MARMARA UNIVERSITY
INSTITUTE FOR GRADUATE STUDIES
IN PURE AND APPLIED SCIENCES

DYNAMICS OF PARTICLES AND FIELDS IN GRAVITATIONAL MONOPOLE SPACE - TIMES

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MASTER THESIS

Department of Physics

Thesis Supervisor

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Muhammed Haluk SEÇUK, a Master of Science student of Marmara University Institute for Graduate Studies in Pure and Applied Sciences, defended his thesis entitled "Dynamics of Particles and Fields In Gravitational Monopole Space-times", on June 17, 2019 and has been found to be satisfactory by the jury members.

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CONTENTS

ABSTRACT

DYNAMICS OF PARTICLES AND FIELDS IN GRAVITATIONAL MONOPOLE SPACE-TIMES

In this thesis, we investigate the dynamics of particles and fields in the Reissner-Nordström-(A)dS-Monopole space-time by solving the Klein-Gordon and Hamilton-Jacobi equations. We found an analytical solution to each of these equations in certain conditions. Furthermore, we investigate the light deflection from a black hole and superradiance on these phenomena in our space-time configuration and discuss the effect of the global monopole.

ÖZET

PARÇACIKLARIN VE ALANLARIN GRAVİTASYONEL MONOPOL UZAY-ZAMANLARDA DİNAMİKLERİ

Bu tez çalışmasında parçacıkların ve alanların davranışını Reissner-Nordström-(A)dS-Monopole uzay-zamanında Klein-Gordon ve Hamilton-Jacobi denklemlerini belirli durumlar için analitik olarak çözerek inceledik. Buna ek olarak bir kara delikten ışığın sapması ve süperışıma kararsızlığını bu uzay-zaman konfigurasyonu için ele aldık ve monopole teriminin etkisini gözlemledik.

SYMBOLS

$\boldsymbol{G_{ab}}$	:	Einstein tensor
$\boldsymbol{\eta_{ab}}$	:	Minkowski metric
$\boldsymbol{T_{ab}}$	:	Energy-momentum tensor
$\boldsymbol{\Lambda}$	:	Cosmological constant
$\boldsymbol{T_{ab}}$	:	Energy-momentum tensor
$\boldsymbol{d\Omega_2^2}$	:	Two dimensional solid angle
$\boldsymbol{T_{ab}}$	:	Energy-momentum tensor
$\boldsymbol{\theta_a}$	:	Orthonormal basis one form
$\boldsymbol{\omega_{ab}}$	:	Connection one forms
$\boldsymbol{\Omega_{ab}}$	:	Curvature two forms
$\boldsymbol{g_{ab}}$	:	Space-time metric
$\boldsymbol{R_{abcd}}$	:	Riemann tensor
$\boldsymbol{F_{ab}}$	:	Faraday tensor
$\boldsymbol{F}$	:	Faraday two form
$\boldsymbol{g}$	:	Determinant of the metric tensor
$\boldsymbol{g^{ab}}$	:	Inverse metric tensor
$\Box$	:	d'Alembert operator
$\boldsymbol{\Phi}$	:	Wave function
$\boldsymbol{D_\mu}$	:	Gauge differential operator
$\boldsymbol{\nabla_\mu}$	:	Covariant derivative
$\boldsymbol{A_\mu}$	:	Vector potential
$\boldsymbol{P_\nu{}^m}$	:	Associated Legendre polynomials
$\boldsymbol{Hl}$	:	Local Heun solutions
$\boldsymbol{S}$	:	The action
$\boldsymbol{\Phi_h}$	:	Electric potential
$\boldsymbol{\overline{\omega}}$	:	Superradiant factor

ABBREVIATIONS

GR	: General Relativity
RKGE	: Radial Klein-Gordon Equation
BH	: Black Hole
RN	: Reissner-Nordström
RN-(A)dS	: Reissner-Nordström (Anti)-de Sitter

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SAYFA

1. INTRODUCTION

One of the four fundamental interactions of nature is the gravitational interaction and this interaction is explained by General Theory of Relativity (GR) [1] which takes its final form in 1916 after nearly ten years of hard work by A. Einstein. This theory determines the structure of the space-time for a given matter-energy distribution by solving a tensor equation called the Einstein equations which consist of a set of a nonlinear, coupled system of differential equations. There are many exact solutions [2] of these equations which helps to understand not only the behavior of this theory but also to determine the physical structure of the space-time generated by the given matter-energy distribution. For example, the first and most important exact solution, the spherically symmetric vacuum solution of Einstein equations, known as the Schwarzschild solution [3], is not only helped to understand the structure of such space-times in this theory but also introduced a new concept in physics, namely the black holes.

The next property of this theory is that this theory also allows us to explore the behavior of other particles or fields in this space-time [4]. In other words, for the particles, we can, in principle, determine how test particles move in this space-time by solving geodesic equations of this space-time. For the case of Schwarzschild solution, investigation of the geodesics actually confirmed Newtonian results of planet motion around stars for weak gravity in the slow-motion limit, namely the Kepler motion of planets. In addition, this investigation yields new results in the strong gravity regime such as the existence of event horizons around black holes in which neither particles nor photons escape from and also the near horizon motion of particles and photons which is very different from the Newtonian motion. The behavior of other fields in the given space-time can be determined by analyzing the appropriate field equations. For electromagnetic fields, we must investigate Lorentz force equations in curved space-time, which is nothing but a geodesic equation with a force field, for scalar fields we have to investigate Klein-Gordon equation in a given space-time.

In order to fully understand the mathematical structure and physical properties of a given solution of Einstein's equations, therefore, the behavior of test fields must also be investigated. In this thesis, thus, we mainly investigate the behavior of particles or fields in a given space-time. This analysis will help us to understand the physical and mathematical structure of this space-time and also the behavior of the particles and fields in this given curved space-time in comparison with flat Minkowski space-time. Since there are many different exact solutions of Einstein equations [2], we have to limit ourselves to a certain class of solutions. In this thesis, therefore, we will first investigate the behavior of test fields in gravitational monopole space-times.

Gravitational monopoles are a special class of topological defects which may be produced in the early universe [5] during the symmetry breaking phase transitions. Depending on the type of broken symmetry, monopoles, cosmic strings, domain walls or textures may be produced [5,6]. For example, monopoles formed when the vacuum manifold involves surfaces which cannot be continuously shrunk to a point whereas strings formed when vacuum manifold contains closed paths that cannot be shrunk to a point. Note that most of the energy of the monopoles are concentrated in a very small volume near the core. According to grand unified theories, monopoles must have produced with a very large amount [7], which conflicts with observations. This monopole problem is solved by the invention of cosmic inflation which dilutes the number of monopoles for a given horizon volume. Before the end of the 20th century, cosmic strings were believed to play an important role in galaxy formation [6]. The observational data obtained at the end of the 20th century showed that the main mechanism for the galaxy formation is the quantum perturbations by the inflation and the contribution of cosmic strings to this phenomenon, if they exist, is not primary. This result overshadowed the importance of topological defects in cosmology. However, there are some theoretical and observational implications [8] that they are still an important field of research.

In this thesis, we will mainly investigate the behavior of other fields in global monopole space-times [9] which are also called as gravitational hedgehogs [10]. We will consider a more general global monopole space-time, which includes mass, electrical charge and a cosmological constant, namely the Reissner-Nordström (Anti)-de Sitter space-time (RN-(A)dS) as given in [10,11]. RN space-time is a generalization of Schwarzschild black hole via the solution of Einstein's equations in an electromagnetic field which is called Einstein-Maxwell equations and introduced separately by Reissner [12] and Nordström [13]. Addition of the cosmological constant promotes the solution to a cosmological level.

The behavior of test particles in this space-time will be investigated by using the Hamilton-Jacobi approach and using this approach, possible trajectories in this space-time will be determined. Moreover, we make use of these trajectories to calculate the deflection angle of light. The next step will be to obtain Klein-Gordon equation in this space-time then we will separate and solve this equation and investigate some physical effects due to the scalar field such as superradiance.

Superradiance, which dates back to 1947's with the leading work of Ginzburg and Frank [15] on the Doppler effect, is fundamentally a radiation amplification process that involves dissipative systems and has significant applications in quantum mechanics and especially in general theory of relativity and astrophysics. In black hole theory, this process can be observed when a scalar or electromagnetic wave is sent from a far distance to the black hole, which has a frequency that has certain threshold value, enhanced by extracting energy at the event horizon. Rotational or electromagnetic energy can be extracted depending on the black hole. The first classical example of superradiant scattering, in black hole space-times, was given by Zel'dovich [14], by inspecting scalar waves that strike upon a rotating cylindrical absorbing object. For detailed review concerning superradiance see [16].

The material of the thesis is categorized as follows. In section one, we explicitly construct the space-time corresponding to a global monopole swallowed by a black hole with mass M, and electrical charge Q in the cosmological background which is called RN-(A)dS global monopole space-time via the solution of Einstein-Maxwell equations with the energy-momentum tensor for a global monopole outside the core.

In section two, we solve, in our thus construct space-time, the Klein-Gordon equation for a scalar field by using an ansatz to separate the equation to its angular and radial parts. To obtain an analytical solution for radial equation we exploit holomorphic and F-homotopic transformations respectively to fit the radial equation to the Heun form, which has well-known solutions. Then we present a polynomial solution to the Heun equation. We also find possible trajectories of a charged particle by solving Hamilton-Jacobi equation, in particular, we find an analytical solution corresponding to photons in terms of Weierstrass elliptic function. By these solutions, we obtain dynamical information about our space-time configuration. We apply these trajectories to physical calculations by considering light deflection from a black hole, in order to do so, we derive a Binet equation for our space-time, solve it for a photon using the perturbative technique and obtain the deflection angle. We separate this calculation into two subsections to observe the effect of the cosmological constant. Section three is devoted to the phenomenon of superradiance. We investigate the instability condition for superradiance for our space-time configuration in two cases. In the first case, we have AdS space behaves effectively as a reflecting box. In the second case, however, we consider a space in the absence of the cosmological constant, as a result, we need to put a reflecting box by hand. In addition, we argue the effect of global monopole to superradiant instability for both cases.

2. CONSTRUCTION OF SPACE-TIME

In this section we will construct the spherical symmetric line element corresponding to the global monopole swallowed by RN-(A)dS black hole. To obtain such a solution we need to solve the Einstein equations with a cosmological constant, which is, in orthonormal basis, given by,

$$G_{ab} + \eta_{ab}\,\Lambda = 8\pi T_{ab} \tag{2.1}$$

. Here the latin indices take the values $a, b = 0, 1, 2, 3$ with

$$\eta_{ab} = diag(-1, 1, 1, 1), \tag{2.2}$$

where G_{ab}, Λ, η_{ab} and T_{ab} are Einstein tensor, the cosmological constant, the Minkowski metric tensor and energy momentum tensor of the physical system respectively.

2.1. The Einstein Tensor

To obtain the Einstein tensor, let us first consider the general line element

$$ds^2 = -A^2(r)dt^2 + B^2(r)dr^2 + C^2(r)d\Omega_2^2. \tag{2.3}$$

Where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the metric of a unit two-sphere. Choosing a suitable orthonormal basis one forms as,

$$\theta^0 = A(r)dt, \qquad \theta^1 = B(r)dr, \qquad \theta^2 = C(r)d\theta, \qquad \theta^3 = C(r)\sin\theta d\phi, \tag{2.4}$$

then one can express the line element in an othonormal basis as,

$$ds^2 = \eta_{ab}\,\theta^a \wedge \theta^b. \tag{2.5}$$

The exterior derivatives of the basis one forms are,

$$d\theta^0 = \frac{A_r}{AB}\,\theta^1 \wedge \theta^0, \quad d\theta^1 = 0, \quad d\theta^2 = \frac{C_r}{BC}\,\theta^1 \wedge \theta^2, \quad d\theta^3 = \frac{C_r}{BC}\,\theta^1 \wedge \theta^3 + \frac{\cot\theta}{C}\,\theta^2 \wedge \theta^3 \tag{2.6}$$

where A_r represents the total derivative with respect to coordinate r, i.e. $A_r = \frac{dA(r)}{dr}$. Now we will compute the connection one forms ${\omega^a}_b$ and curvature two forms ${\Omega^a}_b$ via,

$$d\theta^a + {\omega^a}_b = 0 \tag{2.7}$$

$$d{\omega^a}_b + {\omega^a}_c \wedge {\omega^c}_b = {\Omega^a}_b \tag{2.8}$$

which are called Cartan's first and second structure equations, respectively. The compatibility condition for connection one forms is,

$$\omega_{ab} + \omega_{ba} = 0 \tag{2.9}$$

with $\omega_{ab} = \eta_{ac}\,{\omega^c}_b$. Substituting (2.4) and (2.6) into (2.7) we can compute the non-zero components of connection one forms,

$${\omega^0}_1 = \frac{A_r}{AB}\,\theta^0, \quad {\omega^1}_2 = -\frac{C_r}{BC}\,\theta^2, \quad {\omega^1}_3 = -\frac{C_r}{BC}\,\theta^3, \quad {\omega^2}_3 = -\frac{\cot\theta}{C}\,\theta^3. \tag{2.10}$$

With the aid of (2.8) we obtain the non-zero curvature two form components which are,

$${\Omega^0}_1 = \left(\frac{A_r B_r}{AB^3} - \frac{A_{rr}}{AB^2}\right)\theta^0 \wedge \theta^1, \tag{2.11}$$

$${\Omega^0}_2 = -\left(\frac{A_r C_r}{AB^2C}\right)\theta^0 \wedge \theta^2, \tag{2.12}$$

$${\Omega^0}_3 = -\left(\frac{A_r C_r}{AB^2C}\right)\theta^0 \wedge \theta^3, \tag{2.13}$$

$${\Omega^1}_2 = \left(\frac{C_r B_r}{B^3C} - \frac{C_{rr}}{B^2C}\right)\theta^1 \wedge \theta^2, \tag{2.14}$$

$${\Omega^1}_3 = \left(\frac{C_r B_r}{B^3C} - \frac{C_{rr}}{B^2C}\right)\theta^1 \wedge \theta^3, \tag{2.15}$$

$${\Omega^2}_3 = \frac{1}{C^2}\left(1 - \frac{C_r^2}{B^2}\right)\theta^2 \wedge \theta^3. \tag{2.16}$$

The relation between curvature two forms and Riemann tensor components is given by the following equation,

$$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \theta^c \wedge \theta^d. \tag{2.17}$$

Therefore we have,

$$R^0{}_{101} = \left(\frac{A_r B_r}{AB^3} - \frac{A_{rr}}{AB^2} \right), \tag{2.18}$$

$$R^0{}_{202} = R^0{}_{303} = -\left(\frac{A_r C_r}{AB^2 C} \right), \tag{2.19}$$

$$R^1{}_{212} = R^1{}_{313} = \left(\frac{C_r B_r}{B^3 C} - \frac{C_{rr}}{B^2 C} \right), \tag{2.20}$$

$$R^2{}_{323} = \frac{1}{C^2} \left(1 - \frac{C_r^2}{B^2} \right). \tag{2.21}$$

Ricci tensor is the contraction of the Riemann tensor components, i.e $R_{ab} = R^c{}_{acb}$, thus we have,

$$R_{00} = \left(-\frac{A_r B_r}{AB^3} + \frac{A_{rr}}{AB^2} + 2\frac{A_r C_r}{AB^2 C} \right), \tag{2.22}$$

$$R_{11} = \left(2\frac{C_r B_r}{B^3 C} - 2\frac{C_{rr}}{B^2 C} + \frac{A_r B_r}{AB^3} - \frac{A_{rr}}{AB^2} \right), \tag{2.23}$$

$$R_{22} = R_{33} = \frac{1}{C^2} \left(1 - \frac{C_r^2}{B^2} \right) + \frac{C_r B_r}{B^3 C} - \frac{C_{rr}}{B^2 C} - \frac{A_r C_r}{AB^2 C}. \tag{2.24}$$

Now we have the Ricci tensor components therefore, we can calculate the Ricci scalar, that is $R = R^a_a = \eta^{ab} R_{ab}$, for our case given by

$$R = 2\left(\frac{1}{C^2} \left(1 - \frac{C_r^2}{B^2} \right) + \frac{C_r B_r}{B^3 C} - \frac{C_{rr}}{B^2 C} - \frac{A_r C_r}{AB^2 C} \right) + \frac{A_r B_r}{AB^3} - \frac{A_{rr}}{AB^2}. \tag{2.25}$$

Finally, the Einstein tensor is given by,

$$G_{ab} = R_{ab} - \frac{1}{2} \eta_{ab} R. \tag{2.26}$$

Using (2.22) - (2.24) with (2.25) we obtain the Einstein tensor components,

$$G_{00} = -\frac{2C_{rr}}{B^2 C} + \frac{2B_r C_r}{B^3 C} + \frac{1}{C^2} \left(1 - \frac{C_r^2}{B^2} \right), \tag{2.27}$$

$$G_{rr} = \frac{2A_r C_r}{AB^2 C} - \frac{1}{C^2} \left(1 - \frac{C_r^2}{B^2} \right), \tag{2.28}$$

$$G_{\theta\theta} = G_{\phi\phi} = \frac{A_{rr}}{AB^2} + \frac{C_{rr}}{B^2 C} - \frac{A_r B_r}{AB^3} + \frac{A_r C_r}{AB^2 C} - \frac{B_r C_r}{B^3 C}. \tag{2.29}$$

Hence we completed the geometric part of the equation (2.1). Our next task is to obtain the energy momentum tensor for our space-time configuration.

2.2. The Energy Momentum Tensor

Aim of this section is the identify the energy-momentum tensor for our configuration, which we will achieve this in two parts. First, we will obtain the energy-momentum tensor for global monopole $T_{ab}^{(mon)}$, since we have an electrically charged black hole, we will then calculate the energy-momentum tensor for an electric field produced by a charge Q.

2.2.1. Energy Momentum Tensor for Global Monopole

The simple Lagrangian that can be written for the global monopole, [6, 9, 10], is

$$L_m = \frac{1}{2}\partial_\mu\vec{\psi}\partial_\nu\vec{\psi}g^{\mu\nu} - \frac{1}{4}\lambda\left[\vec{\psi}.\vec{\psi} - \eta^2\right]^2, \tag{2.30}$$

where $\vec{\psi} = (\psi^1, \psi^2, \psi^3)$ is a triplet isovector scalar, that implies the $\vec{\psi}$ is parallel to $\hat{r}$, the unit vector in radial direction and $\vec{\psi}$ is constraint to,

$$\vec{\psi}\,.\,\vec{\psi} = \eta^2, \tag{2.31}$$

therefore,

$$L_m = \frac{1}{2}\partial_\mu\vec{\psi}\partial_\nu\vec{\psi}g^{\mu\nu}. \tag{2.32}$$

In the case of a global monopole configuration we set,

$$\vec{\psi} = +\eta\,\hat{r}, \quad or \quad \vec{\psi} = -\eta\,\hat{r} \tag{2.33}$$

where

$$\hat{r_1} = \sin\theta\cos\theta, \quad \hat{r_2} = \sin\theta\sin\phi, \quad \hat{r_3} = \cos\theta. \tag{2.34}$$

Both forms of (2.33) leads to the same energy-momentum tensor $T_{\mu\nu}^{(mon)}$. As it is seen that (2.33) and (2.34) are a solution of the scalar field equations of motion when the geometry is considered as a spherical symmetric one.

Thus the energy momentum tensor produced from (2.30) is,

$$T^{(mon)}_{\mu\nu} = \partial_\mu\vec{\psi}.\partial_\nu\vec{\psi} - g_{\mu\nu}\left(\frac{1}{2}(\partial_\alpha\vec{\psi}).(\partial_\beta\vec{\psi})g^{\alpha\beta}\right) \tag{2.35}$$

for given $\vec{\psi}$ in (2.33), we obtain,

$$T^{(mon)}_{tt} = -T^{(mon)}_{rr} = \frac{\eta^2}{r^2}, \quad T^{(mon)}_{\theta\theta} = T^{(mon)}_{\phi\phi} = 0. \tag{2.36}$$

2.2.2. Energy Momentum Tensor for An Electric Field

Next issue is to obtain energy-momentum tensor for an electric field produced by a charge Q, which is, in orthomormal basis, given by,

$$T^{(e)}_{ab} = \frac{1}{4\pi}\left(F^c{}_a F_{cb} - \frac{1}{4}\eta_{ab}F_{cd}F^{cd}\right) \tag{2.37}$$

where F_{ab} is the Faraday tensor components for an electric field.

When calculating F_{ab}, it is more adequate to use differential forms. From differential geometric point of view the homogeneous Maxwell equations are,

$$dF = 0, \tag{2.38}$$

$$d\star F = 0 \tag{2.39}$$

where two form F in orthonormal basis, defined by,

$$F = \frac{1}{2}F_{ab}\,\theta^a \wedge \theta^b \tag{2.40}$$

and the operator $\star$ is defined as a linear operator which sends any p-form α, in n-dimensional manifold, to (n-p)-form given by the following,

$$\star\alpha = \frac{1}{p!(n-p)!}\epsilon_{\nu_1\ldots\nu_p\mu_1\ldots\mu_{(n-p)}}\,\alpha^{\nu_1\ldots\nu_p}\,\theta^{\mu_1}\wedge\ldots\wedge\theta^{\mu_{(n-p)}} \tag{2.41}$$

with volume form defined via the Levi-Civita symbol ε and the determinant of metric tensor $detg_{\mu\nu} = g$ as,

$$\epsilon = \frac{1}{n!}\sqrt{|g|}\,\varepsilon_{\mu_1 \dots \mu_n}\,\theta^{\mu_1} \wedge \dots \wedge \theta^{\mu_n}. \tag{2.42}$$

The operator defined in equation (2.41) is called the Hodge dual operator.

Since we can obtain the electric field from an electrical potential, it is appropriate to choose a scalar function for the potential as,

$$A = f(r)\,dt. \tag{2.43}$$

Due to the Poincaré's lemma, i.e, $d^2 = 0$ it satisfies,

$$dA = F \quad \rightarrow \quad dF = 0. \tag{2.44}$$

Since it is an identity, substituting (2.43) to (2.38) we gain no information, on the other hand inserting to (2.39), and using our line element (2.3) with $C(r) = r$ we have,

$$\frac{f_r(r)}{AB} = \frac{const.}{r^2} \tag{2.45}$$

To determine the $const.$ we can use Gauss's total flux theorem, which merely implies $const. = Q$, where Q is the total electric charge of the black hole.

Now we can calculate the non-zero components of Faraday tensor, which are,

$$F_{tr} = -F_{rt} = \frac{Q}{r^2}. \tag{2.46}$$

Plugging (2.46) to (2.37) yields the the energy-momentum tensor for an electric field,

$$T^{(e)}_{ab} = \frac{Q^2}{8\pi r^2}\,diag(1,-1,1,1). \tag{2.47}$$

Now if we want our space-time to be spherically symmetric, without loss of generality we can set $C(r) = r$ in our general line element (2.3). Using this constraint, if one solves the equation (2.1) for given line element (2.3) either with (2.36) or (2.37) one obtains a relation between metric functions given as $A = \frac{1}{B}$. Hence one can safely conclude that the combination of (2.37) with (2.36) is a linear problem. Thus we can write the (2.1) as,

$$G_{ab} + \eta_{ab}\,\Lambda = 8\pi\left(T_{ab}^{(e)} + T_{ab}^{(mon)}\right). \tag{2.48}$$

Using the non-zero Einstein tensor component given by the equations (2.27) - (2.29), with (2.36) and (2.37), calculation of (2.48) yields the unkown metric function as

$$A^2(r) = \frac{1}{B^2(r)} = b^2 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{Q^2}{r^2} \tag{2.49}$$

where $b^2 = 1 - 8\pi\eta^2$. The line element becomes,

$$ds^2 = -\frac{\Delta_r}{r^2}\,dt^2 + \frac{r^2}{\Delta_r}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \tag{2.50}$$

with Δ_r is defined as,

$$\Delta_r = b^2r^2 - 2Mr - \frac{\Lambda}{3}r^4 + Q^2, \tag{2.51}$$

where M and Q are the total mass and the total charge, which are the physical parameters, of the black hole. Equation (2.50) is nothing but the line element implicitly given in the reference [10] which describes the space-time of global monopole swallowed by an RN-(A)dS black hole.

One important note comes from the inspection of the pure global monopole configuration. The line element for such a configuration can be obtained by neglecting the black hole parameters. Therefore line element becomes [6],

$$ds^2 = -(1 - 8\pi\eta^2)\,dt^2 + \frac{1}{(1 - 8\pi\eta^2)}\,dr^2 + r^2d\theta^2 + r^2\sin\theta^2 d\phi^2. \tag{2.52}$$

Re-scaling the t and r variables by,

$$t \to \frac{t}{b} \qquad r \to br \tag{2.53}$$

we can rewrite the global monopole line element as,

$$ds^2 = -dt^2 + dr^2 + (1 - 8\pi\eta^2)\,r^2\left(d\theta^2 + \sin\theta^2 d\phi^2\right). \tag{2.54}$$

The line element (2.54) not only describes the asymptotic behavior of the global monopole outside the core but also states that the pure global monopole space-time is not asymptotically flat, which implies a space with a solid angle. Hence the area of a sphere of a radius r is not $4\pi r^2$, but rather $(1 - 8\pi\eta^2)\, 4\pi r^2$.

Note also that for positive values of b^2, i.e $b^2 > 0$, the equation (2.50) defines a space-time such that $A^2(r) = 0$ at a certain value of r. However, for $b^2 < 0$, r is always a timelike variable and (2.51) can be interpreted as an anisotropic cosmological solution.

3. DYNAMICS OF PARTICLES AND FIELDS

Task of this section is the calculation of the Klein-Gordon and the Hamilton-Jacobi equation, for the line element (2.50) that describes the space-time of a global monopole swallowed the Reissner-Nordström de Sitter black hole, from those we will obtain the information about the behavior of a scalar field and the trajectories of a charged particle respectively.

3.1. The Klein-Gordon Equation

The Klein-Gordon equation for a scalar field Φ which describes the dynamics of a massive scalar electrically charged particle of mass μ and charge e, in a curved space-time is described by the equation

$$\Box\Phi = \mu^2\Phi, \tag{3.1}$$

where,

$$\Box\Phi = \frac{1}{\sqrt{-g}} D_\mu(\sqrt{-g} g^{\mu\nu} D_\nu \Phi), \tag{3.2}$$

and the inverse space-time metric tensor components given by,

$$g^{tt} = -\frac{r^2}{\Delta_r}, \quad g^{rr} = \frac{\Delta_r}{r^2}, \quad g^{\theta\theta} = \frac{1}{r^2}, \quad g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta}, \tag{3.3}$$

with g is the determinant of the metric tensor, i.e $g = det(g_{\mu\nu})$, has the value,

$$g = det(g_{\mu\nu}) = -r^4 \sin^2\theta \tag{3.4}$$

and the gauge differential operator defines as,

$$D_\mu = \nabla_\mu - ieA_\mu, \tag{3.5}$$

where ∇_ν and A_μ, are the covariant derivative and the vector potential defined via,

$$\nabla_\mu T_\nu = \partial_\mu T_\nu - \Gamma^\alpha{}_{\mu\nu}\, T_\alpha, \tag{3.6}$$

$$A = -\frac{Q}{r}\, dt, \tag{3.7}$$

and $\Gamma^\alpha{}_{\mu\nu}$'s are the Levi-Civita connections. With these definitions, calculation of (3.1) yields,

$$\Box\Phi = \left(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \frac{ie}{\sqrt{-g}}A^\mu\left(\partial_\mu\sqrt{-g}\right) - ie(\partial_\mu A^\mu) - 2ieA^\mu\partial_\mu - e^2A_\mu A^\mu\right)\Phi. \tag{3.8}$$

It is easily seen that the calculation of the second and the third term yields zero. Therefore the contribution from the gauge field is,

$$A_\mu A^\mu = g^{\mu\nu}A_\mu A_\nu = g^{tt}A_tA_t = -\frac{Q^2}{\Delta_r}, \tag{3.9}$$

$$A^\mu\partial_\mu = g^{\mu\nu}A_\mu\partial_\nu = g^{tt}A_t\partial_t = \frac{Qr}{\Delta_r}\partial_t, \tag{3.10}$$

and the first term,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) = -\frac{r^2}{\Delta_r}\partial_t^2 + \frac{1}{r^2}\partial_r(\Delta_r\partial_r) + \frac{1}{r^2\sin^2\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2. \tag{3.11}$$

To solve the equation (3.8), we choose the following ansatz,

$$\Phi = R(r)\, S(\theta)\, e^{im\phi}\, e^{-i\omega t}. \tag{3.12}$$

Substituting (3.12) to (3.8) yields consequently, to a separability condition on differential equation(3.8), which means we can write the total differential equation as angular and radial parts separately.

The angular equation is given by,

$$\frac{1}{\sin\theta}\,\partial_\theta\left(\sin\theta\partial_\theta\right)S(\theta) + \left(\nu(\nu+1) - \frac{m^2}{\sin^2\theta}\right)S(\theta) = 0, \tag{3.13}$$

where $\nu(\nu+1)$ is the separation constant. Now changing the variable by $x := \cos\theta$

we obtain,

$$(1-x^2)\frac{d^2}{dx^2}S(x) - 2x\frac{d}{dx}S(x) + \left(\nu(\nu+1) - \frac{m^2}{1-x^2}\right)S(x) = 0. \tag{3.14}$$

The equation (3.14) is known as the associated Legendre differential equation and its solutions are given by,

$$S(\theta) = P_\nu{}^m(\cos\theta), \tag{3.15}$$

where $P_\nu{}^m(x)$ are called associated Legendre polynomials, values of it can be found by using the Rodrigues' formula, that is,

$$P_\nu{}^m(x) = \frac{(-1)^m}{2^\nu\,\nu!}(1-x^2)^{\frac{m}{2}}\frac{d^{\nu+m}}{dx^{\nu+m}}(x^2-1)^\nu. \tag{3.16}$$

And the radial part is,

$$\Delta_r\partial_r\left[\Delta_r\partial_r R\right] - \left[\Delta_r\left(\nu(\nu+1)+\mu^2r^2\right) - (\omega r^2 - eQr)^2\right]R = 0. \tag{3.17}$$

To find an analytic solution of the Eqn. (3.17), comparing to equation (3.14), is less obvious one, to obtain such a solution we need to transform the Eqn. (3.17) to a much more familiar one. And such an endeavor deserves another section.

3.1.1. Solution To The Radial Klein-Gordon Equation

We start by rewriting our metric function coefficient Δ_r as,

$$\Delta_r = -\frac{\Lambda}{3}(r-r_+)(r-r_-)(r-r_c^+)(r-r_c^-). \tag{3.18}$$

We have five regular singularities in our radial differential equation (3.17) located at $r_\pm, r_c^\pm, \infty$. First we apply the homographic substitution via the transformation,

$$z = \left(\frac{r_+ - r_c^-}{r_+ - r_-}\right)\left(\frac{r - r_-}{r - r_c^-}\right). \tag{3.19}$$

Therefore we can write (3.18) in terms of new variable as follows,

$$\Delta_r = -\frac{\Lambda}{3}\frac{Kz_\infty^3 z(z-1)(z-z_r)}{(z_\infty - z)^4}, \tag{3.20}$$

where K defined as,

$$K := \frac{(r_- - r_c^-)^2 (r_+ - r_-)(r_c^+ - r_-)}{z_r}. \tag{3.21}$$

also we have the following definitions and relations,

$$r = \frac{r_- z_\infty - r_c^- z}{z_\infty - z}, \; z_r := z_\infty \left(\frac{r_c^+ - r_-}{r_c^+ - r_c^-} \right), \; z_\infty := \frac{r_+ - r_c^-}{r_+ - r_-}, \tag{3.22}$$

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{z_\infty} \frac{1}{r_- - r_c^-} (z_\infty - z)^2 = \frac{r_+ - r_-}{r_+ - r_c^-} \frac{1}{r_- - r_c^-} (z_\infty - z)^2, \tag{3.23}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} = -2 \, \frac{z_\infty (r_- - r_c^-)}{(r - r_c^-)^3}, \quad \left(\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} \right) \left(\frac{\mathrm{d}z}{\mathrm{d}r} \right)^{-2} = \frac{-2}{z_\infty - z}. \tag{3.24}$$

Applying the homographic transformation (3.19) to the the radial equation (3.17) we have,

$$\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + \left(\left(\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} \right) \left(\frac{\mathrm{d}z}{\mathrm{d}r} \right)^{-2} + \frac{1}{\Delta_r} \frac{\mathrm{d}\Delta_r}{\mathrm{d}z} \right) \frac{\mathrm{d}R}{\mathrm{d}z} + \left(\frac{\mathrm{d}z}{\mathrm{d}r} \Delta_r \right)^{-2} \left((\omega r^2 - eQr)^2 - \Delta_r \left(\nu(\nu+1) + \mu^2 r^2 \right) \right) R = 0. \tag{3.25}$$

Now we will focus on the coefficients separately. First,

$$\left(\left(\frac{\mathrm{d}^2 z}{\mathrm{d}r^2} \right) \left(\frac{\mathrm{d}z}{\mathrm{d}r} \right)^{-2} + \frac{1}{\Delta_r} \frac{\mathrm{d}\Delta_r}{\mathrm{d}z} \right) \frac{\mathrm{d}R}{\mathrm{d}z} = \left\{ \frac{1}{z} + \frac{1}{z-1} + \frac{1}{z - z_r} - \frac{2}{z - z_\infty} \right\} \frac{\mathrm{d}R}{\mathrm{d}z}. \tag{3.26}$$

Second,

$$\frac{1}{\left(\frac{\mathrm{d}z}{\mathrm{d}r} \Delta_r \right)^2} \left[\omega r^2 - eQr \right]^2 = \left[\frac{A'}{z^2} + \frac{B'}{z} + \frac{C'}{(z-1)^2} + \frac{D'}{z-1} + \frac{E'}{(z-z_r)^2} + \frac{H'}{z - z_r} \right], \tag{3.27}$$

where the relevant expansion coefficients are,

$$\sqrt{A'} = \frac{r_-}{k_\Lambda} \frac{[\omega \, r_- - e \, Q]}{(r_- - r_c^-)(r_- - r_+)(r_- - r_c^+)}, \tag{3.28}$$

$$\sqrt{C'} = \frac{r_+}{k_\Lambda} \frac{[\omega \, r_+ - e \, Q]}{(r_+ - r_c^-)(r_+ - r_c^+)(r_+ - r_-)}, \tag{3.29}$$

$$\sqrt{E'} = \frac{r_c^+}{k_\Lambda} \frac{[\omega \, r_c^+ - e \, Q]}{(r_c^+ - r_-)(r_c^+ - r_c^-)(r_c^+ - r_+)}, \tag{3.30}$$

with $k_\Lambda^2 = \frac{\Lambda}{3}$.

Third,

$$-\frac{\mu^2 r^2}{\left(\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 \Delta_r\right)} = \left[\frac{A}{(z_\infty - z)^2} + \frac{B}{z_\infty - z} + \frac{C}{z} + \frac{D}{z-1} + \frac{F}{z - z_r}\right], \tag{3.31}$$

where,

$$A = \frac{3\mu^2}{\Lambda}, \tag{3.32}$$

$$B = \frac{3\mu^2}{\Lambda}\frac{1}{r_- - r_c^-}\left[\frac{(r_c^- + r_-)z_r - 2\,r_-\,z_\infty - 2\,r_-\,z_r\,z_\infty - (r_c^- - 3r_-)z_\infty^2}{(1-z_\infty)(z_r - z_\infty)z_\infty}\right], \tag{3.33}$$

$$C = \frac{3\mu^2}{\Lambda}\frac{1}{(r_+ - r_-)(r_c^+ - r_-)}\frac{r_-^2}{z_\infty}, \tag{3.34}$$

$$D = -\frac{3\mu^2}{\Lambda}\frac{z_r}{(r_+ - r_-)(r_c^+ - r_-)}\frac{1}{z_\infty}\frac{[r_c^- - r_- z_\infty]^2}{(z_r - 1)(z_\infty - 1)^2}, \tag{3.35}$$

$$F = \frac{3\mu^2}{\Lambda}\frac{1}{(r_+ - r_-)(r_c^+ - r_-)}\frac{1}{z_\infty}\frac{(r_c^- z_r - r_- z_\infty)^2}{(z_r - z_\infty)^2(z_r - 1)}, \tag{3.36}$$

and finally,

$$-\frac{\nu(\nu+1)}{\left(\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 \Delta_r\right)} = \left[\frac{\mathscr{B}}{z} + \frac{\mathscr{D}}{z-1} + + \frac{\mathscr{H}}{z - z_r}\right]\nu(\nu+1), \tag{3.37}$$

where $\mathscr{B}, \mathscr{D}$ and $\mathscr{H}$ given by,

$$\mathscr{B} = \frac{1}{k_\Lambda^2}\frac{1}{(r_c^+ - r_c^-)(r_+ - r_-^c)}, \tag{3.38}$$

$$\mathscr{D} = \frac{1}{k_\Lambda^2}\frac{1}{(r_c^+ - r_c^-)(r_+ - r_-)}\frac{1}{(1 - z_r)}, \tag{3.39}$$

$$\mathscr{H} = \frac{1}{k_\Lambda^2}\frac{1}{(r_c^+ - r_c^-)(r_+ - r_-)}\frac{1}{z_r(1 - z_r)}. \tag{3.40}$$

Hence with these results we have transformed the (3.17) of the following form,

$$\begin{gathered}\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + \left\{\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-z_r} - \frac{2}{z - z_\infty}\right\}\frac{\mathrm{d}R}{\mathrm{d}z} + \left[\frac{A}{(z_\infty - z)^2} + \frac{B}{z_\infty - z} + \frac{C}{z} + \frac{D}{z-1} + \frac{F}{z-z_r}\right]R \\ \left[\frac{A'}{z^2} + \frac{B'}{z} + \frac{C'}{(z-1)^2} + \frac{D'}{z-1} + \frac{E'}{(z-z_r)^2} + \frac{H'}{z-z_r}\right]R + \left[\frac{\mathscr{B}}{z} + \frac{\mathscr{D}}{z-1} + + \frac{\mathscr{H}}{z-z_r}\right]\nu(\nu+1)R = 0.\end{gathered} \tag{3.41}$$

However, this form of the radial equation (3.17) is still not a familiar one to us, therefore we must proceed with a further transformation. To achieve this aim, we need to find the exponents of (3.41) with the aid of the indicial equation, which is given by the following equation,

$$I(r) = r^2 + r(p_0 - 1) + q_0 = 0, \tag{3.42}$$

p_0 and q_0 defined as,

$$p_0 = \lim_{z \to z_i} (z - z_i)\, p(z), \tag{3.43}$$

$$q_0 = \lim_{z \to z_i} (z - z_i)^2\, q(z), \tag{3.44}$$

where $p(z)$ and $q(z)$ are the coefficient functions of $\frac{\mathrm{d}R}{\mathrm{d}z}$ and R respectively and z_i's are the relevant singularities, in our case they are $z_i = 0, 1, z_r, z_\infty$. Thus regarding a singularity located at z_∞ we have,

$$I(r) = r^2 - 3r + \frac{3\mu^2}{\Lambda} = 0, \tag{3.45}$$

which has two distinct roots,

$$r^{1,2}_{z_\infty} = \frac{1}{2}\left(3 \pm \sqrt{9 - \frac{12\mu^2}{\Lambda}}\right), \tag{3.46}$$

if we set $\mu^2 = \frac{2}{3}\Lambda$, we have

$$r^{1,2}_{z_\infty} = \{1, 2\}. \tag{3.47}$$

Similarly for $z = 0, 1$ and z_r we obtain the following roots,

$$r^{1,2}_{z=0} = \gamma_1 = \pm i\sqrt{A'}, \quad r^{1,2}_{z=1} = \gamma_2 = \pm i\sqrt{C'}, \quad r^{1,2}_{z=z_r} = \gamma_3 = \pm i\sqrt{E'}. \tag{3.48}$$

Having found the exponents of the (3.41) for a fixed value of the square of the particle mass, i.e $\mu^2 = \frac{2}{3}\Lambda$, we can thus define a transformation of the following form,

$$R = z^{\gamma_1}(z-1)^{\gamma_2}(z - z_r)^{\gamma_3}(z - z_\infty)\tilde{R} \tag{3.49}$$

where we choose one of the roots of the singularity located at $z = z_\infty$ as $r^2_{z_\infty} = 1$ which corresponds to the fixed value of $\mu^2 = \frac{2}{3}\Lambda$. Following results are immediate by derivatives of (3.49) with respect to z,

$$\frac{\mathrm{d}R}{\mathrm{d}z} = \frac{R}{\tilde{R}}\left[\frac{\mathrm{d}\tilde{R}}{\mathrm{d}z} + \left(\frac{\gamma_1}{z} + \frac{\gamma_2}{z-1} + \frac{\gamma_3}{z - z_r} + \frac{1}{z - z_\infty}\right)\tilde{R}\right], \tag{3.50}$$

$$\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} = \frac{R(z)}{\tilde{R}(z)}\left[\frac{\mathrm{d}^2\tilde{R}}{\mathrm{d}z^2} + 2\left(\frac{\gamma_1}{z} + \frac{\gamma_2}{z-1} + \frac{\gamma_3}{z-z_r} + \frac{1}{z-z_\infty}\right)\frac{\mathrm{d}\tilde{R}}{\mathrm{d}z}\right]$$
$$+ \left(\frac{\gamma_1(\gamma_1-1)}{z^2} + \frac{\gamma_2(\gamma_2-1)}{(1-z)^2} + \frac{\gamma_3(\gamma_3-1)}{(z_r-z)^2}\right)R - \frac{2}{z_\infty - z}\left(\frac{\gamma_1}{z} + \frac{\gamma_2}{z-1} + \frac{\gamma_3}{z-z_r}\right)R$$
$$+ 2\left[\frac{\gamma_1\gamma_2}{z(z-1)} + \frac{\gamma_2\gamma_3}{(z-z_r)(z-1)} + \frac{\gamma_3\gamma_1}{z(z-z_r)}\right]R. \quad (3.51)$$

Inserting (3.50) and (3.51) to (3.41) we see that the coefficient of $\frac{1}{z-z_\infty}$ vanishes, from the following equation,

$$\frac{1}{z-z_\infty}\left\{\frac{1}{z_\infty} - \frac{1}{1-z_\infty} - \frac{1}{z_r - z_\infty} - B\right\} = -\frac{1}{z_\infty(r_c^- - r_-)}\left(r_- + r_- + r_c^- + r_c^+\right) = 0 \quad (3.52)$$

due to Vieta's relation, i.e $r_- + r_- + r_c^- + r_c^+ = 0$. Thus we see that for a fixed value of $\mu^2 = \frac{2}{3}\Lambda$ and using appropriate transformation we have removed one of the singularities from radial Klein-Gordon equation (3.41), specifically the z_∞ one.

Now after combining the coefficients of $\frac{\mathrm{d}\tilde{R}}{\mathrm{d}z}$ and $\tilde{R}$, and performing some algebra we have obtained the following differential equation,

$$\frac{\mathrm{d}^2\tilde{R}}{\mathrm{d}z^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-z_r}\right)\frac{\mathrm{d}\tilde{R}}{\mathrm{d}z} + \frac{\alpha\beta z - q}{z(z-1)(z-z_r)}\tilde{R} = 0 \quad (3.53)$$

where,

$$\gamma = 2\gamma_1 + 1, \quad \delta = 2\gamma_2 + 1, \quad \varepsilon = 2\gamma_3 + 1, \quad (3.54)$$

$$\alpha\beta = (\sigma_{12} + \sigma_{23} + \sigma_{31}) - (a(1-z_r) + c)\,, \quad q = \sigma_{31} - z_r(c - \sigma_{31}), \quad (3.55)$$

with

$$\sigma_{ij} = \gamma_i + 2\gamma_i\gamma_j + \gamma_j, \quad \left\{i, j = 1, 2, 3 \quad if \quad i = j \Rightarrow \sigma_{ij} = 0\right\},$$
$$a = F + H' + \mathscr{H}\nu(\nu+1) + \frac{1}{z_r - z_\infty}, \quad b = D + D' + \mathscr{D}\nu(\nu+1) + \frac{1}{1-z_\infty},$$
$$c = C + B' + \mathscr{B}\nu(\nu+1) - \frac{1}{z_\infty}.$$

Note that $a + b + c = 0$.

Equation (3.53) is known as Heun's differential equation, and has the constraint for the parameters as follows,

$$\gamma + \delta + \varepsilon = \alpha + \beta + 1, \tag{3.56}$$

furthermore (3.53) has local solution Heun-local which we denote Hl, to represent it as the following series [17],

$$\tilde{R}(z) = \sum_{n=0}^{\infty} b_n \, z^n \tag{3.57}$$

with the normalisation $b_0 = 1$, given by

$$Hl(z_r, q; \alpha, \beta, \gamma, \delta; z). \tag{3.58}$$

Note that the parameter ε does not appear explicitly in this notational expression, so that the relation (3.56) should be kept in mind.

Heun's differential equation has been investigated extensively by many authors, for further inspection regarding the subject of complex differential equations and their solutions we refer to the literature [20–24], for physical relevance of the problem see [25, 26]. Having said that, however, we can present an analytic solution to the Eqn. (3.17), which we have thus obtained in the Heun form, in terms of hypergeometric Jacobi polynomials, this type of solution was first investigated by *N.Svartholm* [19]. Hence we start with making the following ansatz for the Heun's equation (3.53),

$$\tilde{R}(z) = \sum_{\rho=0}^{\infty} \; b_\rho y_\rho(z) \tag{3.59}$$

where

$$y_\rho(z) = F(-\rho, \rho + \omega; \gamma; z) = \frac{\rho! \, \Gamma(\gamma)}{\Gamma(\rho + \gamma)} P_\rho^{(\gamma-1, \delta-1)}(1 - 2z) \tag{3.60}$$

with $\omega = \delta + \gamma - 1$. The polynomial $y_\rho(z)$ satisfies the following differential equation,

$$y_\rho''(z) + \left[\frac{\gamma}{z} + \frac{\delta}{z-1}\right] y_\rho'(z) - \frac{\rho(\rho + \omega)}{z(z-1)} y_\rho(z) = 0 \tag{3.61}$$

and the recursion relations are given by,

$$z\, y_\rho(z) = \mathscr{X}_\rho\, y_{\rho+1}(z) + \mathscr{Y}_\rho\, y_\rho(z) + \mathscr{Z}_\rho\, y_{\rho-1}(z), \tag{3.62}$$

$$z(z-1)\,\frac{\mathrm{d}}{\mathrm{d}z} y_\rho(z) = \mathscr{X}'_\rho\, y_{\rho+1}(z) + \mathscr{Y}'_\rho\, y_\rho(z) + \mathscr{Z}'_\rho\, y_{\rho-1}(z) \tag{3.63}$$

where ($\rho \in \mathbb{Z}_+$)

$$\begin{cases} \mathscr{X}_\rho = -\frac{(\rho+\omega)(\rho+\gamma)}{(2\rho+\omega)(2\rho+\omega+1)} \\ \mathscr{Y}_\rho = \frac{(\omega-1)(\gamma-\delta)}{2(2\rho+\omega+1)(2\rho+\omega-1)} + \frac{1}{2} \\ \mathscr{Z}_\rho = -\frac{\rho(\rho+\delta-1)}{(2\rho+\omega)(2\rho+\omega-1)} \end{cases} \qquad \begin{cases} \mathscr{X}'_\rho = -\frac{\rho(\rho+\omega)(\rho+\gamma)}{(2\rho+\omega)(2\rho+\omega+1)} \\ \mathscr{Y}'_\rho = \frac{\rho(\rho+\omega)(\gamma-\delta)}{(2\rho+\omega+1)(2\rho+\omega-1)} \\ \mathscr{Z}'_\rho = \frac{\rho(\rho+\omega)(\rho+\delta-1)}{(2\rho+\omega)(2\rho+\omega-1)} \end{cases}$$

with initial values ($\rho = 0$)

$$\begin{cases} \mathscr{X}_0 = -\frac{\gamma}{\omega+1} \\ \mathscr{Y}_0 = \frac{\gamma}{\omega+1} \\ \mathscr{Z}_0 = 0 \end{cases} \qquad \begin{cases} \mathscr{X}'_0 = 0 \\ \mathscr{Y}'_0 = 0 \\ \mathscr{Z}'_0 = 0. \end{cases}$$

For large index values as $\rho \to \infty$, from the the first recursion relation one obtains for $\mathscr{X}_\rho, \mathscr{Y}_\rho, \mathscr{Z}_\rho$,

$$\begin{cases} \mathscr{X}_\rho = -\frac{1}{4} + \frac{1-2\gamma}{8\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right) \\ \mathscr{Y}_\rho = \quad \frac{1}{2} + \mathcal{O}\left(\frac{1}{\nu^2}\right) \\ \mathscr{Z}_\rho = -\frac{1}{4} - \frac{1-2\gamma}{8\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right). \end{cases}$$

Substituting (3.59) into Heun's equation (3.53) we obtain the result that if the series (3.59) is a solution of the Heun's differential equation then the coefficients in the series of Jacobi polynomials, b_ρ, satisfies the following relation,

$$b_{\rho+1} = \mathscr{P}_\rho\, b_\nu + \mathscr{Q}_\rho\, b_{\rho-1}, \tag{3.64}$$

where

$$\mathscr{P}_\rho = -\frac{[\alpha\beta + \rho(\rho+\omega)]\mathscr{Y}_\rho - q - z_r(\rho+\omega)\rho + \varepsilon\mathscr{Y}'_\rho}{[(\rho+1)(\rho+1+\omega)]\mathscr{Z}_{\rho+1} + (\alpha\beta)\mathscr{Z}_{\rho+1} + \varepsilon\mathscr{Z}'_{\rho+1}}, \tag{3.65}$$

$$\mathscr{Q}_\rho = -\frac{[(\rho-1)(\rho-1+\omega) + \alpha\beta]\mathscr{X}_{\rho-1} + \varepsilon\mathscr{X}'_{\rho-1}}{[\alpha\beta + (\rho+1)(\rho+1+\omega)]\mathscr{Z}_{\rho+1} + \varepsilon\mathscr{Z}'_{\rho+1}}. \tag{3.66}$$

And the sequence is preconditioned to,

$$b_1 = \mathscr{P}_0\, b_0 = -\frac{(2+\omega)[\alpha\beta\gamma - q(\omega+1)]b_0}{\delta[(\varepsilon-1)(1+\omega)-\alpha\beta]}. \tag{3.67}$$

Furthermore, when the index $\rho \to \infty$, the coefficients $\mathscr{P}_\rho$ and $\mathscr{Q}_\rho$ behave asymptotically as,

$$\mathscr{P}_\rho = 2 - 4\, z_r - \frac{[2(\gamma+\varepsilon)-5](2\, z_r - 1)}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right), \tag{3.68}$$

$$\mathscr{Q}_\rho = -\left\{1 + \frac{[2(\gamma+\varepsilon)-5]}{\rho}\right\} + \mathcal{O}\left(\frac{1}{\rho^2}\right). \tag{3.69}$$

Invoking the concept of so-called "augmented converges" one can determine the domain of the convergence for the series (3.59) which is expressed in terms of hypergeometric Jacobi polynomials, one obtains that the series is convergent inside the ellipse with foci at $z = 1$ and $z = 0$, pierce through the point $z = z_r$, with a possible exclusion of the line connecting the two foci. Furthermore, if one ensures that the $\Re[\,\varepsilon] < 1$, one obtains the absolute convergences for the series given by the equation (3.59). For a detailed analysis of the asymptotic properties both for the polynomial $y_\rho(z)$ and the coefficient b_r, see [18].

Other complex physical systems such as Kerr-(Anti)de Sitter and Kerr-Newmann-(Anti)de Sitter space-times [25, 26] have studied extensively in the literature. It is shown that the RKGE can be also transformed into the Heun form, using the same mechanical steps that we have used. However, the difference from the sad works lies in the roots of the (3.18), which now encodes the effect of the gravitational monopole. Hence, the parameters of the Heun equation and as a result(3.53) the radial solution has changed. In [26], if one takes the rotation parameter as $a \to 0$ then, one obtains the RKGE solution in the limit $b \to 1$, in our solution.

3.2. The Hamilton-Jacobi Equation

The subject of this section is the derivation of geodesics of our space-time configuration that is given by the line element (2.50), via the Hamilton-Jacobi equation.

It is given by the equation for a massive and electrically charged particle e as,

$$\frac{\partial S}{\partial \tau}+\frac{1}{2}\,g^{\mu\nu}\left(\partial_\mu S-e\,A_\mu\right)\left(\partial_\nu S-e\,A_\nu\right)=0. \tag{3.70}$$

where τ is the proper time, defined via the line element as $ds^2=-d\tau^2$ with $g^{\mu\nu}$ and A_μ are the inverse metric and electric potential given by the equations (3.3) and (3.7) respectively. To solve this equation we consider an action S given by the following,

$$S=\frac{1}{2}\,\epsilon-E\,t+L\,\phi+S_r(r)+S_\theta(\theta) \tag{3.71}$$

where E and L are identified as the total energy and the angular momentum of the particle and ϵ is a constant defined as,

$$\epsilon=-\frac{ds^2}{d\tau^2}, \tag{3.72}$$

and takes different values with respect the following cases,

$$\begin{cases} +1\,for\;time-like\,trajectory\,(massive\,particle), \\ \;\;0\,for\,null\,trajectory\,(massless\,particle), \\ -1\,for\;space-like\,trajectory\,(non-physical). \end{cases}$$

Calculation of the equation (3.70) with (2.50) and (3.7) yields,

$$\frac{\Delta_r}{r^2}\left(\partial_r S\right)^2+\frac{1}{r^2}\left(\partial_\theta S\right)^2+\frac{r^2}{\Delta_r}\left[E-\frac{e\,Q}{r}\right]^2+\frac{L^2}{r^2\sin^2\theta}+\epsilon=0. \tag{3.73}$$

Substitution of equation (3.71) to (3.73) yields the following equations,

$$\left(\partial_\theta S_\theta\right)^2+\frac{L^2}{\sin^2\theta}+\epsilon=-\Delta_r\left(\partial_r S_r\right)^2-\frac{r^4}{\Delta_r}\left[E-\frac{e\,Q}{r}\right]^2, \tag{3.74}$$

with

$$\frac{\mathrm{d}t}{\mathrm{d}\tau}=-\frac{r^2}{\Delta_r}\left\{E-\frac{e\,Q}{r}\right\}, \tag{3.75}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau}=\frac{L^2}{r^2\sin^2\theta}. \tag{3.76}$$

We can separate equation (3.74) by using a separation constant. However we have spherical symmetry in our space-time, therefore we can confine the motion to an equatorial plane, which implies $\theta=\pi/2$. Then if we define the Mino time [27] as

$r^2 d\gamma = d\tau$ equations take the form,

$$\frac{\mathrm{d}r}{\mathrm{d}\gamma} = \sqrt{r^4 \left\{ E - \frac{e\,Q}{r} \right\}^2 - \Delta_r \left(\frac{L^2}{r^2} + \epsilon \right)} = \sqrt{R}, \tag{3.77}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\gamma} = -\frac{r^4}{\Delta_r} \left\{ E - \frac{e\,Q}{r} \right\} = \chi(r), \tag{3.78}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\gamma} = L^2. \tag{3.79}$$

Analytically solution of (3.79) rather trivial, on the other hand integrals in equations (3.77) and (3.78) are very difficult to solve since, they include higher degree polynomials. However, for the sake of demonstration we can solve the integral of (3.77) for a null geodesic, i.e $\varepsilon = 0$ [28]. Here the polynomial degree of $R(r)$ is four. We can solve it by recognising the integral (3.77) as an elliptic type. If one assume such that the polynomial $R(r)$ has only simple poles, one can immediately state that the solutions of (3.91) can be expressed as Weierstrass elliptic functions that is $\wp(\,.\,; g_2\,; g_3\,)$ where g_2 and g_3 are Weierstrass invariants. Now equation (3.77), after taking the square of both sides, can be written as

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\gamma} \right)^2 = R(r) = \sum_{i=0}^{4} s_i\, r^i \tag{3.80}$$

where,

$$s_0 = -Q^2\,L^2, \quad s_1 = 2ML^2, \quad s_2 = e^2\,Q^2 - b^2\,L^2, \quad s_3 = -2Q\,e, \quad s_4 = E^2 + \frac{\Lambda}{3} L^2. \tag{3.81}$$

We start by reducing the degree of fourth order polynomial by exploiting the transformation $r = 1/t + r_p$, where r_p is one of the roots of $R(r)$, which yields,

$$R_3(t) = \sum_{j=0}^{3} a_j\, t^j, \quad \text{where} \quad a_j = \frac{1}{(4-j)!} \frac{\mathrm{d}^{4-j}}{\mathrm{d}r^{4-j}} R(r) \quad \text{evaluated at} \quad r = r_p. \tag{3.82}$$

To obtain the Weierstrass form we further use the transformation:

$$t = \frac{1}{a_3} \left(4z - \frac{a_2}{3} \right). \tag{3.83}$$

Hence we have,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\gamma}\right)^2 = P_3(z) = 4\,z^3 - g_2\,z - g_3 \tag{3.84}$$

with Weierstrass invariants given by,

$$g_2 = \frac{a_2^2}{12} - \frac{a_1\,a_3}{4}, \quad g_3 = \frac{a_1\,a_2\,a_3}{48} - \frac{a_2^2}{216} - \frac{a_0\,a_3^2}{16}. \tag{3.85}$$

Solution to the equation (3.84) is given by Weierstrass elliptic $\wp$-function given as [29],

$$z(\gamma) = \wp(\,\gamma - \gamma_i'\,;\, g_2\,;\, g_3\,) \tag{3.86}$$

where

$$\gamma_i' = \gamma_i + \int_{z_i}^{\infty} \frac{\mathrm{d}z}{\sqrt{4\,y^3 - g_2\,z - g_3}}, \quad \text{with} \quad z_i = \frac{a_3}{4\,(r_i - r_p)} + \frac{a_2}{12} \tag{3.87}$$

Therefore solution to the (3.80) can be expresses in the following form,

$$r = \frac{a_3}{\wp(\,\gamma - \gamma_i'\,;\, g_2\,;\, g_3\,) - b_2/3} + r_p. \tag{3.88}$$

Thus we have obtained a radial solution corresponding to the trajectory of a light-ray in terms of Weierstrass $\wp$-function. Contribution of the gravitational monopole b^2 is encoded in the Weierstrass invariants, as a result, it affects the radial trajectory of the light beam.

3.3. Geodesics

Having determined the equations defining trajectories of a charged particle using Hamilton-Jacobi formalism and discussing some of their possible analytical solutions, in this subsection we will discuss geodesic equations of neutral test particles. This will help us to analyze the effect of the monopole charge on some physically relevant phenomena.

From the Hamilton-Jacobi equations, we find the following equatorial ($\theta = \pi/2$) geodesic equations for a neutral test particle,

$$\frac{dt}{d\lambda} = -\frac{E^2\, r^2}{\Delta_r}, \tag{3.89}$$

$$\frac{d\phi}{d\lambda} = \frac{L}{r^2}, \tag{3.90}$$

$$\frac{dr}{d\lambda} = \sqrt{E^2 - \left(\epsilon + \frac{L^2}{r^2}\right)\frac{\Delta_r}{r^2}}. \tag{3.91}$$

The equation (3.91) can be put into the form

$$\dot{r}^2 = E^2 - V_{eff}, \tag{3.92}$$

where overdot means derivative with respect to the affine parameter, λ, and the effective potential is given by

$$V_{eff} = \left(\epsilon + \frac{L^2}{r^2}\right)\frac{\Delta_r}{r^2} = \left(\epsilon + \frac{L^2}{r^2}\right)\left(b^2 - \frac{2M}{r} - \frac{\Lambda\, r^2}{3} + \frac{Q^2}{r^2}\right). \tag{3.93}$$

Hence, radial geodesic motion is only possible for $E^2 \geq V_{eff}$.

Using the equations (3.90) and (3.91) one obtains an orbit equation as follows

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}\left[E^2 - \left(\frac{L^2}{r^2} + \epsilon\right)\frac{\Delta_r}{r^2}\right]. \tag{3.94}$$

With employing the usual inverse radial distance parameter $u = 1/r$, the orbit equation yields a modified Binet equation of the form

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{L^2} - \left(\frac{\epsilon}{L^2} + u^2\right)\frac{\Delta_r}{r^2}. \tag{3.95}$$

The explicit form of this equation is obtained if we replace the metric function Δ_r, which is

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{L^2} - \left(\frac{\epsilon}{L^2} + u^2\right)\left(b^2 - 2Mu - \frac{\Lambda}{3u^2} + Q^2u^2\right). \tag{3.96}$$

Taking the derivative of both sides with respect to the coordinate ϕ, we obtain a modified Binet equation as follows:

$$\frac{d^2u}{d\phi^2} - \epsilon\frac{M}{L^2} + u = (1 - b^2)u + 3Mu^2 - \epsilon\frac{\Lambda}{3L^2u^3} - Q^2u\left(2u^2 + \frac{\epsilon}{L^2}\right). \tag{3.97}$$

Note that for timelike particles the left-hand side of this equation reduces to Newtonian Binet equation for central force motion for two body problem whereas the right-hand side involves corrections of the GR from monopole charge, mass and angular momentum of the source, cosmological constant and electrical charge of the source. We will analyze this equation for some physically important phenomena such as the deflection of light rays from a gravitational monopole swallowed by a charged black hole with a cosmological constant.

3.3.1. Deflection of light rays

For photons we take $\epsilon = 0$ in equation (3.97) which yields

$$\frac{d^2u}{d\phi^2} + u = \eta'^2 u + 3Mu^2 - 2Q^2u^3. \tag{3.98}$$

Here we have recovered the monopole term by rescalling $b^2 = 1 - 8\pi\eta^2$ and used the abbreviation $\eta' = 8\pi\eta^2$. Note that this equation does not involve the cosmological constant, hence its solution is independent of it. However, the cosmological constant may indirectly affect the light rays. For clarity, we first skip the effect of the cosmological constant to this phenomena and restrict ourselves with a solution describing monopole swallowed by a Reissner-Nordstrom black hole by setting $\Lambda = 0$.

3.3.2. Deflection of light rays from RN-monopole spacetime

We exploit the usual perturbative technique up to second-order terms to determine a solution to the equation (3.98). Hence we consider the following solution ansatz which contains solutions up to $O(\eta'^2, \varepsilon^2, \nu^2)$:

$$\begin{aligned} u(\phi) &= u_0(\phi) + \eta'\, u_b(\phi) + \varepsilon\, u_m(\phi) + \nu\, u_q(\phi) + \eta'^2\, v_b(\phi) + \varepsilon^2\, v_m(\phi) \\ &\quad + \nu^2\, v_q(\phi) + \eta'\varepsilon\, z_{bm}(\phi) + \eta'\nu\, z_{bq}(\phi) + \varepsilon\nu\, z_{mq}(\phi), \end{aligned} \tag{3.99}$$

where the dimensionless perturbation parameters are given by

$$\varepsilon = \frac{M}{R}, \tag{3.100}$$

$$\nu = \frac{Q^2}{R^2}, \tag{3.101}$$

together with $\eta' = 8\pi\eta^2$. Replacing (3.99) into (3.98) we obtain the following set of equations

$$O(1): \quad u_0''(\phi) + u_0(\phi) = 0, \tag{3.102}$$

$$O(\eta'): \quad u_b''(\phi) + u_b(\phi) = u_0(\phi), \tag{3.103}$$

$$O(\varepsilon): \quad u_m''(\phi) + u_m(\phi) = 3R\, u_0^2(\phi), \tag{3.104}$$

$$O(\nu): \quad u_q''(\phi) + u_q(\phi) = -2R^2\, u_0^3(\phi), \tag{3.105}$$

$$O(\eta'^2): \quad v_b''(\phi) + v_b(\phi) = u_b(\phi), \tag{3.106}$$

$$O(\varepsilon^2): \quad v_m''(\phi) + v_m(\phi) = 6R\, u_0(\phi)\, u_m(\phi), \tag{3.107}$$

$$O(\nu^2): \quad v_q''(\phi) + v_q(\phi) = -6R^2\, u_0^2(\phi)\, u_q(\phi), \tag{3.108}$$

$$O(\eta'\varepsilon): \quad z_{bm}''(\phi) + z_{bm}(\phi) = 6R\, u_0(\phi) u_b(\phi) + u_m(\phi), \tag{3.109}$$

$$O(\eta'\nu): \quad z_{bq}''(\phi) + z_{bq}(\phi) = -6R^2\, u_0^2(\phi) u_b(\phi) + u_q(\phi), \tag{3.110}$$

$$O(\varepsilon\nu): \quad z_{mq}''(\phi) + z_{mq}(\phi) = -6R^2\, u_0^2(\phi) u_m(\phi) + 6R\, u_0(\phi) u_q(\phi). \tag{3.111}$$

Solving perturbatively, we find the following solution

$$\begin{aligned} u(\phi) \;=\; & \frac{\cos\phi}{R} + \eta' \frac{\cos(\phi) + \phi\sin(\phi)}{2R} - \varepsilon \frac{\cos(2\phi) - 3}{2R} + \nu \frac{\cos(3\phi) - 12\phi\sin(\phi) - 9\cos(\phi)}{16R} \\ & + \eta'^2 \frac{(5 - 2\phi^2)\cos(\phi) + 6\phi\sin(\phi)}{16R} + \varepsilon^2 \frac{3[20\phi\sin(\phi) + 22\cos(\phi) + \cos(3\phi)]}{16R} \\ & + \nu^2 \frac{(273 - 72\phi^2)\cos(\phi) + 384\phi\sin(\phi) - 36\phi\sin(3\phi) - 48\cos(3\phi) + \cos(5\phi)}{256R} \\ & - \eta'\varepsilon \frac{\phi\sin(2\phi) + 2\cos(2\phi) - 6}{2R} \\ & + \eta'\nu \frac{(24\phi^2 - 91)\cos(\phi) + 2[3\phi(\sin(3\phi) - 19\sin(\phi)) + 5\cos(3\phi)]}{64R} \\ & - \varepsilon\nu \frac{87 - 40\cos(2\phi) - 12\phi\sin(2\phi) + \cos(4\phi)}{16R}. \end{aligned} \tag{3.112}$$

Consider a photon which comes from far away at a distant past ($u = 0, \phi = -\pi/2 - \delta\phi/2$) deflected by the black hole and travels towards far away at distant future

$(u=0, \phi=\pi/2+\delta\phi/2)$ where $\delta\phi$ is the angle of deflection by the black hole. Since the solution (3.112) is symmetric under the transformation $\phi \to -\phi$, it is sufficient to calculate the half deflection angle by feeding $\phi \to \pi/2+\delta\phi/2$ into (3.112) and taking the limit $\delta\phi \to 0$ we find the deflection angle as

$$\delta\phi = \frac{\pi}{2}\eta' + 4\varepsilon - \frac{3\pi}{4}\nu + \frac{3\pi}{8}\eta'^2 + \frac{15\pi}{4}\varepsilon^2 + \frac{105\pi}{64}\nu^2 + 8\eta'\varepsilon - \frac{15\pi}{8}\eta'\nu - 16\varepsilon\nu. \quad (3.113)$$

In terms of parameters of the black hole and the impact parameter R, this expression explicitly becomes

$$\begin{aligned} \delta\phi &= \frac{\pi}{2}\eta' + \frac{4M}{R} - \frac{3\pi Q^2}{4R^2} + \frac{3\pi}{8}\eta'^2 + \frac{15\pi M^2}{4R^2} + \frac{105\pi Q^4}{64R^4} \\ &\quad + \frac{8\eta' M}{R} - \frac{15\pi\eta' Q^2}{8R^2} - \frac{16MQ^2}{R^3}. \end{aligned} \quad (3.114)$$

On the other hand, we can also calculate the minimum distance, r_{min}, that the light can approach the central deflecting object to express the deflection angle. Setting $\phi = 0$ in (3.112), which yields $u = 1/r_{min}$, with the result

$$\begin{aligned} \frac{1}{r_{min}} &= \frac{1}{R}\left(1 + \frac{\eta'}{2} + \frac{M}{R} - \frac{Q^2}{2R^2} + \frac{5\eta'^2}{16} + \frac{69M^2}{16R^2} \right. \\ &\quad \left. + \frac{226Q^4}{256R^4} + \frac{2\eta' M}{R} - \frac{81\eta' Q^2}{64R^2} - \frac{3MQ^2}{R^3}\right). \end{aligned} \quad (3.115)$$

Now we need to invert this equation for r_{min}, then put back into (3.114). It turns out that we will only need the first order correction terms for the parameters, namely we just need the following expression

$$\frac{1}{R} \simeq \frac{1}{r_{min}}\left(1 - \frac{\eta'}{2} - \frac{M}{r_{min}} + \frac{Q^2}{2r_{min}^2}\right). \quad (3.116)$$

Replacing this into (3.114), we find the deflection angle, to the $O(\eta'^2, \varepsilon^2, \nu^2)$, as follows

$$\begin{aligned} \delta\phi &= \frac{\pi}{2}\eta' + \frac{4M}{r_{min}} - \frac{3\pi Q^2}{4r_{min}^2} + \frac{3\pi\eta'^2}{8} + \frac{(15\pi - 16)M^2}{4r_{min}^2} + \frac{57\pi Q^4}{64r_{min}^4} \\ &\quad + \frac{6\eta' M}{r_{min}} - \frac{9\pi\eta' Q^2}{8r_{min}} + \frac{(3\pi - 28)MQ^2}{r_{min}^3}. \end{aligned} \quad (3.117)$$

When we compare this result with the existing solutions in the literature we see that it agrees with the previous results found before. The first term and fourth terms are the effects of the monopole, i. e. the angle deficit, in the photon motion and is compatible with the result given in [33]. The terms first order in the mass of the black hole is the famous result derived by Einstein himself and the second order term in mass is the PPN result, [34–36] and also presented in [37]. The first order effects of the electrical charge are presented in [38] in Brane-world models where a black hole living in a brane embedded in five-dimensional bulk resembles the form of a RN black hole of GR. The second order corrections of the charge to the light deflection is also presented in [39] again for brane world scenario. Our results in (3.117) resemble all of these previous works and our new results are the second order terms involving the monopole term η'.

3.3.3. Deflection of light rays from RN-dS-monopole spacetime

To observe the effect of the cosmological constant to the light deflection we fallow the Rindler and Ishak method [43] which is a special treatment where the source and observer are taken to be static.

Consider now the metric on the equatorial plane $\theta = \pi/2$ for our space-time confifuration,

$$d\ell^2 = g_{ij}dx^i dx^j = \frac{dr^2}{A(r)} + r^2 d\phi^2, \tag{3.118}$$

where the metric function

$$A(r) = 1 - \eta' - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2. \tag{3.119}$$

One can exploit the invariant formula for the cosine between this two coordinate directions d and δ given below to calculate the deflection angle as,

$$\cos\psi = \frac{g_{ij}dx^i \delta x^j}{\sqrt{g_{ij}dx^i dx^j}\sqrt{g_{ij}\delta x^i \delta x^j}}, \tag{3.120}$$

$$d = (dr, d\phi) = (\frac{dr}{d\phi}, 1)d\phi, \tag{3.121}$$

$$\delta = (\delta r, 0). \tag{3.122}$$

Using (3.118) in (3.120) with the relation $\tan\psi = \sqrt{\sec^2\psi - 1}$ one obtains,

$$\tan\psi = \sqrt{A(r)}\, r \left|\frac{dr}{d\phi}\right|^{-1} \quad\to\quad \psi \approx \left[\sqrt{A(r)}\, r \left|\frac{dr}{d\phi}\right|^{-1}\right]_{\phi=\frac{\pi}{2}}. \tag{3.123}$$

From (3.112) we calculate,

$$\begin{aligned}\left[\frac{1}{r}\right]_{\phi=\frac{\pi}{2}} &= \frac{1}{R}\left(\frac{\pi}{4}\,\eta' + 2\,\varepsilon - \frac{3\pi}{8}\,\nu + \frac{3\pi}{16}\,\eta'^2 + \frac{15\pi}{8}\,\varepsilon^2 + \frac{105\pi}{128}\,\nu^2\right.\\ &\quad \left. +4\,\eta'\varepsilon - \frac{15\pi}{16}\eta'\nu - 8\,\varepsilon\nu\right) = \frac{\delta\phi}{2R},\end{aligned} \tag{3.124}$$

$$\begin{aligned}\left[\frac{dr}{d\phi}\right]_{\phi=\frac{\pi}{2}} &= \frac{r^2(\pi/2)}{R}\left(1 - \frac{(2+\pi^2)}{32}\,\eta'^2 + \frac{(18\pi^2-2)}{256}\,\nu^2 + \frac{3}{16}\,\varepsilon^2\right.\\ &\quad \left. +\frac{\pi}{2}\,\eta'\varepsilon + \frac{(91\pi^2-456)}{256}\,\eta'\nu - \frac{3\pi}{4}\,\varepsilon\nu\right) \approx \frac{r^2}{R}.\end{aligned} \tag{3.125}$$

Inserting these results with the weak field form of the metric function $A(r)$ to the Eq. (3.123) we obtain,

$$\psi \approx \sqrt{1 - \eta' - \frac{\Lambda}{3}r^2}\,\frac{R}{r} \approx \frac{\delta\phi}{2} - \frac{\Lambda R^2}{3\,\delta\phi} - \left\{\frac{\pi}{8}\eta'^2 + \frac{\eta' M}{R} - \frac{3\pi\eta' Q^2}{16R^2}\right\}. \tag{3.126}$$

The contribution from the cosmological constant can be written explicitly as,

$$\begin{aligned}\frac{\Lambda R^2}{3\,\delta\phi} =& \frac{64\Lambda R^6}{3}\left[(32\pi R^4)\eta' + (256R^3)M - (48\pi R^2)Q^2 + (24\pi R^4)\eta'^2 + (240\pi R^2)M^2\right.\\ & \left. +(105\pi)Q^4 + (512R^3)\eta' M - (120R^2)\eta' Q^2 - (1024R)MQ^2\right]^{-1}.\end{aligned} \tag{3.127}$$

Note that in the weak field we have an (A)-dS-monopole space-time as a result the term η' further effects the bending angle. In Eqn. (3.127), the second term in brackets is the contribution of the mass which is given in [44], the third term is the first order effect of the charge [45]. Our result generalizes the previous work done by the cited authors via the inclusion of the monopole term.

4. PHENOMENA OF SUPERRADIANCE

As an application of the RKGE which we have obtained in section (3.1) given by the equation (3.17), we investigate the phenomena of superradiance in the background of RN-AdS space-time, which corresponds to the space-time of a massive, charged, static black hole, with global monopole against charged scalar perturbations. The aim of this section is to find an instability condition for our space-time configuration for small mass and charge via solving the radial wave equation (3.17) in the low frequency domain, i.e $(r - r_+) << \frac{1}{\omega}$, by exploiting the asymptotic matching technique. We separate our investigation into two cases such that; the first case corresponds to nonzero values of cosmological constant, specifically $\Lambda < 0$, that can be called natural superradiance. In the second case we will be interested in the absence of cosmological constant, i.e $\Lambda = 0$, which will be the case of artificial superradiance. Before we start our investigation we first discuss the asymptotic behaviour of the scalar field near the horizon and the radial infinity for certain values of the parameters of the scalar field and the black hole.

4.1. The Asymptotic behaviour of the scalar field

Our radial differential equation given by equation (3.17) was,

$$\Delta_r \partial_r \left[\Delta_r \partial_r R(r)\right] + \left[(\omega\, r^2 - e\, Q\, r)^2 - \Delta_r \left(\nu(\nu+1) + \mu^2 r^2\right)\right] R(r) = 0, \tag{4.1}$$

where Δ_r given by (2.51)

$$\Delta_r = b^2 r^2 - 2Mr - \frac{r^4}{\ell^2} + Q^2, \tag{4.2}$$

with

$$b^2 = (1 - 8\pi\eta^2), \qquad \ell = \sqrt{-\frac{3}{\Lambda}}. \tag{4.3}$$

Consider now the tortoise coordinate transformation defined as follows,

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = \frac{r^2}{\Delta_r}, \qquad \bar{R}(r^*) = R(r)\, r. \tag{4.4}$$

Then the equation (3.17) takes the following Schrödinger-like form,

$$\frac{\mathrm{d}^2\bar{R}(r^*)}{\mathrm{d}r^{*2}} + V(r^*)\bar{R}(r^*) = 0, \tag{4.5}$$

where we have defined the effective potential as,

$$V(r^*) = \left(\omega - \frac{e\,Q}{r}\right)^2 - \frac{\Delta_r}{r^4}\left[\nu(\nu+1) + \mu^2 r^2\right] + \frac{2}{r^6}\Delta_r\left(\frac{r^4}{\ell^2} - Mr + Q^2\right). \tag{4.6}$$

4.1.1. Scalar field near the horizon

Near the horizon, which is corresponds to the largest root of Δ_r, $r \to r_+$ such that the coefficient of radial equation (3.17) behaves as $\Delta_r \to 0$, the effective potential becomes,

$$V(r^*) \to (\omega - e\Phi_h)\,, \quad \text{as} \qquad r \to r_+ \qquad \Delta_r \to 0, \tag{4.7}$$

where Φ_h is the electric potential at near the event horizon defined by,

$$\Phi_h = \frac{Q}{r_+}. \tag{4.8}$$

The boundary conditions for the scalar field are the following,

$$\Phi \to 0 \qquad \text{when} \qquad r \to \infty. \tag{4.9}$$

Due to the fact that AdS space behaves effectively as a reflecting mirror. Hence near the horizon we have,

$$\Phi \sim e^{-i\omega t \pm i(\omega - \Phi_h)r^*}, \tag{4.10}$$

where r^* is the tortoise coordinate given by the equation (4.5). Since our investigation is placed in the classical domain, we require only ingoing waves at the horizon, which implies that one must restrict the group velocity of the wave packet to a negative one. Due to the fact that, classically speaking, no information can come out from a static black hole.

4.1.2. Scalar field at the infinity

At the radial infinity there are different asymptotic behaviour for the scalar field depending on the cosmological constant and mass of the scalar field. Hence we will discuss below these different cases, separately.

I - Nonvanishing Cosmological constant case

For nonvanishing cosmological constant, i. e $\ell \neq \infty$, we have

$$V(r^*) \to \infty, \quad \text{as} \qquad r \to \infty \quad (\text{for} \quad \ell \neq \infty) \tag{4.11}$$

which implies that the boundary condition for the scalar field in this case is the following,

$$R \to 0 \qquad \text{when} \qquad r^* \to \infty, \tag{4.12}$$

due to the fact that AdS space behaves effectively as a reflecting mirror.

II - Vanishing Cosmological Constant case

In the absence of cosmological constant, however, the behaviour of the scalar field is very different, since

$$V(r^*) \to \omega^2 - b^2\mu^2, \quad \text{as} \qquad r \to \infty \quad (\text{for} \quad \ell = \infty). \tag{4.13}$$

Hence, for vanishing cosmological constant, and if the scalar field is massive ($\mu \neq 0$), then bound states that are decaying at infinity are possible for the scalar field if $\omega^2 < b^2\mu^2$ with

$$\bar{R}(r^*) \to e^{-\sqrt{b^2\mu^2-\omega^2}\,r^*} \qquad \text{when} \qquad r \to \infty. \tag{4.14}$$

Hence, similar to RN or Kerr black holes, the mass of the scalar field can act as a potential barrier if it satisfies $\omega^2 < b^2\mu^2$. We see that the effect of the monopole term is to reduce the height of the potential barrier by a factor of $b^2 = 1 - 8\pi\eta^2 < 1$. However, it was shown in [40–42] that, unlike Kerr black holes, in the superradiant regime there is no meta stable bound states for RN solution and RN black holes are stable against charged scalar perturbations. Hence we will not pursue the investigation of stability due to mass of the scalar field in this thesis. An open problem will be to investigate the stability of a global monopole swallowed by a charged and rotating black hole, a solution which awaits its discovery, against charged and massive scalar perturbations. However, this solution is not known as far as we know yet.

For the case where the mass of the scalar field vanishes or $\omega^2 \geq b^2\mu^2$, then there is no bound state solutions and the field behaves as

$$R(r^*) \to e^{\pm i\omega_0 r^*} \tag{4.15}$$

where $\omega_0 = \sqrt{\omega^2 - b^2\mu^2}$, with $\omega^2 \geq b^2\mu^2$. For this case the superradiance scattering cannot lead to an instability unless one uses some artificial mechanisms such as surrounding the black hole with a reflective mirror as done in the black hole bomb mechanism.

4.2. Superradiance Condition

Here we derive the superradiance condition for vanishing cosmological constant case. Let us consider a scattering experiment of a monochromatic scalar wave with frequency ω with a wave function of the form $\Phi = \bar{R}e^{-i\omega t + im\phi}$. When a scalar wave is sent from the radial infinity with unit amplitude, and when we consider the black hole horizon as a one way membrane with no flux outside the horizon from the black hole, then the asymptotic form of the solution of the equation (4.5) can be written as

$$\bar{R} \sim \begin{cases} \mathcal{T}e^{-i(\omega - e\Phi_h)r^*} & \text{as } r \to r_h \\ \mathcal{R}e^{i\omega_0 r^*} + e^{-i\omega_0 r^*} & \text{as } r \to \infty \end{cases} \tag{4.16}$$

Here $\mathcal{R}$ and $\mathcal{T}$ are the amplitudes of the reflected and transmitted wawes, respectively. Note that the complex conjugate of $\bar{R}$, in which we will denote as $\bar{R}^\dagger$, should be also a solution of the equation (4.5) since the potential $V(r^*)$ is real and the solutions are invariant under $t \to -t$, $\omega \to -\omega$. Then $\bar{R}$ and $\bar{R}^\dagger$ should be linearly independent and their Wronskian $W = \bar{R}\partial r^* \bar{R}^\dagger - \bar{R}^\dagger \partial r^* \bar{R}$ should be independent of r^*. Calculating Wronskians near the horizon and at the radial infinity and equating them one obtains

$$|\mathcal{R}|^2 = 1 - \frac{\omega - e\Phi_h}{\omega_0} |\mathcal{T}|^2. \tag{4.17}$$

Hence when the superradiant condition

$$\omega < e\Phi_h \tag{4.18}$$

is satisfied, then the amplitude of the scattered wave becomes greater than it is sent. This phenomenon is called as the superradiant scattering. Note that for the AdS case, the condition for superradiance is also the same. This can be derived by the fact that the phase velocity of the waves flowing into the horizon changes sign relative to the group velocity of these waves. Now let us discuss the role of the monopole charge on the superradiant treshold frequency

$$\omega_p = e\Phi_h = e\frac{Q}{r_+}. \tag{4.19}$$

The monopole term, $b^2 = 1 - 8\pi\eta^2 < 1$, affect the superradiance treshold frequency since it changes the location of the outer horizon r_+ which is given by

$$r_+ = \frac{M + \sqrt{M^2 - Q^2 b^2}}{b^2}. \tag{4.20}$$

When the monopole is present, the location of the outer horizon increases relative to the case where the monopole is not present ($b = 1$). Hence the electric potential of the horizon decreases in the presence of the monopole term. Therefore, here we conclude that the presence of the monopole charge reduces the superradiant treshold frequency of the wave. A wave with frequency ω which may trigger the superradiant scattering when the monopole term is absent, may not trigger the superradiant scattering when the monopole term present.

4.3. Case 1 : Superradiant Instability of Global Monopole Configuration in RN-AdS Space-Times ($\Lambda < 0$)

As we have said before, we exploit the asymptotic matching technique to obtain the superradiance instability condition for RN-AdS space-time with global monopole configuration where this technique divides the solution as near and far region solutions [30, 31].

A - Near Region Solution

For small AdS black holes, we have $r_+ << \ell$, in the near region we assume $(r - r_+) << \frac{1}{\omega}, \Lambda \sim 0, r \sim r_+$ and $\Delta_r \sim \Delta$, where

$$\Delta = b^2 r^2 - 2Mr + Q^2 = (r - r_+)(r - r_-), \tag{4.21}$$

$$r_\pm = \frac{M \pm \sqrt{M^2 - Q^2 b^2}}{b^2}, \tag{4.22}$$

we further assume that $\mu^2 r^2 << 1$ in the near region, since we are in the low frequency regime.

Now we will make a change of variable through the following definition,

$$z = \frac{r - r_+}{r - r_-}, \qquad 0 \leq z \leq 1, \tag{4.23}$$

the $z = 0$ correspond now the event horizon $r = r_+$. Using (4.23) we have the following results,

$$\Delta \partial_r = z[r_+ - r_-]\, \partial_z, \quad \Delta = z(r_- - r)^2, \quad (1 - z) = \frac{r_- - r_+}{r_- - r}. \tag{4.24}$$

Therefore Eqn. (3.17) becomes,

$$(1 - z) z\, \partial_z^2\, R + (1 - z)\partial_z\, R + \left\{ \varpi^2 \, \frac{1 - z}{z} - \frac{\nu(\nu + 1)}{1 - z} \right\} R = 0, \tag{4.25}$$

where we idenfined the so-called superradiant factor given by,

$$\varpi = \frac{(\omega - e\Phi_h)}{r_+ - r_-} r_+^2. \tag{4.26}$$

As we have seen in the section (3.1.1), when we transform the radial equation to Heun form, we have used the indicial equation (3.42), consequently we can find the exponent of (4.25) at $z = 0$ and $z = 1$ as

$$\alpha_1 = i\,\varpi, \qquad \alpha_2 = \nu + 1, \tag{4.27}$$

therefore we can define a transformation of the following form,

$$R = z^{i\,\varpi}\,(1-z)^{\nu+1}\,F. \tag{4.28}$$

Substituting (4.28) to (4.25) we obtain,

$$\begin{aligned}(1-z)z\,\partial_z^2\,F + \{(1+2\,i\,\varpi) - [2(\nu+1) + 2\,i\,\varpi + 1]\,z\}\,\partial_z\,F \\ + \left[(\nu+1)^2 + (\nu+1)2\,i\,\varpi\right]\,F = 0.\end{aligned} \tag{4.29}$$

Let us consider now the hypergeometric differential equation given as,

$$(1-z)z\,\partial_z^2\,y(z) + [(\alpha+\beta+1)z - \gamma\,]\,\partial_z\,y(z) + \alpha\beta\,y(z) = 0, \tag{4.30}$$

which has the solution,

$$y(z) = F(\alpha,\beta;\gamma;z) = \sum_{k=0}^{\infty} \frac{(\alpha,k)(\beta,k)}{(\gamma,k)}\,\frac{z^k}{k!}, \qquad |z| < 1, \tag{4.31}$$

where (α, k) is the Pochhammer symbol defined by,

$$(\alpha,k) = \prod_{\mu=0}^{k-1}(\mu+\alpha) = \alpha(\alpha+1)(\alpha+1)...(\alpha+k-1) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad (\alpha,0) = 1, \tag{4.32}$$

observe that

$$F(\alpha,\beta;\gamma;0) = 1. \tag{4.33}$$

We can write a general solution to the hypergeometric differential equation (4.30), in the neighbourhood of $z = 0$ as [32],

$$y(z) = az^{1-\gamma}\,F(1+\alpha-\gamma, \beta+1-\gamma; 2-\gamma; z) + b\,F(\alpha,\beta;\gamma;z). \tag{4.34}$$

By comparison of the equations (4.29) and (4.30) we have,

$$\alpha = \nu + 1 + 2\,i\,\varpi, \qquad \beta = \nu + 1, \qquad \gamma = 2\,i\,\varpi + 1. \tag{4.35}$$

Therefore we can read of the solution (4.25) as,

$$R = A\,(1-z)^{1+\nu}\,z^{-i\varpi}F(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; z)$$
$$+B\,z^{i\varpi}(1-z)^{\nu+1}F(\alpha,\beta;\gamma;z). \tag{4.36}$$

Since we are in the classical limit, there will not be outgoing waves, therefore we have to set the coefficient $B = 0$.

Now we analyse for the large values of r, i.e $z \to 1$, the behaviour of the ingoing wave solution in the near region. To accomplish that we will use the hypergeometric transformation law as $z \to 1 - z$, which is given by,

$$F(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; z) = \frac{\Gamma(-\gamma+2)\Gamma(-\gamma+\alpha+\beta)}{\Gamma(-\gamma+1+\beta)\Gamma(-\gamma+1+\alpha)}(1-z)^{\gamma-\alpha-\beta}$$
$$\times F(-\alpha+1, -\beta-\gamma+1; \gamma-(\alpha+\beta); -z+1) + \frac{\Gamma(-\gamma+2))\Gamma(-\gamma+\beta+\alpha)}{\Gamma(-\alpha+1)\Gamma(-\beta+1)}$$
$$\times F(1-\gamma+\alpha, 1-\gamma+\beta; \alpha+\beta-\gamma+1; -z+1). \tag{4.37}$$

Since in the limit as $z \to 1 \Rightarrow 1 - z \to 0$, we can use (4.33) to write the near region solution of the form,

$$R \sim A\,\Gamma(1-2\,i\,\varpi)\left[\frac{(r_+-r_-)^{-\nu}\,\Gamma(2\nu+1)}{\Gamma(\nu+1)\Gamma(\nu-2\,i\,\varpi+1)}\,r^{\nu} + \frac{(r_+-r_-)^{(\nu+1)}\,\Gamma(-2\nu-1)}{\Gamma(-\nu)\Gamma(-2\,i\,\varpi-\nu)}\,r^{-(\nu+1)}\right]. \tag{4.38}$$

B - Far Region Solution

Here we assume $r - r_+ >> M$, such that the physical parameters of the BH, namely the mass and the charge can be neglected, i.e $M \sim 0, Q \sim 0$. Hence the polynomial (2.51) now becomes,

$$\Delta \sim r^2\left(b^2 + \frac{r^2}{\ell^2}\right), \tag{4.39}$$

thus the radial part of Klein-Gordon equation (3.17) can be written as,

$$\left(b^2+\frac{r^2}{\ell^2}\right)\partial_r^2 R + 2r\left(\frac{b^2}{r^2}+\frac{2}{\ell^2}\right)\partial_r R + \left[\frac{\omega^2}{b^2+\frac{r^2}{\ell^2}} - \frac{\nu(\nu+1)}{r^2} - \mu^2\right]R = 0. \tag{4.40}$$

Note that the equation (4.40) is the radial wave equation for AdS space-time with a global monopole. Moreover, we also observe that the monopole term b^2 in the equation (4.40) does not vanish, which is adequate due to the fact that the monopole space-time is not asymptotically flat. Hence we must keep the monopole term b^2 in the far region approximation.

Let us start our calculation with a coordinate transformation defined as $y = b^2 + \frac{r^2}{\ell^2}$, then we further transform that with $y = b^2 x$. With these transformations equation (4.40) takes the following form,

$$(1-x^2)x\,\partial_x^2 R + \left(1 - \frac{5x}{2}\right)\partial_r - \left\{\frac{\tilde{\omega}^2\ell^2}{4x} + \frac{\lambda(\lambda+1)}{4(1-x)} - \frac{\mu^2\ell^2}{4}\right\} R = 0. \tag{4.41}$$

Where we have defined,

$$\tilde{\omega}^2 = \frac{\omega^2}{b^2}, \qquad \lambda(\lambda+1) = \frac{\nu(\nu+1)}{b^2}. \tag{4.42}$$

Using the indicial equation (3.42) we have obtained the exponents of (4.41) for the singularities located at $x = 0, 1$ as,

$$\beta_1 = \frac{\tilde{\omega}^2\ell^2}{2}, \qquad \beta_2 = \frac{\lambda}{2}. \tag{4.43}$$

Now we make the following ansatz,

$$R = x^{\beta_1}(1-x)^{\beta_2} F. \tag{4.44}$$

Substitution of (4.44) to (4.41) yields,

$$\begin{aligned}(1-x)x\,\partial_x^2 F &+ \left\{(\,2\beta_1 + 1) - x\left[2(\beta_1+\beta_2) + \frac{5}{2}\right]\right\}\partial_x F \\ &- \left[(\beta_1)^2 + (\beta_2)^2 + 2\beta_1\beta_2 + \frac{3}{2}(\beta_2+\beta_2) - \frac{\mu^2\ell^2}{4}\right] F = 0.\end{aligned} \tag{4.45}$$

We define,

$$\alpha' = \beta_1 + \beta_2 + \frac{3}{4} + \frac{1}{4}\sqrt{9+4\mu^2\ell^2} = \frac{\tilde{\omega}\ell}{2} + \frac{\lambda}{2} + \frac{3}{4} + \frac{1}{4}\sqrt{9+4\mu^2\ell^2}, \tag{4.46}$$

$$\beta' = \beta_1 + \beta_2 + \frac{3}{4} - \frac{1}{4}\sqrt{9+4\mu^2\ell^2} = \frac{\tilde{\omega}\ell}{2} + \frac{\lambda}{2} + \frac{3}{4} - \frac{1}{4}\sqrt{9+4\mu^2\ell^2}, \tag{4.47}$$

$$\gamma' = \tilde{\omega}\ell + 1, \tag{4.48}$$

such that,

$$\alpha'\beta' = \left[(\beta_1)^2 + (\beta_2)^2 + 2\beta_1\beta_2 + \frac{3}{2}(\beta_2+\beta_2) - \frac{\mu^2\ell^2}{4}\right]$$

$$= \frac{1}{4}\left(\tilde{\omega} + \lambda + \frac{3+\sqrt{9+4\mu^2\ell^2}}{2}\right) \times \frac{1}{4}\left(\tilde{\omega} + \lambda + \frac{3-\sqrt{9+4\mu^2\ell^2}}{2}\right), \tag{4.49}$$

$$1 + \alpha' + \beta' = 2(\beta_1 + \beta_2) + \frac{5}{2}. \tag{4.50}$$

With these identifications the equation (4.45) therefore becomes,

$$x(x-1)\,\partial_r^2 F + [\,x(\alpha' + 1 + \beta') - \gamma'\,]\,\partial_x F + \beta'\alpha' F = 0, \tag{4.51}$$

which is in the form of hypergeometric differential equation and such an equation admits a solutions in the neighbourhood of $x = \infty$ as [32],

$$\begin{aligned} F(\alpha', \beta'; \gamma'; x) &= C\,x^{-\alpha'} F(\alpha', -\gamma' + 1 + \alpha'; -\beta' + 1 + \alpha'; x^{-1}) \\ &\quad + D x^{-\beta'} F(\beta', -\gamma' + \beta' + 1; -\alpha' + 1 + \beta'; x^{-1}), \end{aligned} \tag{4.52}$$

hence we can write via (4.44),

$$\begin{aligned} R &= C\,x^{\beta_1 - \alpha'}(1-x)^{\beta_2} F(\alpha', -\gamma' + 1 + \alpha'; -\beta' + 1 + \alpha'; x^{-1}) \\ &\quad + D x^{\beta_1 - \beta'}(1-x)^{\beta_2} F(\beta', -\gamma' + \beta' + 1; -\alpha' + 1 + \beta'; x^{-1}). \end{aligned} \tag{4.53}$$

Taking the limit as $x \to \infty$ and using (4.33), we see that the solution behaves as,

$$R \sim (-1^{\beta_2})\left[C\,x^{\frac{-1}{4}\left(3+\sqrt{9+\mu^2\ell^2}\right)} + D\right]. \tag{4.54}$$

However, at infinity AdS space-time behaves as a wall, such that the scalar field Φ vanishes. This implies the restriction that the coefficient D must be zero.

To explore the equation (4.53) corresponding to the small values of r, i.e $x \to 1$, we use the $\frac{1}{x} \to 1 - x$ transformation law of the hypergeometric functions, which is given by,

$$F(\alpha,-\gamma+\alpha+1;-\beta+1+\alpha;\frac{1}{x}) = x^{1+\alpha-\gamma}(x-1)^{-\alpha-\beta+\gamma}\frac{\Gamma(-\beta+1+\alpha)\Gamma(-\gamma+\alpha+\beta)}{\Gamma(-\gamma+1+\alpha)\Gamma(\alpha)}$$
$$\times F(-\beta+1,-\alpha+1;-\alpha+\gamma-\beta;-x+1)+\frac{\Gamma(-\gamma+2))\Gamma(\alpha-\gamma+\beta)}{\Gamma(-\alpha+1)\Gamma(-\beta+1)}$$
$$\times x^{\alpha}\, F(\alpha,\beta;\alpha+\beta+1-\gamma;-x+1). \tag{4.55}$$

Note that, when $x \to 1$ we have $x-1 \to \frac{r^2}{\ell^2 b^2}$. Therefore the far region solution for small values of r is given by,

$$R \sim C\,\Gamma(\alpha'-\beta'+1)\left[(-1)^{\frac{\lambda}{2}}\frac{\Gamma(-\alpha'+\gamma'-\beta')}{\Gamma(1-\beta')\Gamma(\gamma'-\beta')}\left(\frac{r}{\ell b}\right)^{\lambda}\right.$$
$$\left.+(-1)^{\frac{-3\lambda}{2}}\frac{\Gamma(-\gamma'+\alpha'+\beta')}{\Gamma(\alpha')\Gamma(-\gamma'+1+\alpha')}\left(\frac{r}{\ell b}\right)^{-1-\lambda}\right]. \tag{4.56}$$

We observe that the global monopole term, $b^2 = 1-8\pi\eta^2$ effects the far region solution (4.56) as a constant, therefore we can safely apply the boundaries of the pure AdS space-time to analyse (4.56). When $r \to 0$ equation (4.56) diverges due to $r^{-\lambda-1} \to \infty$. To obtain regular solutions we impose the condition as follows,

$$\Gamma(\alpha'-\gamma'+1)=\infty \qquad if \qquad \Gamma(-m)=\infty, \quad m\in\mathbb{Z}_+. \tag{4.57}$$

Thus, the regularity condition (4.57) enables us to interpret m, which takes the values from the nonnegative integer numbers $\mathbb{Z}_+$, as a principal quantum number. Hence we obtain the discrete spectrum as,

$$\omega = \frac{2\,b}{\ell}(m+\sigma). \tag{4.58}$$

For the sake of abbreviation we have defined $\sigma = \lambda/2 + 3/4 + \sqrt{9+4\mu^2\ell^2}/4$.

Now, it is natural to assume that the condition (4.58) can be interpreted as the generator of the frequency spectrum at large distances, due to the fact that at infinity the structure of the RN-AdS black hole is similar to pure AdS background. In addition, one can still observe the effect of the global monopole in (4.58). Having said that however, we should approach the current predicament more cautiously, since the inner boundaries of the AdS and black hole case are different. For pure

AdS space-time we have, $r = 0$ as inner boundary, on the other hand for the black hole case we have $r = r_+$. Hence if one wishes to observe the effect of the black hole, one must take into account of the possibility of tunnelling of the wave through the potential located at $r = r_+$, into the black hole and scattering back. Furthermore, the scattered amplitude of the wave will decrease or grows exponentially and causes the superradiant instability. To sum up, the quasinormal mode frequencies for the black hole case can be written with complex frequencies as follows,

$$\omega_{QM} = \frac{2\,b}{\ell}\,(m + \sigma) + i\delta. \tag{4.59}$$

where δ is possibly a small quantity signalling the effects of the charged black hole having the gravitational monopole.

Exploiting the assumption (4.59), and using the Gamma function relations for small δ we have

$$\frac{1}{\Gamma(\alpha')\Gamma(\alpha' - \gamma' + 1)} \simeq i\,(-1)^{m+1} \frac{m!}{(m + \lambda + \epsilon - 1)!}\frac{\ell}{2b}\delta, \quad \delta << 1, \tag{4.60}$$

where $\epsilon = (3 + \sqrt{9 + 4\mu^2\ell^2})/2$. And

$$\Gamma(-\beta' + 1)\Gamma(-\beta' + \gamma') = \Gamma(-\lambda - 1/2 - m)\Gamma(m + 1 + \sqrt{9 + 4\mu^2\ell^2}/2). \tag{4.61}$$

Using (4.60) and (4.61) in the far region solution given by the equation (4.56) we obtain,

$$\begin{aligned} R \sim C\,\Gamma(\alpha' - \beta' + 1) \Bigg[& \frac{(-1)^{\lambda/2}\Gamma(-\lambda - 1/2)(\ell\, b)^{-\lambda}}{\Gamma(-\lambda - m - 1/2)\Gamma(m + 1 + \sqrt{9 + 4\mu^2\ell^2}/2)}\, r^{\lambda} \\ & + i\,\delta\,(-1)^{-3\lambda/2+m+1} \frac{\Gamma(\lambda + 1/2)}{2} \frac{\ell^{\lambda+2}\, b^{\lambda}\, m!}{(m + \lambda + \epsilon - 1)!}\, r^{-\lambda-1} \Bigg]. \end{aligned} \tag{4.62}$$

Now if we want to be successful at the asymptotic matching procedure of the near region and the far region solutions we need a restriction on λ. The relation between λ and ν is given by,

$$\lambda(\lambda + 1) = \frac{\nu(\nu + 1)}{b^2}, \tag{4.63}$$

where $b^2 = 1 - 8\pi\eta^2$. Taylor expansion of $1/b^2$ yields,

$$\frac{1}{b^2} = 1 + \mathcal{O}(\eta^2), \tag{4.64}$$

neglecting the $\mathcal{O}(\eta^2)$ term we have, $\lambda = \nu$. Therefore we can write the far region solution as,

$$R \sim C\,\Gamma(\alpha' - \beta' + 1)\left[A'\,r^{\nu} + i\,\delta\,B'\,r^{-\nu-1}\right], \tag{4.65}$$

where we set the coefficients of r^{ν} and $r^{-\nu-1}$ to A' and B' respectively. Matching eqn. (4.38) with eqn. (4.65) yields then,

$$\delta \sim (-2\,i)(-1)^{m+1}\frac{\ell^{-2(\nu+1)}b^{-2\nu}}{\Gamma(\nu+1/2)}\,\frac{(r_+ - r_-)^{2\nu+1}}{\Gamma(m+1+\sqrt{9+4\mu^2\ell^2}/2)}\,\frac{\Gamma(-2\nu-1)}{\Gamma(-\nu)}$$
$$\frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}\,\frac{\Gamma(\nu+1-2\,i\,\varpi)}{\Gamma(-\nu-2\,i\,\varpi)}\,\frac{\Gamma(-\nu-1/2)}{\Gamma(-\nu-1/2-m)}\,\frac{\left[m+\nu+\left(1+\sqrt{9+4\mu^2\ell^2}\right)/2\right]!}{m!}. \tag{4.66}$$

Using the Gamma function property given by,

$$\Gamma(1+x) = x\,\Gamma(x), \tag{4.67}$$

we obtain,

$$\delta \sim \; -\xi(\omega - e\,\Phi_h)\,\frac{r_+^2(r_+ - r_-)^{2\nu}}{\ell^{2(\nu+1)}\,b^{2\nu}\sqrt{\pi}} \tag{4.68}$$

where ξ defined as,

$$\xi \equiv \; \frac{(\nu!)^2\left[m+\nu+\left(1+\sqrt{9+4\mu^2\ell^2}\right)/2\right]!}{m!\,(2\nu+1)!\,(2\nu)!}\,\frac{2^{\nu+2-m}(2\nu+1+m)!!}{(2\nu+1)!!(2\nu-1)!!}$$
$$\times\left[\prod_{s=1}^{\nu}(s^2+4\,\varpi^2)\right][\Gamma(m+1+\sqrt{9+4\mu^2\ell^2}/2)]^{-1}, \tag{4.69}$$

with $\varpi = \left[r_+^2/(r_+ - r_-)\right](\omega - e\,\Phi_h)$ and $\omega = (2b/\ell)(m+\sigma)$. Now we have,

$$\delta \propto -\left(\Re[\omega_{QM}] - \,e\,\Phi_h\right). \tag{4.70}$$

Hence,

$$\Re[\omega_{QM}] < \,e\,\Phi_h, \qquad for \qquad \delta > 0, \tag{4.71}$$

$$\Re[\omega_{QM}] > \,e\,\Phi_h, \qquad for \qquad \delta < 0. \tag{4.72}$$

The scalar field has dependence of ω as,

$$\Phi \propto e^{-i\omega_{QM}t} = exp\left[-i\,\Re(\omega_{QM})t + \delta t\right] \tag{4.73}$$

Equation (4.73) implies with the condition (4.71), that the amplitude of the scalar field grows exponentially and causes instabilities. However one should bear in mind the effect of the global monopole term b^2. The physical choice of the global monopole term is $b^2 = 1 - 8\pi\eta^2 > 0$. Furthermore if we took η^2 as a positive number, i.e $\eta^2 > 0$, then $0 < b^2 < 1$. To observe the net effect of the global monopole let us graph the condition (4.71).

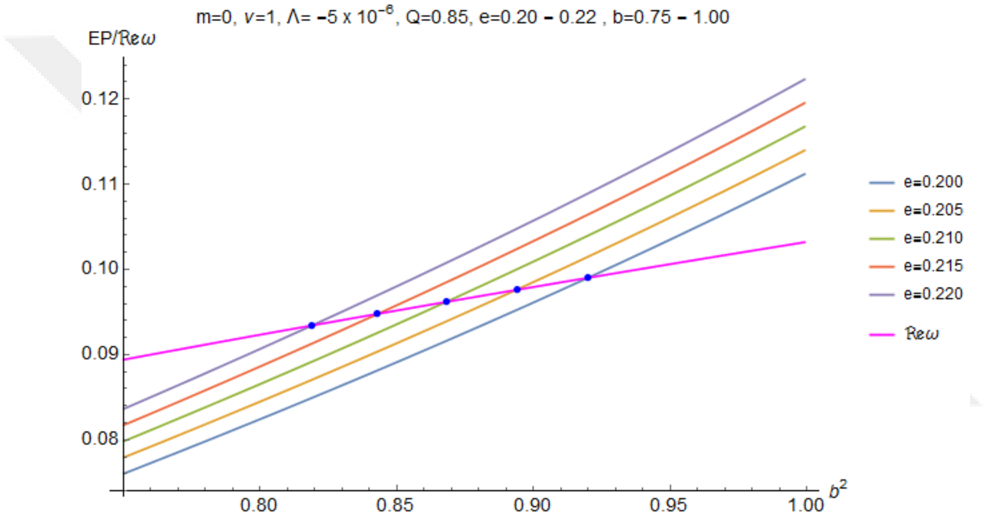

Figure 4.1 Graph of $e\Phi_h/\Re(\omega_{QN})$-b^2 for $m = 0, \nu = 1, \Lambda = -5 \times 10^{-6}, Q = 0.85$ with different particle charge (e) values. As b^2 becomes smaller, which corresponds to larger values of η, we see that the condition $\delta > 0$ starts to not hold. Instability condition only holds above the intersection points for these choosen parameter values.

We see from both (4.71) and figure (4.1) that the monopole term causes an augmentation on the outer horizon consequently decreases the value of the electric potential and it makes difficult the instability condition $\delta > 0$ to hold.

Moreover, inspection of equation (4.68) regarding the effect of global monopole shows a growth in δ, therefore, we observe a decrease in τ, since in superradiant instability, the time scale is given by $\tau_{AdS} = 1/\delta = 1/\Im[\omega_{QM}] \propto \ell^{2(\nu+1)}b^{2\nu}$. To see more clearly, we listed the graphs below regarding the effect of the global monopole to the different values of Q, Λ and ν.

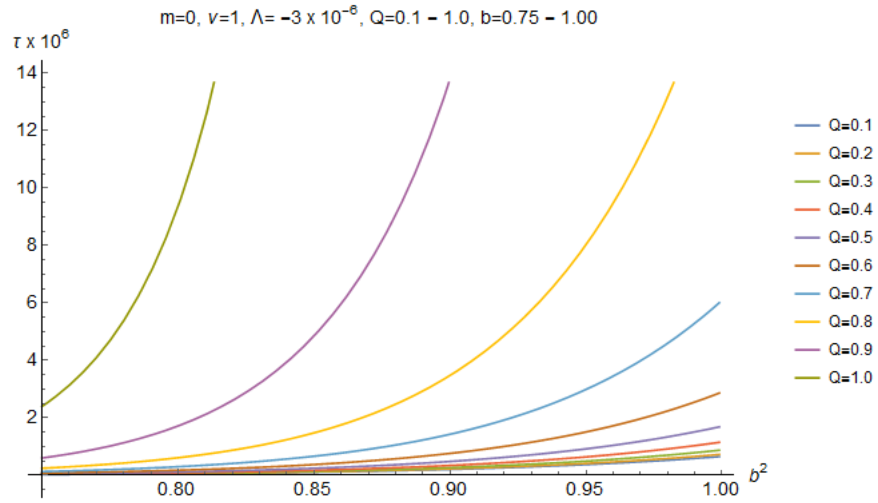

Figure 4.2 Graph of of $\tau - b^2$ for $Q = 0.1 - 1.0$. For $m = 0, \nu = 1, \Lambda = -3 \times 10^{-6}$.

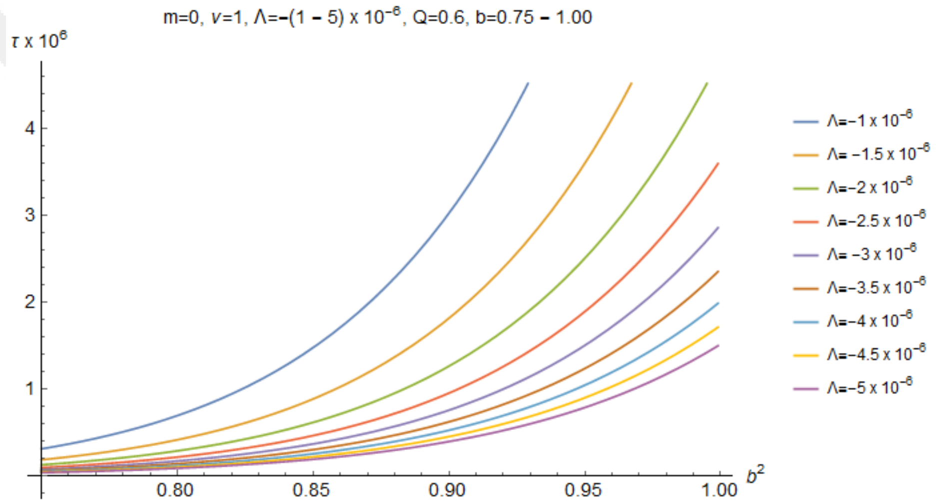

Figure 4.3 Graph of of $\tau - b^2$ for $\Lambda = -(1 \times 10^{-6} - 5 \times 10^{-6})$. For $m = 0, \nu = 1, Q = 0.6$.

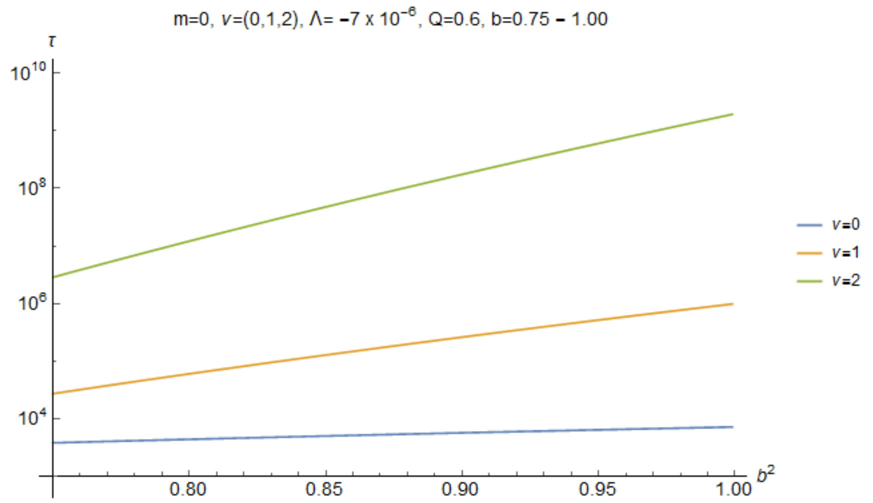

Figure 4.4 Graph of of $\tau - b^2$ for $\nu = 0, 1, 2$. The time scale changes drastically and effect of the monopole is seen better at the higher values of it. At $\nu = 0$ (blue) we see reletively a straight line however, at $\nu = 2$ (green) the change is at order 10^2.

In figures (4.2), (4.3) and (4.4) one can see that as the monopole term increase, the time scale decreases. Hence, we conclude that in RN-AdS black hole with global monopole configuration the onset of superradiant instability decreases with the monopole term nevertheless instability occurs, it will grow slower in comparison with the case when the monopole term is absent. Thus, the RN-AdS black holes having a gravitational monopole are more stable against superradiance instability then the same black holes having no gravitational monopole.

4.4. Case 2 : Black Hole Bomb and Superradiant Instabilities of Global Monopole Configuration in RN Space-Times ($\Lambda = 0$)

In this section we discuss the instability condition in the absence of the cosmological constant Λ. As before, we will use the asymptotic matching technique to obtain the instability condition in addition to so-called mirror condition which will become clear in the process of calculation. Inspection of the near region solution yields the same equation since we have set the cosmological constant to zero in the case one, namely,

$$R \sim A\,\Gamma(-2\,\varpi\,i+1)\left[\frac{(r_+-r_-)^{-\nu}\,\Gamma(2\nu+1)}{\Gamma(\nu+1)\Gamma(\nu-2\,i\,\varpi+1)}\,r^{\nu}+\frac{(r_+-r_-)^{(\nu+1)}\,\Gamma(-2\nu-1)}{\Gamma(-\nu)\Gamma(-2\,i\,\varpi-\nu)}\,r^{-(\nu+1)}\right]. \tag{4.74}$$

Hence all that remains is to find the far region solution. Here, as before, we assume $M \sim 0, Q \sim 0$, where M and Q is the mass and the charge of the black hole. And the polynomial Δ_r now becomes $\Delta \sim b^2 r^2$. Thus, RKGE (3.17) is given by,

$$\partial_r^2 R + \frac{2}{r}\partial_r R + \left\{\varpi^2 - \frac{\lambda(\lambda+1)}{r^2}\right\} R = 0, \tag{4.75}$$

where $\varpi^2 = \omega^2/b^4 - \mu^2/b^2$ and $\nu(\nu+1)/b^2 = \lambda(\lambda+1)$. Equation (4.75) admits a general solution in terms of the Bessel function [32] given by,

$$R = r^{-1/2}\left[\alpha\,J_{\nu+1/2}(\varpi\,r) + \beta\,J_{-\nu-1/2}(\varpi\,r)\right], \tag{4.76}$$

Here we used the restriction (4.64). For small values of r it reduces to [32],

$$R \sim \frac{\alpha\,(\varpi/2)^{\nu+1/2}}{\Gamma(\nu+3/2)}\,r^{\nu} + \frac{\beta\,(\varpi/2)^{-\nu-1/2}}{\Gamma(-\nu+1/2)}\,r^{-\nu-1}. \tag{4.77}$$

Applying the similar mechanical steps that we have performed for the matching procedure in the previous case, we obtain the corresponding condition for the equations (4.77) and (4.74) given as,

$$\frac{\beta}{\alpha} = 2\,i\,\overline{\omega}\,\frac{(-1)^{\nu}}{(2\nu+1)}\left(\frac{\nu!}{(2\nu-1)!!}\right)^{2}\frac{(r_{+}-r_{-})^{2\nu+1}}{(2\nu)!(2\nu+1)!}\left[\prod_{k=1}^{\nu}(k^{2}+4\,\overline{\omega}^{2})\right](\varpi)^{2\nu+1} \tag{4.78}$$

where $b^2 = 1 - 8\pi\eta^2$ and $\overline{\omega}$ is the superradiant factor given by the equation (4.26). Notice that, we have used the approximation (4.63) in order to matching to work. In addition we have set the coefficient of the near region solution A, of the following

$$A = \alpha\,\frac{(r_{+}-r_{-})^{\nu}}{\Gamma(\nu+3/2)}\,\frac{\Gamma(\nu+1)}{\Gamma(2\nu+1)}\,\frac{\Gamma(\nu-2\,i\,\overline{\omega}+1)}{\Gamma(1-2\,i\,\overline{\omega})}\,(\varpi/2)^{\nu+1/2}\,, \tag{4.79}$$

to obtain the (4.78).

The main difference between the *case I* and *case II* is the fact that in *case I* we have the AdS space-time which behaves effectively as a reflecting box. In the *case II* however we put a mirror by hand at the far region located at a radius $r = r_0$, as a result the scalar field must vanish at the surface of the mirror. Hence we have an additional condition between the amplitudes α and β due to the fact that equation (4.76) vanishes. Therefore we have,

$$\frac{\beta}{\alpha} = -\frac{J_{\nu+1/2}(\varpi\,r_0)}{J_{-\nu-1/2}(\varpi\,r_0)}, \tag{4.80}$$

for small values of particle mass $\mu^2 << 1$ it yields,

$$\frac{\beta}{\alpha} = -\frac{J_{\nu+1/2}(r_0\,\omega/b^2)}{J_{-\nu-1/2}(r_0\,\omega/b^2)}. \tag{4.81}$$

Calling equation (4.78) we obtain,

$$\begin{aligned}\frac{J_{\nu+1/2}(r_0\,\omega/b^2)}{J_{-\nu-1/2}(r_0\,\omega/b^2)} &= i\,(-1)^{\nu+1}\,\overline{\omega}\,\frac{2}{2\nu+1}\left[\frac{\nu!}{(2\nu-1)!!}\right]^{2} \\ &\times\frac{(r_{+}-r_{-})^{2\nu+1}}{(2\nu)!(2\nu+1)!}\left[\prod_{k=1}^{\nu}(k^{2}+4\,\overline{\omega}^{2})\right]\left(\frac{\omega}{b^{2}}\right)^{2\nu+1}.\end{aligned} \tag{4.82}$$

As a solution to equation (4.82), we use the approximations $\omega << 1$ and $\Re(\omega) >> \Im(\omega)$, which are adequate for our problem. With these approximations, the $R.H.S$ of the equation (4.82) can be safely set to zero, the result is therefore,

$$J_{\nu+1/2}(r_0\,\omega/b^2) = 0, \tag{4.83}$$

which has real solutions [32]. If we label to solutions of (4.83) as,

$$j_{\nu+1/2,\,s} = \frac{\omega\,r_0}{b^2}, \tag{4.84}$$

where s is a non-negative integer number, i.e $s \in \mathbb{Z}_+$. As a complete solution to (4.83) assume that,

$$\omega \sim \frac{b^2}{r_0}\left[j_{\nu+1/2,\,s} + i\,\tilde{\delta}\right] \qquad \tilde{\delta} << 1. \tag{4.85}$$

Hence, under these assumptions we have,

$$\begin{aligned}\frac{J_{\nu+1/2}(j_{\nu+1/2,\,s} + i\,\tilde{\delta})}{J_{-\nu-1/2}(j_{\nu+1/2,\,s} + i\,\tilde{\delta})} &= i\,(-1)^{\nu+1}\,\varpi\,\frac{2}{2\nu+1}\left[\frac{\nu!}{(2\nu-1)!!}\right]^2 \\ &\times \frac{(r_+ - r_-)^{2\nu+1}}{(2\nu)!(2\nu+1)!}\left[\prod_{k=1}^{\nu}(k^2 + 4\,\varpi^2)\right](\omega/b^2)^{2\nu+1}.\end{aligned} \tag{4.86}$$

The Taylor expansion of the $L.H.S$ of the equation (4.86) for small values of $\tilde{\delta}$ gives,

$$\frac{J_{\nu+1/2}(i\,\tilde{\delta} + j_{\nu+1/2,\,s})}{J_{-\nu-1/2}(i\,\tilde{\delta} + j_{\nu+1/2,\,s})} \sim i\,\frac{J'_{\nu+1/2}(j_{\nu+1/2,\,s})}{J_{-\nu-1/2}(j_{\nu+1/2,\,s})}\,\tilde{\delta}. \tag{4.87}$$

The values of the expression in the $R.H.S$ of the equation (4.87) can be found in [32]. Via the presence of the mirror located at $r = r_0$, the frequencies of the scalar field therefore are,

$$\omega_{BQN} \simeq \frac{b^2}{r_0}j_{\nu+1/2,\,s} + i\,\delta. \tag{4.88}$$

Substitution of (4.87) to (4.86) with (4.88) yields the following result for δ,

$$\delta = -\vartheta\,(-1)^{\nu}\,\frac{J_{-\nu-1/2}(j_{\nu+1/2,\,s})}{J'_{\nu+1/2}(j_{\nu+1/2,\,s})}\,\frac{\left(j_{\nu+1/2,\,s}\,b^2/r_0\right) - e\,\Phi_h}{(r_0)^{2\nu+2}}\,(b)^2 \tag{4.89}$$

where,

$$\vartheta \equiv \frac{2}{2\nu+1}\left[\frac{\nu!}{(2\nu-1)!!}\right]^2\left(\frac{r_+^2(r_+ - r_-)^{2\nu}}{(2\nu)!(2\nu+1)!}\right)\left[\prod_{k=1}^{\nu}(k^2 + 4\,\varpi^2)\right]\left(j_{\nu+1/2,\,s}\right)^{2\nu+1}. \tag{4.90}$$

Therefore we have,

$$\delta \propto -\left(\Re[\omega_{BQN}] - e\,\Phi_h\right). \tag{4.91}$$

The case $\Re[\omega_{BQN}] < e\,\Phi_h$ corresponds positive values of δ, i.e $\delta > 0$ as a result the scalar field Φ grows exponentially and causes instabilities. Note that for large values of r_0, we obtain small values of δ, hence the assumption $\Re(\omega) >> \Im(\omega)$ is remains valid.

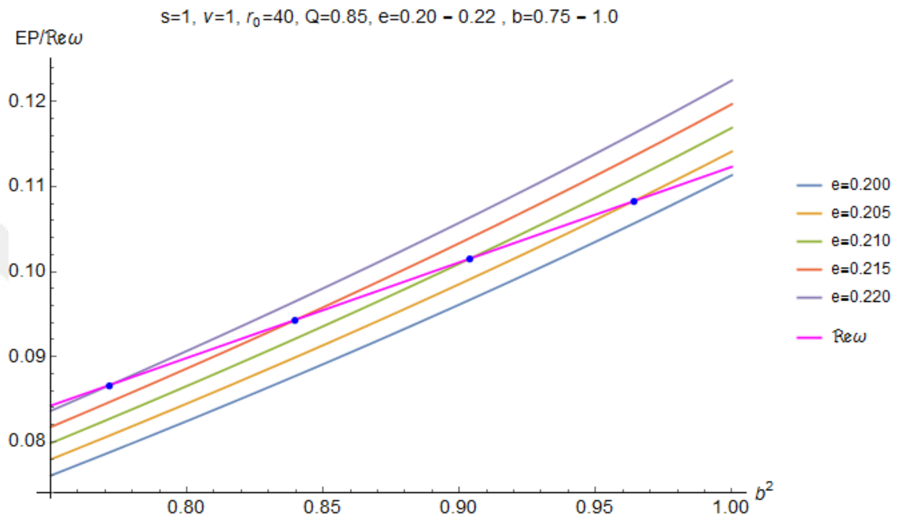

Figure 4.5 Graph of $e\Phi_h/\Re(\omega_{BQN})$-$b^2$ for different particle charge values. $\Re[\omega_{BQN}]$ is climbing more steeply in comparison to the figure 4.1. This is because the coupling of the monopole term is different in each case. Superradiant condition holds above the intersection points.

The results that we have obtained for both cases are very similar. The main difference lies in the fact that monopole term affects the real part of the frequency modes of the black hole bomb by a factor b due to calculation procedure, as a result, we may say that the chances of instability to occur is more likely in comparison with the AdS space. In addition if we look at the time scale i.e $\tau_{bhb} = 1/\delta = 1/\Im[\omega_{BQM}] \propto 1/r_0^{2\nu+2}$, we observe that monopole term does not couple with the mirror radius as it was in the AdS space hence time scale therefore τ_{bhb} will only change by a factor of $(1-8\pi\eta^2)$ since monopole terms has no depence in ν. We conclude that to obtain more accurate results concerning the comparison of superradiant instability in RN-AdS space-time with black hole bomb in RN spacetime numerical analysis is also needed.

5. RESULTS

In this thesis, we have studied the dynamics of test particles and scalar fields in RN-(A)dS-monopole space-time configuration by solving the Klein-Gordon and Hamilton Jacobi equations and investigate possible applications.

In section 2 we briefly construct the desired space-time by solving Einstein's equations in the orthonormal basis. To obtain the geometrical information of the spherical symmetric system, which is encoded in the Einstein tensor, we use the Cartan's structure equations. Then we re-derive the Energy-momentum tensor both for a gravitational monopole outside the core and an electrically charged black hole. The solution to the Einsteins equations enables us to find the line element for our space-time configuration [9].

In section 3 we have found an analytic solution to the RKGE for a fixed value of scalar field mass in RN-dS-monopole space-time in terms of hypergeometric Jacobi polynomials via a transformation of the radial equation to the Heun form as for the Kerr-AdS [25] and the Kerr-Newmann-AdS [26] space-times. Furthermore, we have obtained the equations that govern the trajectories of a charged test particle and light-ray invoking the Hamilton-Jacobi equation. We found an analytical solution for a photon by the elliptic integral method [28, 29]. Trajectory equations for a photon lead us to investigate the light deflection from an RN-Monopole black hole. The solution of the orbit equation by a perturbative method in the second order of black hole parameters and monopole contribution enables us to find a generalized deflection angle. Later, we carry discussion to RN-(A)dS space-time to observe the effect of the cosmological constant. With the aid of Rindler and Ishak method [43,44], we obtained the contribution of the cosmological constant and the monopole term to the light deflection in the weak field limit.

Furthermore, in section 4, we exploited the asymptotic matching technique to inspect the stability conditions [30,31] of both RN-(A)dS-monopole and RN-monopole black hole against charged scalar perturbations and found that the global monopole effects the onset of superradiant instability in a negative way by coupling with the outer horizon of the black hole. However, the way of the coupling is different in each case. Due to that fact the behaviour of the real part of the quasi-normal modes are different for RN-AdS-monopole and and RN-monopole cases, which can be seen in figure (4.5), the line corresponding to the $\Re(\omega_{BQN})$ is climbing more steeply compare to $\Re(\omega_{QN})$ given in figure 4.5. The time scale of the scalar field is also affected by global monopole and causes the instability to grow slower in the RN-AdS-monopole space-time, yet in RN-monopole case time scale only changes by a factor b^2 and does not have a dependence on ν. This can be explained by the asymptotic behavior of the AdS monopole space-time. Since the RN-AdS-monopole space-time is not asymptotically flat and as a result one can see that there is a natural coupling between AdS radius ℓ and monopole term η^2. In the RN-monopole space-time, however, the asymptotic behavior of the space only produces an angle deficit and there is no coupling between mirror radius and b. Therefore, the monopole term has no influence on the mirror radius.

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ÖZGEÇMİŞ

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