certain eigenvalues, (ω, k) , from which corresponding eigenfunctions, $\phi_i(y)$, can be calculated.

In dimensional terms, the varicose and sinuous dispersion relations are, respectively:

$$D \equiv \frac{M_2 - \tanh[k\delta_w(1-\pi)]}{1 - M_2 \tanh[k\delta_w(1-\pi)]} - \frac{\rho_1 [M_1 - \tanh(k\delta_w\pi)]}{\rho_2 [1 + M_1 \tanh(k\delta_w\pi)]} - (1 - \frac{\rho_1}{\rho_2}) \frac{U_2 - U_1}{k\delta_w(U_\pi - \frac{\omega}{k})} + \quad (2.5)$$

$$+ \sigma \frac{k}{\rho_2 (U_\pi - \frac{\omega}{k})^2} = 0,$$

$$D \equiv \frac{N_2 - \tanh[k\delta_w(1-\pi)]}{1 - N_2 \tanh[k\delta_w(1-\pi)]} - \frac{\rho_1 [N_1 - \tanh(k\delta_w\pi)]}{\rho_2 [1 + N_1 \tanh(k\delta_w\pi)]} - (1 - \frac{\rho_1}{\rho_2}) \frac{U_2 - U_1}{k\delta_w(U_\pi - \frac{\omega}{k})} + \quad (2.6)$$

$$+ \sigma \frac{k}{\rho_2 (U_\pi - \frac{\omega}{k})^2} = 0,$$

where U_π is the flow velocity at the interfaces:

$$U_\pi = \pi U_2 + (1 - \pi) U_1,$$

$$\text{and } M_1 = \coth[k(h_1 - \frac{\delta_w}{2})] + \frac{U_2 - U_1}{k\delta_w(U_1 - \frac{\omega}{k})}, \quad M_2 = N_2 = -\coth[k(h_2 - \frac{\delta_w}{2})] +$$

$$+ \frac{U_2 - U_1}{k\delta_w(U_2 - \frac{\omega}{k})}, \quad \text{and } N_1 = \tanh[k(h_1 - \frac{\delta_w}{2})] + \frac{U_2 - U_1}{k\delta_w(U_1 - \frac{\omega}{k})}.$$

Using as reference length scale h_1 , as reference velocity scale $U_m = (U_1 + U_2)/2$, as density scale $\rho_m = (\rho_1 + \rho_2)/2$ the non-dimensional parameters can be constructed: $h^* = h_2/h_1$ is the confinement parameter, $\delta_w^* = \delta_w/h_1$ is the shear layer thickness, $\Lambda^* = (U_1 - U_2)/(U_1 + U_2)$ is the velocity ratio, $S^* = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$ the density ratio and $\Sigma^* = \sigma/(h_1 \rho_m U_m^2)$ is the surface tension parameter. Furthermore, the dimensionless variables are $k^* \equiv kh_1$ and $\omega^* \equiv \omega h_1/U_m$. The varicose and sinuous

dispersion relations (equations 2.5 and 2.6) become:

$$D^* \equiv (1 - S^*) \frac{M_2^* - \tanh[k^* \delta_w^* (1 - \pi)]}{1 - M_2^* \tanh[k^* \delta_w^* (1 - \pi)]} - (1 + S^*) \frac{[M_1^* - \tanh(k^* \delta_w^* \pi)]}{[1 + M_1^* \tanh(k^* \delta_w^* \pi)]} + \frac{4S\Lambda^*}{k^* \delta_w^* [1 + \Lambda^* (1 - 2\pi)] - \frac{\omega^*}{k^*}} + \frac{k^* \Sigma^*}{[1 + \Lambda^* (1 - 2\pi) - \frac{\omega^*}{k^*}]^2} = 0, \quad (2.7)$$

$$D^* \equiv (1 - S^*) \frac{N_2^* - \tanh[k^* \delta_w^* (1 - \pi)]}{1 - N_2^* \tanh[k^* \delta_w^* (1 - \pi)]} - (1 + S^*) \frac{[N_1^* - \tanh(k^* \delta_w^* \pi)]}{[1 + N_1^* \tanh(k^* \delta_w^* \pi)]} + \frac{4S\Lambda^*}{k^* \delta_w^* [1 + \Lambda^* (1 - 2\pi)] - \frac{\omega^*}{k^*}} + \frac{k^* \Sigma^*}{[1 + \Lambda^* (1 - 2\pi) - \frac{\omega^*}{k^*}]^2} = 0, \quad (2.8)$$

where M_1^* , M_2^* , N_1^* and N_2^* are: $M_1^* = \coth[k^*(1 - \frac{\delta_w^*}{2})] - \frac{2\Lambda^*}{k^* \delta_w^* (1 + \Lambda^* - \frac{\omega^*}{k^*})}$, $M_2^* = N_2^* = -\coth[k^*(h - \frac{\delta_w^*}{2})] - \frac{2\Lambda^*}{k^* \delta_w^* (1 - \Lambda^* - \frac{\omega^*}{k^*})}$, and $M_1^* = \tanh[k^*(1 - \frac{\delta_w^*}{2})] - \frac{2\Lambda^*}{k^* \delta_w^* (1 + \Lambda^* - \frac{\omega^*}{k^*})}$.

Starting from this model, we can construct several simplified models, neglecting the effect of either the surface tension, either the shear layer thickness or both. When Σ^* is settled to zero, the dispersion relations for both modes are easy to obtain. It is enough to neglect the terms containing the surface tension in eqs. (2.7) and (2.8). This model has been analyzed by Juniper [55] and the reader is referred to this publication for the corresponding dispersion relations. When $\delta_w \rightarrow 0$, the limit of the non-dimensional dispersion relations becomes:

$$D^* \equiv (1 + S^*)(1 + \Lambda^* - \frac{\omega^*}{k^*})^2 \coth(k^*) + (1 - S^*)(1 - \Lambda^* - \frac{\omega^*}{k^*})^2 \coth(k^* h) - k^* \Sigma^* = 0, \quad (2.9)$$

$$D^* \equiv (1 + S^*)(1 + \Lambda^* - \frac{\omega^*}{k^*})^2 \tanh(k^*) + (1 - S^*)(1 - \Lambda^* - \frac{\omega^*}{k^*})^2 \coth(k^* h) - k^* \Sigma^* = 0, \quad (2.10)$$

which are the dispersion relations obtained by Juniper [54] and Rees & Juniper [90]. The model becomes a top hat jet/wake with the presence of a surface tension on the interfaces. Note that, without shear layer thickness δ_w^* , the parameter π loses its meaning. Also for eqs. (2.9) and (2.10), the surface tension can be set to zero, and in this way, we obtain the dispersion relations of the model initially analyzed by Juniper & Candel [57]. In general, wake flows have negative velocity ratio. while jet flows have positive one. Particularly, for $0 < |\Lambda^*| < 1$ a co-flow configuration is modeled, otherwise if $|\Lambda^*| > 1$, it is a counter-flow jet/wake.

The non-dimensional framework is used in most of this paper and, thus, the asterisks from now on are omitted.

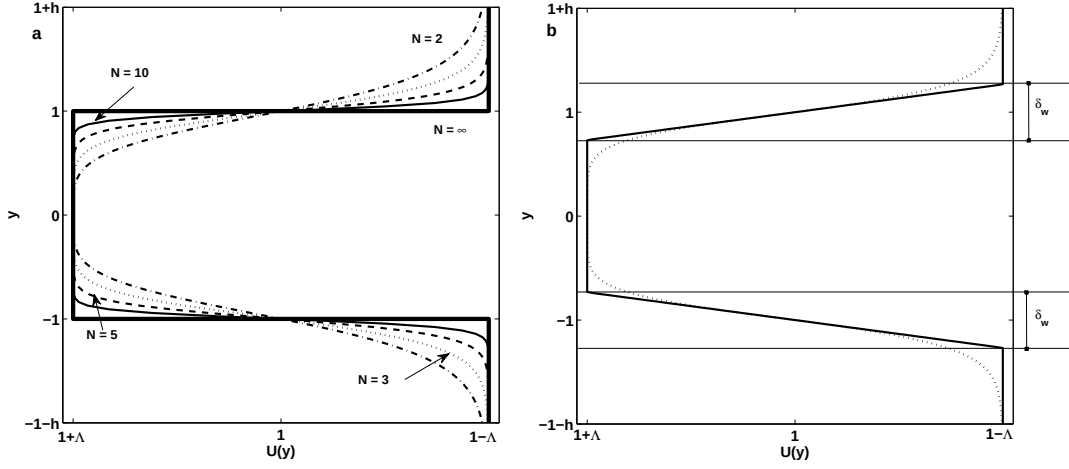


Figure 2.2: (a) Velocity profiles for the model given by equations (2.11): the thick continuous line depicts $N = \infty$, the thin continuous line $N = 10$, the dashed line $N = 5$, the dotted line $N = 3$, and the dash-dotted one $N = 2$. (b) Approximation of the continuous profile (dotted line) with a piecewise broken line (continuous line). The corresponding shear layer thickness δ_w has been calculated using eq. (2.16).

2.2.1.2 Continuous profile

Since the reference scales are the same of the piecewise broken-line model described in previous section 2.2.1.1, the dimensionless notation is directly used. The non-dimensional velocity profile of the second model $U(y)$ is a non-dimensional three-parameter family continuous wake, already studied by Monkewitz [75] without confinement, which is defined for $y \leq |1 + h|$ by the following equations,

$$\begin{aligned} U(y) &= 1 - \Lambda + 2\Lambda F(y), \\ F(y) &= \frac{1}{1 + \sinh^{2N}[|y| \operatorname{arcsinh}(1)]}, \\ \Lambda &= \frac{U_1 - U_2}{U_1 + U_2}, \end{aligned} \quad (2.11)$$

where U_1 and U_2 are the velocities respectively for $y = 0$ and $y \rightarrow \infty$. Note that this time U_2 is the velocity for $y \rightarrow \infty$ for $h = \infty$. The three dimensionless parameters are: Λ , the velocity ratio, h , the confinement parameter, and N , which determines the profile shape. The second derivative of the basic velocity profile $U''(y)$ is zero at $y \pm 1$ and $U(\pm 1) = (U_1 + U_2)/2 = 1$. According to Rayleigh's criterion, the presence of these inflection points allows to the flow to be unstable [28].

Figure 2.11a shows some velocity profiles for different $N = \infty, 10, 5, 3, 2$, high-

lighting the role of the shape parameter N : the profiles become steeper as N increases towards $N = \infty$, for which equations (2.11) yield a top hat wake. It is worthwhile noting that the wall velocity $U(1+h)$ is different from U_2 for $N \neq \infty$, i.e. for smooth profiles. In practice, we limit the shape parameters to values such that the wall velocity departs from U_2 by less than 10%.

In absence of an analytical solution for the flow described by equations (2.11), a spectral Chebyshev collocation method is used in order to solve eq. (2.3) (see Trefethen [101], for a detailed account on this method). Through this numerical discretization, the differential operators become differentiation matrices acting on velocity profiles at collocation points y_i

$$\frac{d^n}{dy^n} \rightarrow D_{ij}^n \equiv \mathbf{D}^n, \quad U(y_i) \rightarrow U\delta_{ij} \equiv \mathbf{U}, \quad \text{and} \quad \frac{d^n U}{dy^n}(y_i) \rightarrow \mathbf{D}^n \mathbf{U}. \quad (2.12)$$

With this numerical scheme, the Rayleigh equation can be rearranged into a generalized eigenvalue problem

$$\mathbf{A}\phi = \omega\mathbf{B}\phi, \quad (2.13)$$

which associates at each wavenumber, a set of eigenfrequencies. The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\mathbf{A} = k\mathbf{U}(\mathbf{D}^2 - k^2\mathbf{I}) - k\mathbf{D}^2\mathbf{U}, \quad (2.14a)$$

$$\mathbf{B} = \mathbf{D}^2 - k^2\mathbf{I}. \quad (2.14b)$$

The implicit relationship $\mathcal{D}(\omega, k)$ represents the dispersion relation of the model.

Since the Rayleigh equation (2.3) is a second order differential equation, eq. (2.13) has to be coupled to two boundary conditions. The perturbations have been divided into varicose and sinuous ones [28] in order to operate in an half domain $0 \leq y \leq 1+h$ and, therefore, with a reduced size of differentiation matrices. Two different boundary conditions have to be imposed: for varicose perturbations, a Dirichlet condition $\phi = 0$ is prescribed on both boundaries, while for the sinuous ones, a Dirichlet condition is applied on the upper boundary $y = 1+h$ and, a Neumann one, $d\phi/dy = 0$, on the lower boundary $y = 0$.

The MATLAB function *eig* is used to solve the eigenvalue problem (2.13) coupled with varicose or sinuous boundary conditions, and, thus, to analyze the temporal and spatio-temporal stability properties.

Furthermore, an analogy between this model and the piecewise broken line profile without surface tension and with uniform density illustrated in section 2.2.1.1 can be created. Recall that N measures the shape of the velocity profile (as seen in fig. 2.2a) and originates in a shear layer or vorticity thickness δ_w , as depicted in figure 2.2b. Steeper profiles associated to larger N , create smaller shear layer thicknesses, which can be estimated using the formula [75]

$$\delta_w = \frac{U_1 - U_2}{|dU/dy|_{max}}. \quad (2.15)$$

The maximum value of $|dU/dy|_{max}$ is reached for the inflection points $y = \pm 1$, thus, when $N \rightarrow \infty$,

$$\delta_w \sim \frac{2^{1/2}}{N \cdot \operatorname{arcsinh}(1)} = \frac{1.6046}{N}. \quad (2.16)$$

In this manner, the piecewise broken line model could be revisited as an approximation of the continuous line model with the same shear layer thickness δ_w measured by means of eq. (2.16). In sections 2.3.2 and 2.5.4, we compare respectively the temporal and spatio-temporal stability properties of the two models in order to measure the predictive power of the broken line model.

2.2.2 Summary

Before starting to analyze the temporal and the spatio-temporal stability of the previously described models, a summary of studies conducted in this thesis and by other authors is needed in order to provide a guideline.

Concerning the temporal stability analysis (section 2.3), the top hat profile, i.e. the model with $\delta_w = 0$, with or without surface tension, is analyzed in section 2.3.1. Both cases, i.e the piecewise broken line profile and the continuous one, including the shear layer thickness and assuming no surface tension at the interfaces are analyzed in section 2.3.2. Finally in section 2.3.3 the temporal stability of the model including both shear layer thickness and surface tension is illustrated.

The same models under the perspective of the counterpropagating Rossby waves (CRWs) are illustrated in section 2.4, with the notable exception of the continuous profile model. Particularly, the piecewise broken line excluding or including the surface tension is discussed in sections 2.4.1 and 2.4.3. Starting from these models, the vanishing shear layer thickness case, i.e. the limits obtained for $\delta_w \rightarrow 0$ are discussed in sections 2.4.4.1 and 2.4.4.2, respectively.

Concerning the spatio-temporal stability (section 2.5), the top hat profile ($\delta_w = 0$ and $\Sigma = 0$) was analyzed by Juniper (2006) [57]. The spatio-temporal properties of the top hat with the addition of the surface tension was analyzed by Rees & Juniper (2009) [90]. The addition of a vanishing shear layer thickness was studied by Juniper (2007) [54] by means of a full impulse response analysis. The model with a complete range of shear layer thickness, both continuous and broken-line ones, is analyzed in sections 2.5.3 and 2.5.4. This complements the analysis of Rees & Juniper (2010) [91] who have analyzed a continuous profile including shear layer thickness in addition to viscous effects. No informations about the spatio-temporal properties of the general model is included in the thesis. Finally, in section 2.5.6 the CRW analysis is applied to the broken line profile in order to explain some spatio-temporal behaviours of this profile.

The complete summary of the local stability analysis conducted either in this thesis or in other works is illustrated in table 2.1.

Model		Temporal stability	CRW	Spatio-temporal stability
Top hat profile	$\delta_w = 0, \Sigma = 0$	2.3.1	2.4.4.1	Juniper(2006) [54]
Top hat profile w. surface tension	$\delta_w = 0$	2.3.1	2.4.4.2	Rees & Juniper (2009) [90]
Broken line profile	$\Sigma = 0$	2.3.2	2.4.1	Juniper (2007) [54] + 2.5.4
Continuous profile		2.3.2	Not Applicable	2.5.3
Broken line profile w. surface tension		2.3.3	2.4.3	To Be Done

Table 2.1: Summary of the local stability analysis for the model described in section 2.1.

2.3 Temporal stability

This section is totally devoted to the temporal behaviours of the models described in section 2.2. Generally, in order to analyze the temporal stability, the wavenumber k is taken real. Each solution j of dispersion relations $\omega^j(k) = \omega_r^j(k) + i \cdot \omega_i^j(k)$, is called a temporal branch, while ω_r^j and ω_i^j are respectively the angular frequency and temporal growth rate. If there exists a particular wavenumber $\bar{k}$ and a particular index j for which $\omega_i^j(\bar{k}) > 0$, the basic flow is linearly unstable, since at least one perturbation is amplified. Conversely, when all temporal modes are neutrally stable or damped, the flow is linearly stable. This section is composed of three parts: each one is devoted to discuss the temporal stability of a particular model. In section 2.3.1 the top hat profile excluding or including surface tension ($\delta_w = 0$) is analyzed, while in section 2.3.2 jet/wake flows including only shear layer thickness are examined ($\Sigma = 0$). Finally in section 2.3.3 the temporal stability of flows including both surface tension and shear layer thickness is discussed.

2.3.1 Confined plane jets and wakes with surface tension

We start this analysis examining the top hat profile (see section 2.2.1.1) adding surface tension. In this case, the dispersion relations, for both modes, varicose and sinuous, eqs. (2.9) and (2.10) can be expressed as an explicit function $\omega(k) = f(k) \pm \sqrt{g(k)}$, where

$$f(k) = k \frac{(1+S)(1+\Lambda) \coth(k) - (1-S)(1-\Lambda) \coth(kh)}{(1+S) \coth(k) + (1-S) \coth(kh)}, \quad (2.17a)$$

$$g(k) = -\frac{4k^2\Lambda^2(1+S)(1-S) \coth(k) \coth(kh)}{[(1+S) \coth(k) + (1-S) \coth(kh)]^2} + \frac{k^3\Sigma}{(1+S) \coth(k) + (1-S) \coth(kh)}, \quad (2.17b)$$

for the varicose mode, while for the sinuous mode, one should substitute $\tanh(k)$ to $\coth(k)$ in (2.17). Owing the square root, it has two values of ω for each value of k , and the stability of the flow can be immediately determined by the sign of $g(k)$.

It is then possible to eliminate the influence of Λ by observing that the growth rate is proportional to Λ , once the surface tension parameter has been rescaled as Σ/Λ^2 . This rescaling of the surface tension is equivalent to the use as reference velocity of $U_d = (U_1 - U_2)/2$, i.e the velocity difference between the two fluids, rather than the velocity average. It is then natural that the growth rate becomes proportional to the shear between the fluids. Using U_d as reference velocity, it becomes clear why the non-dimensional temporal growth rate is independent of Λ . Physically, with this reference velocity, changing the non-dimensional velocity is equivalent to changing only the observer's reference frame, without modifying the shear strength. It is then well known that the growth rates in temporal stability theory are independent of the frame of reference. In the sequel, Λ is therefore set to one, without loss of generality.

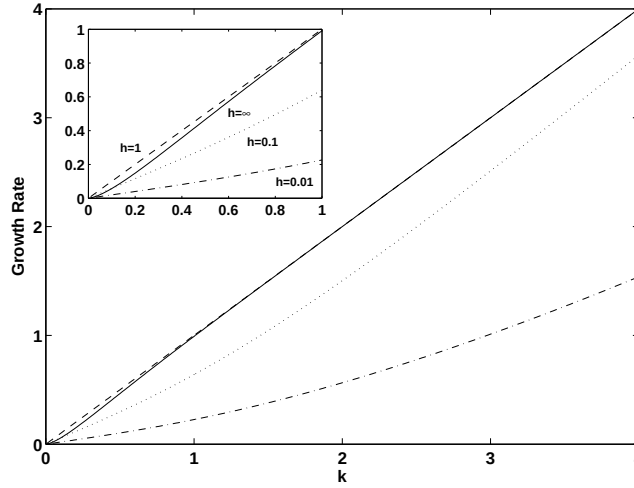


Figure 2.3: Growth rate for the varicose mode of KH instability ($\Sigma = 0$). The continuous line represents $h = \infty$, the broken line $h = 1$, the dotted line $h = 0.1$ and dash-and-dot line $h = 0.01$; $S = 0$.

2.3.1.1 Effect of confinement

Throughout this section, the density ratio is set to $S = 0$, the focus being set on the influence of confinement.

Effect of confinement without surface tension

Considering the pure Kelvin-Helmholtz instability, with zero surface tension, it is known that there is no stabilizing parameter and that both modes are therefore unstable for arbitrary high wavenumbers. Figure 2.3 and 2.4 show how the growth rate varies with the introduction of confinement, respectively for varicose and sinuous modes. There is a substantial difference between the modes, since for the varicose mode, we note (especially in the inset) that for small k , unbounded flow ($h = \infty$) is not the most unstable condition, the introduction of confinement being destabilizing in a narrow region of the (k, h) -plane. Interestingly, note that $h = 1$ corresponds to the point where the varicose mode is both at its most temporally unstable (see figure 2.3) and at its most spatio-temporally stable (as seen in figure 11 of Juniper (2006) [54]). In contrast, for the sinuous mode, confinement is everywhere stabilizing. This contrasts with the influence of confinement on the spatio-temporal growth rate of the sinuous mode, which can be increased by confinement (see figure 14 in Juniper (2006) [54]). In consequence, for pure Kelvin-Helmoltz instability, confinement is generally stabilizing, except for the varicose mode at small wavenumbers.

Effect of surface tension in absence of confinement

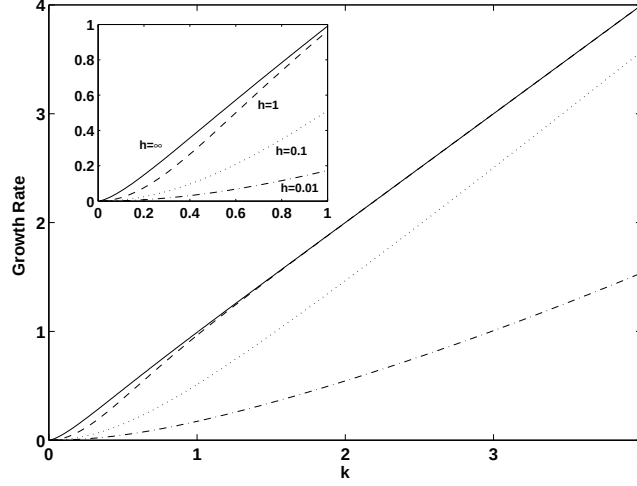


Figure 2.4: Growth rate for the sinuous mode of KH instability ($\Sigma = 0$). The continuous line represents $h = \infty$, the broken line $h = 1$, the dotted line $h = 0.1$ and dash-and-dot line $h = 0.01$; $S = 0$.

Let us now introduce surface tension, which is known to introduce a cut-off wavenumber stabilizing small wavelengths, and consider first the unconfined flow. Figure 2.5 recalls results for several values of the surface tension parameter Σ , together with the asymptotic behavior of the dispersion relation (2.17b) when $k \gg 1$:

$$\omega_i = \sqrt{(1 + S)(1 - S)k^2 - \frac{\Sigma k^3}{2}}. \quad (2.18)$$

This formula clearly illustrates the competition of the destabilizing Kelvin-Helmholtz instability of the vorticity sheet model ($\omega_i \sim k$) and the stabilizing capillary effect, i.e. second term in (2.18).

When the surface tension goes to zero, the maximum growth rate occurs at larger and larger k (see figure 2.5), ensuring accurate predictions of equation (2.18). Conversely, when the surface tension parameter becomes too high, this expression is not relevant anymore, as seen for the case $\Sigma = 5$ in figure 2.5. In this latter case, the cut-off number decreases strongly (as seen for $\Sigma = 5$), which causes only very long wavelengths to become unstable, thereby violating the parallel flow approximation. In the sequel, we focus therefore our attention to surface tension parameter values less than unity ($\Sigma \leq 1$), but still using the equation (2.17b).

Effect of surface tension and confinement

Let us now look at the influence of confinement in presence of surface tension.

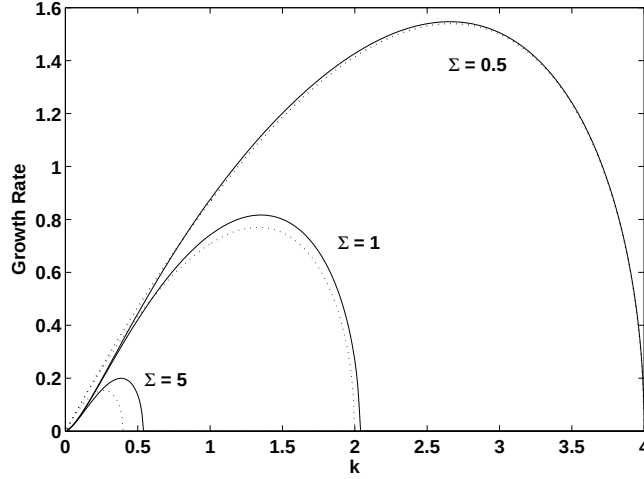


Figure 2.5: Growth rate in absence of confinement for the varicose mode of jet instability for various values of the surface tension parameter; the solid line is the exact growth rate given by eq. (4) and the dashed line is approximation (5); $S = 0$.

Figures 2.6 and 2.7 depict the evolution of the growth rate obtained from the dispersion relations (2.9) and (2.10) for various values of the confinement parameter h , for the varicose and sinuous mode, respectively, and for $\Sigma = 1$. We observe a strong destabilizing influence of the confinement for intermediate values around $h \sim 0.1$. Furthermore, the cut-off wavenumber increases as the confinement becomes stronger. However, a stronger confinement has the expected stabilizing effect.

In order to better illustrate the complex influence of confinement on the jet instability, figure 2.8 and 2.9 depict isocontours of the growth rate ω_i for varicose and sinuous modes in the (k, h) -plane, for $\Sigma = 1$. The isocontours are equally spaced by 0.05.

In the limit of unconfined flow ($h \rightarrow \infty$) the typical temporal growth rate of Kelvin-Helmoltz instability with surface tension is recovered, with a maximum amplification at finite wavenumber $k = 1.35$ and cut-off at $k = 2.03$ (compare with figure 2.6) for the varicose mode and respectively $k = 1.33$, $k = 1.96$ for the sinuous one. However, we note for both symmetries the existence of a central region in the (k, h) -plane of increased instability, centered around $k = 1.93, h = 0.192$ for the varicose and $k = 2.07, h = 0.156$ for the sinuous one. Furthermore, the range of unstable wavenumbers widens as the confinement is increased. Interestingly, though the influence of confinement on the sinuous mode without surface tension is always stabilizing (see figure 2.4), the addition of surface tension completely modifies the picture and the confinement becomes

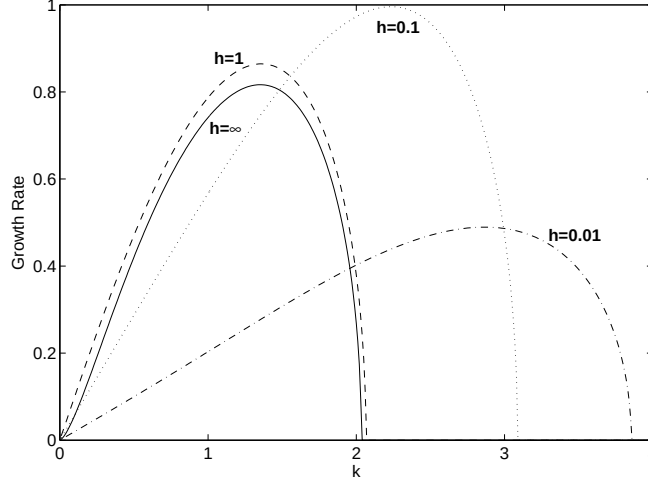


Figure 2.6: Growth rate for the varicose mode with surface tension $\Sigma = 1$. The continuous line represents $h = \infty$, the broken line $h = 1$, the dotted line $h = 0.1$ and $- \cdot$ line $h = 0.01$; $S = 0$.

destabilizing in a considerable region of the (k, h) -plane, exactly like the varicose mode.

The natural question is now to determine if these results are robust with respect to a change in surface tension, as shown previously in the unconfined limit for $\Sigma \leq 1$. We therefore introduce an *ad hoc* rescaling, inspired from relation (2.18)

$$k_{\Sigma} = k\Sigma, \quad h_{\Sigma} = \frac{h}{\Sigma} \quad \text{and} \quad \omega_{i\Sigma} = \omega_i\Sigma. \quad (2.19)$$

The validity of this change of scales is confirmed by figure 2.10, where it is seen that for the cases $\Sigma = 0.01$ (dotted line) and $\Sigma = 0.1$ (broken line), the contours are nearly identical for the larger part of the domain (only near the h -axis there is a slight difference). At fixed k , the rescaling is remarkable for small Σ , whilst for $\Sigma = 1$ (continuous line) the contours are close, but not equal. A similar behavior for increasing Σ was noted previously in figure 2.5 for $h = \infty$.

In fact, the validity of this rescaling (2.19) at low surface tension parameter Σ can be rigorously ascertained by considering following exact change of variables:

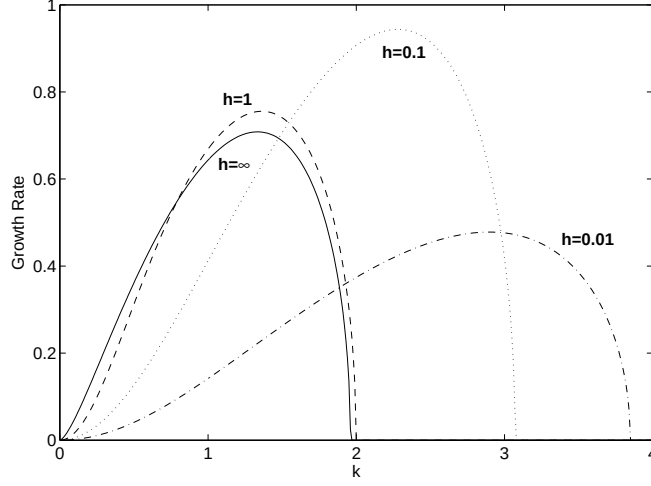


Figure 2.7: Growth rate for the sinuous mode with surface tension $\Sigma = 1$. The continuous line represents $h = \infty$, the broken line $h = 1$, the dotted line $h = 0.1$ and $- \cdot$ line $h = 0.01$; $S = 0$.

$$\begin{aligned}
k_{\Sigma} &= k\Sigma, \\
h_{\Sigma} &= \frac{h}{\Sigma}, \\
\omega_{i,\Sigma}^2 &= \omega_i^2 \Sigma^2 + \frac{4k_{\Sigma}^2(1+S)\coth(\frac{k_{\Sigma}}{\Sigma})(1-S)\coth(k_{\Sigma}h_{\Sigma})}{[(1+S)\coth(\frac{k_{\Sigma}}{\Sigma}) + (1-S)\coth(k_{\Sigma}h_{\Sigma})]^2} + \\
&\quad - \frac{4k_{\Sigma}^2(1+S)\coth(k_{\Sigma})(1-S)\coth(k_{\Sigma}h_{\Sigma})}{[(1+S)\coth(k_{\Sigma}) + (1-S)\coth(k_{\Sigma}h_{\Sigma})]^2} - \frac{k_{\Sigma}^3}{(1+S)\coth(\frac{k_{\Sigma}}{\Sigma}) + (1-S)\coth(k_{\Sigma}h_{\Sigma})} + \\
&\quad + \frac{k_{\Sigma}^3}{(1+S)\coth(k_{\Sigma}) + (1-S)\coth(k_{\Sigma}h_{\Sigma})}.
\end{aligned} \tag{2.20}$$

As Σ goes to zero, it can be shown that the four last terms compensate two by two.

A physical explanation for the rescaling (2.19) can be proposed in the light of the recent paper of Rees and Juniper (2009) [90]. The change of variables by means of (2.19) is equivalent to the use as reference length the surface tension length-scale $L_{\sigma} = \sigma/(\rho_m U_m^2)$ instead of the non-dimensional inner flow width h_1 . With this rescaling, the term $1/\Sigma = h_1/L_{\sigma}$ represents a comparison between different lengthscales. A small value of Σ therefore implies $h_1 \gg L_{\sigma}$. Physically, this means that the surface tension lengthscale is smaller than h_1 and thus that the perturbations due to surface tension do not interact with those of the symmetric

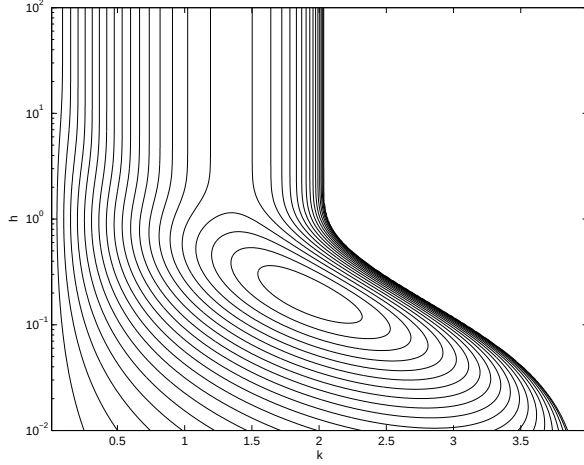


Figure 2.8: Growth rate for the varicose mode on the (k, h) -plane; $\Sigma = 1$, $S = 0$. The isocontours are equally spaced by 0.05. Note the central region where the instability is maximal.

shear layer. This is equivalent to having $k \gg 1$ in equation (4b) and explains the remarkable scaling for small Σ in figure 2.10.

2.3.1.2 Effect of density ratio

The goal of this paragraph is to consider the influence of the density ratio on the destabilization due to confinement and in particular on the maximum region identified in figures 2.8 and 2.9. The surface tension parameter is set to $\Sigma = 1$, in accordance with the rescaling proposed in last section. We also focus our attention on the varicose mode, the sinuous mode behaving in very similar manner. In absence of confinement, the influence of the density ratio on the growth rate is analyzed in figure 2.11. It is seen that the range of unstable wavenumbers widens as the density ratio is increased with a region of maximum growth centered around $S = -0.05$, $k = 1.35$. The superimposed curve in figure 2.11 joins the maxima with respect to k at fixed S .

This maximum growth curve, without confinement, is reproduced in figure 2.12 as a thin continuous line, together with similar curves for several values of the confinement parameter $h = \{1, 0.1, 0.01\}$. As h is decreased, the overall shape of the curve remains similar, displaying however an increasing maximum growth rate at an increasing value of the density ratio S . Note also that when the inner and outer flow widths are equal, i.e $h = 1$, the curve (the dashed line in figure 2.12) is perfectly symmetric and therefore the maximum of growth rate occurs for $S = 0$. The envelope curve of maximal growth (over k and h) as a function of S is also plotted. Note its monotonic increase.

In addition, a typical growth rate isocontour plot in the (k, h) -plane is depicted in figure 2.13 for $S = 0.8$. In comparison to figure 2.8, the central maximum region

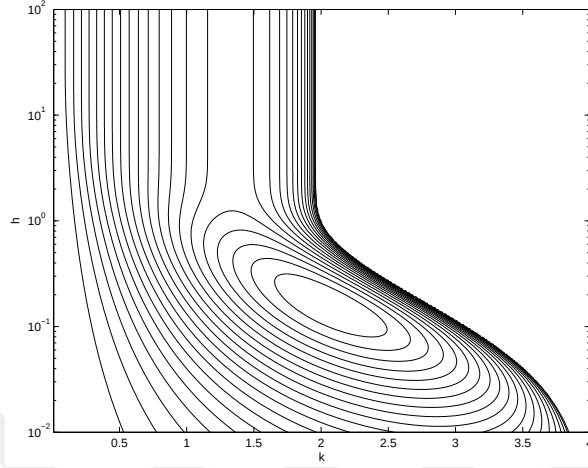


Figure 2.9: Growth rate for the sinuous mode on the (k, h) -plane; $\Sigma = 1$, $S = 0$. The isocontours are equally spaced by 0.05. Note the central region where the instability is maximal.

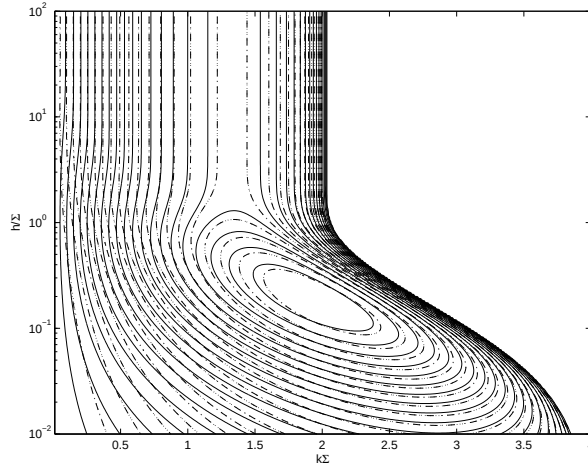


Figure 2.10: Comparison between different values of Σ on $(\Sigma k, h/\Sigma)$ -plane for the varicose mode. The continuous line represent the values of $\Sigma\omega_i$ for $\Sigma = 1$, the broken line for $\Sigma = 0.1$, the dotted line for $\Sigma = 0.01$. The isocontours for every value of Σ are equally spaced by 0.05. $S=0$.

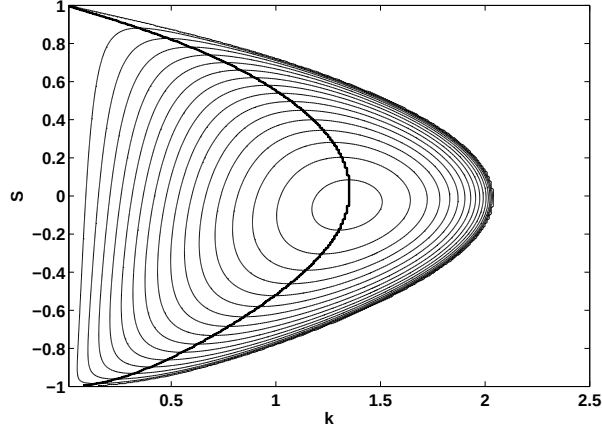


Figure 2.11: Growth rate in the (k, S) -plane for the unconfined configuration. The thick continuous line depicts the position of the maximum growth rate for each S . The isocontours are equally spaced by 0.05.

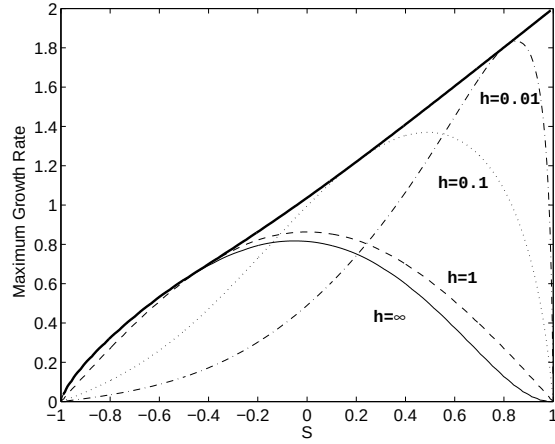


Figure 2.12: Growth rate maxima versus density ratio. The thick continuous line depicts the general maximum growth rate for all kind of confinement, the continuous line depicts the maximum growth rate for the unbounded configuration, the dashed line $h = 1$, the dotted line $h = 0.1$ and the dashed-dotted line is $h = 0.01$.

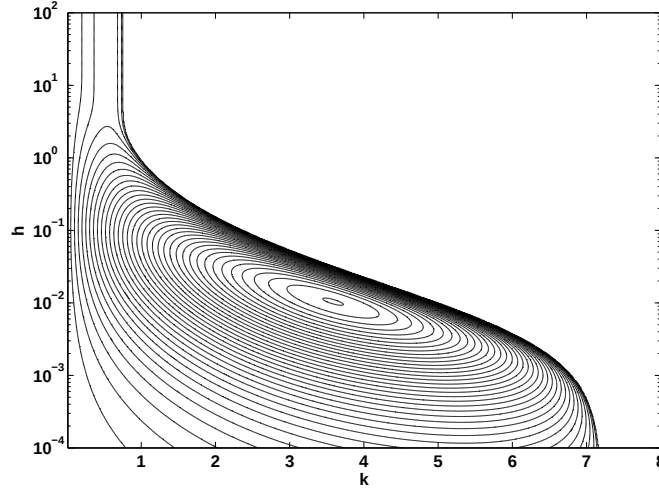


Figure 2.13: Growth rate of varicose mode on (k, h) -plane, $\Sigma = 1$, $S = 0.8$. The isocontours are equally spaced by 0.05. This figure is to be compared with figure 2.8.

is still observed, with however an higher growth rate increase in comparison to the unconfined case and at smaller values of h .

2.3.2 Confined plane jets and wakes with finite shear layer thickness

Now we continue our discussion including the shear layer thickness and excluding the surface tension from the model. See section 2.2 to recall the description of the models analyzed in the present section. The continuous profile (eq. 2.11) is analyzed first and subsequently compared with the piecewise broken line model with corresponding shear layer thickness calculated by means of eq. (2.16).

The temporal stability of the particular case $N = \infty$ has been studied in section 2.3.1. Remember that a small destabilizing influence of confinement on long waves for varicose mode was noted, in contrast to the sinuous mode, for which no destabilizing influence was found. Fig. 2.14 shows the growth rate for different amounts of confinement. The thickness of the shear layer is set to $\delta_w = 0.5$, corresponding to $N = 3.209$, according to eq. (2.16). Note that the growth rates in the temporal stability theory are independent of the reference frame, as seen in section 2.3.1 therefore settling $\Lambda = -1$ does not alter the generality of the results. In comparison to the results for the infinitely thin vorticity sheet model (see figures 2.3 and 2.4), the addition of a finite shear layer thickness stabilizes large wavenumbers but it does not modify the influence of confinement, which is only active at small wavenumbers. The small destabilizing effect on the varicose perturbations at small wavenumber is

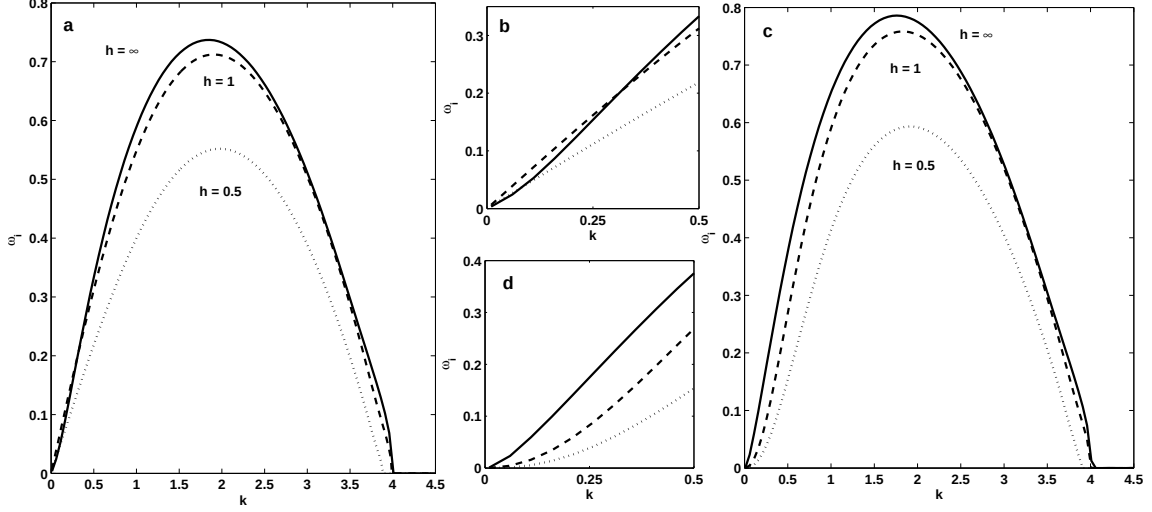


Figure 2.14: Temporal growth rate for varicose (a and b) and sinuous (c and d) mode for continuous profile with shape parameter $N = 3.209$ and $\Lambda = -1$. The continuous line represents $h = \infty$, the dashed line $h = 1$, and the dotted one $h = 0.5$. Figs. b and d are respectively a detail of Figs. a and c. Note in figure b the small destabilizing effect of confinement at small wavenumbers for varicose mode.

evidenced in Fig. 2.14b, while for the sinuous instability, the unconfined case is the most temporally unstable configuration (Fig. 2.14d).

Introducing the piecewise broken line, in figure 2.15 the same shear layer thicknesses are illustrated also for this mode in order to compare the temporal stability of the models. The broken line approximation is seen to be excellent at small wavenumbers, and therefore to share the same influence of confinement as the continuous profiles. A similar value of maximal growth rate for both modes is furthermore obtained, but the cut-off numbers change significantly with respect to continuous profile, as noticed by Esch [29] for unconfined mixing layers.

In figure 2.16, the phase speed of the (a) varicose and (b) sinuous mode of two models for different confinement ratios is compared. Note that the phase speed grows when adding confinement to the flow. Furthermore, the frequencies of the continuous profile are well approximated by the broken line when the confinement is sufficiently weak, while for $h \leq 0.5$ the approximation appears to break down.

2.3.3 Confined plane jets and wakes with both surface tension and shear layer thickness

In section 2.3.1 we have illustrated that for the top hat including the surface tension at the interfaces, there exists only one unstable mode for each kind of symmetry, varicose

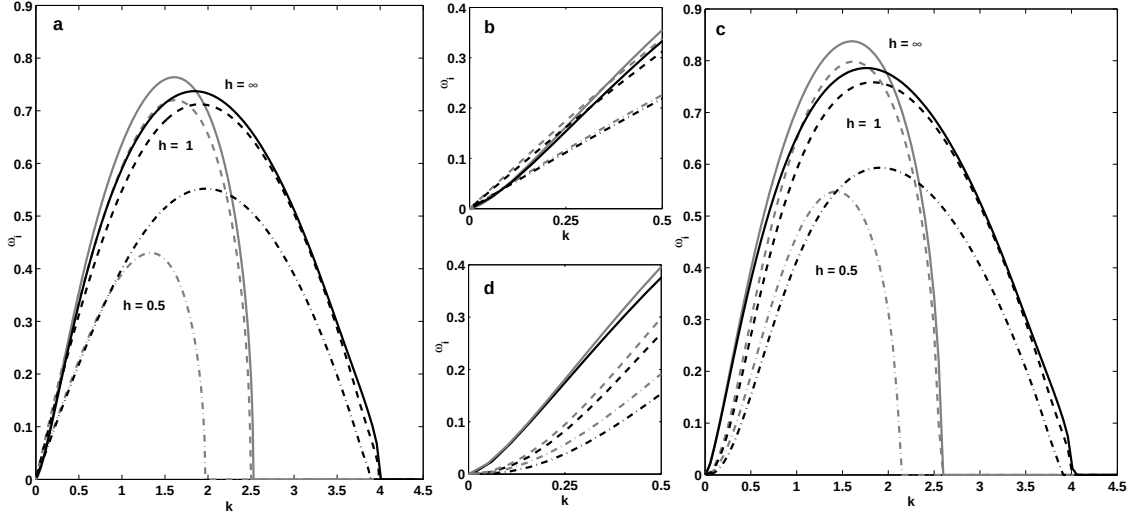


Figure 2.15: Comparison of the temporal growth rates between continuous profile (bold black lines) and broken-line one (thin gray lines) for varicose (a and b) and sinuous (c and d) mode with shear layer thickness $\delta_w = 0.5$ and $\Lambda = -1$. The continuous line represents $h = \infty$, the dashed line $h = 1$, and the dash-and-dot one $h = 0.5$. Figs. b and d are respectively a detail of Figs. a and c.

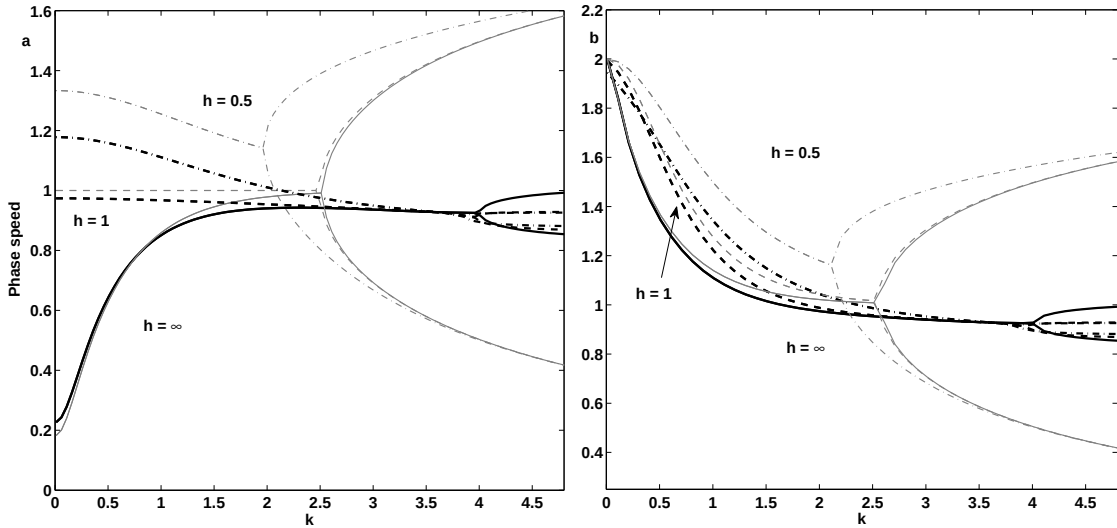


Figure 2.16: Comparison of the phase speed of the (a) varicose and the (b) sinuous mode between continuous profile (bold black lines) and broken-line one (thin gray lines) with shear layer thickness $\delta_w = 0.5$ and $\Lambda = -1$. The continuous line represents $h = \infty$, the dashed line $h = 1$, and the dash-and-dot one $h = 0.5$.

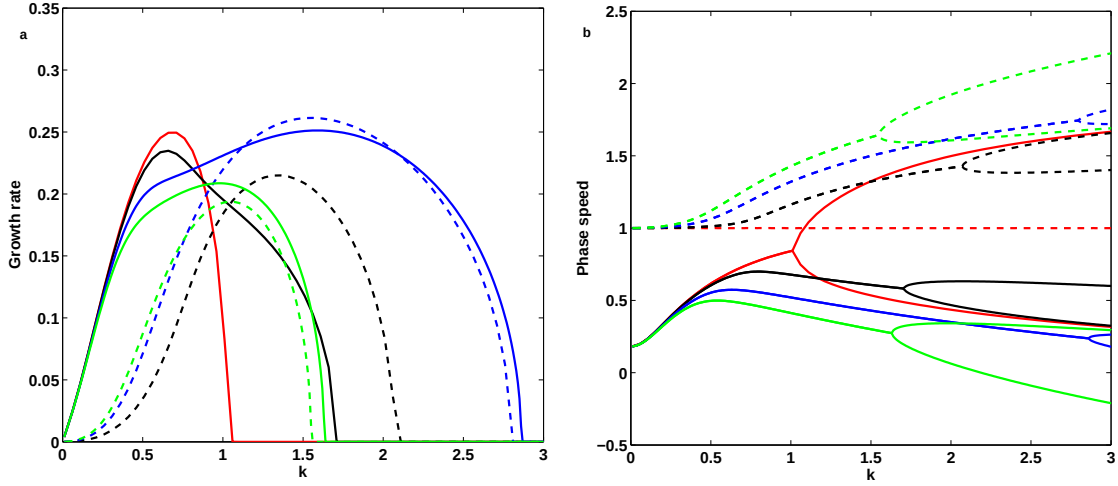


Figure 2.17: (a) Growth rate and (b) phase speed for the varicose mode as a function of the wavenumber k for $h = \infty$, $\delta_w = 1$, $S = 0$, $\pi = 0.5$, and $\Lambda = -1$. Red lines depict $\Sigma = 0$, black ones $\Sigma = 0.1$, blue $\Sigma = 0.5$, and green $\Sigma = 1$. The continuous lines depict the δ_w -mode and the dashed ones the Σ -mode.

and sinuous. Similar result was obtained for the piecewise broken line model in section 2.3.2. In the former case, the Kelvin-Helmholtz (KH) instability (section 2.3.1.1) was damped by the effect of surface tension, represented by the parameter Σ , while in the latter one by the presence of a finite shear layer thickness δ_w . Concerning the confinement, the main difference between the two models was the destabilizing effect noticed when the surface tension is added, conversely when shear layer thickness is added, the effect of confinement is unvaried with respect to Kelvin-Helmholtz instability (section 2.3.2), being mainly stabilizing.

A natural question is what will indeed occur adding both parameters together, surface tension and shear layer thickness in the model. First of all, examining the dispersion relations (2.7) and (2.8), they can be rearranged as a fourth order equation, which it means that four solutions $\omega_i(k)$ are found. Generally we expect to have, thus, at most two different unstable modes, when both parameters, δ_w and Σ are non-zero.

In figure 2.17 the (a) growth rate and the (b) phase speed for the varicose mode as a function of the wavenumber are shown for different values of Σ with $\delta_w = 1$, $S = 0$, and $\pi = 0.5$.

The red line depicts the case, already analyzed in section 2.3.2, without surface tension, where only one unstable mode is present. Adding the surface tension, a second unstable mode appears while at the same time the first one is modified, as larger wavenumbers are destabilized, see the case with $\Sigma = 0.1$ (black lines). In order to distinguish the modes, we call δ_w -mode the mode already present when the surface

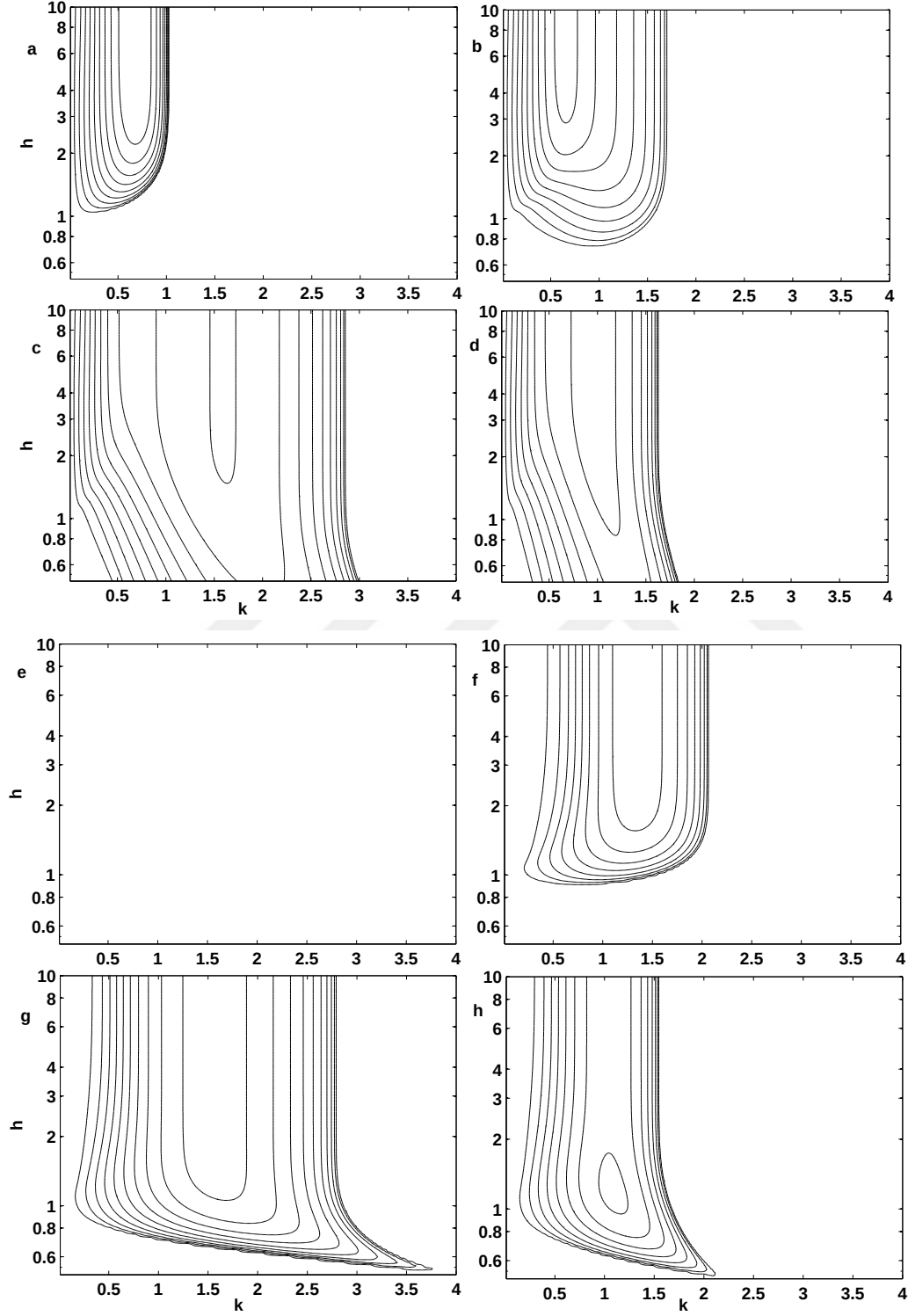


Figure 2.18: Contours of the temporal growth rate of the varicose mode for the (a,b,c,d) δ_w -mode and the (e,f,g,h) Σ -mode in the (k,h) -plane for $\delta_w = 1$, $S = 0$, $\pi = 0.5$ and (a,e) $\Sigma = 0$, (b,f) $\Sigma = 0.1$, (c,g) $\Sigma = 0.5$, and (d,h) $\Sigma = 1$. The contours are equally spaced by 0.025 with maximum value 0.25.

tension is zero, and Σ -mode the other one, appearing when the surface tension is introduced. Focusing on the δ_w -mode, observe in fig. 2.17a, that small wavenumbers are stabilized as the surface tension is introduced, following the intuition and also the results given in section 2.3.1. However the effect of surface tension for larger wavenumbers is counterintuitive, since a larger range of wavenumber is destabilized as a small surface tension is added. A similar counterintuitive destabilization due to a small surface tension was recently noticed by Tammisola *et al.* [100] in their linear global analysis of two phases-wakes. However, the stabilizing influence of the surface tension finally takes over for larger wavenumbers as the value of Σ is further increased. The cut-off wavenumber k_c decreases, however still remaining higher than in the case with zero-surface tension. The Σ -mode is further destabilized at small wavenumbers when surface tension is increased, while the effect on large wavenumbers is similar to that of the δ_w -mode. It should be noted that the sinuous mode has a similar behaviour as the varicose one. Furthermore, note in figure 2.17b that the phase speed of the δ_w -mode is enhanced by the addition of the surface tension, while for the Σ -mode it is reduced.

Figure 2.18 introduces the confinement for the (a,b,c,d) δ_w -mode and the (e,f,g,h) Σ -mode for different values of the surface tension: (a,e) $\Sigma = 0$, (b,f), $\Sigma = 0.1$, (c,g) $\Sigma = 0.5$, and (d,h) $\Sigma = 1$. First of all, we can extend for each confinement the destabilizing influence of surface tension noted for the unconfined configuration for both modes. The δ_w -mode does not present a particular influence of confinement, except for small wavenumbers where the growth rate is enhanced by confinement as noticed in sections 2.3.1.1 and 2.3.2 respectively for the top hat profile and the models including only the shear layer thickness. Conversely for the Σ -mode the destabilizing influence of confinement increases with surface tension. Finally for $\Sigma = 1$, we note the existence of a region where the instability is maximal, reminiscent of figure 7 (see section 2.3.1).

In order to understand the role played by the density ratio S and the position of the interfaces π , we illustrate in figure 2.19 the behaviour of the maximum of the growth rate in the (k,h) -plane with respect to (a) S and (b) π for both δ_w -mode and Σ -mode. The continuous line depicts the δ_w -mode, while the dashed one the Σ -mode. Focusing on the influence of density ratio, we note that both the modes have an optimal density ratio for which the temporal instability is maximal, $S \sim 0.25$ for the δ_w -mode and $S \sim -0.4$ for the Σ -mode. Conversely examining the influence of the position of the interfaces, only the Σ -mode highlights a clear maximum of the instability for $\pi \sim 0.25$. For the δ_w -mode, a small peak is noticed for $\pi = 0.75$. Another interesting information is that for $S = 0$ and $\pi = 0.5$ two modes have the same maximal value. Moreover for this configuration ($\Sigma = 1$, $\delta_w = 1$, $\pi = 0.5$, and $S = 0$) two modes coincide for each value of the confinement parameter.

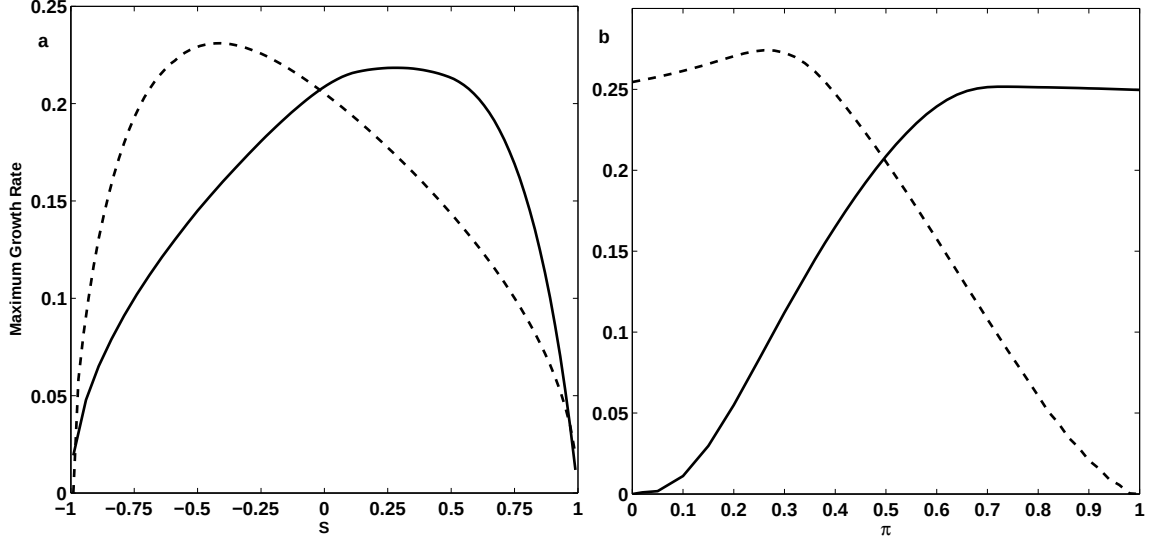


Figure 2.19: Maximum growth rate of varicose perturbations vs (a) density ratio S , (b) interfaces position π for $\delta_w = 1$, $\Sigma = 1$, and (a) $\pi = 0.5$, (b) $S = 0$. The continuous lines depict δ_w -mode, while the dashed ones Σ -mode.

2.4 Interaction between counterpropagating Rossby waves and capillarity waves in shear instability

In this section we will examine the temporal stability of each piecewise broken line-model flow described in section 2.2.1.1 from the point of view of counterpropagating Rossby wave (CRW) perspective. In the general model including both surface tension and shear layer thickness, two types of waves are created: the classical CRWs due to the vorticity gradient [13, 48] of the base flows and the capillarity waves (CWs), due to the action of surface tension.

First of all, in order to understand carefully the role played by both wave types, we examine two simplified confined models in section 2.4.1 and 2.4.2, respectively, where these waves act alone and cannot interact together. The first model is the piecewise broken line profile without surface tension, where only CRWs are created by vorticity gradients, while the second one consists in a flow with constant basic vorticity profile, where the surface tension is present at the interfaces, thus, creating two CWs. Afterwards CRWs and CWs are coupled in section 2.4.3 and therefore the general model flow is discussed. Furthermore in section 2.4.4, the limit for vanishing shear layer thickness including or not the surface tension is discussed in order to revisit the temporal stability of the top hat jet/wake ($\delta_w = 0$) at the light of the CRWs perspective.

2.4.1 Counterpropagating Rossby waves in confined jet/wakes flows

The model analyzed in this section is the piecewise broken line profile described in section 2.2.1.1 in absence of the surface tension at the interfaces. Hence the nondimensional basic velocity is given by

$$U(y) = \begin{cases} 1 - \Lambda & \text{for } |y| \geq 1 + \frac{\delta_w}{2} \\ 1 + \frac{2\Lambda}{\delta_w}(|y| - 1) & \text{for } 1 - \frac{\delta_w}{2} < |y| < 1 + \frac{\delta_w}{2} \\ 1 + \Lambda & \text{for } |y| \leq 1 - \frac{\delta_w}{2} \end{cases} \quad (2.21)$$

and the non-dimensional basic vorticity profile $Q = -dU/dy$, is

$$Q(y) = \begin{cases} 0 & \text{for } |y| \geq 1 + \frac{\delta_w}{2} \\ \frac{2\Lambda}{\delta_w} & \text{for } 1 - \frac{\delta_w}{2} < y < 1 + \frac{\delta_w}{2} \\ -\frac{2\Lambda}{\delta_w} & \text{for } -1 - \frac{\delta_w}{2} < y < -1 + \frac{\delta_w}{2} \\ 0 & \text{for } |y| \leq 1 - \frac{\delta_w}{2} \end{cases} \quad (2.22)$$

In particular, (x, y) are the streamwise and cross-stream Cartesian coordinates associated with the velocity $\mathbf{v} = (U + u', v')$, where the capital letters and the primes indicate respectively the basic state and perturbation. The model is symmetric with respect to the x -axis, and we can examine only positive y , allowing to decompose the perturbations into varicose and sinuous ones. The varicose mode is equivalent to study a mixing layer analyzed by Heifetz *et al.* (1999) [45] and Heifetz & Methven (2005) [46], however adding confinement. Linearizing the vorticity equation $\frac{Dq}{Dt}$ with respect to the basic state gives:

$$\left[\frac{\partial}{\partial t} + U(y) \frac{\partial}{\partial x} \right] q' = -v' \frac{\partial Q(y)}{\partial y} = v' m \left[\delta(y - 1 + \frac{\delta_w}{2}) - \delta(y - 1 - \frac{\delta_w}{2}) \right] \quad (2.23)$$

where $m = -\frac{2\Lambda}{\delta_w}$ and δ symbolizes the Dirac delta function.

Applying the CRW perspective [45], we assume that the Fourier-spectrum of the vorticity can be written

$$\hat{q}' = \left(q_1(k, t) \delta\left[y - 1 + \frac{\delta_w}{2}\right] + q_2(k, t) \delta\left[y - 1 - \frac{\delta_w}{2}\right] \right). \quad (2.24)$$

Since the vorticity is given by the Laplacian of the stream perturbation we have $\hat{q}' = L\hat{\psi}'$, where the operator $L = [d^2/dy^2 - k^2]$. The Green function of L for the varicose mode, which imposes Dirichlet conditions $\partial_x \psi' = 0$ at the boundaries $y = 0$ and $y = 1 + h$, is given by

$$G(y, y') = -\frac{1}{k \sinh[k(1 + h)]} \begin{cases} \sinh(ky') \sinh[k(1 + h - y)] & y' \leq y \leq 1 + h \\ \sinh[k(1 + h - y')] \sinh(ky) & 0 \leq y \leq y', \end{cases} \quad (2.25)$$

while for the sinuous mode, for which a Neumann condition $\partial_{x,y}\psi' = 0$ on the axis $y = 0$ and a Dirichlet one at the wall $y = 1 + h$ are imposed,

$$G(y, y') = -\frac{1}{k \cosh[k(1+h)]} \begin{cases} \cosh(ky') \sinh[k(1+h-y)] & y' \leq y \leq 1+h \\ \sinh[k(1+h-y')] \cosh(ky) & 0 \leq y \leq y'. \end{cases} \quad (2.26)$$

From here on, only the sinuous perturbation is developed, since for the varicose one, the reasoning is identical, with the only difference that eq. (2.25) should be used. Thus, by means of the Green function (2.26), the spectrum solution of eq. (2.23) can be found :

$$\begin{aligned} & - \text{ for } y \geq 1 + \frac{\delta_w}{2} \\ & \quad \hat{\psi}'(y) = -q_1(k, t) \frac{\cosh[k(1-\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & \quad \quad - q_2(k, t) \frac{\cosh[k(1+\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] \\ & - \text{ for } 1 - \frac{\delta_w}{2} \leq y \leq 1 + \frac{\delta_w}{2} \\ & \quad \hat{\psi}'(y) = -q_1(k, t) \frac{\cosh[k(1-\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & \quad \quad - q_2(k, t) \frac{\sinh[k(h-\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky) \\ & - \text{ for } y \leq 1 - \frac{\delta_w}{2} \end{aligned} \quad (2.27)$$

$$\begin{aligned} \hat{\psi}'(y) &= -q_1(k, t) \frac{\sinh[k(h+\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky) + \\ & \quad - q_2(k, t) \frac{\sinh[k(h-\frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky), \end{aligned}$$

where ψ' satisfies $u' = \frac{\partial \psi'}{\partial y}$, $v' = -\frac{\partial \psi'}{\partial x}$, i.e. in Fourier space $\hat{u}' = \partial_y \hat{\psi}'$ and $\hat{v}' = ik \hat{\psi}'$, and $q_1(k, t)$, $q_2(k, t)$ represent the counter-propagating Rossby waves at $y = 1 \pm \frac{\delta_w}{2}$. Substituting equations (2.24) and (2.27) into eq. (2.23), one is left with

$$\begin{aligned} & \dot{q}_1 \delta(y - 1 + \frac{\delta_w}{2}) + \dot{q}_2 \delta(y - 1 - \frac{\delta_w}{2}) + ikU(y) [q_1 \delta(y - 1 + \frac{\delta_w}{2}) + q_2 \delta(y - 1 - \frac{\delta_w}{2})] = \\ & = [-imq_1 \frac{\cosh[k(1-\frac{\delta_w}{2})] \sinh[k(1+h-y)]}{\cosh[k(1+h)]} - imq_2 \frac{\sinh[k(h-\frac{\delta_w}{2})] \cosh[ky]}{\cosh[k(1+h)]}] \cdot \\ & \quad \cdot [\delta(y - 1 + \frac{\delta_w}{2}) - \delta(y - 1 - \frac{\delta_w}{2})], \end{aligned} \quad (2.28)$$

where the notation $\dot{q}_i$ denotes dq_i/dt .

Calculating eq. (2.28) at the edges of CRW ($1 \pm \frac{\delta_w}{2}$), the evolution of vorticity perturbation is obtained:

$$\dot{q} = Mq, \quad \text{where} \quad q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} -ikc_1 & -i\gamma \\ i\gamma & -ikc_2 \end{bmatrix}, \quad (2.29)$$

with

$$\gamma = -\frac{2\Lambda}{\delta_w} \cdot \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 - \frac{\delta_w}{2})]}{\sinh[k(1+h)]}, \quad (2.30)$$

$$c_1 = \frac{1-\Lambda}{\delta_w} \cdot \frac{\cosh[k(1 - \frac{\delta_w}{2})] \sinh[k(h + \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} + \\ + (1+\Lambda)(1 - \frac{\cosh[k(1 - \frac{\delta_w}{2})] \sinh[k(h + \frac{\delta_w}{2})]}{k\delta_w \cosh[k(1+h)]}), \quad \text{and} \quad (2.31)$$

$$c_2 = \frac{1+\Lambda}{\delta_w} \cdot \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 + \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} + \\ + (1-\Lambda)(1 - \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 + \frac{\delta_w}{2})]}{k\delta_w \cosh[k(1+h)]}). \quad (2.32)$$

This perturbation can be regarded as two vorticity edge waves $q_1 = Q_1(k, t)e^{i\epsilon_1(k, t)}$, $q_2 = Q_2(k, t)e^{i\epsilon_2(k, t)}$ where Q_1, Q_2 are the amplitudes and ϵ_1, ϵ_2 the phases. Replacing these relationships into eq. (2.29) the evolution of counter-propagating Rossby waves is obtained:

$$\dot{Q}_1 = \gamma Q_2 \sin \epsilon, \quad \dot{Q}_2 = \gamma Q_1 \sin \epsilon \quad (2.33)$$

$$\dot{\epsilon}_1 = -kc_1 - \gamma \frac{Q_2}{Q_1} \cos \epsilon, \quad \dot{\epsilon}_2 = -kc_2 + \gamma \frac{Q_1}{Q_2} \cos \epsilon. \quad (2.34)$$

The interaction between the CRWs is governed by coefficient γ , while $\epsilon = \epsilon_2 - \epsilon_1$ is the CRW phase difference. If $\gamma = 0$, the waves would propagate with their intrinsic phase speeds c_1 and c_2 given by equations (2.31) and (2.32). In the locked configuration the CRW growth rates (2.33), have to be equal, $\dot{Q}_1 = \dot{Q}_2$, thus, if $\sin \epsilon \neq 0$, also their amplitudes Q_1 and Q_2 should be equal. Concerning the phase difference ϵ the locked configuration occurs when $\dot{\epsilon}_1 = \dot{\epsilon}_2$, consequently from equations (2.34) is giving:

$$\cos \epsilon = \frac{k(c_2 - c_1)}{2\gamma} \quad \text{and} \quad \sin \epsilon = \sqrt{1 - \frac{k^2(c_2 - c_1)^2}{4\gamma^2}}. \quad (2.35)$$

In such manner, we have decomposed the shear instability into two different waves which interact each other. It is possible now to analyze which is the influence of the confinement on the singular waves, without considering the interaction between them in order to interpret some stability properties of confined jet/wakes. Such a goal of section 2.5.6 where we will try to interpret some results obtained with the spatio-temporal analysis by means of the physical properties of the individual CRWs.

2.4.2 Capillarity Waves

In order to study the capillarity waves (CWs) associated to the sinuous mode, a model with constant basic vorticity, i.e. without vorticity gradients has to be constructed.

The dimensionless basic velocity profile of CW model is:

$$U(y) = 1 + \frac{2\Lambda}{\delta_w}(y - 1) \quad (2.36)$$

in the confined domain $0 \leq y \leq 1 + h$. The basic vorticity is thus constant, $Q = \frac{-2\Lambda}{\delta_w}$, with an uniform basic pressure P . Two fluids compose the CW model and a non-dimensional surface tension Σ is present at their interface. The position of the interface is governed by the parameter π , see section 2.2.

The presence of the surface tension creates a jump of the pressure at the interface. Linearizing around the uniform basic pressure P and decomposing in normal mode solutions (see Huerre & Rossi, chapter 4 [51]), the jump of the perturbation of the pressure can be written

$$[\tilde{p}(y)]_{y_\pi^-}^{y_\pi^+} = \Sigma k^2 \eta, \quad (2.37)$$

where η is the y -deformation of the interface. As already obtained in eqs. (2.2a) and (2.2b), decomposing in normal modes the linearized 2D Euler equations, we obtain

$$-ik\tilde{p} = ik(U(y) - c)\tilde{u} + \frac{\partial U(y)}{\partial y}\tilde{v}, \quad (2.38)$$

$$\frac{\partial \tilde{p}}{\partial y} = -ik(U(y) - c)\tilde{v}, \quad (2.39)$$

where c is the complex phase velocity ω/k [51]. In the light of the jump experienced by the pressure at the interface illustrated by eq. (2.37) deriving eq. (2.38) with respect to y and introducing the normal mode solution of the streamfunction $\psi'(x, y, t) = \phi(y)e^{i(kx - \omega t)}$:

$$\frac{\partial \tilde{p}}{\partial y} = \frac{\partial^2 U(y)}{\partial y^2} \phi - (U(y) - c) \frac{\partial^2 \phi}{\partial y^2} + \Sigma k \eta \delta[y - y_\pi], \quad (2.40)$$

where we have used that $\tilde{u} = \frac{\partial \phi}{\partial y}$ and $\tilde{v} = -ik\phi$. Equating eqs. (2.39) and (2.40) and moreover considering that $\eta = -\frac{i\tilde{v}}{k(U(y) - c)}$, we obtain

$$(U(y) - c)^2(\phi'' + k^2\phi) = -i\tilde{v}\Sigma k\delta[y - y_\pi]. \quad (2.41)$$

The first term of eq. (2.41) corresponds to the decomposition in normal modes of $\frac{1}{k^2} \frac{D^2 q'}{Dt^2}$, thus

$$\frac{1}{k^2} \frac{D^2 q'}{Dt^2} = -i\tilde{v}\Sigma k\delta[y - y_\pi]. \quad (2.42)$$

The Fourier spectrum solution of (2.42) can be assumed as

$$\hat{q}' = q_3(k, t)\delta[y - y_\pi]. \quad (2.43)$$

By means of the Green function (2.26) the perturbation streamfunction for the CW model is obtained

- for $y \geq y_\pi$

$$\hat{\psi}'(y) = -q_3(k, t) \frac{\cosh[k(1 + \delta_w(\pi - \frac{1}{2}))]}{k \cosh[k(1 + h)]} \sinh[k(1 + h - y)]$$

- for $y < y_\pi$

$$\hat{\psi}'(y) = -q_3(k, t) \frac{\sinh[k(h - \delta_w(\pi - \frac{1}{2}))]}{k \cosh[k(1 + h)]} \cosh(ky). \quad (2.44)$$

Substituting eqs. (2.43) and (2.44) into eq. (2.42) calculated at the interfaces, one can obtain remembering that $v' = -\frac{\partial \psi'}{\partial x}$:

$$\begin{aligned} (\ddot{q}_3 + 2ikU_\pi \dot{q}_3 - k^2 U_\pi^2 q_3) = \\ = -q_3 \Sigma k^3 \frac{\cosh[k(1 + \delta_w(\pi - \frac{1}{2}))] \cdot \sinh[k(h - \delta_w(\pi - \frac{1}{2}))]}{\cosh[k(1 + h)]} \delta[y - y_\pi], \end{aligned} \quad (2.45)$$

where U_π is the dimensionless velocity at the interfaces $U_\pi = (1 - \pi)(1 + \Lambda) + \pi(1 - \Lambda)$. Setting $\dot{q}_3 = q_4$, we can write in the matricial form:

$$\dot{q} = Mq, \quad q = \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 1 \\ k^2 U_\pi^2 - \Sigma k^3 \gamma_3 & -2ikU_\pi \end{bmatrix}. \quad (2.46)$$

where

$$\gamma_3 = \frac{\cosh[k(1 + \delta_w(\pi - \frac{1}{2}))] \cdot \sinh[k(h - \delta_w(\pi - \frac{1}{2}))]}{\cosh[k(1 + h)]}.$$

In order to obtain the CWs $\tilde{q}_3$ and $\tilde{q}_4$, one shall diagonalize the matrix M . Calling V the matrix of the eigenvectors, the CW vector is given by $\tilde{q} = V^{-1}q$, and, thus

$$\dot{\tilde{q}} = V^{-1}MV\tilde{q} = \tilde{M}\tilde{q}, \quad \tilde{M} = \begin{bmatrix} i(\sqrt{\Sigma k^3 \gamma_3} - kU_\pi) & 0 \\ 0 & -i(\sqrt{\Sigma k^3 \gamma_3} + kU_\pi) \end{bmatrix} \quad (2.47)$$

As well as the pure CRW model, the waves $\tilde{q}_3$ and $\tilde{q}_4$ can be written in the exponential form:

$$\tilde{q}_3(k, t) = Q_3(k, t)e^{i\epsilon_3}, \quad \tilde{q}_4(k, t) = Q_4(k, t)e^{i\epsilon_4}. \quad (2.48)$$

Replacing equation (2.48) in the eq. (2.47)

$$\dot{Q}_3 = \dot{Q}_4 = 0, \quad \epsilon_3 = \sqrt{\Sigma k^3 \gamma_3} - kU_\pi \quad \text{and} \quad \epsilon_4 = -\sqrt{\Sigma k^3 \gamma_3} - kU_\pi. \quad (2.49)$$

The amplitudes Q_3 and Q_4 are constant and the growth rates $\dot{Q}_3$ and $\dot{Q}_4$ are zero. This fact means that the flow is stable whatever parameters. Both CW phase speeds are composed by the sum of two terms: one is identical in two CWs, kU_π , while one is opposed, $\sqrt{\Sigma k^3 \gamma_3}$. These terms represent the components due to the transport and to the action of the surface tension, respectively.

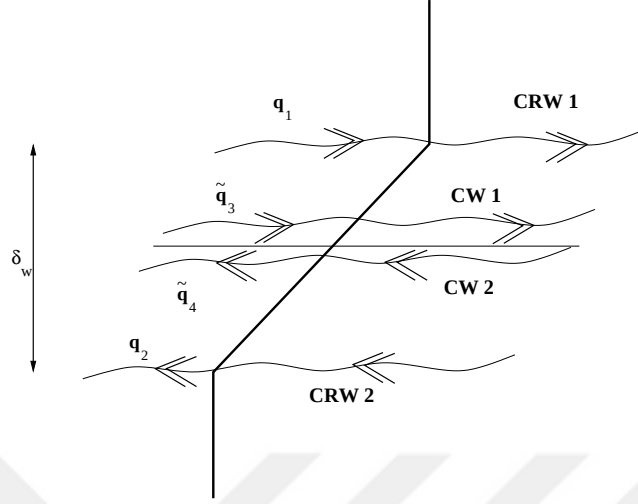


Figure 2.20: Sketch of the waves interacting in the piecewise broken line model including surface tension.

2.4.3 Interaction between CRW and CW

Now, we are ready to study the piecewise broken line model described in section 2.2.1.1 including the surface tension where both types of waves, CRWs and CWS, are present. In figure 2.20 we illustrate a sketch of the waves interacting in such model.

The vorticity perturbation is supposed to be the superposition of equations (2.23) and (2.42), which represent those of the CRWs and CWS models, respectively,

$$\hat{q}' = \left(q_1(k, t) \delta \left[y - 1 + \frac{\delta_w}{2} \right] + q_2(k, t) \delta \left[y - 1 - \frac{\delta_w}{2} \right] + q_3(k, t) \delta [y - y_\pi] \right). \quad (2.50)$$

Again by means of the Green function (2.26), the perturbation streamfunction is obtained:

- for $y \geq 1 + \frac{\delta_w}{2}$

$$\begin{aligned} \hat{\psi}'(y) = & -q_1(k, t) \frac{\cosh[k(1 - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & -q_2(k, t) \frac{\cosh[k(1 + \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & -q_3(k, t) \frac{\cosh[k(1 + \delta_w(\pi - \frac{1}{2}))]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] \end{aligned}$$

- for $1 + \delta_w(\pi - \frac{1}{2}) \leq y \leq 1 + \frac{\delta_w}{2}$

$$\begin{aligned} \hat{\psi}'(y) = & -q_1(k, t) \frac{\cosh[k(1 - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & -q_2(k, t) \frac{\sinh[k(h - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky) \\ & -q_3(k, t) \frac{\cosh[k(1 + \delta_w(\pi - \frac{1}{2}))]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] \end{aligned}$$

- for $1 - \frac{\delta_w}{2} \leq y \leq 1 + \delta_w(\pi - \frac{1}{2})$

$$\begin{aligned}\hat{\psi}'(y) = & -q_1(k, t) \frac{\cosh[k(1 - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \sinh[k(1+h-y)] + \\ & -q_2(k, t) \frac{\sinh[k(h - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky) + \\ & -q_3(k, t) \frac{\sinh[k(h - \delta_w(\pi - \frac{1}{2}))]}{k \cosh[k(1+h)]} \cosh(ky)\end{aligned}\quad (2.51)$$

- for $y \leq 1 - \frac{\delta_w}{2}$

$$\begin{aligned}\hat{\psi}'(y) = & -q_1(k, t) \frac{\sinh[k(h + \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky) + \\ & -q_2(k, t) \frac{\sinh[k(h - \frac{\delta_w}{2})]}{k \cosh[k(1+h)]} \cosh(ky) + \\ & -q_3(k, t) \frac{\sinh[k(h - \delta_w(\pi - \frac{1}{2}))]}{k \cosh[k(1+h)]} \cosh(ky).\end{aligned}$$

Substituting eq. (2.51) into eq. (2.50) and defining, as well as the CW model (section 2.4.2), $\dot{q}_3 = q_4$:

$$\dot{q} = Mq, \quad q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}, \quad \text{and} \quad (2.52)$$

$$M = \begin{bmatrix} -im\alpha_1 - ik(1 + \Lambda) & -im\alpha_2 & -im\alpha_3 & 0 \\ im\beta_1 & im\beta_2 - ik(1 - \Lambda) & im\beta_3 & 0 \\ 0 & 0 & 0 & 1 \\ -\Sigma k^3 \gamma_1 & -\Sigma k^3 \gamma_2 & k^2 U_\pi^2 - \Sigma k^3 \gamma_3 & -2ikU_\pi \end{bmatrix},$$

$$\begin{aligned}\text{where } \alpha_1 = & \frac{\cosh[k(1 - \frac{\delta_w}{2})] \sinh[k(h + \frac{\delta_w}{2})]}{\cosh[k(1+h)]}, \alpha_2 = \beta_1 = \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 - \frac{\delta_w}{2})]}{\cosh[k(1+h)]}, \alpha_3 = \gamma_1 = \\ & \frac{\cosh[k(1 - \frac{\delta_w}{2})] \sinh[k(h - \delta_w(\pi - \frac{1}{2}))]}{\cosh[k(1+h)]}, \beta_2 = \frac{\cosh[k(1 + \frac{\delta_w}{2})] \sinh[k(h - \frac{\delta_w}{2})]}{\cosh[k(1+h)]}, \beta_3 = \gamma_2 = \frac{\sinh[k(h - \frac{\delta_w}{2})] \cosh[k(1 + \delta_w(\pi - \frac{1}{2}))]}{\cosh[k(1+h)]}, \\ \gamma_3 = & \frac{\cosh[k(1 + \delta_w(\pi - \frac{1}{2}))] \sinh[k(h - \delta_w(\pi - \frac{1}{2}))]}{\cosh[k(1+h)]}.\end{aligned}$$

Focusing on the last two equations, they can be rewritten

$$\dot{q}_{34} = M_{34}q_{34} + G. \quad (2.53)$$

Note that both vector q_{34} and the matrix M_{34} are equal to vector q and matrix M , respectively, of CW model, see eq. (2.46), and $G = \begin{bmatrix} 0 \\ -\Sigma k^3 \alpha_3 q_1 - \Sigma k^3 \beta_3 q_2 \end{bmatrix}$. Transforming by means of the same matrix of the eigenvectors V of the CW model, it is obtained:

$$\dot{\tilde{q}}_{34} = \widetilde{M}_{34}\tilde{q}_{34} + \widetilde{G}, \quad (2.54)$$

where $\tilde{q}_{34}$ and $\tilde{M}_{34}$ are equal to $\tilde{q}$ and $\tilde{M}$, respectively, of eq. (2.47) and $\tilde{G}$ is

$$\tilde{G} = P^{-1}G = \begin{bmatrix} i\sqrt{\frac{\Sigma k^3}{4\gamma_3}}(\alpha_3 q_1 + \beta_3 q_2) \\ -i\sqrt{\frac{\Sigma k^3}{4\gamma_3}}4\gamma_3(\alpha_3 q_1 + \beta_3 q_2) \end{bmatrix}. \quad (2.55)$$

Now, the system (2.52) can be rearranged implementing $\tilde{q}_3$ and $\tilde{q}_4$.

$$\begin{aligned} \tilde{\dot{q}} &= \tilde{M}\tilde{q}, \quad \tilde{q} = \begin{bmatrix} q_1 \\ q_2 \\ \tilde{q}_3 \\ \tilde{q}_4 \end{bmatrix}, \quad \text{and} \\ \tilde{M} &= \begin{bmatrix} -im\alpha_1 - ik(1 + \Lambda) & -im\alpha_2 & -im\alpha_3 & -im\alpha_3 \\ im\alpha_2 & im\beta_2 - ik(1 - \Lambda) & im\beta_3 & im\beta_3 \\ i\sqrt{\frac{\Sigma k^3}{4\gamma_3}}\alpha_3 & i\sqrt{\frac{\Sigma k^3}{4\gamma_3}}\beta_3 & -ikU_\pi + i\sqrt{\Sigma k^3\gamma_3} & 0 \\ -i\sqrt{\frac{\Sigma k^3}{4\gamma_3}}\alpha_3 & -i\sqrt{\frac{\Sigma k^3}{4\gamma_3}}\beta_3 & 0 & -ikU_\pi - i\sqrt{\Sigma k^3\gamma_3} \end{bmatrix}. \end{aligned} \quad (2.56)$$

The real part of eigenvalues of the matrix $\tilde{M}$ represents the growth rates of different modes of the general model, while the imaginary part is its phase speed. Decomposing the waves

$$\begin{aligned} q_1 &= Q_1(k, t)e^{i\epsilon_1(k, t)}, \quad q_2 = Q_2(k, t)e^{i\epsilon_2(k, t)}, \quad \tilde{q}_3(k, t) = Q_3(k, t)e^{i\epsilon_3}, \quad \text{and} \\ \tilde{q}_4(k, t) &= Q_4(k, t)e^{i\epsilon_4}, \end{aligned} \quad (2.57)$$

we obtain the system

$$\dot{Q}_1 = m\alpha_2 Q_2 \sin(\epsilon_2 - \epsilon_1) + m\alpha_3 Q_3 \sin(\epsilon_3 - \epsilon_1) + m\alpha_3 Q_4 \sin(\epsilon_4 - \epsilon_1), \quad (2.58)$$

$$\dot{Q}_2 = m\alpha_1 Q_1 \sin(\epsilon_2 - \epsilon_1) - m\beta_3 Q_3 \sin(\epsilon_3 - \epsilon_2) - m\beta_3 Q_4 \sin(\epsilon_4 - \epsilon_2), \quad (2.59)$$

$$\dot{Q}_3 = \sqrt{\frac{\Sigma k^3}{4\gamma_3}}\alpha_3 Q_1 \sin(\epsilon_3 - \epsilon_1) + \sqrt{\frac{\Sigma k^3}{4\gamma_3}}\beta_3 Q_2 \sin(\epsilon_3 - \epsilon_2), \quad (2.60)$$

$$\dot{Q}_4 = -\sqrt{\frac{\Sigma k^3}{4\gamma_3}}\alpha_3 Q_1 \sin(\epsilon_4 - \epsilon_1) - \sqrt{\frac{\Sigma k^3}{4\gamma_3}}\beta_3 Q_2 \sin(\epsilon_4 - \epsilon_2), \quad (2.61)$$

$$\begin{aligned} \dot{\epsilon}_1 &= -k(1 + \Lambda) - m(\alpha_1 + \alpha_2 \frac{Q_2}{Q_1} \cos(\epsilon_2 - \epsilon_1) + \\ &\quad + \alpha_3 \frac{Q_3}{Q_1} \cos(\epsilon_3 - \epsilon_1) + \alpha_3 \frac{Q_4}{Q_1} \cos(\epsilon_4 - \epsilon_1)), \end{aligned} \quad (2.62)$$

$$\begin{aligned} \dot{\epsilon}_2 &= -k(1 - \Lambda) + m(\beta_2 + \alpha_2 \frac{Q_1}{Q_2} \cos(\epsilon_2 - \epsilon_1) + \\ &\quad + \beta_3 \frac{Q_3}{Q_2} \cos(\epsilon_3 - \epsilon_2) + \beta_3 \frac{Q_4}{Q_2} \cos(\epsilon_4 - \epsilon_2)), \end{aligned} \quad (2.63)$$

$$\dot{\epsilon}_3 = -kU_\pi + \sqrt{\Sigma k^3\gamma_3} + \sqrt{\frac{\Sigma k^3}{4\gamma_3}}\frac{Q_1}{Q_3}\alpha_3 \cos(\epsilon_3 - \epsilon_1) + \sqrt{\frac{\Sigma k^3}{4\gamma_3}}\frac{Q_2}{Q_3}\alpha_3 \cos(\epsilon_3 - \epsilon_2), \quad (2.64)$$

$$\dot{\epsilon}_4 = -kU_\pi - \sqrt{\Sigma k^3\gamma_3} - \sqrt{\frac{\Sigma k^3}{4\gamma_3}}\frac{Q_1}{Q_4}\beta_3 \cos(\epsilon_4 - \epsilon_1) - \sqrt{\frac{\Sigma k^3}{4\gamma_3}}\frac{Q_2}{Q_4}\beta_3 \cos(\epsilon_4 - \epsilon_2). \quad (2.65)$$

2.4.4 Vanishing shear layer thickness limit

In the previous sections 2.4.1 and 2.4.3, the CRW perspective for the model including a shear layer thickness has been elucidated, first excluding and second including the surface tension. We have seen how by means of the CRW perspective it is possible to obtain the same eigenvalues of the dispersion relations calculated by means of normal modes of the temporal stability (section 2.3). In order to also revisit the models where the shear layer thickness is not present, a vanishing shear layer thickness has to be considered, this means to calculate the limit $\delta_w \rightarrow 0$ of the systems (2.29) and (2.56), which corresponds to the case excluding and including, respectively, the surface tension.

2.4.4.1 Top hat jet/wake in absence of surface tension

For an unconfined mixing layer, a similar limiting model was analyzed by Heifetz *et al.* [47]. We use a different method to obtain the system which represents the top hat jet/wake without surface tension at the light of a recent book of Haragus & Iooss (2011) [39], see their chapter 3. First of all, we develop by means of a Taylor expansion for $\delta_w \rightarrow 0$ the matrix M of the eq. (2.29)

$$m_{11} = i \left(\frac{2\Lambda A}{\delta_w} + \Lambda k(B - C) - k(1 + \Lambda) - \Lambda \frac{D - A}{2} k^2 \delta_w \right), \quad (2.66a)$$

$$m_{12} = i \left(\frac{2\Lambda A}{\delta_w} - \Lambda k + \Lambda \frac{D + A}{2} k^2 \delta_w \right), \quad (2.66b)$$

$$m_{21} = i \left(-\frac{2\Lambda A}{\delta_w} + \Lambda k - \Lambda \frac{D + A}{2} k^2 \delta_w \right), \quad \text{and} \quad (2.66c)$$

$$m_{22} = i \left(-\frac{2\Lambda A}{\delta_w} + \Lambda k(B - C) - k(1 - \Lambda) + \Lambda \frac{D - A}{2} k^2 \delta_w \right), \quad (2.66d)$$

where $A = \frac{\cosh(k) \sinh(kh)}{\cosh[k(1+h)]}$, $B = \frac{\cosh(k) \cosh(kh)}{\cosh[k(1+h)]}$, $C = \frac{\sinh(k) \sinh(kh)}{\cosh[k(1+h)]}$, and $D = \frac{\sinh(k) \cosh(kh)}{\cosh[k(1+h)]}$.

Now, the matrix M can be written in the form

$$M = \frac{1}{\delta_w} L = \frac{1}{\delta_w} (L_0 + \delta_w L_1 + \delta_w^2 L_2), \quad (2.67)$$

with

$$L_0 = 2i\Lambda A \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad (2.68)$$

which is a non-diagonalizable matrix. The normal form of L_0 is the Jordan block

$$J_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (2.69)$$

However, we have to find the normal form of L , which is composed by $J = J_0 + \delta_w J_1 + \delta_w^2 J_2$. J_1 and J_2 can be found by means of the relationship $J_0^* J_{1,2} = J_{1,2} J_0^*$ [39], which gives

$$J_{1,2} = \begin{bmatrix} \alpha_{1,2} & 0 \\ \beta_{1,2} & \alpha_{1,2} \end{bmatrix}. \quad (2.70)$$

The calculations to obtain $\alpha_{1,2}$ and $\beta_{1,2}$ are detailed in the Appendix.

The matrix J finally can be written:

$$J = i \begin{bmatrix} \Lambda(B - C) - k & \frac{2\Lambda A}{\delta_w} \\ -2k^2 \delta_w D & \Lambda(B - C) - k \end{bmatrix}, \quad (2.71)$$

and its eigenvalues, which are independent on δ_w , are

$$\lambda_{1,2} = ik\Lambda(B - C) - ik \pm 2k\Lambda\sqrt{AD}, \quad (2.72)$$

which correspond to the explicit dispersion relation $i\omega(k)$ (2.10), where $i\omega(k) = f(k) \pm \sqrt{g(k)}$ found in section 2.3.1 when setting the surface tension $\Sigma = 0$.

2.4.4.2 Top hat jet/wake in presence of surface tension

If surface tension is added to the piecewise broken line profile, the two-equations system (2.29) is transformed in a four-equations system (2.56) as shown in section (2.4.3). In order to pass from the general model to the top hat profile including the surface tension, we have to compute the limit of the four-equations system by means of the same procedure illustrated in the previous section 2.4.4.1 [39].

Also in this case we can develop with a Taylor expansion the elements of the matrix $\tilde{M}$ (2.56). Note that the elements m_{11} , m_{12} , m_{21} , and m_{22} are equal to those of eqs. (2.66), while the others are

$$m_{13} = m_{14} = i \left(\frac{2\Lambda A}{\delta_w} - \Lambda k C + \frac{\Lambda k^2 A}{4} \delta_w \right), \quad (2.73a)$$

$$m_{23} = m_{24} = i \left(-\frac{2\Lambda A}{\delta_w} + \Lambda k B - \frac{\Lambda k^2 A}{4} \delta_w \right), \quad (2.73b)$$

$$m_{31} = i \left(\frac{\sqrt{\Sigma k^3 A}}{2} - \frac{k \tanh(k) \sqrt{\Sigma k^3 A}}{4} \delta_w \right), \quad (2.73c)$$

$$m_{32} = i \left(\frac{\sqrt{\Sigma k^3 A}}{2} - \frac{k \coth(kh) \sqrt{\Sigma k^3 A}}{4} \delta_w \right), \quad (2.73d)$$

$$m_{33} = i(-k + \sqrt{\Sigma k^3 A}), \quad m_{44} = i(-k - \sqrt{\Sigma k^3 A}), \quad (2.73e)$$

$$m_{41} = i \left(-\frac{\sqrt{\Sigma k^3 A}}{2} + \frac{k \tanh(k) \sqrt{\Sigma k^3 A}}{4} \delta_w \right), \quad \text{and} \quad (2.73f)$$

$$m_{42} = i \left(-\frac{\sqrt{\Sigma k^3 A}}{2} + \frac{k \coth(kh) \sqrt{\Sigma k^3 A}}{4} \delta_w \right). \quad (2.73g)$$

Hence, we can reinterpret the terms in δ_w as a perturbation of a matrix L_0 :

$$\tilde{M} = \frac{1}{\delta_w} L = \frac{1}{\delta_w} (L_0 + \delta_w L_1 + \delta_w^2 L_2). \quad (2.74)$$

Therefore:

$$L_0 = 2i\Lambda A \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (2.75)$$

$$L_1 = \frac{i}{2\Lambda A} \begin{bmatrix} \Lambda k(B - C) - k(1 + \Lambda) & -\Lambda k & -\Lambda k C & -\Lambda k C \\ \frac{\Lambda k}{\sqrt{\Sigma k^3 A}} & \Lambda k(B - C) - k(1 - \Lambda) & \frac{\Lambda k B}{\sqrt{\Sigma k^3 A}} & \frac{\Lambda k B}{\sqrt{\Sigma k^3 A}} \\ -\frac{2}{\sqrt{\Sigma k^3 A}} & -\frac{2}{\sqrt{\Sigma k^3 A}} & -k + \sqrt{\Sigma k^3 A} & 0 \\ 0 & 0 & 0 & k - \sqrt{\Sigma k^3 A} \end{bmatrix}, \quad (2.76)$$

$$\text{and } L_2 = \frac{i}{2\Lambda A} \begin{bmatrix} -\Lambda \frac{D-A}{2} k^2 & \Lambda \frac{D+A}{2} k^2 & \frac{\Lambda k^2 A}{4} & \frac{\Lambda k^2 A}{4} \\ -\Lambda \frac{D+A}{2} k^2 & \Lambda \frac{D-A}{2} k^2 & -\frac{\Lambda k^2 A}{4} & -\frac{\Lambda k^2 A}{4} \\ -\frac{k \tanh(k) \sqrt{\Sigma k^3 A}}{4} & -\frac{k \coth(kh) \sqrt{\Sigma k^3 A}}{4} & 0 & 0 \\ \frac{k \tanh(k) \sqrt{\Sigma k^3 A}}{4} & \frac{k \coth(kh) \sqrt{\Sigma k^3 A}}{4} & 0 & 0 \end{bmatrix}. \quad (2.77)$$

The normal form of L is $J = J_0 + \delta_w J_1 + \delta_w^2 J_2$, where J_i are found such that $J_i J_0 = J_0 J_i$ and correspond to

$$J_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad (2.78)$$

$$J_{1,2} = \begin{bmatrix} \alpha_{1,2} & 0 & 0 & 0 \\ \beta_{1,2} & \alpha_{1,2} & W_{1,2} & Z_{1,2} \\ \varepsilon_{1,2} & 0 & \gamma_{1,2} & D_{1,2} \\ \eta_{1,2} & 0 & C_{1,2} & \varphi_{1,2} \end{bmatrix}. \quad (2.79)$$

All the computations are illustrated in the Appendix. Here, we report directly the eigenvalues given by the linear form $J_M = \frac{1}{\delta_w} J_0 + J_1 + \delta_w J_2$ of $\tilde{M}$ when $\delta_w \rightarrow 0$

$$\lambda_{1,2} = i\Lambda k(B - C) - ik \pm 2k\Lambda \sqrt{AD - \frac{\Sigma k A}{\Lambda^2}} \quad \text{and} \quad \lambda_{3,4} = ik. \quad (2.80)$$

The first two eigenvalues $\lambda_{1,2}$ are equal to $i\omega(k)$, where $\omega(k) = f(k) \pm \sqrt{g(k)}$ corresponds to explicit dispersion relation eq. (2.10) of the top hat profile with surface tension, see section 2.3.1.

2.5 Spatio-temporal analysis of a confined wake including shear layer thickness

This section is devoted to spatio-temporal stability analysis of velocity profiles including shear layer thickness, both continuous and broken-line profile. The spatio-temporal stability analysis for the top hat profile excluding or including the surface tension have been already conducted by Juniper (2006) [54] and Rees & Juniper (2009) [90], respectively. Furthermore, spatio-temporal analyses of profiles including shear layer thickness have been conducted by Juniper (2007) [55] on a vanishing shear layer thickness flow and Rees & Juniper (2010) [91] on viscous jet/wakes. In a round geometry the influence of the shear layer thickness in confined jet/wakes was examined still by Juniper (2008) [56], noticing that there exists a significant similarity with the planar geometry.

The outline of the section is as follows. In sections 2.5.1 and 2.5.2 a brief description on the procedures used in our spatio-temporal stability is illustrated. While the absolute/convective transition curves as well as absolute frequency and complex wavenumber of the mode prevailing at the C/A transition are discussed in section 2.5.3, in section 2.5.4, the continuous profiles are approximated by means of piecewise broken-line profile and are shown to share similar spatiotemporal properties. In addition, a comparison between the frequencies obtained in section 2.5.3 and those experimentally measured in the case of confined cylinder wakes is included in section 2.5.5. Finally an application of the counter-propagating Rossby waves (CRWs) perspective to the spatio-temporal stability analysis is illustrated in section 2.5.6.

2.5.1 Saddle points in the complex k -plane

The analysis of spatio-temporal stability looks into absolute or convective nature of instability. The Briggs-Bers criterion [14, 6, 49] requires the determination of temporal growth rate of the dominant mode at the location of an impulsive source. From the analysis of the large-time asymptotic impulse response [51] the complex group velocity $d\omega/dk$ of the dominant mode is equal to zero. In the complex k -plane, it corresponds to a saddle point k_0 of the dispersion relations $\omega(k)$ obtained in section 2.2, named absolute wavenumber. The value of ω at the saddle point is called absolute frequency ω_0 . If the imaginary part of the absolute frequency, i.e the absolute growth rate $\omega_{0,i}$, is positive, the flow is absolutely unstable, provided the saddle point results from coalescence of a downstream and upstream mode [51]. In other words, the two spatial branches $k(\omega)$ which pinch to form the saddle point, should be indeed emanate from a k^+ branch and a k^- branch, respectively. With k^+ (k^-), one indicates the spatial branch coming out from a partition of a generalized spatial mode, which stands in the complex ω -plane above the maximal temporal growth rate $\omega_{i,max}$, located in the complex k -plane above (below) the k_r -axis. From here on, the saddle points counting for the determination of the nature of the instability, are named simply k^+/k^- , to

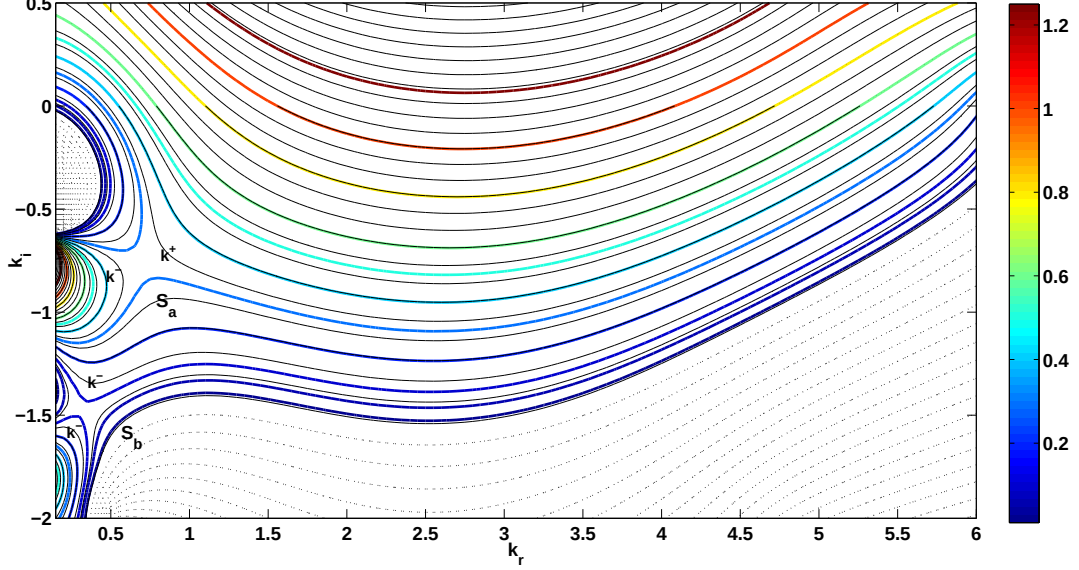


Figure 2.21: Contours of ω_i in the k -complex plane for $N = 5$, $h = 1.5$ and $\Lambda = -1$ for continuous profile model. Black continuous contours depict positive values of ω_i , while dashed black contours depict negative ones. There are two positive saddle points (S_a and S_b), which depict two unstable modes, however only the first one is a k^+/k^- saddle point. The colored contour lines help to understand the values of ω_i and to estimate the nature of saddle points.

differentiate with k^-/k^- , which are the coalescence of two k^- spatial branches, and do not contribute in the impulse response.

An example of a k^+/k^- (S_a) and of a k^-/k^- (S_b) is depicted in figure 2.21. Here, in the k -complex plane contours of ω_i are traced: positive values with continuous lines, negative ones with dotted lines. Observe the colored contours, which depict some significant values of ω_i . The darker contour, which represents a value of $\omega_i > \omega_{i,max}$, is split into two parts: the first one located in the upper complex k -plane, the second one in the lower plane. They depict an example of a k^+ and k^- spatial branch, respectively. Deforming these spatial branches, thus, decreasing ω_i , the k^+ (k^-) branch enters (remains) in the lower complex k -plane, as noticed by following the colored contours, until to join the saddle point S_a , which is, thus, a k^+/k^- . Conversely, S_b is a k^-/k^- , since no k^+ branch joins it. Even if the imaginary part of ω_0 of S_b would happen to be higher than that of S_a , it would not count to estimate the instability nature of the flow.

The first complete map is simple to construct for broken-line model having found explicitly the solution. Conversely for continuous line model, eq. (2.13) gives N_i eigenvalues, where N_i is the number of Chebyshev collocation points y_i . The physical

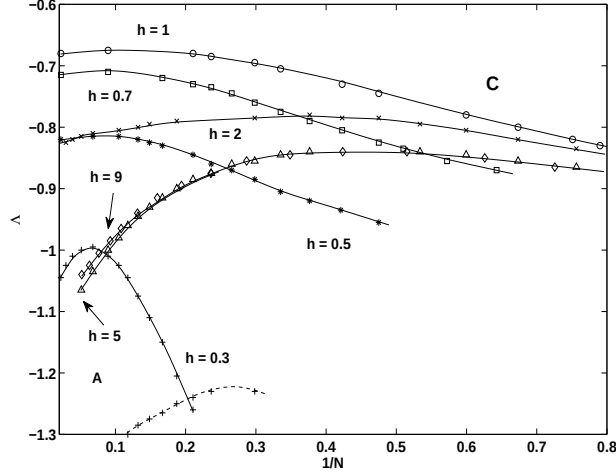


Figure 2.22: Convective-absolute limits in the plane (N^{-1}, Λ) for several confinements for the continuous profile model. $-\diamond-$ depicts $h = 9$, $-\triangle-$ $h = 5$, $-\times-$ $h = 2$, $-\circ-$ $h = 1$, $-\square-$ $h = 0.7$, $-*-$ $h = 0.5$ and $-+-$ $h = 0.3$. The latter case is distinguished by two different lines, a continuous line and a dashed one, which represent two different saddle points. The regions of absolutely and convectively instability are marked by A and C. Every limit line presents a maximum of absolutely instability for an optimal shape parameter $N_{max}(h)$.

eigenvalues, and particularly the eigenvalue which depicts the unstable mode shall to be discriminated from others. In the temporal stability, this particular eigenvalue ω_t and his corresponding eigenvector ϕ_t can be simply determined, as made in section 2.3. Hence, for each k_r on the real axis eq. (2.13) becomes $\mathbf{M}_r \phi = \omega \phi$, where $\mathbf{M}_r = \mathbf{B}_r^{-1} \mathbf{A}_r$. Adding a small imaginary part ($\delta_w k_i$) to real wavenumber is equivalent to perturb the matrix $\mathbf{M}_r$. As noted by Chomaz [18], one may proceed to an expansion procedure: $\phi' \sim \phi_t + \epsilon \delta \phi$, $\omega' \sim \omega_t + \epsilon \delta \omega$. Expanding the relation $(\mathbf{M}_r + \epsilon \Delta \mathbf{M}) \phi' = \omega' \phi'$ at order ϵ , one is left with

$$\delta \omega = \frac{\langle \psi_t | \Delta \mathbf{M} \phi_t \rangle}{\langle \psi_t | \phi_t \rangle}, \quad (2.81)$$

where ψ_t is the adjoint eigenvector of ω_t , defined as $\mathbf{M}_t^* \phi_t = \omega_t^* \psi_t$. Among the eigenvalues obtained by eq. (2.13) for the perturbed wavenumber $k' = k_r + i \delta_w k_i$, the eigenvalue closest to ω' is chosen. Adding again δk_i to k' and repeating the procedure, a map of the complex k -plane can be traced (see figure 2.21 for an example).

2.5.2 Following a saddle point: Deissler's method

To determine the location of a saddle point k_0 and his absolute frequency ω_0 , it is not necessary to trace complete maps of $\omega(k)$ for all the parameter configurations.

An iterative numerical procedure can be used to track the absolute saddle point, following that of Deissler [26]: saddle points are computed by fitting, with Matlab function *fsolve*, a generic quadratic expression of the form

$$\omega(k) = \omega_0 + r(k - k_0)^2, \quad (2.82)$$

on absolute growth rate $\omega(k_j)$ obtained for three different wavenumber k_j chosen on a circle of diameter d around an initial guess value of absolute wavenumber $k_{0,n}$. The three unknowns (ω_0, k_0, r) are computed and new wavenumbers are chosen in the same way around just calculated $k_{0,n+1}$. The procedure is repeated until both k_0 and ω_0 become stationary within the desired tolerance, which in the present thesis has been chosen of four significant digits. The initial value $k_{0,0}$ is obtained by constructing a complete map of k for one single set of parameter and locating the dominant saddle point, its absolute frequency $\omega_{0,0}$ and its absolute wavenumber $k_{0,0}$. Slightly perturbing a single parameter (h , N/δ_w or Λ) the saddle point location $k_{0,n+1}$ would be close to $k_{0,n}$ and, thus, it needs only to choose a large enough diameter in order that the circle around $k_{0,n}$ contains $k_{0,n+1}$.

2.5.3 Convective/absolute instability transition for the continuous velocity profile

In accordance with Monkewitz & Nguyen [74], the varicose mode has shown no sign of absolute instability for $\Lambda < -1$ and $h = \infty$ and for different values, it has always been seen to be less absolutely unstable than the sinuous one. Hence, in this section we focus on the sinuous instability. Let us however mention the study of Healey (2007) [42] for an exhaustive analysis and original interpretation of the influence of confinement on the varicose mode of confined wake flows, viewed therein as asymmetrically confined mixing layers.

2.5.3.1 On the plane (N^{-1}, Λ)

Figure 2.22 gathers the convective-absolute limit curves for different confinement parameter h in the plane (N^{-1}, Λ) for the continuous profile model. Each curve delineates absolutely unstable flows at velocity ratios below the curve from convectively unstable flows for velocity ratios above the curve. Note that the limit curve for the unconfined case, which was obtained by Monkewitz [75], is very similar to our $h = 9$ case, depicted by diamonds in figure 2.22.

The combined influence of confinement h and shape parameter N is further analyzed in figure 2.22. For each value of the confinement h , the limit curve shows a maximum of absolute instability $\Lambda_{max}^{C/A}(h)$ for a given of shape parameter $N_{max}(h)$. While $\Lambda_{max}^{C/A}(h)$ is non-monotonic, $N_{max}^{-1}(h)$ increases as the confinement is increased, *i.e* as h decreases. Physically, this means that the shear layer thickness of the wake should be fine enough and commensurate to the confinement h to take full benefit of the destabilization through the confinement.

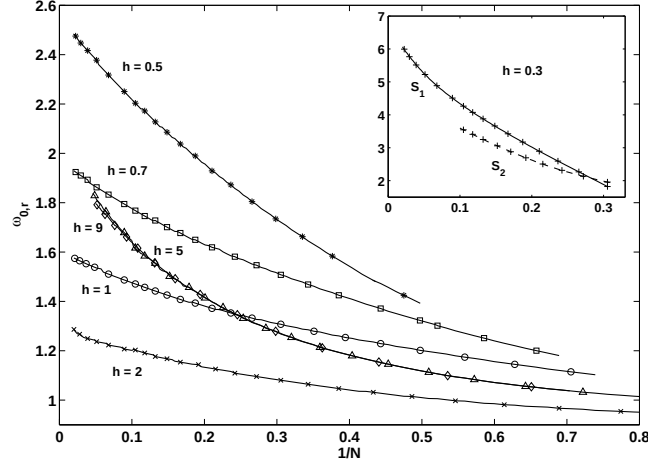


Figure 2.23: Absolute frequencies ($\omega_{0,r}$) of dominating mode at C/A transition in function of N^{-1} : The legend is the same of figure 2.22. The broken-line approximates well the frequencies of continuous profile for any analyzed confinement. On the top the case $h = 0.3$ is illustrated. Dashed and dotted lines depict the unstable modes described respectively by saddle points S_1 and S_2 .

Strikingly, absolute instability is most favored for $h = 1$ for any N , and the parameters $h = 1$ and $N^{-1} \sim 0.1$ correspond to the most absolutely unstable configuration. In the case of very strong confinement $h = 0.3$, a mode-crossing is observed, where the dominant saddle point switches for $N \sim 5$. We have verified that both saddle points, labeled S_1 and S_2 are indeed admissible k^+/k^- pinching points.

Let us examine the detailed characteristics of the dominating saddle point along the boundaries between convective and absolute instability regions. Both confinement and shear layer thickness influence the absolute frequency $\omega_{0,r}$ and the saddle-point position in the complex k -plane, i.e. the absolute wavenumber k_0 .

Figure 2.23 illustrates absolute frequencies at C/A transition as a function of the inverse of the shape parameter N^{-1} for different confinements. It should be noted that an increase in the shear layer thickness, which is proportional to N^{-1} , always yields a decrease in absolute frequencies. Starting from the unconfined case, an increase in confinement first slowly reduces the absolute frequency until it starts to increase when the confinement is further increased beyond $h < 2$. This non monotonic behavior will be more clearly analyzed in the $(h, \omega_{0,r})$ -plane (figure 2.26).

Figure 2.24 gathers the real (a) and imaginary (b) part of absolute wavenumber k_0 of the dominating mode at the C/A boundary. Monotonic trends are observed as the shear layer thickness N^{-1} is varied, but a non monotonic behavior is observed as h is varied, which deserves an analysis in the $(h, k_{0,r/i})$ planes (figure 2.27).

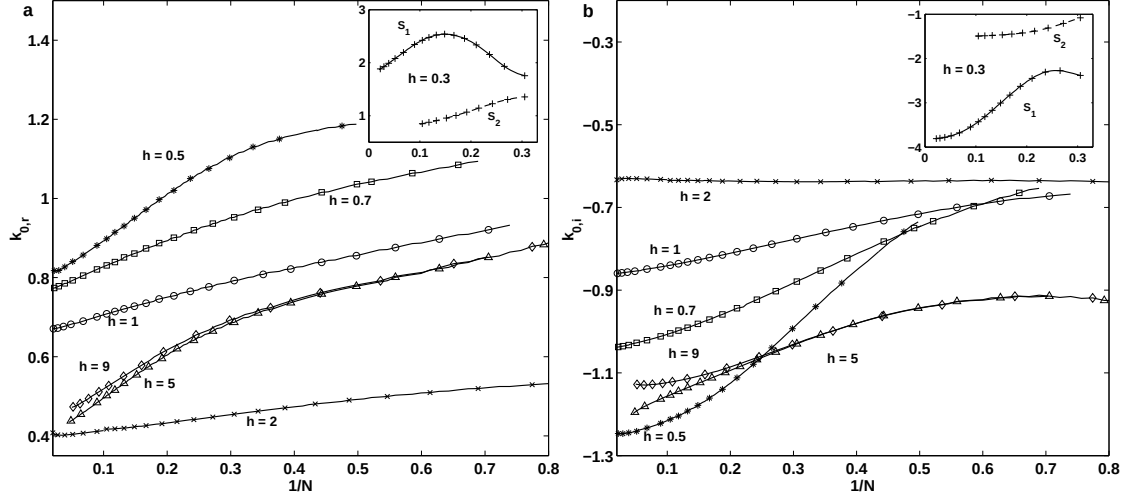


Figure 2.24: Absolute wavenumber k_0 of dominating mode at C/A transition in function of N^{-1} : (a) real part, (b) imaginary part. The legend is the same of figure 2.22. On top of both figures the case $h = 0.3$ is illustrated. Dashed and dotted lines depict the unstable modes described respectively by saddle points S_1 and S_2 .

2.5.3.2 On the plane (Λ^{-1}, h)

We have shown in the previous section 2.5.3.1 that at a given confinement, no mode crossing happened, with the exception of the most confined case, $h = 0.3$. However, prescribing rather the shear layer thickness, one may wonder if there are mode crossings at the C/A boundary as the confinement is continuously varied. To answer this question, the C/A transitions are constructed in the plane (Λ^{-1}, h) for different values of N and illustrated in figure 2.25. The chosen values are $N = \infty, 10, 5, 3$ and 2 . The circles depict the case for $N = \infty$, which corresponds to profile studied by Juniper [54]. Each limit shows a peak at moderate confinement, as discovered by Juniper [54]. Starting from $h_{max} = 1$ for $N = \infty$, h_{max} grows slightly as N is increased. Furthermore, this transition curve is composed by five different unstable saddle points, at the origin of four mode crossings. As the shear layer thickness is increased, the modes tend to merge, until for $N = 2$ (depicted by crosses), only one mode crossing remains. This shows that the addition of a finite shear layer thickness is breaking the $(h, 1/h)$ symmetry existing in the vanishing shear layer thickness limit, in agreement with Rees and Juniper [91]. While we also observe that confined flows are stabilized as the shear layer thickness is increased, our profiles do not allow the consideration of the strong confinement - large shear layer thickness limit, as explained in section 2.2.1.2.

In contrast, figure 2.25 shows that a reasonable increase in the shear layer thickness at large h enhances considerably the absolutely character of the instability. This holds

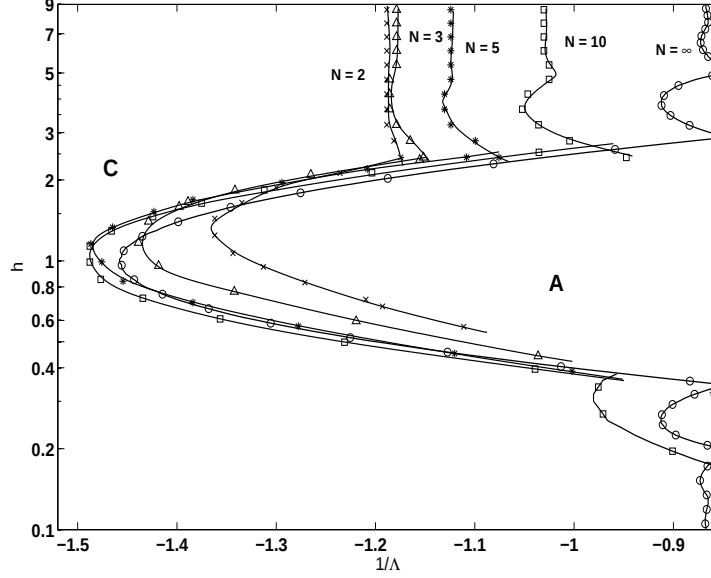


Figure 2.25: C/A transitions in the plane (Λ^{-1}, h) for different shape parameters. $-\circ-$ depicts $N = \infty$, $-\square-$ $N = 10$, $-*-$ $N = 5$, $-\triangle-$ $N = 3$, and $-\times-$ $N = 2$. The regions of absolute and convective instability are marked by A and C.

however only as long as $N < N_{max}(h)$, as already seen in figure 2.22. This influence of shear layer thickness was already noticed by Monkewitz [75] in absence of confinement and by Rees and Juniper [91]. Interestingly, at moderate confinement, there is little influence of shear layer thickness on the C/A boundary.

The absolute frequencies $\omega_{r,0}$ at the C/A transition are gathered in figure 2.26. The mode switchings are here clearly evidenced by the discontinuities in the curves. When the shear layer thickness is small, a strong dependence with respect to confinement is noted, with a non monotonic behavior already noticed in figure 2.23. For example, focusing on $N = 10$, the frequency stagnates around $\omega_{0,r} \sim 1.85$ as the confinement is increased from $h = 9$ to $h = 2.5$, then jumps to $\omega_{0,r} \sim 1.5$ at $h = 2.5$, from where it grows until $\omega_{0,r} \sim 6.5$ as the confinement is further increased from $h = 2.5$ to $h = 0.2$. We speculate that this increase in the absolute frequency when the flow is confined could be related to the increase in Strouhal numbers in confined wake flows [94, 59, 25]. This aspect will be focused in section 2.5.5.

Furthermore figure 2.27 gathers real (a) and imaginary (b) part of absolute wavenumbers at C/A transition with respect to the confinement parameter h . The mode crossings are again clearly evidenced. At large confinement, h does not have a significant effect onto the wavenumbers. Conversely, for lower values of the confinement parameter, the position of the dominating saddle points is strongly affected by confinement. The lowest $k_{0,r}$ values are attained close to the upper mode switching around $h \sim 2.5$,

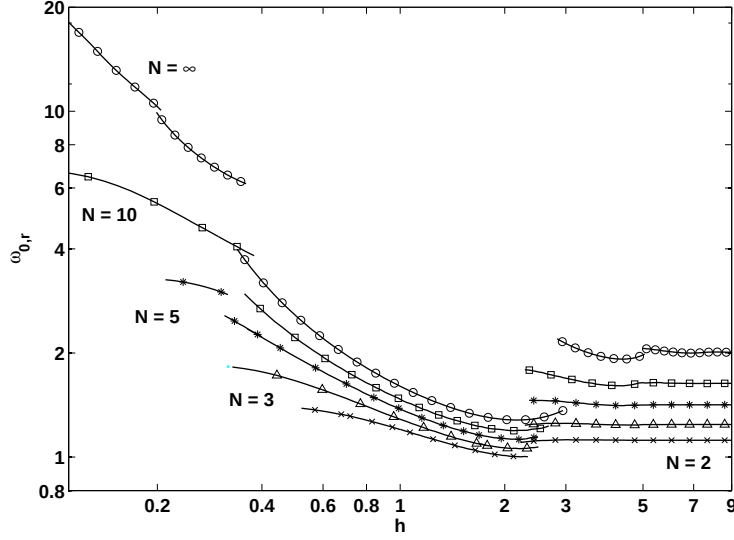


Figure 2.26: Absolute frequencies ($\omega_{0,r}$) of dominating mode at C/A transition in log-scale with respect of h . The legend is the same of figure 2.25.

pointing to large characteristic wavelengths in this parameter range.

2.5.4 Approximating the continuous profile with the piecewise broken-line

In this section, we focus on the comparison between the continuous profile (see equation 2.11) and his approximation constituted by a piecewise broken line (see equation 2.21).

The convective/absolute transition lines shown in figure 2.22 are rescaled to their corresponding δ_w by means of eq. (2.16) in order to ease comparison of the models. From here on, in all comparative figures, dashed lines depict the limit curves of the continuous profiles, while continuous lines depict those of broken-line ones. Figure 2.28 illustrates the comparison between the models of (a) the C/A transitions, (b) the absolute frequencies $\omega_{0,r}$, (c) real part $k_{0,r}$ and (d) imaginary part $k_{0,i}$ of its absolute wavenumber k_0 with respect to shear layer thickness δ_w . Focusing first on figure 2.28a and at small shear layer thicknesses, the C/A transition curves for the broken-line model are seen to fit well those of the continuous profiles. When the shear layer thickness grows, the quality of broken-line approximation however deteriorates and the resulting convective/absolute transition is seen to differ, as noticed for large δ_w in figure 2.28a.

The absolute frequencies of both models are also in good agreement (figure 2.28b), especially at small shear layer thicknesses, where the approximation of the broken-

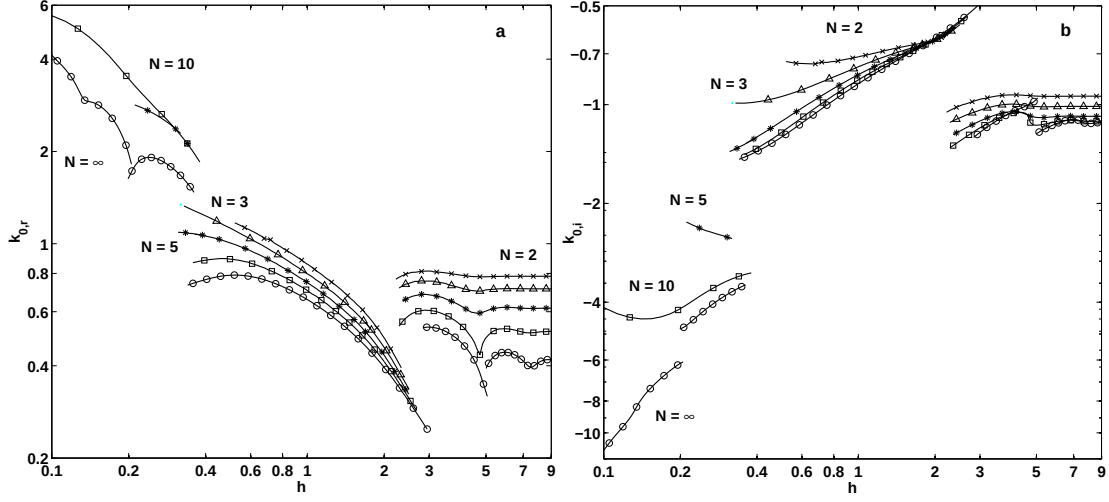


Figure 2.27: Comparison of absolute wavenumber k_0 of dominating mode at C/A transition in log-scale with respect of h : real (a) and imaginary (b) part. The legend is the same of figure 2.25.

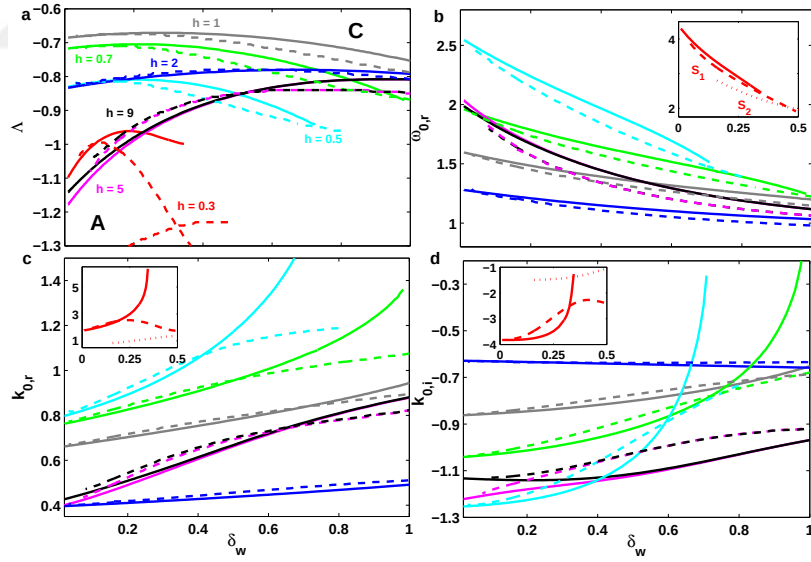


Figure 2.28: Comparison of (a) C/A transitions in the plane (δ_w, Λ) , (b) absolute frequencies $(\omega_{0,r})$ (c) real and (d) imaginary part of absolute wavenumber k_0 with respect to shear layer thickness δ_w : dashed lines depict continuous profile model, while continuous lines depict broken-line one. The black line depicts $h = 9$, magenta $h = 5$, blue $h = 2$, gray $h = 1$, green $h = 0.7$, cyan $h = 0.5$ and red one $h = 0.3$. In (a) the regions of absolutely and convectively instability are marked by A and C.

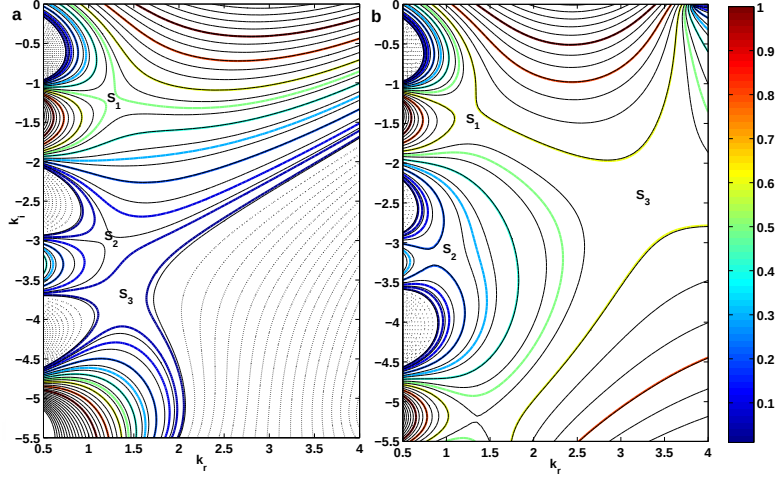


Figure 2.29: Contours of growth rate ω_i , in the complex k -plane, $h = 0.5$, $\Lambda = -1.2$ for (a) the continuous profile model with $N = 5$ and for (b) the broken-line model with corresponding $\delta_w = 0.321$. Black continuous contours depict positive values of ω_i , while dashed black contours depict negative ones. Note the significant difference of location of saddle point S_3 .

line model is particularly accurate irrespective of the confinement. Note moreover that for $h = 0.3$, the absolute frequency of the broken-line model is similar to that of the S_1 saddle involved in the mode crossing of the continuous profile. However, no mode crossing is observed for the broken-line model, where the second saddle point S_2 is not encountered.

Comparing the absolute wavenumbers (figures 2.28c and 2.28d), the accuracy of the approximating broken profile is remarkable at small values of the shear layer thickness, whatever the value of the confinement. However, at large values of the shear layer thickness, the agreement is less satisfactory. At strong confinement ($h = 0.7$, $h = 0.5$, and $h = 0.3$), the locations of the dominating saddle point in the complex wavenumber plane start to behave very differently: as δ_w increases, the saddle point of the broken-line model moves towards the k_r -axis.

This might be better understood by looking at figure 2.29, which compares the dispersion relation map of the two models (on the left (a), continuous profile model, on the right (b), broken-line model) for the same set of parameters: $h = 0.5$, $\Lambda = -1.2$ and $N = 5$ or corresponding shear layer thickness $\delta_w = 0.321$. The location of saddle point S_3 , which represents the so-called shear layer mode [72], is the main difference. For the continuous model, this mode is never the most absolutely unstable and its influence onto the dominating mode is negligible. In contrast, in the broken line model, S_3 can directly influence the dominating mode S_1 , as shown in figure 2.29b, enhancing its instability and eventually pushing it towards the k_r axis. Furthermore,

for the broken-line model, S_3 loses its physical meaning when $\Lambda = -1$ [76], since its absolute wavenumber becomes: $k_{0,r} = +\infty$ and $k_{0,i} = 0$. For $\Lambda > -1$, it stays on the k_r -axis at a finite value of k_r but it loses its contribution to the impulse response, as explained in detail by Ortiz et al. [76] in shear layers.

For completeness, the accuracy of the broken line approximation is evaluated by comparing its C/A transition curves in the plane (Λ^{-1}, h) to the corresponding continuous profile with the same shear layer thickness in figure 2.30a. The agreement is again satisfactory, except at strong confinement. In particular, at moderate values of the shear layer thickness, the values h_{max} and Λ_{max} , for which the absolute instability is maximal, are similar. On the other hand, at large δ_w , growing discrepancies start to appear. At strong confinement, the approximation fails for any shear layer thickness. However, this fact is not surprising since the transition lines at strong confinement are affected by the presence of the shear layer mode, as seen in figure 2.29.

For completeness, the absolute frequencies, the real and the imaginary part of absolute wavenumbers are finally compared in figures 2.30b, 2.30c, and 2.30d, with good overall agreement.

2.5.5 Experimental predictions

An application of this spatio-temporal study is to draw a comparison with the experiments conducted on confined wakes around obstacles. Richter & Naudasher [94] have found that the Strouhal number increases with confinement for the wake past a cylinder. Their analysis is however devoted to Reynolds number $Re \sim 10^5 - 10^6$, thus to turbulent wakes where three-dimensional effects cannot be neglected, in contrast to the present purely two-dimensional analysis. Confined cylinder wakes have also been analyzed experimentally at lower Reynolds numbers: Davis *et al.* [25] have analyzed a wake past a square cylinder for $500 < Re < 1850$ and Rehimí *et al.* [93] have examined a confined circular cylinder wake for Reynolds number close to critical $Re_c = 108$, i.e. $118 < Re < 277$. Comparing a single confinement configuration $h = 2$ to the unconfined wake, the latter authors have also concluded that the Strouhal number was increased if walls are present. A similar result was obtained by Davis *et al.* [25] for two different confinement configurations, $h = 2$ and $h = 5$ and noticed by Camarri and Giannetti [15] in their direct numerical simulations of a confined flow around a square cylinder.

All these authors define the Strouhal number as $St = 4\omega h_1 / 3\bar{U}$, thus referring it to the diameter of the cylinder $2h_1$ and to maximal value of their inlet profile, which correspond to $U_{max} = 3\bar{U}/2$, where $\bar{U}$ is the mean flow rate. The absolute frequencies for $h = 9$, $h = 5$ and $h = 2$ shown in figure 2.23 have been rescaled accordingly to provide figure 2.31. The Strouhal number is seen to grow as the confinement is increased, irrespective of the shear layer thickness. The range of the values of the Strouhal number ($0.251 < St < 0.263$) measured in the experiments of Rehimí *et al.* [93] is highlighted in figure 2.31. The agreement with the predictions of the local

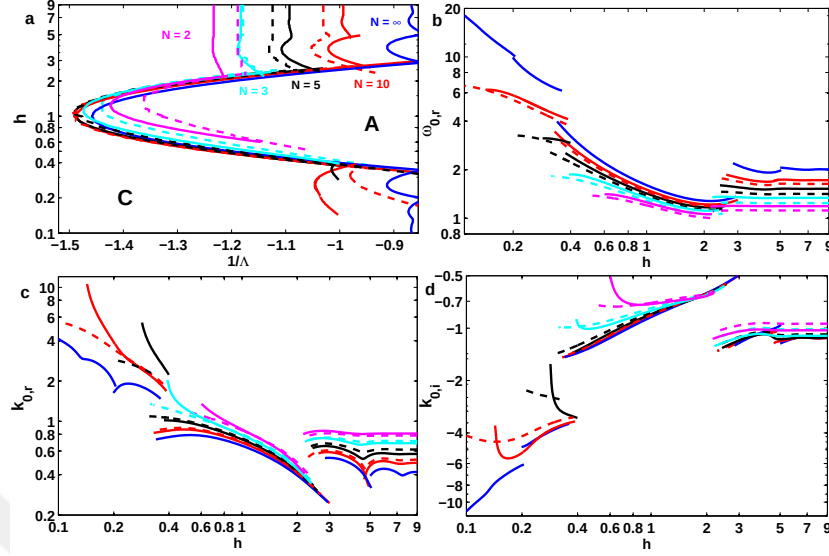


Figure 2.30: Comparison of (a) C/A transitions in the plane (Λ^{-1}, h) , (b) absolute frequencies $(\omega_{0,r})$ (c) real and (d) imaginary part of absolute wavenumber k_0 with respect to confinement h : dashed lines depict continuous profile model, while continuous lines depict broken-line one. Blue line depicts $N = \infty$ ($\delta_w = 0$), red $N = 10$ ($\delta_w = 0.160$), black $N = 5$ ($\delta_w = 0.321$), cyan $N = 3$ ($\delta_w = 0.535$), and magenta $N = 2$ ($\delta_w = 0.802$). In (a) the regions of absolutely and convectively instability are marked by A and C.

spatio-temporal theory is reasonable, if one bears in mind that we have neglected the influence of the Reynolds number and no slip at the wall. Figure 2.31 further tentatively suggests that the shear layer thickness of the velocity profiles past the cylinder could be close to $\delta_w = 0.16$, corresponding to $N^{-1} \sim 0.1$.

Although Davis *et al.* [25] did not provide the exact values of their Strouhal number, they were shown to be encompassed between $0.16 < St < 0.185$ for $h = 2$ and $0.145 < St < 0.165$ for $h = 5$. While the former values clearly lie below our predictions, our analysis can still give some insight regarding the difference between the Strouhal numbers of these two studies at the same confinement parameter $h = 2$, by considering the different geometry of the obstacle. Indeed, the development of the boundary layer on the edge of the square cylinder results in a larger shear layer thickness than in the case of a circular cylinder [74]. Since the Strouhal number decreases as a function of N^{-1} as illustrated in figure 2.31, this could help explaining the lower frequencies measured by Davis *et al.* [25].

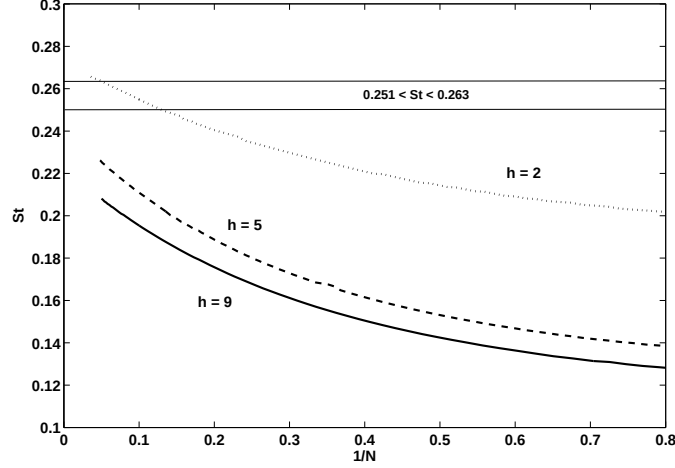


Figure 2.31: Strouhal number vs N^{-1} for three different confinement configurations, $h = 9$, $h = 5$, and $h = 2$, depicted respectively by continuous, dashed and dotted lines. The Strouhal numbers are referred to $2h_1$ and $3\bar{U}/2$. Note that the Strouhal increases with the confinement. The range of the Strouhal number obtained by Rehimi *et al.* [93] for $116 < Re < 277$ is highlighted between thin continuous lines.

2.5.6 Applying the CRW perspective to the spatio-temporal analysis

In section 2.4.1 we have discussed how the temporal stability of the piecewise broken line profile can be explained as the interaction of two counterpropagating Rossby waves (CRWs). However only Bishop & Heifetz (2000) have tried to use the CRW perspective to explain the spatio-temporal behaviour, in particular for the baroclinic Eady's model. The objective of the present section is to establish a connection between the CRWs perspective and the spatio-temporal analysis, particularly with the reinforcement of the absolutely unstable character of the flow at a medium confinement for the sinuous mode, noticed by Juniper (2006) [54] for the top hat wake profile and here confirmed when a finite shear layer thickness is included (section 2.5.3.2).

The CRW approach is based on the destabilizing cooperation or interaction of two waves, that can however also be individually analyzed at first by means of eq. (2.29). The natural question is how the variations on the confinement parameter modify the behaviour of such waves, considered freely to evolve without interacting each other. To do that, it is necessary to neglect the interaction term γ , according to

$$M_0 = \begin{bmatrix} -ikc_1 & 0 \\ 0 & -ikc_2 \end{bmatrix}. \quad (2.83)$$

The eigenvalues $\lambda_{1,2}^0 = \{-ikc_1, -ikc_2\}$ are pure imaginary, and $c_i = \lambda_i^0/k$ represents

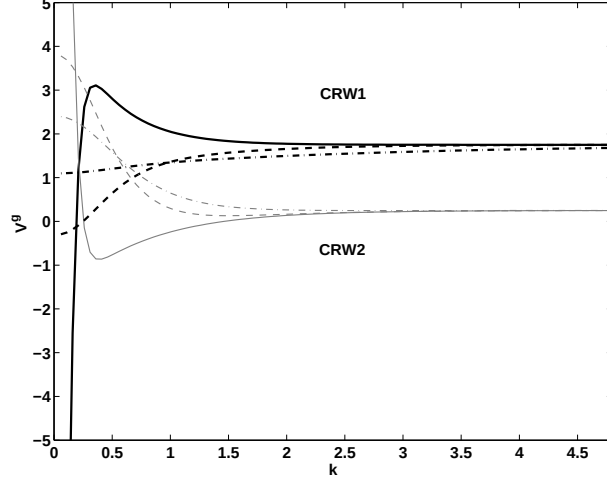


Figure 2.32: Group velocity v_i^g of uncoupled CRWs in the system represented by eq. (2.83) for several confinements, $\delta_w = 0.535$ and $\Lambda = -0.75$. The continuous line depicts $h = 9$, the dashed one $h = 1$, and the dash-and-dot one $h = 0.5$. Black bold lines represents the first CRW, while gray thin lines the second one.

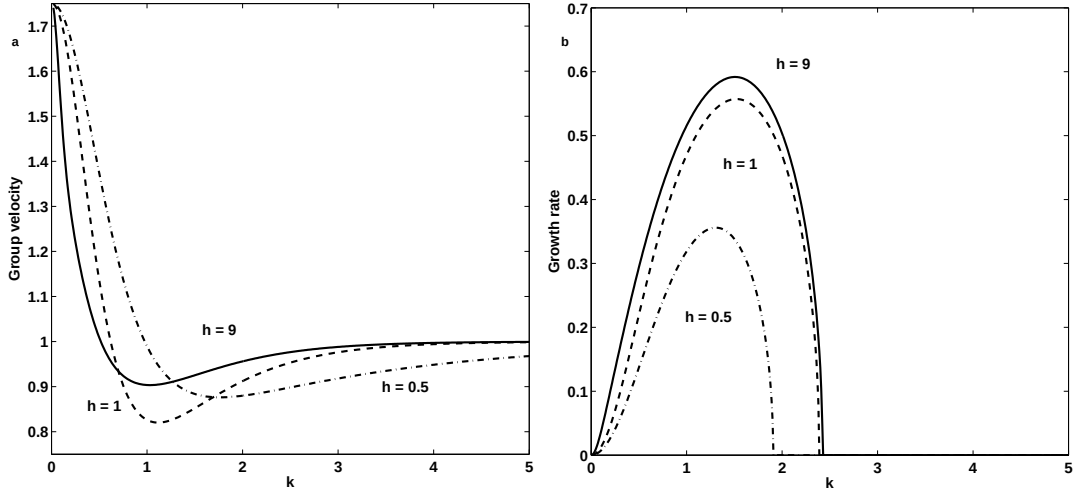


Figure 2.33: (a) Mean group velocity $(v_1^g + v_2^g)/2$ and (b) growth rate of coupled CRWs in the system represented by eq. (2.29) for several confinements, $\delta_w = 0.535$ and $\Lambda = -0.75$. The continuous line depicts $h = 9$, the dashed one $h = 1$, and the dash-and-dot one $h = 0.5$.

the phase speed of the i -CRW. In addition, another interesting information can be extrapolated from eq. (2.29): the group velocity of the two uncoupled CRWs $v_{1,2}^g = -d(kc_i)/dk$. As is possible to observe in figure 2.32, the counterpropagating Rossby waves have very contrasted group velocities with limiting values at $k \rightarrow \infty$ the velocities at both interfaces. Only at weak confinement is the evolution of the group velocity not monotonous towards the interface velocity, starting with a large group velocities in absolute value at low wavenumbers ($CRW1 > 0$, $CRW2 < 0$), before going through a stationary point. At medium to strong confinement, the evolution is monotonous. The influence of confinement smoothes the variations and levels out the group velocities towards their asymptotic value. In other words, the influence of confinement largely depends on the wavenumber regime and is much stronger at small wavenumbers.

Figure 2.33a illustrates the behaviour of the mean group velocity of CRWs $(v_1^g + v_2^g)/2$ with respect to the wavenumber for different confinement configurations, $h = 9$, $h = 1$ and $h = 0.5$ and $\delta_w = 0.535$. Each curve shows a minimum, whose both value and position vary as function of the confinement. This minimum corresponds to the slowest velocity of the compound wavepacket. Note that the minimum minimorum mean is attained for $h = 1$.

This mean group velocity, as well as the mean phase velocity, very naturally appear when calculating the coupled eigenvalues of matrix M

$$\lambda_{1,2} = \frac{-ik(c_1 + c_2)}{2} \pm \sqrt{k^2 c_1 c_2 + \gamma^2 - \frac{k^2(c_1 + c_2)^2}{4}}. \quad (2.84)$$

It is seen that, in the unstable range of wavenumbers, where the term under the square root is positive, the imaginary part of $\lambda_{1,2}$ is precisely given by the average of the eigenvalues of M_0 . Furthermore in figure 2.33b, the growth rates of the coupled CRW system (2.29) are depicted with the same set of parameters of figure 2.33a. The confinement only has a damping effect, as already noticed in figure 2.15d.

Considering the dispersion resulting from the coupled CRW, of the form $\omega(k) = f(k) + i\sqrt{g(k)}$, it is however in general impossible (with the noticeable exception of non dispersive waves where f is proportional to k) to separate the advection of the wavepacket by the term f from the spreading due to the growth of the wavepacket, and the spatio-temporal dynamics can only be understood by an analysis of the dispersion relation in the complex k plane. In the present case, it is however tempting, as a consequence of the physical interpretation associated to the two CRWs, to approximate the velocity of the edge of the wavepacket as the sum of the mean group velocity of the real waves and a second contribution associated to the spreading due to the pure growth of the wavepacket, as depicted in figure 2.34. In mathematical term, this conjecture suggests to approximate the minimal velocity of the wavepacket as

$$v^- = \min[(v_1^g + v_2^g)/2] + v_s, \quad (2.85)$$

where v_s is the symmetric spreading of the pure growth.

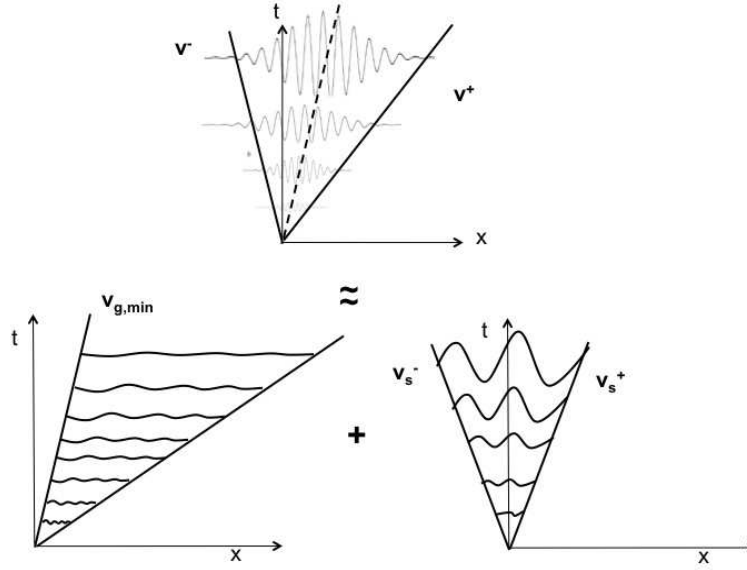


Figure 2.34: Physical interpretation to approximate the velocity of the edge of the wavepacket. The group velocity of the wavepacket is viewed as the sum of an advection term connected to CRWs and a term associated to the spreading of the growth of the wavepacket.

If one reminds that the growth rate does almost not depend on h (at least for $h \geq 1$), v_s can be assumed to be almost independent of h and the dependance of v^- as a function of h dominated by $\min[(v_1^g + v_2^g)/2]$. This fact can be observed in figure 2.35, where the spatio-temporal diagram of the Fourier inverse transform of $i\sqrt{g(k)}$ computed with Matlab is illustrated for different confinement (a) $h = 9$, (b) $h = 1$ and (c) $h = 0.5$ prescribing the other parameters. Such quantity could give an idea of the spreading term v_s . Only for $h = 0.5$ the spatio-temporal behaviour seems slightly different from other two cases, sign of the negligible dependence of the spreading term on the confinement. Further credit is given to this interpretation by noticing that among the three confinements chosen in figure 2.33a, only the $h = 1$ case, which has the lowest mean CRW group velocity, is absolutely unstable (see the continuous cyan line in figure 2.30a).

This fact can be also discerned in figure 2.36 where the spatio-temporal diagram of the log-vorticity perturbation $\log_{10}(q'(x, t))$ in the plane (x, t) for $\Lambda = -0.75$, $\delta_w = 0.535$ and (a) $h = 9$ and (b) $h = 1$. These diagrams are always obtained by means of a Fourier inverse transform of the matrix M computed with Matlab. It is clear that for $h = 9$ the flow is convectively unstable, while for $h = 1$ it is absolutely unstable.

In order to understand how the minimal group velocity of the uncoupled CRWs influences the C/A transition, figure 2.37 illustrates the comparison between the C/A

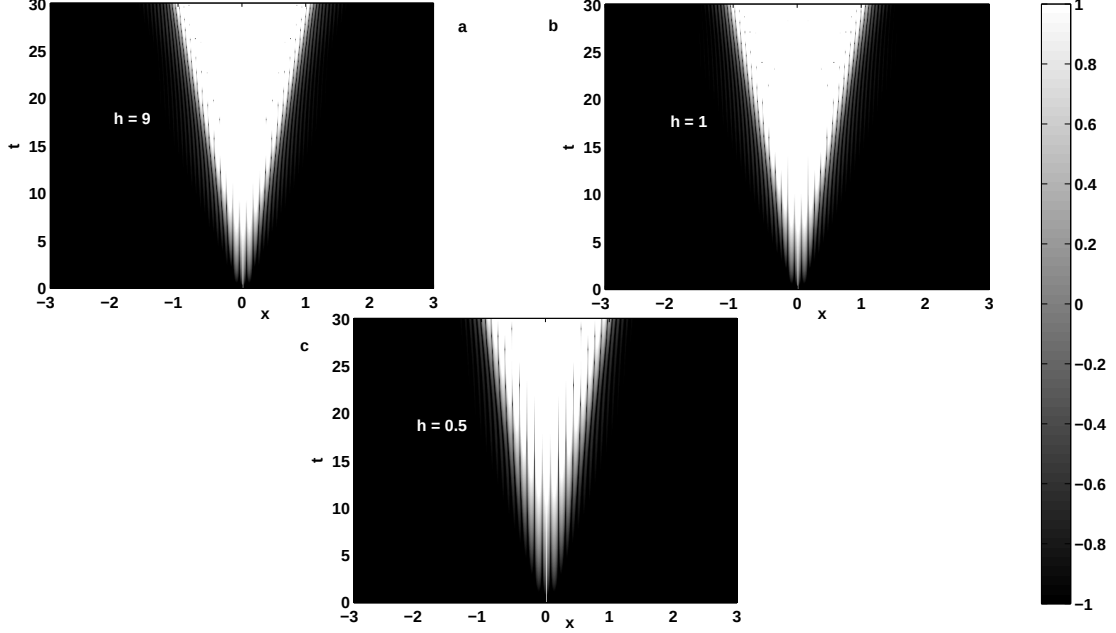


Figure 2.35: Spatio-temporal diagram of the Fourier inverse transform of $i\sqrt{g(k)}$ which represents the spreading term in the plane (x, t) in log-scale for $\Lambda = -0.75$, $\delta_w = 0.535$ and (a) $h = 9$, (b) $h = 1$, and (c) $h = 0.5$. The dependence on the confinement of the spreading is negligible.

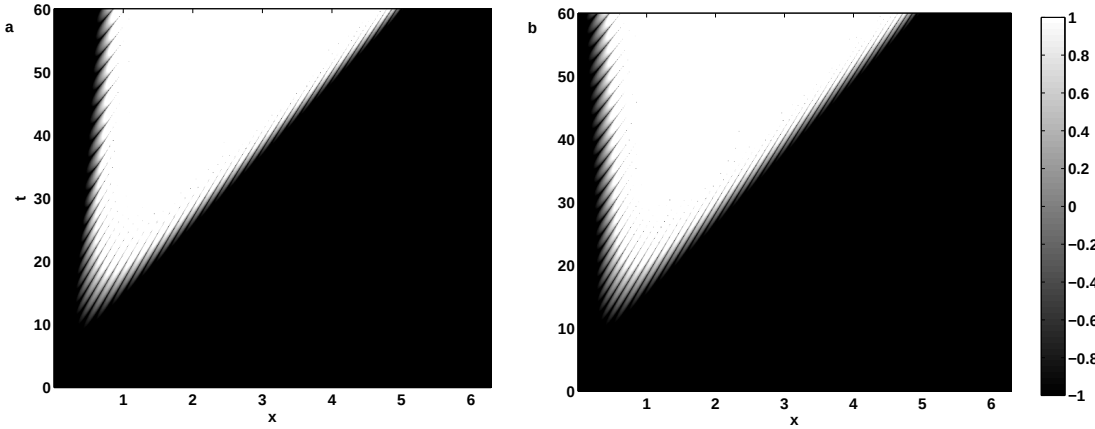


Figure 2.36: Spatio-temporal diagram of the log-vorticity perturbation $\log_{10}(q'(x, t))$ in the plane (x, t) for $\Lambda = -0.75$, $\delta_w = 0.535$, and (a) $h = 9$, (b) $h = 1$. In fig. a the unstable wavepacket is advected away, while in fig. b it grows everywhere.

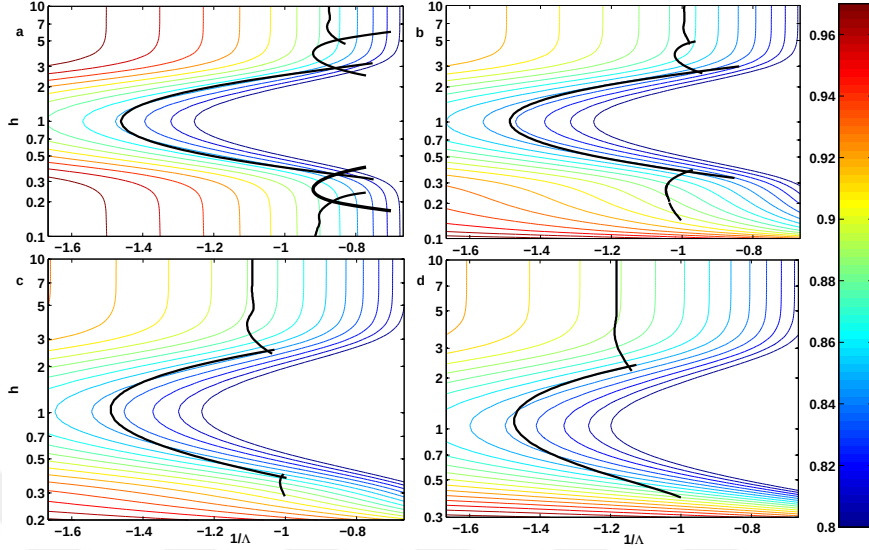


Figure 2.37: Contours of the minimal group velocity v_{min}^g compared to the C/A transition obtained in section 2.5.3 in the plane (Λ^{-1}, h) for different values of the shear layer thickness: (a) $\delta_w = 0.01$, (b) $\delta_w = 0.160$, (c) $\delta_w = 0.321$, and (d) $\delta_w = 0.535$. The black bold line depicts the C/A transition, while the colored lines represent the isocontours of the minimal group velocity.

limit and the isocontours of $v_{g,min}$ in the plane (Λ^{-1}, h) for different values of the shear layer thickness (a) $\delta_w = 0.01$, (b) $\delta_w = 0.160$, (c) $\delta_w = 0.321$, and (d) $\delta_w = 0.535$. The contours present a clear maximum at moderate confinement for each value of the shear layer thickness. Note that specially when $\delta_w \rightarrow 0$, the behaviour of the contours fits very well the C/A transition at medium confinement. This excellent fit allows to determine that the dependence on the confinement of the part of the group velocity due to the interaction of the CRWs is negligible, in particular when the shear layer thickness is small enough.

Therefore, we can conclude that the maximum of the instability at moderate confinement noticed by Juniper [54], confirmed in his more recent analyses [55, 90, 91] and in this thesis (section 2.5.3.2) is principally due to the influence of the confinement on the group velocities of the uncoupled CRWs.

Chapter 3

Direct numerical simulations of two-dimensional confined wakes



3.1 Introduction

In the second chapter of this thesis, the destabilizing influence of the confinement on several wake models by means of a local stability analysis was shown and commented. In particular, both temporal and spatio-temporal analysis have been conducted in section III and V, respectively. Furthermore, the results obtained by Juniper & Candel [57], Juniper [54, 55, 56] and Rees & Juniper [90, 91] on confined jets/wakes have been also discussed in order to give a wide panorama of this counterintuitive effect due to the confinement. Particularly, focusing on the inviscid parallel top hat profile [57, 54], they have discovered that the absolutely unstable nature of these flows is exacerbated when the aperture of the channel approximately equals twice the width of the wake. Broadly speaking, an absolute instability is able to withstand the mean downstream advection and can give rise to self-sustained oscillations [50]. According to this scenario, the conclusions of Juniper (2006) [54] lead to the expectation that the global instability of wakes is favored by a confinement of the same size as the width of the wake, giving birth to vortex shedding even in the presence of consistent co-flow: the confinement-mediated absolute instability was interpreted as the main reason for the increased mixing in presence of a recess. This result was confirmed also in the second chapter of this thesis.

It was shown subsequently that in many cases the self-sustained oscillations reached saturation in the form of a so-called non-linear global mode, described analytically by Pier *et al.* (2001) [85] and Couairon & Chomaz (1999) [21] in the context of weakly non-parallel flows (*i.e.* flows whose variations in the longitudinal direction are slow compared to the dominant wavelengths of the instability). Besides its qualitative predictive power, the weakly non parallel nonlinear front theory is also capable to predict quantitatively the vortex shedding frequency, depending on the nature of the instability of the inlet profile: if the flow is absolutely unstable at the inlet, the shedding frequency is given by its absolute frequency [see 21], otherwise, it is given by the absolute frequency prevailing at the location of the onset of the absolute region [see 85]. It is remarkable that these coherent oscillating structures, though fully nonlinear, are governed by a linear frequency selection criterion. The weakly non-parallel theory has given satisfying results for different spatially evolving open flows, as synthetic wakes [84], spiral vortex breakdown [33], detached boundary layers [70] or axisymmetric wakes behind afterbodies [72].

The purpose of the present chapter is to check the validity of this scenario by relaxing the parallel flow assumption using direct numerical simulations (DNS) of confined planar wakes. Such an approach has been, to the best of our knowledge, only recently followed by Tammisola *et al.* [99] in range of Reynolds number $100 < Re < 400$. By comparing two values of confinement ($h = 1$ and $h = 9$, where h has the same definition of chapter 2, see section 2.2), no destabilizing effect due to confinement was discovered in their global linear and nonlinear simulations. Note also that Tammisola *et al.* [99] impose only no slip boundary conditions at the walls in contrast to our analysis where also a free slip condition is considered, in order to

make contact with available theoretical predictions. In the unconfined setting, our approach is reminiscent of those of Hammond & Redekopp (1997) [37] and Chomaz (2004) [17], or that of Pier & Huerre (2001) [84] who added an extra pressure gradient.

Using direct numerical simulations (DNS), the purpose is to study the influence of confinement on the stability of plane wakes. The objective is to determine if the theoretical analyses illustrated in the second chapter and in Juniper's papers [see 54, 55, 56], which show a strong effect of confinement on spatiotemporal stability properties of the flow, can be confirmed by our more realistic DNSs. Using the DNS indeed allows to relax two restrictive hypothesis of the theoretical studies: the absence of viscosity and the parallel character of the flow. The outline of the chapter is as follows: section 3.2 describes the flow configuration and gives details on the numerical method implemented in the DNS. The global stability regions implementing free slip (FS) and no slip (NS) conditions at the walls in the velocity ratio-confinement ratio plane are examined in sections 3.3 and 3.4, respectively, and compared to the convective/absolute transition prevailing at the inlet. Furthermore, within these sections, the focus is set on the non parallel variations of the base flow, especially concerning the free slip-wake flows. Finally, the influence of confinement on the Strouhal number is examined in section 3.5.

3.2 Flow configuration and numerical method

3.2.1 Physical problem

Let us consider the flow of a viscous fluid in a 2D channel, using two different inlet conditions.

The first one is the velocity profile studied by Juniper [54] (figure 3.1, continuous line) allowed to evolve downstream. In section 3.3 the flow is assumed free to slip on the side walls, as shown in figures 3.2a, while in section 3.4 no slip conditions are imposed at the lateral boundaries, see figure 3.2b.

More precisely, a flow with thickness $2h_1$ and uniform velocity U_1 sandwiched by two identical flows with thickness h_2 and uniform velocity U_2 is prescribed at the inlet of the channel. As reference velocity and length scales, we choose the average velocity $(U_1 + U_2)/2$ and the half-thickness of wake h_1 , respectively. The dimensionless parameters are then the Reynolds numbers $Re = (U_1 + U_2)h_1/2\nu$, the *confinement parameter* $h = h_2/h_1$ and the *velocity ratio* $\Lambda = (U_1 - U_2)/(U_1 + U_2)$. For a wake, the velocity ratio is negative $\Lambda < 0$ and $-1 < \Lambda < 0$ corresponds to co-flow while $\Lambda < -1$ characterizes counter-flow ($U_1 < 0$). The dimensionless entry profile, at $x = 0$, is written:

$$\begin{aligned} U(y) &= 1 + \Lambda & \text{for } 0 \leq |y| < 1, \\ U(y) &= 1 - \Lambda & \text{for } 1 < |y| \leq 1 + h. \end{aligned} \tag{3.1}$$

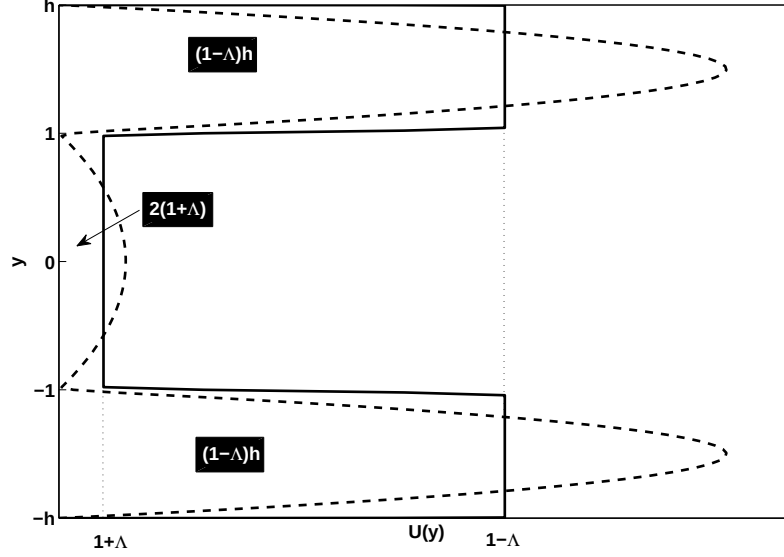


Figure 3.1: The entry profiles used in our DNS in this chapter. The continuous line depicts the top hat profile, while the dashed line depicts the set of Poiseuille flows. In the black squares the flowrate is indicated for each flow.

While this profile is examined for two different side-boundary conditions, free slip (FS) and no slip (NS), the second inlet profile considered in this study is only analyzed with NS conditions at the walls. For the first inlet profile, eq. 3.2, the streamwise velocity is piecewise constant, while for the second one it is piecewise parabolic and moreover continuous. For the latter inlet profile, we enforce a set of Poiseuille flows such that

$$\begin{aligned}
 U(y) &= -\frac{6(1+\Lambda)}{h^2}(h+1-(h+2)|y|+y^2) \quad \text{for } 0 \leq |y| < 1, \\
 U(y) &= \frac{3}{2}(1-\Lambda)(1-y^2) \quad \text{for } 1 < |y| \leq 1+h.
 \end{aligned} \tag{3.2}$$

Note that for $|y| < 1$, the mean flow velocity is $1 + \Lambda$, while for $|y| > 1$ it is $(1 - \Lambda)$ for both profiles. Particularly, the mean flow velocity, given by $\bar{U} = \frac{\int_{-1-h}^{1+h} U(y) dy}{\int_{-1-h}^{1+h} dy}$ is preserved and equal to $1 + \Lambda \frac{1-h}{1+h}$.

The study has been carried out for various values of $(\Lambda, h) = [-1.17, -0.40] \times [0.2, 9]$, and two different Reynolds number: $Re = 100$ and $Re = 500$.

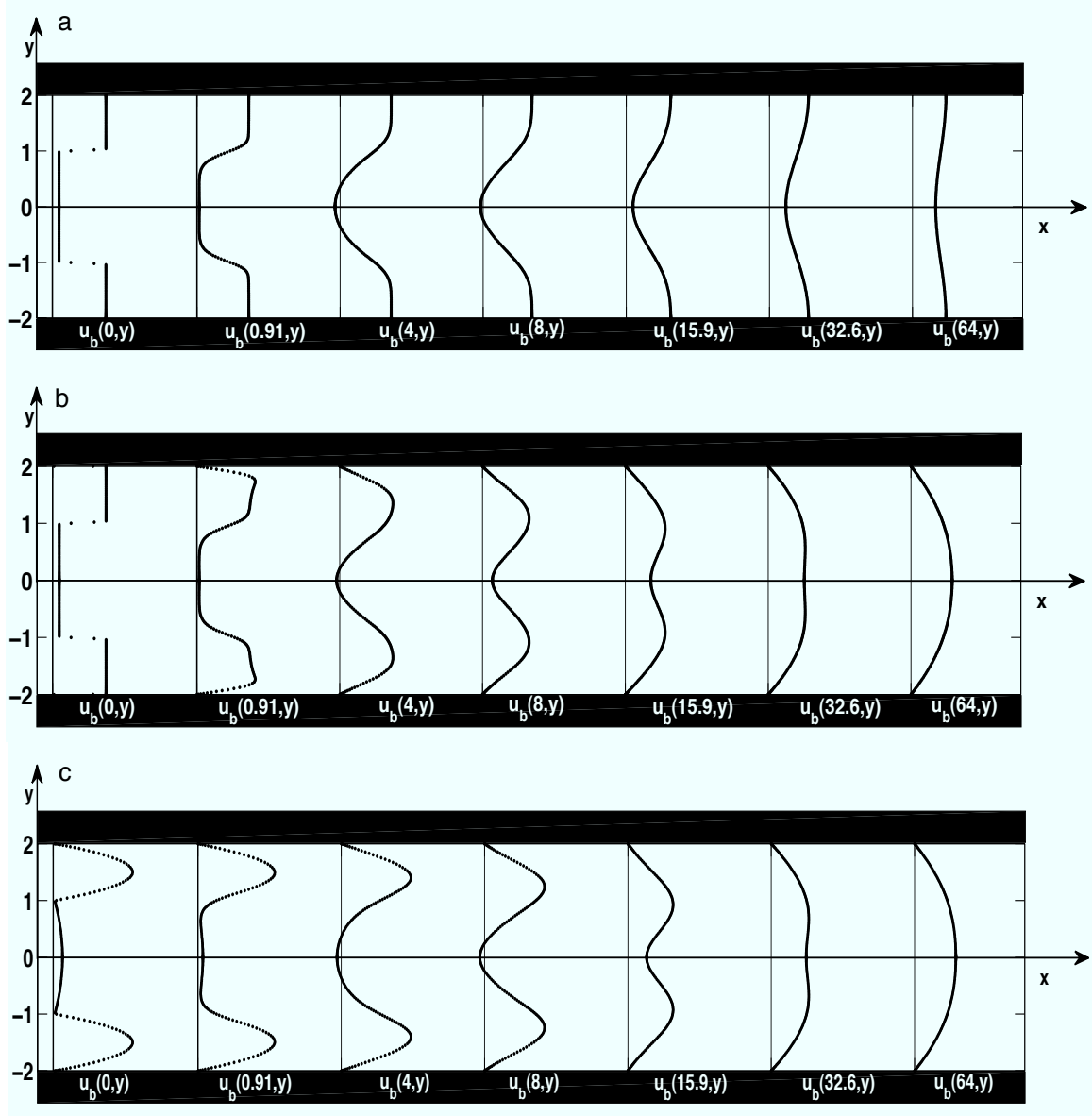


Figure 3.2: Evolution of the base flow for $Re = 100$, $h = 1$, and $\Lambda = -0.811$ for three models: (a) free slip conditions and (b) no slip conditions with plug flow at inlet, (c) no slip conditions with Poiseuille flows at the inlet.

3.2.2 Numerical set-up: direct numerical simulation

In dimensionless form, the Navier-Stokes system can be written :

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + Re^{-1} \Delta \mathbf{u} \quad \text{in } \Omega \times (0, t_f), \quad (3.3)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (3.4)$$

$$B(\mathbf{u}) = \mathbf{g} \quad \text{on } \Gamma \times (0, t_f), \quad (3.5)$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \quad (3.6)$$

where t (resp. t_f) is the (resp. final) time, Ω the computational domain and Γ its boundary, $\mathbf{u} = (u, v)$ the velocity, p the pressure, Re the Reynolds number. The boundary operator B and the function $\mathbf{g}$ are used to express the boundary conditions. The domain is rectangular, say $\Omega = (0, L) \times (-H, H)$, where the values of L and $H = 1 + h$ vary according to configuration. Our computations are carried out at the Mésocentre of the Observatory of Côte Azur with the *SVVLES* code which has been developped in Nice for a few years. We recall hereafter the basic ingredients of this software.

A domain decomposition technique with conforming meshes is used in the elongated longitudinal direction and in each subdomain we use a Chebyshev collocation approximation in both x -streamwise and y -crossflow directions. Between the different non-overlapping subdomains, say Ω_s , $1 \leq s \leq S$, the transmission conditions express the C^1 continuity of the numerical approximation u_N of some unknown variable u , so that u_N belongs to the functional space V_N such that:

$$V_N = \{v \in C^1(\Omega), v|_{\Omega_s} \circ \mathbf{f}_s \in \mathbb{P}_N(-1, 1)^2, s = 1 \cdots S\}, \quad (3.7)$$

where $\mathbf{f}_s : (-1, 1)^2 \rightarrow \Omega_s$ maps the reference domain to the physical one and with $\mathbb{P}_N$ for the set of polynomials of degree N_x in x and N_y in y . More precisely, in each subdomain u_N is formulated as:

$$u_N|_{\Omega_s} \circ \mathbf{f}_s = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} u_{ij}^s l_i l_j, \quad (3.8)$$

where l_i and l_j are the Lagrange polynomials based on the Chebyshev-Gauss-Lobatto points $-\cos(\pi i/N_x)$, $0 \leq i \leq N_x$, and $-\cos(\pi j/N_y)$, $0 \leq j \leq N_y$, respectively. Let us recall that the collocation method consists of enforcing that u_N solves the PDE at all internal collocation points, i.e. such that $0 < i < N_x$ and $0 < j < N_y$, whereas the boundary conditions are imposed at the boundary grid points, see e.g. Peyret (2001) [82] for details. Generally, $N_x = 100$ is settled for each subdomain, while N_y varies according to H , between $90 < N_y < 118$.

3.2.2.1 Temporal scheme

The temporal scheme combines a semi-Lagrangian method to handle the convective terms and a projection method to enforce the divergence-free constituents. It shows three steps [22]: a transport step, a diffusion step and finally a projection step.

Transport step

The approximation of the material derivative, $D_t \mathbf{u} = \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}$, with a BEQ (backward Euler of order Q) [12] schema yields, with α_q , $q = 0, \dots, Q$ a set of a given coefficients:

$$D_t \mathbf{u}(t_{n+1}) = \frac{1}{\Delta t} \left(\alpha_0 \mathbf{u}^{n+1} + \sum_{q=1}^{q=Q} \alpha_q \tilde{\mathbf{u}}^{n+1-q} \right) + O(\Delta t^Q) \quad (3.9)$$

with $\mathbf{u}^{n+1} \approx \mathbf{u}$ and $\tilde{\mathbf{u}}^{n+1-q} \approx \mathbf{u}(\chi(\mathbf{x}, t_{n+1}; t_{n+1-q}), t_{n+1-q})$, where $\chi(\mathbf{x}, t_{n+1}; t)$ solves the characteristics equation stemming from $(\mathbf{x}, t_{n+1})$. Therefore, the natural approach to compute the $\tilde{\mathbf{u}}^{n+1-q}$ is to determine for each mesh-point x_i the value of velocity $\mathbf{u}$ at the times $t_n, t_{n-1}, \dots, t_{n+1-Q}$ and at the points $\chi(\mathbf{x}_i, t_{n+1}; t)$, $q = 1, \dots, Q$. However the method of characteristic cannot be used in the framework of standard spectral methods since high-order interpolations would be too expensive and, furthermore, would yield numerical instabilities. In order to overcome this difficulty, an Operator Integration Factor (OIF) semi-Lagrangian method [109] is used. The main idea is to transport the $\mathbf{u}(\chi(\mathbf{x}, t_{n+1}; t_{n+1-q}))$ at the mesh points, so, that interpolations-extrapolations are only needed in time. This requires to solve a set of Q problems involving an advection equation for each variable. The RK4 (fourth order Runge-Kutta) algorithm is used to this end. In the *SVVLES* code one uses $Q = 2$, so that $\alpha_0 = 3/2$, $\alpha_1 = -2$ and $\alpha_2 = 1/2$.

Diffusion step

The diffusion step is handled implicitly using a Schur complement technique [31] and a fast direct solver in each subdomain. Its aim is to compute a provisional velocity $\mathbf{u}^*$ such that:

$$(Re^{-1} \nabla^2 - \frac{\alpha_0}{\delta t}) \mathbf{u}^* = \mathbf{f}^n + 1 \quad \text{in } \Omega, \quad (3.10)$$

$$B(\mathbf{u}^*) = \mathbf{g}, \quad (3.11)$$

where the eq. (3.11) depicts the boundary condition and

$$\mathbf{f}^{n+1} = \frac{1}{\Delta t} \sum_{q=1}^{q=Q} \alpha_q \tilde{\mathbf{u}}^{n+1-q} + \nabla p^* \quad (3.12)$$

with p^* a provisional pressure. The Goda scheme [36] $p^* = p^n$ is implemented in the framework of a second order approximately $Q = 2$.

Projection step

To derive from $\mathbf{u}^*$ a divergence-free velocity field $\mathbf{u}$, one solves the Darcy problem [12]:

$$\mathbf{u}^{n+1} + \nabla\phi = \mathbf{u}^* \quad \text{in } \Omega, \quad (3.13)$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0 \quad \text{in } \Omega, \quad (3.14)$$

$$\mathbf{u}^{n+1} \cdot \mathbf{n}|_{\Gamma} = \mathbf{u}_k \cdot \mathbf{n} \quad (3.15)$$

and then update the pressure field $p^{n+1} = p^n + \alpha_0\phi/\delta t$. The solution of the Darcy problem is not straightforward. Following Azaiez *et al.* [1] and Botella [12] a unique grid $\mathbb{P}_N$ - $\mathbb{P}_{N-2}$ strategy is used, where $\mathbb{P}_N$ is the space of the polynomials of maximum degree N_1 in x and N_2 in y , i.e. $N = (N_1, N_2)$. In practice, this means that the polynomial spaces of the pressure and of the velocity components are chosen different, so that no boundary conditions are required for the pressure which is completely defined by its values at the inner grid-points. Therefore, in the monodomain case, the polynomial interpolant of the pressure is 2 degrees less than for the velocity components in the non-homogenous directions. In the frame of a multi-domain decomposition in the streamwise x -direction, to enforce the natural C^1 continuity of the pressure at the subdomain-interfaces we use, in each subdomain s , $1 \leq s \leq S$, the pressure polynomial space $P_{N'}$ such that for $s = 1$ or $s = S$: $N' = (N_1 - 1, N_2 - 2)$, while if $1 < s < S$: $N' = (N_1, N_2 - 2)$.

3.2.2.2 Boundary conditions

In section 3.3 at $y = \pm H$ we use a standard free slip (FS) boundary condition, while in section 3.4 a no slip (NS) condition is implemented. In contrast, the conditions used at the inlet and at the outlet of the computational domain are more subtle, as detailed below. Especially, the choice of a relevant outflow boundary condition (OBC) is required in the present study.

The inflow profiles U described in the introduction displays a discontinuity at $y = \pm 1$, which requires to be smoothed out since we use a spectral method. In order to avoid or, at least, weaken the Gibbs phenomenon [77], we use a filtered entry profile, regularizing the initial profile according to:

$$\tilde{U}_i = \frac{1}{4}(U_{i-1} + 2U_i + U_{i+1}), \quad (3.16)$$

where U_i (and $\tilde{U}_i$) are the values at the grid-points of the y -axis. In the frame of a spectral Fourier approximation, it can be shown that this smoothing in physical space is equivalent to a raised cosine filtering, see *e.g.* Pasquetti *et al.* [81]. Note that once applied to the profile in (3.2), the filter does change only the values of grid-points next to $y = \pm 1$, if FS conditions are considered, otherwise the values close to the lateral boundaries are also modified.

At the downstream end of the computational domain ($x = L$) a non-reflexion OBC is implemented. Both the so-called $u - \omega$ and $u - v$ exit conditions [31] have been tried out and we have adopted the first one for accuracy reasons. The $u - \omega$ condition implies that the component normal to the outlet, u , and the vorticity, $\omega = \partial_x v - \partial_y u$, are transported at the average fluid velocity $\bar{U}$:

$$(\partial_t + \bar{U}\partial_x)u = (\partial_t + \bar{U}\partial_x)\omega = 0. \quad (3.17)$$

As the consequence of the global coupling introduced by the spectral method, it is important to "localize" the effect of the OBC. To this end, using the method of characteristics, these conditions are implemented as follows :

$$u^{n+1}(L, t^{n+1}) = u^n(L - \tau\bar{U}, t^n), \quad (3.18)$$

$$\omega^{n+1}(L, t^{n+1}) = \omega^n(L - \tau\bar{U}, t^n), \quad (3.19)$$

where n is the time index. Note that as a result of equation (3.18), equation (3.19) amounts to implement a Neumann condition on $\partial_x v$ (using again a Lagrangian approach).

In the most critical cases, the $u - \omega$ OBC has been compared with a special treatment in an additional subdomain, say the *sponge subdomain*, where the flow is softly enforced to come back at a constant meanflow velocity before applying the OBC. A penalization technique is used to this end, by complementing the momentum equation with an appropriate forcing term $\mathbf{F}$. In the sponge subdomain, of length ΔL , the divergence-free velocity is solution of:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + Re^{-1} \Delta \mathbf{u} + \mathbf{F}, \quad (3.20)$$

$$\mathbf{F} = f\left(\frac{x-L}{\Delta L}\right)(\bar{U}\mathbf{e}_x - \mathbf{u}), \quad (3.21)$$

where $\mathbf{e}_x$ is the unit vector in x -direction. The function f should be chosen to increase smoothly from 0 to its maximum value, that one can relate to the time-step τ . Using a $O(1/\tau)$ maximal value is *e.g.* well adapted to model an obstacle with a smoothed penalization like technique, see Pasquetti *et al.* [81]. In practice we have used:

$$f(X) = \frac{0.1}{\tau} \sin^2\left(\frac{\pi}{2}X\right), \quad 0 < X < 1.$$

At the end of the extended computational domain, the fluctuations with respect to the mean flow are then less than 10^{-5} . Using the sponge subdomain together with the $u - \omega$ OBC is less reflexive than using the latter alone. It has been implemented for such configurations for which the dependence of the flow on the exit conditions should be excluded carefully.

3.2.2.3 Initial condition

The initial condition $\mathbf{u}_0(x, y)$ consists in the superposition of the velocity entry profile $\tilde{U}(y)$ and a random perturbation $\mathbf{u}'(x, y)$ of amplitude 10^{-3} :

$$\mathbf{u}_0(x, y) = \tilde{U}(y)\mathbf{e}_x + \mathbf{u}'(x, y). \quad (3.22)$$

This perturbation is added in order to break the varicose symmetry $y \leftrightarrow -y$ of the base flow velocity profile $\mathbf{u}_b(x, y)$, which is a steady solution of the Navier Stokes equations (3.3 - 3.6). The base flow velocity profile prevails when the flow is stable. Breaking the varicose symmetry allows the sinuous mode to develop faster and, thus, a gain in term of computational time. The different entry profiles studied in this chapter have been previously described in section 3.2.1.

3.2.2.4 Base flow

In order to study the influence of the non-parallel feature of the base flow $\mathbf{u}_b$, we have to determine its variations in downstream direction. For unstable flows, displaying self-sustained oscillations, the base flow cannot be obtained as the end-result of the DNS. However, it may be determined by exploiting the varicose symmetry of the base flow. This amounts to take advantage of the analogy between the varicose evolution of a wake and an asymmetric mixing layer (*i.e.* when the mixing layer does not lie exactly halfway between the plates), with the same flow parameters: Re and Λ . Note that confined asymmetric mixing layers are globally stable in the range of parameters analyzed in the paper [43]. The base flow is then simply obtained by time marching of the DNS of a mixing layer located at $y = 1$ and quenched between two plates at $y = 0$ and $y = H$. The free slip condition at the lower plate eliminates any sinuous perturbation and the DNS converges to the base flow. Some examples of base flows are illustrated in figure 3.2.

3.2.2.5 3D simulations

With the aim to analyze the effect of the third dimension on the results, in the range of parameters analyzed in this paper, several 3D simulations have been carried out. The computational domain is then $(0, L) \times (-H, H) \times (-E/2, E/2)$, where E is the thickness of the domain, and the flow is assumed homogeneous in z -direction.

The numerical approximation for x and y direction is analogous to 2D case, as previously described. Conversely, for z -direction, a Fourier spectral approximation is implemented. As a result, equation 3.8 writes now:

$$u_N|_{\Omega_s} \circ \mathbf{f}_s = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} \sum_{k=-N_z/2}^{N_z/2} \hat{u}_{ij,k}^s l_i l_j e_k \quad (3.23)$$

where N_z is the number of grid-points in z direction and e_k is the trigonometric polynomial of degree k , *i.e.* $e_k(Z) = e^{ikZ}$ with $i^2 = -1$ and $Z \in (0, 2\pi)$. Of course,

because u_N is real then $\hat{u}_{ij,k}^s$ and $\hat{u}_{ij,-k}^s$ are complex conjugates. Here again we assume that each mapping $\mathbf{f}_s$ may be expressed as the product of 1D mappings and moreover that it is linear in spanwise direction, so that FFT may be used.

A standard collocation Chebyshev - Fourier Galerkin pseudo-spectral method is implemented. Since in Fourier space $\partial_Z \equiv ik$ and $\partial_{ZZ} \equiv -k^2$ and thanks to using a pseudo-spectral method, i.e. the non-linear terms are computed in physical space, using the Fourier expansion of each variable the 3D problem splits into a set of 2D-like problems for the real and imaginary parts of the Fourier spectra, $0 \leq k \leq N_z/2$, of the velocity and of the pressure. Each of these problems may thus be solved with an approach similar to the 2D one.

At the outlet of the domain the 3D form of the u - ω OBC writes [30]:

$$(\partial_t + \bar{U}\partial_x)u = 0, \quad (\partial_t + \bar{U}\partial_x)\omega_{yz} = 0, \quad (3.24)$$

where ω_{yz} is the projection of the vorticity vector onto the outlet boundary. With $\mathbf{u} = (u, v, w)$, when taking into account the condition on u , the condition on ω resumes to a transport condition of $\partial_x v$ and $\partial_x w$ at the mean flow velocity.

The initial condition $\mathbf{u}_0(x, y, z)$ of three-dimensional DNS is given by the final saturated state of the corresponding 2D simulation $\mathbf{u}_f(x, y)$ plus a random perturbation $\mathbf{u}'(x, y, z)$ of small amplitude, typically 10^{-3} , to induce a 3D flow.

3.3 Non-linear global stability with free slip conditions

This section is devoted to DNS of a wake flow having as inlet profile the top hat described in section 3.2.1 and allowed to slip at the side walls. In particular, in sections 3.3.1 and 3.3.2 the results of the simulations are provided for $Re = 100$ and $Re = 500$, respectively. In section 3.3.3, the effect of third dimension on the stability is discussed. Generally, the length of the computational domain is $L = 75$, but in order to study the effect of the exit condition on the flow stability, in some cases we have verified that the behaviors were identical when using $L = 34$, $L = 101.5$ and $L = 152$. Only for the cases presenting a vacillation front we have noticed differences, as explained in section 3.3.2.1-3.3.2.3. For the nominal length we have generally used 8 subdomains, but for the greatest one or when the vortices are subject to vortex pairing instability, we have also used 16 subdomains. The time step τ is adapted to the resolution, typically, $\tau = 2.5 \cdot 10^{-3}$, and corresponds to a CFL coefficient ($:= \max(u)\tau / \min(h_x)$, h_x space step in x -direction) of approximately 0.7.

3.3.1 $Re = 100$

Figure 3.3 shows the vorticity fields at time $t = 125$ for three different set of parameters h and Λ . For $h = 1$ and $\Lambda = -0.739$, see figure 3a, this duration was found

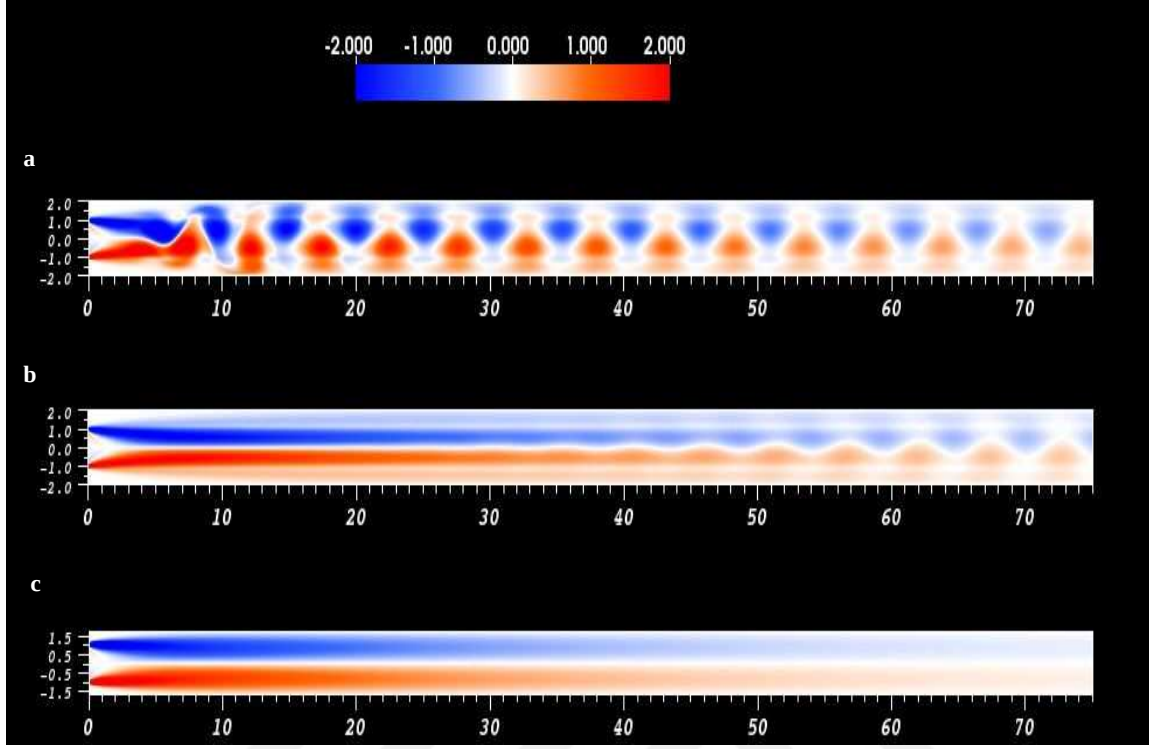


Figure 3.3: Instantaneous vorticity fields for $t = 125$ and $Re = 100$, for three parameter configurations; (a): $h = 1$ and $\Lambda = -0.739$; (b): $h = 1$ and $\Lambda = -0.667$; (c): $h = 0.7$ and $\Lambda = -0.739$. The negative vorticity values are encoded in blue, while the positive ones in red. The color-scale is the same for each flow configuration.

sufficient for the flow to settle in the saturated regime: an alterned vortex street typical of wake flow forms at $x \approx 5$ and extends downstream. The two by two comparison of the figures allows to identify tendencies: fixing the confinement parameter h , the influence of co-flow parameter Λ appears comparing figures a) ($\Lambda = -0.739$) and b) ($\Lambda = -0.667$). A growth of the wake inner velocity U_1 (and thereby an increase of Λ) pushes downstream the perturbation. Instead, if we fix the co-flow parameter $\Lambda = -0.739$, an increase of the confinement from $h = 1$ (a) to $h = 0.7$ (c) seems also to advect the perturbation. However, figure 3.3 represents only three instantaneous snapshots and these tendencies have to be more systematically confirmed by a more exhaustive analysis of the velocity fields, both in space and in time, constructing spatiotemporal diagrams.

In order to better understand the spatiotemporal dynamics, the absolute value of the vertical velocity component along the x -axis, $|v(x, 0, t)|$, is represented in gray colors on a diagram $x-t$, as shown in figures 3.4a and 3.4b. These figures correspond to the parameters of figures 3.3a and 3.3b. Figure 3.4a shows that for $h = 1$ and $\Lambda = -0.739$, the flow is unstable and reaches an oscillating self-sustained state, or global cycle, while the figure 3.4b shows that for the same confinement parameter $h = 1$, but a faster inner fluid such that $\Lambda = -0.667$, the wake is globally stable.

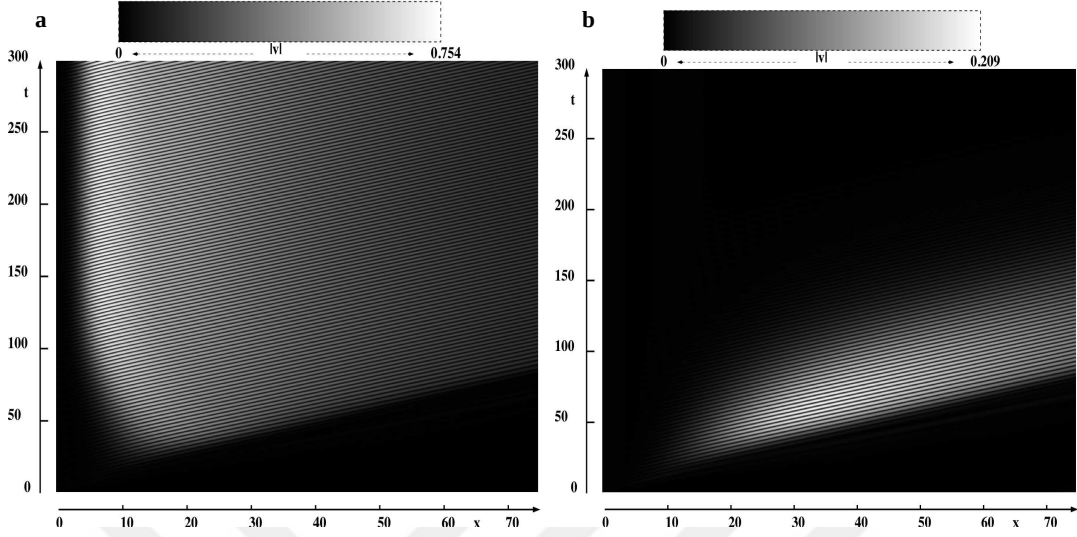


Figure 3.4: Spatiotemporal diagram of absolute value of the velocity vertical component v at $y = 0$ for $h = 1$, $Re = 100$, and (a) $\Lambda = -0.739$, (b) $\Lambda = -0.667$. (a) The flow is globally unstable: the instability persists in the domain. (b) The flow is globally stable: the instability is advected downstream and disappears.

For the unstable configuration (figure 3.4a), one observes a time-independent wave front located at $x_f \approx 5$. A cut in the $x-t$ plane for $x > x_f$, *i.e.* in the unstable region (light region), shows a periodic behavior. This oscillating behavior with a wave front localized in space is typical of non-linear global modes described by Pier & Huerre (2001) [84] for a synthetic wake or by Chomaz (2004) [17] for a viscous wake, even though for weaker Reynolds number.

In contrast to this globally unstable case, for the stable flow (figure 3.4b) we note the presence of a short transient, resulting from the progressive advection of an unstable wave packet. Ultimately the flow ends up becoming stable, *i.e.* the perturbation decreases to zero everywhere. The limit between the black part (where the flow comes back to the rest state) and the striped part represents the wave front [see 18]. If this wave front progresses with a positive velocity, like shown in figure 3.4b, and moves downwards to the outlet of the domain, the flow tends to be stable in the whole domain. It is a typical case where the instability is everywhere convective [50]. The flow is globally stable, but strongly amplifies incoming perturbations, and so behaves like a noise-amplifier [17]. A similar behavior is observed for the parameters corresponding to figure 3.3c, the increase of confinement being now the cause of the flow stabilization.

Such spatiotemporal diagrams can be constructed for a wide range of parameters, in order to determine the stable and unstable regions on a $(-\Lambda, h)$ plane. The results are gathered in figure 3.5. Note that for each simulated configuration, the CPU time t_c has been included between $48 < t_c < 240$ hours. According to intuition, an increase

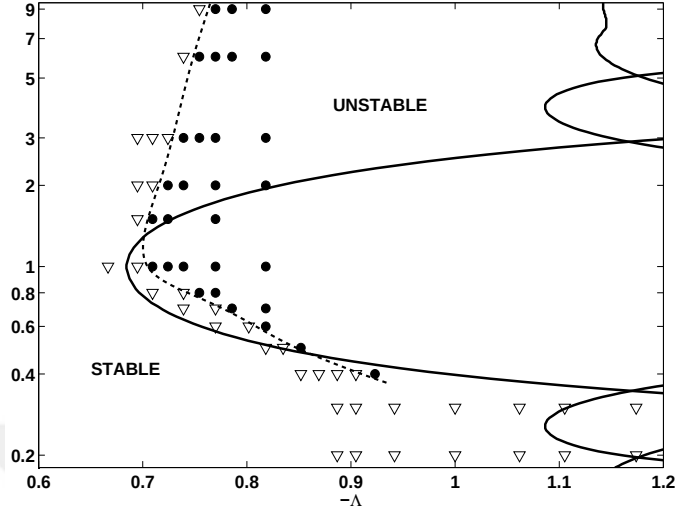


Figure 3.5: Stability diagram on the $(-\Lambda, h)$ plane, for $Re = 100$. The circles and the triangles represent respectively globally unstable and globally stable configurations. The continuous line depicts the limit obtained by Juniper [54] for an inviscid flow. Note that not all simulated configurations have been represented in order to avoid confusion.

of the co-flow parameter, by means for instance of the inner flow velocity, tends to render the flow convectively unstable and, thus, globally stable. The influence of confinement is however more complex. The highest co-flow parameters sustaining global instability are obtained for $h \approx 1$, when the inner and the outer flows have the same thickness, while stronger and weaker confinement tend to stabilize the flow. This counter-intuitive fact has been indeed predicted by Juniper, as seen in figure 3.5, where his theoretical limit obtained for inviscid flow has been reproduced from Juniper (2006) [54] (continuous line). Note that, according to the recent study of Rees & Juniper (2010) [91], the theoretical curve prevailing for the viscous instabilities of a top-hat profile at $Re = 100$ is likely to be shifted to higher values of the velocity ratio.

According to Juniper's predictions, weaker co-flows are required to destabilize the flow at both strong and weak confinements, $h \ll 1$ and $h \gg 1$. Whereas Juniper predicted that counterflow was required for instability at $h > 2.5$, our calculations show however a weaker stabilization, with global instability even observed with consequent co-flow ($-\Lambda > 0.76$) at $h = 9$. Note also that for very strong confinement, $h < 0.4$, globally unstable flows were not encountered, in contrast to Juniper's predictions.

We can conclude to a significant effect of viscosity for this Reynolds number. A higher Reynolds number is therefore analyzed in next section in order to approach the inviscid case studied by Juniper (2006) [54].

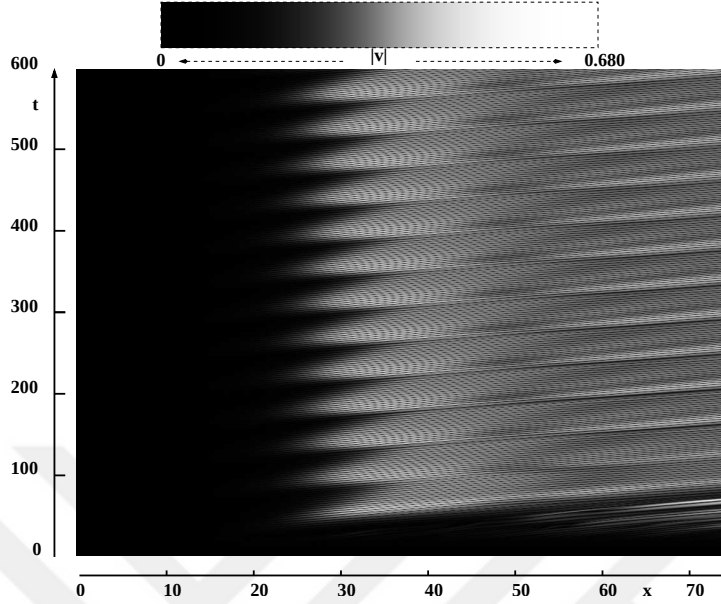


Figure 3.6: Spatiotemporal diagram of absolute value of the velocity vertical component (v) at $y = 0$ for $h = 6$, $\Lambda = -0.600$ and $Re = 500$. The instability persists in the domain, but is characterized by a low frequency modulation.

3.3.2 $Re = 500$

As for $Re = 100$, the stability of the wake flows can be determined as a function of the co-flow and confinement parameters by inspection of spatio-temporal diagrams. In addition to the typical behaviors depicted in figure 4 for $Re = 100$ where the flow is either stable or settles into a nonlinear limit cycle and displays a steady front (a situation that we will refer to as Von Kármán Instability (VKI)), we have also observed much more complicated behaviors for certain parameter settings. A typical example is provided by the spatiotemporal diagram depicted in figure 6 for $h = 6$ and $\Lambda = -0.600$. Although the wave front is not advected towards the outlet, it is not fixed in space and in time, but vacillates around a mean position, displaying in this particular case a periodic low frequency modulation. This behavior is in strong contrast with spatiotemporal diagrams typical of VKI instability (figure 3.4a) where the front is steadily fixed at a further upstream location. We will refer to it in the sequel as Front Vacillation Instability (FVI).

3.3.2.1 Vacillations

Figure 3.7a shows time Fourier spectra of a vertical velocity probe $v(x = 23.15, y = 0, t)$ for three different configurations. All of them are unstable, as seen in figure 3.10, but the weak confinement-weak co-flow case ($h = 6$, $\Lambda = -0.6$) is characterized by the FVI. The stronger confinement case, $h = 1$, depicts a classical self-sustained

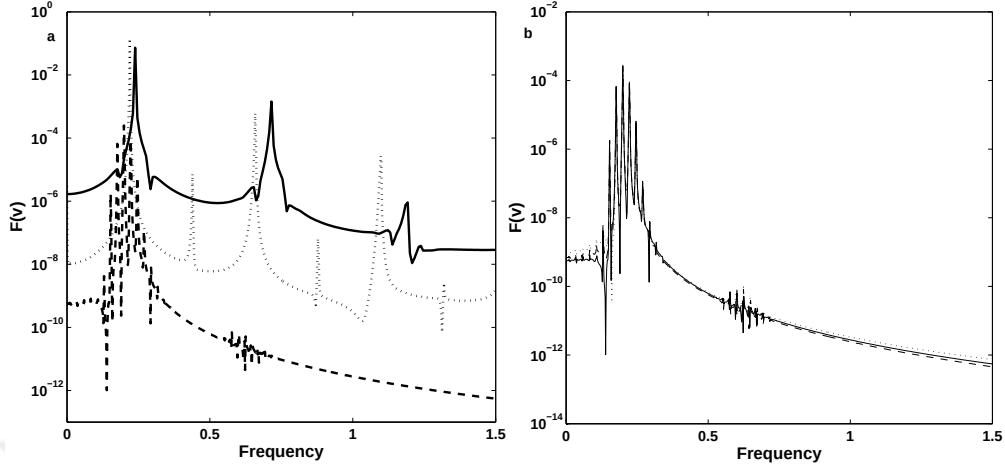


Figure 3.7: (a) Fourier spectra of $v(x = 23.15, y = 0, t)$ for three different unstable configurations at $Re = 500$. Continuous line: $h = 6, \Lambda = -0.667$; Dotted line: $h = 1, \Lambda = -0.667$; Dashed line: $h = 6, \Lambda = -0.600$. The latter configuration shows a low frequency modulation. The case with stronger confinement $h = 1$ only shows the main high frequency, $f \approx 0.2$. (b) Fourier spectra for $h = 6, \Lambda = -0.600$ and $Re = 500$. Continuous line: $L = 75$; Dotted line: $L = 101.5$; Dashed line: $L = 75$ plus an additional sponge subdomain. The low frequency modulation does not change when varying the domain length or the outflow treatment. Note that for the y -axis the scale is modified between figures (a) and (b).

nonlinear global mode instability (VKI), similar to that observed in figure 3.4a, with the front wave fixed in space and in time and, therefore, presents only a main high frequency, $f \approx 0.2$. For $h = 6, \Lambda = -0.667$, the flow first displays a transient low frequency modulation before reaching a limit cycle. The spectral analysis, carried out on the established state, reveals peaks located at odd multiple of the main frequency.

For $h = 6, \Lambda = -0.6$ one observes a broadened spiky peak, resulting from the low frequency modulation observed in figure 3.6 through an enriched beating mechanism between several equispaced frequency peaks. The gap between two consecutive peaks equals to $\Delta f \sim 0.025$, which corresponds to a modulation period $T \sim 40$, as revealed from figure 3.6. The observed modulations are reminiscent of a Benjamin-Feir like sideband instability [98] of the nonlinear wave prevailing in the downstream region of the flow.

The first natural question is to enquire if these vacillation modes are a robust feature or if they are highly sensitive to discretization, box length or outflow boundary conditions. We have checked the grid-convergence by using a value of N_y as high as $N_y = 270$. Depending on the value of N_y , the amplitude and period of the low frequency modulation are slightly modified, but the modulation is robust and persists in the domain. We have checked the influence of the exit condition on this

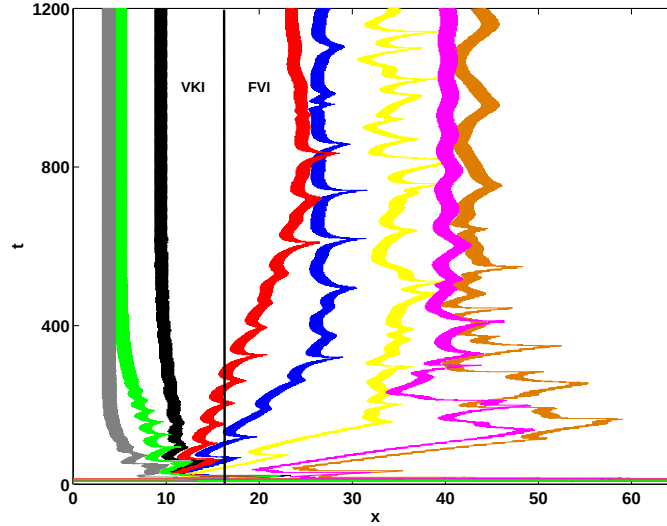


Figure 3.8: Wave front positions for $h = 2$ and different velocity ratio: gray line depicts $\Lambda = 0.695$, green $\Lambda = -0.681$, black, $\Lambda = -0.667$, red $\Lambda = -0.653$, blue $\Lambda = -0.639$, yellow $\Lambda = -0.600$, magenta $\Lambda = -0.538$, and brown -0.504 . The jump between the black and red curves delimits the limit between VKI and FVI.

feedback mechanism: the instability is still present, displaying the same low frequency modulation, as shown in figure 3.7b. Furthermore, the location of the oscillating wave-front is seen to vary within a few % as the length or exit conditions are modified. While this excludes a numerical instability due to the outlet conditions, it does not clearly identify the governing feedback mechanisms in play.

3.3.2.2 Front location

While the spectral signatures of VKI and FVI differ significantly, as discussed above, they can also be discriminated based upon the location of the front. For small enough co-flow parameter, flows experiment von Kármán instability with a front located in vicinity of the inlet. As the co-flow parameter is increased, the steady front slowly migrates downstream until a sudden transition happens towards a vacillating front located much further downstream. This scenario is readily observed on figure 3.8 which displays for $h = 2$ the front locations in the (x, t) plane for several values of the velocity ratio. The location of the front x_f is defined at any time as the first point where the y -component of the velocity on the centerline reaches the value of $v(x_f, 0, t) = 10^{-3}$. As the VKI/FVI transition is passed, the front location experiences an important jump downstream as it simultaneously starts to vacillate. This singular behaviour is illustrated in figure 3.9 where the position of the front x_f is depicted as a function of the velocity ratio prescribing the confinement. One can note

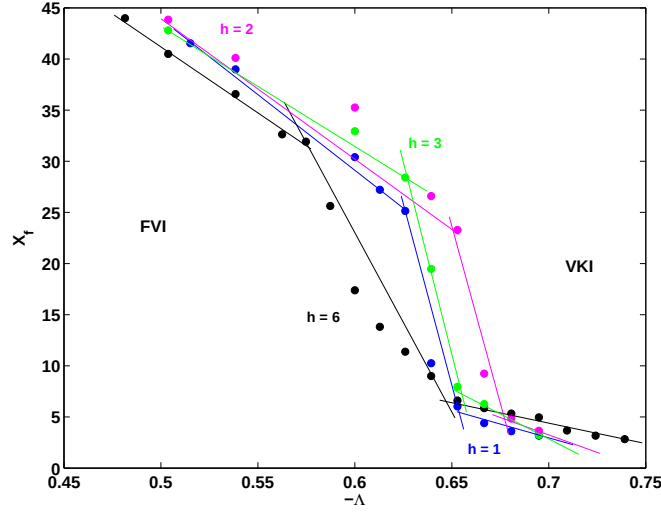


Figure 3.9: Position of the front x_f with respect to $-\Lambda$ for different confinement. The black line depicts $h = 6$, the green $h = 3$, the magenta $h = 2$, and the blue $h = 1$. Each confinement shows a jump which delimitates the threshold between VKI and FVI.

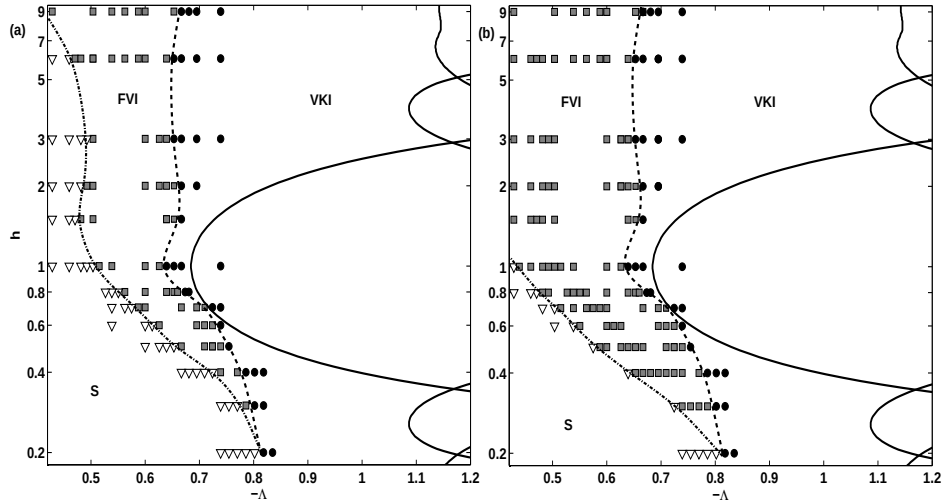


Figure 3.10: Stability diagram on the $(-\Lambda, h)$ plane, for $Re = 500$ and (a) $L = 75$ and (b) $L = 152$. The circles and the triangles represent globally VKI unstable and globally stable configurations, respectively, while the squares depict the FV globally unstable modes. The continuous line depicts the limit obtained by Juniper [54] for an inviscid flow. While the VKI/FVI limit does not depend on the size of the domain L , the FVI/S one is significantly influenced by its variation. Note that some simulated configurations have not been represented in order to avoid confusion.

that, for each confinement, starting from weak co-flow the position of the front x_f linearly increases in the region characterized by the classical Von Kármán instability. When the global modes associated to VKI start to be deactivated and thus, those characterized by vacillations do dominate, the position of the front increases suddenly creating a jump. After this transition due to the interaction of the modes, the position of the front returns to a linear increase, however with a different slope than in the VKI region. Moreover, the slope of the jump depends on the confinement, in particular, stronger confinements are characterized by steeper jumps. We have used in this study the co-flow parameter at which this jump takes place as the threshold for VKI/FVI transition and monitored its dependance as a function of the confinement ratio. However, it should be noted that for larger h , the position of the jump cannot be easily discerned due to the smoothness of the curve, as seen considering the black curve in figure 3.9, i.e. the curve for $h = 6$. For this reason, for $h \geq 3$, we estimate the error on the position of the limit to be 10%.

3.3.2.3 Stability diagram

Figure 3.10 gathers the results for $Re = 500$ for two different sizes of the domain (a) $L = 75$ and (b) $L = 152$ at $Re = 500$ and delineates three distinct regimes in the $(\Lambda - h)$ plane: the von Kármán (VKI) (black filled circles), the front vacillation instability (FVI) (grey filled squares) and the region where the flow is stable (S) (empty triangles). Comparing the stability diagrams of figure 3.10, one can note that the transition curve FVI/VKI does not depend on the size of the domain L (nor on the OBC). Conversely although, as shown in 3.3.2.1, the FVI is a robust feature, it is less clear for the FVI/S transition which is largely sensitive to the computational domain length L but not to the nature of the OBC. Particularly when the length of the domain is increased, the unstable character of vacillations is enhanced. This behaviour is also confirmed by further tests with $L = 34$ and $L = 101.5$. Furthermore, a larger amount of confinement seems to have a damping influence on vacillations, since the FVI region becomes narrower for strong confinement, and, finally does not exist for $h = 0.2$.

Comparing figure 3.10a to figure 3.5, the stabilizing role of viscosity can be identified: a higher co-flow parameter is required to stabilize the flow at $Re = 500$ than at $Re = 100$. The transition curve is therefore shifted to lower values of $-\Lambda$. Another effect of lowering the viscosity is to extend the unstable region towards very strong confinements (for $h \leq 0.4$).

Furthermore, one can note that the FVI/VKI transition is rather close to [54]’s inviscid limit and bears some resemblance with the A/C instability threshold obtained by [90] and [?] by means of a local stability analysis of profiles including shear layer thickness. Since one can conjecture that a local analysis is not able to predict the instability connected to vacillations, see section ?? for further details, it is more relevant to compare Juniper’s threshold to the FVI/VKI limit rather than to the S/FVI one.

As for $Re = 100$ (figure 3.5), a maximum in the VKI/FVI transition curve is observed for a confinement value of order 1 for this higher value of the Reynolds number, as predicted by Juniper’s inviscid analysis. The transition curve between VKI and FVI does however not strictly follow the inviscid prediction pertaining to the inlet flow. The viscous transition curve for the top hat profile at inlet at $Re = 500$ is likely to be only slightly shifted to higher values of the velocity ratio, since in presence of a slip boundary condition, the conclusion drawn from Monkewitz (1988) [75] that the absolute/convective properties of unconfined wake instabilities becomes almost independent of Re for $Re > 200$ are expected to remain valid, a trend recovered by Rees & Juniper (2010) [91]. It is noteworthy that some convectively unstable inlet velocity profiles, following the theoretical studies of Juniper, are seen to yield globally unstable flows. To be more precise, the inlet flows associated to FVI-S transition line are always convectively unstable, excluding the flow dynamics to be governed by the inlet-driven global modes described by Couairon&Chomaz (1999) [21]. There is no contradiction because the parallel flow $\mathbf{u}_0(x, y)$ is not a solution of the Navier-Stokes equations so that the steady solution is a non-parallel flow $\mathbf{u}_b(x, y)$. The non-parallel character of the flow seems to play a key role and weakly non-parallel steep nonlinear global modes [85] are still plausible candidates for the VKI modes. This point is analyzed in more details in section 3.3.4.

3.3.3 3D simulations

The results shown in sections 3.3.1 and 3.3.2 were obtained for a two-dimensional domain and are therefore only fully justified if the influence of the third dimension is negligible. The objective of this section is to confirm that this is indeed the case. To this end, 3D computations have been carried out in a domain of thickness $E = 8$ equal to four times the inner wake width. This was suggested by the fact that for the wake of a cylinder, a channel depth equal to four times the inner wake length was found sufficient to see 3D structures developing [108]. For the initial condition, we have used the result obtained at the end of a 2D computation, plus an additional perturbation. Two kinds of perturbation have been tried, fully random or E -periodic, with an amplitude of 10^{-2} . Note that the inlet boundary condition remains the steady 2D velocity profile.

Figure 3.11a shows a spatiotemporal behavior of the absolute value of the z -component of velocity, $|w|$, for $h = 1$, $\Lambda = -0.667$ and $Re = 500$ at a specific point of the (y, z) plane. The periodic sinuous perturbation of the initial condition develops a 3D effect on the flow, depicted by an increase of $|w|$. However, the 3D perturbation is advected away and the flow comes back to a two-dimensional saturated state, or at least to a quasi-2D state such that the influence of the z -direction may be neglected. Using a random forcing rather than a periodic perturbation yields the same conclusion. It could be useful to remember that the configuration analyzed in figure 3.11a is globally unstable. However, the same behavior of w has been observed for different 3D simulations, varying confinement and velocity ratio resulting

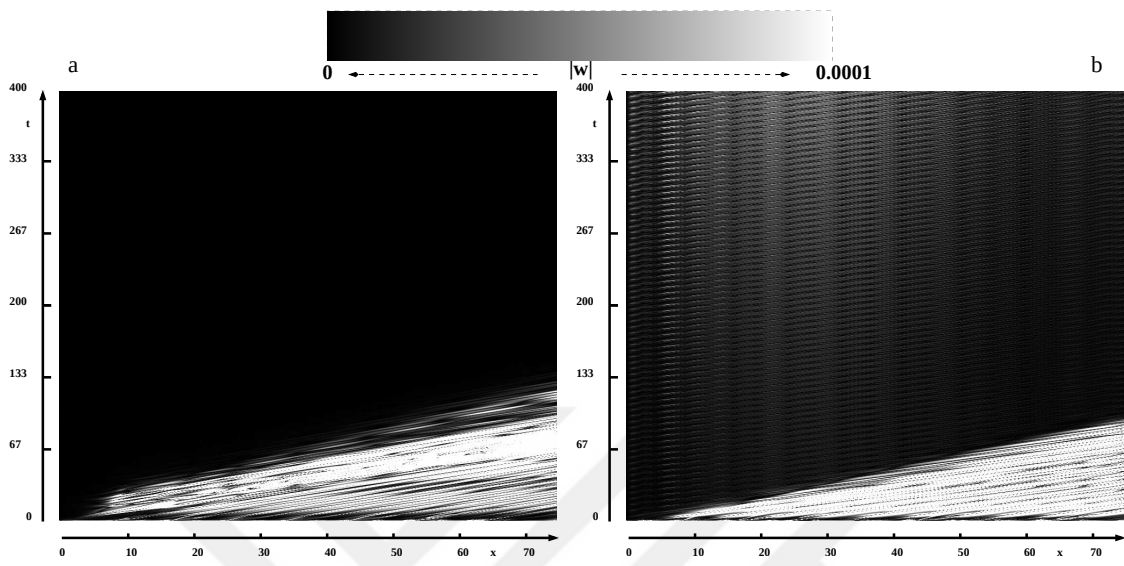


Figure 3.11: Spatiotemporal diagram of absolute value of z component of velocity (w) for $Re = 500$, $h = 1$, and (a) $\Lambda = -0.667$, (b) $\Lambda = -1$, with (y, z) coordinates equal to $y = -1.335$ and $z = -1$. In (a) The 3D perturbation is advected away, while in (b) the initial 3D perturbation persists.

in both stable and unstable configurations close to the limit shown in figure 3.10. Such computations are however costly, owing to their 3D nature and to the required simulation time to ensure the absence of amplification of any residual perturbation.

To confirm that the influence of the third dimension is in any cases negligible, we have considered a situation a priori favorable to the emergence of a 3D flow. This is the case for $h = 1$, $\Lambda = -1$, for which $U_1 = 0$ guarantees no inner flow at the inlet of the channel. Figure 3.11b shows the corresponding spatio-temporal diagram, clearly showing a well established 3D flow. However, in the considered domain, such that $L = 75$, the amplitude of the spanwise component w remains three order of magnitude smaller than the mean streamwise velocity. Considering that the 3D effects are detectable only far from the limit shown in figure 3.10, it is therefore reasonable to assume that the 2D approach is fully justified in the present study.

3.3.4 Weakly non parallel evolution of the base flow

3.3.4.1 Influence of velocity ratio

In the base flow depicted in figure 3.2a, the velocity profile is seen to be modified by the effect of viscosity. Viscous effects tend to flatten the profile as a consequence of momentum diffusion and the flow is no-longer parallel. This effect causes the velocity ratio $\Lambda(x) = (U_{\min} - U_{\max}) / (U_{\min} + U_{\max})$ to vary along the domain. Since a constant

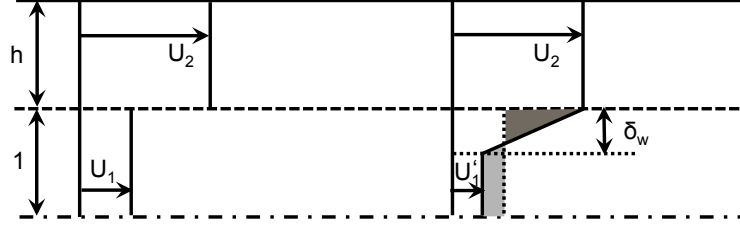


Figure 3.12: Physical mechanism at the origin of the wake deceleration

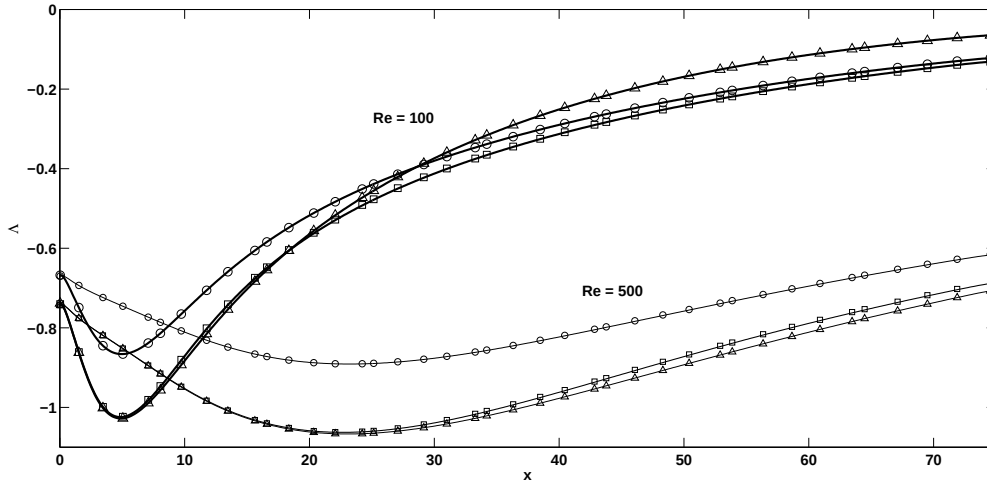


Figure 3.13: Evolution of the velocity ratio parameter Λ along the domain. The bold lines represent the configurations for $Re = 100$, the thin lines for $Re = 500$. $-\square-$ depicts $h = 1$ and $\Lambda = -0.739$ a), $-\triangle-$ depicts $h = 0.7$ and $\Lambda = -0.739$ b), and $-\circ-$ depicts $h = 1$ and $\Lambda = -0.667$ c). The three configurations show a minimum Λ at $x_{\min} \approx 5$ at $Re = 100$ and at $x_{\min} \approx 25$ at $Re = 500$.

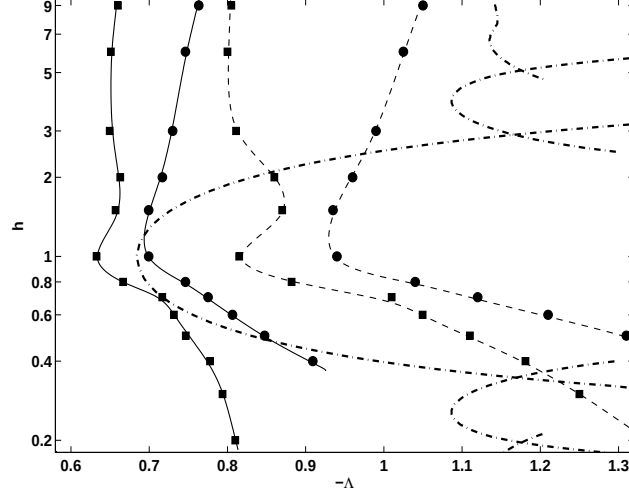


Figure 3.14: Comparison between the limit curves for the two different Reynolds numbers. The continuous lines depict the limit lines referred to the entry velocity ratio Λ , while dashed ones represent the limit referred to $\Lambda_{\min}$. Circles and squares depict the two Reynolds numbers $Re = 100$ and $Re = 500$, respectively. [54] limit curve is depicted with the dash-and-dot line, which represents limit curve for a parallel flow at $x = 0$.

velocity profile corresponds to $\Lambda = 0$ and, remembering that for a wake $\Lambda < 0$, it could be expected that Λ monotonically increases with x . Figure 3.2a, obtained for an unstable configuration, on the contrary suggests that the co-flow parameter Λ may have a non-monotonous evolution along the domain: the velocity profile displays a clear velocity defect at $x = 2$ indicating that, although the initial configuration at $x = 0$ is a co-flow, a counterflow is present in a consistent part of the domain.

This wake retardation can be interpreted as the consequence of the advection-diffusion of the vorticity layer formed by the mixing layer. At the inlet, the vorticity distribution is infinitely thin but it then evolves downstream under the joined action of diffusion and advection, which yields to the development of boundary layers on both sides of the mixing layer. Since the advection is much stronger in the outer stream than in the inner wake, the external boundary layer is much thinner than the inner one and we may at leading order assume that the shear layer is only growing in the diffusion-dominated inner region while it does not penetrate in the advection dominated outer region. As seen in figure 3.12, this oversimplified assumption can only remain valid as the shear layer thickness remains less than the inner stream half width 1. However this figure readily makes clear that the central flow has to decelerate to ensure mass conservation.

Using the notation of figure 3.12 and applying the conservation of mass at a given

streamwise position where the shear layer thickness equals δ_w , one obtains:

$$U'_1 = U_1 \frac{2 - \delta_w U_2 / U_1}{2 - \delta_w} \quad (3.25)$$

This condition is satisfied if the two grey shaded areas are of equal surface areas. The velocity ratio also evolves accordingly

$$\Lambda' = \frac{\Lambda}{1 - \delta_w(1 - \Lambda)/2}, \quad (3.26)$$

and therefore decreases with increasing δ_w .

More quantitatively, figure 3.13 shows the evolution of the velocity ratio $\Lambda = (U_{\min} - U_{\max}) / (U_{\min} + U_{\max})$ along the horizontal direction for the three configurations considered in figure 3.3 and for the two different values of Reynolds number. One observes a minimum of the velocity ratio for all configurations. For $Re = 100$, the minima are reached at $x_{\min} \approx 5$, while for $Re = 500$ they are attained at $x_{\min} \approx 25$. These minima depend above all on the velocity ratio, while the confinement has not a significant influence. When $x > x_{\min}$, the velocity ratio starts to grow and tends to zero at large enough x . For a given flow, the minimal velocity ratio represents the locally most absolutely unstable region in the domain.

Thus, two different significant velocity ratios may be considered for a given configuration: the entry velocity ratio, $\Lambda(0)$, and the minimal velocity ratio, $\Lambda_{\min}$. Following the steep nonlinear front criteria of Pier *et al.* (2001) [85], which ensures global instability as soon as absolute instability is attained somewhere in the domain, we can reinterpret the limit curves obtained in figures 3.5 and 3.10 referring to $\Lambda_{\min}$. Figure 3.14 compares the limit curves for different Reynolds number as a function of both significant velocity ratio and the curve obtained by Juniper (2006) [54] for an inviscid flow (continuous line). Further assuming $Re = 500$ is high enough for viscous effects to be not dominant (see discussion in section 3.3.2), we would therefore expect that Juniper's results are closest to the limit curve for $Re = 500$ and $\Lambda_{\min}$, because this velocity ratio enhances at most the locally absolute instability for the non-parallel wake. This is not true however and the difference between the two curves is significant, invalidating at first sight a global stability prediction based on a steep nonlinear global mode anchored at the convective/absolute transition location.

3.3.4.2 Influence of shear layer thickness

Another interesting difference between our simulations and the analysis of [54] may help to further explain this contrast: the absence of symmetry with respect to $h = 1$ in logarithmic scale for the limit curves obtained with DNS, while this symmetry is present in the top-hat wake model. At $Re = 500$, the vacillations at weak confinements strongly break the symmetry, see figure 3.14, but the FVI/VKI limit, at $Re=500$, and the VKI/S one, at $Re=100$, are also not symmetric. The symmetry noticed by Juniper [54, 56] is due to the absence of a finite shear layer thickness, while the

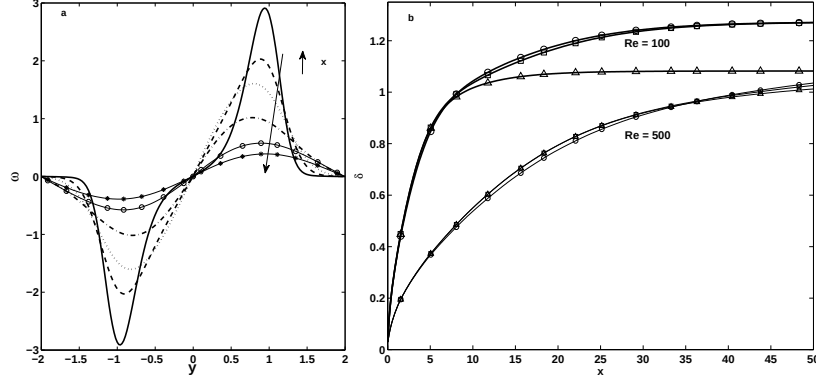


Figure 3.15: (a) Vorticity profiles of the base flow for $h = 1$, $\Lambda = -0.739$ and $Re = 100$. The profiles are given for $x = \{2.26, 4.79, 8.09, 17.27, 33.90, 48.50\}$, the amplitudes are decreasing with respect to x . The arrow depicts the evolution of the base flow along the x -direction. The vorticity profile at $x = 0$, which is a regularization of a pair of opposite Dirac-delta functions, is not shown since its peaks are out of scale. (b) Evolution of the shear layer thickness δ_w along the domain. The bold lines represent the configurations for $Re = 100$, the thin lines for $Re = 500$. $-\square-$ depicts $h = 1$ and $\Lambda = -0.739$ a), $-\triangle-$ depicts $h = 0.7$ and $\Lambda = -0.739$ b), and $-\circ-$ depicts $h = 1$ and $\Lambda = -0.667$ c) (same as figure 9b).

presently simulated flows clearly exhibit a shear layer thickness, as shown in figure 3.2a. Figure 3.15a illustrates the vorticity profiles of the base flow for $h = 1$ and $\Lambda = -0.739$. At each x -location, the vorticity profile depicts two peaks around $y = \pm 1$, indicating the inflection points of the corresponding velocity profiles, which are at the origin of the destabilization of the wake [28].

The influence of the finite shear layer thickness on the strength of the inflection point instability has been addressed in the literature: while it is well known that the maximum temporal growth-rate is proportional to the amplitude of the vorticity extremum, it has also been shown [75, 91] that the shear layer thickness $\delta_w = (U_{\max} - U_{\min})/|dU/dy|_{\max}$ [75], which is inversely proportional to the vorticity extremum, also strongly modifies the absolute/convective properties of a parallel wake flow. In particular, using a family of profiles parametrized by the shear layer thickness, Monkewitz (1988) [75] has shown that there exists an optimal value of the shear layer thickness $\delta_{w,opt}(h = \infty) \sim 0.7$ for which absolute instability is maximal for inviscid unconfined wakes and noticed a weak influence of Re when $Re > 100$. We have shown in the second chapter (see figure 2.22) that this optimal value is indeed decreasing when confinement is added to Monkewitz's inviscid model. According to our analysis, for $h = 1$ and $h = 0.7$, one has $\delta_{w,opt}(h = 1) \sim 0.2$ and $\delta_{w,opt}(h = 0.7) \sim 0.15$, respectively. We also noted that the absolutely unstable character of the flow was strongly altered when δ_w was less than this optimal shear layer thickness.

Figure 3.15b shows the evolution of shear layer thickness δ_w along the horizontal direction. One observes a monotonically increasing behavior for all configurations. For both confinement ratios under consideration $h = 0.7$ and $h = 1$, when the flow reaches the $\Lambda_{\min}$ -region, the shear layer thickness is much larger ($\delta_w \sim 0.85$ as seen in figure 3.15b) than $\delta_{w,opt}(h)$ respectively equal to 0.15 and 0.2. The local velocity profile does not reach simultaneously the velocity ratio and shear layer thickness best promoting absolute instability. This mismatch of optimal Λ and δ_w may elucidate the difference between our $Re = 500$ FVI/VKI transition curve transition curve and the theoretical absolute/convective limit relative to the minimal velocity ratio $\Lambda_{\min}$ reached throughout the domain rather than the direct influence of viscosity on the parallel absolute/convective transition threshold.

3.3.4.3 Fitting the profiles extracted from DNS

In previous sections 3.3.4.1 and 3.3.4.2, local velocity profiles extracted from the base flow along the domain have been computed together with their local characteristic parameters: velocity ratio $\Lambda(x)$ and shear layer thickness $\delta_w(x)$. It was worthwhile to try a comparison with the confined Monkewitz's profiles described in section 2.2.1.2, for which the convective (C)/absolute (A) instability limits have been obtained.

It is then possible to plot the locus of the base flow profile in the (δ_w, Λ) plane, as x is varied from the inlet to the outlet. Figure 3.16 shows this locus for $h = 1$ and four values of the entry velocity ratio Λ_e . In addition, the bold line represents the C/A transition curve duplicated from figure 2.22. The configurations depicted by crosses and asterisks were shown to be globally VKI and their locus of the base flow is seen to go through a consistent absolutely unstable region. The two other cases, which were shown to be globally FVI, are depicted by circles and squares. While for $\Lambda_e = -0.538$, the flow is convectively unstable throughout the whole domain, the FVI flow obtained for $\Lambda_e = -0.6$ is seen to be marginally absolutely unstable in some portion of the flow.

The overall agreement between local C/A stability theory and nonlinear global instability is therefore excellent, in view of the differences in viscosity and the assumption of the fitted profiles. The influence of viscosity was shown by Monkewitz [75] to lower the C/A transition and a similar behavior is expected in this confined situation, which may suppress the incursion of the locus for $\Lambda_e = -0.6$ into the absolutely unstable region.

Finally, we have determined the frequency of the shedding vortex by means of a Fourier analysis for $\Lambda = -0.667$ and $\Lambda = -0.739$ to be respectively $\omega_{NL} = 1.44$ and $\omega_{NL} = 1.47$. In the latter case the absolute frequency at the onset of the absolute region (for $\delta_w \sim 0.12$) is $\omega_{C/A} = 1.51$, while in the former one is $\omega_{C/A} = 1.57$ at the entry (for $\delta_w \rightarrow 0$). The errors are, therefore, respectively less than 5% and 8%, suggesting the reliability of the local stability analysis of Monkewitz's family of profiles. According to Meliga *et al.* [72], there might be a substantial difference between absolute growth rates pertaining to exact profiles extracted from velocity

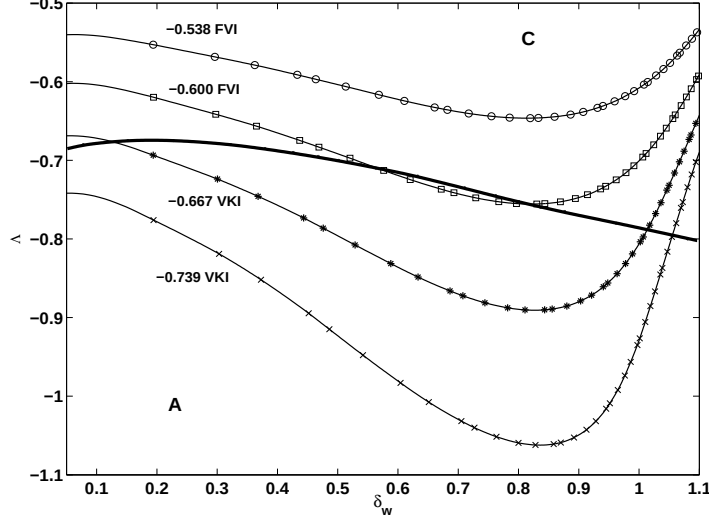


Figure 3.16: Comparison between the inviscid stability limit for the confined Monke-witz's profile (bold line) and the results of DNS for $h = 1$, $Re = 500$ and different entry velocity ratio on the plane (δ_w, Λ) . $- \times -$ depicts $\Lambda_0 = -0.739$, $- * -$ $\Lambda_0 = -0.667$, $- \square -$ $\Lambda_0 = -0.600$, and $- \circ -$ $\Lambda_0 = -0.538$.

field and to fitted ones.

3.3.4.4 Local stability analysis

We have seen in sections 3.3.1 and 3.3.2.3 that there exists several differences between the theoretical limit obtained by Juniper [54] for an inviscid top hat profile and the limit obtained by DNS. However in section 3.3.4.2 the mismatch between Λ_{min} and δ_{opt} is conjectured to have a more relevant role to explain this gap than the direct influence of viscosity.

In order to confirm this intuition, we examine the local stability of several velocity profiles extracted from the base flow at different axial locations. Assuming a normal mode ansatz $e^{i(kx - \omega t)}$, with ω the complex frequency and k the complex wavenumber, the spatiotemporal stability analysis is based on the resolution of the Orr-Sommerfeld equation [28], which can be discretized and rearranged in the form of a generalized eigenvalue problem [91] and solved using MATLAB's function *eig*. The discretization is conducted on the same collocation points y_i as those used in the DNS, *i.e.* the Chebyshev's grid points with the same mapping. In order to ensure the convergence of the procedure, different number of collocation points N_y were tested, between 80 and 130. Dominant eigenvalues are affected by an error less than 5%. A local dispersion relation, relating the complex frequency ω to the local complex wavenumber k is thereby obtained.

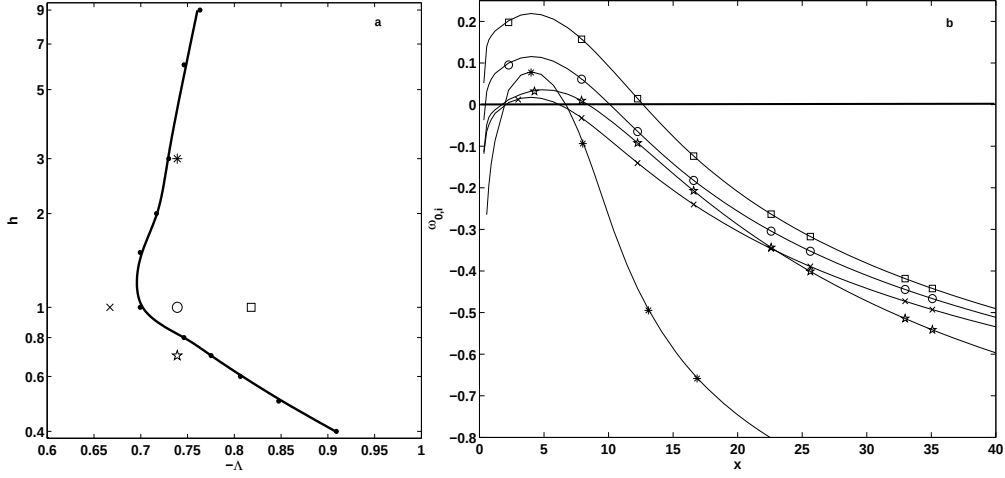


Figure 3.17: (a) Location of the following configurations with respect to global unstable/stable limit curve for $Re = 100$: $\circ$ represents $h = 1$ and $\Lambda = -0.739$, $\square$ represents $h = 1$ and $\Lambda = -0.818$, $\times$ represents $h = 1$ and $\Lambda = -0.667$, $\star$ represents $h = 0.7$ and $\Lambda = -0.739$, and $*$ represents $h = 3$ and $\Lambda = -0.739$ and (b) behavior of the absolute growth rate $\omega_{0,i}$ along the domain for such configurations. The numerical error-bound on the absolute growth rate is 5%.

As reviewed in Huerre&Rossi (1998) [51], the spatiotemporal stability analysis consists of finding the absolute mode corresponding to the admissible saddle point of highest imaginary part of the frequency ω_i in the complex wavenumber k -plane. The value of imaginary part of frequency at the saddle point is called absolute growth rate $\omega_{0,i}$ and it determines the nature of local instability of velocity profile: convective if $\omega_{0,i} < 0$ or absolute if $\omega_{0,i} > 0$. The iterative numerical procedure introduced by Deissler (1987) [26] was used to determine the evolution of $\omega_{0,i}(x)$ with respect to the streamwise coordinate. See section 2.5.2 for further details.

Figures 3.17a and 3.18a depict, for $Re = 100$ and $Re = 500$, the position in the plane $(-\Lambda, h)$ of the configurations for which local stability analysis were conducted, while the evolution of the corresponding absolute growth rate $\omega_{0,i}(x)$ along the domain are shown in figures 3.17b and 3.18b, respectively.

For $Re = 100$, the globally unstable configurations, depicted by asterisks, circles and squares have a significant absolutely unstable region (with positive $\omega_{0,i}$). An absolutely unstable region exists also for globally stable flows (marked by cross and star), with however both a less extended absolute region and a lower maximum value of absolute growth rate than those obtained for globally unstable configurations. This illustrates that the existence of a local region of absolute instability of finite extent is not a sufficient condition for global instability. The flow being far from parallel at $Re = 100$, the local analysis cannot predict accurately its global stability under the weakly nonparallel flow assumption. The qualitative trends are however correctly

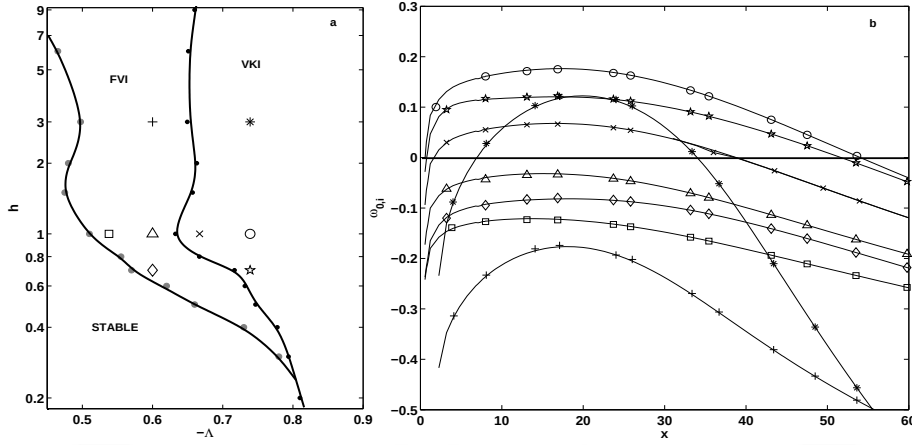


Figure 3.18: (a) Location of the following configurations with respect to global unstable/stable limit curve for $Re = 500$: $-\circ-$ represents $h = 1$ and $\Lambda = -0.739$, $-\times-$ $h = 1$ and $\Lambda = -0.667$, $-\triangle-$ $h = 1$ and $\Lambda = -0.600$, $-\square-$ $h = 1$ and $\Lambda = -0.538$, $-\star-$ $h = 0.7$ and $\Lambda = -0.739$, $-\diamond-$ $h = 0.7$ and $\Lambda = -0.600$, $-*-$ $h = 3$ and $\Lambda = -0.739$, and $-+-$ $h = 3$ and $\Lambda = -0.600$ and (b) behavior of the absolute growth rate $\omega_{0,i}$ along the domain for such configurations. The numerical error-bound on the absolute growth rate is 5%. For profiles closer to inlet the error is slightly larger, but still less than 10%.

captured.

Conversely, for $Re = 500$, only the globally VKI flows present a region for which positive absolute growth rates are found. This shows that the local stability analysis seems to correctly predict the appearance of the VKI in this lower viscosity case. This is not surprising since flows at $Re = 500$ can be considered weakly non-parallel and, thus, closer to the asymptotic limit of weakly non-parallel approximation, imposed by the local stability analysis.

However, note that the maximum value of $\omega_{0,i}$ is negative for all globally FVI configurations. The front vacillation instability is therefore observed when the flow is convectively unstable throughout the domain and cannot be predicted by the steep nonlinear global mode theory. This shows that the existence of a finite region of absolute instability is not a necessary condition for global instability of FVI type.

Furthermore, for VKI cases we have analyzed the shedding frequencies obtained from DNS resorting to the Fourier analysis already described in section 3.3.2 (see figure 3.7). Following the theoretical limit curve [54], if the inlet profiles are convectively unstable these frequencies have to be compared with the absolute frequency $\omega_{0,r}$ at the onset of absolutely instability region as predicted by Pier *et al.* (2001) [85], otherwise, if it is absolute unstable, they have to be compared with $\omega_{0,r}$ at the inlet [21].

Table 3.1 reports the comparison between the two analyses. This time, the agree-

Re	h	Λ	ω GL	ω LOC	X	ERROR %
100	3	-0.739	1.19	1.23	2.0	3.3
100	1	-0.739	1.28	1.49	0.5	14.1
100	1	-0.818	1.45	1.56	0	7.1
500	3	-0.739	1.74	1.30	6.4	33.8
500	1	-0.739	1.47	1.55	0.5	5.2
500	1	-0.667	1.44	1.47	1.6	2.0
500	0.7	-0.739	1.28	1.89	0.7	32.2

Table 3.1: Comparison between the frequencies obtained by local and global analysis.

ment for $Re = 100$ is remarkable, while for $Re = 500$ only for the cases with $h = 1$ do the predictions not fail. We can hypothesize that the presence of the vacillations, also if they are partially hidden by the Von Kármán global mode, modifies the value of the frequency in the direct numerical simulations, leading to the failure of the local stability analysis prediction.

3.3.5 Towards the understanding of vacillations

In this section the surprising phenomenon of vacillations is focused on in order to try to give an interpretation of this mechanism. Two analyses are conducted at weak confinement: the first one is a local stability analysis discussed in section 3.3.5.1, while the second one is a linear global analysis illustrated in section 3.3.5.2.

3.3.5.1 Vacillations and local stability analysis

The local stability analysis on the top hat profile and, moreover, on both broken-line and continuous profiles including shear layer thickness conducted by Juniper [54, 55], Rees and Juniper [91] and also in the second chapter of this thesis cannot predict the surprising mechanism of vacillations (see in particular section 3.3.4.3). For the local co-flow parameter values reached by the base flow at typical inlet velocity ratios $\Lambda = -0.55$, where FVI is observed, the flow is, whatever wake model used to fit the local profile, convectively unstable. However, these profiles are only a fit to the exact velocity profiles, and the failure of the local stability analysis to predict the FV instability for strong co-flow wakes has to be confirmed by a local stability analysis using profiles extracted by the DNS, as explained in section 3.3.5.1. In section 3.3.4.4 we have already conducted a local stability analysis which seems to confirm this failure. Now, we want to focus on weak confinement configurations since they represent these cases for which the discrepancy between local/global stability analysis seems to be the most significant.

In figures 3.19 and 3.20 we report the results of the local stability analysis conducted on two weak confinement cases, i.e. $h = 3$ and $h = 6$, respectively. In figure

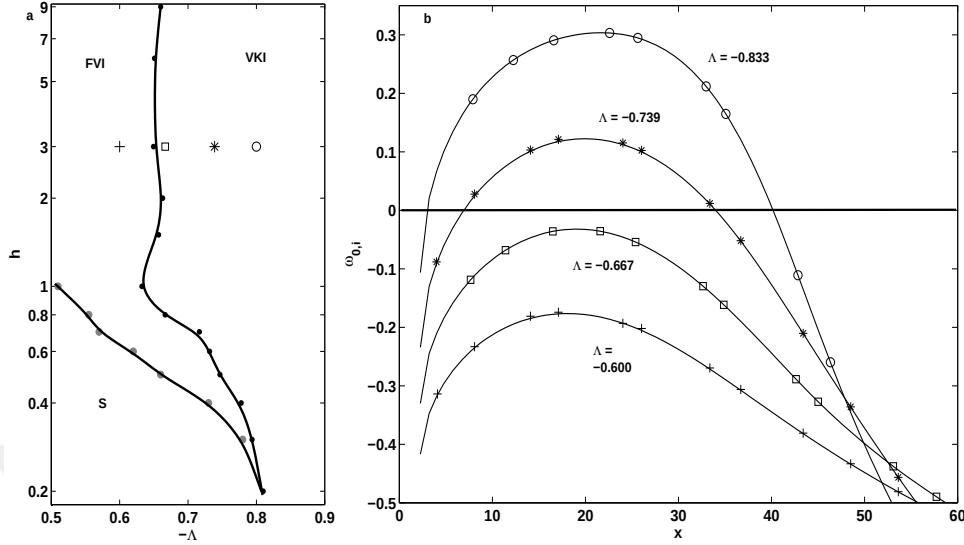


Figure 3.19: (a) Location of configurations examined in (b) with respect to global unstable/stable limit curve. (b) Behavior of absolute growth rate $\omega_{0,i}$ for $Re = 500$ along the domain. $- \circ -$ represent $h = 3$ and $\Lambda = -0.833$, $- * -$ $h = 3$ and $\Lambda = -0.739$, $- \square -$ $h = 3$ and $\Lambda = -0.667$, and $- + -$ $h = 3$ and $\Lambda = -0.600$. The numerical error-bound on the absolute growth rate is 5%. For profiles closer to inlet the error is slightly larger, but still less than 10%.

3.19b, the behaviour of absolute growth rate $\omega_{0,i}$ is illustrated along the domain for four different values of the entry velocity ratio $\Lambda = \{-0.833, -0.739, -0.667, -0.600\}$. The case of $\Lambda = -0.600$ where FVI is observed does not present an absolute unstable region for any x . However for the case $\Lambda = -0.667$, which is characterized by VKI, the flow also remains convectively unstable throughout the domain. Similar behaviour was found for $h = 6$, see figure 3.20b. The absence of an absolute unstable region for $\Lambda = -0.667$ for both confinements could be due to the lack of clear transition for $h \geq 3$ (see figure 3.9). In conclusion, the instability associated to vacillations is not predicted by local stability analysis, since only VKI cases do present an absolute region, except when they are too close to the VKI/FVI threshold.

3.3.5.2 Vacillations and linear global modes

We have seen in previous section 3.3.5.1 that the local stability analysis fails to predict the instability characterized by vacillations. In order to further understand if this mechanism is due to non-linearity effect, we relax the weakly non parallel assumption and resort to a linear global analysis. This study was conducted by Philippe Meliga by means of a FreeFEM code [71]. Linearizing the 2D Navier-Stokes equations (3.3) and (3.4) around the steady base flow solution $\mathbf{q}_b = (\mathbf{u}_b, p_b)$ (see

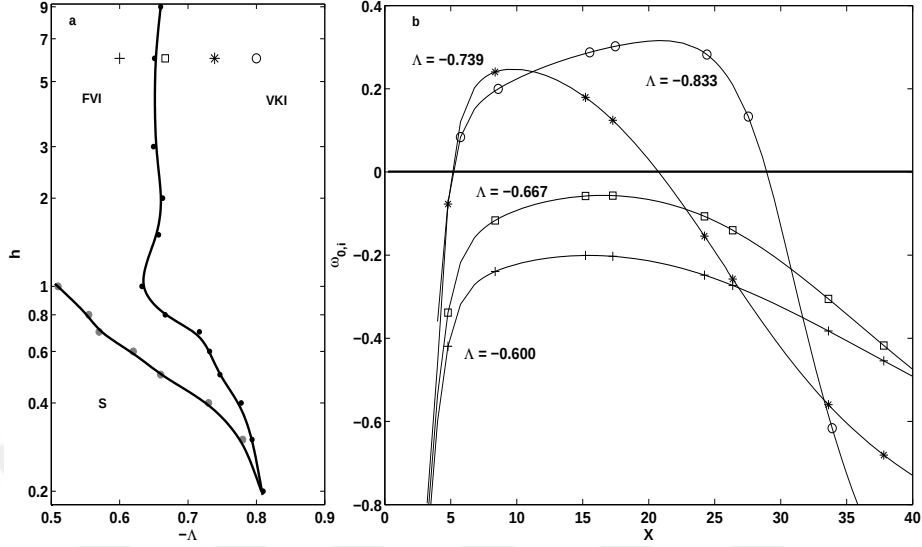


Figure 3.20: (a) Location of configurations examined in (b) with respect to global unstable/stable limit curve. (b) Behavior of absolute growth rate $\omega_{0,i}$ for $Re = 500$ along the domain. $-o-$ represent $h = 6$ and $\Lambda = -0.833$, $-*-$ $h = 6$ and $\Lambda = -0.739$, $-\square-$ $h = 6$ and $\Lambda = -0.667$, and $-+-$ $h = 6$ and $\Lambda = -0.600$. The numerical error-bound on the absolute growth rate is 5%. For profiles closer to inlet the error is slightly larger, but still less than 10%.

section 3.2.2.4) and neglecting the nonlinear terms, we obtain

$$\mathbf{B}\partial_t \mathbf{q}_d - \mathbf{L}\mathbf{q}_d = \mathbf{0}, \quad (3.27)$$

where $\mathbf{q}_d = (\mathbf{u}_d, p_d)$ is the solution of the disturbance equations and $\mathbf{B}$ and $\mathbf{L}$ are linear operators:

$$\mathbf{B} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \mathbb{C}(\cdot, \mathbf{u}_b) - Re^{-1}\Delta & \nabla \\ \nabla \cdot & 0 \end{pmatrix} \quad (3.28)$$

with $\mathbb{C}(a, b)$ is the advection operator $\nabla \mathbf{a} \cdot \mathbf{b} + \nabla \mathbf{b} \cdot \mathbf{a}$. Developing in normal modes the eigenmodes take the form

$$\mathbf{q}_d = \hat{\mathbf{q}}(x, y)e^{(\omega_r + i\omega_i)t}, \quad (3.29)$$

where $\hat{\mathbf{q}}(x, y)$ is the so-called global mode where the real and imaginary part of the eigenvalue, ω_r and ω_i , are the pulsation and the growth rate. Replacing eq. (3.29) into eq. (3.27), we can obtain a generalized eigenvalue problem for $\omega_r + i\omega_i$. For a detailed review on this analysis, see Meliga *et al.* [71].

The same configurations studied in figs. 3.19 and 3.20 have been analyzed with some additions. In figure 3.21a the global eigenvalues for $h = 3$, $Re = 500$ and $L = 75$ are shown in the plane (ω_r, ω_i) . If ω_i is negative the global mode associated

to eigenvalue ω is stable, otherwise, it is unstable. Two kinds of eigenvalues can be distinguished: the first ones are equally distributed along a semi-arch, while the second ones obtained for $\Lambda = \{-0.833, -0.739\}$, seem more isolated. We could suppose that the latter ones are associated to VKI and the former ones to FVI. The linear global stability seems to predict the appearance of vacillations, however it underestimates their strength, since in figure 3.21a the critical velocity ratio Λ_c , i.e. the velocity ratio which discriminates between globally unstable and stable configuration, is $\Lambda_c = -0.633$ (white circles), while by means of DNS the FVI/S limit was obtained for $\Lambda \approx -0.500$, see figure 3.10a.

Moreover in figure 3.21b several values of Λ are examined with a weaker confinement $h = 6$: only $\Lambda = -0.400$ was found globally stable with the DNS, $\Lambda = \{-0.490, -0.600\}$ correspond to a FVI, while the other values correspond to a VKI, as shown in figure 3.10a. Now, the FVI cases are globally unstable and seem to be characterized by vacillations mode. Furthermore, the critical velocity ratio Λ_c is equal to $\Lambda_c = -0.490$, closer to the value obtained with DNS: $\Lambda \approx -0.465$.

It should be noted, however, that which display a well characterized VKI, e.g. for $h = 3$ and both $\Lambda = \{-0.739, -0.667\}$, do not present the isolated global mode, which means that also Von Kármán instability is underestimated by linear global stability analysis. However, the vacillations are strictly connected to the wave front, especially its position, and Pier *et al.* [85] have shown the importance of non-linearity on the wave front. The remarkable difference between DNS and linear analysis, hence, does not constitute a surprise. Furthermore, semi-arc global modes are sensitive to a variation of the length L of the domain, as noticed in section 3.3.2.3 for configuration characterized by vacillations in DNS.

Also the results of the local analysis (compare figures 3.19 and 3.20 with 3.21a and 3.21b, respectively) are significantly different with those of linear global one, further sign of global nature of vacillations modes. In conclusion of these preliminary tests, the nonlinearity is seen to have a strong influence on the phenomenon of vacillations, but it is not the main factor, since also for a linear global stability a significant difference with the local analysis is noticed.

3.4 Non-linear global stability with no slip conditions

In section 3.3 we have implemented free slip conditions at the walls in order to mimic with our direct numerical simulations the model examined by Juniper & Candel [57] and Juniper [54], as well as some refined models studied in this thesis by means of a local stability analysis (see chapter 2), relaxing however two restrictive hypotheses: the inviscid and parallel nature of the base flow. Furthermore, in this manner we have isolated the effect of blockage from the influence of boundary layers which is associated to the creation of vorticity at the walls.

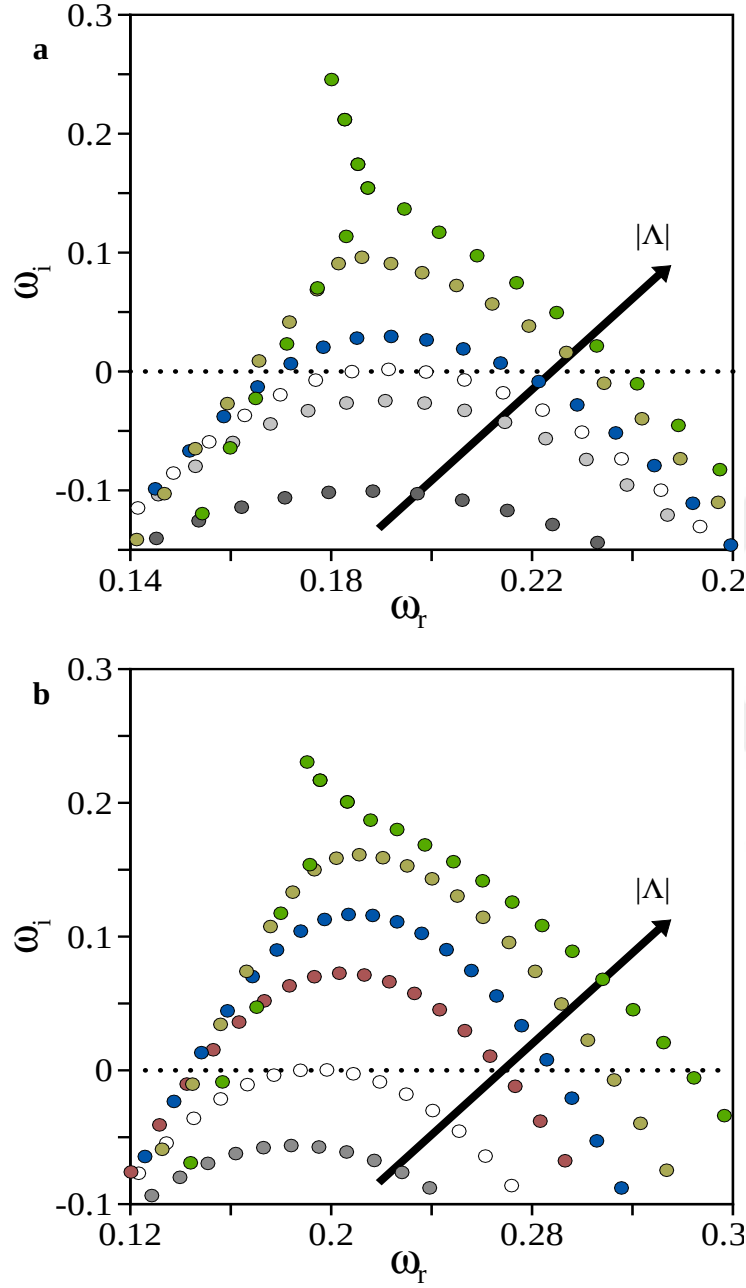


Figure 3.21: Linear global growth rate vs Strouhal number for $Re = 500$, $L = 75$, (a) $h = 3$ and (b) $h = 6$ for different values of Λ . The color code is as follows: colored symbols depict unstable branches, grey symbols depict stable branches, while white ones corresponds to the critical velocity ratio Λ_c . (a) From lower to higher values of ω_i , we have $\Lambda = \{-0.500, -0.600, -0.633, -0.667, -0.739, -0.833\}$. The critical velocity ratio is $\Lambda_c = -0.633$. (b) From lower to higher values of ω_i , we have $\Lambda = \{-0.400, -0.490, -0.600, -0.667, -0.739, -0.833\}$. The critical velocity ratio is $\Lambda_c = -0.490$.

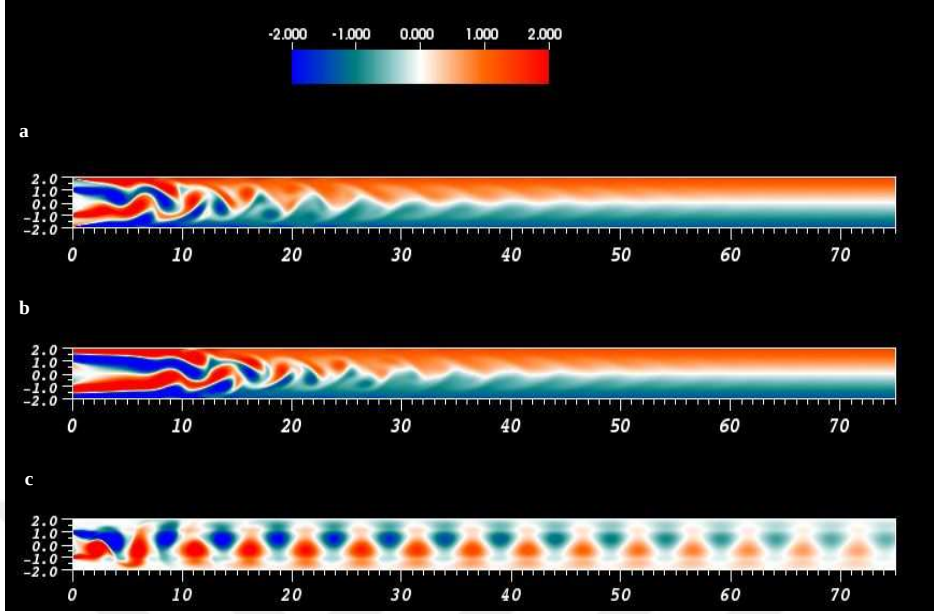


Figure 3.22: Instantaneous vorticity fields for $t = 600$ and $Re = 100$, $h = 1$ and $\Lambda = -0.818$ for the three wake models examined in the present thesis: (a) NS-plug flows; (b) NS-Poiseuille flows (c) FS-plug flows. The negative vorticity values are encoded in blue and the positive ones in red. The color-scale is the same for each flow configuration

However, a flow implementing no slip (NS) conditions at the walls is closer to reality than flows with free slip (FS) conditions. For this reason, in the present section NS conditions have been introduced. Two different inlet profiles have been analyzed, plug flows (or top hat profile), already analyzed with FS conditions and Poiseuille flows, which are described in detail in section 3.2.1. From here on, the former inlet profile will be called simply plug flows (or top hat) and the latter one Poiseuille flows. Note that a similar analysis implementing NS conditions was conducted by Tammisola *et al.* [99].

3.4.1 $Re = 100$

Figure 3.22 illustrates the instantaneous vorticity fields for three wake models examined in the present thesis for the same parameter configuration: $Re = 100$, $h = 1$ and $\Lambda = -0.818$. Each flow presents a classical Von Kármán vortex street peculiar to unstable wake flows. One can observe in figures 3.22a and c, which depict NS- and FS-plug flows, respectively, that the vortex street forms at $x \approx 5$, while for Poiseuille flows, it starts at $x \approx 10$, see figure 3.22b. In the case implementing NS conditions at the walls, a large amount of vorticity is created at the boundaries and interacts with central vortices. Interestingly, comparing NS- and FS-plug flows one observes an inversion of the Von Kármán street in the former model (figure 3.22a). This inversion was already noticed for confined cylinder wakes [96] and studied in detail by

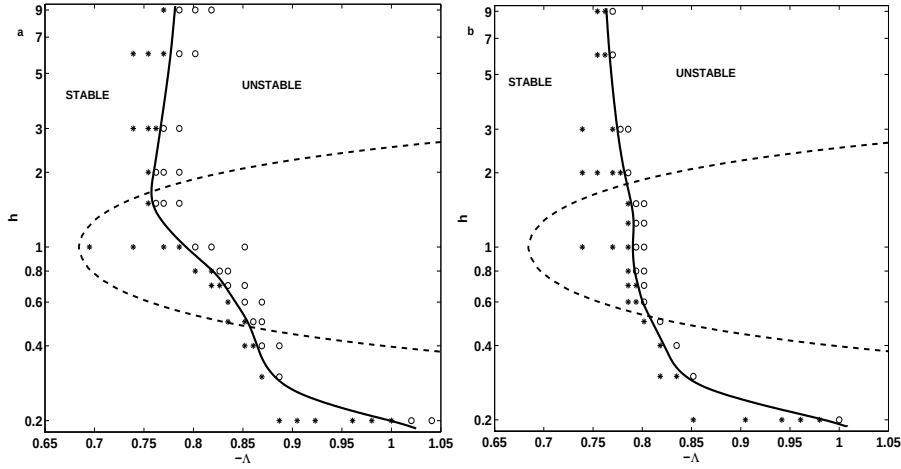


Figure 3.23: Limit between stable and unstable regions into the plane $(-\Lambda, h)$ at $Re = 100$ for (a) the plug flows, (b) the Poiseuille flows. The limit depicted by a continuous line is compared with the C/A limit obtained by Juniper (2006) [54] using a parallel inviscid FS-top hat profile, which is depicted by a dashed line. The circles depict the globally stable configurations, while the asterisks depict the globally unstable ones.

Camarri & Gianetti (2007) [15] who have discovered that the presence of vorticity in the incoming flow is at the origin of this phenomenon, see section 1.2.1.

In section 3.3 we have constructed the spatio-temporal diagrams of the absolute value of the vertical component of the velocity v at the centerline, see e.g figure 3.4. The unstable/stable nature of such a configuration can clearly be distinguished by means of these diagrams. The procedure is unchanged with respect to FS conditions.

As in the FS conditions case, such spatio-temporal diagrams can be constructed for a wide range of parameters ($0.695 \leq -\Lambda \leq 1.05$ and $0.2 \leq h \leq 9$), in order to determine the stable and unstable regions on a $(-\Lambda, h)$ plane for both entry profiles. The results obtained at $Re = 100$ are gathered in figure 3.23 for (a) plug flows and (b) Poiseuille flows. The globally stable configurations are depicted with a circle, while the globally unstable ones with an asterisk. The limit line between stable and unstable regions (dashed line) is compared with the inviscid theoretical limit line obtained by Juniper [54] for the top hat profile. Juniper's theoretical curve is quite different from the limit obtained for the NS-plug flows, however both thresholds show a maximum, $h = 1$ for Juniper's theoretical limit, and $h \sim 1.8$ for our simulations. Also Tammisola *et al.* [99] have analyzed as entry profile the viscous plug flow with no slip conditions using DNS and linear global analysis. They have compared two confinement values $h = 9$ and $h = 1$ in the plane Λ^{-1} and Re and did not notice the effect of confinement since for $h = 1$ the flow is more stable than for $h = 9$, see figure 3.23a. In contrast, the limit pertaining to Poiseuille flows in figure 3.23b does not present a maximum, thus $h = 9$ is the most unstable confinement configuration

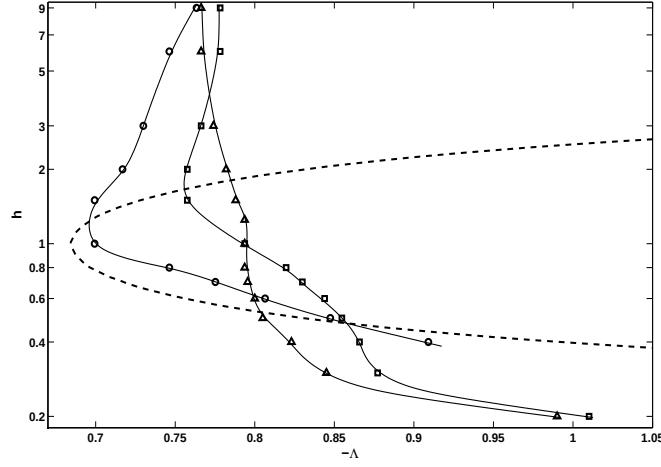


Figure 3.24: Comparison between the limits of the stable/unstable regions in the plane $(-\Lambda, h)$ at $Re = 100$ for the FS-plug flows ($-\circ-$), the NS-plug flows ($-\square-$), and for Poiseuille flows ($-\triangle-$). The C/A limit obtained by Juniper (2006) [54] using a parallel inviscid FS-top hat profile is depicted by a dashed line

presently studied. For both entry profiles, stronger confinements have a significant stabilizing effect on the flow.

In figure 3.24, thresholds obtained for plug flows and Poiseuille flows with NS conditions are compared, in the plane $(-\Lambda, h)$, with the FS-plug flows at $Re = 100$, see section 3.3.1. Implementing NS conditions instead of FS ones has a stabilizing effect, however only for $h \geq 0.5$. This stabilizing influence of no slip conditions is not surprising since they create boundary layers at the side walls, which may stabilize the wake. However, at very strong confinement $h < 0.5$ no slip conditions have a counterintuitive destabilizing influence on the flow, specially for $h < 0.4$, since for the FS-plug flow no unstable configurations could be found in this case. When adding confinement to wake flows, the boundary layers are closer to the vortex sheet and a stabilizing influence due to confinement can be assumed. However in section 3.3.1 we have noticed that without boundary layers the confinement has a destabilizing effect for medium values of h . These two effects are mixed for NS-plug flows, creating a slightly destabilizing influence for medium confinement, as seen in figure 3.23a.

3.4.2 $Re = 500$

In section 3.3.2, when increasing the Reynolds number up to $Re = 500$ we have encountered another global instability which differs from the classical Von Kármán one, characterized by vacillations. The relevant question is then if this surprising instability will survive when implementing NS conditions at the walls. Figures 3.25 and 3.26 show that both plug and Poiseuille flows present these vacillations. In these figures

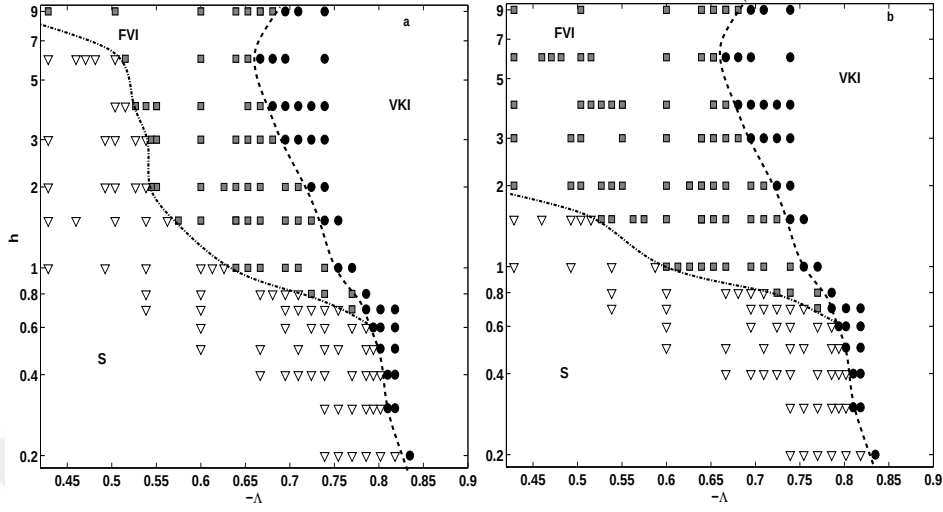


Figure 3.25: Stability diagram on the $(-\Lambda, h)$ plane, for $Re = 500$ and (a) $L = 75$, (b) $L = 152$ for the plug flow implementing no slip conditions. The circles and the triangles represent globally VKI unstable and globally stable configurations, respectively, while the squares depict the FVI globally unstable modes. Note that some simulated configurations have not been represented in order to avoid confusion.

the limits in the plane $(-\Lambda, h)$ between FVI (front vacillation instability)/VKI (Von Kármán instability) and FVI/S (stable) are illustrated for two different sizes of the domain: (a) $L = 75$ and (b) $L = 152$. The circles and the triangles represent globally VKI unstable and globally stable configurations, respectively, while the squares depict the FVI ones. Also for these wake models, the FVI/VKI limit is determined examining the behaviour of the position of the wave front as a function of the velocity ratio, as explained in section 3.3.2.3 for FS-plug flows. Remember that the velocity ratio of the VKI/FVI threshold is chosen when the wave front location experienced a significant jump (see figure 3.9).

The limit between FVI/VKI for NS-plug flows, which, as seen in sections 3.3.5.1 and 3.3.4.4, could be generally predicted by local stability analysis, presents a value of the confinement for which the global instability is maximal, i.e. $h = 6$. As seen for FS-plug flows (figure 3.10), this limit does not change when varying the size of the domain. In contrast, the FVI/S limit is modified by variations of L , as is evident by comparing figure 3.25a and 3.25b. Furthermore, comparing figures 3.10 and 3.25, we can extrapolate the stabilizing influence of NS conditions at the walls for both instabilities. In particular, for $h \leq 0.6$ the vacillations do not appear for both sizes of the domain, while with FS conditions the absence of vacillations was only noticed for $h = 0.2$. The confinement has always a stabilizing effect on FVI/S thresholds in the range examined in the present numerical study.

For Poiseuille flows, the limit between FVI/VKI is slightly sensitive to confinement

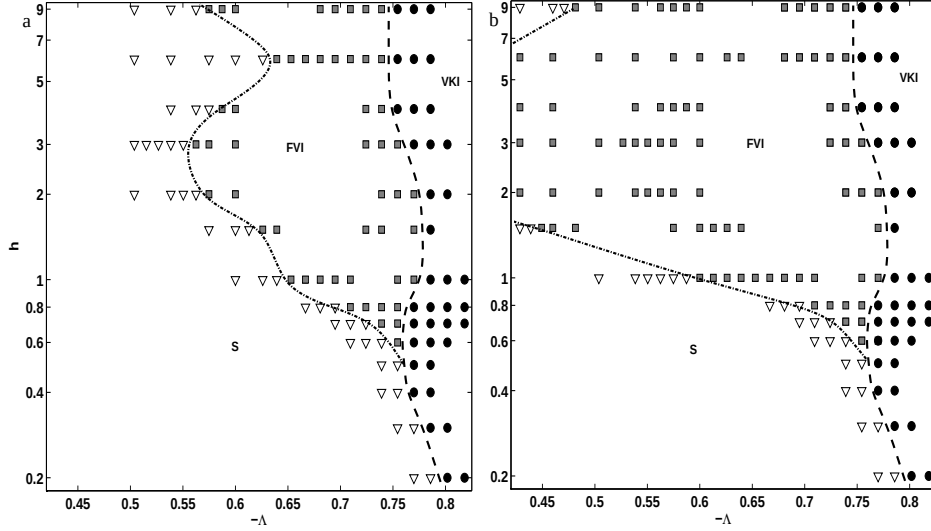


Figure 3.26: Stability diagram on the $(-\Lambda, h)$ plane, for $Re = 500$ and (a) $L = 75$, (b) $L = 152$ for Poiseuille flows. The circles and the triangles represent globally VKI unstable and globally stable configurations, respectively, while the squares depict the FVI globally unstable modes. Note that some simulated configurations have not been represented in order to avoid confusion.

also for $Re = 500$. From $h = 9$ until $h = 0.2$, the velocity ratio at the FVI/VKI limit varies between $-0.74 < \Lambda < -0.8$. Similarly to plug flows, the FVI/S limit has a significant dependence with respect to the size of the domain. Moreover this threshold has a consistent dependence on the confinement and there exists an optimal value h_{opt} for which the unstable character of vacillations is maximal, e.g. $h_{opt} \approx 3$ for $L = 75$. For confinement stronger than $h = 0.6$, no vacillations were found. The persistence of vacillations when implementing NS conditions and varying the inlet profile suggests the strength of this surprising phenomenon.

In figure 3.27, the comparison between the FVI/VKI limits of the three models is illustrated for $Re = 500$. The stabilizing influence of boundary layers created by no slip conditions are evident for each confinement value. Furthermore, note that the Juniper's inviscid theoretical limit [54] (depicted by the dashed line) is very different from the one of NS-plug flows.

3.4.3 Evolution of the base flow

In figures 3.2b and 3.2c the spatial evolutions of the base flow for NS-plug flows and for Poiseuille flows, respectively, are presented prescribing the following set of parameters $Re = 100$, $h = 1$, and $\Lambda = -0.811$. These evolutions have been computed by means of the *ad hoc* numerical set-up described in section 3.2.2.4. The variations of the FS-plug base flows and his local properties have been fully commented in

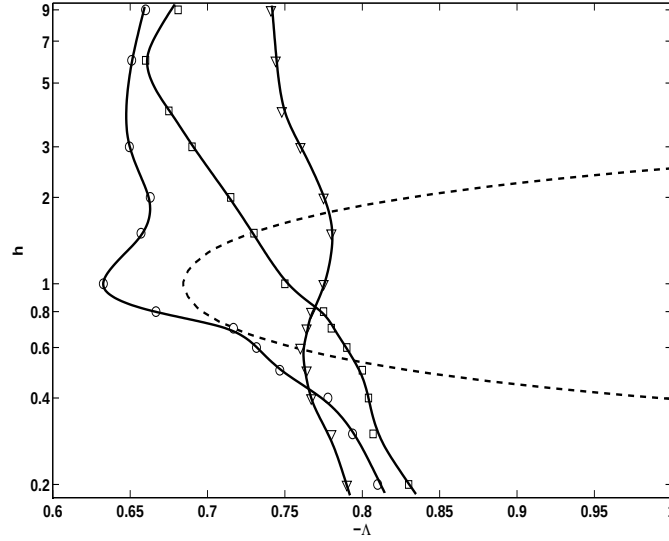


Figure 3.27: Comparison between the limits of the stable/unstable regions in the plane $(-\Lambda, h)$ at $Re = 500$ for the FS-plug flows $(-\circ-)$, the NS-plug flows $(-\square-)$ flows, and for Poiseuille flows $(-\triangle-)$. The C/A limit obtained by Juniper (2006) [54] using a parallel inviscid FS-top hat profile is depicted by a dashed line.

section 3.3.4 and in the second chapter. We have seen that local velocity ratio $\Lambda(x)$ presents a minimal value, which points the most locally unstable region in the domain, while the shear layer thickness $\delta_w(x)$ grows monotonically along the domain. However, it was illustrated that there exists an optimal shear layer thickness depending on confinement for which the absolute character of the instability is enhanced.

For the NS base flows the local velocity ratio Λ , is defined according to $\Lambda(x) = (U_0(x) - U_{max}(x))/(U_0(x) - U_{min}(x))$, where U_0 is the longitudinal velocity at $y = 0$, $U_0(x) = u(x, 0)$. This definition encompasses the previously introduced one for FS-boundary conditions, since in that $U_0(x) = U_{min}(x)$. The behaviour of the (a) local velocity ratio $\Lambda(x)$ and (b) $U_0(x)$ for three cases (FS-/NS-plug, Poiseuille) is depicted in figure 3.28. Each configuration presents a region of counterflow. The entry profile which creates the largest region of counterflow is the FS-plug flows. The influence of the velocity ratio is preserved: a more consistent co-flow stabilizes the wake, see figure 3.3.4.1.

In contrast, the definition of local shear layer thickness δ_w (see section 3.3.4.2) for the base flow with NS conditions loses its meaning. The shear layers still diffuse under the action of viscosity, but interact with the boundary layers. For the plug flows, the NS conditions at the walls create boundary layers evolving along the x -direction, as seen in figure 3.2b. The boundary layers are expected to stabilize the flow, but the interaction between these vorticity layers is delicate to interpret. Conversely, for Poiseuille flows, the boundary layers are already imposed at the inlet as well as two

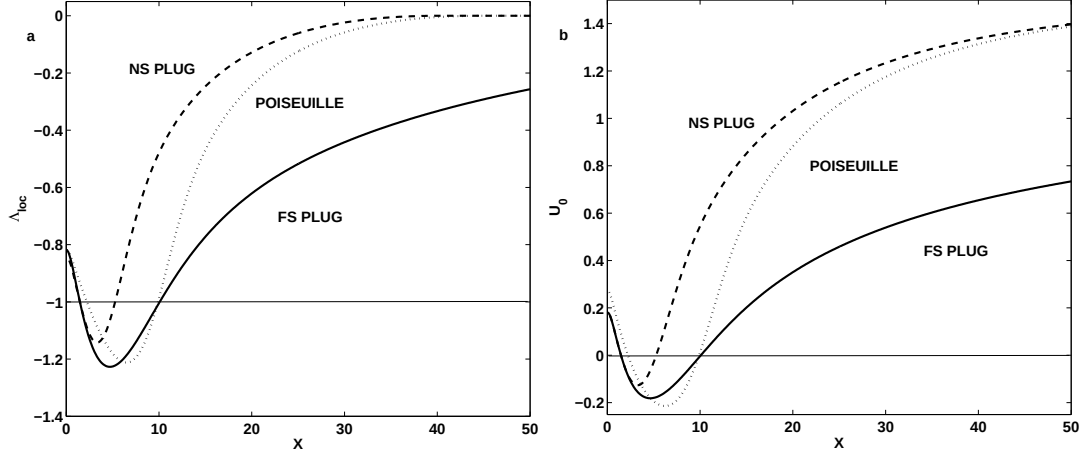


Figure 3.28: Behaviour of the local velocity ratio(a) and the velocity at $y = 0$ along the domain for $h = 1$, $\Lambda = -0.811$ and $Re = 100$. Continuous lines depict the FS-plug flows, dashed ones the NS-plug flows, and dotted ones the Poiseuille flows.

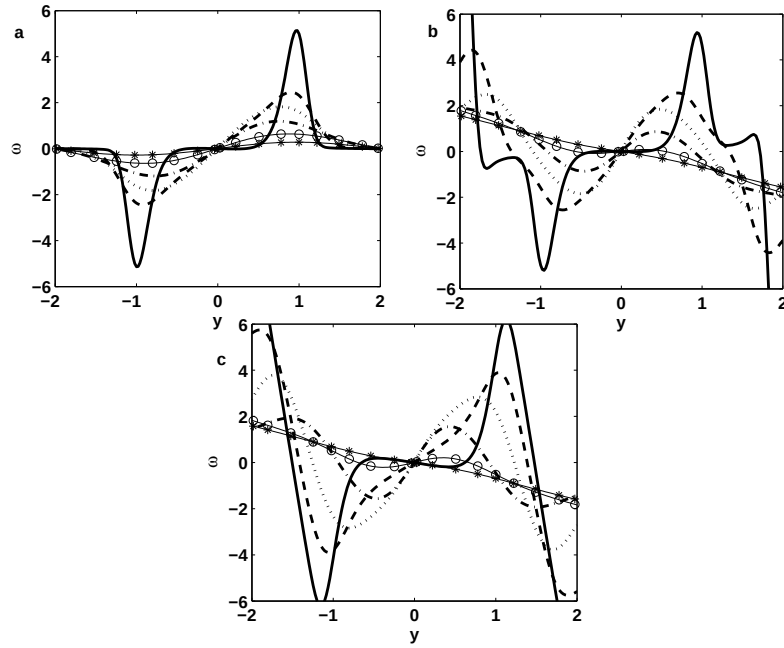


Figure 3.29: Vorticity profiles for $h = 1$, $\Lambda = -0.811$ and $Re = 100$. The profiles are given for $x = \{0.91, 4, 8, 15.9, 32.6, 64\}$, the amplitudes are decreasing with respect to x . (a) FS-plug flows, (b) NS-plug flows, and (c) Poiseuille flows.

shear layers (see figure 3.2c) and their evolutions along the domain are then subtle and fundamental to understand local properties. Flowing downwards, the base flows with NS conditions tend to a Poiseuille flow and thus the profiles of the two models, see figure 3.2, nearby coincide at $x = 64$.

Figure 3.29 depicts the vorticity profile at the abscissa selected in figure 3.2 for (a) FS-plug flows (a), (b) NS-plug flows, and (c) Poiseuille flows. Note that when implementing NS conditions, the vorticity of plug flows at $x = 0.91$ presents six peaks, sign of six inflection points of the corresponding velocity profile. Flowing downwards, the six peaks become four, and finally two. For $x \rightarrow \infty$, no peak of the vorticity will be present. In contrast, for FS conditions the vorticity profiles have only two peaks. The number of peaks is a signal of the formation and the evolution of the shear layers. The interaction between them is subtle and beyond the scope of the present thesis. However such profiles, which can hardly be modeled by profile families available in the literature, can provide the groundstone for a weakly non parallel local stability analysis.

3.4.4 Local stability analysis

In the same spirit described in section 3.3.4.4, a spatio-temporal local stability analysis is conducted on the velocity profiles extracted from DNS of the base flow of several configurations, varying the confinement parameter and the velocity ratio.

The chosen configurations for $Re = 100$ are illustrated for the NS-plug flows and the Poiseuille flows respectively in figure 3.30a and 3.31a compared to the corresponding stable/unstable limit. Remember that if the imaginary part of the absolute growth rate $\omega_{0,i} > 0$, the local velocity profile is absolutely unstable, otherwise if $\omega_{0,i} < 0$, it is convectively unstable. Focusing on figure 3.30, we can see how the unstable configurations present a large region for which the imaginary part of the absolute growth rate is positive. Such a region is present also for the stable configurations closer to the limit, however it is smaller and protrudes less in the positive ω_i plane. For other stable configurations the absolute growth rate remains negative. However, for $Re = 100$, the evolution of the base flow does not fulfill the condition of weakly non parallelism, thus, pointing out an error in the global properties predictions, as already explained in section 3.3.4.4. Finally, note that in figure 3.31, obtained for Poiseuille flows, only unstable configurations show an absolute region in the domain.

Furthermore in table 3.2 we compare the non-linear global frequencies obtained by a Fourier spectral analysis with the frequencies computed by spatio-temporal local analysis. The frequencies implementing FS conditions are also reported. As explained in section 3.3.4.4, if the inlet profile is convectively unstable, the local frequency is taken at the onset of the absolute instability region [85], otherwise, we have reported the local frequency at the inlet [21]. One can note that, only one case, i.e. $h = 1$, $\Lambda = -0.739$ for free slip-plug flows, shows an error larger than 10%. We can conclude that for $Re = 100$ the predictions of the frequencies given by local stability analysis

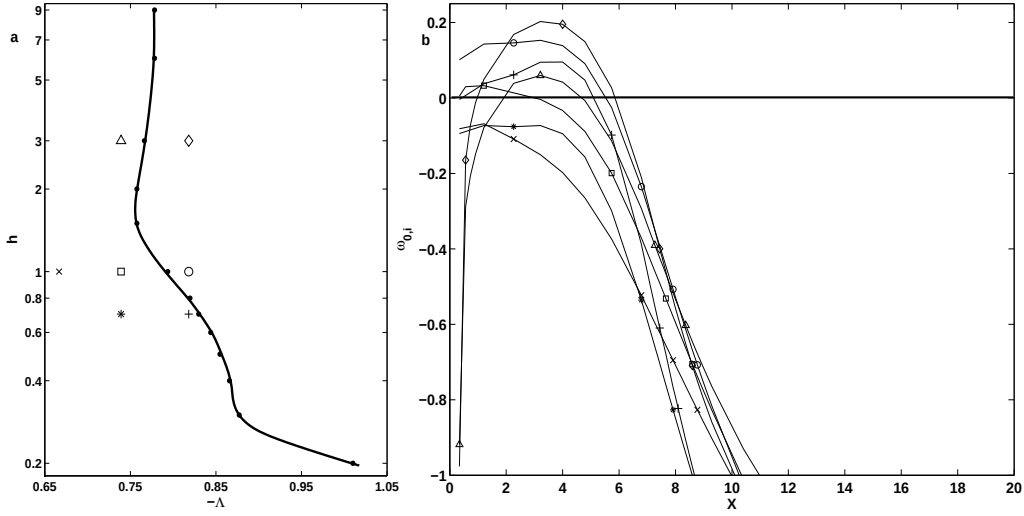


Figure 3.30: (a) Location of configurations examined in (b) with respect to global unstable/stable limit curve. (b) Behavior of local absolute growth rate $\omega_{0,i}$ for the plug flow at $Re = 100$ along the domain. $-\square-$ represents $h = 1$ and $\Lambda = -0.739$, $-\circ-$ $h = 1$ and $\Lambda = -0.818$, $-\times-$ $h = 1$ and $\Lambda = -0.667$, $-*$ $h = 0.7$ and $\Lambda = -0.739$, $-\triangle-$ $h = 3$ and $\Lambda = -0.739$, and $-\diamond-$ $h = 3$ and $\Lambda = -0.818$. The numerical error-bound on the absolute growth rate is 5%.

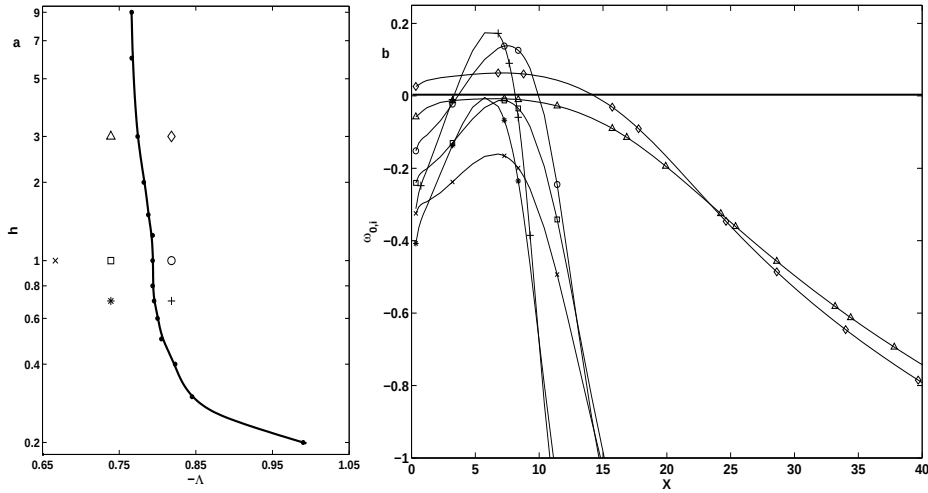


Figure 3.31: (a) Location of configurations examined in (b) with respect to global unstable/stable limit curve. (b) Behavior of local absolute growth rate $\omega_{0,i}$ for the Poiseuille flows for $Re = 100$ along the domain. $-\circ-$ represents $h = 1$ and $\Lambda = -0.739$, $-\square-$ $h = 1$ and $\Lambda = -0.818$, $-\times-$ $h = 1$ and $\Lambda = -0.667$, $-*$ $h = 0.7$ and $\Lambda = -0.739$, $-\triangle-$ $h = 3$ and $\Lambda = -0.739$, and $-\diamond-$ $h = 3$ and $\Lambda = -0.818$. The numerical error-bound on the absolute growth rate is 5%.

MODEL	BC	h	Λ	ω GL	ω LOC	X	ERROR %
PLUG	free slip	3	-0.739	1.19	1.23	2.0	3.3
PLUG	free slip	1	-0.739	1.28	1.49	0.5	14.1
PLUG	free slip	1	-0.818	1.45	1.56	0	7.1
PLUG	no slip	3	-0.818	1.46	1.34	1.0	9.0
PLUG	no slip	1	-0.818	1.61	1.63	0	1.2
POISEUILLE	no slip	3	-0.818	1.31	1.29	0	1.6
POISEUILLE	no slip	1	-0.818	1.60	1.54	3.6	3.9
POISEUILLE	no slip	0.7	-0.818	1.60	1.55	3.4	3.2

Table 3.2: Comparison between the frequencies obtained by local and global analysis for $Re = 100$.

are surprisingly satisfactorily.

For $Re = 500$, figures 3.32b and 3.33b illustrate the absolute growth rate for several configurations, for NS-plug flows and for the Poiseuille flows, respectively. In figures 3.32a and 3.33a, the FVI/VKI thresholds are depicted in order to understand if such a case is globally characterized by a front vacillations instability or by a classical Von Kármán vortex street. Since local stability analysis are not able to predict the unstable nature of the flow characterized by vacillations (see sections 3.3.5.1 and 3.3.4.4), then, one can expect that the absolute growth rate will remain negative throughout the domain in these cases. For the NS-plug flows (figure 3.32b) the configuration depicted by circles, i.e. $h = 1$ and $\Lambda = -0.739$ presents a region with positive absolute growth rate $\omega_{0,i}$, despite its FVI nature. However, one should note in figure 3.32a that this configuration is very close to the limit, which may account for this inconsistency. The same problem occurs for $h = 0.7$ and $\Lambda = -0.739$, depicted by crosses in figure 3.33b, which displays a narrow absolutely unstable region.

Furthermore in table 3.3, the comparison between global/local frequency is reported for VKI cases at $Re = 500$. In contrast to $Re = 100$, several predictions present a large inaccuracy, as noticed for FS conditions in section 3.3.4.4. In conclusion, we may conjecture that the presence of vacillations could take a significant part in the partial failure of local stability analysis to predict the global frequency.

3.5 Strouhal number

In this section, we analyze the influence of the confinement on the Strouhal number. In cylinder wakes, generally, it was shown that if the confinement was enforced the Strouhal number increased (see also section 2.5.5). Both numerical [25, 15] and experimental [25, 93] studies have noticed this effect. Prescribing the Reynolds number at $Re = 100$, in order to avoid the presence of vacillations, and the velocity ratio at $\Lambda = -0.739$, the behaviour of the Strouhal number is illustrated in figure

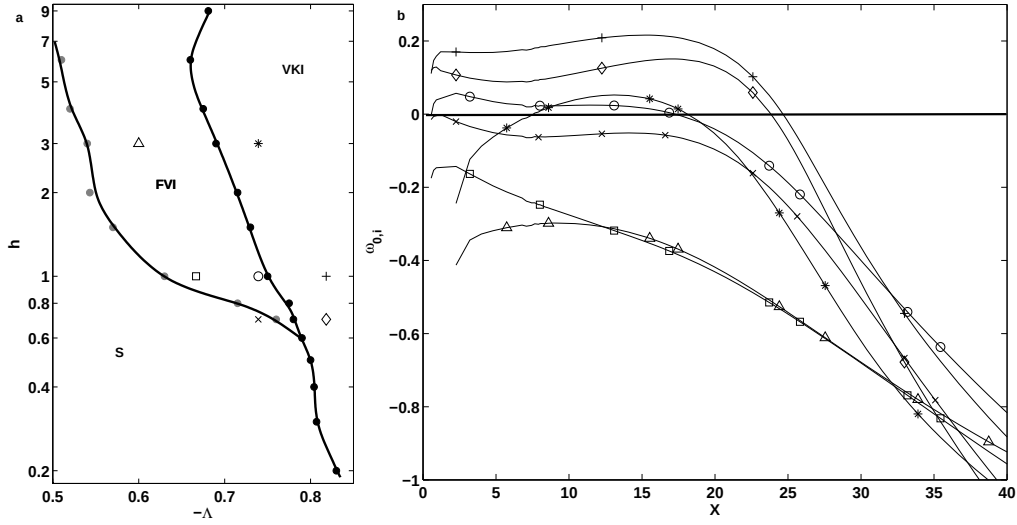


Figure 3.32: (a) Location of configurations examined in (b) with respect to FVI/VKI limit curve. (b) Behavior of local absolute growth rate $\omega_{0,i}$ for the plug flow at $Re = 500$ along the domain. $-\square-$ represents $h = 1$ and $\Lambda = -0.667$, $-\circ-$ $h = 1$ and $\Lambda = -0.739$, $-\times-$ $h = 1$ and $\Lambda = -0.818$, $-\diamond-$ $h = 0.7$ and $\Lambda = -0.739$, $-\ast-$ $h = 0.7$ and $\Lambda = -0.818$, $-\ast-$ $h = 3$ and $\Lambda = -0.739$, and $-\triangle-$ $h = 3$ and $\Lambda = -0.600$. The numerical error-bound on the absolute growth rate is 5%.

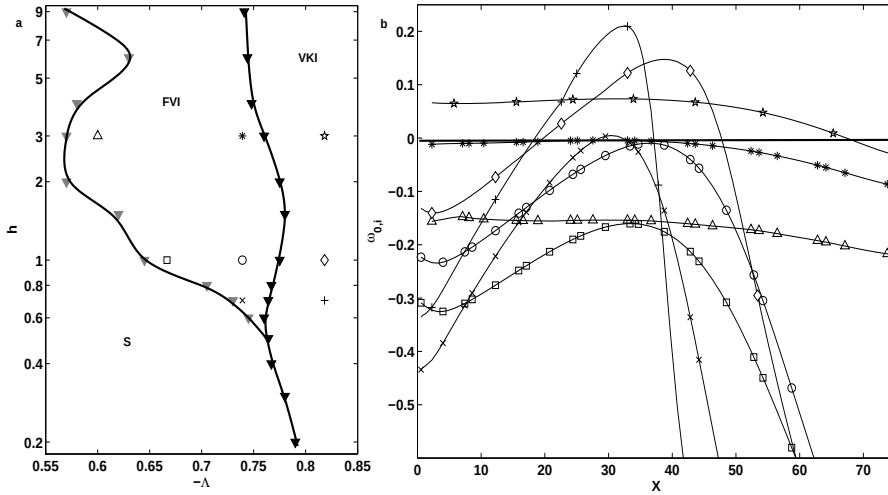


Figure 3.33: (a) Location of configurations examined in (b) with respect to FVI/VKI limit curve. (b) Behavior of local absolute growth rate $\omega_{0,i}$ for the Poiseuille flows for $Re = 100$ along the domain. $-\circ-$ represents $h = 1$ and $\Lambda = -0.739$, $-\square-$ $h = 1$ and $\Lambda = -0.818$, $-\times-$ $h = 1$ and $\Lambda = -0.667$, $-\ast-$ $h = 0.7$ and $\Lambda = -0.739$, $-\triangle-$ $h = 3$ and $\Lambda = -0.739$, and $-\diamond-$ $h = 3$ and $\Lambda = -0.818$. The numerical error-bound on the absolute growth rate is 5%.

MODEL	BC	h	Λ	ω GL	ω LOC	X	ERROR %
PLUG	free slip	3	-0.739	1.74	1.30	6.4	33.8
PLUG	free slip	1	-0.739	1.47	1.55	0.5	5.2
PLUG	free slip	1	-0.667	1.44	1.47	1.6	2.0
PLUG	free slip	0.7	-0.739	1.28	1.89	0.7	32.2
PLUG	no slip	3	-0.739	1.68	1.39	7.5	21.6
PLUG	no slip	1	-0.818	1.57	1.63	0	3.7
PLUG	no slip	0.7	-0.818	1.80	2.03	0	11.3
POISEUILLE	no slip	3	-0.818	1.53	1.27	0	20.5
POISEUILLE	no slip	1	-0.818	1.56	1.56	19.5	0.0
POISEUILLE	no slip	0.7	-0.818	1.59	1.54	18.4	3.2

Table 3.3: Comparison between the frequencies obtained by local and global analysis for $Re = 500$.

3.34a depicted by dashed line for three models: FS-plug flows (circles), NS-plug flows (squares), and Poiseuille flows (triangles). The Strouhal number seems to slightly increase with respect to confinement when NS conditions are implemented at the walls, while with FS conditions no particular tendency is noticed. Comparisons are provided with the wake of a cylinder and one can notice significant differences.

However, for confined cylinder wakes, Re and St are generally based on the mean velocity $\bar{U}$ and the diameter of the cylinder $2h_1$, in contrast with our definition. From now on, we refer to Re_c , for the Reynolds number, and St_c for the Strouhal number based on these reference values. First of all, the Reynolds number Re_c depends on the confinement, see figure 3.34b, which illustrates the ratio Re_c/Re as a function of confinement. Note that Re_c decreases when confinement is increased. Furthermore, the dependence of St_c on the confinement radically differs from that of St , as observed in figure 3.34a. The Strouhal number St_c (depicted by continuous lines) is significantly enhanced by confinement for all wake model flows. It should be noticed that for cylinder wakes the Strouhal number St_c is known to increase with the Reynolds number [108], implying that the damping effect of confinement on Re_c should not favor the growth of the Strouhal number St_c .

In conclusion, the strong increase of the St_c number as confinement is increased is an effect of the blockage alone, since it is not modified by either the presence of boundary layers or difference in the inlet profile.

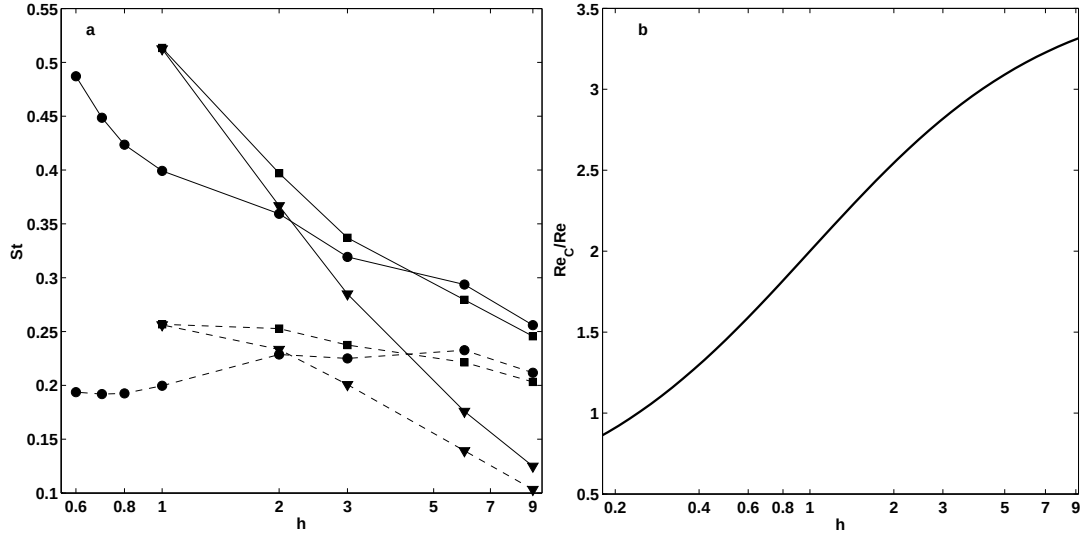


Figure 3.34: (a) Strouhal number vs confinement for the FS-plug flows ($-\circ-$), NS-plug flows ($-\square-$), for Poiseuille flows ($-\triangle-$) for $Re = 100$ and $\Lambda = -0.739$. The continuous line depicts the Strouhal number St_c and the dashed one the Strouhal number St based on the reference scales used in this thesis. (b) The ratio between the Reynolds number Re_c and the Reynolds number Re used in this thesis as a function of confinement h .



Chapter 4

Influence of confinement on turbulent wake flows



4.1 Introduction

In the previous chapter, we examined the influence of confinement on laminar wakes at medium Reynolds number, i.e. $Re = 100$ and $Re = 500$ imposing a top hat profile at the inlet. For $Re = 100$, a medium confinement promotes the instability, especially if free slip conditions are implemented at the walls. When implementing no slip conditions such a destabilizing effect is damped, but persists.

These results help to understand the physics of a recessed coaxial injector [35, 57], as explained in the introduction of this thesis (chapter 1), however it is worthwhile to ask what will be the influence of the confinement on a more realistic situation involving a turbulent flow. The quality of the mixing is indeed strictly connected to the intensity of the turbulence, and, thus we have to analyze the behaviour of the flows of chapter 3 when increasing the Reynolds number, up to e.g. $Re = 5000$.

Such high-Reynolds number flows are out of reach of direct numerical simulations, thus, large eddy simulations (LES) are needed. Stabilizing the computations by means of a spectral vanishing viscosity (SVV) technique constitutes a LES approach of interest when spectral methods are concerned. Following this SVV-LES approach, we can study how the turbulent behaviour of wake flows is modified by strengthening confinement. The numerical method used in this chapter implementing LES based on a SVV technique has been already tested to compute cylinder/sphere wakes, stratified wakes [80] or for the Ahmed's body flow [73].

In literature there are no traces of experimental or numerical studies concerning obstacle-free turbulent wake flows submitted to confinement. It seems indeed that only some analyses on turbulent confined cylinder wakes may be found. For $10^4 < Re < 10^7$, the experiments on confined circular cylinder wakes conducted by Richter & Naudascher (1976) [94] have shown that the influence of confinement is destabilizing. Same results were obtained by Kim *et al.* (2004) [59] by means of LES of confined square cylinder wakes. In particular, these studies show that both the turbulence intensity and the Strouhal number are enhanced as the confinement is increased. Early Bearman & Zdravkovich (1978) have conducted experiments on semiconfined sphere wakes at $Re = 45000$. They found that adding a plate close to the cylinder seems to regularize the vortex street for certain values of confinement.

In this chapter, we focus on the top hat profile, already analyzed in chapter 3, implementing no slip conditions at the side walls. This model is chosen, since it is the closest to a real confined wake exiting from a coaxial injector. However, some results on the other wake flows analyzed in chapter 3, i.e. free slip-top hat profile and the set of Poiseuille flows, are also reported. The outline of the chapter is as follows: section 4.2 describes the models and the numerical method, while the influence of the confinement on the turbulent properties of the wake flows is described in section 4.3.

4.2 Model and numerical method

The numerical method used for our SVV-LES is similar to the DNS one, described in detail in section 3.2.2. Some modifications, however, are required: the diffusion step of the temporal scheme (see section 3.2.2.1) is modified by adding a spectral vanishing viscosity (SVV) term in order to stabilize the computations. In section 4.2.1 we briefly describe the model and its numerical set-up, while section 4.2.2 is devoted to the SVV stabilization technique. Finally, the procedure to compute the dissipation rate of the turbulent kinetic energy is reported in section 4.2.3.

4.2.1 Model and numerical set-up

We have described two different inlet profiles for the DNS in section 3.2: a set of plug flows (or top hat) simulated with both conditions at the walls: free slip (FS) and no slip (NS), and a set of Poiseuille flows, obviously examined only with NS conditions. These profiles are also used as inlet conditions for the LES. As previously in this chapter, we call our models plug flows and Poiseuille flows in order to simplify the text. The study principally focuses on NS-plug flows, however also FS-plug flows and Poiseuille flows are briefly commented in section 4.3.4. We remember the three non-dimensional parameters: the Reynolds number $Re = (U_1 + U_2)h_1/2\nu$, the confinement ratio $h = h_2/h_1$ and the velocity ratio $\Lambda = (U_1 - U_2)/(U_1 + U_2)$. In this study, the velocity ratio $\Lambda = -0.739$ and the Reynolds number $Re = 5000$ are prescribed and we analyze the influence of the confinement parameter h for three different values: $h = 3$, $h = 1$, and $h = 0.7$.

The computational domain is the same as in the previous chapter, to detect 3D structures in the wake flow: $\Omega = (0, L) \times (-H, H) \times (-E/2, E/2)$, where E is the thickness of the domain, and the flow is assumed homogeneous in the z -direction. The length of the domain is settled at $L = 75$, and H varies with respect to the confinement, i.e. $H = 1 + h$, E is settled to four times the inner wake, thus, $E = 8$. We have generally used 8 subdomains, and in each of them the polynomial approximation degree in the x - and z -directions are settled to $N_x = 100$ and $N_z = 32$ (64 grid points), respectively. In contrast the polynomial approximation degree in the y -direction varies according to H , i.e. $N_y = 90$ for $h = 0.7$ and $h = 1$, while $N_y = 100$ for $h = 3$.

At the outlet of the domain, we have compared the 3D form of the u - v and the u - ω OBC [30]. The latter one is described in section 3.2.2.2, while the former one writes

$$(\partial_t + \bar{U}\partial_x)u = 0, \quad (\partial_t + \bar{U}\partial_x)v = 0. \quad (4.1)$$

We have chosen the u - v type condition, since they guaranteed more stability compared to u - ω one in 3D form.

The initial condition $\mathbf{u}_0(x, y, z)$ of three-dimensional LES is given by the final saturated state $\mathbf{u}_f(x, y)$ of the corresponding 2D simulations, i.e. same confinement and velocity ratio, for $Re = 100$ (see chapter 3). A random perturbation $\mathbf{u}'(x, y, z)$,

of small amplitude, typically 10^{-3} , is added to induce a 3D flow, so that

$$\mathbf{u}_0(x, y, z) = \mathbf{u}_f(x, y) + \mathbf{u}'(x, y, z). \quad (4.2)$$

4.2.2 Spectral vanishing viscosity

Now, in order to compute the provisional velocity implementing the SVV, we complete equation (3.10) with a SVV stabilizing operator S :

$$\left(Re^{-1} \nabla^2 + S - \frac{\alpha_0}{\delta t} \right) \mathbf{u}^* = \mathbf{f}^{n+1} \quad \text{in } \Lambda, \quad (4.3)$$

$$\mathbf{u}^*|_{\Gamma} = \mathbf{u}^{n+1}|_{\Gamma} = \mathbf{u}_{\Gamma}. \quad (4.4)$$

The definition of the operator S is given according to Maday *et al.* [68], who have resolved in the interval $(-1, 1)$ the inviscid 1D Burgers equation by means of the spectral Legendre method. In this case, with N for the degree of the polynomial approximation, we have

$$Su = \epsilon_N \partial_x Q(\partial_x u), \quad (4.5)$$

where Q is the operator such that, with L_k the Legendre polynomial of degree k :

$$Q\phi = \sum_{k=0}^N \hat{Q}_k \hat{\phi}_k L_k, \quad \forall \phi \quad \phi = \sum_{k=0}^N \hat{\phi}_k L_k, \quad (4.6)$$

with ϵ_N a coefficient of order $\epsilon_N = O(N^{-1})$, $\hat{Q}_k = 0$, if $k \leq m_N$ and $1 \geq \hat{Q}_k > 0$ if $k > m_N$. Our choice for m_N , the threshold frequency, was $m_N = \sqrt{N}$, while $m_N = N/2$, which is used by Karamanos & Karniadakis (2000) [58] was not found enough stabilizing. As shown by numerical experiments a smooth variation for $\hat{Q}_k$ yields better results for $m_N < k \leq N$, thus, as in Maday *et al.* [68], we use

$$\hat{Q}_k = e^{-(\frac{N-k}{m_N-k})^2}, \quad k > m_N. \quad (4.7)$$

As proposed by Pasquetti (2006) [78] in our multidimensional framework, such a SVV term is written

$$Su_i^* = \epsilon_N \nabla \cdot Q(\nabla u_i^*) \quad (4.8)$$

with u_i^* any component of $\mathbf{u}^*$ and where Q applies independently to each component of ∇u_i^* . An extension of the initial 1D definition can be discussed, specially for complex geometries, see Xu & Pasquetti [110]. Combining the viscous and the SVV term, one obtains

$$\nu \partial_j (\partial_j u_i) + \epsilon_N \partial_j (Q \partial_j u_i) = \nu \partial_j (\mathbb{I} + \epsilon_N / \nu Q) \partial_j u_i = \nu \partial_j \tilde{\partial}_j u_i \quad (4.9)$$

$$\text{with } \tilde{\partial}_j = (\mathbb{I} + \epsilon_N / \nu Q) \partial_j. \quad (4.10)$$

4.2.3 Dissipation rates of the turbulent kinetic energy

Following Pasquetti (2006) [79], we have computed the mean dissipation rate of the turbulent kinetic energy as described in Pope (2000) [86], except the fact that we start from the SVV-stabilized incompressible Navier-Stokes equation:

$$\partial_t \bar{u}_i + \bar{u}_j \partial_j \bar{u}_i = -\partial_i \bar{p} + \nu \partial_j \tilde{\partial}_j \bar{u}_i, \quad (4.11)$$

$$\partial_j \bar{u}_j = 0, \quad (4.12)$$

where summation over repeated indices is assumed, with $\tilde{\partial}_j$ as defined in equation (4.10) and where $\bar{u}_i$ and $\bar{p}$ stand for the SVV-LES approximations of the i -component of the velocity and of the pressure divided by the density, respectively. The over-bar is a classical LES notation to express the fact that at the end we will only compute some filtered quantities, even if we have not followed the usual LES methodology, based on the filtered Navier-Stokes equations.

In order to obtain an equation for the specific kinetic energy of the filtered velocity, $e_f = \bar{u}_i \bar{u}_i / 2$, we multiply eq. (4.11) by $\bar{\mathbf{u}}$ and introducing the tensor $\tilde{S}_{ij} = (\tilde{\partial}_j \bar{u}_i + \partial_i \bar{u}_j) / 2$ which is associated to the work of the viscous and sub-grid forces

$$\bar{D}_t e_f + \partial_j (\bar{u}_j \bar{p} - 2\nu \bar{u}_i \tilde{S}_{ij}) = -\epsilon, \quad (4.13)$$

with $\bar{D}_t = \partial_t + \bar{u}_j \partial_j$ and where appears the dissipation term

$$\epsilon = \nu (\partial_j \bar{u}_i \tilde{\partial}_j \bar{u}_i + \partial_j (\bar{u}_i \tilde{\partial}_i \bar{u}_j)). \quad (4.14)$$

Assuming that ∂_j and $\tilde{\partial}_j$ commute, with the continuity equation and thanks to the symmetry of the $\tilde{S}_{ij}$, one can obtain:

$$\epsilon = 2\nu S_{ij} \tilde{S}_{ij}. \quad (4.15)$$

Without SVV stabilization, thus, $\tilde{\partial}_j = \partial_j$ and $\tilde{S}_{ij} = S_{ij}$, we recover the usual form of the dissipation term [86]. However, it should be noted that $S_{ij} S_{ij} \geq 0$, whereas ϵ , from eq. (4.15), may be negative, giving then the possibility of a local transfer of energy from the non-resolved to the resolved scales (backscatter phenomenon).

From eq. (4.15), it is possible to compute the dissipation rate of the turbulent kinetic energy. Hence, denoting with $\langle . \rangle$ a statistical mean,

$$\langle \epsilon \rangle = 2\nu \langle S_{ij} \rangle \langle \tilde{S}_{ij} \rangle + \varepsilon, \quad (4.16)$$

where the first term in the right hand side is the dissipation due to the mean flow and ε the dissipation rate of the turbulent kinetic energy:

$$\varepsilon = 2\nu (\langle S_{ij} \tilde{S}_{ij} \rangle - \langle S_{ij} \rangle \langle \tilde{S}_{ij} \rangle). \quad (4.17)$$

The validity of this method was especially tested for the stratified wake of a cylinder [79].

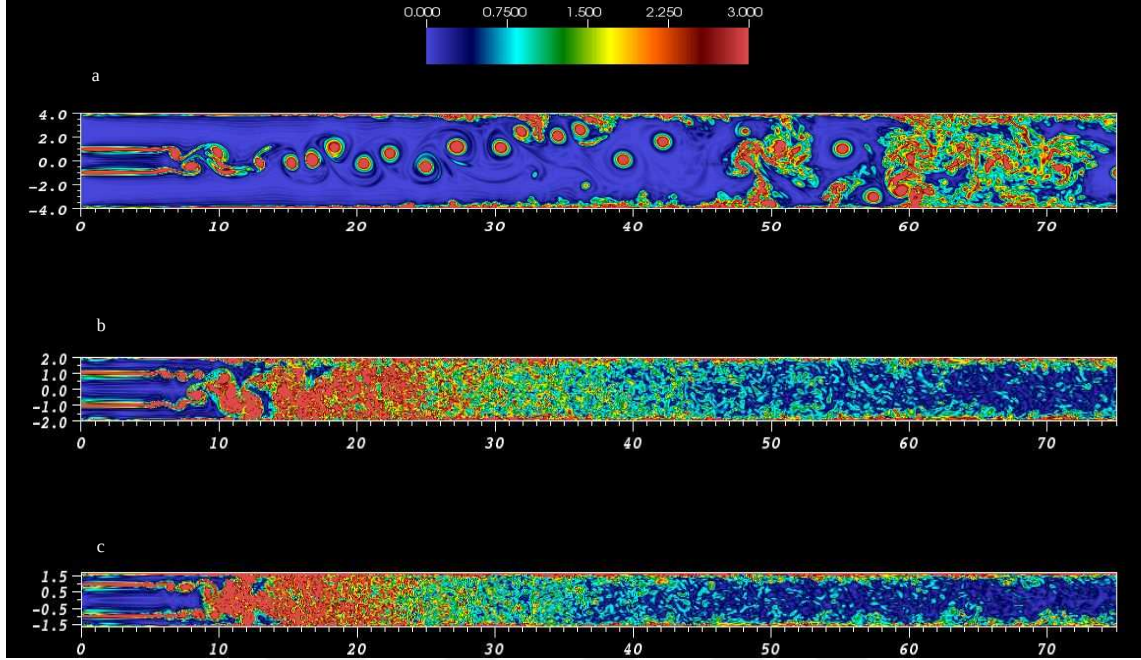


Figure 4.1: Instantaneous modulus $|\omega(x, y, 0)|$ of the vorticity at $t = 400$ for (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$ at $Re = 5000$.

4.3 Confined turbulent wakes implementing no slip conditions at the walls

In this section we investigate the influence of the confinement on several properties of the turbulence at $Re = 5000$ using as inlet profile of the LES the plug flows described in section 3.2.1 implementing no slip (NS) conditions at the lateral boundaries. Some comments on the computations of the statistical quantities as velocity profile, turbulent kinetic energy and dissipation rate are required: they have been calculated when the transient state due to initial random perturbation has evacuated from the domain and the turbulent flow is fully developed, in practice for $100 < t \leq t_f$. Generally, the final time of computations was $t_f = 400$. For each investigated configurations a CPU time of 2000 hours has been necessary. The computations have been carried out at both the IDRIS and the Mésocentre SIGAMM of the Observatory of Côte Azur.

4.3.1 Instantaneous flow field

We start this analysis showing in figure 4.1 the instantaneous modulus of the vorticity $|\omega|$ in the (x, y) plane at $z = 0$ at the final time of the computations $t_f = 400$ for different confinement configurations: (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$ for $Re = 5000$. Here, one can note that the position of the turbulent front depends on the confinement parameter: it is located upstream when the confinement is strengthened.

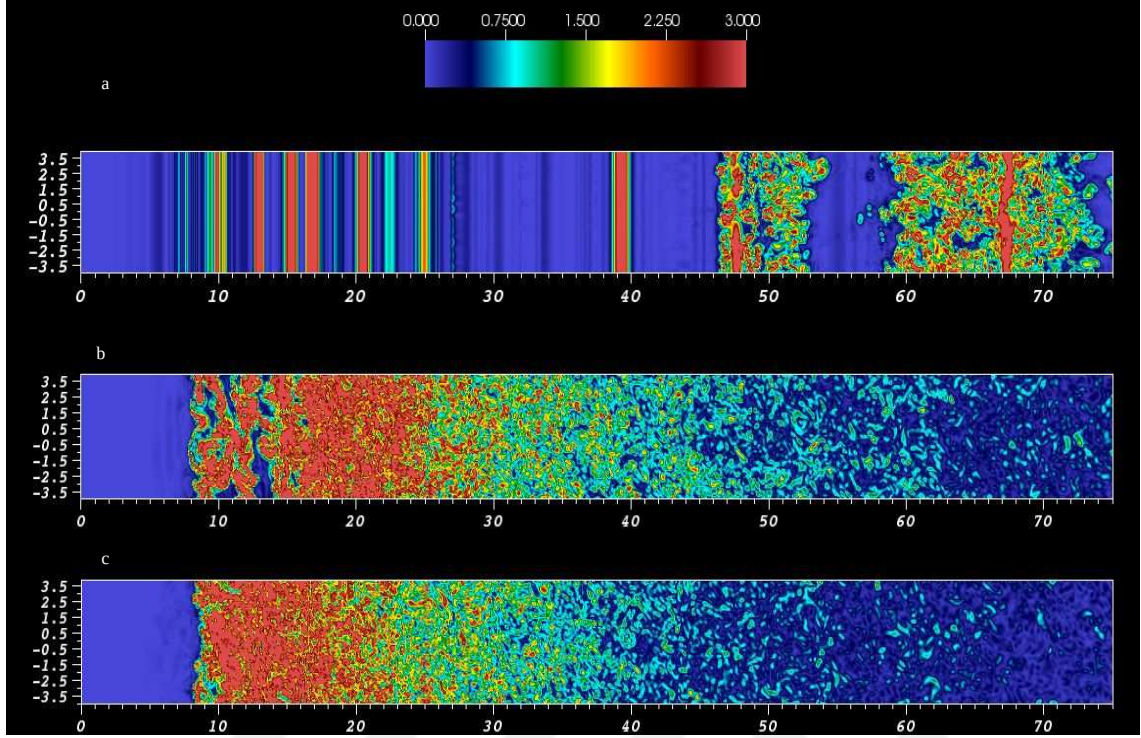


Figure 4.2: Instantaneous modulus $|\omega(x, 0, z)|$ of the vorticity at $t = 400$ for (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$ at $Re = 5000$.

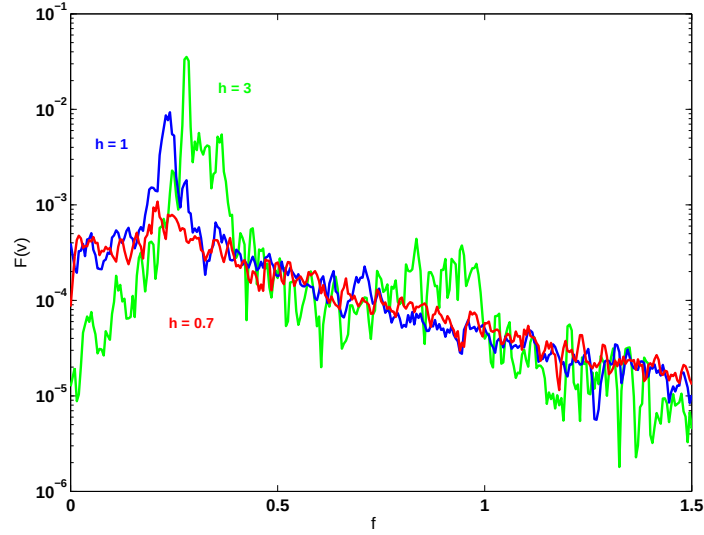


Figure 4.3: Fourier spectra of crossflow component of velocity $v(x = 12.25, y = 0)$. Note that we have averaged the spectra between several Fourier mode. The green line depicts $h = 3$, the blue one $h = 1$, and the red one $h = 0.7$. The Strouhal number of the dominant frequency decreases with respect to confinement.

h	Re	St	Re_c	St_c
3	5000	0.280	13695	0.409
1	5000	0.240	10000	0.480
0.7	5000	0.210	8696	0.483

Table 4.1: Summary of Reynolds number and the Strouhal number obtained in our study.

While for $h = 1$ and $h = 0.7$ the flow seems to be fully 3D turbulent, for $h = 3$ the turbulence seems to be characterized by a peculiar two-/three-dimensional behavior. This hypothesis is confirmed by figure 4.2, where we illustrate the instantaneous modulus $|\omega|$ of the vorticity in the plane (x, z) for $y = 0$ at the same final time $t_f = 400$. Looking to figure 4.2a it is evident that in the first part of the domain, i.e. until $x \approx 45$, the effect of the third dimension can be neglected, enforcing the conjecture of a two-dimensional nature of the turbulence in such region.

Furthermore note that in figure 4.1a, the structures around the vortices upstream of the turbulent front suggest the presence of the direct cascade of enstrophy, which is peculiar to a two-dimensional turbulence. These structures are not detected for stronger confinements. This fact could be interpreted as the confinement promotes fully 3D turbulence. However this hypothesis has to be confirmed by means of further analyses.

4.3.2 One-dimensional velocity spectra

The computation of the one-dimensional longitudinal velocity spectra by means of a Fourier analysis is important for two reasons: the first one is the computation of the Strouhal number in order to estimate the influence of the confinement on the dominant frequency (section 4.3.2.1), while the other argument is a comparison with the Kolmogorov spectrum typical of isotropic turbulent flows (section 4.3.2.2). In the latter section the behaviour of the third order-velocity structure functions is also analyzed.

4.3.2.1 Strouhal number

The influence of confinement on the shedding frequency for the wake around a cylinder has been already examined. Particularly Richter & Naudascher [94] have found that the Strouhal number increases, when enforcing the confinement until $h = 1$ for turbulent flows such that $10^4 < Re < 10^7$, in their experiments on a confined circular cylinder. A similar tendency was found by Kim *et al.* [59] by comparing $h = 4$ and $h = \infty$ results in their large eddy simulations of a confined turbulent flow past a square cylinder at $Re = 3000$. However, the cylinder wakes are quite different from

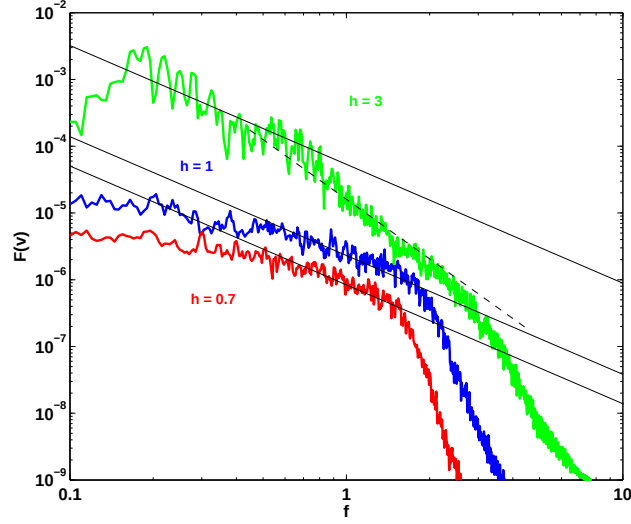


Figure 4.4: Fourier spectra of the crossflow component of velocity $v(x = 51.85, y = 0)$. Note that we have averaged the spectra between several z . The green line depicts $h = 3$, the blue one $h = 1$, and the red one $h = 0.7$. Thin lines represent the slopes of $-5/3$ (continuous) and -3 (dashed).

the geometry considered in our simulations for which no previous experiments or numerical simulations on turbulence was found in the literature.

In figure 4.3 the spectra of the crossflow component of the velocity v at $x = 12.25$, $y = 0$ are illustrated with respect to the dimensionless frequency f with reference value U_m/h_1 , for examined confinements. Note that the spectra are averaged between several z in order to improve the statistical sample. The main peak is damped when strengthening the confinement and the Strouhal number $St = fh_1/U_m$ decreases with respect to confinement. Hence, it seems that the dependence of the Strouhal number on the confinement is opposite of that one discerned for confined cylinder wakes. Instead, for lower Reynolds number, i.e. $Re = 100$, the Strouhal number seems to slightly increase, as noted in section 3.5. However, as explained in that section, for confined cylinder wakes, generally Re and St are referred to mean flow rate $\bar{U}$ and diameter of the cylinder $2h_1$, in contrast with our definition. From now, the Reynolds and Strouhal number rescaled by means of these reference scales are named respectively Re_c and St_c .

Table 4.1 illustrates the values of Re , St , Re_c and St_c for the three different confinements. The new Reynolds number Re_c increases with respect of confinement and, generally the new Strouhal number St_c increases as a function of Re_c [108]. Despite this fact, the Strouhal number St_c increases when enforcing the confinement as noticed for cylinder wakes, sign of the enhancing power of the confinement on the dominant frequency. This tendency was already encountered in section 3.5 for

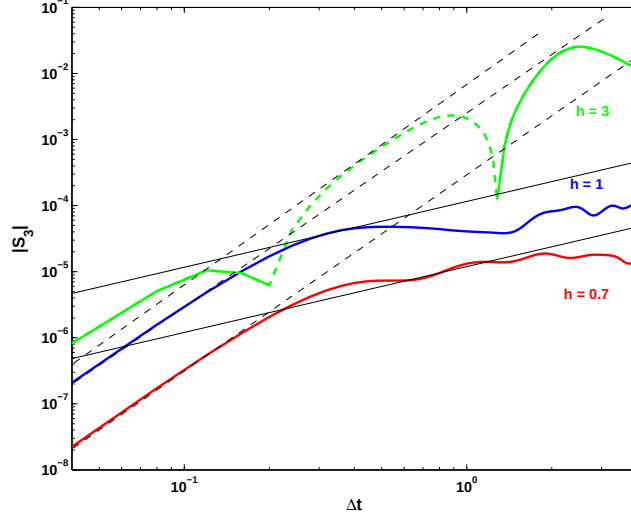


Figure 4.5: Absolute value of the third order structure function S_3 versus time shift Δt for $u(x = 51.85, y = 0)$. The color legend is the same of figure 4.3.2.1. While the continuous lines depict the negative values of S_3 , the dashed ones depict the positive values. Thin lines represent the slopes of 1 (continuous) and 3 (dashed). Note that we have averaged the spectra between several Fourier modes.

$Re = 100$, for the same value of the velocity ratio $\Lambda = -0.739$.

4.3.2.2 Kolmogorov scales and structure function

In figure 4.4 the one dimensional crossflow velocity spectra are still represented, but now in log-scale and computed at a different point of the domain, $x = 51.85$, $y = 0$. The most confined cases, i.e. $h = 1$ and $h = 0.7$ present a classical inertial subrange, where the Kolmogorov direct energy cascade takes place peculiar to a fully 3D turbulent flow [86]. In the inertial subrange, which however is not very large, the slope of the spectra fits satisfactorily with the expected value of $-5/3$ (continuous black line). Towards small scales, the two curves show a subgrid regime analogous to the dissipative regime, predicted by Kolmogorov's hypotheses [86]. In contrast of this behaviour, the spectrum of the less confined case, i.e. $h = 3$, (green line) seems to present two different tendencies in the inertial subrange, which is a reminiscence of the spectra of a two-dimensional turbulent flow [63]. In the large scales range the spectrum has a slope close to $-5/3$, which means that a cascade of energy with a constant flux takes place, while for smaller scales the slope of the spectrum rather approximates the value of -3 (dashed black lines), see Kraichnan (1967,1971) [61, 62]. Following Kraichnan & Montgomery (1980) [63], this behaviour of the spectrum is a reminiscence of an inverse cascade of enstrophy, further hint of a 2D nature of the turbulence for $h = 3$.

In order to confirm this intuition, we can compute the third order-velocity structure function of the velocity $S_3(\mathbf{l}) = \langle \{[\mathbf{u}(\mathbf{x} + \mathbf{l}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \mathbf{l}/l\}^3 \rangle$, which can distinguish between 2D and 3D turbulent flows [5]. By means of the Taylor's hypothesis [86] and considering $\mathbf{u}\Delta t = \mathbf{l}$, the third order structure function may be computed as a function of the time shift Δt , $S_3(\Delta t) = \langle [u(\mathbf{x}, t + \Delta t) - u(\mathbf{x}, t)]^3 \rangle$, in the same point of the spectra ($x = 51.85$ and $y = 0$), always averaging between several z . In an homogeneous and isotropic turbulent flow, the sign of the third order-velocity structure function can determine the direction of the energy cascade. If it is negative, there is a transfer of turbulent kinetic energy from the large scales towards the small scales, sign of a fully 3D turbulence, otherwise if positive the energy is transferred from the small scales towards the large ones, peculiar to a two-dimensional turbulent flow. Both inverse and direct cascade of energy have a slope of $S_3(\Delta t)$ in log-scale equal to 1 [5]. In 2D turbulence Bernard (1999) [5] has also shown that in the range where the direct enstrophy cascade takes place, the third order-velocity structure function $S_3(\Delta t)$ is still positive and its slope in log-scale fits 3. This behaviour was confirmed by Lindborg (1999) [66] in atmospheric turbulence and recently by Boffetta & Musacchio (2010) [10] for forced 2D Navier-Stokes equations. Furthermore, in the subgrid regime the behaviour of $|S_3|$ approximates the theoretical slope of 3 for both 2D- and 3D-turbulence [5].

In figure 4.5, the behaviour of the absolute value $|S_3(\Delta t)|$ is reported in log-scale for the three confinement configurations. The continuous lines mean negative values of S_3 , while dashed lines mean positive ones. The slopes of 3 and 1 are depicted by black continuous and dashed lines, respectively. One can note that for $h = 1$ and $h = 0.7$ the third order structure function is still negative, according to the hypothesis of a fully 3D turbulent flow, while for $h = 3$, a region for which S_3 is positive is encountered. Considering that our wake turbulent flows do not respect the conditions of homogeneity and isotropicity, we can conclude that the behaviour of S_3 approximates quite satisfactorily the theoretical slopes. In particular, while in the subgrid ranges the three configurations have a similar slope of 3, in the inertial regime their behaviour is different. The fully 3D turbulent cases present a clear slope of 1 which represents the energy cascade, while for $h = 3$, the slope of S_3 is close to 3, sign of an enstrophy cascade. However the presence of a range with positive values of $S_3(\Delta t)$ for $h = 3$ confirms the hypothesis of 2D turbulence for such case. Furthermore, if one associates f to $1/\Delta t$, the positive range of the third order-velocity structure function corresponds roughly to the inertial subrange where the direct cascade of enstrophy takes place, see figure 4.4. Finally, the behaviors of both the spectra and the third order-velocity structure function suggest that a stronger confinement favors the formation of fully 3D turbulent flow.

4.3.3 Mean flow and turbulence statistics

In this section the statistical quantities of the turbulence are analyzed. Moreover, these quantities are averaged over the periodic spanwise direction, except the dissi-

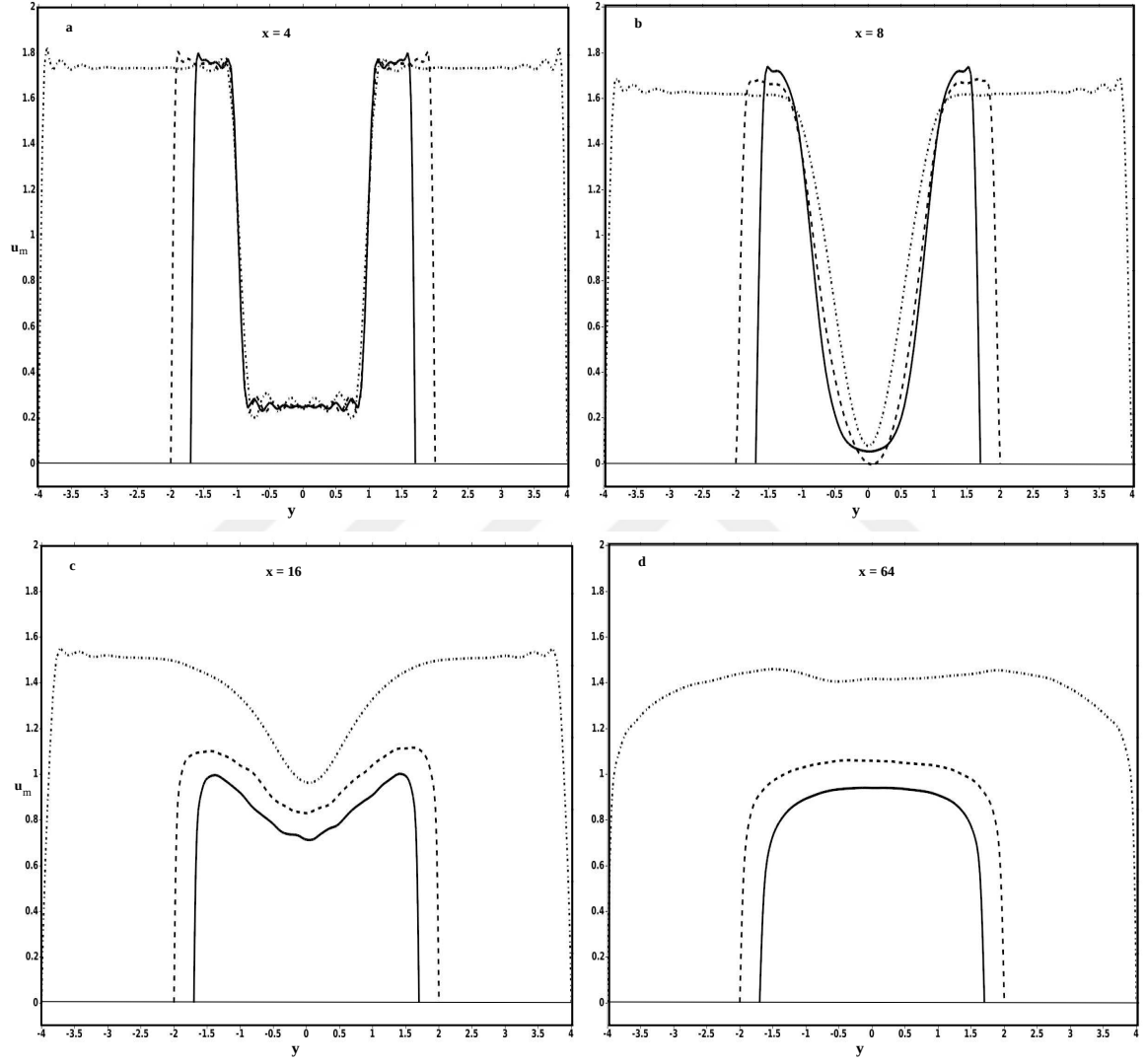


Figure 4.6: Streamwise component of the mean flow velocity profile $u_m(x, y)$ as a function of the crossflow direction y at different abscissa of the domain: (a) $x = 4$, (b) $x = 8$, (c) $x = 16$, and (d) $x = 64$. The continuous line depicts $h = 0.7$, the dashed line $h = 1$, and the dotted one $h = 3$.

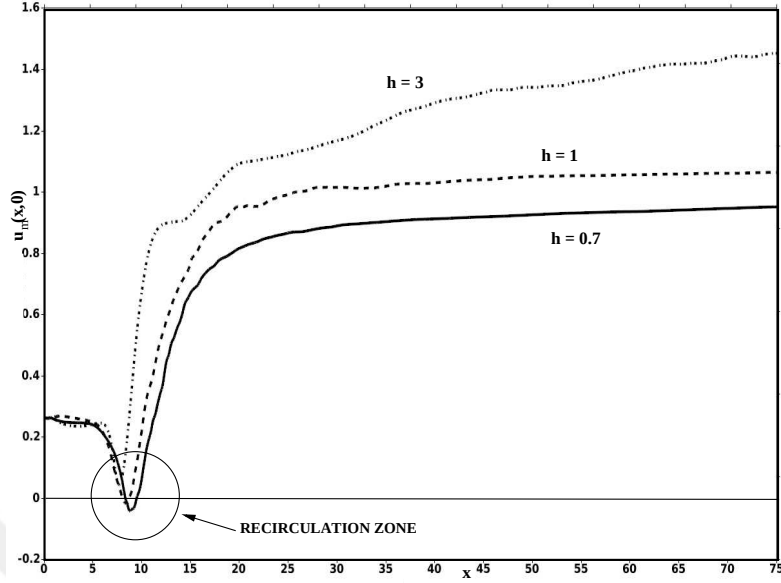


Figure 4.7: Streamwise component of the mean flow velocity profile $u_m(x, 0)$ as a function of the longitudinal direction x . The continuous line depicts $h = 0.7$, the dashed line $h = 1$, while the dotted one $h = 3$. Note that the recirculation region is present only for $h = 0.7$ and $h = 1$.

pation rates which are computed at $z = 0$.

4.3.3.1 Mean flow profiles

In figure 4.6, the profiles of the streamwise component of the mean flow velocity $u_m(x, y)$ at different abscissa x of the domain are illustrated: (a) $x = 4$, (b) $x = 8$, (c) $x = 16$, and (d) $x = 64$. From now, the dash-and-dot line depicts $h = 3$, the dashed one $h = 1$, and the continuous one $h = 0.7$. For $x = 64$, close to the exit, we can see that the profiles are close to a plug flow, especially for stronger confinements. Thus, the three flows can be considered statistically saturated at the exit of the domain.

Despite that the inlet profiles are co-flow wakes, in figure 4.6b for $h = 1$ the velocity is negative at the centerline. In order to measure the length of the recirculation region and to also reveal its existence for other confinements, figure 4.7 illustrates the value of the mean streamwise velocity $u_m(x, 0)$ for $y = 0$ as a function of the x -direction. First of all, one can note that the weakest analyzed confinement $h = 3$ does not present a recirculation zone, in contrast to $h = 1$ and $h = 0.7$. Since $h = 0.7$ has a larger region of recirculation, one can conclude that the confinement seems to promote the formation of a recirculation bubble in the domain.

4.3.3.2 Turbulent kinetic energy

The behaviour of the turbulent front illustrated in figures 4.1 and 4.2 may be interpreted as a destabilizing effect due to confinement. However, we have to confirm this intuition by analyzing the turbulent kinetic energy profiles along the domain. Figures 4.8 and 4.9 illustrate the behaviour of the turbulent kinetic energy E_k as a function of the crossflow direction y for several values of the abscissa x . In particular, figure 4.8 illustrates (a) $x = 2.06$, (b) $x = 4$, (c) $x = 8$ and (d) $x = 16$, while figure 4.9 (a) $x = 32$ and (b) $x = 64$. Note that the scale is modified between different figures.

Close to the inlet, as is possible to see in figure 4.8, the flows present a larger turbulent kinetic energy than downwards and the peaks around $y = \pm 1$ have a similar value for each confinement. Instead, for $x = 8$ (figure 4.8c), the higher peaks are those of stronger confinements, a further hint of a growth of the intensity of the turbulence associated to an increasing confinement. However, flowing downstream, the most confined flows loose a large part of their kinetic energy, as noticed in figure 4.9. Hence, in the final part of the domain the turbulent character for $h = 3$ is higher than the other ones, as already noticed in the snapshots of figures 4.1 and 4.2.

One could arrive to the same conclusion when analyzing figure 4.10, which illustrates the behaviour of the turbulent kinetic energy E_k along the streamwise direction x , for four different values of y : (a) $y = 0$, (b) $y = 1.04$, (c) $y = 1.34$, and (d) $y = 1.52$. At the centerline $y = 0$ (figure 4.10a) the case with $h = 3$ presents the highest turbulent kinetic energy. However, E_k at the centerline is negligible if compared to this one at the y -values closer to the boundaries (figure 4.10). Furthermore, for $y = 1.34$ and $y = 1.52$ (figure 4.10) the configurations with the strongest confinement present the highest peaks, which means that the confinement enhances the turbulent character of the flow. However, only close to the entry this destabilizing influence is noticed.

One may think that the boundary layers dissipate rapidly the turbulent kinetic energy, thus, the less confined wake flow is the most turbulent in the final part of the domain. One may also conjecture that (i) once the turbulent wake flow has become similar to a standard turbulent channel flow, the velocity fluctuations are much less important and (ii) that this occurs farther downstream for the least confined flow. Finally, it is clear that the mean flow velocity and the width of the channel increase with h , so that a Reynolds number based on these quantities is much higher for the least confined flow.

4.3.3.3 Dissipation rates

By means of equation (4.17), the dissipation rates ε of the turbulent kinetic energy can be computed and the role of the confinement on this quantity discovered. Figure 4.11 illustrates the behaviour of $Re \cdot \varepsilon$ as a function of x -direction. Moreover the dissipation rate is calculated for $y = 0$ and $z = 0$. Clearly the maximum of dissipation rate ε is enhanced when the confinement is increased. Another effect should be noticed: while for the two strongest confinements the large part of the dissipation occurs close to

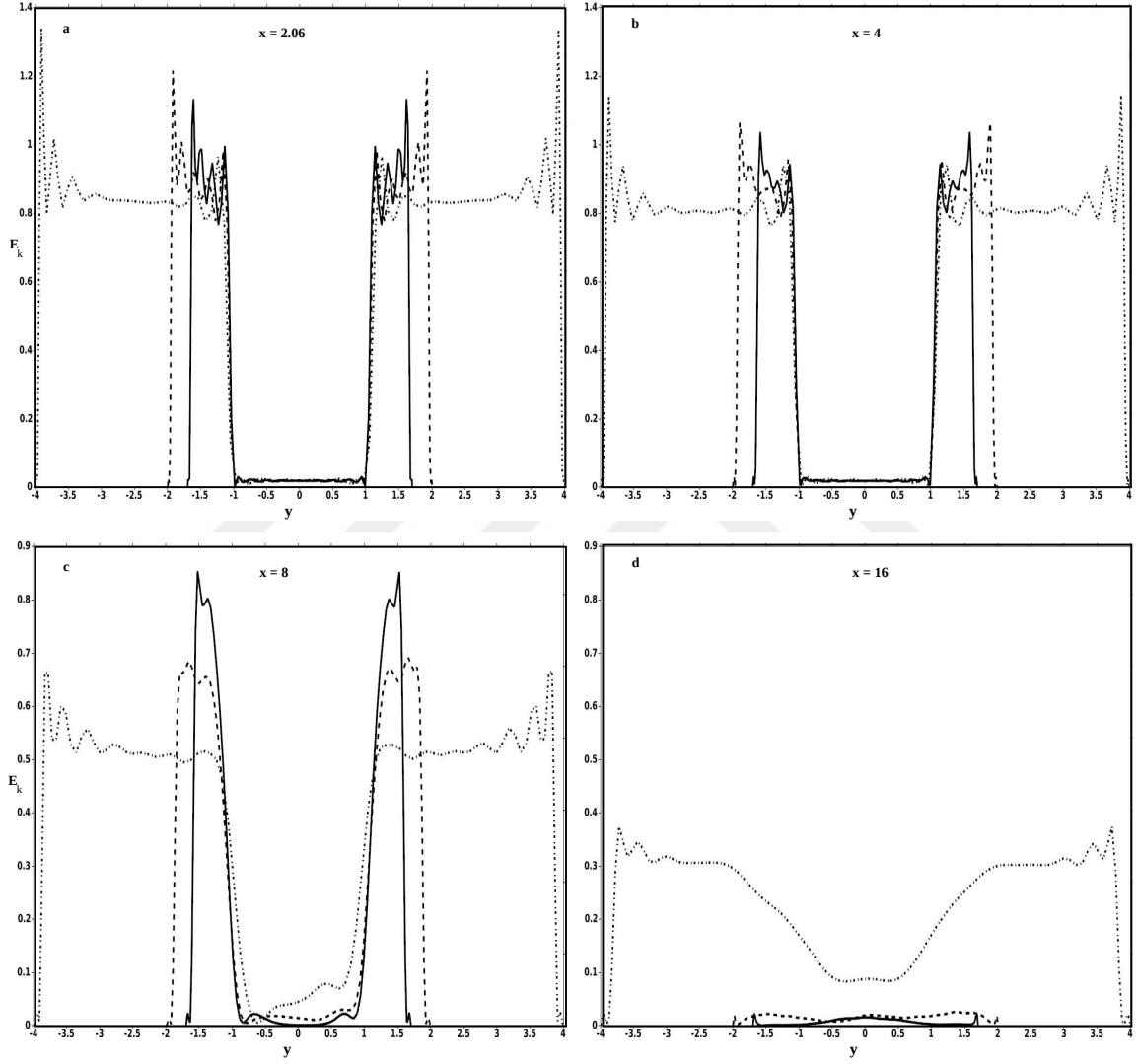


Figure 4.8: Turbulent kinetic energy E_k with respect to crossflow direction y at different abscissa of the domain: (a) $x = 2.04$, (b) $x = 4$, (c) $x = 8$, and (d) $x = 16$. The continuous line depicts $h = 0.7$, the dashed line $h = 1$, while the dotted one $h = 3$. Note that E_k is averaged along the spanwise direction z .

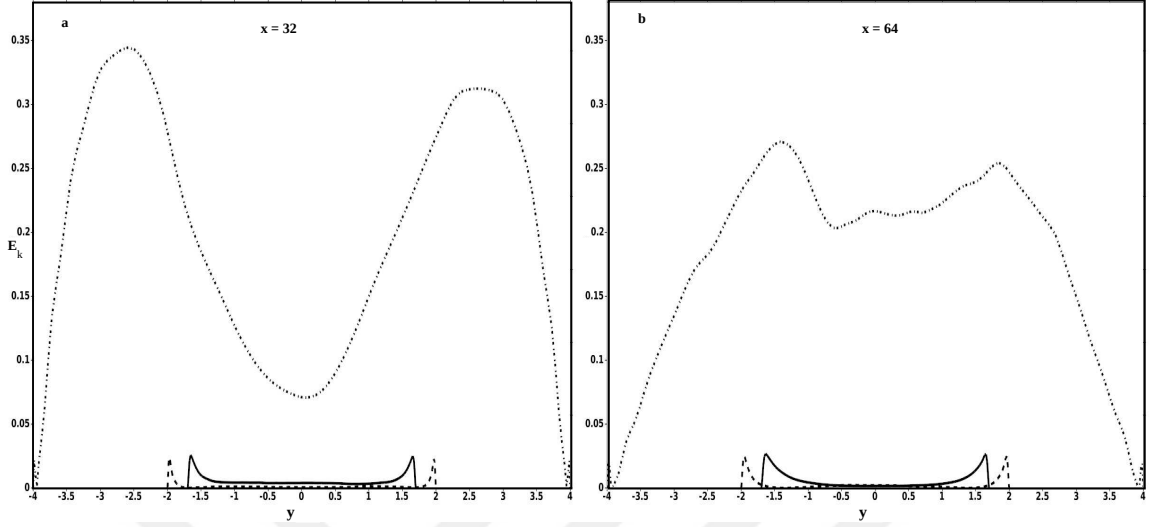


Figure 4.9: Turbulent kinetic energy E_k with respect to crossflow direction y at different abscissa of the domain: (a) $x = 32$ and (b) $x = 64$. The continuous line depicts $h = 0.7$, the dashed line $h = 1$, while the dotted one $h = 3$. Note that E_k is averaged along the spanwise direction z .

the inlet, for $h = 3$, the dissipation persists along the domain. It is clear that the explanations provided for the turbulent kinetic energy also apply for the dissipation rates.

4.3.4 Other wake flow models

This section is devoted to describe the influence of confinement on the turbulence for the other wake models described in section 3.2.1. The FS-plug flows are analyzed in section 4.3.4.1, in order to reveal the role of boundary layers at the side walls, while in section 4.3.4.2 the results for the set of Poiseuille flows are reported. Finally, a comparison between the 2D/3D character of the turbulence is illustrated in section 4.3.4.3 for the three wake flow models.

4.3.4.1 Plug flows with free slip conditions

In this section, we still analyze the plug flows but implementing free slip conditions at the walls. Previously, the formation of boundary layers due to NS conditions have been supposed to enhance the turbulent character of the flow. Especially, they could give to confinement a destabilizing power, since the boundary layers are moved closer to the shear layers occurring around $y \pm 1$.

For FS-plug flows and for three different values of the confinement parameter, (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$, figures 4.12 and 4.13 illustrate the instantaneous modulus $|\omega|$ of vorticity in the planes (x, y) at $z = 0$ and (x, z) at $y = 0$, respectively,

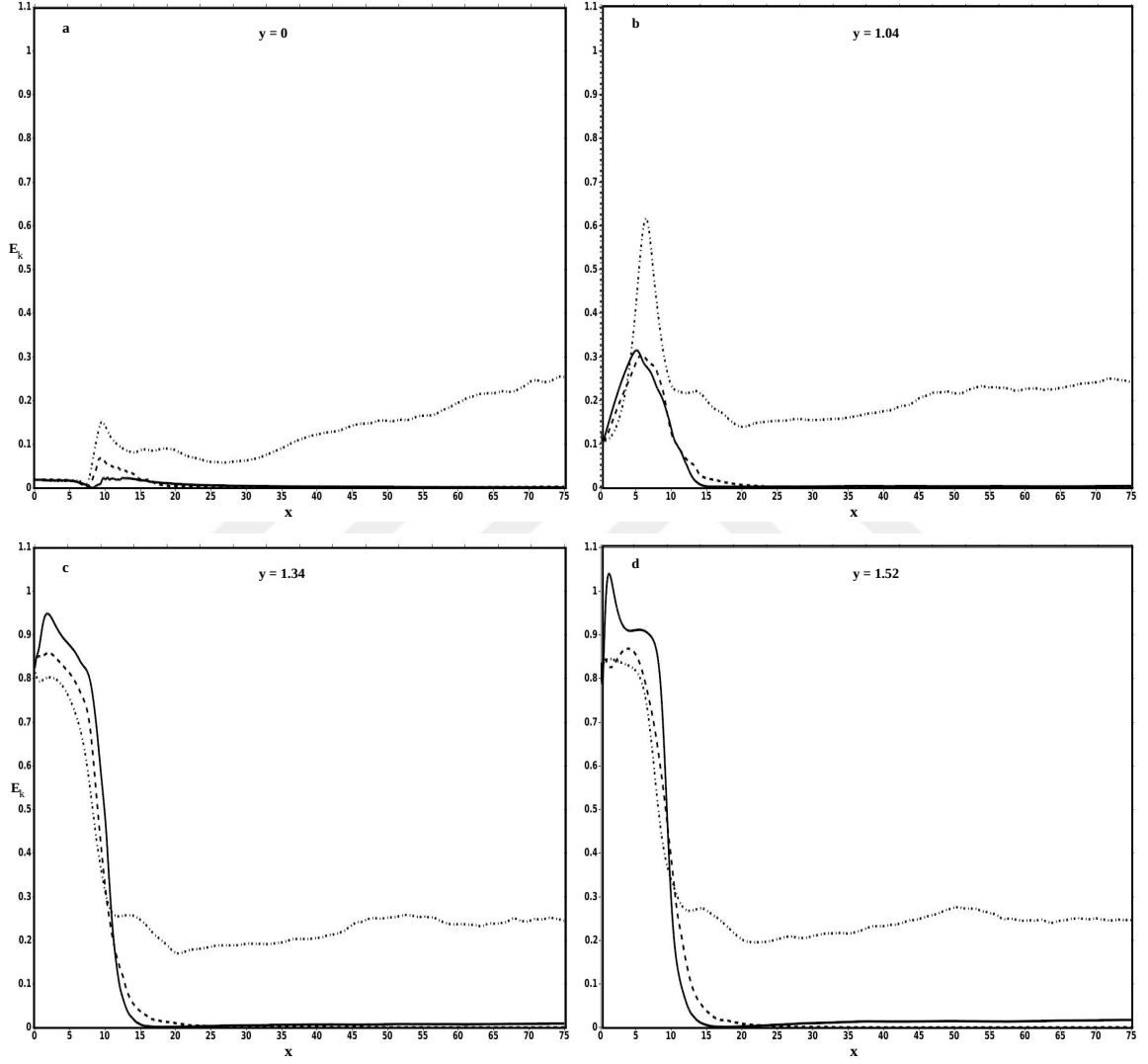


Figure 4.10: Turbulent kinetic energy E_k with respect to streamwise direction x at different y -values of the domain: (a) $y = 1.34$ and (b) $y = 1.52$. The continuous line depicts $h = 0.7$, the dashed line $h = 1$, while the dotted one $h = 3$. Note that E_k is averaged along the spanwise direction z .

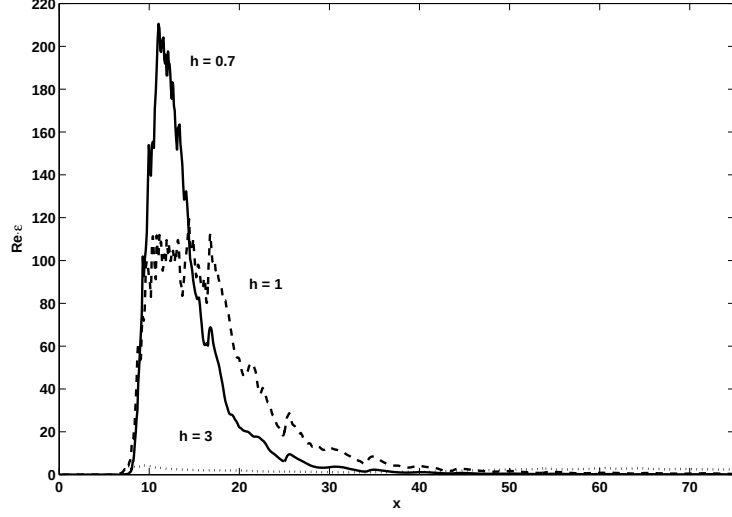


Figure 4.11: Dissipation rate of turbulent kinetic energy multiplied by Re as a function of longitudinal direction x for $y = 0$ at $z = 0$. The continuous line depicts $h = 0.7$, the dashed line $h = 1$, while the dotted one $h = 3$.

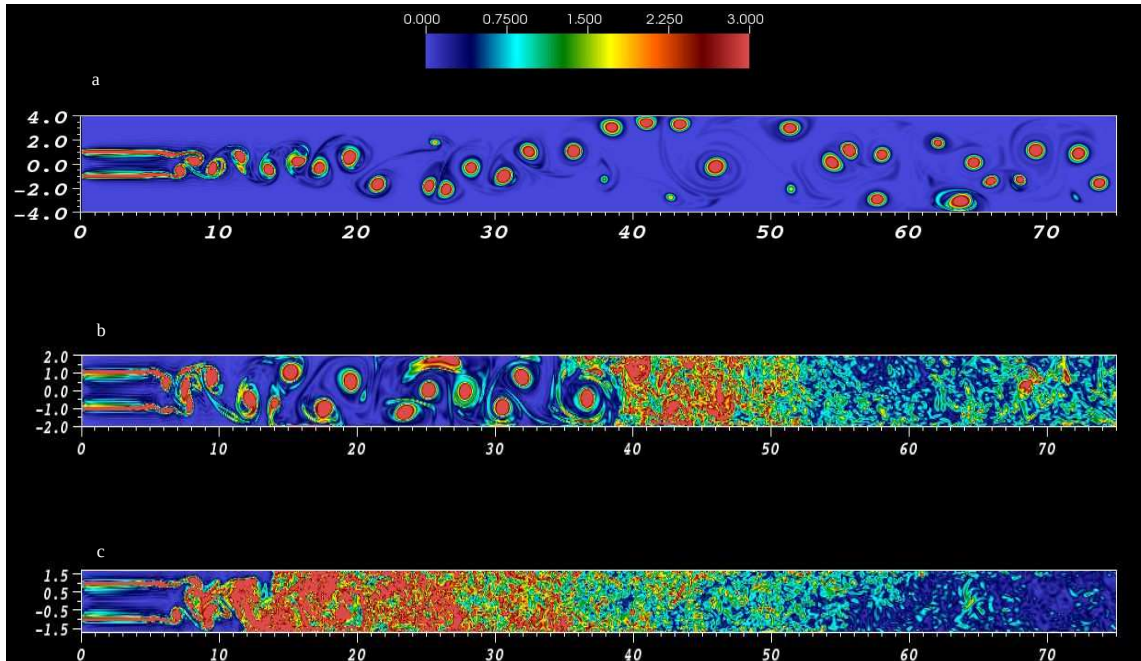


Figure 4.12: Instantaneous modulus $|\omega(x, y, 0)|$ of the vorticity at $t_f = 400$ for FS-plug flows: (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$.

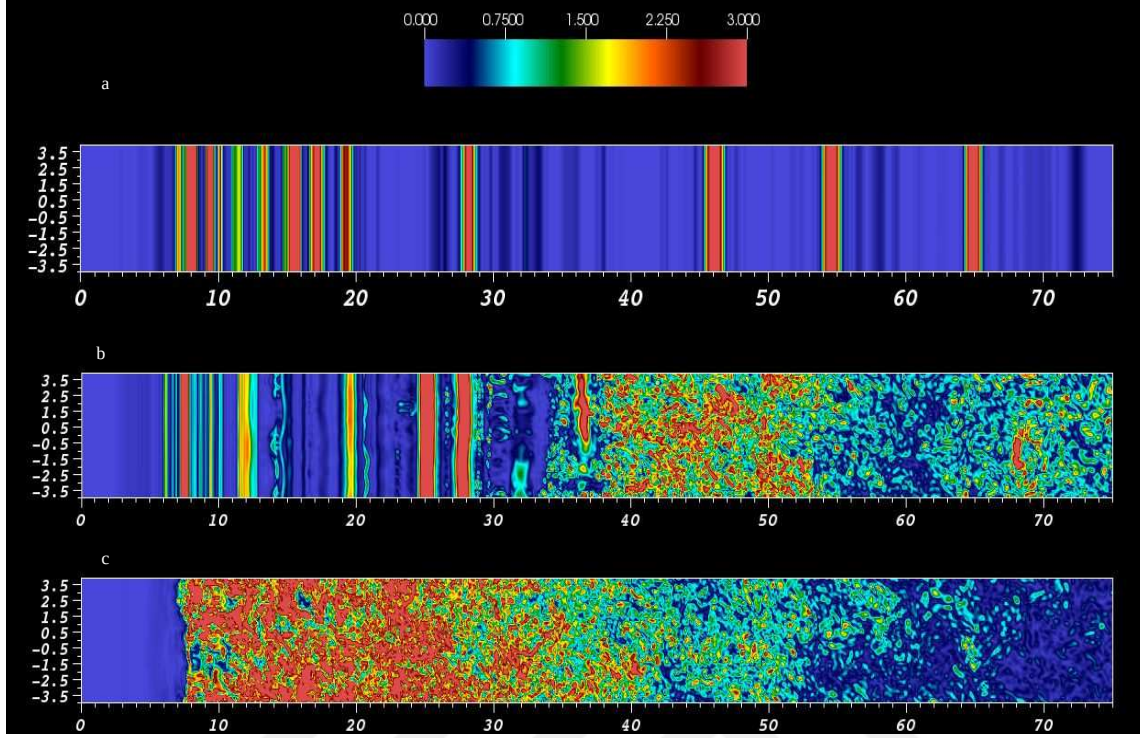


Figure 4.13: Instantaneous modulus $|\omega(x, 0, z)|$ of the vorticity at $t_f = 400$ for FS-plug flows: (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$.

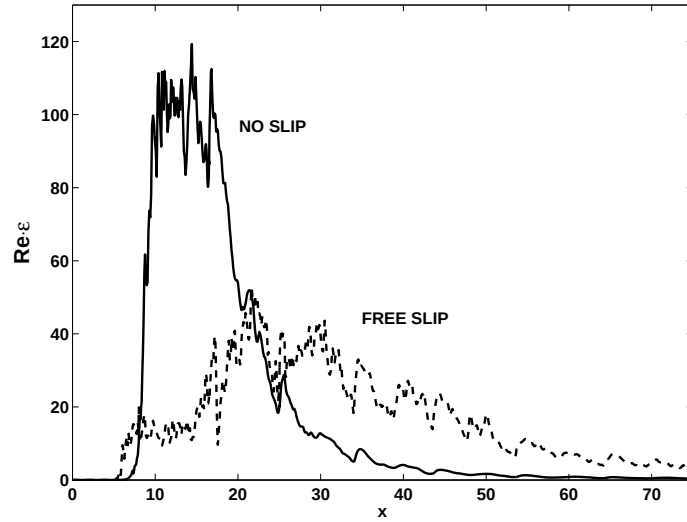


Figure 4.14: Dissipation rate of turbulent kinetic energy ε multiplied by Re as a function of longitudinal direction x for $y = 0$ at $z = 0$ for the plug flows with $h = 1$. The continuous line depicts no slip conditions, while dashed one free slip conditions.

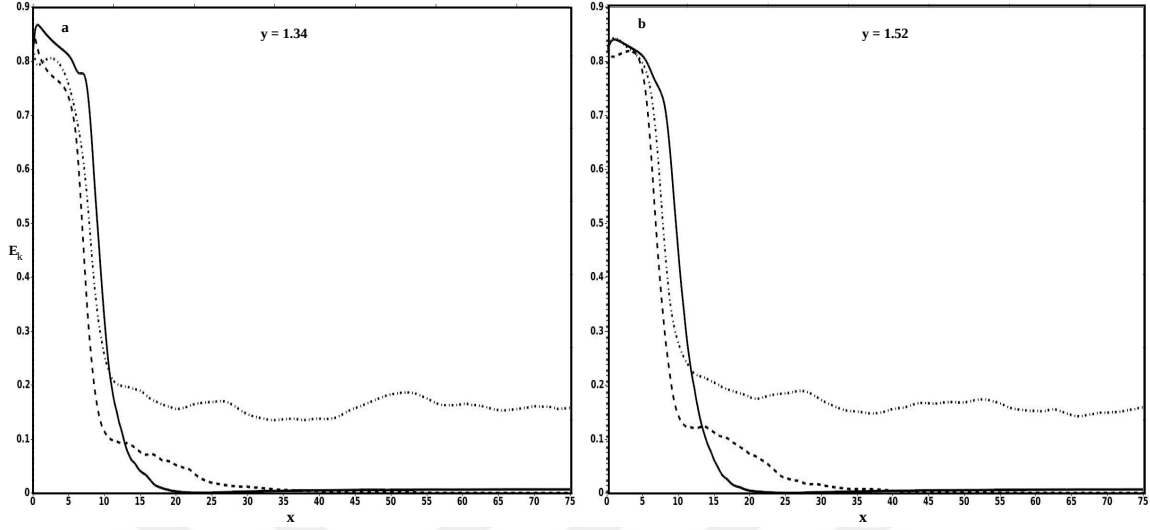


Figure 4.15: Turbulent kinetic energy for the FS-plugs flows with respect to streamwise direction x at (a) $y = 1.34$ and (b) $y = 1.52$. The continuous line depicts $h = 0.7$, the dashed line $h = 1$, and the dotted one $h = 3$. Note that E_k is averaged along the spanwise direction z .

at the final time of computations $t_f = 400$. Comparing with NS-plug flows (figures 4.1 and 4.2), we can note that for each configuration the NS conditions seem to enhance the 3D character of the turbulence, e.g. for $h = 1$ the turbulent front is moved downstream. Moreover for the weakest confinement $h = 3$, no clear turbulent front is detected and the flow seems to be characterized only by two-dimensional turbulence. However, even without boundary layer the confinement seems to promote a fully three-dimensional turbulence. The analysis of the one-dimensional velocity spectra and the third-order structure function of the velocity can be found for $h = 3$ in section 4.3.4.3, in figure 4.19.

In figure 4.14 the dissipation rate ε computed by using eq. (4.17) is shown for $h = 1$ for FS- and NS-plug flows, which are depicted by dashed and continuous line, respectively. Note that ε is multiplied by the value of the Reynolds number. One can note that the dissipation is stronger for NS conditions than for the FS conditions.

Finally, figure 4.15 shows the behaviour of the turbulent kinetic energy in the streamwise direction x at (a) $y = 1.34$ and (b) $y = 1.52$ for FS-plug flows. The growth in turbulent kinetic energy with respect to confinement is also clear in that case.

The comparison between FS/NS conditions suggests the role of the boundary layers at the walls. First of all, their presence enhances the 3D turbulent nature of the flows and subsequently their interaction with the shear layers increases the development of the turbulence itself. However, the role of the boundary layers cannot explain alone the growth in turbulent kinetic energy due to the confinement, since also

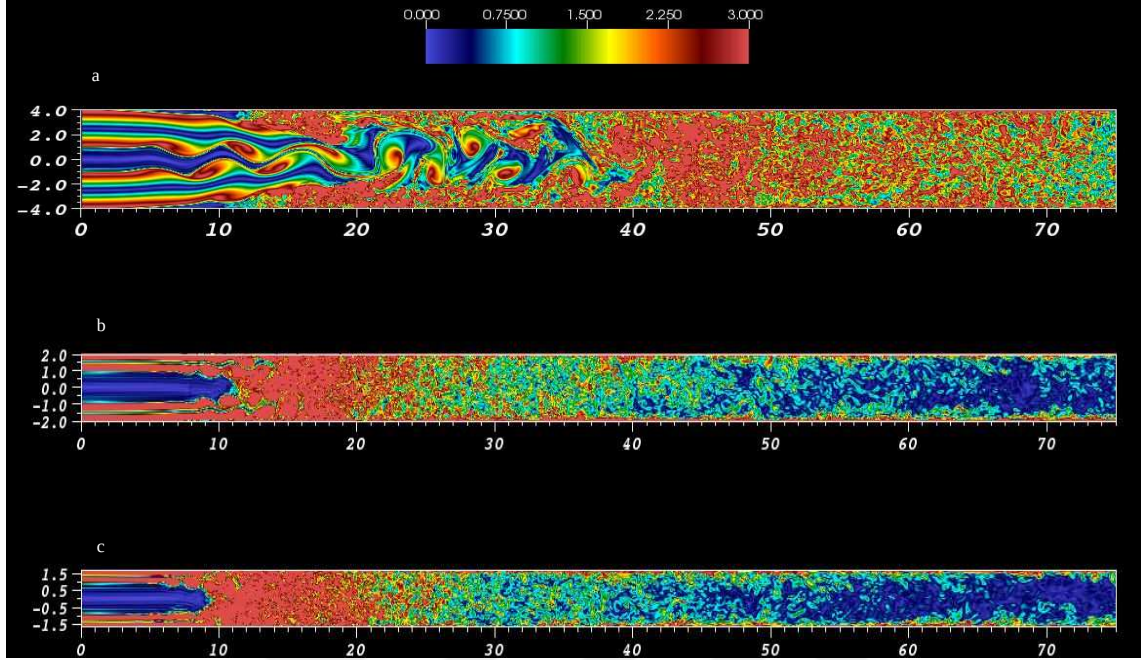


Figure 4.16: Instantaneous modulus $|\omega(x, y, 0)|$ of the vorticity at $t_f = 400$ for the set of Poiseuille flows: (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$ for $Re = 5000$

for FS-plug flows, the turbulent character of the wakes is enhanced by confinement. Remember that for lower Reynolds number (see chapter 3), $Re = 100, 500$, we have seen that when implementing FS conditions instead of NS conditions, the flows are destabilized. Thus when boundary layers occur a significant difference is noticed between laminar and turbulent wake flows.

4.3.4.2 Poiseuille flows

In this section we analyze the turbulent characteristics of the wake flow using as inlet profile the set of Poiseuille flows described in section 3.2.1. Figures 4.16 and 4.17 show the instantaneous modulus $|\omega|$ of the vorticity in the plane (x, y) at $z = 0$ and (x, z) for $y = 0$, respectively, at the final time of computations $t_f = 400$ for the three different confinement configurations: (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$. For Poiseuille flows each confinement configuration seems to favor a fully three-dimensional turbulence. In next section, 4.3.4.3, in particular in figure 4.19, a confirmation of this intuition is provided. This fact is not surprising, since its base flow is characterized by the presence of two shear layers close to $y = \pm 1$ (figure in section 3.2.1c), whereas for plug flows only one shear layer is present (figure in section 3.2.1b). Hence the presence of more vorticity is expected at the inlet. Moreover, Poiseuille inlet profiles are certainly not realistic for turbulent flows and therefore can induce high perturbations and finally turbulence.

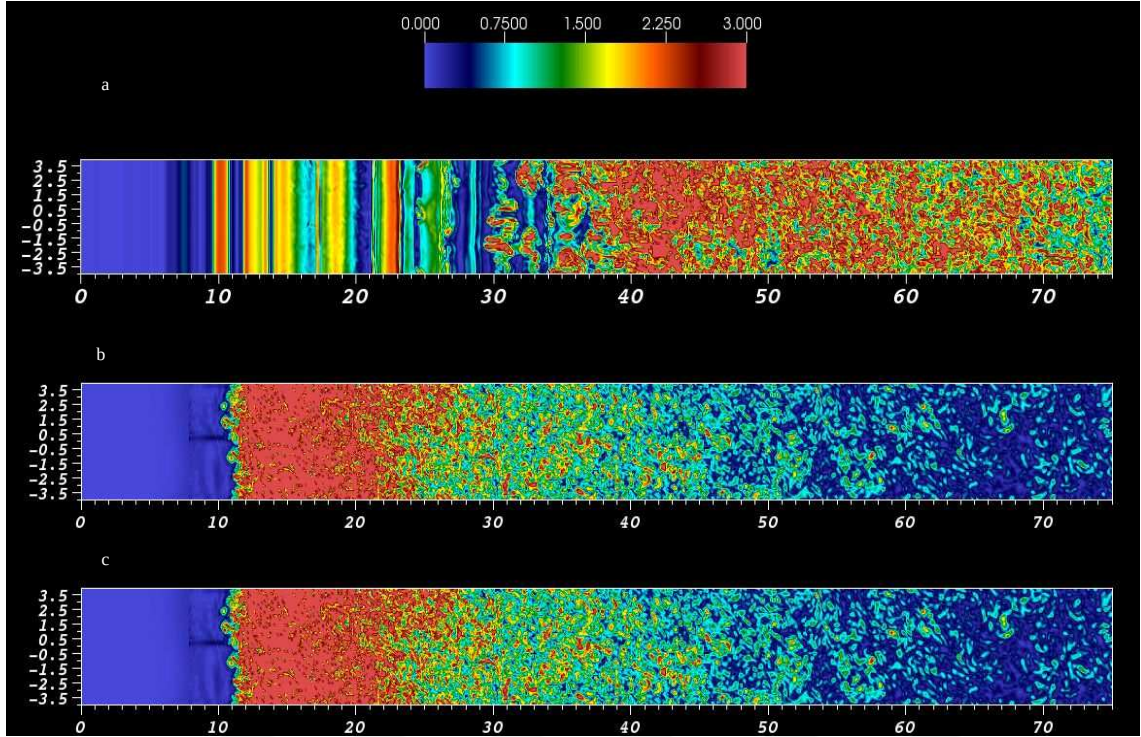


Figure 4.17: Instantaneous modulus of the vorticity $|\omega(x, 0, z)|$ at $t_f = 400$ for the set of Poiseuille flows: (a) $h = 3$, (b) $h = 1$, and (c) $h = 0.7$ for $Re = 5000$.

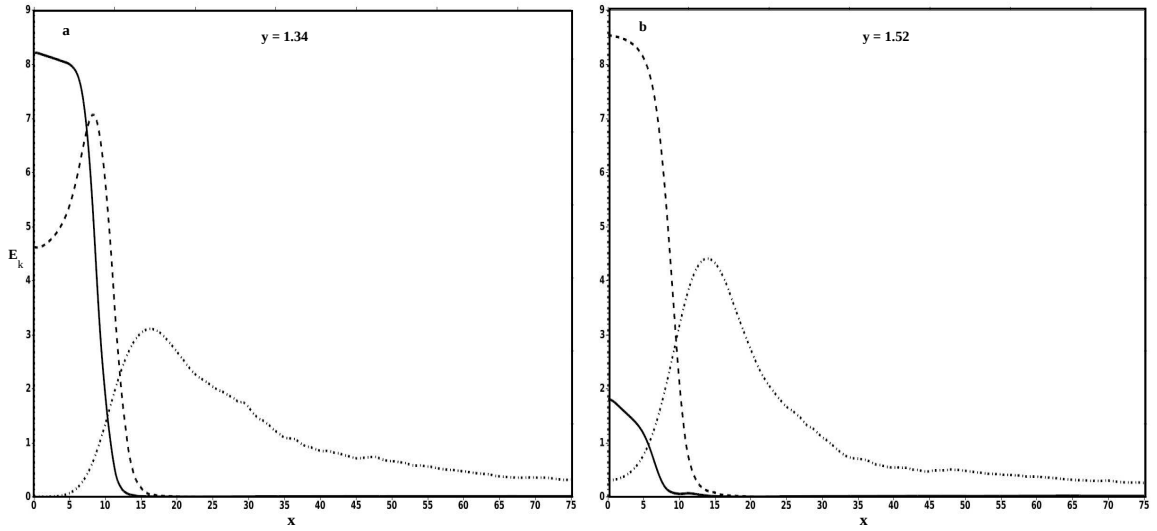


Figure 4.18: Turbulent kinetic energy E_k with respect to streamwise direction x for Poiseuille flows at (a) $y = 1.34$ and (b) $y = 1.52$. The continuous line depicts $h = 0.7$, the dashed line $h = 1$, and the dotted one $h = 3$. Note that E_k is averaged along the spanwise direction z .

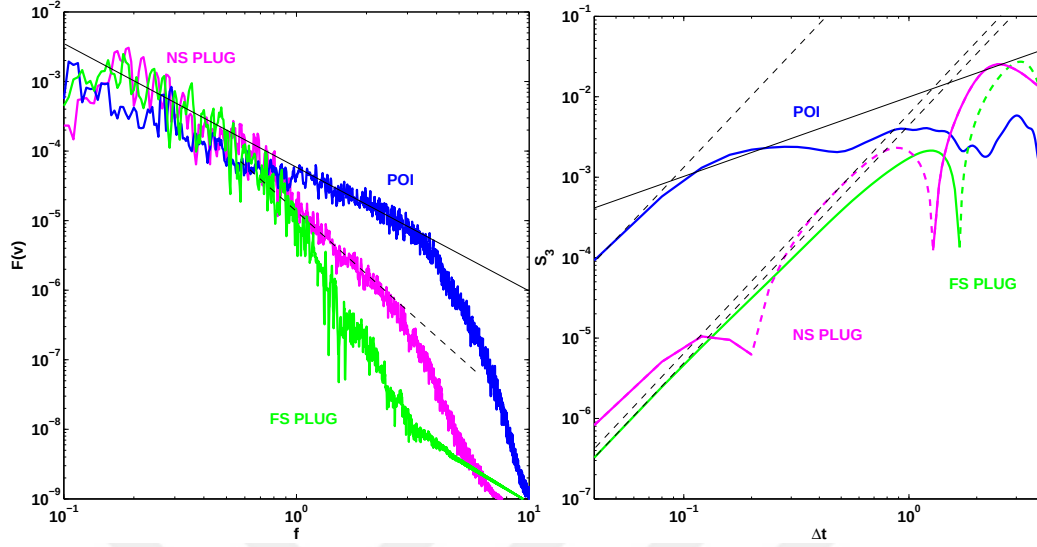


Figure 4.19: (a) One-dimensional velocity spectra and (b) absolute value of the third order structure function S_3 for the three wake models versus time shift for $v(x = 51.58, y = 0)$ for $h = 3$: FS-plug flows (green line), NS-plug flows (magenta line), and Poiseuille flows (blue line). Note that we have averaged the spectra between several z . (a) Thin lines represent the slope of $-5/3$ (continuous) and -3 (dashed). (b) The continuous lines depict the negative values of S_3 , while dashed ones depict the positive values. Thin lines represent the slope of 1 (continuous) and 3 (dashed).

The influence of confinement on the turbulent kinetic energy is more complex. However also for Poiseuille flows, there is a region close to the inlet for which the turbulent kinetic energy is enhanced by confinement, as shown in figure 4.18. Here, the turbulent kinetic energy is represented with respect to streamwise direction x for two different y : (a) $y = 1.34$, and (b) $y = 1.52$. In the former case, the strongest confinement $h = 0.7$ presents the highest values of the turbulent kinetic energy, while in the latter one the medium confinement $h = 1$ is close to the inlet the most turbulent. Also in that case, for large x , the turbulent kinetic energy of the weakest confined flow is less dissipated.

4.3.4.3 2D vs 3D turbulence

A brief panorama on the two-/three-dimensional character of the turbulence for the different models is illustrated in this section for confinement parameters: $h = 3$ and $h = 1$.

First of all, the fully 3D character of the turbulence for Poiseuille flows, even for $h = 3$, is suggested by figure 4.19, where (a) the one-dimensional velocity spectra and (b) the absolute value of the third order velocity structure function $|S_3(\Delta t)|$ are shown for the three models. The third order-velocity structure function (figure

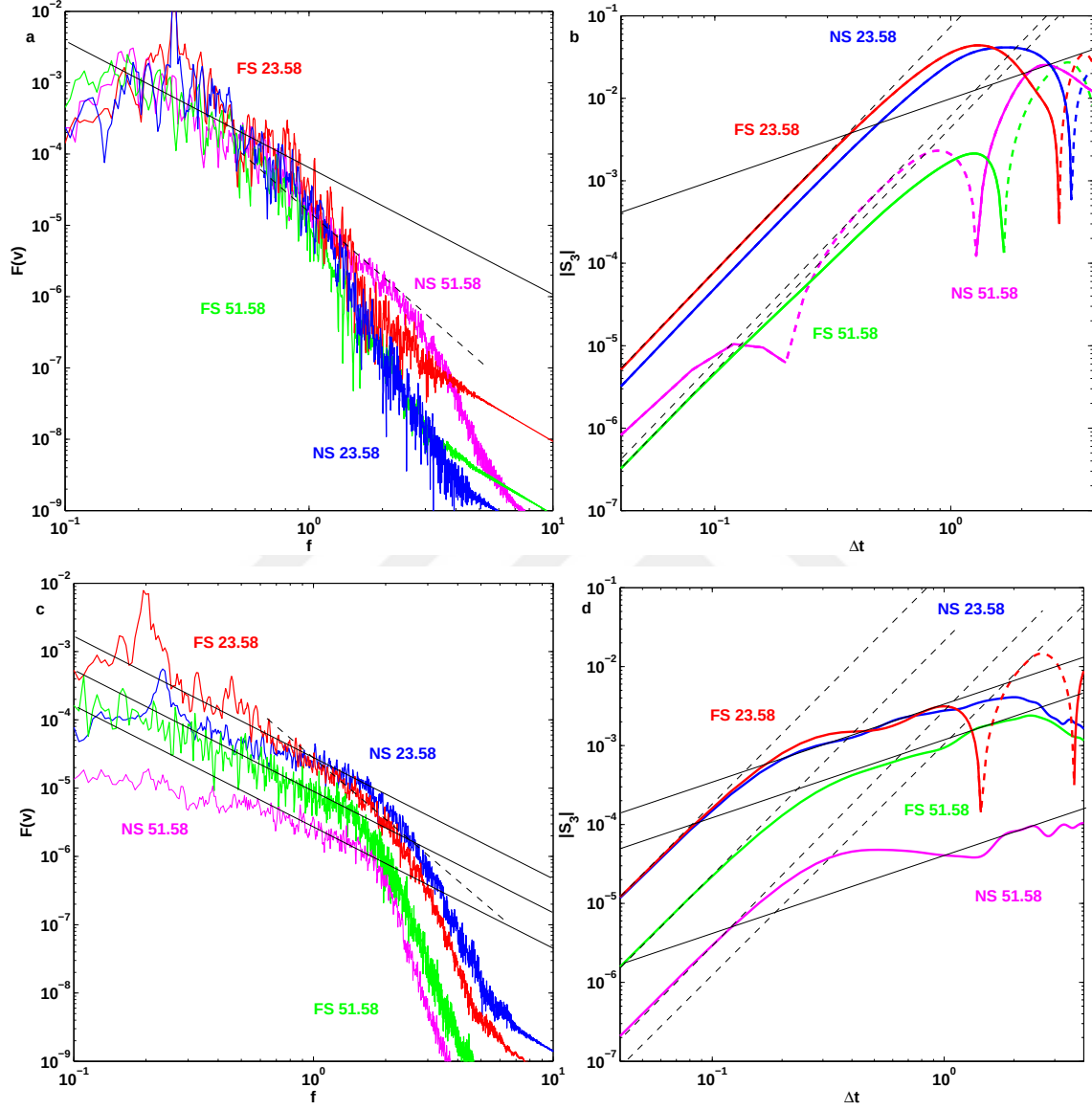


Figure 4.20: (a,c) One-dimensional velocity spectra and (b,d) absolute value of the third order structure function S_3 of plug flows versus time shift with (a,b) $h = 3$ and (c,d) $h = 1$. In these figures the two quantities are measured at two different abscissa $x = 23.58$ (blue: NS, red: FS) and $x = 51.58$ (magenta: NS and green: FS). (a,c) Thin lines represent the slopes of $-5/3$ (continuous) and -3 (dashed). (b,d) The continuous lines depict the negative values of S_3 , while dashed ones depict the positive values. Thin lines represent the slopes of 1 (continuous) and 3 (dashed).

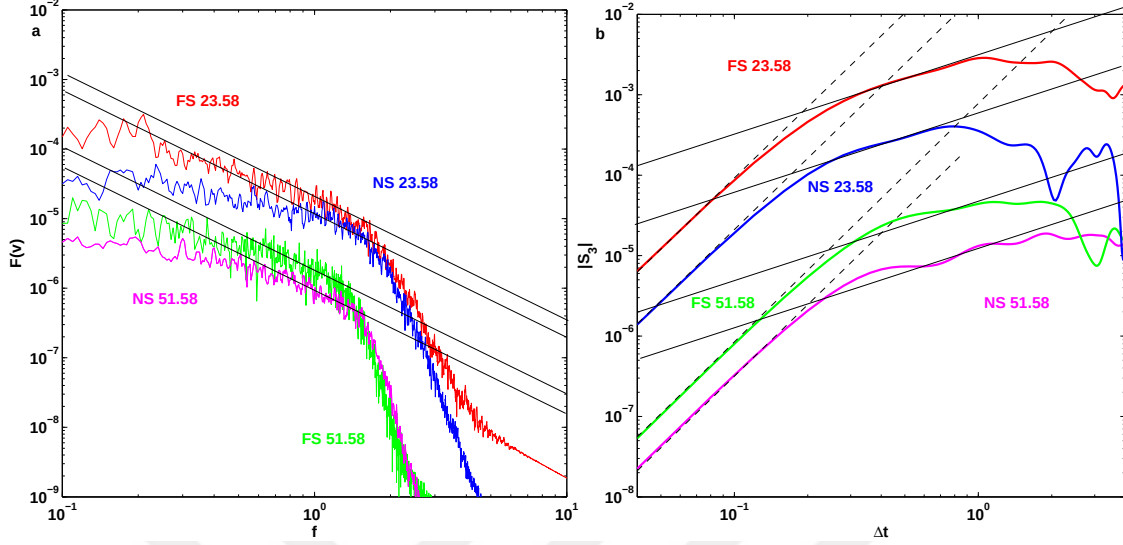


Figure 4.21: (a) One-dimensional velocity spectra and (b) absolute value of the third order structure function S_3 of plug flows versus time shift for $h = 0.7$. The legend is the same of figure 4.20.

4.19b) is still negative only for Poiseuille flows (blue line), and moreover the spectrum is characterized by a large inertial subrange, thus, by the presence of a direct cascade of the energy [86].

Conversely, both plug flows-models present a zone where $S_3(\Delta t)$ is positive, thus, the flow could show some 2D turbulent features. However, comparing the plug flows between FS/NS conditions, we can distinguish some differences. For FS-plug flows (green lines), only the largest scales are characterized by a positive third order-velocity structure function. Its spectrum does not show a direct cascade of the enstrophy, in contrast to NS-plug flows. Also at medium scales it seems that a dissipative subrange is already present and thus, that the effect of viscosity is not negligible, characteristic peculiar to laminar flows. The role of the boundary layers seems clear: their presence allows to 2D- and 3D-turbulent character to grow.

In order to confirm this conjecture, in figure 4.20 we illustrate the behaviour of (a,c) the velocity spectra and (b,d) the absolute value of the third order structure function of both NS- and FS-plug flows at two different abscissa, i.e. $x = 23.58$ and $x = 51.58$ for (a,b) $h = 3$ and (c,d) $h = 1$.

For $h = 3$ (figures 4.20 a,b), close to the inlet, i.e. $x = 23.58$, the spectra and the structure functions of the two models are very similar, sign that the effect of boundary layers is not yet relevant. Hence, both cases seem to be laminar. However, at $x = 51.58$ the NS-model shows both a two- and three-dimensional turbulent character, as already described in section 4.3.2.2, while for the FS-model the laminar nature seems to persist. For $h = 1$ (figures 4.20 c,d), at both abscissa the flows present a fully turbulent nature. However, for FS-plug flows, close to the inlet, an enstrophy

cascade occurs, sign of a two-dimensional turbulence. This fact is confirmed also by the positive range of the third order structure function, see figure 4.20d. The 2D turbulent character is lost downwards, as illustrated by the spectrum and the structure function at $x = 51.58$. In this case, the spectrum presents only a direct energy cascade (slope $-5/3$) and the structure function is always negative. However, we repeat that the models are far from the condition of isotropic and homogenous flows, thus, a perfect agreement of the curves of $S_3(\Delta t)$ in the inertial subrange with the slope of 1 cannot be expected, especially for cases where a change of sign occurs.

Finally in figure 4.21, the behaviour of (a) the velocity spectra and (b) the absolute value of the third order structure function are shown for $h = 0.7$. In this case, for both models and both abscissa the turbulence is only characterized by 3D features, since no cascade of entrophy is detected and moreover S_3 is always negative.

In conclusion, the interaction between the vortex street and the vorticity created at the boundaries seems to have a relevant role in the formation of turbulence, both in 2D and 3D regimes. For instance for $h = 3$, in the first part of the domain the FS- and NS-plug flows have a similar laminar behaviour, but when the boundary layers start to interact with the vortex street, 2D and 3D turbulent features are only detected in the latter model.

Chapter 5

Conclusions and perspectives



The main objective of the present thesis was to examine the influence of the confinement on some wake models by means of various kinds of analyses. Particularly, the results of this dissertation would have the aim to reveal the mechanism at the origin of the destabilizing influence encountered when recessing a coaxial injector of the rockets, noticed by Gill (1978) [35] and already confirmed by analyses of Juniper & Candel (2003) [57] and Juniper (2006) [54]. Different approaches have been used: inviscid parallel wakes have been investigated in the framework of the local stability theory (chapter 2), and more realistic viscous spatially developing wakes, both in laminar and turbulent regimes, have then been examined by means of direct numerical simulations (chapter 3) and large-eddy simulations (chapter 4), respectively.

5.1 Local stability analysis

Two models have been considered in order to determine the effect of confinement on wake flows: a piecewise broken-line profile including shear layer thickness and surface tension, and an uniform wake flow with a continuous profile, already used by Monkewitz [75] in the unconfined configuration. The local properties of these models have been examined by using three different approaches: temporal stability, spatio-temporal stability, and their reinterpretation by means of the counterpropagating Rossby waves (CRW) perspective.

The first part of the local stability analysis was devoted to a temporal stability analysis describing how the addition of both the shear layer thickness and the surface tension on confined top hat profile [54] modifies the effect of confinement. While the shear layer thickness appears to have a negligible influence, we have shown that when including surface tension, a right amount of confinement has a strong destabilizing influence, both for the varicose and sinuous modes, see figures 2.6 and 2.7, respectively. Including both parameters, the appearance of a second mode is observed and a counterintuitive destabilization due to the surface tension is moreover noticed.

Concerning the spatio-temporal analysis, we have extended Monkewitz's analysis [75] dedicated to the spatio-temporal stability of inviscid two-dimensional wakes by adding confinement to his model flow. We have focused our analysis on the joined influence of confinement and shear layer thickness on convectively/absolutely instability transition of the sinuous mode. For each value of the confinement parameter, we have shown that there is an optimal shear layer thickness for which the absolute instability is maximal, as seen in figure 2.22, where the overall maximum is seen to be attained at $h \sim 1$ and $\delta_w \sim 0.2$.

As seen in figure 30a, our main finding is that the strong destabilization at $h \sim 1$ discovered by Juniper [54] is almost insensitive to the addition of a finite shear layer thickness and constitutes a robust feature of confined wakes. Exhaustive properties have been given regarding frequency and wavenumber selection along the convective/absolute instability transition curves.

The broken line approximation of the family of continuous profiles yields satisfactory temporal and spatio-temporal stability results when the shear layer thickness is small enough. As δ_w grows, the influence of the shear layer mode, which is not well captured by the broken line model, starts to be felt, specially at strong confinement or large shear layer thickness. Hence the broken line approximation breaks down.

These results further help to interpret the influence of confinement on the Strouhal number experimentally measured in confined cylinder flows. While we have found a good agreement when comparing the frequencies computed in this study with those experimentally measured by Rehimi *et al.* [93], it was not possible to estimate the exact influence of the shear layer thickness in absence of provided velocity profiles past the cylinder.

Furthermore, we have also revisited the temporal stability of the confined piecewise broken-line profile including or excluding the surface tension in the light of the counterpropagating Rossby waves (CRW) perspective [13, 48]. To this end, we have introduced the capillarity waves (CWs) and studied their interaction with the CRWs on jet/wake flows. The Kelvin-Helmoltz instability has been also illustrated under the CRW perspective by considering a vanishing shear layer thickness, both including or excluding the surface tension in the model, recovering the classical stability results.

Last, the spatio-temporal stability analysis has been revisited by using the CRW perspective. In particular, we have tried to explain the destabilizing effect noticed at moderate confinement by Juniper [54] (that was recovered when adding the shear layer thickness). We have concluded that this destabilization, triggered by the confinement, is due to the dependance on the confinement of the group velocity of the CRWs taken freely to move, so without interacting one with each other, see figure 2.37.

5.2 Direct numerical simulations

In chapter 3, the global stability of laminar non-parallel wake flows, solutions of the Navier-Stokes equations, was studied, implementing both free slip (FS) and no slip (NS) conditions. Concerning FS conditions, the stability is studied by varying the velocity ratio Λ , the confinement ratio h and the Reynolds number Re . We have constructed a stability diagram for $Re = 100$ and $Re = 500$ in order to analyze the effect of viscosity on such flows. These diagrams demonstrate the destabilizing influence of a moderate confinement (order one) according qualitatively to the theoretical studies of Juniper (2006) [54]. The main difference with Juniper's results is the strong instability at weak confinements and sustained co-flow for $Re = 500$, which takes the form of a vacillating front. An increase in viscosity, as analyzed at $Re = 100$, stabilizes the flow and quenches this instability, yielding thus moderate confinement to be the most unstable flow configuration, as predicted by Juniper. The discrepancies between Juniper's theory and results of DNS are attributed to the non-parallel character of the base flow. A shear layer thickness is created by the viscous diffusion which modifies the base flow velocity profile, thus allowing the velocity ratio to vary

along the domain. Two different comparisons with the local stability analysis have been commented: the first one makes use fitted profiles, i.e. the confined Monkewitz's velocity profile [75] (see chapter 2), while the second one makes use extracted local velocity profiles from the base flow. In both cases, the local stability analysis has given satisfying predictions, specially for $Re = 500$, for which the base flow is closer to the conditions of parallelism. However, the existence of a region of absolute instability was shown neither to be a sufficient nor a necessary condition for global instability. The vacillating instability in particular could not be related to local properties.

We have studied two different inlet profiles implementing no slip conditions: the top hat profile (or plug flow) already analyzed with FS conditions and a set of Poiseuille flows. At $Re = 100$, for the plug flows the confinement has a destabilizing influence at moderate values, while it is still stabilizing for Poiseuille flows. At $Re = 500$, for plug flows there is a small destabilizing effect at $h = 6$, otherwise the confinement always stabilizes the flow for both models. Comparing FS- and NS-plug flows, we have seen that normally the presence of boundary layers is stabilizing. However, at $Re = 100$ and $h < 0.5$, a counterintuitive destabilization due to boundary layers was noticed. Also in that case a local stability analysis conducted on the extracted local velocity profiles from the base flows has given satisfying prediction.

Furthermore, at $Re = 500$ an interesting and surprising phenomenon was revealed conducting these simulations: for several cases and for all wake models, although the wave front does not exit from the domain, it is not fixed in space and in time, but vacillates around a mean position, destabilizing also strong co-flow wakes, see figure 3.6 for an example. For this reason, we have distinguished between two different instabilities at $Re = 500$: the Von Kármán instability (VKI), when the front is fixed in space and in time and front vacillation instability (FVI), when it vacillates. While the local linear stability analysis fails to predict the destabilization due to vacillations, a linear global analysis conducted by Philippe Meliga shows several unstable global modes which seem to be associated to vacillations. However, the linear global analysis furthermore presents remarkable differences with the DNS, which is a sign of a significant role of the nonlinearity in this phenomenon. Finally, while the limit between FVI and stable configurations does not depend on the size of the domain, the limit between FVI/VKI is largely sensible on variations of its length.

The influence of the confinement on the Strouhal number was also analyzed. We have noticed a significant difference between the dependence on confinement of the Strouhal number referred to scales used in this thesis, i.e. half-width of the wake and average velocity, and the Strouhal number referred to scales normally used in confined cylinder wakes, i.e. width of the wake and mean flow velocity. In the latter case, the confinement enhances the value of the Strouhal number, similarly to what is observed for confined cylinder wakes.

5.3 Large-eddy simulations

By means of large-eddy simulations (SVV-LES code) we have also studied the influence of confinement on turbulent obstacle-free wakes. We have shown that confinement enhances the turbulent character of the flow considering several quantities as one-dimensional spectra, third-order velocity structure function, turbulent kinetic energy and dissipation rate. In particular, we have seen that adding confinement enhances the three-dimensional nature of the turbulence and, moreover, that the kinetic turbulent energy is increased as well as the dissipation rate.

Comparing FS- and NS-plug flows, it was noticed that the presence of boundary layers increases the strength of the turbulence and consequently the dissipation rate, an opposite effect with respect to laminar flows where the implementation of NS conditions at the walls stabilizes wake flows.

Finally, also in the turbulent regime the influence of the confinement on the Strouhal number was examined, illustrating a remarkable difference when referring St to the scales used in the present thesis or to the usual scales of confined cylinder wakes. In the former case, the Strouhal number was enhanced by confinement, while in the latter case, it is decreased.

5.4 Perspectives

The question is how the results presented in this study could be useful for future analyses. It is clear that the main limitation of the present thesis is the absence of experimental studies concerning the geometry described in the previous chapters. Experiments can be achieved on several aspects treated in the present thesis. Concerning the temporal stability presented in section 2.3, an analogous situation of Holmboe's tube [87], where temporal stability theory directly applies, could be experimentally created in order to analyze the combined influence of surface tension and confinement. In order to confirm the suggestion given by section 2.5.5, it would be also interesting to conduct new experiments on confined cylinder wakes, to provide measurement of the shear layer thickness created by different geometries of the cylinder and comparisons with the present results, in the same spirit as Monkewitz & Nguyen [74] and Unal & Rockwell [105] for unconfined cylinder flows. However we suggest that the most useful experiments to conduct would be devoted to confined wakes exiting from a coaxial injector [57] for both laminar and turbulent regimes with the aim to unveil if the destabilizing effect of the confinement noticed in the present thesis is confirmed for several quantities: appearance of the laminar instability and of coherent structures along the spanwise direction, transition to turbulence and, finally, strength of the turbulence itself.

Another aspect which has to be explained is the phenomenon of vacillations. Several theories can be formulated about this mechanism. The growth of the unstable character of vacillations at weak confinement could suggest that the role of the wall

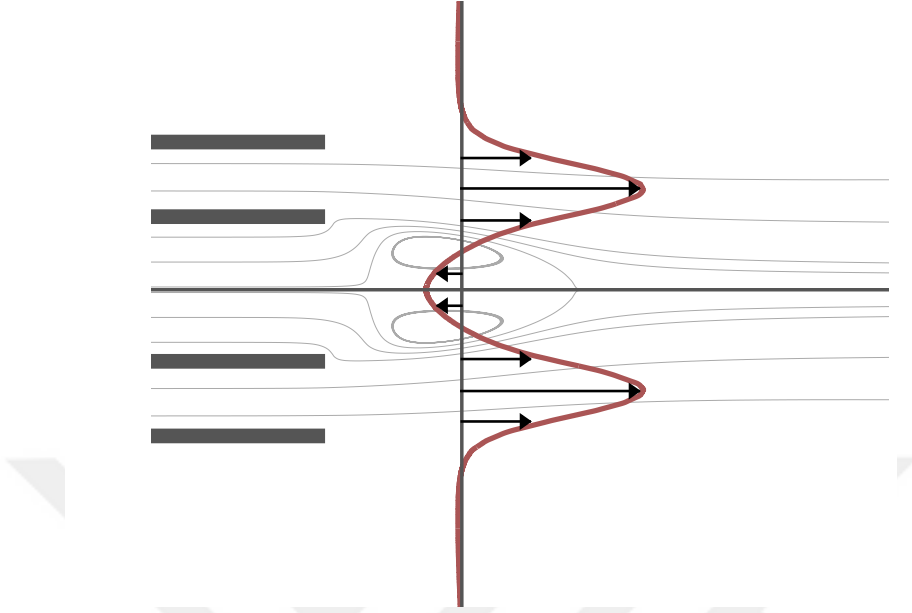


Figure 5.1: Sketch of a geometry mimicking a real coaxial injector.

normal-reflecting modes discovered by Healey [42] cannot be excluded. Perhaps a resonance between two convectively unstable modes as the von Kármán and the wall normal-reflecting ones could create several global unstable modes, as shown in figure 3.21 in the linear global analysis conducted by Philippe Meliga. Another interpretation could be based on the remarkable dependence of the persistence of vacillations on variations of the length of the domain. Interpreting the exit conditions as a soft obstacle, the vacillations would be due to streamwise-reflecting modes growing upwards along the domain, as e.g. found by Chomaz *et al.* (1988) [19] for blunt body-wakes with a second blunt body located downstream. However these are only speculations and studies devoted to understand this phenomenon would be very useful. In particular, an analysis of the negative k_r -plane could be interesting to understand the role of the Healey's reflecting modes [42], while more focused linear as well as non linear global analyses could illustrate the connection between vacillations and length of the domain.

Another limitation of the present thesis on modeling coaxial injector is the planar geometry, since a round geometry is closer to reality. Despite that Juniper (2008) [56] has shown an evident similitude between planar and round wakes concerning the local stability analysis, this fact has to be confirmed by means of more realistic direct numerical simulations and large-eddy simulations of turbulent cylindrical wake flows. However, we can also try to imagine simulations mimicking a real injector geometry. In figure 5.1 we illustrate a conjectural model which could be used for new numerical simulations, a reminiscence of the work of Rehab *et al.* [92], but implementing the confinement.

For local stability analysis we have discussed in chapter 2 about surface tension

and slightly different densities. These aspects have been deeply analyzed by Rees & Juniper (2009) [90], but no simulations have been carried out in this thesis considering two-phase fluids. This work coupled to the present study suggest the existence of an optimal value of surface tension and density ratio to maximize the instability. Recently, Tammisola *et al.* [100] have conducted a linear global analysis considering confined wakes including surface tension for two Reynolds number $Re = 100$ and $Re = 316$. They have however not examined variations of the surface tension coupled to those of the confinement. In conclusion, simulations in the same spirit of those of Tammisola *et al.* [100] considering density ratio and surface tension can be conducted and compared with results obtained in the present thesis and by Rees & Juniper (2009) [90]. Also the effect of these parameters on the strength of the turbulence at high Reynolds number is an important aspect.

Finally, in chapter 4, the character of the turbulence has been distinguished in two-dimensional and three-dimensional settings. Although the confinement seems to favor the formation of a fully-developed 3D turbulence, it would be interesting to study the role of the confinement in a more theoretical fully 2D domain. The goal would be to discover if the strength of two-dimensional turbulence is favored by confinement thus excluding artificially the formation of 3D turbulence.



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Appendix A

CRW perspective: calculations



A.1 Vanishing shear layer thickness

A.1.1 Piecewise broken line without surface tension

In this section, we report all the computations related to section 2.4.4.1. Note that we have followed the procedure described by Haragus & Iooss (2011) [39] in their chapter 3.

Remembering that

$$M = \frac{1}{\delta_w}(L_0 + \delta_w L_1 + \delta_w^2 L_2) \quad (\text{A.1})$$

the notation is simplified as follows: $\frac{kB}{2A} = B'$, $\frac{kC}{2A} = C'$, $\frac{k^2 D}{8A} = D'$, $\frac{k^2 A}{8A} = A'$, $\frac{k}{2\Lambda A} = U'$, and $\frac{k}{2A} = k'$. Hence, L_0 , L_1 and L_2 become:

$$L_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad (\text{A.2})$$

$$L_1 = \begin{bmatrix} (B' - C') - k' - U' & -k \\ k' & (B' - C') + k' - U' \end{bmatrix}, \quad \text{and} \quad (\text{A.3})$$

$$L_2 = \begin{bmatrix} -2(D' - A') & 2(D' + A') \\ -2(D' + A') & 2(D' - A') \end{bmatrix}, \quad (\text{A.4})$$

respectively, where each matrix is multiplied by $2i\Lambda A$. The matrix L_0 has one eigenvalue, $\lambda_0 = 0$ with multiplicity 2 and, therefore, one eigenvector ξ_{10} such that $L_0 \xi_{10} = 0$ and one *generalized* eigenvector ξ_{20} such that $L_0 \xi_{20} = \xi_{10}$. These eigenvectors are

$$\xi_{10} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \xi_{20} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (\text{A.5})$$

respectively. The adjoint matrix L_0^* of L_0 is

$$L_0^* = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad (\text{A.6})$$

and its eigenvectors

$$\xi_{10}^* = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \xi_{20}^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (\text{A.7})$$

Note that the first one ξ_{10}^* is the generalized adjoint eigenvector and that $\langle \xi_{i0} \cdot \xi_{j0}^* \rangle = \delta_{i,j}$. The normal form of $L = L_0 + \delta_w L_1 + \delta_w^2 L_2$ is given by $J = J_0 + J_1 \delta_w + J_2 \delta_w^2$, where J_0 is a Jordan block, while J_1 and J_2 are

$$J_{1,2} = \begin{bmatrix} \alpha_{1,2} & 0 \\ \beta_{1,2} & \alpha_{1,2} \end{bmatrix}, \quad (\text{A.8})$$

as reported in section 2.4.4.1. We suppose that new bases $\xi_1(\delta_w)$, $\xi_2(\delta_w)$ are close to eigenvectors ξ_{10} and ξ_{20} and calling X the matrix with eigenvectors $\xi_1(\delta_w)$ and $\xi_2(\delta_w)$ we have $LX = XJ$, which gives

$$L\xi_1(\delta_w) = \alpha(\delta_w)\xi_1(\delta_w) + \beta(\delta_w)\xi_2(\delta_w) \quad \text{and} \quad (\text{A.9a})$$

$$L\xi_2(\delta_w) = \xi_1(\delta_w) + \alpha(\delta_w)\xi_2(\delta_w). \quad (\text{A.9b})$$

Expanding equations (A.9) until to the second order of δ_w , we obtain:

$$(L_0 + \delta_w L_1 + \delta_w^2 L_2)(\xi_{10} + \delta_w \xi_{11} + \delta_w^2 \xi_{12}) = \quad (\text{A.10a})$$

$$= \delta_w [(\alpha_1 + \delta_w \alpha_2)(\xi_{10} + \delta_w \xi_{11} + \delta_w^2 \xi_{12}) + (\beta_1 + \delta_w \beta_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22})],$$

$$(L_0 + \delta_w L_1 + \delta_w^2 L_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22}) = \quad (\text{A.10b})$$

$$= (\xi_{10} + \delta_w \xi_{11} + \delta_w^2 \xi_{12}) + \delta_w (\alpha_1 + \delta_w \alpha_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22}).$$

We can only consider the first order of δ_w :

$$L_0 \xi_{11} + L_1 \xi_{10} = \alpha_1 \xi_{10} + \beta_1 \xi_{20}, \quad (\text{A.11a})$$

$$L_0 \xi_{21} + L_1 \xi_{20} = \xi_{11} + \alpha_1 \xi_{20}. \quad (\text{A.11b})$$

Computing the scalar product of both sides of the eq. (A.11a) with ξ_{20}^* and taking out the real constants (α_1 and β_1), we can write:

$$\beta_1 = \langle L_1 \xi_{10}, \xi_{20}^* \rangle + \langle L_0 \xi_{11}, \xi_{20}^* \rangle. \quad (\text{A.12})$$

Furthermore, we can simplify remembering

$$\langle L_0 \xi_{11}, \xi_{20}^* \rangle = \langle \xi_{11}, L_0^* \xi_{20}^* \rangle = 0. \quad (\text{A.13})$$

Therefore, $\beta_1 = 0$. Conversely in order to obtain α_1 , an estimate of ξ_{11} is needed

$$\xi_{11} = \alpha_1 \xi_{20} + \widetilde{\xi_{11}} + \alpha_{11} \xi_{10}, \quad (\text{A.14})$$

where $\widetilde{\xi_{11}}$ is perpendicular to ξ_{10}^* and α_{11} is arbitrary and so fixed equal to zero. Replacing eq. (A.14) into eq. (A.11b) and always computing the scalar product with ξ_{20}^* , the following expression is obtained for α_1 :

$$2\alpha_1 = \langle L_1 \xi_{20}, \xi_{20}^* \rangle - \langle \widetilde{\xi_{11}}, \xi_{20}^* \rangle. \quad (\text{A.15})$$

Now, we substitute eq. (A.14) into the eq. (A.11a) to obtain $\widetilde{\xi_{11}}$

$$L_0 \widetilde{\xi_{11}} = -L_1 \xi_{10} \quad (\text{A.16})$$

and, therefore:

$$\widetilde{\xi_{11}} = \begin{bmatrix} -B' + C' + U' \\ 0 \end{bmatrix}. \quad (\text{A.17})$$

Replacing eq. (A.17) into eq. (A.15) $\alpha_1 = B' - C' - U'$ is computed. Hence, we can write ξ_{11} from eq. (A.14):

$$\xi_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (\text{A.18})$$

On the other hand, starting from eq. (A.11b), it is possible to obtain ξ_{21}

$$\xi_{21} = \begin{bmatrix} k' \\ 0 \end{bmatrix} \quad (\text{A.19})$$

and conclude the calculations concerning the first order of δ_w .

Now, we can only consider the terms of eqs. (A.10a) with δ_w^2 :

$$L_0\xi_{12} + L_2\xi_{10} + L_1\xi_{11} = \alpha_2\xi_{10} + \beta_2\xi_{20} + \alpha_1\xi_{11} + \beta_1\xi_{21}, \quad (\text{A.20a})$$

$$L_0\xi_{22} + L_2\xi_{20} + L_1\xi_{21} = \xi_{21} + \alpha_2\xi_{20} + \alpha_1\xi_{21}. \quad (\text{A.20b})$$

From eq. (A.20a) it is possible to obtain β_2 , computing the scalar product of both sides with ξ_{20}^*

$$\beta_2 = \langle L_2\xi_{10}, \xi_{20}^* \rangle + \langle L_1\xi_{11}, \xi_{20}^* \rangle - \alpha_1 \langle \xi_{11}, \xi_{20}^* \rangle = -8D'. \quad (\text{A.21})$$

To obtain α_2 , we have to estimate ξ_{12} as well as the first order:

$$\xi_{12} = \alpha_2\xi_{20} + \widetilde{\xi}_{12} + \alpha_{21}\xi_{10}, \quad (\text{A.22})$$

where $\widetilde{\xi}_{12}$ is perpendicular to ξ_{10}^* . For α_1 the following expression is reached

$$2\alpha_2 = \langle L_2\xi_{20}, \xi_{20}^* \rangle + \langle L_1\xi_{21}, \xi_{20}^* \rangle - \langle \widetilde{\xi}_{12}, \xi_{20}^* \rangle - \alpha_1 \langle \xi_{21}, \xi_{20}^* \rangle. \quad (\text{A.23})$$

Replacing eq. (A.22) into eq. (A.11b)

$$L_0\widetilde{\xi}_{12} = -L_2\xi_{10} - L_1\xi_{11} + \beta_2\xi_{20} \quad (\text{A.24})$$

is obtained and, therefore

$$\widetilde{\xi}_{12} = \begin{bmatrix} -4D' \\ 0 \end{bmatrix} \quad (\text{A.25})$$

is computed. Substituting eq. (A.25) into eq. (A.23) we can calculate $\alpha_2 = -4D' + k'(B' - C' - U') + 4D' - k'(B' - C' - U') = 0$. Therefore, ξ_{12} is given

$$\xi_{12} = \begin{bmatrix} -4D' \\ 0 \end{bmatrix} \quad (\text{A.26})$$

from eq. (A.22). Finally, we can find ξ_{22} from eq. (A.20b):

$$\xi_{22} = \begin{bmatrix} -2(D' + A') + k'^2 \\ 0 \end{bmatrix}. \quad (\text{A.27})$$

Therefore the normal form J_M of the matrix M is

$$J_M = 2i\Lambda A \begin{bmatrix} B' - C' - U' & \frac{1}{\delta_w} \\ -8\delta_w D' & B' - C' - U' \end{bmatrix}. \quad (\text{A.28})$$

The eigenvalues of J_M are

$$\lambda_{1,2} = 2i\Lambda A(B' - C' - U' \pm \sqrt{-8D'^2}), \quad (\text{A.29})$$

which if are written with the old notation

$$\lambda_{1,2} = ik\Lambda(B - C) - ik \pm 2k\Lambda\sqrt{AD}. \quad (\text{A.30})$$

A.1.2 Piecewise broken line including surface tension

In this section, we report all the computations related to section 2.4.4.2. In order to simplify the notation we call $\frac{kB}{2A} = \frac{k \coth(kh)}{2} = B'$, $\frac{kC}{2A} = \frac{k \tanh(k)}{2} = C'$, $\frac{k^2 D}{8A} = D'$, $\frac{k^2 A}{8A} = A'$, $\frac{k}{2\Lambda A} = U'$, $\frac{k}{2A} = k'$, and $\frac{\sqrt{\Sigma k^3 A}}{4\Lambda A} = f'$. Hence, L_0 , L_1 , and L_2 become

$$L_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (\text{A.31})$$

$$L_1 = \begin{bmatrix} (B' - C') - k' - U' & -k' & -C' & -C' \\ k' & (B' - C') + k' - U' & B' & B' \\ f' & f' & -U' + 2f' & 0 \\ -f' & -f' & 0 & -U' - 2f' \end{bmatrix}, \quad (\text{A.32})$$

$$\text{and } L_2 = \begin{bmatrix} -2(D' - A') & 2(D' + A') & A' & A' \\ -2(D' + A') & 2(D' - A') & -A' & -A' \\ -C' f' & -B' f' & 0 & 0 \\ C' f' & B' f' & 0 & 0 \end{bmatrix}, \quad (\text{A.33})$$

where each matrix is multiplied by $2i\Lambda A$. The matrix L has one eigenvalue $\lambda_0 = 0$ with multiplicity 4, therefore has one eigenvector ξ_{10} such that $L\xi_{10} = 0$ with multiplicity 2, one generalized eigenvector ξ_{20} such that $L\xi_{20} = \xi_{10}$, and finally two eigenvectors ξ_{30} and ξ_{40} such that $L\xi_{30} = 0$ and $L\xi_{40} = 0$:

$$\xi_{10} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_{20} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_{30} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \xi_{40} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}. \quad (\text{A.34})$$

The adjoint of L_0 is L_0^* :

$$L_0^* = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \quad (\text{A.35})$$

and its eigenvectors (ξ_{i0}^*) have to be such that $\langle \xi_{i0} \cdot \xi_{j0}^* \rangle = \delta_{i,j}$:

$$\xi_{10}^* = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_{20}^* = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \xi_{30}^* = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \xi_{40}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad (\text{A.36})$$

where ξ_{10}^* is the generalized one. Remembering that the linear form J of L is given by $J = J_0 + \delta_w J_1 + \delta_w^2 J_2$, where

$$J_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad (\text{A.37})$$

$$J_{1,2} = \begin{bmatrix} \alpha_{1,2} & 0 & 0 & 0 \\ \beta_{1,2} & \alpha_{1,2} & W_{1,2} & Z_{1,2} \\ \varepsilon_{1,2} & 0 & \gamma_{1,2} & D_{1,2} \\ \eta_{1,2} & 0 & C_{1,2} & \varphi_{1,2} \end{bmatrix}, \quad (\text{A.38})$$

it is possible to write the relationship between L and J , $LX = XJ$, with X the matrix of eigenvectors ξ_{i0} of L . From this relationship we can find

$$(L_0 + \delta_w L_1 + \delta_w^2 L_2)(\xi_{10} + \delta_w \xi_{11} + \delta_w^2 \xi_{12}) = \delta_w[(\alpha_1 + \delta_w \alpha_2)(\xi_{10} + \delta_w \xi_{11} + \delta_w^2 \xi_{12}) + (\beta_1 + \delta_w \beta_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22}) + (\varepsilon_1 + \delta_w \varepsilon_2)(\xi_{30} + \delta_w \xi_{31} + \delta_w^2 \xi_{32}) + (\eta_1 + \delta_w \eta_2)(\xi_{40} + \delta_w \xi_{41} + \delta_w^2 \xi_{42})], \quad (\text{A.39a})$$

$$(L_0 + \delta_w L_1 + \delta_w^2 L_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22}) = (\xi_{10} + \delta_w \xi_{11} + \delta_w^2 \xi_{12}) + \delta_w(\alpha_1 + \delta_w \alpha_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22}), \quad (\text{A.39b})$$

$$(L_0 + \delta_w L_1 + \delta_w^2 L_2)(\xi_{30} + \delta_w \xi_{31} + \delta_w^2 \xi_{32}) = \delta_w[(W_1 + \delta_w W_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22}) + (\gamma_1 + \delta_w \gamma_2)(\xi_{30} + \delta_w \xi_{31} + \delta_w^2 \xi_{32}) + (C_1 + \delta_w C_2)(\xi_{40} + \delta_w \xi_{41} + \delta_w^2 \xi_{42})], \quad (\text{A.39c})$$

$$(L_0 + \delta_w L_1 + \delta_w^2 L_2)(\xi_{40} + \delta_w \xi_{41} + \delta_w^2 \xi_{42}) = \delta_w[(Z_1 + \delta_w Z_2)(\xi_{20} + \delta_w \xi_{21} + \delta_w^2 \xi_{22}) + (D_1 + \delta_w D_2)(\xi_{30} + \delta_w \xi_{31} + \delta_w^2 \xi_{32}) + (\varphi_1 + \delta_w \varphi_2)(\xi_{40} + \delta_w \xi_{41} + \delta_w^2 \xi_{42})]. \quad (\text{A.39d})$$

Considering only the first order of δ_w

$$L_0 \xi_{11} + L_1 \xi_{10} = \alpha_1 \xi_{10} + \beta_1 \xi_{20} + \varepsilon_1 \xi_{30} + \eta_1 \xi_{40}, \quad (\text{A.40a})$$

$$L_0 \xi_{21} + L_1 \xi_{20} = \xi_{11} + \alpha_1 \xi_{20}, \quad (\text{A.40b})$$

$$L_0 \xi_{31} + L_1 \xi_{30} = W_1 \xi_{20} + \gamma_1 \xi_{30} + C_1 \xi_{40}, \quad (\text{A.40c})$$

$$L_0 \xi_{41} + L_1 \xi_{40} = Z_1 \xi_{20} + D_1 \xi_{30} + \varphi_1 \xi_{40}, \quad (\text{A.40d})$$

is given. In order to calculate β_1 , we compute the scalar product for both sides of the eq. (A.40a) with ξ_{20}^* . Taking out of the scalar product, the real constants (α_1 , β_1 , etc.):

$$\beta_1 = \langle L_1 \xi_{10}, \xi_{20}^* \rangle + \langle L_0 \xi_{11}, \xi_{20}^* \rangle \quad (\text{A.41})$$

is given. Remembering that $\langle L_0 \xi_{11}, \xi_{20}^* \rangle = \langle \xi_{11}, L_0^* \xi_{20}^* \rangle$, we can obtain $\beta_1 = 0$. Replacing into the scalar product ξ_{20}^* with ξ_{30}^* and ξ_{40}^* , we can calculate $\varepsilon_1 = 0$ and $\eta_1 = 0$, respectively. As well as in section A.1.1, we have to estimate ξ_{11} in order to obtain α_1 :

$$\xi_{11} = \alpha_1 \xi_{20} + \widetilde{\xi}_{11} + \alpha_{11} \xi_{10} + \alpha_{13} \xi_{30} + \alpha_{14} \xi_{40}, \quad (\text{A.42})$$

where $\widetilde{\xi}_{11}$ is perpendicular to ξ_{10}^* , ξ_{30}^* , and ξ_{40}^* . α_{11} is arbitrary and can be settled equal to zero. Substituting eq. (A.42) into eq. (A.40b), it is possible to obtain α_{13} and α_{14} by means of the same procedure:

$$\alpha_{13} = \langle L_1 \xi_{20}, \xi_{30}^* \rangle = -f', \quad (\text{A.43})$$

$$\alpha_{14} = \langle L_1 \xi_{20}, \xi_{40}^* \rangle = f'. \quad (\text{A.44})$$

Conversely for α_1

$$2\alpha_1 = \langle L_1 \xi_{20}, \xi_{20}^* \rangle - \langle \widetilde{\xi}_{11}, \xi_{20}^* \rangle \quad (\text{A.45})$$

is given. To obtain $\widetilde{\xi}_{11}$ we substitute eq. (A.42) into eq. (A.40a)

$$L_0 \widetilde{\xi}_{11} = -L_1 \xi_{10} \quad (\text{A.46})$$

and, therefore

$$\widetilde{\xi}_{11} = \begin{bmatrix} -B' + C' + U' \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A.47})$$

is obtained. Replacing eq. (A.47) into eq. (A.45) $\alpha_1 = B' - C' - U'$ is obtained. From eq. (A.42), finally, we can write ξ_{11} :

$$\xi_{11} = \begin{bmatrix} 0 \\ 0 \\ f' \\ -f' \end{bmatrix}. \quad (\text{A.48})$$

Starting from eq. (A.40b), it is possible to obtain ξ_{21} :

$$\xi_{21} = \begin{bmatrix} k' \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{A.49})$$

Computing the scalar product with the suitable adjoint eigenvector ξ_{20}^* , ξ_{30}^* , and ξ_{40}^* , we obtain from eq. (A.40c):

$$W_1 = \langle L_1 \xi_{30}, \xi_{20}^* \rangle = -2f', \quad (\text{A.50})$$

$$\gamma_1 = \langle L_1 \xi_{30}, \xi_{30}^* \rangle = -U' + f', \quad (\text{A.51})$$

$$C_1 = \langle L_1 \xi_{30}, \xi_{40}^* \rangle = f', \quad (\text{A.52})$$

respectively. Replacing W_1 , γ_1 , and C_1 into eq. (A.40c), ξ_{31} is computed:

$$\xi_{31} = \begin{bmatrix} k' - B' \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{A.53})$$

Applying the same procedure to eq. (A.40d):

$$Z_1 = \langle L_1 \xi_{40}, \xi_{20}^* \rangle = 2f', \quad (\text{A.54})$$

$$D_1 = \langle L_1 \xi_{40}, \xi_{30}^* \rangle = -f', \quad (\text{A.55})$$

$$\varphi_1 = \langle L_1 \xi_{40}, \xi_{40}^* \rangle = -U' - f', \quad \text{and} \quad (\text{A.56})$$

$$\xi_{41} = \begin{bmatrix} k' - B' \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{A.57})$$

In order to obtain the corrections due to the second order of δ_w , we only consider the terms with δ_w^2 in eq. (A.39):

$$L_0 \xi_{12} + L_2 \xi_{10} + L_1 \xi_{11} = \alpha_2 \xi_{10} + \beta_2 \xi_{20} + \varepsilon_2 \xi_{30} + \eta_2 \xi_{40} + \alpha_1 \xi_{11} + \beta_1 \xi_{21} + \varepsilon_1 \xi_{31} + \eta_1 \xi_{41}, \quad (\text{A.58a})$$

$$L_0 \xi_{22} + L_2 \xi_{20} + L_1 \xi_{21} = \xi_{21} + \alpha_2 \xi_{20} + \alpha_1 \xi_{21}, \quad (\text{A.58b})$$

$$L_0 \xi_{31} + L_1 \xi_{30} + L_1 \xi_{31} = W_2 \xi_{20} + \gamma_2 \xi_{30} + C_2 \xi_{40} + W_1 \xi_{21} + \gamma_1 \xi_{31} + C_1 \xi_{41}, \quad (\text{A.58c})$$

$$L_0 \xi_{41} + L_1 \xi_{40} + L_1 \xi_{41} = Z_2 \xi_{20} + D_2 \xi_{30} + \varphi_2 \xi_{40} + Z_1 \xi_{21} + D_1 \xi_{31} + \varphi_1 \xi_{41}. \quad (\text{A.58d})$$

We can calculate β_2 , ε_2 , and η_2 , computing the scalar product with ξ_{20}^* , ξ_{30}^* and ξ_{40}^* , respectively, for both sides of eq. (A.58a)

$$\beta_2 = \langle L_2 \xi_{10}, \xi_{20}^* \rangle + \langle L_1 \xi_{11}, \xi_{20}^* \rangle - \alpha_1 \langle \xi_{11}, \xi_{20}^* \rangle = 4f'^2 - 8D', \quad (\text{A.59})$$

$$\begin{aligned} \varepsilon_2 &= \langle L_2 \xi_{10}, \xi_{30}^* \rangle + \langle L_1 \xi_{11}, \xi_{30}^* \rangle - \alpha_1 \langle \xi_{11}, \xi_{30}^* \rangle = \\ &= -f'[B' - C' - B' + C'] - 2f' = 2f'^2, \end{aligned} \quad (\text{A.60})$$

$$\begin{aligned}\eta_2 &= \langle L_2 \xi_{10}, \xi_{40}^* \rangle + \langle L_1 \xi_{11}, \xi_{40}^* \rangle - \alpha_1 \langle \xi_{11}, \xi_{40}^* \rangle \\ &= -f'[B' - C' - B' + C'] + 2f' = -2f'^2.\end{aligned}\quad (\text{A.61})$$

In order to obtain α_2 , an estimate of ξ_{12} is required as well as order one:

$$\xi_{12} = \alpha_2 \xi_{20} + \widetilde{\xi}_{12} + \alpha_{21} \xi_{10} + \alpha_{23} \xi_{30} + \alpha_{24} \xi_{40}, \quad (\text{A.62})$$

where $\widetilde{\xi}_{12}$ is perpendicular to ξ_{10}^* , ξ_{30}^* and ξ_{40}^* . The coefficient α_{21} is arbitrary and settled equal to zero. Substituting eq. (A.62) into eq. (A.58b), α_{23} and α_{24} are obtained

$$\alpha_{23} = \langle L_2 \xi_{20}, \xi_{30}^* \rangle + \langle L_1 \xi_{21}, \xi_{30}^* \rangle - \alpha_1 \langle \xi_{21}, \xi_{30}^* \rangle = f'(C' - k'), \quad (\text{A.63})$$

$$\alpha_{24} = \langle L_2 \xi_{20}, \xi_{40}^* \rangle + \langle L_1 \xi_{21}, \xi_{40}^* \rangle - \alpha_1 \langle \xi_{21}, \xi_{40}^* \rangle = -f'(C' - k'). \quad (\text{A.64})$$

Conversely for α_2

$$2\alpha_2 = \langle L_2 \xi_{20}, \xi_{20}^* \rangle + \langle L_1 \xi_{21}, \xi_{20}^* \rangle - \langle \widetilde{\xi}_{12}, \xi_{20}^* \rangle - \alpha_1 \langle \xi_{21}, \xi_{20}^* \rangle \quad (\text{A.65})$$

is given. Replacing eq. (A.62) into eq. (A.40b), we compute

$$L_0 \widetilde{\xi}_{12} = -L_2 \xi_{10} - L_1 \xi_{11} + \beta_2 \xi_{20} + \varepsilon_2 \xi_{30} + \eta_2 \xi_{40} \quad (\text{A.66})$$

and, therefore:

$$\widetilde{\xi}_{12} = \begin{bmatrix} -4D' \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (\text{A.67})$$

is obtained. Substituting eq. (A.67) into eq. (A.65) $\alpha_2 = -4D' + k'(B' - C' - U') + 4D' - k'(B' - C' - U') = 0$ is calculated. From eq. (A.62), finally, ξ_{12} is given

$$\xi_{12} = \begin{bmatrix} -4D' \\ 0 \\ -f'(C' - k') \\ f'(C' - k') \end{bmatrix}. \quad (\text{A.68})$$

From eq. (A.58b), we can find ξ_{22}

$$\xi_{22} = \begin{bmatrix} -2(D' + A') + k'^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{A.69})$$

Starting from eqs. (A.58c) and (A.58d) and computing the scalar product with the suitable adjoint eigenvector ξ_{20}^* , ξ_{30}^* , or ξ_{40}^* each coefficient of L_2 and moreover the

correction of eigenvectors ξ_{32} and ξ_{42} due to the second order of δ_2 are obtained:

$$W_2 = \langle L_2 \xi_{30}, \xi_{20}^* \rangle + \langle L_1 \xi_{31}, \xi_{20}^* \rangle - W_1 \langle \xi_{21}, \xi_{20}^* \rangle - \gamma_1 \langle \xi_{31}, \xi_{20}^* \rangle + \\ - C_1 \langle \xi_{41}, \xi_{20}^* \rangle = -4D' + (k' - B')(B' - C') + 2f'B', \quad (\text{A.70})$$

$$\gamma_2 = \langle L_2 \xi_{30}, \xi_{30}^* \rangle + \langle L_1 \xi_{31}, \xi_{30}^* \rangle - W_1 \langle \xi_{21}, \xi_{30}^* \rangle - \gamma_1 \langle \xi_{31}, \xi_{30}^* \rangle + \\ - C_1 \langle \xi_{41}, \xi_{30}^* \rangle = f'(C' - k' + B'), \quad (\text{A.71})$$

$$C_2 = \langle L_2 \xi_{30}, \xi_{40}^* \rangle + \langle L_1 \xi_{31}, \xi_{40}^* \rangle - W_1 \langle \xi_{21}, \xi_{40}^* \rangle - \gamma_1 \langle \xi_{31}, \xi_{40}^* \rangle + \\ - C_1 \langle \xi_{41}, \xi_{40}^* \rangle = -f'(C' - k' + B'), \quad (\text{A.72})$$

$$Z_2 = \langle L_2 \xi_{40}, \xi_{20}^* \rangle + \langle L_1 \xi_{41}, \xi_{20}^* \rangle - Z_1 \langle \xi_{21}, \xi_{20}^* \rangle - D_1 \langle \xi_{31}, \xi_{20}^* \rangle + \\ - \varphi_1 \langle \xi_{41}, \xi_{20}^* \rangle = -4D' + (k' - B')(B' - C') - 2f'B', \quad (\text{A.73})$$

$$D_2 = \langle L_2 \xi_{40}, \xi_{30}^* \rangle + \langle L_1 \xi_{41}, \xi_{30}^* \rangle - Z_1 \langle \xi_{21}, \xi_{30}^* \rangle - C_1 \langle \xi_{31}, \xi_{30}^* \rangle + \\ - \varphi_1 \langle \xi_{41}, \xi_{30}^* \rangle = f'(C' - k' + B'), \quad (\text{A.74})$$

$$\varphi_2 = \langle L_2 \xi_{40}, \xi_{40}^* \rangle + \langle L_1 \xi_{41}, \xi_{40}^* \rangle - Z_1 \langle \xi_{21}, \xi_{40}^* \rangle - D_1 \langle \xi_{31}, \xi_{40}^* \rangle + \\ - \varphi_1 \langle \xi_{41}, \xi_{40}^* \rangle = -f'(C' - k' + B'), \quad (\text{A.75})$$

$$\xi_{32} = \begin{bmatrix} k'(k' - B') - 2D' + A' \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad (\text{A.76})$$

$$\xi_{42} = \begin{bmatrix} k'(k' - B') - 2D' + A' \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (\text{A.77})$$

Furthermore, defining $R_1 = -4D' + (k' - B')(B' - C')$, $R_2 = 2f'B'$ and $Q = f'(C' - k' + B')$ the matrices J_1 and J_2 are

$$J_1 = \begin{bmatrix} B' - C' - U' & 0 & 0 & 0 \\ 0 & B' - C' - U' & -2f' & 2f' \\ 0 & 0 & -U' + f' & -f' \\ 0 & 0 & f' & -U' - f' \end{bmatrix} \quad \text{and} \quad (\text{A.78})$$

$$J_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4f'^2 - 8D' & 0 & R_1 + R_2 & R_1 - R_2 \\ -2f'^2 & 0 & Q & Q \\ -2f'^2 & 0 & -Q & -Q \end{bmatrix}, \quad (\text{A.79})$$

respectively. The eigenvalues of the normal form $J_M = \frac{1}{\delta_w} J_0 + J_1 + \delta_w J_2$ of $\widetilde{M}$ have to be calculated. Since it has two zero-elements we take in account the first row of $J_M - \lambda I$ to compute its determinant:

$$2i\Lambda A \text{Det}(J_M - \lambda I) = [(B' - C' - U' - \lambda)^2 - (4s'f'^2 - 8D')] \cdot \text{Det}(N_1) + \\ - 2f'^2 \cdot \text{Det}(N_2) + 2f'^2 \cdot \text{Det}(N_3) = 0, \quad (\text{A.80})$$

where $Det(N_i)$ are respectively the determinants of the following matrices 2×2

$$N_1 = \begin{bmatrix} -U' + f' + \delta_w Q - \lambda & -f' + \delta_w Q \\ f' - \delta_w Q & -U' - f' - \delta_w Q - \lambda \end{bmatrix}, \quad (\text{A.81})$$

$$N_2 = \begin{bmatrix} -2f' + \delta_w R_1 + \delta_w R_2 & 2f' + \delta_w R_1 - \delta_w R_2 \\ f' - \delta_w Q & -U' - f' - \delta_w Q - \lambda \end{bmatrix}, \quad \text{and} \quad (\text{A.82})$$

$$N_3 = \begin{bmatrix} -2f' + \delta_w R_1 + \delta_w R_2 & 2f' + \delta_w R_1 - \delta_w R_2 \\ -U' + f' + \delta_w Q - \lambda & -f' + \delta_w Q \end{bmatrix}. \quad (\text{A.83})$$

Hence, eq. (A.80) becomes

$$\begin{aligned} & 2i\Lambda A[(U' + \lambda)^2 - 4f'Q\delta_w] \cdot \left(\lambda - (B' - C' - U') - \sqrt{4f'^2 - 8D'} \right) \cdot \\ & \cdot \left(\lambda - (B' - C' - U') + \sqrt{4f'^2 - 8D'} \right) + 4f'\delta_w[R_1(U' + \lambda) - 4f'^2Q - 2f'^2R_1] + \\ & + 4f'R_2Q\delta_w^2. \end{aligned}$$

Computing the limit for $\delta_w \rightarrow 0$

$$\begin{aligned} & 2i\Lambda A(U' + \lambda)^2 \cdot \left(\lambda - (B' - C' - U') - \sqrt{4f'^2 - 8D'} \right) \cdot \\ & \cdot \left(\lambda - (B' - C' - U') + \sqrt{4f'^2 - 8D'} \right) \end{aligned} \quad (\text{A.84})$$

is given. Therefore, the eigenvalues are $\lambda_{1,2} = 2i\Lambda A[B' - C' - U' \pm \sqrt{4f'^2 - 8D'}]$ and $\lambda_{3,4} = -2i\Lambda AU'$. Finally, we return to the old notation

$$\lambda_{1,2} = i\Lambda k(B - C) - ik \pm 2k\Lambda \sqrt{AD - \frac{\Sigma k A}{\Lambda^2}}$$

and $\lambda_{3,4} = ik$, which correspond to the two eigenvalues obtained in eq. (A.29) corrected by the surface tension and two transport waves, respectively.