OPTION PRICING UNDER DELAY EFFECT

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ABSTRACT

OPTION PRICING UNDER DELAY EFFECT

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In many fields like physics, ecology, biology, economics, engineering, and financial mathematics, events often don't have an immediate effect. Instead, they impact future situations. To understand how these systems work and behave, we use stochastic delay differential equations (SDDEs) which are obtained by adding information from past events into stochastic differential equations (SDEs). Thus, SDDEs are gaining attention because they can better reflect real-life situations. Some numerical methods for SDDEs have been developed because it's often very difficult, and sometimes impossible, to find exact solutions using stochastic calculus. The most known methods are Euler Maruyama and Milstein methods. Recently, researchers in economics and finance have been studying option pricing for systems with time delays, which can be either random or fixed. We aim to understand the general structure of SDDEs while solving them when the time delay is fixed and then use the delayed dynamics for option pricing. The pricing of European vanilla, American vanilla, European foreign exchange and European exchange options whenever underlying dynamics follow delayed geometric Brownian motion (GBM) and European vanilla where the asset price follows the delayed Heston model are considered. Some numerical implementations are carried out to see the effect of delay term on the pricing process.

Keywords: Stochastic delay differential equations, SDDE, option pricing with delay,

delayed GBM, delayed Heston model

ÖZ

GECİKME ETKİSİYLE OPSİYON FİYATLAMA

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Fizik, ekoloji, biyoloji, ekonomi, mühendislik, finansal matematik gibi birçok alanda olaylar genellikle anında etki göstermez. Bunun yerine, gelecekteki durumları etkilerler. Bu sistemlerin nasıl çalıştığını ve davrandığını anlamak için, stokastik diferansiyel denklemlere (SDE) geçmiş olaylardan gelen bilgileri ekleyerek elde edilen stokastik gecikmeli diferansiyel denklemler (SDDE) kullanılır. Bu nedenle, SDDE'ler gerçek hayatı daha iyi yansıtabildikleri için giderek daha fazla ilgi çekmektedir. Stokastik kalkülüs kullanılarak tam çözümler bulmak genellikle çok zor ve bazen imkânsız olduğundan, SDDE'ler için bazı sayısal yöntemler geliştirilmiştir. En bilinen yöntemler Euler Maruyama ve Milstein yöntemleridir. Son zamanlarda, ekonomi ve finans alanında, zaman gecikmelerinin rastgele ya da sabit olabileceği sistemler için opsiyon fiyatlandırması üzerine araştırmalar yapılmaktadır. Bu tez, sabit bir gecikme süresi olduğunda SDDE'lerin genel formlarını anlamayı, bu denklemleri çözmeyi ve daha sonra bu denklemleri opsiyon fiyatlandırması için kullanmayı amaçlamaktadır. Gecikmeli geometrik Brown hareketini (GBM) takip eden dinamikler altında Avrupa tipi vanilla, Amerikan tipi vanilla, Avrupa tipi döviz, Avrupa tipi takas opsiyonlarının ve varlık fiyatının gecikmeli Heston modelini takip ettiği Avrupa tipi vanilla opsiyonlarının fiyatlandırılması ele alınmıştır. Ardından gecikme teriminin fiyatlandırma sürecine etkisini görmek için bazı sayısal uygulamalar yapılmıştır.

Anahtar Kelimeler: Stokastik gecikmeli differensiyal denklemler, SDDE, Gecikmeli opsiyon fiyatlaması, gecikmeli geometrik Brown hareketi, gecikmeli Heston modeli

To My Ayşes and Family...

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CHAPTER 1

INTRODUCTION

The study of stochastic differential equations (SDEs) has become a crucial part of the modeling of various phenomena in most fields, such as finance, biology, physics, etc., since it includes randomness compared to ordinary differential equations (ODEs). In financial mathematics, SDEs are important in modeling underlying asset price dynamics. In such models, it is assumed that the system satisfies the principle of causality, which implies that the system's future state is determined just by its current state, without any dependence on past states [24]. Moreover, Mao states that the efficient market hypothesis is a foundational assumption for asset pricing models. According to this hypothesis, historical information is fully examined and already implemented into the current stock price. Markets react immediately to new information about underlying assets, resulting in random price movements.

However, real-world phenomena often show delays because of various factors such as transaction lags, information dissemination delays, and other temporal effects not captured by standard SDEs [35]. Transaction lags mean the time delay between initiating and completing a transaction. For example, after placing a trade order, there can be a lag before it is executed because of market conditions, processing time or other factors. The time delay between when new information (such as earnings reports, news about underlying assets or economic data) becomes available and fully reflected in market prices is known as information dissemination delay. This can happen because it takes time for the information to spread and be analyzed by all market participants. Thus, the causality principle and efficient market hypothesis make the model constructed by an SDE only an approximation of real situations. An additional term representing time delay derived from the system's history can be incorporated into the model to create a more realistic model. The stochastic delay differential equations (SDDEs) are constructed with this extension of SDEs. Actually, SDDEs combine the randomness of SDEs with the memory effects of delay differential equations (DDEs). The necessity of SDDEs is clear in modeling various scientific phenomena where delays play a significant role such as modeling various financial instruments and markets.

There are two main reasons for our study of SDDEs in the context of financial modeling. First, SDDEs provide a more realistic representation of the temporal dynamics of financial markets. Second, incorporating delay elements can lead to more realistic pricing and hedging financial derivatives, thereby improving risk management strategies.

In Chapter 2, a general overview of SDDEs is provided. We then discuss how solutions to SDDEs can be obtained, highlighting methods (namely Itô formula) and challenges associated with these equations. To illustrate these concepts, we present some examples. These examples demonstrate the impact of delays on the behavior of stochastic processes and provide a basis for understanding more complex models. For more detailed information and proofs, see [5, 7, 24, 29, 38, 2].

Chapter 3 is dedicated to some of the numerical methods to solve SDDEs. Because of the complexity of SDDEs, analytical solutions are rarely obtained and so numerical methods are essential. We focus on two widely used methods: the Euler Maruyama and Milstein. Both methods are extensions of their counterparts used for standard SDEs, adapted to handle the additional challenges posed by delays. For the detailed information and proofs, we refer to [4, 5, 7, 8, 9, 25, 38, 33].

Chapter 4 focuses on applying SDDEs in financial modeling, specifically in option pricing whenever stock price follows delayed GBM. We derive value formulas for various European call options, including vanilla, foreign exchange and exchange options both with and without delays. These value formulas are crucial for understanding how delays affect option prices and for developing effective pricing strategies. [6, 30, 1, 3, 26, 28, 22] give more information.

Chapter 5 provides the implementation of the option pricing models using Julia programming. Julia is known for its high performance and ease of use. This implementation shows how the theoretical models discussed in Chapter 4 can be translated into executable code. This chapter serves as a bridge between theoretical developments and their applications. The effect of delay terms, initial paths, number of simulations, and stock prices are examined in those applications.

Chapter 6 examines the Heston model with and without delay cases for the European type option pricing. Since Heston model does not have analytical solution Monte Carlo is considered for option pricing [18, 11, 37, 20, 36]. To see the effect of delay and initial path on the valuation process some numerical implementations are considered.

In Chapter 7, we provide a conclusion and some future works.

CHAPTER 2

STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

This chapter aims to provide a general overview of stochastic delay differential equations (SDDEs). SDDEs extend the framework of stochastic differential equations (SDEs) by incorporating delays into the system, making them a powerful tool for modeling processes where past states influence future dynamics. Thus, it will be easier to understand the fundamental properties of SDDEs whenever the behavior of SDEs understood. For more detailed information and proofs about SDEs, [15, 21, 24, 31] can be seen.

SDDEs combine the randomness of SDEs with the memory effects of delay differential equations (DDEs). This means that the future state of the system depends not only on its current state and random perturbations but also on its past states. This addition of delay terms helps to create a more accurate representation of systems where historical data has significant impacts on future behavior.

The analysis of SDDEs involves both theoretical and numerical approaches. Theoretically, Itô's lemma from stochastic calculus is used. Numerically, especially the Euler Maruyama and Milstein methods for SDDEs are developed to approximate solutions.

Overall, SDDEs provide a more realistic framework for understanding and predicting the behavior of complex systems influenced by both randomness and time delays. This makes them invaluable in both theoretical research and practical applications where delays and randomness cannot be ignored. For more detailed information and proofs about SDDEs, see [5, 7, 24, 29, 38, 2].

2.1 Genaral Framework for SDDEs

Let's introduce a general form of SDDEs, starting with vector-valued SDDEs and then focusing on real-valued SDDEs for simplicity.

Definition 2.1. We take an m-dimensional Wiener process $W(t)$ on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The following equation

$$\begin{aligned} &dX(t) = f(t, X(t), X(t-\tau))dt + g(t, X(t), X(t-\tau))dW(t),\ t \in [0, T] \\ &X(t) = \varphi(t),\ t \in [-\tau, 0] \end{aligned} \tag{2.1}$$

defines an SDDE where f and g are $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ valued functions, respectively. The initial path φ is continuous $\mathbb{R}^n$-valued $\mathcal{F}_0$-measurable function on $[-\tau, 0]$ where τ is positive delay term.

We set $n = m = 1$ to work on the real-valued SDDEs in this thesis.

Remark 2.1. The stochastic integral form of (2.1) is

$$X(t) = \varphi(0) + \int_0^t f(u, X(u), X(u-\tau))du + \int_0^t g(u, X(u), X(u-\tau))dW(u).$$

Definition 2.2. If $X(t)$ satisfies (2.1) almost surely, adapted $(\mathcal{F}_t)_{0\leq t\leq T}$-measurable and continuous, then it is called a strong solution.

Definition 2.3. The functions f and g satisfy the local Lipschitz condition for a positive constant K if they satisfy

$$|f(t,x_1,y_1)-f(t,x_2,y_2)|+|g(t,x_1,y_1)-g(t,x_2,y_2)| \leq K(|x_1-x_2|+|y_1-y_2|) \tag{2.2}$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and any $t \in \mathbb{R}^+$.

Definition 2.4. The functions f and g satisfy the weakly Lipschitz condition if

$$|f(t,x,y_1) - f(t,x,y_2)| + |g(t,x,y_1) - g(t,x,y_2)| \leq K|y_1 - y_2|$$

is satisfied for a positive constant K, any $y_1, y_2 \in \mathbb{R}$ and any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$.

Definition 2.5. If the functions f and g satisfy

$$|f(t,x,y)|^2 + |g(t,x,y)|^2 \leq L(1 + |x|^2 + |y|^2)$$

for a positive constant L and for all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, they satisfy the linear growth condition.

After these definitions, we are ready for the existence and uniqueness theorem.

Theorem 2.1 (Existence and Uniqueness Theorem)**.** *If the local Lipschitz and linear growth conditions are satisfied by the functions f and g,then* (2.1) *admits a pathwise unique strong solution for all $t \geq -\tau$ where τ, $T > 0$. Furthermore, the solution satisfies*

$$E\left(\sup|X(t)|^2\right) < \infty,\, t \in [-\tau, T].$$

The proof in [29] depends on the Picard iterations. We need to apply Itô formula to find the solution after discussing the existence of solution.

Theorem 2.2 (Itô's Lemma)**.** *Let $X(t)$ be an Itô process defined by*

$$dX(t) = f(t, X(t))\, dt + g(t, X(t))\, dW(t)$$

where $W(t)$ is a Brownian motion. Let $F(t, x)$ be twice continuously differentiable function in x and once continuously differentiable in t. The differential of $F(t, X(t))$ satisfies

$$\begin{aligned} dF(t, X(t)) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\ &= \left(\frac{\partial F}{\partial t} + f(t, X(t))\frac{\partial F}{\partial x} + \frac{1}{2}g(t, X(t))^2\frac{\partial^2 F}{\partial x^2}\right) dt \\ &\qquad + g(t, X(t))\frac{\partial F}{\partial x}dW(t). \end{aligned}$$

Itô's formula can be extended to functions of multiple Itô processes or to functions of more than one variable.

Theorem 2.3 (Itô's Lemma for two process)**.** *Let $X(t)$ and $Y(t)$ be Itô processes defined by*

$$\begin{aligned} dX(t) &= f_X(t, X(t), Y(t))\, dt + g_X(t, X(t), Y(t))\, dW_1(t) \\ dY(t) &= f_Y(t, X(t), Y(t))\, dt + g_Y(t, X(t), Y(t))\, dW_2(t) \end{aligned}$$

where W_1 and W_2 are correlated Brownian motions with the correlation coefficient ρ. Let $F(t, x, y)$ be twice continuously differentiable in x and y, and once continuously differentiable in t. Then, Itô's formula is given by:

$$\begin{aligned} dF(t, X(t), Y(t)) &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX(t) + \frac{\partial F}{\partial y}dY(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX(t))^2 \\ &\quad + \frac{\partial^2 F}{\partial x \partial y}dX(t)dY(t) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(dY(t))^2 \end{aligned}$$

where

$$\frac{\partial^2 F}{\partial x \partial y} dX(t)dY(t) = \rho g_X(t, X(t), Y(t)) g_Y(t, X(t), Y(t)) \frac{\partial^2 F}{\partial x \partial y} dt.$$

To find the solution process, proceed step-by-step and apply Itô's formula in intervals of equal step-size τ starting from the initial point. To see the solution way better, we consider the following example.

Example 2.1. We assume that trading occurs continuously over time and stock returns respond to information received at a previous point τ. This means that the trading asset depends on historical information. Thus the stock price process is modeled by an SDDE, which is obtained by adding a linear delay into the most known geometric Brownian motion (GBM) model;

$$\begin{aligned} dS(t) &= (a_0 S(t) + a_1 S(t-\tau) + a_2)\, dt \\ &\qquad + (b_0 S(t) + b_1 S(t-\tau) + b_2)\, dW(t),\ t \in [0, T] \\ S(t) &= \varphi_1(t),\ t \in [-\tau, 0] \end{aligned} \tag{2.3}$$

where the delay term τ is positive fixed number and the coefficients are in $\mathbb{R}$. Assume that $\varphi_1(t) : [-\tau, 0] \to \mathbb{R}$ is a continuous initial function on its domain. Let's check conditions of existence and uniqueness theorem where $f(t, x, y) = a_0 x + a_1 y + a_2$ and $g(t, x, y) = b_0 x + b_1 y + b_2$.

- Lipschitz condition: Let's define I_1 and I_2 as;

$$\begin{aligned} I_1 &= |f(t, x_1, y_1) - f(t, x_2, y_2)| = |a_0 (x_1 - x_2) + a_1 (y_1 - y_2)| \\ I_2 &= |g(t, x_1, y_1) - g(t, x_2, y_2)| = |b_0 (x_1 - x_2) + b_1 (y_1 - y_2)|. \end{aligned}$$

  Then, we get:

$$\begin{aligned} I_1 + I_2 &\leq (|a_0| + |b_0|) |x_1 - x_2| + (|a_1| + |b_1|) |y_1 - y_2| \\ &\leq K (|x_1 - x_2| + |y_1 - y_2|) \end{aligned}$$

  for some $K \geq \max\{|a_0| + |b_0|, |a_1| + |b_1|\}$.

- Linear growth condition:

$$\begin{aligned} |f(t, x, y)|^2 + |g(t, x, y)|^2 = \left(a_0^2 + b_0^2\right) x^2 + \left(a_1^2 + b_1^2\right) y^2 + \left(a_2^2 + b_2^2\right) \\ + 2 (a_0 a_1 + b_0 b_1) xy + 2 (a_0 a_2 + b_0 b_2) x \\ + 2 (a_1 a_2 + b_1 b_2) y. \end{aligned}$$

Note that

$$
\begin{aligned}
(x-y)^2 \geq 0 &\Rightarrow x^2+y^2 \geq 2xy \\
(x-1)^2 \geq 0 &\Rightarrow x^2+1 \geq 2x \\
(y-1)^2 \geq 0 &\Rightarrow y^2+1 \geq 2y.
\end{aligned}
$$

Thus

$$
\begin{aligned}
2\left(a_0a_1+b_0b_1\right)xy &\leq \left(a_0a_1+b_0b_1\right)\left(x^2+y^2\right) \\
2\left(a_0a_2+b_0b_2\right)x &\leq \left(a_0a_2+b_0b_2\right)\left(x^2+1\right) \\
2\left(a_1a_2+b_1b_2\right)y &\leq \left(a_1a_2+b_1b_2\right)\left(y^2+1\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
|f(t,x,y)|^2+|g(t,x,y)|^2 &\leq (\overbrace{a_0^2+b_0^2+a_0a_1+b_0b_1+a_0a_2+b_0b_2}^{C_1})x^2 \\
&\quad + (\overbrace{a_1^2+b_1^2+a_0a_1+b_0b_1+a_1a_2+b_1b_2}^{C_2})y^2 \\
&\quad + (\overbrace{a_2^2+b_2^2+a_0a_2+b_0b_2+a_1a_2+b_1b_2}^{C_3})\,1 \\
&\leq C(1+x^2+y^2)
\end{aligned}
$$

for some $C \geq \max\{C_1, C_2, C_3\}$.

So, there exists pathwise unique solution to (2.3). To find the solution on $[0,T]$, consider method of steps with step size τ.

- For $t \in [0,\tau]$; $S(t-\tau) = \varphi_1(t-\tau)$ since $-\tau \leq t-\tau \leq 0$. Thus, SDDE becomes

$$
\begin{aligned}
dS(t) &= \left(a_0S(t)+a_1\varphi_1(t-\tau)+a_2\right)dt \\
&\qquad + \left(b_0S(t)+b_1\left(\varphi_1(t-\tau)+b_2\right)dW(t)\right.
\end{aligned}
$$

Then corresponding stochastic integral and solution are;

$$
\begin{aligned}
S(t) &= S(0)+\int_0^t \left(a_0S(u)+a_1\varphi_1(u-\tau)+a_2\right)du \\
&\qquad + \int_0^t \left(b_0S(u)+b_1\varphi_1(u-\tau)+b_2\right)dW(u) \\
&:= \varphi_2(t)
\end{aligned}
$$

- For $t \in [\tau, 2\tau]$; $S(t-\tau) = \varphi_2(t-\tau)$ since $0 \leq t-\tau \leq \tau$. Thus, SDDE becomes

  $$dS(t) = (a_0 S(t) + a_1 \varphi_2(t-\tau) + a_2)\, dt + (b_0 S(t) + b_1 \varphi_2(t-\tau) + b_2)\, dW(t)$$

  Then corresponding stochastic integral and solution are;

$$\begin{aligned} S(t) = \varphi_2(\tau) &+ \int_\tau^t (a_0 S(u) + a_1 \varphi_2(u-\tau) + a_2) du \\ &+ \int_\tau^t (b_0 S(u) + b_1 \varphi_2(u-\tau) + b_2) dW(u) \\ := \varphi_3(t) \end{aligned}$$

This procedure can be repeated on $[i\tau, (i+1)\tau]$ for $i = 2, 3, 4, \dots$ recursively and construct the solution for this SDDE, which is called as method of steps.

Let's find corresponding expected value of $S(t)$ where the stochastic integral of $S(t)$ for any $t \in [0, T]$ is

$$S(t) = S(0) + \int_0^t f(u, S(u), S(u-\tau)) du + \int_0^t g(u, S(u), S(u-\tau)) dW(u) \quad (2.4)$$

where

$$\begin{aligned} f\,(t, S(t), S(t-\tau)) &= a_0 S(t) + a_1 S(t-\tau) + a_2, \\ g\,(t, S(t), S(t-\tau)) &= b_0 S(t) + b_1 S(t-\tau) + b_2. \end{aligned}$$

Taking the expectation of (2.4) and setting $E(S(t)) = m(t)$, we get

$$\begin{aligned} E(S(t)) = E(S(0)) &+ E\left(\int_0^t f(u, S(u), S(u-\tau)) du\right) \\ &+ E\left(\int_0^t g(u, S(u), S(u-\tau)) dW(u)\right), \\ m(t) = m(0) &+ \int_0^t (a_0 m(u) + a_1 m(u-\tau) + a_2) du, \end{aligned}$$

since expectation satisfies the linearity property and $\int_0^t g(u, S(u), S(u-\tau)) dW(u)$ is martingale, its expectation is zero.

While using the Fundamental Theorem of Calculus, taking the derivative of that equation with respect to t, we obtain

$$\begin{aligned} m'(t) &= a_0 m(t) + a_1 m(t-\tau) + a_2,\ t \in [0, T], \\ m(t) &= E(\varphi_1(t)),\ t \in [-\tau, 0]. \end{aligned}$$

Taking $a_0 = -3, a_1 = 2e^{-1}, a_2 = 3 - 2e^{-1}$ and $\varphi_1(t) = 1 + e^{-t}$ so that our example is the extended version of Example 3.4 given in [38];

$$\begin{aligned} &m'(t) = -3m(t) + 2e^{-1}m(t-\tau) - 2e^{-1} + 3,\ t \in [0, T], \\ &m(t) = e^{-t} + 1,\ t \in [-\tau, 0]. \end{aligned} \tag{2.5}$$

We solve (2.5) iteratively.

For $t \in [0, \tau]$: our equation becomes

$$\begin{aligned} &m'(t) = -3m(t) + 3 + 2e^{\tau - t - 1}, \\ &m(0) = 2 \end{aligned}$$

and corresponding solution turns out to be

$$m(t) = e^{-3t}(1 - e^{\tau-1}) + e^{\tau-t-1} + 1.$$

For $t \in [\tau, 2\tau]$: the equation in (2.5) becomes

$$\begin{aligned} &m'(t) = -3m(t) + 2e^{3\tau-3t-1} - 2e^{4\tau-3t-2} + 2e^{2\tau-t-2} + 3, \\ &m(\tau) = e^{-3\tau} - e^{-2\tau-1} + e^{-1} + 1, \end{aligned}$$

where the solution is

$$\begin{aligned} m(t) = 1 + e^{2\tau-t-2} + 2te^{-3t}(e^{3\tau-1} - e^{4\tau-2}) + e^{-3t}(1 + e^{4\tau-2}(-1 + 2\tau) \\ + e^{3\tau-1}(1 - 2\tau) - e^{\tau-1}). \end{aligned}$$

To sum up, the solution becomes

$$m(t) = \begin{cases} 1 + e^{-t}, & t \in [-\tau, 0], \\ e^{-3t}(1 - e^{\tau-1}) + e^{\tau-t-1} + 1, & t \in [0, \tau], \\ 1 + e^{2\tau-t-2} + 2te^{-3t}(e^{3\tau-1} - e^{4\tau-2}) & \\ +e^{-3t}(1 + e^{4\tau-2}(2\tau - 1) + e^{3\tau-1}(-2\tau + 1) - e^{\tau-1}), & t \in [\tau, 2\tau]. \end{cases}$$

Setting $T = 2$ and $\tau = 1$, the mean function becomes

$$m(t) = 1 + e^{-t},\ -1 \le t \le 2.$$

Finding a closed form solution to SDDE is not easy and generally not possible. Thus, we need numerical methods to find an approximate solution.

CHAPTER 3

NUMERICAL METHODS FOR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

Finding analytical solutions for an SDDE is generally difficult because of their complexity and the stochastic nature of these processes. Therefore, numerical methods are important to find approximate solutions. Moreover, practitioners can simulate and analyze the behavior of these stochastic systems with the help of such numerical methods. Euler Maruyama and Milstein are the most commonly used numerical methods. For the detailed information and proofs [4, 7, 5, 8, 9, 25, 38, 33].
This chapter focuses on these numerical methods for SDDEs while providing some definitions and an illustrative example.

We consider the general form of SDDE:

$$\begin{aligned} &dX(t) = f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dW(t),\, t \in [0, T], \\ &X(t) = \varphi(t),\, t \in [-\tau, 0]. \end{aligned} \tag{3.1}$$

To find an approximate solution, consider a partition of the time interval $[0, T]$, into N pieces $0 = t_0 < t_1 < \cdots < t_N = T$ so that for any $n = 0, 1, 2, \ldots, N-1$, step size for time is $\Delta t_{n+1} = t_{n+1} - t_n$ and the increment of standard Brownian motion is $\Delta W_{n+1} = W(t_{n+1}) - W(t_n) = W(t_{n+1} - t_n) = W(\Delta t_{n+1})$. Since $W(t)$ is a continuous process satisfing stationary increment, independent increment properties and $W(t) \sim N(o, t)$. According to the Central Limit Theorem,

$$W_t - W_u = W_{t-u} \sim N(0, \sqrt{t-u})$$

for $0 \leq u \leq t$. Therefore,

$$\Delta W_{n+1} = \sqrt{t_{n+1} - t_n} Z_{n+1} = \sqrt{\Delta t_{n+1}} Z_{n+1}$$

for some random variable $Z_{n+1} \sim N(0,1)$.

- Note that N-partition of the interval $[0, T]$ with equal length h means $h = T/N$ and $t_n = nh$ for all $n = 0, 1, \dots, N$.
- N_τ is a positive integer number such that $N_\tau h = \tau$ and thus we have $(N + N_\tau)$-partition of $[-\tau, T]$.
- The increments of time and Brownian motion are:

$$\begin{aligned} \Delta t_{n+1} &= h, \\ \Delta W_{n+1} &= \Delta W(h) = \sqrt{h} Z_{n+1}, \end{aligned}$$

  for some $Z_{n+1} \sim N(0,1)$ where $n = 0, 1, 2, \dots, N-1$.
- Let $\tilde{X}_n$ be an approximation of the solution of (3.1), using a stochastic explicit one step method where ϕ is an increment function, then it must satisfy:

$$\begin{aligned} &\tilde{X}_{n+1} = \tilde{X}_n + \phi(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}),\ 0 \leq n \leq N-1, \\ &\tilde{X}_{n-N_\tau} = \varphi(t_n - \tau),\ 0 \leq n \leq N_\tau. \end{aligned} \tag{3.2}$$

3.1 Euler Maruyama Method for SDDE

The increment function ϕ in (3.2) with uniform step size h is:

$$\phi(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}) = f(\tilde{X}_n, \tilde{X}_{n-N_\tau})h + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1}. \tag{3.3}$$

The corresponding approximation of strong solution according to (3.2) for the Euler Maruyama method is

$$\begin{aligned} \tilde{X}_{n+1} &= \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})h + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1}, \\ &= \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})h + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\sqrt{h} Z_{n+1}, \end{aligned}$$

where $Z_{n+1} \sim N(0,1)$ and for all indices $n - N_\tau \leq 0$, we define $\tilde{X}_{n-N_\tau} := \varphi(t_n - \tau)$.

Theorem 3.1 (Theorem 7 in [4]). *Let the functions f and g in* (3.1) *satisfy the conditions of the existence and uniqueness theorem. Then the Euler Maruyama method is consistent with order $p_1 = 2$ in the mean and order $p_2 = 1$ in the mean square sense.*

The complete proof can be found in [4].

3.2 Milstein Method for SDDE

According to [33], the increment function ϕ in stochastic explicit one-step method for Milstein with uniform step size h on the interval $[-\tau, T]$ is:

$$\begin{aligned}\phi(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}) = \tilde{f}h + \tilde{g}\Delta W_n + \frac{1}{2}\tilde{g}\frac{\partial \tilde{g}}{\partial \tilde{X}_n}[(\Delta W_n)^2 - h] \\ + \tilde{g}\frac{\partial \tilde{g}}{\partial \tilde{X}_{n-N_\tau}}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} dW(s_2 - \tau)dW(s_1)\end{aligned}$$

where $\tilde{f} = f(t_n, \tilde{X}_n, \tilde{X}_{n-N_\tau})$ and $\tilde{g} = g(t_n, \tilde{X}_n, \tilde{X}_{n-N_\tau})$. Then, the corresponding Milstein scheme is

$$\begin{aligned}\tilde{X}_{n+1} = \tilde{X}_n + \tilde{f}h + \tilde{g}\Delta W_n + \frac{1}{2}\tilde{g}\frac{\partial \tilde{g}}{\partial \tilde{X}_n}[(\Delta W_n)^2 - h] \\ + \tilde{g}\frac{\partial \tilde{g}}{\partial \tilde{X}_{n-N_\tau}}\int_{t_n}^{t_{n+1}}\int_{t_n}^{s_1} dW(s_2 - \tau)dW(s_1)\end{aligned}$$

for all $n - N_\tau \geq 0$ and for all indices $n - N_\tau \leq 0$, we define $\tilde{X}_{n-N_\tau} := \varphi(t_n - \tau)$ where $n = 0, 1, 2, \ldots, N-1$.

We consider an example of applying numerical methods to simulate an approximate solution.

Example 3.1. For the SDDE in Example 2.1, we take $a_0 = -3$, $a_1 = 2e^{-1}$, $a_2 = 3 - 2e^{-1}$, $b_0 = b_1 = b_2 = 0.5, \tau = 1$, $T = 2$ and $\varphi(t) = 1 + e^{-t}$. Then the corresponding equation satisfied by the stock process in (2.3) becomes

$$\begin{aligned}dS(t) &= (-3S(t) + 2e^{-1}S(t-1) + 3 - 2e^{-1})\,dt \\ &\quad +0.5\,(S(t) + S(t-\tau) + 1)\,dW(t),\ t \in [0, 2], \\ S(t) &= 1 + e^{-t},\ t \in [-1, 0];\end{aligned} \tag{3.4}$$

and corresponding mean function in (2.5) satisfies

$$\begin{aligned}m'(t) &= -3m(t) + 2e^{-1}m(t-1) + 3 - 2e^{-1},\ t \in [0, 2], \\ m(t) &= 1 + e^{-t},\ t \in [-1, 0].\end{aligned} \tag{3.5}$$

Solving this sample delayed ODE, we obtain

$$m(t) = 1 + e^{-t},\ t \in [-1, 2]. \tag{3.6}$$

To simulate our solution process, we take the time step of $dt = 0.01$ and the number of simulations as 1000. In Figure 3.1, sample paths to our SDDE are obtained

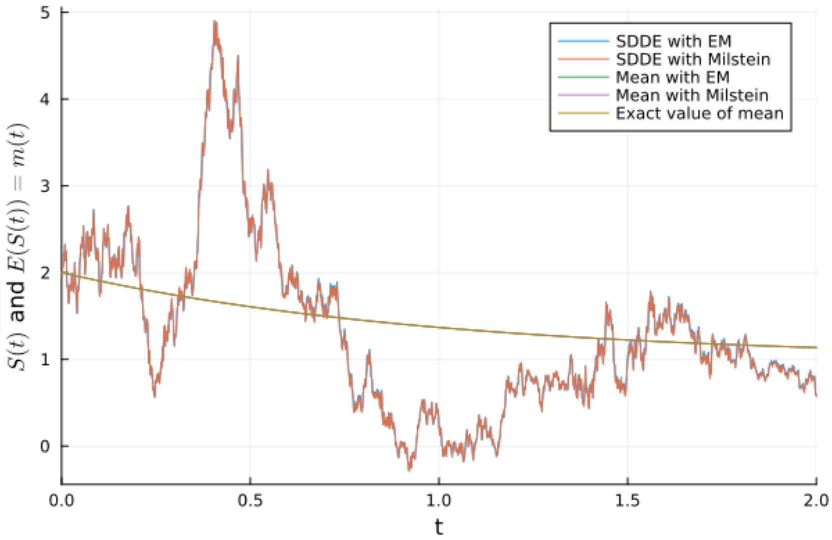


Figure 3.1: Sample path and its mean function with Euler Maruyama and Milstein methods

by Euler Maruyama and Milstein methods while using the same random numbers. Moreover, the expectation of this process using that numerical methods and exact solution of mean function are obtained. The graphs are almost the same.

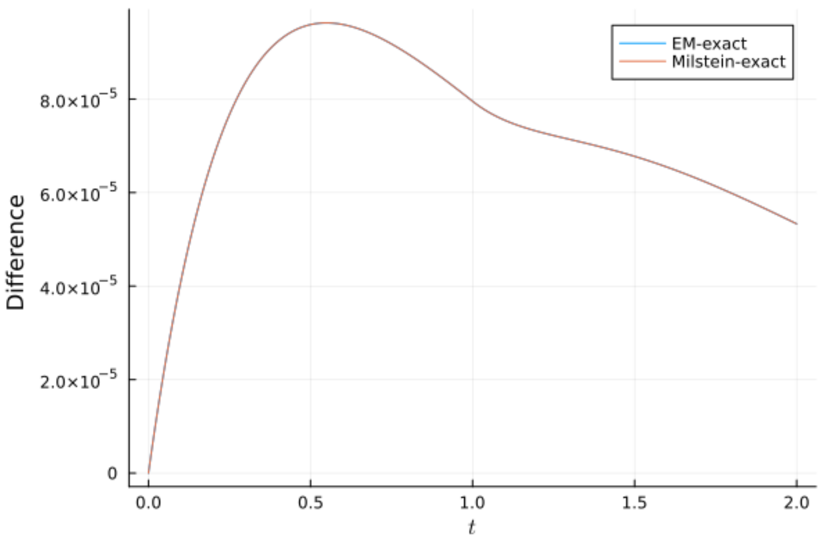


Figure 3.2: Difference in the exact solution of mean functions and its approximations

When we consider the absolute error (difference) between the exact solution of mean and approximate solutions in Figure 3.2, Euler Maruyama and Milstein methods fit the model since the error is too small and close to zero. Since the mean function does not include any randomness, its graphs obtained by numerical methods are the same.

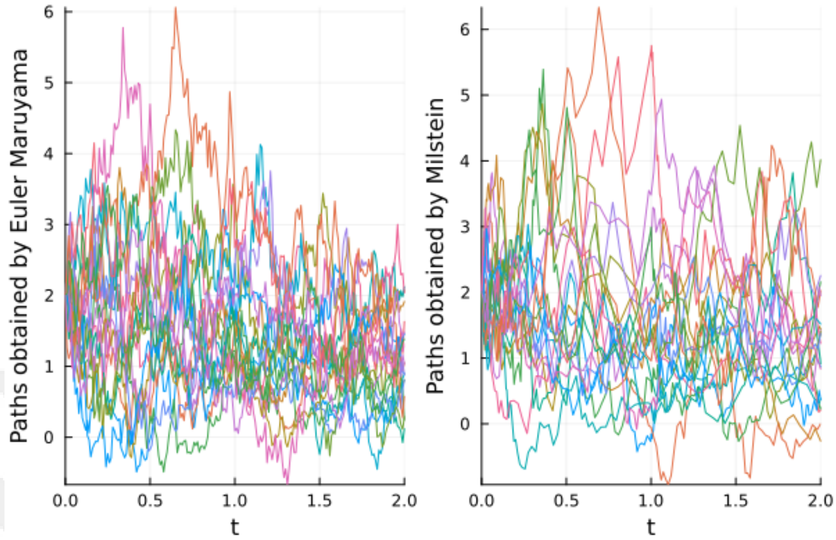


Figure 3.3: Multi paths obtained by numerical methods

In Figure 3.3, 20 paths for $S(t)$ are obtained by using these numerical methods.

Table 3.1: Mean value obtained with different methods

$m(m(t))$	$m(S_{\text{EM}}, 1000)$	$m(S_{\text{Mil}}, 1000)$	$m(E_{\text{EM}}(S(t)))$	$m(E_{\text{Mil}}(S(t)))$
1.4324	1.4497	1.4496	1.4322	1.4322

In Table 3.1, the mean value is obtained with different methods. $m(m(t))$ represents the mean value obtained from the exact solution of expectation in (3.6). $m(S_{\text{EM}}, 1000)$ and $m(S_{\text{Mil}}, 1000)$ represent the mean value obtained from simulation of 1000 paths of $S(t)$ in (3.4) with Euler Maruyama and Milstein methods while $m(E_{\text{EM}}(S(t)))$ and $m(E_{\text{Mil}}(S(t)))$ represent the mean value obtained from simulation of (3.5) with Euler Maruyama and Milstein method. As it is seen from the table, Euler Maruyama and Milstein's methods for $S(t)$ give approximate values to the real value $m(m(t)) = 1.4324$.

Now, use SDDEs and their numerical solutions for pricing options.

CHAPTER 4

OPTION PRICING

Options are financial derivatives that give investors the right to buy or sell an underlying asset at a predetermined price within a predetermined time interval. The call options give the right to buy, while the put options give the right to sell underlying assets [19].

The pricing of options is influenced by several factors. The key components are the underlying asset's price, the strike price, time to expiration, volatility of the underlying asset, interest rate and dividend yields. Several models and methods have been developed to estimate the fair price of options. The most common one is the Black-Scholes-Merton model, introduced by Fischer Black, Myron Scholes, and Robert Merton in the early 1970s [6]. This model gives a framework for calculating the theoretical price of European Vanilla options while assuming constant volatility and interest rates where the underlying asset follows GBM. Apart from the analytical solution of the Black-Scholes-Merton model, some numerical models like the Monte Carlo method also give alternative approaches to pricing options [30, 1]. The dividend yields, changing volatility, and changing interest rates can easily be incorporated into this numerical model for more realistic situations.

In some phenomena where immediate execution or information is not guaranteed, delay has an important impact. Thus, the delay must also be considered as one of the main components of pricing the option [35]. The studies related to the delay effect try to understand how delays in trading or receiving information affect the option's value. These delays can change risks and profits. Thus, pricing models must be adjusted to account for the time delay. This adjustment helps provide a more accurate valuation as illustrated in [3, 26, 28, 22].

In this chapter, we consider some call options with and without delay effect where stock price processes follow GBM, namely European Vanilla Options, American Vanilla Options, European Foreign Exchange Options, and European Exchange options. The corresponding valuation formulas apart from American Vanilla Options are provided.

4.1 European Vanilla Option Pricing with GBM

4.1.1 European Vanilla Option Pricing without Delay

The Black-Scholes-Merton Model assumes the price of the stock S follows GBM with constant drift μ and volatility σ for the given maturity T, namely,

$$\begin{aligned} &dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ t \in [0, T] \\ &S(0) = s_0, \end{aligned}$$

where $W(t)$ is a standard Brownian motion and s_0 is the given initial stock price. The Black-Scholes-Merton model is a widely used option pricing model. It provides the theoretical value of options using current stock prices s_0, the option's strike price K, risk free rate r, time to maturity T and volatility σ under the following assumptions:

- It is a European-type option which can be exercised at maturity only,
- Market is efficient,
- There is no transaction cost,
- The risk free rate and volatility of the stock are known and constant,
- The returns of the underlying assets are normally distributed.

Theorem 4.1 (See [6]). *If the stock price S satisfies the following equation*

$$\begin{aligned} &dS(t) = rS(t)dt + \sigma S(t)dW(t),\ t > 0, \qquad (4.1) \\ &S(0) = s_0, \end{aligned}$$

under risk neutral probability measure Q and conditions of Black-Scholes-Merton Model, then the value of European call and put options, namely V_C and V_P respectively, are given by

$$V_C(S,K,t) = S(t)\phi(d_1) - Ke^{-r(T-t)}\phi(d_2), \tag{4.2}$$

$$V_P(S,K,t) = Ke^{-r(T-t)}\phi(-d_2) - S(t)\phi(-d_1), \tag{4.3}$$

where

$$d_1 = \frac{\ln(\frac{S(t)}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln(\frac{S(t)}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and ϕ is the density of the standard normal distribution which is defined as

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}du,\ x \in \mathbb{R}.$$

4.1.2 European Vanilla Option Pricing with Delay

European vanilla option pricing model with delayed GBM is examined to understand the effect of delay in the determination of the price. We also consider single and multi-delay in underlying assets and provide corresponding value formulas.

4.1.2.1 European Vanilla Option Pricing with a Single Delay

Arriojas et al. (2007) [3] have obtained the fair price formula of the European Vanilla call option for any time $t \leq T$ while showing market is complete and arbitrage free. We extend that work for the put option and show put-call parity is satisfied also under delay effect.

The market includes a riskless asset $B(t)$ and a single stock $S(t)$ so that $B(t) = e^{rt}$ for the risk free rate r and the stock price satisfies:

$$\begin{aligned} &dS(t) = \mu S(t-a)S(t)dt + g(S(t-b))S(t)dW(t),\ t \in [0,T], \\ &S(t) = \varphi(t),\ t \in [-L,0] \end{aligned} \tag{4.4}$$

where μ and T are positive real numbers, a and b are delay terms with $L = \max\{a, b\}$, g is a continuous function and φ is $\mathcal{F}_0$-measurable initial path with $\varphi(0) > 0$.

Theorem 4.2 (Theorem 4 in [3])**.** *Let the asset price S satisfies* (4.4) *with $\varphi(0) > 0$ and $g(t) \neq 0$ whenever $t \neq 0$. Let r be the positive risk free rate and Q be the risk neutral probability measure. Then the value of European vanilla call option $V_C(t)$ where K is the strike price, T is the maturity and $l = \min\{a, b\}$ is:*

- *For any $t \in [T - l, T]$:*

$$V_C(t) = S(t)\phi(\beta_1(t)) - Ke^{-r(T-t)}\phi(\beta_2(t)), \tag{4.5}$$

where

$$\beta_1(t) = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r + \frac{1}{2} g(S(u-b))^2) du}{\sqrt{\int_t^T g(S(u-b))^2 du}},$$

$$\beta_2(t) = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r - \frac{1}{2} g(S(u-b))^2) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

- *For all $T > l$ and $t < T - l$:*

$$V_C(t) = e^{rt} E_Q\Bigg(H\Bigg(e^{-r(T-l)} S(T-l), -\tfrac{1}{2}\int_{T-l}^T g(S(u-b))^2 du, \int_{T-l}^T g(S(u-b))^2 du \Bigg) \Bigg| \mathcal{F}_t \Bigg), \tag{4.6}$$

where

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2}\phi(\alpha_1(x, m, \sigma)) - Ke^{-rT}\phi(\alpha_2(x, m, \sigma))$$

and

$$\alpha_1(x, m, \sigma) = \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m + \sigma^2\right],$$

$$\alpha_2(x, m, \sigma) = \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m\right],$$

for $\sigma, x \in \mathbb{R}^+$, $m \in \mathbb{R}$.

The hedging strategy for $t \in [T - l, T]$ is given by

$$\pi_S(t) = \phi(\beta_1(t)),$$
$$\pi_B(t) = -Ke^{-r(T-t)}\phi(\beta_2(t)).$$

Detailed proof can be found in Appendix A.

Corollary 4.3. *Assume that the conditions in Theorem 4.2 are satisfied. Let $V_P(t)$ be the value of a European put option. Then the value of the put option is:*

- *For all $t \in [T-l, T]$:*

$$V_P(t) = Ke^{-r(T-t)}\phi(-\beta_2(t)) - S(t)\phi(-\beta_1(t)) \tag{4.7}$$

where β_1 and β_2 are the same as in Theorem 4.2.

- *For all $T > l$ and $t < T-l$:*

$$\begin{aligned} V_P(t) = e^{rt}E_Q\Bigg(H\Bigg(& e^{-r(T-l)}S(T-l), -\tfrac{1}{2}\textstyle\int_{T-l}^{T} g(S(u-b))^2du, \\ & \textstyle\int_{T-l}^{T} g(S(u-b))^2du\Bigg)\Bigg|\mathcal{F}_t\Bigg), \end{aligned} \tag{4.8}$$

where

$$H(x, m, \sigma^2) = Ke^{-rT}\phi(-\alpha_2(x, m, \sigma)) - xe^{m+\sigma^2/2}\phi(-\alpha_1(x, m, \sigma))$$

and α_1 and α_2 are the same as in Theorem 4.2.

Proof. Actually, in this proof, we use the same ideas in the proof of Theorem 4.2 while considering the payoff function as $h_P(S(T)) = (K - S(T), 0)^+ = -h_C(S(T))$. The solution of stock price $S(t)$ which satisfies (4.4);

$$\tilde{S}(T) = e^{-rt}S(t)e^{m+\sigma y} = \tilde{S}(t)e^{m+\sigma y}$$

from (A.1) where $\sigma^2 := \displaystyle\int_t^T g(S(u-b))^2du$ and $m := \dfrac{-1}{2}\displaystyle\int_t^T g(S(u-b))^2du.$ The corresponding value of the European put option is

$$\begin{aligned} V_P(S(t), t) &= e^{rt}E_Q\left[\frac{(K - S(T), 0)^+}{e^{rT}}\middle| F_t\right] \\ &= e^{rt}E_Q\left[\left(Ke^{-rT} - \tilde{S}(T), 0\right)^+ \middle| F_t\right] \\ &= e^{rt}E_Q\left[\left(Ke^{-rT} - \tilde{S}(t)e^{m+\sigma y}, 0\right)^+ \middle| F_t\right]. \end{aligned}$$

For $x := \tilde{S}(t)$, define the function H as

$$
\begin{aligned}
H(x,m,\sigma^2) &:= E_Q\left[\left(Ke^{-rT} - xe^{m+\sigma Y} -, 0\right)^+ \mid F_{T-l}\right), \\
&= E_Q\left[\left(Ke^{-rT} - xe^{m+\sigma Y}, 0\right)^+\right], \text{ since payoff is } F_{T-l}\text{-measurable} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(Ke^{-rT} - xe^{m+\sigma y}, 0\right)^+ e^{-y^2/2} dy,
\end{aligned}
$$

where

$$
Ke^{-rT} - xe^{m+\sigma y} > 0 \text{ so that } y < -\beta_2 := -\frac{\ln\left(\frac{x}{K}\right) + rT + m}{\sigma}.
$$

Thus;

$$
\begin{aligned}
H(x,m,\sigma^2) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\beta_2} \left(Ke^{-rT} - xe^{m+\sigma y}\right) e^{-y^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\beta_2} Ke^{-rT} e^{-y^2/2} dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\beta_2} xe^{m+\sigma y - y^2/2} dy \\
&= Ke^{-rT}\phi(-\beta_2) - x\phi(-\beta_1)
\end{aligned}
$$

where $\beta_1 = \sigma + \beta_2$. Then, the value of the European put option under the Black-Scholes-Merton setting with delay effect for any $t \in [0,T]$ is

$$
V_P(S,t) = e^{rt} E_Q\left[H(x,m,\sigma^2) \mid F_t\right].
$$

Case 1: When $t \in [T-l, T]$; since H is F_{T-l}-measurable and $F_{T-l} \subset F_t$;

$$
\begin{aligned}
V_P(t) &= e^{rt} E_Q\left(H(x,m,\sigma^2) \mid F_t\right) \\
&= e^{rt} H(x,m,\sigma^2) \\
&= Ke^{-r(T-t)}\phi(-\beta_2) - S(t)\phi(-\beta_1).
\end{aligned}
$$

Case 2: When $T > l$ and $t < T-l$, then consider to write $S(T)$ in terms of $S(T-l)$;

$$
\begin{aligned}
V_P(t) = e^{rt} E_Q\Bigg(H\Bigg(\tilde{S}(T-l), -\frac{1}{2}\int_{T-l}^{T} g(S(u-b))^2 du, \\
\int_{T-l}^{T} g(S(u-b))^2 du\Bigg)\Bigg| F_t\Bigg)
\end{aligned}
$$

where

$$
\begin{aligned}
&H(x,m,\sigma^2) = Ke^{-rT}\phi(-\alpha_2(x,m,\sigma)) - xe^{m+\sigma^2/2}\phi(-\alpha_1(x,m,\sigma)), \\
&\alpha_1 = \frac{\ln\left(\frac{x}{K}\right) + m + rT + \sigma^2}{\sigma}, \\
&\alpha_2 = \frac{\ln\left(\frac{x}{K}\right) + m + rT}{\sigma} = \alpha_1 - \sigma.
\end{aligned}
$$

So, this completes the proof. □

Corollary 4.4. *Let the conditions in Theorem 4.2 be satisfied for the stock S with strike price K. Then put-call parity is satisfied also under delay effect i.e.*

$$V_C(t) - V_P(t) = S(t) - Ke^{-r(T-t)}.$$

Proof. From Theorem 4.2 and Corollary 4.3, pricing formulas are provided in two cases. Thus, it also proves that parity is satisfied in both cases.

- When $t \in [T-l, T]$ (i.e. $l > T$);

$$\begin{aligned} V_C(t) &= S(t)\phi(\beta_1(t)) - Ke^{-r(T-t)}\phi(\beta_2(t)), \\ V_P(t) &= Ke^{-r(T-t)}\phi(-\beta_2(t)) - S(t)\phi(-\beta_1(t)). \end{aligned}$$

  Then,

$$\begin{aligned} V_C(t) - V_P(t) &= S(t)\phi(\beta_1) - Ke^{-r(T-t)}\phi(\beta_2) \\ &\qquad - \left[Ke^{-r(T-t)}\phi(-\beta_2) - S(t)\phi(-\beta_1)\right] \\ &= S(t)\left[\phi(\beta_1) + \phi(-\beta_1)\right] - Ke^{-r(T-t)}\left[\phi(\beta_2) + \phi(-\beta_2)\right] \\ &= S(t) - Ke^{-r(T-t)} \end{aligned}$$

  since ϕ is the density of the standard normal distribution.

- When $t < T - l$ (i.e. $l < T$);

$$\begin{aligned} V_C(t) &= e^{rt}E_Q(xe^{m+\sigma^2/2}\phi(\alpha_1) - Ke^{-rT}\phi(\alpha_2)|\mathcal{F}_t), \\ &= E_Q(S(t)\phi(\alpha_1) - Ke^{-r(T-t)}\phi(\alpha_2)|\mathcal{F}_t), \\ V_P(t) &= e^{rt}E_Q(Ke^{-rT}\phi(-\alpha_2) - xe^{m+\sigma^2/2}\phi(-\alpha_1)|\mathcal{F}_t) \\ &= E_Q(Ke^{-r(T-t)}\phi(-\alpha_2) - S(t)\phi(-\alpha_1)|\mathcal{F}_t), \end{aligned}$$

  since $m = -\sigma^2/2$ and $x = \tilde{S}(t)$. Then,

$$\begin{aligned} V_C(t) - V_P(t) &= E_Q(S(t)\phi(\alpha_1) - Ke^{-r(T-t)}\phi(\alpha_2)|\mathcal{F}_t) \\ &\qquad - E_Q(Ke^{-r(T-t)}\phi(-\alpha_2) - S(t)\phi(-\alpha_1)|\mathcal{F}_t) \\ &= E_Q(S(t)(\phi(\alpha_1) + \phi(-\alpha_1)) \\ &\qquad - Ke^{-r(T-t)}(\phi(\alpha_2) + \phi(-\alpha_2))|\mathcal{F}_t) \\ &= E_Q\left(S(t) - Ke^{-r(T-t)}|\mathcal{F}_t\right) \\ &= S(t) - Ke^{-r(T-t)}. \end{aligned}$$

So this completes the proof and using the same logic, one can also prove the put-call parity for the other options namely FX and exchange options under delay effect. □

4.1.2.2 European Vanilla Option Pricing with Multi-Delay

We are extending the study of Arriojas et al. [3] for the several delays while assuming that the price process for the stock $S(t)$ at time t satisfies the following SDDE:

$$\begin{aligned} dS(t) &= f(S_{t-a_1}, S_{t-a_2}, ..., S_{t-a_n})S(t)dt \\ &\qquad + g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})S(t)dW(t), \quad t \in [0,T] \\ S(t) &= \varphi(t), \quad t \in [-L, 0], \end{aligned} \tag{4.9}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$.

- S_{t-a_i} means $S(t-a_i)$ for any $i = 1, 2, ..., n$ and S_{t-b_j} means $S(t-b_j)$ for any $j = 1, 2, ..., m$,
- a_i's and b_j's are positive fixed delays where $L = \max\{a_1, ..., a_n, b_1, ...b_m\}$,
- $g : \mathbb{R}^m \to \mathbb{R}$ is a continuous function,
- $W(t)$ is standard Wiener process,
- $\varphi(t) : [-L, 0] \to \mathbb{R}^+$ is $\mathcal{F}_0$-measurable initial path so that $\varphi(0) > 0$ a.s.

We will show that the above model admits pathwise unique solutions.

Theorem 4.5. *Assume that $S(t)$ satisfies* (4.9) *and functions f and g satisfy the linear growth and local Lipschitz conditions. Then* (4.9) *has a pathwise unique solution S for the given initial path. Moreover, $S(t) > 0$ almost surely for all $t \geq 0$ whenever the initial path $\varphi(t) > 0$ for all $t \in [-L, 0]$. Furthermore, if $\varphi(t) \geq 0$ a.s, then $S(t) \geq 0$ for all $t \geq 0$.*

Proof. Let $l := \min\{a_1, a_2, ..., a_n, b_1, b_2, ..., b_m\}$. Then, when we rewrite the equation for $t \in [0, l]$, we get

$$\begin{aligned} dS(t) &= S(t)f(\varphi_{t-a_1}, \varphi_{t-a_2}, ..., \varphi_{t-a_n})dt \\ &\qquad + S(t)g(\varphi_{t-b_1}, \varphi_{t-b_2}, ..., \varphi_{t-b_m})dW(t),\ t \in [0, l] \\ S(t) &= \varphi(t),\ t \in [-L, 0] \end{aligned}$$

Define the semimartingale process for $t \in [0, l]$ as

$$N(t) := \int_0^t f(\varphi_{u-a_1}, \varphi_{u-a_2}, ..., \varphi_{u-a_n}) du + \int_0^t g(\varphi_{u-b_1}, \varphi_{u-b_2}, ..., \varphi_{u-b_m}) dW(u),$$

where the quadratic variation of N is

$$[N, N]_t = \int_0^t g(\varphi_{u-b_1}, \varphi_{u-b_2}, ..., \varphi_{u-b_m})^2 du.$$

Then, the SDDE becomes

$$dS(t) = S(t)dN(t),\ t \in [0, l],$$
$$S(t) = \varphi(t),\ t \in [-L, 0],$$

where the unique solution for any $t \in [0, l]$ is obtained by Doléans Dade exponent as

$$\begin{aligned} S(t) &= \varphi(0) \exp\left[N(t) - \frac{1}{2}[N, N]_t\right] \\ &= \varphi(0) \exp\left[\int_0^t f(\varphi_{u-a_1}, \varphi_{u-a_2}, ..., \varphi_{u-a_n}) du \right. \\ &\left. + \int_0^t g(\varphi_{u-b_1}, \varphi_{u-b_2}, ..., \varphi_{u-b_m}) dW(u) - \frac{1}{2}\int_0^t g(\varphi_{u-b_1}, \varphi_{u-b_2}, ..., \varphi_{u-b_m})^2 du\right]. \end{aligned}$$

From this equation, it is easily seen that if $\varphi(t) > 0$ a.s, then $S(t) > 0$ for all $t \in [0, l]$. While using the induction method, one can find a solution for any $t \in [kl, (k+1)l]$ where $(k+1)l \leq T$ and show $S(t) > 0$ whenever $\varphi(t) > 0$. □

Our purpose is to derive the fair price formula for the European Vanilla option written on the stock S with exercise price K and maturity T with the assumption of no dividend and no transaction costs. Assume that S satisfies (4.9) while the riskless asset B satisfies $B(t) = e^{rt}$ where r is the rate of return and $r > 0$.

Theorem 4.6. *We consider the market consisting of riskless asset $B(t)$ and stock $S(t)$. Let $r > 0$ be the risk free rate and the asset price S satisfy* (4.9) *for $\varphi(0) > 0$ and $g(t) \neq 0$ whenever $t \neq 0$. Then, the market is arbitrage-free.*

Proof. Let us find an equivalent martingale measure with the help of Girsanov's theorem to show market is arbitrage free. Consider the discounted stock price for any

$t \in [0,T]$, $\tilde{S}(t) = \dfrac{S(t)}{B(t)} = e^{-rt}S(t)$ and take derivative where $S(t)$ satisfies (4.9)

$$\begin{aligned} d\tilde{S}(t) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ &= -re^{-rt}S(t)dt + e^{-rt}\big[f(S_{t-a_1}, S_{t-a_2}, ..., S_{t-a_n})S(t)dt \\ &\qquad + g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})S(t)dW(t)\big] \\ &= \tilde{S}(t)\bigg[\left(f(S_{t-a_1}, S_{t-a_2}, ..., S_{t-a_n}) - r\right) dt \\ &\qquad + g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})dW(t)\bigg]. \end{aligned} \tag{4.10}$$

According to Theorem 4.5, $g(S_{t-b_1}, S_{t-b_2}, \ldots, S_{t-b_m})$ is different from zero since $\varphi(t) > 0$ for all $t \in [0,T]$ and $g(u) \neq 0$ whenever $u = (u_1, u_2, \ldots, u_m) \neq 0$. Define

$$\Sigma(t) := -\frac{f(S_{t-a_1}, S_{t-a_2}, ..., S_{t-a_n}) - r}{g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})}, \quad \text{for all } t \in [0,T]$$

which is $\mathcal{F}_{t-l}$ measurable predictable process such that $\int_0^T |\Sigma(u)|^2 du < \infty$. Define

$$Q_T := \exp\left\{ -\int_0^T \Sigma(u)dW(u) - \frac{1}{2}\int_0^T |\Sigma(u)|^2 du \right\}$$

so that $E_P(Q_T) = 1$. Then, according to Girsanov theorem there exists probability measure Q defined by $dQ = Q_T dP$ and standard Brownian motion $\tilde{W}$ under Q defined by

$$\tilde{W}(t) := W(t) - \int_0^t \Sigma(u)du \quad \text{for all } t \in [0,T];$$

or equivalently, we have

$$d\tilde{W}(t) = dW(t) - \Sigma(t)dt,$$

which implies $dW(t) = d\tilde{W}(t) + \Sigma(t)dt$. Arranging the discounted asset price process in (4.10) under the measure Q, we get

$$\begin{aligned}
d\tilde{S}(t) &= \tilde{S}(t)\left[(f(S_{t-a_1}, S_{t-a_2}, ..., S_{t-a_n}) - r)\,dt + g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})dW(t)\right] \\
&= \tilde{S}(t)\Big[(f(S_{t-a_1}, S_{t-a_2}, ..., S_{t-a_n}) - r)\,dt \\
&\qquad\qquad + g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})(d\tilde{W}(t) + \Sigma(t)dt)\Big] \\
&= \tilde{S}(t)g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})\Big[\frac{f(S_{t-a_1}, S_{t-a_2}, ..., S_{t-a_n}) - r}{g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})}dt \\
&\qquad\qquad + d\tilde{W}(t) + \Sigma(t)dt\Big] \\
&= \tilde{S}(t)g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})\left[-\Sigma(t)dt + d\tilde{W}(t) + \Sigma(t)dt\right] \\
&= \tilde{S}(t)g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})d\tilde{W}(t).
\end{aligned}$$

So, the discounted asset price $\tilde{S}(t)$ is Q-martingale since drift term is zero which implies that Q is an equivalent martingale (risk neutral) measure. Therefore, the market is arbitrage free and the proof is completed. □

Remark 4.1. Note that under the risk neutral measure Q, the stock price satisfies

$$dS(t) = rS(t)dt + g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})S(t)d\tilde{W}(t). \tag{4.11}$$

Theorem 4.7. *If the market satisfies the conditions that are defined in Theorem 4.6, then it is complete.*

Proof. Consider the discounted asset price process under Q and apply Itô formula to $\ln \tilde{S}(t)$:

$$\begin{aligned}
\ln \tilde{S}(t) &= \ln \tilde{S}(0) + \int_0^t \frac{1}{\tilde{S}(u)} d\tilde{S}(u) - \frac{1}{2}\int_0^t \frac{1}{\tilde{S}^2(u)} d[\tilde{S}, \tilde{S}]_u \\
&= \ln \varphi(0) + \int_0^t g(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})d\tilde{W}(u) \\
&\qquad - \frac{1}{2}\int_0^t g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du
\end{aligned}$$

Thus, for any $t \in [0, T]$, it follows that

$$\begin{aligned}
\tilde{S}(t) = \varphi(0)\exp\Big[&\int_0^t g(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})d\tilde{W}(u) \\
&- \frac{1}{2}\int_0^t g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du\Big]
\end{aligned} \tag{4.12}$$

Now, let X be any contingent claim which is an integrable non negative $\mathcal{F}_t^S$ measurable random variable where $\mathcal{F}_t^S = \mathcal{F}_t^{\tilde{S}} = \mathcal{F}_t^{\tilde{W}} = \mathcal{F}_t^W$ for any $t \geq 0$ according to definition of $\tilde{S}$ and $\tilde{W}$. Consider Q-martingale process $M(t)$ for any $t \in [0, T]$ which is defined as

$$M(t) := E_Q(e^{-rT} X \mid \mathcal{F}_t^S) = E_Q(e^{-rT} X \mid \mathcal{F}_t^{\tilde{W}}).$$

Then there exists $(\mathcal{F}_t^{\tilde{W}})$ predictable process h_0 such that $\int_0^T h_0^2(u)du < \infty$ and

$$M(t) = E_Q(e^{-rT} X \mid \mathcal{F}_t^{\tilde{W}}) = E_Q(e^{-rT} X) + \int_0^t h_0(u) d\tilde{W}(u)$$

by Martingale Representation theorem. Define the hedging strategy by

$$\begin{aligned}
\pi_S(t) &= \frac{h_0(t)}{\tilde{S}(t) g(S_{t-b_1}, S_{t-b_2}, ..., S_{t-b_m})}, \\
\pi_B(t) &= M(t) - \pi_S(t)\tilde{S}(t),
\end{aligned}$$

for a portfolio $\{(\pi_S(t), \pi_B(t)) \mid t \in [0, T]\}$ where $\pi_S(t)$ and $\pi_B(t)$ represent the amount of stock and bond in the portfolio respectively. Then the value of the portfolio at any time $t \in [0, T]$ is given by

$$V(t) = \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t).$$

This value process implies that

$$\begin{aligned}
dV(t) &= e^{rt} dM(t) + M(t) d(e^{rt}) \\
&= \pi_B(t) d(e^{rt}) + \pi_S(t) dS(t);
\end{aligned}$$

in other words, $\{(\pi_S(t), \pi_B(t)) \mid t \in [0, T]\}$ is self-financing strategy and $V(T) = e^{rT}M(T) = X$. Thus, X is attainable and market consisting of $B(t)$ and $S(t)$ is complete. □

Theorem 4.8. *Let the asset price S satisfy* (4.9) *with $\varphi(0) > 0$ and $g(t) \neq 0$ whenever $t \neq 0$. Let $r > 0$ be the risk free rate and Q be the risk neutral probability measure. Assume that $V(t)$ is the value of a European call option with maturity time T and strike price K. Let ϕ be the standard normal distribution function so that*

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \; x \in \mathbb{R}.$$

Then the value of the European option is

- *Case 1:* *If* $t \in [T-l, T]$ *with* $l := \min\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$*:*

$$V(t) = S(t)\phi(\beta_1(t)) - Ke^{-r(T-t)}\phi(\beta_2(t)), \tag{4.13}$$

where

$$\beta_1 = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r + \frac{1}{2}g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du}{\sqrt{\int_t^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du}},$$

$$\beta_2 = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r - \frac{1}{2}g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du}{\sqrt{\int_t^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du}}.$$

- *Case 2:* *If* $T > l$ *and* $t < T - l$*:*

$$V(t) = e^{rt}E_Q\Bigg(H\Bigg(e^{-r(T-l)}S(T-l), -\frac{1}{2}\int_{T-l}^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du, \int_{T-l}^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du\Bigg)\Bigg|\mathcal{F}_t\Bigg) \tag{4.14}$$

where

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2}\phi(\alpha_1(x, m, \sigma)) - Ke^{-rT}\phi(\alpha_2(x, m, \sigma))$$

and

$$\alpha_1(x, m, \sigma) = \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m + \sigma^2\right],$$

$$\alpha_2(x, m, \sigma) = \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m\right]$$

for $\sigma, x \in \mathbb{R}^+$*,* $m \in \mathbb{R}$*.*

The hedging strategy for $t \in [T-l, T]$ *is given by*

$$\pi_S(t) = \phi(\beta_1(t)),$$
$$\pi_B(t) = -Ke^{-r(T-t)}\phi(\beta_2(t)).$$

Proof. For any $t \in [0, T]$ define the following equalities

$$x := \tilde{S}(t)$$
$$m := -\frac{1}{2}\int_t^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du$$
$$\sigma^2 := \int_t^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m})du$$

so that $\sigma^2 = -2m$. Note that $\int_t^T g(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m}) d\tilde{W}u$ is normally distributed with mean zero and variance $\sigma^2 = \int_t^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m}) du$. So, consider any $Y \sim N(0,1)$ then

$$\int_t^T g(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m}) d\tilde{W}u \sim \sigma Y.$$

With these observations, discounted asset price at maturity T in terms of any time $t > 0$ can be written as;

$$\begin{aligned}\tilde{S}(T) = \tilde{S}(t) \exp \bigg[\int_t^T & g(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m}) d\tilde{W}u \\ & - \frac{1}{2} \int_t^T g^2(S_{u-b_1}, S_{u-b_2}, ..., S_{u-b_m}) du \bigg] \\ = x e^{m+\sigma Y}. &\end{aligned}$$

Note that $\tilde{S}(T)$ is $\mathcal{F}_{T-l}$ measurable where $l := \min\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m\}$, since $t \le u \le T$ implies that

$$t - b_i \le u - b_i \le T - b_i \le T - l \quad \text{for any} \quad i = 1, 2, \ldots, m.$$

Then the value of European call option at any time $t \in [0, T]$ with the pay off function $(S(T) - K)^+$ under the risk neutral probability measure Q is

$$\begin{aligned} V(t) &= e^{-r(T-t)} E_Q \left((S(T) - K)^+ \mid \mathcal{F}_t \right) \\ &= e^{rt} E_Q \left((\tilde{S}(T) - K e^{-rT})^+ \mid \mathcal{F}_t \right) \\ &= e^{rt} E_Q \left(E_Q \left((\tilde{S}(T) - K e^{-rT})^+ \mid \mathcal{F}_{T-l} \right) \mid \mathcal{F}_t \right), \text{ by Tower property} \\ &= e^{rt} E_Q \left(E_Q \left((x e^{m+\sigma y} - K e^{-rT})^+ \mid \mathcal{F}_{T-l} \right) \mid \mathcal{F}_t \right). \end{aligned}$$

Now, define a new function H as $H(x, m, \sigma^2) := E_Q \left((x e^{m+\sigma y} - K e^{-rT})^+ \mid \mathcal{F}_{T-l} \right)$ so that $V(t) = e^{rt} E_Q \left(H(x, m, \sigma^2) \mid \mathcal{F}_t \right)$. Then

$$\begin{aligned} H(x, m, \sigma^2) &= E_Q \left((\tilde{S}(T) - K e^{-rT})^+ \mid \mathcal{F}_{T-l} \right) \\ &= E_Q \left((\tilde{S}(T) - K e^{-rT})^+ \right) \text{ since } \tilde{S}(T) \text{ is } \mathcal{F}_{T-l} \text{ measurable} \\ &= E_Q \left((x e^{m+\sigma Y} - K e^{-rT})^+ \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (x e^{m+\sigma y} - K e^{-rT})^+ e^{-y^2/2} dy. \end{aligned}$$

Finding y values so that $xe^{m+\sigma y} - Ke^{-rT} > 0$;

$$\begin{aligned} xe^{m+\sigma y} - Ke^{-rT} > 0 &\implies xe^{m+\sigma y} > Ke^{-rT} \\ &\implies \ln x + m + \sigma y > \ln K - rT \\ &\implies \ln \frac{x}{K} + m + rT > -\sigma y \\ &\implies y > -\frac{\ln \frac{x}{K} + m + rT}{\sigma}. \end{aligned}$$

Define

$$\alpha_2(x, m, \sigma) := \frac{\ln \frac{x}{K} + m + rT}{\sigma}$$

so that

$$(xe^{m+\sigma y} - Ke^{-rT})^+ = \begin{cases} xe^{m+\sigma y} - Ke^{-rT}, & \text{if } y > -\alpha_2, \\ 0, & \text{if } y \leq -\alpha_2. \end{cases}$$

Then,

$$\begin{aligned} H(x, m, \sigma^2) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (xe^{m+\sigma y} - Ke^{-rT})^+ e^{-y^2/2} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\alpha_2}^{\infty} (xe^{m+\sigma y} - Ke^{-rT}) e^{-y^2/2} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\alpha_2}^{\infty} xe^{m+\sigma y - y^2/2} dy - Ke^{-rT} \int_{-\alpha_2}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\alpha_2}^{\infty} xe^{m+\sigma y - y^2/2} e^{-\sigma^2/2+\sigma^2/2} dy - Ke^{-rT}\phi(\alpha_2) \\ &= xe^{m+\sigma^2/2} \int_{-\alpha_2}^{\infty} \frac{e^{-(y-\sigma)^2/2}}{\sqrt{2\pi}} dy - Ke^{-rT}\phi(\alpha_2) \\ &= xe^{m+\sigma^2/2} \int_{-\alpha_2-\sigma}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} dz - Ke^{-rT}\phi(\alpha_2), \text{ for } y - \sigma = z \\ &= xe^{m+\sigma^2/2}\phi(\alpha_2 + \sigma) - Ke^{-rT}\phi(\alpha_2). \end{aligned}$$

Define $\alpha_1(x, m, \sigma) := \alpha_2(x, m, \sigma) + \sigma = \dfrac{\ln \frac{x}{K} + m + rT + \sigma^2}{\sigma}$. So, the function H is defined as

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2}\phi(\alpha_1) - Ke^{-rT}\phi(\alpha_2).$$

We already know that $V(t) = e^{rt} E_Q(H(x, m, \sigma^2) \mid \mathcal{F}_t)$, where H is $\mathcal{F}_{T-l}$-measurable, hence,

- for $t \in [T-l, T]$;

$$\begin{aligned} V(t) &= e^{rt} E_Q(H(x, m, \sigma^2) \mid \mathcal{F}_t) \\ &= e^{rt} H(x, m, \sigma^2), \quad \text{since} \quad \mathcal{F}_{T-l} \subset \mathcal{F}_t \\ &= e^{rt} \left(x e^{m+\sigma^2/2} \phi(\alpha_1) - K e^{-rT} \phi(\alpha_2) \right) \\ &= S(t)\phi(\alpha_1) - K e^{-r(T-t)} \phi(\alpha_2), \end{aligned}$$

- for $t < T - l$ whenever $T > l$;

$$\begin{aligned} V(t) &= e^{rt} E_Q(H(x, m, \sigma^2) \mid \mathcal{F}_t) \\ &= e^{rt} E_Q(x e^{m+\sigma^2/2} \phi(\alpha_1) - K e^{-rT} \phi(\alpha_2) \mid \mathcal{F}_t). \end{aligned}$$

Using the parameters x, m and σ^2, we can easily show that $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, which completes the proof. □

Proposition 4.9. *Consider the stock price process $S(t)$ which satisfies the multi delay SDE in* (4.11) *under risk neutral probability measure Q. Let $g(S_{t-b_1}, S_{t-b_2}, \ldots, S_{t-b_m})$ be equal to $\tilde{\sigma}$ for some positive constant. Then, the value formula for the stock price (*(4.13)-(4.14)*) and Black-Scholes-Merton formula with volatility term $\tilde{\sigma} > 0$ in* (4.2) *are the same.*

Proof. Note that with that choice of function g, our equation in (4.11) becomes:

$$\begin{aligned} &dS(t) = rS(t)dt + \tilde{\sigma} S(t) dW(t), \quad t \in [0, T] \\ &S(t) = \varphi(t), \quad t \in [-L, 0], \end{aligned}$$

and corresponding equations in ((4.13)-(4.14)) become:

Case 1: If $t \in [T-l, T]$ with $l := \min\{a_1, a_2, \ldots, a_n\}$:

$$V(t) = S(t)\phi(\beta_1(t)) - K e^{-r(T-t)} \phi(\beta_2(t)), \tag{4.15}$$

where

$$\begin{aligned} \beta_1 &= \frac{\log(\frac{S(t)}{K}) + (r + \frac{1}{2}\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}}, \\ \beta_2 &= \frac{\log(\frac{S(t)}{K}) + (r - \frac{1}{2}\tilde{\sigma}^2)(T-t)}{\tilde{\sigma}\sqrt{T-t}}. \end{aligned}$$

Note that d_1 and d_2 in (4.2) are exactly the same as β_1 and β_2 respectively.

Case 2: If $T > l$ and $t < T - l$:

$$V(t) = e^{-r(T-l-t)} S(T-l)\phi(\alpha_1) - K e^{-r(T-t)}\phi(\alpha_2)$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{\tilde{\sigma}\sqrt{l}}\left[\log\left(\frac{S(T-l)}{K}\right) + (r + \frac{1}{2}\tilde{\sigma}^2)l\right],\\
\alpha_2 &= \frac{1}{\tilde{\sigma}\sqrt{l}}\left[\log\left(\frac{S(T-l)}{K}\right) + (r - \frac{1}{2}\tilde{\sigma}^2)l\right].
\end{aligned}$$

Using the Markov property of solution process S and rewrite the valuation formulas in (4.2) as:

$$\begin{aligned}
V(T-l) &= S(T-l)\phi(d_1) - K e^{-rl}\phi(d_2) \qquad (4.16)\\
d_1 &= \frac{1}{\tilde{\sigma}\sqrt{l}}\left[\log\left(\frac{S(T-l)}{K}\right) + (r + \frac{1}{2}\tilde{\sigma}^2)l\right],\\
d_2 &= \frac{1}{\tilde{\sigma}\sqrt{l}}\left[\log\left(\frac{S(T-l)}{K}\right) + (r - \frac{1}{2}\tilde{\sigma}^2)l\right].
\end{aligned}$$

Since the value of the option is martingale, at any time $t > 0$, equation in (4.16) can be written as

$$\begin{aligned}
V(t) &= e^{-r((T-l)-t)} E_Q\left[V(T-l) \mid \mathcal{F}_t\right]\\
&= e^{-r((T-l)-t)} E_Q\left[S(T-l)\phi(d_1) - K e^{-rl}\phi(d_2) \mid \mathcal{F}_t\right]\\
&= e^{-r(T-l-t)} S(T-l)\phi(d_1) - K e^{-r(T-t)}\phi(d_2).
\end{aligned}$$

Note that value formulas are exactly the same where $d_1 = \alpha_1$ and $d_2 = \alpha_2$. □

To find the corresponding partial differential equation (PDE) of the SDDE in (4.11) where $\tilde{S}(t) = e^{-rt}S(t)$, define the value and discounted value processes for any time $t \in [0, T]$ as

$$\begin{aligned}
V(t) &:= F(t, S(t)) = e^{-r(T-t)} E_Q((S(T) - K)^+ \mid \mathcal{F}_t)\\
\tilde{V}(t) &= e^{-rt} V(t) := \tilde{F}(t, \tilde{S}(t)).
\end{aligned}$$

Then, by derivation of the discounted value process, it follows that

$$
\begin{aligned}
d\tilde{V}(t) &= d\tilde{F}(t,\tilde{S}(t)) \\
&= -re^{-rt}F(t,S(t))dt + e^{-rt}dF(t,S(t)) \\
&= -re^{-rt}F(t,S(t))dt + e^{-rt}\left[\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S_t}dS(t) + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}d[S,S]_t\right] \\
&= -re^{-rt}F(t,S(t))dt + e^{-rt}\left[\frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S_t}rS(t)dt\right. \\
&\qquad + \frac{\partial F}{\partial S_t}S(t)g(S_{t-b_1},S_{t-b_2},...,S_{t-b_m})d\tilde{W}_t \\
&\qquad \left. + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}S^2(t)g^2(S_{t-b_1},S_{t-b_2},...,S_{t-b_m})dt\right] \\
&= e^{-rt}\left[-rF + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial S_t}rS(t) + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}S^2(t)g^2(S_{t-b_1},...,S_{t-b_m})\right]dt \\
&\qquad + e^{-rt}\frac{\partial F}{\partial S_t}S(t)g(S_{t-b_1},S_{t-b_2},...,S_{t-b_m})d\tilde{W}_t
\end{aligned}
$$

Since $\tilde{F}(t,\tilde{S}(t))$ is martingale under Q, drift term must be zero. So this implies

$$
-rF + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial S_t}rS(t) + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}S^2(t)g^2(S_{t-b_1},S_{t-b_2},...,S_{t-b_m}) = 0
$$

As a result, the corresponding PDE is:

$$
\begin{aligned}
&\frac{\partial F}{\partial t} = rF - \frac{\partial F}{\partial S_t}rS(t) - \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}S^2(t)g^2(S_{t-b_1},S_{t-b_2},...,S_{t-b_m}) \qquad (4.17)\\
&F(T,S(T)) = (S(T)-K)^+.
\end{aligned}
$$

This PDE also includes delay terms and it can also be used for pricing.

4.2 American Vanilla Option Pricing with GBM

American options can be exercised at any time up to maturity. This feature makes their pricing process more complex than the pricing European options, that can only be exercised at maturity. In American options, we need to find the optimal time to exercise. Because of this early exercise property, finding an analytical solution for the value process is generally not possible. Thus, numerical methods are needed. The commonly used method is the least square Monte Carlo method (LSMC) which was developed by Longstaff and Schwartz (2001) [23]. This method depends on the iteration procedure. First, one creates multiple paths of the underlying asset and then, for all these paths, applies least square regression in each step while considering backward in time to find the continuation value. Having the continuation value, one determines whether the option is exercised or not. The detailed information and application of the method can be found in [23, 12].

In our model, we consider the pricing of American call and put options whenever asset price S satisfies GBM and delayed GBM which are

- for without delay case, S satisfies

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t)dW(t),\ t > 0, \\ S(0) &= s_0, \end{aligned}$$

- for with delay case, S satisfies

$$\begin{aligned} dS(t) &= rS(t)dt + g(S(t-\tau))S(t)dW(t),\ t \in (0, T], \\ S(t) &= \varphi(t), \quad t \in [-\tau, 0]. \end{aligned}$$

4.3 Foreign Exchange Option Pricing with GBM

European foreign exchange (FX) options are financial derivatives that grant the holder the right, but not the obligation, to exchange currency amounts at a predetermined exchange rate on a specific date. These options are essential for hedging against currency risk and speculating on currency movements. Related works are in [17, 14, 13].

4.3.1 Foreign Exchange Option Pricing without delay

Consider a European type FX option with F being the current exchange rate, that is, the domestic currency price of one unit of foreign currency, K being the specified exchange rate and T is the specified date. The terminal payoff function for the call option is $h(F, K, T) = (F(T) - K, 0)^+$. Let the domestic and foreign money market accounts be M_d and M_f respectively, then these risk free assets and current exchange rate F satisfy the following equations:

$$\begin{aligned} dM_d &= r_d M_d dt \\ dM_f &= r_f M_f dt \\ dF(t) &= F(t)[\mu dt + \sigma dZ_F] \end{aligned} \tag{4.18}$$

where r_d and r_f are risk free rates for domestic and foreign money market accounts, μ constant drift coefficient and σ constant volatility term. We consider M_d as numeraire and find the corresponding SDE for F under risk neutral measure Q_d. First, we define

$$X(t) = \frac{F(t)M_f(t)}{M_d(t)},$$

where $F(t)M_f(t)$ represents price in terms of domestic currency and $X(t)$ represents discounted price process for foreign exchange in terms of the domestic currency. We find the corresponding martingale process for $X(t)$ under Q_d while specifying the corresponding Brownian motion. For that purpose, corresponding SDE for $1/M_d(t)$ whit the help of Itô formula, (for $1/x$) we find that

$$\begin{aligned} d\left[\frac{1}{M_d}\right] &= \frac{-1}{M_d^2} dM_d + \frac{1}{2}\frac{2}{M_d^3} d[M_d, M_d]_t \\ &= \frac{-1}{M_d^2} M_d r_d dt + \frac{1}{M_d^3} 0dt \\ &= \frac{-r_d}{M_d} dt. \end{aligned}$$

Then corresponding SDE for $X(t)$;

$$\begin{aligned} dX(t) &= \frac{1}{M_d} d\left(F(t)M_f(t)\right) + F(t)M_f(t) d\left[\frac{1}{M_d}\right] + d\left[FM_f, \frac{1}{M_d}\right]_t \\ &= \frac{1}{M_d}\left[M_f dF + F dM_f\right] + FM_f d\left[\frac{1}{M_d}\right] \\ &= \frac{1}{M_d}\left[M_f F \mu dt + M_f F \sigma dZ_F + FM_f r_f dt\right] + FM_f \frac{-r_d}{M_d} dt \\ &= \frac{FM_f}{M_d}\left[\left(\mu + r_f - r_d\right) dt + \sigma dZ_F\right] \\ &= X\sigma \left[\left(\frac{\mu + r_f - r_d}{\sigma}\right) dt + dZ_F\right]. \end{aligned}$$

Note that for $\gamma := \dfrac{\mu + r_f - r_d}{\sigma}$, $\quad dW_d := \gamma dt + dZ_F$ defines Brownian motion according to Girsanov's theorem under Q_d. Thus, $X(t)$ satisfies $dX(t) = \sigma X dW_d$ which implies its a martingale process and Q_d is risk neutral measure. Now consider the corresponding SDE for F under Q_d where $dZ_F = dW_d - \gamma dt$:

$$\begin{aligned} dF(t) &= F(t)\left[\mu dt + \sigma dZ_F\right] \\ &= F(t)\left[\mu dt + \sigma\left[dW_d - \gamma dt\right]\right] \\ &= F(t)\left[\left(r_d - r_f\right)dt + \sigma dW_d\right] \\ &= F(t)\left[\delta_F dt + \sigma dW_d\right] \quad \text{for} \quad \delta_F := \left(r_d - r_f\right) \end{aligned} \tag{4.19}$$

Having found the governing SDE for $F(t)$ under risk neutral measure Q_d, we are ready for finding solution for $F(t)$ applying by Itô formula (for $\ln x$)

$$\begin{aligned} \ln F(T) &= \ln F(t) + \int_t^T \frac{1}{F(u)} dF(u) + \frac{1}{2}\int_t^T \frac{-1}{F^2(u)} d[F,F]_u \\ &= \ln F(t) + \int_t^T \left[\delta_F du + \sigma dW_d(u)\right] - \frac{1}{2}\int_t^T \sigma^2 du \\ &= \ln F(t) + \delta_F(T-t) + \sigma(W_d(T) - W_d(t)) - \frac{\sigma^2}{2}(T-t) \\ &= \ln F(t) + \delta_F \tilde{t} + \sigma W_d(\tilde{t}) - \frac{\sigma^2}{2}\tilde{t} \quad \text{where} \quad \tilde{t} := T - t \\ &= \ln F(t) + \left(\delta_F - \frac{\sigma^2}{2}\right)\tilde{t} + +\sigma W_d(\tilde{t}). \end{aligned}$$

Since $W_d(\tilde{t}) \overset{d}{=} \sqrt{\tilde{t}}Y$ for any $Y \sim N(0,1)$;

$$F(T) = F(t)\exp\left\{\left(\delta_F - \frac{\sigma^2}{2}\right)\tilde{t} + \sigma\sqrt{\tilde{t}}y\right\}. \tag{4.20}$$

Theorem 4.10. *Let the domestic and foreign money market accounts and current exchange rate satisfies* (4.18). *Consider the European call foreign exchange option where the payoff function is* $h(F,K,T) = (F(T) - K, 0)^+$. *For the choice of* M_d *as numeraire, the corresponding value of that option is*

$$V\left(F,K,t\right) = F(t)e^{-r_f(T-t)}N\left(d_1\right) - Ke^{-r_d(T-t)}N\left(d_2\right) \tag{4.21}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{F(t)}{K}\right) + \left(r_d - r_f + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\ln\left(\frac{F(t)}{K}\right) + \left(r_d - r_f - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Proof. According to above computations, exchange rate satisfies (4.19) and then corresponding solution is like in (4.20) under the risk neutral measure Q_d whenever M_d

is taken as numeraire. Then the corresponding option value is:

$$\begin{aligned} V(F,K,t) &= M_d(t)E_{Q_d}\left[\left.\frac{h(F,K,T)}{M_d(T)}\right| F_t\right] \\ &= e^{-r_d(T-t)}E_{Q_d}\left[h(F,K,T) \mid F_t\right] \text{ since } M_d(t)=e^{r_d t} \\ &= e^{-r_d\tilde{t}}E_{Q_d}\left[(F-K,0)^+ \mid F_t\right] \text{ for } \tilde{t}=T-t. \end{aligned}$$

Use the ideas used in the proof of valuation formula for Black-Scholes-Merton;

$$\begin{aligned} V\left(F,K,t\right) &= e^{-r_d\tilde{t}}E_{Q_d}\left[h(F,K,T) \mid F_t\right] \\ &= \frac{e^{-r_d\tilde{t}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(F,K,T)e^{-y^2/2}dy \\ &= \frac{e^{-r_d\tilde{t}}}{\sqrt{2\pi}}\int_{-d_2}^{\infty} (F(T)-K)e^{-y^2/2}dy \\ &= \frac{e^{-r_d\tilde{t}}}{\sqrt{2\pi}}\int_{-d_2}^{\infty} F(t)e^{(\delta_F-\frac{\sigma^2}{2})\tilde{t}+\sigma\sqrt{\tilde{t}}y}e^{-y^2/2}dy - Ke^{-r_d\tilde{t}}N\left(d_2\right) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} F(t)e^{-r_f\tilde{t}}e^{\frac{-1}{2}(\sigma^2\tilde{t}-2\sigma\sqrt{\tilde{t}}y+y^2)}dy - Ke^{-r_d\tilde{t}}N\left(d_2\right) \\ &= \frac{1}{\sqrt{2\pi}}\int_{\sigma\sqrt{\tilde{t}}+d_2}^{-\infty} F(t)e^{-r_f\tilde{t}}e^{-v^2/2}(-dv) - Ke^{-r_d\tilde{t}}N\left(d_2\right) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} F(t)e^{-r_f\tilde{t}}e^{-v^2/2}dv - Ke^{-r_d\tilde{t}}N\left(d_2\right) \\ &= F_te^{-r_f\tilde{t}}N\left(d_1\right) - Ke^{-r_d\tilde{t}}N\left(d_2\right) \end{aligned}$$

where $v := \sigma\sqrt{\tilde{t}} - y$, $d_1 := d_2 + \sigma\sqrt{\tilde{t}}$ and

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{F}{K}\right) + \left(r_d - r_f + \frac{\sigma^2}{2}\right)\tilde{t}}{\sigma\sqrt{t}} \\ d_2 &= \frac{\ln\left(\frac{F}{K}\right) + \left(r_d - r_f - \frac{\sigma^2}{2}\right)\tilde{t}}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{\tilde{t}}. \end{aligned}$$

The proof is completed with the back substitution of $\tilde{t} = T - t$. □

4.3.2 Foreign Exchange Option Pricing with delay

The work in [3] is extended for foreign exchange options with delay. Let domestic money market account and foreign money market account namely M_d and M_f satisfy the following equations;

$$\begin{aligned} dM_d &= M_d r_d dt \quad \text{so that} \quad M_d(t) = e^{r_d t} \\ dM_f &= M_f r_f dt \quad \text{so that} \quad M_f(t) = e^{r_f t} \end{aligned} \tag{4.22}$$

where r_d is the domestic risk free rate, and r_f is the foreign risk free rate. Let the exchange rate satisfies the following SDDE;

$$\begin{aligned} &dF = F(f(t-a)dt + g(F(t-b))dZ_F),\ t \in [0,T] \\ &F(t) = \varphi(t),\ t \in [-L,0] \end{aligned} \tag{4.23}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{0\le t\le T}$, for any positive constant fixed delays a and b so that $L = \max(a,b)$, the functions f and g are continuous, Z_F is one dimensional standard Brownian motion and $\varphi(t) : [-L,0] \to \mathbb{R}^+$ is $\mathcal{F}_0$-measurable initial path.

Theorem 4.11. *Let the exchange rate $F(t)$ satisfy* (4.23) *where the functions f and g satisfy the linear growth and local Lipschitz conditions. Then, it has a pathwise unique solution F for the given initial path. Moreover, $F(t) > 0$ almost surely for all $t \ge 0$ whenever the initial path $\varphi(t) > 0$ for all $t \in [-L,0]$.*

Proof. Since linear growth and local Lipschitz conditions are satisfied, solution for the given SDDE exists according to Existence and Uniqueness theorem. To find that solution process, we consider step by step solution for any $t \in [0,l]$ where $l := \min\{a,b\}$. Writing the equation on that interval as

$$\begin{aligned} &dF(t) = F(t)[f(\varphi_{t-a})dt + g(\varphi_{t-b})dZ_F(t)], \quad t \in [0,l] \\ &F(t) = \varphi(t), \quad t \in [-L,0] \end{aligned}$$

We define the semimartingale process N for $t \in [0,l]$ to be

$$N(t) := \int_0^t f(\varphi_{u-a})du + \int_0^t g(\varphi_{u-b})dZ_F(u)$$

so that the quadratic variation becomes

$$[N,N]_t = \int_0^t g(\varphi_{u-b})^2 du.$$

Consequently, the SDDE can be rewritten as

$$\begin{aligned} &dF(t) = F(t)dN(t), \quad t \in [0,l], \\ &F(t) = \varphi(t), \quad t \in [-L,0], \end{aligned}$$

where the unique solution for any $t \in [0, l]$ according to Doléans Dade exponent is

$$\begin{aligned} F(t) &= \varphi(0) \exp\left[N(t) - \frac{1}{2}[N, N]_t\right] \\ &= \varphi(0) \exp\left[\int_0^t f(\varphi_{u-a}) du + \int_0^t g(\varphi_{u-b}) dZ_F(u) - \frac{1}{2}\int_0^t g(\varphi_{u-b})^2 du\right]. \end{aligned}$$

Since the exponential function is always positive if $\varphi(t) > 0$ a.s., then $F(t) > 0$ for all $t \in [0, l]$. Hence using method of steps, solution for any $t \in [kl, (k+1)l]$ where $(k+1)l \leq T$ can be computed. □

To find the fair price formula for the foreign exchange option, we need the corresponding risk neutral probability measure, namely Q_d, whenever M_d is numeraire while showing that the market is arbitrage-free.

Theorem 4.12. *Let $M_d(t)$, and $M_f(t)$ be riskless assets satisfying* (4.22) *and $F(t)$ be current exchange rate satisfying* (4.23). *If $M_d(t)$ is taken as numeraire with measure Q_d generated by Brownian motion $W_d(t)$ which is equal to,*

$$dW_d = dZ_F + \frac{f(F(t-a)) + r_f - r_d}{g(F(t-b))} dt,$$

for a nonzero function g then Q_d is risk neutral probability measure and thus, market is arbitrage-free.

Proof. Define $X(t) = \dfrac{F(t)M_f(t)}{M_d(t)}$, where $F(t)M_f(t)$ represents price in terms of domestic currency and $X(t)$ represents discounted price process for foreign exchange in terms of the domestic currency. Note that $d\left(\dfrac{1}{M_d}\right) = \dfrac{-r_d}{M_d} dt$, then the corresponding

SDDE of $X(t)$ is

$$
\begin{aligned}
dX(t) &= \frac{1}{M_d(t)} d\left(F(t)M_f(t)\right) + F(t)M_f(t) d\left(\frac{1}{M_d(t)}\right) + d\left[FM_f, \frac{1}{M_d}\right]_t \\
&= \frac{1}{M_d(t)}\left[M_f(t)dF(t) + F(t)dM_f(t)\right] + F(t)M_f(t)d\left(\frac{1}{M_d(t)}\right) \\
&= \frac{1}{M_d(t)}[M_f(t)F(t)f(F(t-a))dt \\
&\qquad\qquad + M_f(t)F(t)g(F(t-b))dZ_F(t) + F(t)M_f(t)r_f dt] \\
&\qquad\qquad + F(t)M_f(t)\frac{-r_d}{M_d(t)}dt \\
&= \frac{F(t)M_f(t)}{M_d(t)}\left[\left(f(F(t-a)) + r_f - r_d\right)dt + g(F(t-b))dZ_F(t)\right] \\
&= X(t)g(F(t-b))\left[\left(\frac{f(F(t-a)) + r_f - r_d}{g(F(t-b))}\right)dt + dZ_F(t)\right].
\end{aligned}
$$

Define $\gamma(t) := -\frac{f(F(t-a)) + r_f - r_d}{g(F(t-b))}$ for all $t \in [0,T]$ which is $\mathcal{F}_{t-l}$ measurable predictable process so that $\int_0^T |\gamma(u)|^2 du < \infty$. Define

$$
Q_T := \exp\left\{-\int_0^T \gamma(u)dZ_F(u) - \frac{1}{2}\int_0^T |\gamma(u)|^2 du\right\}
$$

so that $E_Q(Q_T) = 1$. Then, according to Girsanov theorem there exists probability measure Q_d defined by $dQ_d = Q_T dQ$ and standard Brownian motion W_d under Q_d defined by

$$
W_d(t) := Z_F(t) - \int_0^t \gamma(u)du \quad \text{for all } t \in [0,T].
$$

This means:

$$
dW_d(t) = dZ_F(t) - \gamma(t)dt = dZ_F + \frac{f(F(t-a)) + r_f - r_d}{g(F(t-b))}dt, \tag{4.24}
$$

so that $X(t)$ satisfies the following equation under Q_d:

$$
dX(t) = X(t)g(F(t-b))dW_d(t), \tag{4.25}
$$

which is a martingale process. Thus, Q_d is a risk neutral measure, and the market is arbitrage-free. □

Arranging the SDDE satisfied by exchange rate F under Q_d with

$$
dZ_F = dW_d - \frac{f(F(t-a)) + r_f - r_d}{g(F(t-b))}dt
$$

according to (4.24) we obtain

$$
\begin{aligned}
dF(t) &= F(t)\left[f(F(t-a))dt + g(F(t-b))dZ_F\right] \\
&= F(t)\left[f(F(t-a))dt + g(F(t-b))\left[dW_d - \frac{f(F(t-a)) + r_f - r_d}{g(F(t-b))}dt\right]\right] \\
&= F(t)\left[(r_d - r_f)\,dt + g(F(t-b))dW_d\right] \\
&= F(t)\left[\delta_F dt + g(F(t-b))dW_d\right] \qquad (4.26)
\end{aligned}
$$

for $\delta_F := (r_d - r_f)$.

Theorem 4.13. *If the market satisfies the conditions defined in Theorem 4.12, then it is complete.*

Proof. Let Y be any contingent claim which is an integrable nonnegative $\mathcal{F}_t^{\tilde{F}}$ measurable random variable where $\tilde{F} := FM_f$. Note that $\mathcal{F}_t^{\tilde{F}} = \mathcal{F}_t^{M_d} = \mathcal{F}_t^{W_d} = \mathcal{F}_t^{Z_F}$ for any $t \geq 0$ by definition of $\tilde{F}$ and W_d. Consider Q_d-martingale process $M(t)$ for any $t \in [0, T]$ which is defined as

$$M(t) := E_{Q_d}(e^{-r_d T} Y \mid \mathcal{F}_t^{\tilde{F}}) = E_{Q_d}(e^{-r_d T} Y \mid \mathcal{F}_t^{W_d}).$$

Then there exists $\mathcal{F}_t^{W_d}$ predictable process h_0 such that $\int_0^T h_0^2(u)du < \infty$ and

$$M(t) = E_{Q_d}(e^{-r_d T} Y \mid \mathcal{F}_t^{W_d}) = E_{Q_d}(e^{-r_d T} Y) + \int_0^t h_0(u)dW_d(u) \qquad (4.27)$$

by Martingale Representation theorem. Also, we can write the process $M(t)$ as

$$M(t) = \pi_{\tilde{F}} X(t) + \pi_{M_d}, \qquad (4.28)$$

where $\pi_{\tilde{F}}(t)$ and $\pi_{M_d}(t)$ represent the amount of $\tilde{F}$ and M_d in a portfolio respectively. Therefore, from (4.27), (4.28) and (4.25) we get

$$
\begin{aligned}
dM(t) &= h_0(t)dW_d(t) \\
&= \pi_{\tilde{F}} dX(t) \\
&= \pi_{\tilde{F}} X(t) g(F(t-b))dW_d(t).
\end{aligned}
$$

This implies the hedging strategy for the portfolio $\{(\pi_{\tilde{F}}(t), \pi_{M_d}(t)) \mid t \in [0, T]\}$ to be

$$
\begin{aligned}
\pi_{\tilde{F}}(t) &= \frac{h_0(t)}{X(t)g(F(t-b))}, \\
\pi_{M_d}(t) &= M(t) - \pi_{\tilde{F}}(t)X(t).
\end{aligned}
$$

Hence, the value of the portfolio at any time $t \in [0, T]$ is given by

$$V(t) = \pi_{M_d}(t)e^{r_d t} + \pi_{\tilde{F}}(t)\tilde{F}(t) = e^{r_d t}M(t),$$

for which we have

$$\begin{aligned} dV(t) &= e^{r_d t}dM(t) + M(t)d(e^{r_d t}) \\ &= \pi_{M_d}(t)d(e^{r_d t}) + \pi_{\tilde{F}}(t)d\tilde{F}(t), \end{aligned}$$

so that $\{(\pi_{\tilde{F}}(t), \pi_{M_d}(t)) \mid t \in [0, T]\}$ is self-financing strategy and

$$V(T) = e^{r_d T}M(T) = e^{r_d T}\frac{Y}{e^{r_d T}} = Y.$$

Consequently, Y is attainable and the market consisting of $M_d(t)$ and $\tilde{F}(t)$ is complete. □

Before providing a valuation formula for the FX option under delay effect, we find a solution for $F(t)$ for any $t \in [0, T]$, using Itô formula, for $\ln x$, which satisfies (4.26) under risk neutral probability measure Q_d :

$$\begin{aligned} \ln F(T) &= \ln F(t) + \int_t^T \frac{1}{F(u)}dF(u) + \frac{1}{2}\int_t^T \frac{-1}{F^2(u)}d[F, F]_u \\ &= \ln F(t) + \int_t^T [\delta_F du + g(F(u-b))dW_d(u)] - \frac{1}{2}\int_t^T g(F(u-b))^2 du \\ &= \ln F(t) + \delta_F(T-t) + \int_t^T g(F(u-b))dW(u) \\ &\qquad - \frac{1}{2}\int_t^T g(F(u-b))^2 du. \end{aligned}$$

Note that $\int_t^T g(F(u-b))dW_d(u) \sim N\left(0, \int_t^T g(F(u-b))^2 du\right)$. Defining

$$\tilde{t} := T - t, \quad m := -\frac{1}{2}\int_t^T g(F(u-b))^2 du, \quad \sigma^2 := \int_t^T g(F(u-b))^2 du.$$

We see that, $\int_t^T g(F(u-b))dW_d(u) \overset{d}{=} \sigma Y$ for any $Y \sim N(0, 1)$ and

$$F(T) = F(t)e^{\delta_F \tilde{t}}e^{m+\sigma Y}. \tag{4.29}$$

Theorem 4.14. *Consider a European call foreign exchange option under delay effect where the current exchange rate, domestic money market account and foreign money market account satisfy* (4.22) *and* (4.23). *For the predetermined exchange rate* K *with the payoff function*

$$h(F,K,T) = (F(T) - K, 0)^{+},$$

the corresponding value of the option is:

- ***Case 1:** for* $t \in [T-l, T]$*;*

$$V(F,K,t) = F(t)e^{-r_f(T-t)}N(d_1) - Ke^{-r_d(T-t)}N(d_2), \tag{4.30}$$

  where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{x}{K}\right) - m + r_d T}{\sigma} \\ d_2 &= \frac{\ln\left(\frac{x}{K}\right) + m + r_d T}{\sigma} = d_1 - \sigma \\ x &= \widetilde{F}(t)e^{-r_f(T-t)} \\ m &= -\frac{1}{2}\int_t^T g(F(t-b))^2 du \\ \sigma^2 &= \int_t^T g(F(t-b))^2 du. \end{aligned}$$

- ***Case 2:** for* $T > l$ *and* $t < T - l$*;*

$$V(F,K,t) = e^{r_d t}E_{Q_d}\left[H\left(\tilde{F}(T-l)e^{-r_f\tau}, -\frac{1}{2}\int_{T-l}^{T} g^2 du, \int_{T-l}^{T} g^2 du\right)\middle|\, F_t\right], \tag{4.31}$$

  where

$$\begin{aligned} H\left(\gamma,\omega,\sigma^2\right) &= \gamma N(d_1) - Ke^{-r_d T}N(d_2), \\ \gamma &= \tilde{F}(T-l)e^{-r_f\tau}, \\ \omega &= -\frac{1}{2}\int_{T-l}^{T} g(F(u-b))^2 du, \\ \sigma^2 &= \int_{T-l}^{T} g(F(u-b))^2 du, \\ d_1 &= \frac{\ln\left(\frac{\gamma}{K}\right) - \omega + r_d T}{\sigma}, \\ d_2 &= \frac{\ln\left(\frac{\gamma}{K}\right) + \omega + r_d T}{\sigma} = d_1 - \sigma. \end{aligned}$$

Proof. Consider the value of this option with numeraire M_d for any $t \in [0, T]$ using (4.29), we calculate

$$
\begin{aligned}
V(F, K, t) &= M_d(t) E_{Q_d}\left[\left.\frac{h(F, K, T)}{M_d(T)}\right| F_t\right] \\
&= e^{r_d t} E_{Q_d}\left[\left(e^{-r_d T} F(T) - e^{-r_d T} K, 0\right)^+ \mid F_t\right] \\
&= e^{r_d t} E_{Q_d}\left[(e^{-r_d T} F(t) e^{\delta_F \tilde{t}} e^{m+\sigma y} - e^{-r_d T} K, 0)^+ \mid Ft\right] \\
&= e^{r_d t} E_{Q_d}\left[\left(x e^{m+\sigma y} - e^{-r_d T} K, 0\right)^+ \mid F(t)\right]
\end{aligned}
$$

where $\tilde{t} = T - t$ and $x = F(t) e^{-r_d t} e^{-r_f \tilde{t}}$. Note that $F(T)$ is F_{T-l} measurable and hence

$$
\begin{aligned}
H\left(x, m, \sigma^2\right) &:= E_{Q_d}\left(\left(e^{-r_d T} F(T) - e^{-r_d T} K, 0\right)^+ \mid F_{T-l}\right) \\
&= E_{Q_d}\left(\left(e^{-r_d T} F(T) - e^{-r_d T} K, 0\right)^+\right) \\
&= E_{Q_d}\left(\left(x e^{m+\sigma y} - e^{-r_d T} K, 0\right)^+\right) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(x e^{m+\sigma y} - e^{-r_d T} K, 0\right)^+ e^{-y^2/2} dy,
\end{aligned}
$$

since

$$
\begin{aligned}
x e^{m+\sigma y} - e^{-r_d T} K > 0 &\Rightarrow \ln x + m + \sigma y > -r_d T + \ln K \\
&\Rightarrow \sigma y > -\left(\ln x - \ln K + m + r_d T\right) \\
&\Rightarrow y > -d_2 := -\frac{\ln\left(\frac{x}{K}\right) + m + r_d T}{\sigma}.
\end{aligned}
$$

Therefore, further calculations yields:

$$
\begin{aligned}
H\left(x, m, \sigma^2\right) &= \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} x e^{-\frac{1}{2}\left(-2m+\sigma y+y^2\right)} dy - \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-r_d T} K e^{-y^2/2} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} x e^{-\frac{1}{2}\left(-2m+\sigma y+y^2\right)} dy - K e^{-r_d T} N(d_2) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} x e^{-\frac{1}{2}\left(\sigma^2+\sigma y+y^2\right)} dy - K e^{-r_d T} N(d_2), \text{ for } -2m = \sigma^2 \\
&= \frac{1}{\sqrt{2\pi}} \int_{\sigma+d_2}^{-\infty} x e^{-v^2/2} (-dv) - K e^{-r_d T} N(d_2), \text{ for } \sigma - y := v \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} x e^{-v^2/2} dv - K e^{-r_d T} N\left(d_2\right) \\
&= x N\left(d_1\right) - K e^{-r_d T} N\left(d_2\right).
\end{aligned}
$$

Thus for any $t \in [0, T]$, the value of foreign exchange option under delay effect with M_d numeraire is;

$$V(F, K, t) = e^{r_d t} E_{Q_d} \left[E_{Q_d} \left(x e^{m+\sigma y} - e^{r_d T} K \right)^+ |F_{T-l}| \, F_t \right].$$
$$= e^{r_d t} E_{Q_d} \left[H(x, m, \sigma) \mid F_t \right],$$

where

$$H(x, m, \sigma^2) = xN(d_1) - Ke^{-r_d T} N(d_2),$$

and

$$\begin{aligned}
x &= F(t) e^{-r_d t} e^{-r_f \tilde{t}} = \widetilde{F}(t) e^{-r_f \tilde{t}} \\
m &= -\frac{1}{2} \int_t^T g(F(t-b))^2 du \\
\sigma^2 &= \int_t^T g(F(t-b))^2 du \quad \text{so that} \quad m = \frac{-\sigma^2}{2} \\
d_1 &= \frac{\ln\left(\frac{x}{K}\right) - m + r_d T}{\sigma} \\
d_2 &= \frac{\ln\left(\frac{x}{K}\right) + m + r_d T}{\sigma} = d_1 - \sigma.
\end{aligned}$$

Case 1: for $t \in [T-l, T]$; since H is F_{T-l}-measurable and $F_{T-l} \subset F_t$, the valuation formula becomes

$$\begin{aligned}
V(F, K, t) &= e^{r_d t} E_{Q_d}(H(x, m, \sigma^2) \mid F_t) \\
&= e^{r_d t} H\left(x, m, \sigma^2\right) \\
&= e^{r_d t} \left[xN(d_1) - Ke^{-r_d T} N(d_2) \right] \\
&= F(t) e^{-r_f \tilde{t}} N(d_1) - Ke^{-r_d \tilde{t}} N(d_2).
\end{aligned}$$

Case 2: for $T > l$ and $t < T - l$, we consider to write $F(T)$ in (4.29) in terms of $F(T-l)$, i.e., instead of t in that formula write $T - l$:

$$\begin{aligned}
\ln F(T) = \ln F(T-l) &+ \int_{T-l}^T \frac{1}{F(u)} dF(u) + \frac{1}{2} \int_{T-l}^T \frac{-1}{F^2(u)} d[F, F]_u \\
= \ln F(T-l) &+ \delta_F(l) + \int_{T-l}^T g(F(u-b)) dW_d(u) \\
&- \frac{1}{2} \int_{T-l}^T g(F(u-b))^2 du
\end{aligned}$$

Since $\int_{T-l}^{T} g(F(u-b))dW_d(u) \sim N\left(0, \int_{T-l}^{T} g(F(u-b))^2 du\right)$, we have

$$\int_{T-l}^{T} g(F(u-b))dW_d(u) \stackrel{d}{=} \sigma Y$$

for any $Y \sim N(0,1)$ where

$$\omega := -\frac{1}{2}\int_{T-l}^{T} g(F(u-b))^2 du, \quad \sigma^2 := \int_{T-l}^{T} g(F(u-b))^2 du.$$

Then,

$$F(T) = F(T-l)e^{\delta_F l} e^{\omega+\sigma Y}.$$

Thus, corresponding value of option when $T > l$ and $t < T-l$ becomes

$$V(F,K,t) = e^{r_d t} E_{Q_d}\left[H\left(\tilde{F}(T-l)e^{-r_f \tau}, -\frac{1}{2}\int_{T-l}^{T} g^2 du, \int_{T-l}^{T} g^2 du\right)\middle| F_t\right]$$

where

$$\begin{aligned} H\left(\gamma,\omega,\sigma^2\right) &= \gamma N\left(d_1\right) - Ke^{-r_d T} N\left(d_2\right) \\ \gamma &= \tilde{F}(T-l)e^{-r_f l} \\ \omega &= -\frac{1}{2}\int_{T-l}^{T} g(F(u-b))^2 dU \\ \sigma^2 &= \int_{T-l}^{T} g(F(u-b))^2 du \\ d_1 &= \frac{\ln\left(\frac{\gamma}{K}\right) - \omega + r_d T}{\sigma} \\ d_2 &= \frac{\ln\left(\frac{\gamma}{K}\right) + \omega + r_d T}{\sigma} = d_1 - \sigma. \end{aligned}$$

The proof is completed. □

4.4 European Exchange Option Pricing with GBM

Exchange options allow investors to exchange one asset for another at a predetermined date. These options are crucial in managing risks and optimizing returns in dynamic market conditions. The value formula for the exchange options without delay under the setting of the Black-Scholes-Merton model was derived by William

Margrabe in 1978 [27]. Some extensions of the work without delay can be found in [10, 32]. To improve the model, delay terms are added to the system. As an extension of Arriojas et al. [3], Lin et al. in [22] provide a value formula for exchange options under delay effect where assets follow GBM.

4.4.1 Exchange Option Pricing without delay

Consider a European call exchange option with asset S_1 and S_2 so that exchange S_2 with S_1 at maturity T with the terminal payoff

$$h(S_1, S_2, T) = (S_1(T) - S_2(T), 0)^+.$$

Let the assets satisfy the following SDEs under risk neutral probability measure:

$$\begin{aligned} dS_1 &= S_1[\delta_{S_1} dt + \sigma_1 dW_1],\ t \in [0, T], \\ S_1(0) &= s_1, \\ dS_2 &= S_2[\delta_{S_2} dt + \sigma_2 dW_2],\ t \in [0, T], \\ S_2(0) &= s_2, \end{aligned} \tag{4.32}$$

where $\delta_{S_i} = r - q_i$ for $i = 1, 2$, r is risk free rate, $q_1,\ q_2$ are the divident yields, $\sigma_1,\ \sigma_2$ are constant volatility and the Wiener processes W_1 and W_2 are correlated with the correlation coefficient ρ.

Theorem 4.15 (See [27]). *Consider the European call exchange option with two assets, S_1 and S_2, satisfying* (4.32)*. Then the corresponding value of the option for any $t \in [0, T]$ is*

$$V(S_1, S_2, t) = e^{-q_1 \tilde{t}} S_1(t)\phi(d_1) - e^{-q_2 \tilde{t}} S_2(t)\phi(d_2), \tag{4.33}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_1(t)}{S_2(t)} + (q_2 - q_1 + \frac{\sigma^2}{2})\tilde{t}}{\sigma\sqrt{\tilde{t}}}, \\ d_2 &= \frac{\ln \frac{S_1(t)}{S_2(t)} + (q_2 - q_1 - \frac{\sigma^2}{2})\tilde{t}}{\sigma\sqrt{\tilde{t}}} = d_1 - \sigma\sqrt{\tilde{t}}. \end{aligned}$$

Proof. First, we define $S = \frac{S_1}{S_2}$ where the corresponding payoff function can be rewritten

$$h(T) = S_2(T)\left(\frac{S_1(T)}{S_2(T)} - 1, 0\right)^+ = S_2(T)\left(S(T) - 1, 0\right)^+$$

and the corresponding SDE turns to satisfy

$$dS(t) = \frac{1}{S_2}dS_1 + S_1 d\left(\frac{1}{S_2}\right) + d\left[S_1, \frac{1}{S_2}\right]_t. \tag{4.34}$$

Hence the corresponding SDE for $1/S_2(t)$, using Itô formula for $1/x$ can be obtained as follows:

$$\frac{1}{S_2(t)} = \frac{1}{S_2(0)} + \int_0^t \frac{-1}{S_2^2(u)}dS_2(u) + \frac{1}{2}\int_0^t \frac{2}{S_2^3(u)}d\left[S_2, S_2\right]_u,$$

so that

$$\begin{aligned} d\left(\frac{1}{S_2(t)}\right) &= \frac{-1}{S_2(t)}[\delta_{S_2}dt + \sigma_2 dW_2] + \frac{1}{S_2(t)}\sigma_2^2 dt \\ &= \frac{1}{S_2(t)}\left[\left(-\delta_{S_2} + \sigma_2^2\right)dt - \sigma_2 dW_2\right]. \end{aligned}$$

The SDE for $S(t) = S_1(t)/S_2(t)$ according to (4.34) therefore satisfies:

$$\begin{aligned} dS(t) &= \frac{1}{S_2}S_1[\delta_{S_1}dt + \sigma_1 dW_1] + S_1\frac{1}{S_2}[(-\delta_{S_2} + \sigma_2^2)dt - \sigma_2 dW_2] + \rho S_1\sigma_1\frac{-\sigma_2}{S_2}dt \\ &= S[(\delta_{S_1} - \delta_{S_2} + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 dW_1 - \sigma_2 dW_2] \\ &= S[(\delta_S + \sigma_2^2 - \rho\sigma_1\sigma_2)dt + \sigma_1 dW_1 - \sigma_2 dW_2] \\ &= S[\delta_S dt + \sigma_1[(dW_1 - \rho\sigma_2 dt] - \sigma_2[(dW_2 - \sigma_2 dt]] \end{aligned}$$

where $\delta_S := \delta_{S_1} - \delta_{S_2} = q_2 - q_1$. To find the corresponding value formula, we consider the change of numeraire according to [16]. Taking $S_2(t)e^{q_2 t}$ as numeraire where

$$S_2(t) = S_2(0)e^{(\delta_{S_2} - \frac{-\sigma_2^2}{2})t + \sigma_2 W_2(t)}$$

by analytic solution of GBM, we see that

$$\frac{dQ_2}{dQ} := \frac{S_2(t)e^{q_2 t}}{S_2(0)}e^{-rt} = e^{\frac{-\sigma_2^2}{2}t + \sigma_2 W_2(t)}$$

is Radon Nikodym derivative where $\gamma = -\sigma_2$ in Girsanov's theorem and

$$d\tilde{W}_1 := dW_1 - \rho\sigma_2 dt \text{ and } d\tilde{W}_2 := dW_2 - \sigma_2 dt$$

are Brownian Motions under measure Q_2 by Girsanov's theorem. So $S(t)$ satisfies the following equation under Q_2:

$$dS(t) = S[\delta_S dt + \sigma_1 d\tilde{W}_1 - \sigma_2 d\tilde{W}_2].$$

Note that $\sigma_1 d\tilde{W}_1 - \sigma_2 d\tilde{W}_2 \sim N(0, \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$. Then for any Wiener process W

$$\sigma_1 d\tilde{W}_1 - \sigma_2 d\tilde{W}_2 \stackrel{d}{=} \sigma dW$$

where $\sigma^2 := \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$. So,

$$dS(t) = S(t)[\delta_S dt + \sigma dW], \quad \text{where} \quad \delta_S = q_2 - q_1.$$

Furthermore, for $S(T)$, using Itô formula, we find that under Q_2;

$$\begin{aligned}
\ln S(T) &= \ln S(t) + \int_t^T \frac{1}{S(u)} dS(u) - \frac{1}{2}\int_t^T \frac{1}{S^2(u)} d\left[S, S\right]_u \\
&= \ln S(t) + \int_t^T (\delta_S du + \sigma dW(u)) - \frac{1}{2}\int_t^T \sigma^2 du \\
&= \ln S(t) + \left(\delta_S - \frac{\sigma^2}{2}\right)(T - t) + \sigma\left(W(T) - W(t)\right),
\end{aligned}$$

where $(W(T) - W(t)) \sim N\left(0, T - t\right)$. Then defining $\tilde{t} := T - t$, and $Y \sim N(0, 1)$ we can write $(W(T) - W(t)) \stackrel{d}{=} \sqrt{\tilde{t}}Y$. Thus

$$\begin{aligned}
\ln S(T) &= \ln S(t) + \left(\delta_S - \frac{\sigma^2}{2}\right)\tilde{t} + \sigma\sqrt{\tilde{t}}Y, \\
S(T) &= S(t)\exp\left\{\left(\delta_S - \frac{\sigma^2}{2}\right)\tilde{t} + \sigma\sqrt{\tilde{t}}Y\right\}.
\end{aligned}$$

Therefore, the corresponding exchange option value with numeraire $S_2(t)e^{q_2 t}$ becomes

$$\begin{aligned}
V(S_1, S_2, t) &= S_2(t)e^{q_2 t} E_{Q_2}\left[\left.\frac{h(T)}{S_2(T)e^{q_2 T}}\right| F_t\right] \\
&= S_2(t)e^{-q_2 \tilde{t}} E_{Q_2}\left[(S(T) - 1, 0)^+ \mid F_t\right].
\end{aligned}$$

Note also that

$$\begin{aligned}
S(T) - 1 > 0 &\Rightarrow S(t)\exp\left\{\left(\delta_S - \frac{\sigma^2}{2}\right)\tilde{t} + \sigma\sqrt{\tilde{t}}Y\right\} - 1 > 0 \\
&\Rightarrow \ln S(t) + \left(\delta_S - \frac{\sigma^2}{2}\right)\tilde{t} + \sigma\sqrt{\tilde{t}}Y > \ln 1 = 0 \\
&\Rightarrow Y > -d_2 := -\frac{\ln S(t) + \left(\delta_S - \frac{\sigma^2}{2}\right)\tilde{t}}{\sigma\sqrt{\tilde{t}}}.
\end{aligned}$$

Then, further calculations yields:

$$
\begin{aligned}
V(S_1,S_2,t) &= S_2(t)e^{-q_2\tilde{t}}\frac{e^{r\tilde{t}}}{e^{r\tilde{t}}}E_{Q_2}\left[(S(T)-1,0)^+ \mid F_t\right]\\
&= \frac{S_2(t)e^{\delta_{S_2}\tilde{t}}}{e^{r\tilde{t}}}\left[\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}(S(T)-1)e^{-y^2/2}dy\right]\\
&= \frac{S_2(t)e^{\delta_{S_2}\tilde{t}}}{e^{r\tilde{t}}}\left[\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}S(t)e^{(\delta_S-\frac{\sigma^2}{2})\tilde{t}+\sigma\sqrt{\tilde{t}}y}e^{-y^2/2}dy-\phi\left(d_2\right)\right]\\
&= \frac{S_2(t)e^{\delta_{S_2}\tilde{t}}}{e^{r\tilde{t}}}\left[\frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}S(t)e^{\delta_S\tilde{t}}e^{\frac{-1}{2}(\sigma^2\tilde{t}-2\sigma\sqrt{\tilde{t}}y+y^2)}dy-\phi\left(d_2\right)\right]\\
&= \frac{S_2(t)e^{\delta_{S_2}\tilde{t}}}{e^{r\tilde{t}}}\left[\frac{1}{\sqrt{2\pi}}\int_{\sigma\sqrt{\tilde{t}}+d_2}^{-\infty}S(t)e^{\delta_S\tilde{t}}e^{-v^2/2}(-dv)-\phi\left(d_2\right)\right]\\
&= \frac{S_2(t)e^{\delta_{S_2}\tilde{t}}}{e^{r\tilde{t}}}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1}S(t)e^{\delta_S\tilde{t}}e^{-v^2/2}dv-\phi\left(d_2\right)\right]\\
&= \frac{S_2(t)e^{\delta_{S_2}\tilde{t}}}{e^{r\tilde{t}}}\left[S(t)e^{\delta_S\tilde{t}}\phi(d_1)-\phi\left(d_2\right)\right]\\
&= e^{-r\tilde{t}}\left[S_1(t)e^{\delta_{S_1}\tilde{t}}\phi(d_1)-S_2(t)e^{\delta_{S_2}\tilde{t}}\phi(d_2)\right]\\
&= e^{-q_1\tilde{t}}S_1(t)\phi(d_1)-e^{-q_2\tilde{t}}S_2(t)\phi(d_2)
\end{aligned}
$$

for the choice of $v := \sigma\sqrt{\tilde{t}}-y$ and $d_1 := \sigma\sqrt{\tilde{t}}+d_2$, where

$$
\begin{aligned}
d_1 &= \frac{\ln\frac{S_1(t)}{S_2(t)}+(\delta_{S_1}-\delta_{S_2}+\frac{\sigma^2}{2})\tilde{t}}{\sigma\sqrt{\tilde{t}}},\\
d_2 &= \frac{\ln\frac{S_1(t)}{S_2(t)}+(\delta_{S_1}-\delta_{S_2}-\frac{\sigma^2}{2})\tilde{t}}{\sigma\sqrt{\tilde{t}}} = d_1-\sigma\sqrt{\tilde{t}}.
\end{aligned}
$$

The proof is completed for this $\tilde{t} = T-t$. □

Now, we consider adding the delay term to that model.

4.4.2 Exchange Option Pricing with delay

This section depends on the work in [22], which is deal with exchange option under delay effect. We consider a market consisting of three assets $B(t)$, $S_1(t)$ and $S_2(t)$, where they represent the prices of risk free assets and two underlying assets at time t, respectively. They satisfy the following:

$$
\begin{aligned}
&dS_i(t) = \mu_i S_i(t-a_i)S_i(t)dt + g_i(S_i(t-b_i))S_i(t)d\hat{W}_i(t),\ t\in(0,T],\\
&S_i(t) = \varphi_i(t),\ t\in[-L_i,0],
\end{aligned}
\tag{4.35}
$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}_t = \{\mathcal{F}_t^1, \mathcal{F}_t^2\}_{0\leq t\leq T}$ satisfing the usual conditions. The terms in the equations are:

- r is risk free rate,
- μ_i , a_i and b_i are positive constants, where $L_i = \max\{a_i, b_i\}$ for $i = 1, 2$,
- ρ is the correlation coefficient between W_i's,
- $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function,
- the $\hat{W_i}(t)$ are a one-dimensional standard Brownian motions for $i = 1, 2$,
- $\varphi_i(t) : [-L_i, 0] \to \mathbb{R}^+$ is $\mathcal{F}_0$-measurable random variables such that $\varphi_i(0) > 0$ a.s. for $i = 1, 2$.

According to Arrijoes [3], the equation in (4.35) has a unique solution

$$S_i(t) = \varphi_i(0)\exp\Bigg[\int_0^t (\mu_i S_i(u-a_i) - \frac{1}{2}g_i^2(S_i(u-b_i)))du + \int_0^t g_i(S_i(u-b_i))d\hat{W}_i(u)\Bigg]$$

for any $t \in [0, l_i]$ where $l_i = \min\{a_i, b_i\}$ for $i = 1, 2$. While using the induction method, one can find a solution for any $t \in [kl_i, (k+1)l_i]$ where $(k+1)l_i \leq T$. From this equation, it is easily seen that if $\varphi_i(t) > 0$ a.s, then $S_i(t) > 0$ for all $t \in [0, T]$.

With the multi-dimensional Girsanov theorem, stock price processes in (4.35) can be written as

$$\begin{aligned} dS_i(t) &= rS_i(t)t + g_i(S_i(t-b_i))S_i(t)dW_i(t), \quad t \in (0, T], \\ S_i(t) &= \varphi_i(t), \quad t \in [-L_i, 0], \end{aligned} \tag{4.36}$$

where the W_i are standard Brownian motion under the risk neutral measure Q with correlation coefficient ρ.

Lemma 4.16. *Let $S_1(t)$ and $S_2(t)$ be the prices of two underlying assets at time t satisfying* (4.36). *If $S_2(t)$ is taken as a numeraire with the associated measure Q_2 then*

$$\begin{aligned} \tilde{W_1}(t) &= W_1(t) - \rho\int_0^t g_2(S_2(u-b_2))du, \\ \tilde{W_2}(t) &= W_2(t) - \int_0^t g_2(S_2(u-b_2))du, \end{aligned} \tag{4.37}$$

define Brownian motions under Q_2 equivalent to the risk neutral measure Q.

The proof can be found in [22].

Defining $S(t) = S_1/S_2$ as the case in without delay and finding corresponding SDDE as

$$dS(t) = \frac{1}{S_2}dS_1 + S_1 d\left(\frac{1}{S_2}\right) + d\left[S_1, \frac{1}{S_2}\right],$$

we calculate the corresponding SDDE for $1/S_2(t)$ using Itô formula for $1/x$ as follows:

$$\begin{aligned}
\frac{1}{S_2(t)} &= \frac{1}{S_2(0)} + \int_0^t \frac{-1}{S_2^2(u)} dS_2(u) + \frac{1}{2}\int_0^t \frac{2}{S_2^3(u)} d\left[S_2, S_2\right]_u \\
d\left(\frac{1}{S_2(t)}\right) &= \frac{-1}{S_2(t)}[rdt + g_2 dW_2] + \frac{1}{S_2(t)} g_2^2 dt \\
&= \frac{1}{S_2(t)}\left[\left(-r + g_2^2\right)dt - g_2 dW_2\right]
\end{aligned}$$

for $g_1 = g_1(S_1(t-b_1))$ and $g_2 = g_2(S_2(t-b_2))$. Furthermore,

$$\begin{aligned}
dS(t) &= \frac{1}{S_2} S_1[rdt + g_1 dW_1] + S_1 \frac{1}{S_2}[(-r + g_2^2)dt - g_2 dW_2] + \rho \frac{S_1 g_1(-g_2)}{S_2} dt \\
&= S[(r - r + g_2^2 - \rho g_1 g_2)dt + g_1 dW_1 - g_2 dW_2]. \\
&= S[g_1[(dW_1 - \rho g_2 dt] - g_2[(dW_2 - g_2 dt]] \\
&= S[g_1 d\tilde{W}_1 - g_2 d\tilde{W}_2],
\end{aligned}$$

where $d\tilde{W}_1 := dW_1 - \rho g_2 dt$ and $d\tilde{W}_2 := dW_2 - g_2 dt$ are Brownian Motions under Q_2 according to Lemma 4.16.

$g_1 d\tilde{W}_1 - g_2 d\tilde{W}_2$ is normally distributed with mean 0 and variance g^2 where $g^2 := g_1^2 + g_2^2 - 2\rho g_1 g_2$. Hence,

$$dS(t) = S(t) g dW$$

for $W \sim N(0,1)$ since $g_1 d\tilde{W}_1 - g_2 d\tilde{W}_2 \overset{d}{=} g dW.$

The solution $S(T)$ using Itô with $\ln x$ can be found as follows:

$$\begin{aligned}
\ln S(T) &= \ln S(t) + \int_t^T \frac{1}{S(u)} dS(u) - \frac{1}{2}\int_t^T \frac{1}{S^2(u)} d[S,S]_u \\
&= \ln S(t) + \int_t^T g dW(u) - \frac{1}{2}\int_t^T g^2 du
\end{aligned}$$

so that

$$S(T) = S(t)\exp\left\{-\frac{1}{2}\int_t^T g^2 du + \int_t^T g dW(u)\right\},$$

where $\int_t^T gdW(u) \sim N\left(0, \int_t^T g^2 du\right)$.

Define $x := S(t)$, $m := -\frac{1}{2}\int_t^T g^2 du$ and $\sigma^2 := \int_t^t g^2 du$ so that $\sigma^2 = -2m$ and for any $Y \sim N(0,1)$, $\int_t^T gdW(u) \overset{d}{=} \sigma Y$. Thus

$$S(T) = xe^{m+\sigma Y}. \tag{4.38}$$

Theorem 4.17. *Consider the European call exchange option under delay effect with two assets, S_1 and S_2, satisfying* (4.36) *with the payoff function*

$$h(S_1, S_2, T) = (S_1(T) - S_2(T), 0)^+.$$

Then the corresponding value of option for any $t \in [0,T]$:

- ***Case 1:** for $t \in [T-l, T]$;*

$$V(S_1, S_2, t) = S_1(t)\phi(d_1) - S_2(t)\phi(d_2) \tag{4.39}$$

where

$$d_1 = \frac{\ln x + m + \sigma^2}{\sigma}$$
$$d_2 = \frac{\ln x + m}{\sigma} = d_1 - \sigma.$$

with $x = S(t)$, $m = -\frac{1}{2}\int_t^T g^2 du$ and $\sigma^2 = \int_t^t g^2 du$.

- ***Case 2:** for $T > l$ and $t < T - l$;*

$$V(S_1, S_2, t) = S_2(t) E_{Q_2}\left[H(S(T-l), -\frac{1}{2}\int_{T-l}^T g^2 du, \int_{T-l}^T g^2 du) \middle| F_t \right] \tag{4.40}$$

where

$$H\left(\gamma, \omega, \sigma^2\right) = \gamma\phi(\beta_1) - \phi(\beta_2)$$
$$\gamma = S(T-l) = \frac{S_1(T-l)}{S_2(T-l)}$$
$$\beta_1 = \frac{\ln\gamma + \omega + \sigma^2}{\sigma}$$
$$\beta_2 = \frac{\ln\gamma + \omega}{\sigma} = \beta_1 - \sigma.$$

Proof. Consider the value of that option with numeraire $S_2(t)$;

$$\begin{aligned} V(S_1,S_2,t) &= S_2(t)E_{Q_2}\left[\left.\frac{h(T)}{S_2(T)}\right| F_t\right] \\ &= S_2(t)E_{Q_2}\left[(S(T)-1,0)^+|F_t\right] \\ &= S_2(t)EQ_2\left[\left(xe^{m+\sigma Y}-1,0\right)^+\left|F_t\right.\right] \end{aligned} \tag{4.41}$$

by (4.38) and note that $\frac{h(T)}{S_2(T)}$ is F_{T-l} measurable, where $l = \min\{b_1, b_2\}$. We then define:

$$\begin{aligned} H(x,m,\sigma^2) &:= E_{Q_2}\left(\left(xe^{m+\sigma Y}-1,0\right)^+|F_{T-l}\right) \\ &= E_{Q_2}\left(\left(xe^{m+\sigma Y}-1,0\right)^+\right) \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(xe^{m+\sigma y}-1,0\right)^+ e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}\left(xe^{m+\sigma y}-1\right) e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} xe^{m+\sigma y}e^{-y^2/2}dy - \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty} xe^{-(\sigma^2-2\sigma y+y^2)/2}dy - \phi(d_2)) \quad \text{since} \quad m=-\sigma^2/2 \\ &= \frac{1}{\sqrt{2\pi}}\int_{(\sigma+d_2)}^{-\infty} xe^{-v^2/2}(-dv) - \phi\left(d_2\right) \quad \text{where} \quad v=\sigma-y \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} xe^{-v^2/2}dv - \phi\left(d_2\right) \quad \text{where} \quad d_1=\sigma+d_2 \\ &= x\phi(d_1)-\phi(d_2). \end{aligned}$$

Therefore, the formula for the value of the option for any $t \in (0,T)$ can be calculated as follows:

$$\begin{aligned} V(S_1,S_2,t) &= S_2(t)E_{Q_2}\left[\left(xe^{m+\sigma y}-1,0\right)^+|F_t\right] \\ &= S_2(t)E_{Q_2}\left[E_{Q_2}\left(\left(xe^{m+\sigma y}-1,0\right)^+|F_{T-l}\right)|F_t\right] \\ &= S_2(t)E_{Q_2}\left[E_{Q_2}\left(\left(xe^{m+\sigma y}-1,0\right)^+\right)|F_t\right] \\ &= S_2(t)E_{Q_2}\left[H(x,m,\sigma^2)|F_t\right] \\ &= S_2(t)E_{Q_2}\left[(x\phi\left(d_1\right)-\phi\left(d_2\right))|F_t\right] \end{aligned} \tag{4.42}$$

where $x = S(t) = \frac{S_1(t)}{S_2(t)}$, $m = -\frac{1}{2}\int_t^T g^2du$, $\sigma^2 = \int_t^t g^2du$, $d_1 = \dfrac{\ln x + m + \sigma^2}{\sigma}$ and $d_2 = \dfrac{\ln x + m}{\sigma} = d_1 - \sigma$.

Case 1: for $t \in [T-l, T]$; since H is F_{T-l}-measurable and $F_{T-l} \subset F_t$, (4.42) turns to be

$$\begin{aligned} V(S_1, S_2, t) &= S_2(t) H(x, m, \sigma^2) \\ &= S_2(t) \left[x\phi\left(d_1\right) - \phi\left(d_2\right) \right] \end{aligned}$$

where $x = S(t) = \frac{S_1(t)}{S_2(t)}$ and hence,

$$V(S_1, S_2, t) = S_1(t)\phi\left(d_1\right) - S_2(t)\phi\left(d_2\right).$$

Case 2: for $T > l$ and $t < T - l$, consider to write $S(T)$ in terms of $S(T-l)$ in (4.41) where

$$S(T) = S(T-l) \exp\left\{ -\frac{1}{2} \int_{T-l}^{T} g^2 du + \int_{T-l}^{T} g dW(u) \right\}.$$

Defining the parameters

$$\gamma = S(T-l),\ \omega = \frac{1}{2} \int_{T-l}^{T} g^2 du,\ \sigma^2 = \int_{T-l}^{T} g^2 du$$

we get:

$$\begin{aligned} V(S_1, S_2, t) &= S_2(t) E_{Q_2} \left[H(S(T-l), -\frac{1}{2} \int_{T-l}^{T} g^2 du, \int_{T-l}^{T} g^2 du) \middle| \, F_t \right] \\ &= S_2(t) E_{Q_2} [H\left(\gamma, \omega, \sigma^2\right) \big| \, F_t], \end{aligned}$$

where

$$\begin{aligned} H\left(\gamma, \omega, \sigma^2\right) &= \gamma\phi\left(\beta_1\right) - \phi\left(\beta_2\right) \\ \beta_1 &= \frac{\ln \gamma + \omega}{\sigma} + \sigma \\ \beta_2 &= \frac{\ln \gamma + \omega}{\sigma}. \end{aligned}$$

This completes the proof. □

CHAPTER 5

NUMERICAL IMPLEMENTATIONS

For numerical implementations and simulations we use Julia programming. Julia is a high-level, high-performance programming language designed for technical and numerical computing. It combines the ease of use of languages like Python with the speed of languages like C. It has a growing ecosystem of packages for various applications, including finance, statistics, data science, machine learning and more. Moreover, fortunately it also includes packages for solving complex mathematical problems, including SDEs and SDDEs. The "`DifferentialEquations.jl`", along with its specialized sub-packages like "`StochasticDiffEq.jl`" and "`StochasticDelayDiffEq.jl`", provides robust tools for researchers and practitioners in various fields.

In this chapter, we use "`StochasticDelayDiffEq.jl`" packages of Julia for the implementations of our models that we provide in Chapter 4 for pricing the options and compare values obtained from MC while considering the convergence of MC method. The value formulas for European options under delay effect is provided in two cases, either $\tau \geq T$ or $\tau < T$, where τ represents the maximum positive time delay. When we examine those formulas, we have an exact solution actually whenever $\tau \geq T$ and the semi-closed formula for $\tau < T$. In that semi-closed forms in equation (4.14), (4.31) and (4.40) we need to apply Monte Carlo to compute expectation of function H; create $nsim_1$ number of paths in interval $[0, T-l]$ and then for each path, create $nsim_2$ number of paths in the interval $[T-l, T]$ to obtain the corresponding value. Thus, we use the Monte Carlo method from the very beginning whenever $\tau < T$. In our implementations, we check the effect of initial path and delay term to model.

5.1 Implementation of European Vanilla Call Option with Delayed GBM

According to the work of Arrijoes et. al. in [3], stock price under delay effect satisfies the following SDDE;

$$\begin{aligned} &dS(t) = rS(t)dt + g(S(t-\tau))S(t)dW(t),\ t \in (0,T], \\ &S(t) = \varphi(t),\ t \in [-\tau, 0], \end{aligned} \tag{5.1}$$

under the risk neutral measure Q. Take $g(S(t-\tau)) = \sigma + \tau \exp\left\{\dfrac{-S(t-\tau)}{\alpha}\right\}$ so that whenever delay goes to zero, model turn to without delay case for some positive constant α. We use the parameter α which is taken to be $\alpha = S(0)$ in most cases to control the effect of large or small asset prices. To see the effect of the delay term in our model, we consider the multiplication of τ with $\exp\left\{\dfrac{-S(t-\tau)}{\alpha}\right\}$ so that this multiplier is bounded and not so big.

We simulate $2^{14} = 16384$ sample paths of the stock with the choice of $r = 0.05$, $\sigma = 0.2$, $dt = 0.01$, $T = 1.0$, $K = 1.0$ for the valuation of European vanilla call option and search for the effect of delay term, stock price and initial path on the valuation process.

In Table 5.1; we take delay terms as $\{0, 0.001, 0.1, 0.25, 0.5, 1.0, 1.25, 1.5, 2.0\}$ and initial paths as $\varphi_1(t) = e^t$, $\varphi_2(t) = 2 - e^t$ and $\varphi_3(t) = 1.0$. Note that initial paths have different characteristics (increasing, decreasing and constant) but have the same initial value, $S(0) = 1.0$. MC_i represents the value obtained by using Monte Carlo method, V_i represents formula price obtained from (4.13), P is the value of option for the no-delay case and $|CI_i|$ represents the length of confidence interval for the initial paths $\varphi_i(t)$ for $i = 1, 2, 3$. We consider the change in Monte Carlo prices, formula prices (whenever $\tau \geq T$) and confidence intervals. It is seen that MC_i is close to V_i value whenever the valuation formula is applicable for $i = 1, 2, 3$. Moreover for the decreasing delay terms, the values MC_i's are getting closer to P the case without delay and for too small delay terms MC_i values are almost same since in that case initial paths don't have much effect on the process. However, the difference between

Table 5.1: Vanilla Call Option Values for Different Delays and Initial Paths

τ	MC_1	V_1	MC_2	V_2	MC_3	V_3	P	$\lvert CI_1\rvert$	$\lvert CI_2\rvert$	$\lvert CI_3\rvert$
2.0	0.6288	0.2334	0.2338	0.1910	0.3758	0.2815	-	0.1127	0.0140	0.0309
1.5	0.4757	0.4089	0.2181	0.1881	0.3107	0.2685	-	0.0513	0.0126	0.0220
1.25	0.3909	0.3428	0.2108	0.1824	0.2772	0.2422	-	0.0336	0.0120	0.0182
1.0	0.3090	0.2794	0.2034	0.1765	0.2432	0.2156	-	0.0219	0.0113	0.0148
0.5	0.1830	-	0.1679	-	0.1741	-	-	0.0094	0.0084	0.0088
0.25	0.1403	-	0.1384	-	0.1393	-	-	0.0065	0.0064	0.0065
0.1	0.1185	-	0.1184	-	0.1184	-	-	0.0052	0.0052	0.0052
0.001	0.1051	-	0.1051	-	0.1051	-	-	0.0046	0.0046	0.0046
0.0	0.1046	-	0.1046	-	0.1046	-	0.1045	0.0045	0.0045	0.0045

MC_i values increases whenever the delay term increases because of the effect of initial paths. Similarly, $|CI_i|$ values are decreasing and getting almost equal to each other with the decrease in delay terms.

In Figure 5.1, the price of the option for the different initial paths φ_i is obtained while using the Monte Carlo method where delay terms are taken in $[0, 2]$. Note that although $\varphi_2(t) \geq \varphi_3(t) \geq \varphi_1(t)$ for any $t \in [-\tau, 0]$, we have $MC_2 \leq MC_3 \leq MC_1$ where MC_i represents the Monte Carlo value in the path φ_i since φ_1 is increasing, φ_2 is decreasing and φ_3 is constant functions.

In Figure 5.2, we consider the convergence of MC for the initial path $\varphi(t) = 1.0$. In

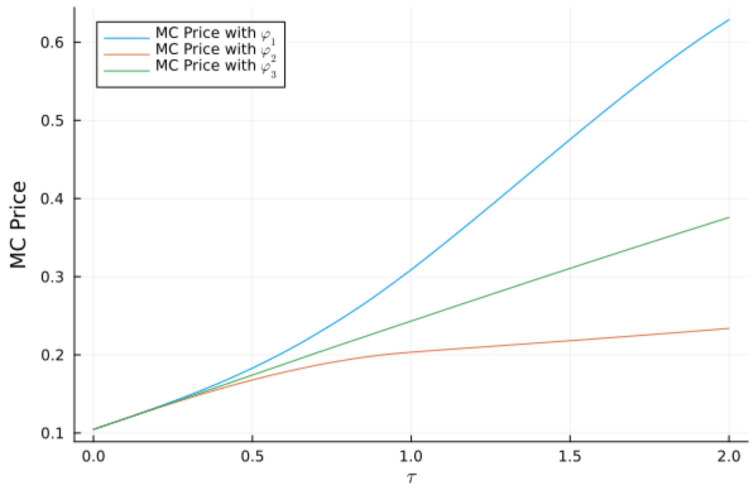
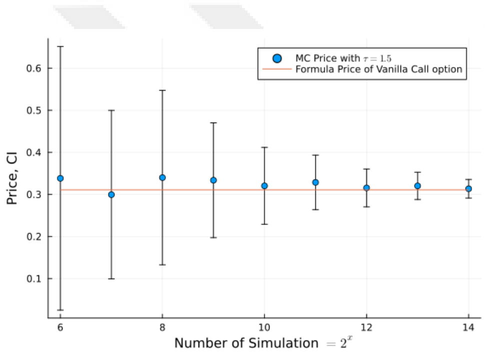
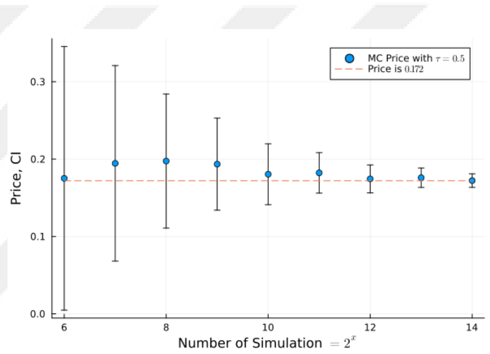


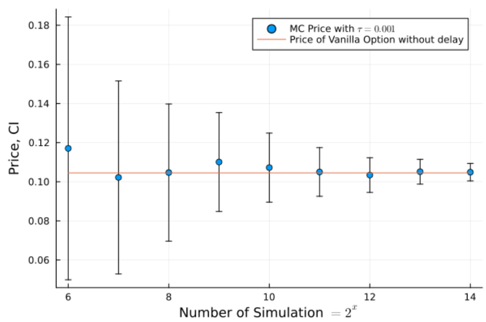
Figure 5.1: European vanilla call option price change with respect to delay term for the different choice of initial path



(a) $\tau = 1.5 > T$



(b) $\tau = 0.5 < T$



(c) $\tau = 0.001 \approx 0$

Figure 5.2: Convergence of Monte Carlo for Vanilla Call Option

Figure 5.2a and Figure 5.2c, Monte Carlo method for $\tau = 1.5$ and $\tau = 0.001$ converges to formula price V and exact value P without delay with shrinking confidence intervals as the number of simulation increases $\{2^6, 2^7, \cdots, 2^{14}\}$. In Figure 5.2b, MC price converges to our reference price 0.172, which is the option's value according to the Monte Carlo method with 2^{14} number of simulation.

Similar results can also be obtained for the put options.

In Figure 5.3, the effect of the stock price on the put and call option is seen where the

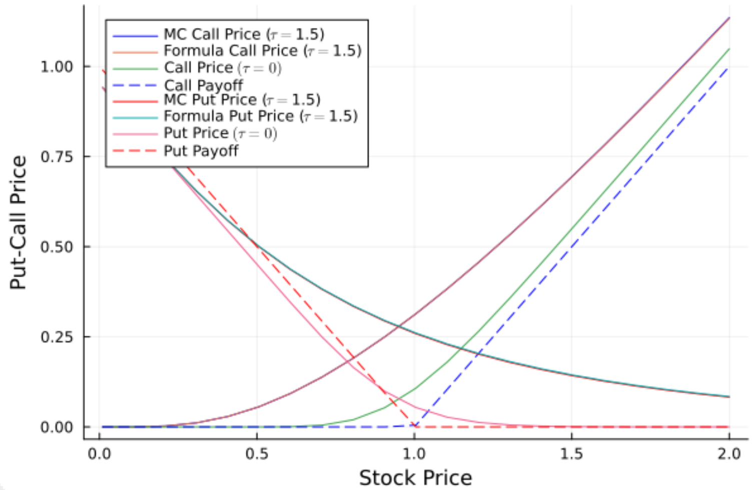
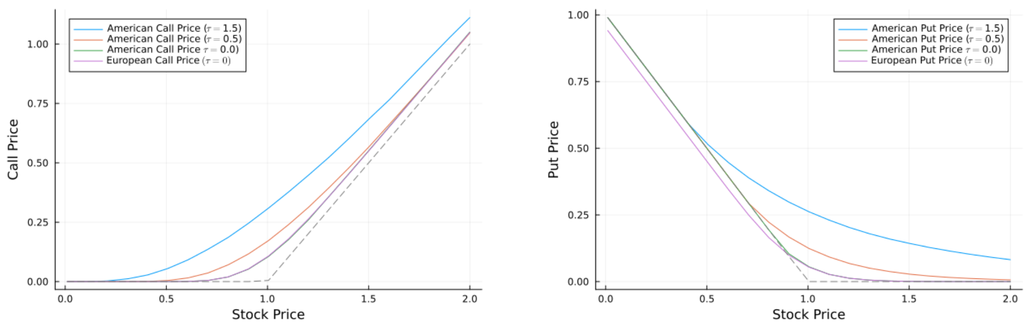


Figure 5.3: European vanilla call/put option prices change with respect to stock price

delay is taken as 1.5 and the initial path is $\varphi(t) = s_0$ for $s_0 \in [0.01, 2.0]$. Note that formula prices and MC prices are almost the same. Prices of the options are bigger whenever there is a pronounced delay.

5.2 Implementation of American Vanilla Call Option with Delayed GBM

Like in the implementation of European vanilla call option with delay, stock price satisfies (5.1) where $g(S(t-\tau)) = \sigma + \tau \exp\left\{\dfrac{-S(t-\tau)}{\alpha}\right\}$. We simulate $2^{14} = 16384$ sample path of the stock with the choice of $r = 0.05$, $\sigma = 0.2$, $dt = 0.01$, $T = 1.0$, $K = 1.0$, $\alpha = S(0)$ for the valuation and search for the effect of delay term and stock price on the valuation process for the initial path $\varphi(t) = S(0)$.



(a) (b)

Figure 5.4: American vanilla call/put option prices change with respect to stock price

From the Figure 5.4, the value of the options increases with the increase in delay term. If the underlying asset does not pay dividends, the American call option will never be exercised before expiration. That's why its value is the same as the European call option. Since we take dividend yield $q = 0$ in our model, the American call option value is equal to the European call option value for all stock prices in Figure 5.4a whenever $\tau = 0$. However, the American put value is greater than or equal to the European put value for $\tau = 0$ in Figure 5.4b because of the possibility of early exercise. If the underlying asset price drops significantly, the put holder may prefer to sell it rather than wait until expiration.

5.3 Implementation of European FX Call Option with Delayed GBM

According to the previous chapter current exchange rate F satisfies the following SDDE under risk neutral probability measure Q;

$$
\begin{aligned}
dF(t) &= F(t)\left[\delta_F dt + g(F(t-\tau))dW_d\right],\ t \in [0,T] \\
F(t) &= \varphi(t),\ t \in [-\tau, 0]
\end{aligned}
$$

where $\delta_F = (r_d - r_f)$. Again take $g(F(t-\tau)) = \sigma + \tau \exp\left\{\dfrac{-F(t-\tau)}{\alpha}\right\}$ and simulate $2^{14} = 16384$ sample paths with the choice of $r_f = 0.05$, $r_d = 0.06$, $\sigma = 0.2$, $dt = 0.01$, $T = 1.0$, $F(0) = 1.0$, $K = 1.0$, $\alpha = F(0)$.

We examine the effect of delay term and initial path on the valuation process while making different choices for them. Moreover, the convergence of the Monte Carlo method is considered while increasing the number of simulations up to 2^{14}.
In Figure 5.5, the effect of the delay term and initial paths are seen. As in European Vanilla call option implementation, delay terms are taken in $[0, 2]$ and initial paths are taken as $\varphi_1(t) = e^t$, $\varphi_2(t) = 2 - e^t$ and $\varphi_3(t) = 1.0$ which have different characteristic but same initial value. It is seen that whenever the delay term gets bigger, the difference between MC prices increases. Moreover, the characteristic of the initial path in the interval $[-\tau, 0]$ affects the MC price characteristic in $[0, T]$.

In Figure 5.6, we consider the convergence of MC for the initial path $\varphi(t) = S(0) = 1.0$. In Figure 5.6a and Figure 5.6c, convergence of method for $\tau = 1.5$ and $\tau =$

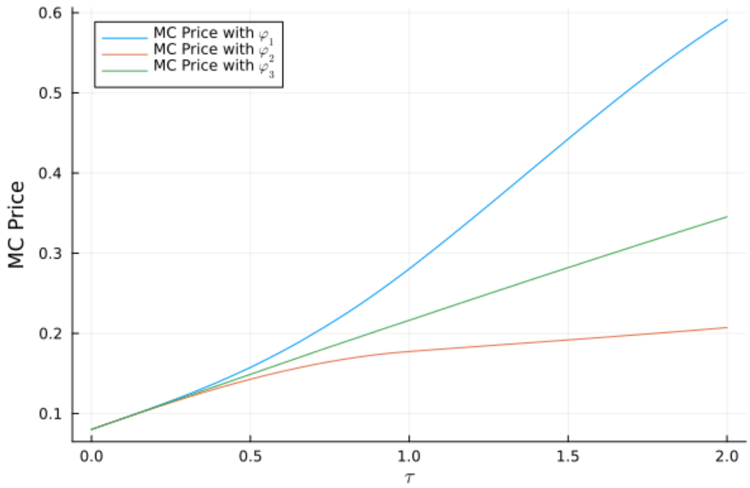
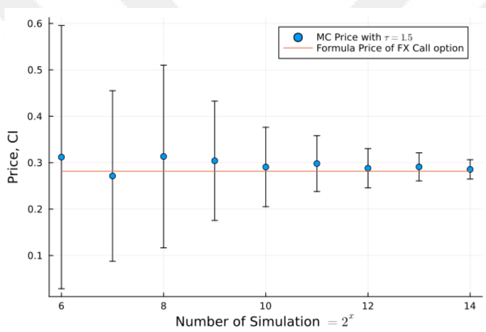
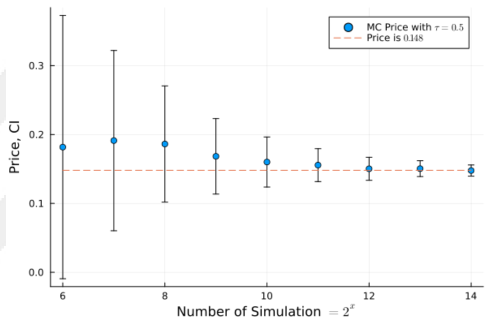


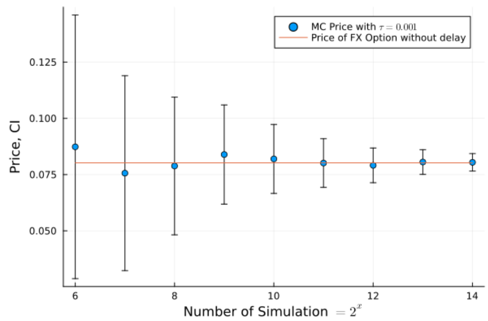
Figure 5.5: European FX call option price change with respect to delay term for the different choice of initial path



(a) $\tau = 1.5 > T$



(b) $\tau = 0.5 < T$



(c) $\tau = 0.001 \approx 0$

Figure 5.6: Convergence of Monte Carlo for FX Call Option

0.001 to formula price V and exact value P without delay respectively are shown by drawing confidence intervals with different number of simulation. In Figure 5.6b, convergence of the method is clear when number of simulation 2^{14}, the reference price 0.148 is considered.

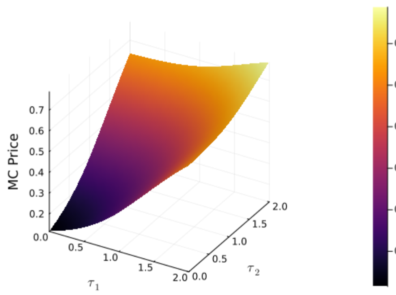
5.4 Implementation of European Exchange Call Option with Delayed GBM

For the Exchange option, assets satisfy the following SDDEs:

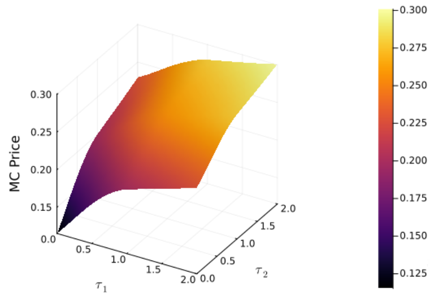
$$
\begin{aligned}
dS_i(t) &= rS_i(t)dt + g_i(S_i(t-b_i))S_i(t)d\hat{W}_i(t),\ t \in (0,T], \\
S_i(t) &= \varphi_i(t),\ t \in [-L_i, 0],
\end{aligned}
$$

where $L_i = \min\{a_i, b_i\}$ for $i = 1, 2$ under risk neutral probability measure Q. We take $g_i(S_i(t-b_i)) = \sigma_i + b_i \exp\left\{\dfrac{-S_i(t-b_i)}{\alpha_i}\right\}$ so that whenever delay terms go to zero diffusion term $g_i \to \sigma_i$ and model turns to without delay case.

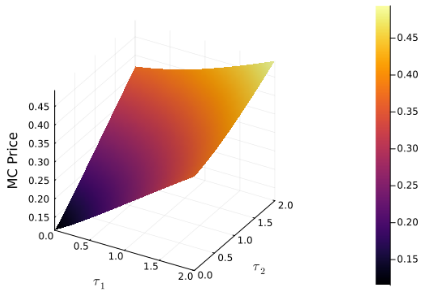
We simulate $2^{14} = 16384$ sample paths of the stock prices again while taking $r = 0.05$, $\sigma_1 = 0.2, \sigma_2 = 0.21, \rho = 0$, $dt = 0.01$, $T = 1.0$, $S_1(0) = 1.0$, $S_2(0) = 1.0$, $\alpha_1 = S_1(0)$, $\alpha_2 = S_2(0)$.



(a) $\varphi_1(t) = [e^t, e^t]$



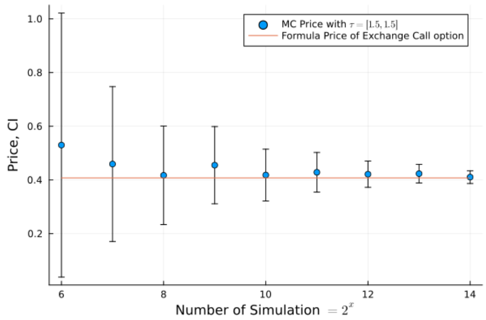
(b) $\varphi_2(t) = [2-e^t, 2-e^t]$



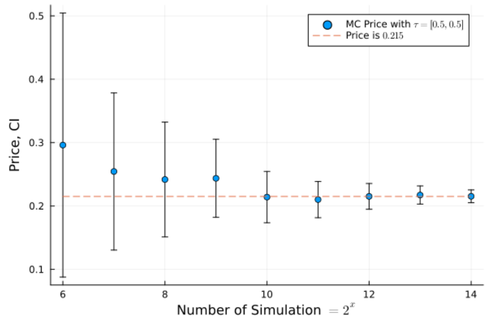
(c) $\varphi_3(t) = [1.0, 1.0]$

Figure 5.7: European exchange call option price change with respect to delay term for the different choice of initial path

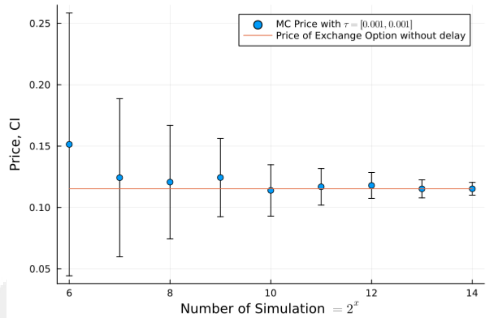
In Figure 5.7, the price of the option for the different initial paths is obtained while using the Monte Carlo method where delay terms are taken in $[0,2] \times [0,2]$ again. It is seen that $MC_2 \leq MC_3 \leq MC_1$ where MC_i represents the Monte Carlo price in the path φ_i although $\varphi_1(t) \leq \varphi_3(t) \leq \varphi_2(t)$ for any $t \in [-L_i, 0]$.



(a) $\tau = [1.5, 1.5] > T$



(b) $\tau = [0.5, 0.5] < T$



(c) $\tau = [0.001, 0.001] \approx 0$

Figure 5.8: Convergence of Monte Carlo for Exchange Call Option

In Figure 5.8, we consider the convergence of MC for the initial path $\varphi_3(t) = [1.0, 1.0]$. The results are same as the other options, Monte Carlo method converges with the increase in number of simulation. In Figure 5.8a and Figure 5.8c, MC prices converge to the formula price V and value P without delay respectively. Moreover, in Figure 5.8b, MC price converges to the reference price 0.215.

CHAPTER 6

HESTON MODEL FOR PRICING THE EUROPEAN VANILLA OPTION

The Heston model is widely used for option pricing and financial derivatives. It improves the Black-Schole-Merton model by allowing for stochastic volatility, better capturing market phenomena such as volatility clustering and the volatility smile. It considers the non-log normal distribution of the asset returns and the leverage effect. Since there is no closed-form solution, option pricing requires numerical methods, such as finite difference methods or Monte Carlo simulations. This section considers the stochastic volatility without delay and provides its partial differential equation (PDE). Then, we consider adding delay terms into the model to see the effect of past information, and also, their PDEs are provided. [18, 11, 37, 20, 36] are some references.

6.1 European Vanilla Option Pricing without delay under Heston model

The Heston stochastic volatility model is defined by the following stochastic process:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{\nu(t)}S(t)dZ_1(t), \\ d\nu(t) &= K(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)}dZ_2(t), \\ dZ_1(t)dZ_2(t) &= \rho dt, \end{aligned} \tag{6.1}$$

where

- $Z_1(t)$ is the standard Brownian motion of the asset price,
- $Z_2(t)$ is the standard Brownian motion of the asset's price variance,

- ρ is correlation coefficient,
- $\nu(t)$ is volatility of the asset price,
- σ is the volatility of the volatility $\nu(t)$,
- μ is a deterministic drift term,
- θ is the long-term price variance,
- K is the rate of reversion to the long-term price variance.

Note that the SDE in (6.1) is under any probability measure P. When we search the corresponding partial differential equation (PDE) satisfied by (6.1), we firstly write the system under risk neutral probability measure Q. In general, if $X(t)$ satisfies the following SDE under the probability measure P;

$$dX(t) = f(t, X(t))dt + g(t, X(t))dZ(t)$$

then the corresponding SDE under Q for the process X is

$$dX(t) = [f(t, X(t)) - \lambda g(t, X(t))]dt + g(t, X(t))dW(t)$$

where λ is market price of risk, W is standard Brownian motion under Q. So, the stock price process $S(t)$ and volatility process $\nu(t)$ in (6.1) satisfy the following processes under martingale measure Q;

$$\begin{aligned} dS(t) &= \left[\mu S(t) - \lambda_1 \sqrt{\nu(t)} S(t)\right] dt + \sqrt{\nu(t)} S(t) dW_1(t) \\ d\nu(t) &= \left[K(\theta - \nu(t)), -\lambda_2 \sigma \sqrt{\nu(t)}\right] dt + \sigma \sqrt{\nu(t)} dW_2(t), \end{aligned}$$

where $W_1(t)$, $W_2(t)$ are standard Brownian motion under Q and λ_1, λ_2 are market price of risk. From the Black-Scholes-Merton model, it is known that $\lambda_1 = \dfrac{\mu - r}{\sqrt{\nu(t)}}$ by Girsanov's theorem. Then the stock price process $S(t)$ satisfies the following process under Q;

$$\begin{aligned} dS(t) &= \left[\mu S(t) - \frac{\mu - r}{\sqrt{\nu(t)}} \sqrt{\nu(t)} S(t)\right] dt + \sqrt{\nu(t)} S(t) dW_1(t), \\ &= rS(t)dt + \sqrt{\nu(t)} S(t) dW_1(t), \end{aligned}$$

where r is the risk free interest rate. So, our Heston model under risk neutral probability measure Q is

$$\begin{aligned} dS(t) &= rS(t)dt + \sqrt{\nu(t)}S(t)dW_1(t), \\ d\nu(t) &= \left[K(\theta - \nu(t)) - \lambda_2\sigma\sqrt{\nu(t)}\right]dt + \sigma\sqrt{\nu(t)}dW_2(t), \\ dW_1(t)dW_2(t) &= \rho dt. \end{aligned} \tag{6.2}$$

The corresponding value of European call and put options with strike price K and maturity T is given by

$$\begin{aligned} V_C(t) &= e^{rt}E_Q\left(\frac{(S(T)-K,0)^+}{e^{rT}}\Big|F_t\right), \\ V_P(t) &= e^{rt}E_Q\left(\frac{(K-S(T),0)^+}{e^{rT}}\Big|F_t\right). \end{aligned}$$

There is no closed-form solution of $S(T)$ using stochastic calculus; we cannot solve this valuation process explicitly. So, our question is what is the PDE satisfied by this system to solve this system explicitly and find the pricing formula using the PDE approach. Let the fair price satisfied by this system be $V(t,S,\nu)$. Then the discounted price under measure Q becomes $\tilde{V}(t,S,\nu) = e^{-rt}V(t,S,\nu)$ since discounted prices are martingale under risk neutral measure. Then, with the help of Itô formula we obtain

$$\begin{aligned} d\tilde{V}(t) &= -re^{-rt}Vdt + e^{-rt}dV \\ &= -re^{-rt}Vdt + e^{-rt}\left[\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \nu}d\nu + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\nu S^2 d\left[S,S\right]\right. \\ &\qquad \left. + \frac{1}{2}\frac{\partial^2 V}{\partial \nu^2}d\left[\nu,\nu\right] + \frac{\partial^2 V}{\partial\nu\partial S}d[\nu,S]\right] \\ &= e^{-rt}\left[-rV + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\nu S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \nu^2}\sigma^2\nu + \frac{\partial^2 V}{\partial\nu\partial S}\rho\sigma\nu S\right]dt \\ &\qquad + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \nu}d\nu \\ &= e^{-rt}\left[-rV + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\nu S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \nu^2}\sigma^2\nu + \frac{\partial^2 V}{\partial\nu\partial S}\rho\sigma\nu S\right. \\ &\qquad \left. + \frac{\partial V}{\partial S}rS + \frac{\partial V}{\partial \nu}K_\lambda\right]dt + \frac{\partial V}{\partial S}\sqrt{\nu}SdW_1 + \frac{\partial V}{\partial \nu}\sqrt{\nu}\sigma dW_2 \end{aligned}$$

where $K_\lambda := K(\theta-\nu) - \lambda_2\sigma\sqrt{\nu(t)}$ for simplicity. Since discounted asset price is a martingale, we obtain the following equation:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\nu S^2 + \frac{\partial V}{\partial \nu}K_\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial \nu^2}\sigma^2\nu + \frac{\partial^2 V}{\partial\nu\partial S}\rho\sigma\nu S - rV = 0$$

So, the corresponding PDE and boundary conditions of the equation (6.2) are:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\nu S^2 + \frac{\partial V}{\partial \nu}K_\lambda + \frac{1}{2}\frac{\partial^2 V}{\partial \nu^2}\sigma^2\nu + \frac{\partial^2 V}{\partial \nu \partial S}\rho\sigma\nu S - rV = 0$$
$$V(T,S,\nu) = (S-K)^+$$
$$V(t,0,\nu) = 0$$
$$\lim_{S\to\infty} \frac{\partial V}{\partial S}(t,S,\nu) = 1$$
$$\lim_{\nu\to\infty} V(t,S,\nu) = S$$
$$\frac{\partial V(t,S,0)}{\partial t} + \frac{\partial V(t,S,0)}{\partial S} rS + \frac{\partial V(t,S,0)}{\partial \nu}K_\lambda + -rV(t,S,0) = 0.$$

However, there is no analytical solution for this PDE, either.

Since no closed-form solutions exist, we will use numerical methods and the Monte Carlo method to find the approximate value of the option.

6.2 European Vanilla Option Pricing with delay under Heston model

Let τ_1 represents the time delay for the stock price S and τ_2 represents the time delay for the volatility of the asset price ν for the Heston model so that $L = \max\{\tau_1, \tau_2\}$. The terms r, κ, λ, σ, ρ, W_1 and W_2 are same as without delay case. g_i's are continuous functions, $\varphi_1(t)$ is initial path for S and $\varphi_2(t)$ is initial path for ν. More on Heston model and its use in option pricing can be found in [36, 20]. We consider the addition of delay into the diffusion term of the model in a multiplicative way, like in delayed GBM, since the direct impact of delay in the drift is not seen under risk neutral probability measure Q. Below, three different ways of adding delay are considered:

- **Delayed Heston Model-1:** We create our delayed Heston model-1 while adding stock price delay into stock price process and volatility delay into volatility process. The processes satisfy the following SDDEs:

$$\begin{aligned}
&dS(t) = rS(t)dt + \sqrt{\nu(t)}S(t)g_1(S(t-\tau_1))dW_1(t),\\
&d\nu(t) = \left[\kappa(\theta - \nu(t)) - \lambda\sigma\sqrt{\nu(t)}\right]dt + \sigma\sqrt{\nu(t)}g_2(\nu(t-\tau_2))dW_2(t),\\
&\varphi(t) = (\varphi_1(t), \varphi_2(t)),\ t \in [-L, 0]\\
&dW_1(t)dW_2(t) = \rho dt.
\end{aligned} \tag{6.3}$$

- **Delayed Heston Model-2:** Consider the addition of volatility delay into stock price process and stock price delay into volatility process. The corresponding processes satisfy the following SDDEs:

$$
\begin{aligned}
&dS(t) = rS(t)dt + \sqrt{\nu(t)}S(t)g_2(\nu(t-\tau_2))dW_1(t),\\
&d\nu(t) = \left[\kappa(\theta-\nu(t)) - \lambda\sigma\sqrt{\nu(t)}\right]dt + \sigma\sqrt{\nu(t)}g_1(S(t-\tau_1))dW_2(t),\\
&\varphi(t) = (\varphi_1(t),\varphi_2(t)),\ t\in[-L,0]\\
&dW_1(t)dW_2(t) = \rho dt.
\end{aligned}
\tag{6.4}
$$

- **Delayed Heston Model-3:** Consider just one time delay, the addition of volatility delay into stock price process. The corresponding processes satisfy the following SDDEs:

$$
\begin{aligned}
&dS(t) = rS(t)dt + \sqrt{\nu(t)}S(t)g_2(\nu(t-\tau_2))dW_1(t),\\
&d\nu(t) = \left[\kappa(\theta-\nu(t)) - \lambda\sigma\sqrt{\nu(t)}\right]dt + \sigma\sqrt{\nu(t)}dW_2(t),\\
&\varphi(t) = (\varphi_1(t),\nu(0)),\ t\in[-\tau_2,0]\\
&dW_1(t)dW_2(t) = \rho dt.
\end{aligned}
\tag{6.5}
$$

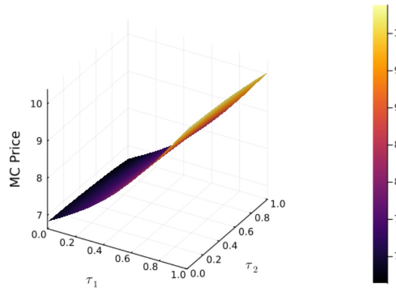
Since it is not possible to obtain an analytical solution for S and the value process in those three cases, we will use the Monte Carlo method to find the approximate value of the option and the effect of delay terms in those models.

6.3 Numerical Implementation of Delayed Heston Model

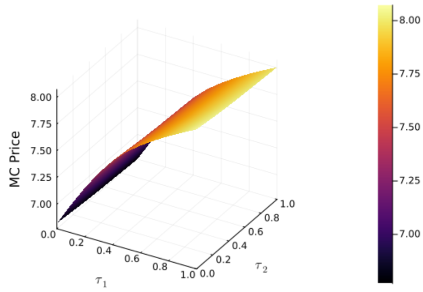
Since there is no closed-form solution for the Heston model, we consider just the Monte Carlo method to find the corresponding value for the different values of τ's. We choose our parameters to be the same as the example without delay case in [34].

We simulate 2^{14} path of the stock price and volatility for the choice of $r = 0.03$, $\kappa = 5.0$, $\theta = 0.05$, $\sigma = 0.5$, $\lambda_2 = 0$, $\rho = -0.8$, $T = 0.5$, $S(0) = 100.0$, $\nu(0) = 0.05$, $K = 100.0$, $dt = 0.001$ and different values of τ's in $[0, 1.0] \times [0, 1.0]$.

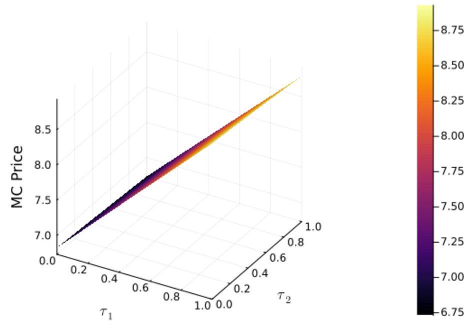
In Figure 6.1, we see the effect of the initial path and delay terms for pricing if the processes satisfy (6.3). Whenever delay terms get bigger, the difference between option prices with different initial paths differs. Like in the European vanilla call



(a) $\varphi_1(t) = [S(0)e^t, \nu(0)e^t]$



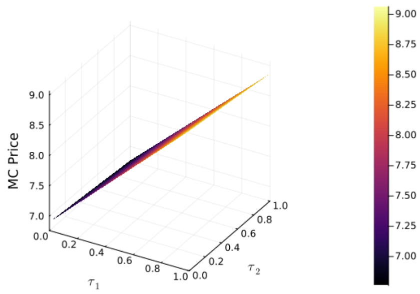
(b) $\varphi_2(t) = [S(0)(2 - e^t), \nu(0)(2 - e^t)]$



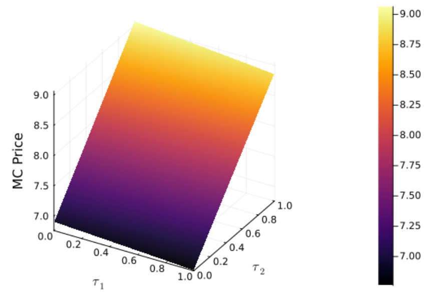
(c) $\varphi_3(t) = [S(0), \nu(0)]$

Figure 6.1: European vanilla call option price change with respect to delay term for the different choice of initial path with delayed Heston model-1

option with delayed GBM, the price of the option is highest whenever the initial path is increasing Figure 6.1a.



(a) Heston model-1



(b) Heston model-2

Figure 6.2: European vanilla call option price with delayed Heston model-1 and model-2 change with respect to delay term for the constant initial path $\varphi(t) = [S(0), \nu(0)]$.

In Figure 6.2, we see the effect of model and delay terms for pricing. In both models, the MC prices increase with the increase in τ_1 and τ_2. The surface in Figure 6.2a shows the nearly linear trend in the increasing prices, which implies a less volatile response to changes in delay. However, the MC price surface in Figure 6.2b has more varied curvature. The surface is not as smooth as in Figure 6.2a which implies more sensitivity to changes in delay.

We take increasing initial path $\varphi_1(t) = [S(0)e^t, \nu(0)e^t]$, decreasing initial path $\varphi_2(t) =$

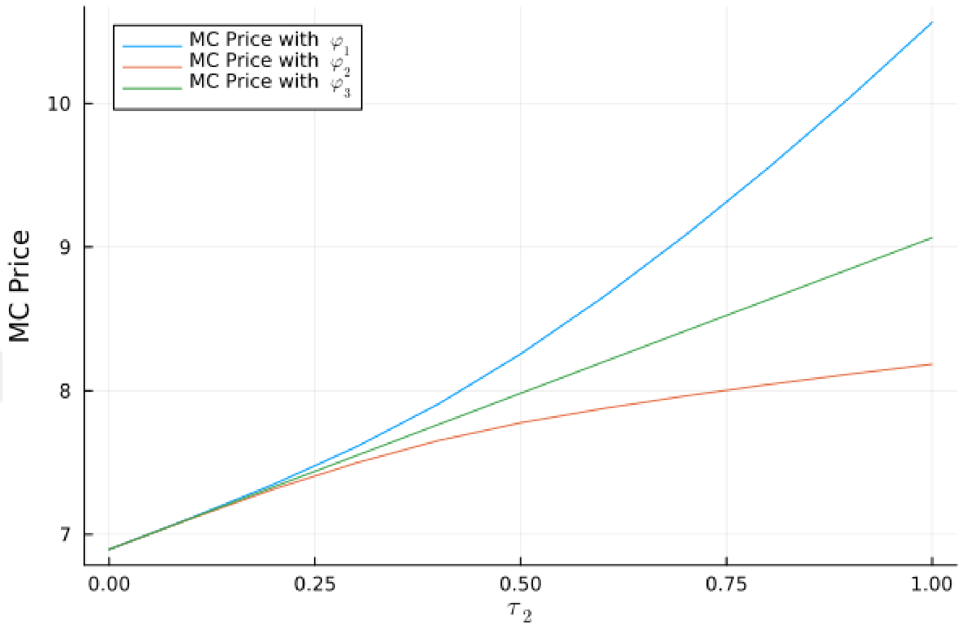
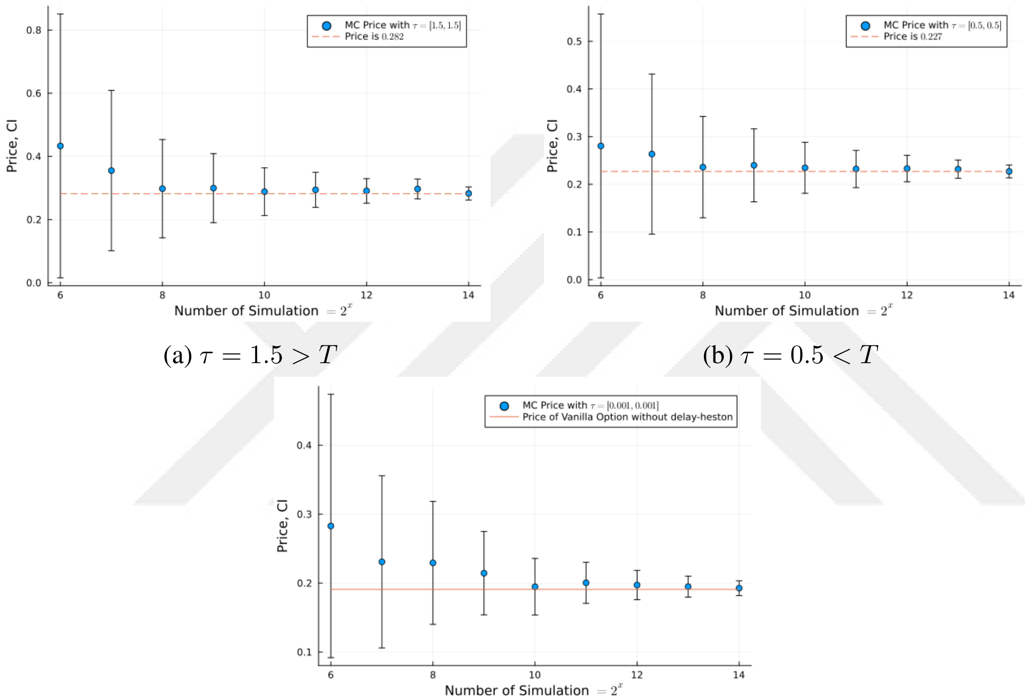


Figure 6.3: European vanilla call option price change with respect to delay term for the different choice of initial path with delayed Heston model-3

$[S(0)(2-e^t), \nu(0)(2-e^t)]$ and constant initial path $\varphi_3(t) = [S(0), \nu(0)]$ in Figure 6.3 for the delayed Heston model-3 in (6.5). The difference between MC prices decreases whenever the delay term decreases. Moreover, the characteristic of the initial path gets more important if the delay term is bigger since the effect of it is seen more in that case.

The length of confidence intervals is getting smaller with the increase in a number of simulations in Figure 6.4, which verifies the convergence of the Monte Carlo method.



(a) $\tau = 1.5 > T$

(b) $\tau = 0.5 < T$

(c) $\tau = 0.001 \approx 0$

Figure 6.4: Convergence of Monte Carlo for Heston Model-1 Call Option

CHAPTER 7

CONCLUSION

This thesis aims to contribute to the growing body of knowledge on SDDEs and their applications in financial modeling.

In Chapter 2, the general form of equations under delay effect, conditions for existence and uniqueness of the solution, and the way of solving these equations are considered.

Because of the difficulty in finding analytical solutions, numerical methods, namely Euler Maruyama and Milstein methods, are considered in Chapter 3. Moreover, to see the effectiveness of these methods, an example is provided.

In Chapter 4, option pricing with delayed GBM, which is our motivation to work with SDDE, is examined for European and American call types of vanilla options, European foreign exchange options, and European exchange options with and without delays.

In Chapter 5, implementations of these models using Julia programming are considered. A comparison of option prices for the different delays is made, and convergence in the Monte Carlo method while considering confidence intervals is examined.

In Chapter 6, we consider the Heston model with and without delay cases. Since it does not have closed form solution, numerical methods are considered for the pricing of European Vanilla call option.

For future works, parameter estimation can be a challenging work for practitioners to mimic the market parameters in question for more realistic applications.

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APPENDIX A

PROOF OF SOME THEOREMS

Theorem A.1 (Theorem 4 in [3]). *Assume that the stock price S satisfies equation* (4.4) *with conditions $\varphi(0) > 0$ and $g(u) \neq 0$ whenever $u \neq 0$. Let $r > 0$ be the risk free rate and Q be the risk neutral probability measure. Let $V(t)$ be the pricing formula of a European call option which is written on the stock S with strike price K and maturity time T. Then there exist two cases for the value of the option:*

- *For all $t \in [T-l, T]$ for $l = \min\{a, b\}$:*

$$V(t) = S(t)\phi(\beta_1(t)) - Ke^{-r(T-t)}\phi(\beta_2(t)),$$

where

$$\beta_1 = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r + \frac{1}{2}g(S(u-b))^2)du}{\sqrt{\int_t^T g(S(u-b))^2 du}},$$

$$\beta_2 = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r - \frac{1}{2}g(S(u-b))^2)du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

- *For all $T > l$ and $t < T - l$:*

$$V(t) = e^{rt}E_Q\Bigg(H\Bigg(e^{-r(T-l)}S(T-l), -\frac{1}{2}\int_{T-l}^T g(S(u-b))^2 du, \int_{T-l}^T g(S(u-b))^2 du\Bigg)\Bigg|\mathcal{F}_t\Bigg),$$

where

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2}\phi(\alpha_1(x, m, \sigma)) - Ke^{-rT}\phi(\alpha_2(x, m, \sigma)),$$

and

$$\alpha_1(x, m, \sigma) = \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m + \sigma^2\right],$$

$$\alpha_2(x,m,\sigma) = \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + rT + m\right]$$

for $\sigma, x \in \mathbb{R}^+$, $m \in \mathbb{R}$.

The hedging strategy for $t \in [T-l, T]$ *is given by*

$$\pi_S(t) = \phi(\beta_1(t)),$$
$$\pi_B(t) = -Ke^{-r(T-t)}\phi(\beta_2(t)).$$

Proof. The corresponding equation of (4.4) under risk neutral probability measure Q is:

$$\frac{dS(t)}{S(t)} = rdt + g(S(t-b))dW(t), \quad 0 < t \leq T$$
$$S(t) = \varphi(t) \quad t \in [-L, 0]$$

where $L = \max\{a, b\}$. According to Itô formula for $\ln x$ corresponding solution is;

$$f(S(t)) = f(S(0)) + \int_0^t \frac{\partial f}{\partial S} dS(u) + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial S^2} d\left[S,S\right]_u$$
$$\ln S(t) = \ln S(0) + \int_0^t \frac{1}{S(u)} S(u)\left[rdu + g(S(u-b))dW(u)\right]$$
$$+ \frac{1}{2}\int_0^t \frac{-1}{S^2(u)} S^2(u) g(S(u-b))^2 du.$$

Thus, we find the corresponding solution as

$$S(t) = S(0)\exp\left[\int_0^t rdu + \int_0^t g(S(u-b))dW(u) - \frac{1}{2}\int_0^t g(S(u-b))^2 du\right].$$

Using Markow property, it can be written as

$$S(T) = S(t)\exp\left[\int_t^T rdu + \int_t^T g(S(u-b))dW(u) - \frac{1}{2}\int_t^T g(S(u-b))^2 du\right]$$
$$= S(t)e^{r(T-t)}\exp\left[\int_t^T g(S(u-b))dW(u) - \frac{1}{2}\int_t^T g(S(u-b))^2 du\right].$$

Since $\int_t^T g(S(u-b))dW(u) \sim N\left(0, \int_t^T g(S(u-b))^2 du\right)$, we define

$$\tilde{t} := T - t, \quad \sigma^2 := \int_t^T g(S(u-b))^2 du \quad \text{and} \quad m := \frac{-1}{2}\int_t^T g(S(u-b))^2 du.$$

Thus, for any $Y \sim N(0,1)$, we get $\int_t^T g(S(u-b))dW(u) \stackrel{d}{=} \sigma Y$. So,

$$\begin{aligned} S(T) &= S(t)e^{r\tilde{t}}e^{m+\sigma y} \\ e^{-rT}S(T) &= e^{-rT}e^{r\tilde{t}}S(t)e^{m+\sigma y} \\ \tilde{S}(T) &= e^{-rt}S(t)e^{m+\sigma y} = \tilde{S}(t)e^{m+\sigma y}. \end{aligned} \tag{A.1}$$

We consider the corresponding value of the European call option with the risk free asset, $B(t) = e^{rt}$, as numeraire ;

$$\begin{aligned} V(S(t),t) &= B(t)E_Q\left[\left.\frac{(S(T)-K,0)^+}{B(T)}\right| F_t\right] \\ &= e^{rt}E_Q\left[\left.\frac{(S(T)-K,0)^+}{e^{rT}}\right| F_t\right] \\ &= e^{rt}E_Q\left[\left.\left(\tilde{S}(T)-e^{-rT}K,0\right)^+\right| F_t\right] \\ &= e^{rt}E_Q\left[\left.\left(\tilde{S}(t)e^{m+\sigma y}-Ke^{-rT},0\right)^+\right|F_t\right]. \end{aligned}$$

The discounted payoff function $\dfrac{(S(T)-K,0)^+}{e^{rT}} = \left(\tilde{S}(T)-Ke^{-rT},0\right)^+$ is always F_{T-l} measurable, since $t \in [0,T]$ implies $t-b \in [-b,T-b] \subset [-b,T-l]$ where $l = \min\{a,b\}$. So, for $x := \tilde{S}(t)$

$$\begin{aligned} H(x,m,\sigma^2) &:= E_Q\left[\left(xe^{m+\sigma Y}-Ke^{-rT},0\right)^+ \mid F_{T-l}\right) \\ &= E_Q\left[\left(xe^{m+\sigma Y}-Ke^{-rT},0\right)^+\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(xe^{m+\sigma y}-Ke^{-rT},0\right)^+ e^{-y^2/2}dy \end{aligned}$$

where

$$\begin{aligned} xe^{m+\sigma y}-Ke^{-rT} > 0 &\Rightarrow xe^{m+\sigma y} > Ke^{-rT} \\ &\Rightarrow \ln x + m + \sigma y > \ln K - rT \\ &\Rightarrow y > -\beta_2 := -\frac{\ln\left(\frac{x}{K}\right)+rT+m}{\sigma} \end{aligned}$$

This implies

$$
\begin{aligned}
H(x,m,\sigma^2) &= \frac{1}{\sqrt{2\pi}}\int_{-\beta_2}^{\infty}\left(xe^{m+\sigma y}-Ke^{-rT}\right)e^{-y^2/2}dy\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\beta_2}^{\infty}xe^{m+\sigma y-y^2/2}dy-\frac{1}{\sqrt{2\pi}}\int_{-\beta_2}^{\infty}Ke^{-rT}e^{-y^2/2}dy\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\beta_2}^{\infty}xe^{\frac{-(\sigma-y)^2}{2}}dy-Ke^{-rT}\phi(\beta_2)\quad \text{since}\quad m=-\sigma^2/2\\
&= \frac{1}{\sqrt{2\pi}}\int_{\sigma+\beta_2}^{-\infty}xe^{-u^2/2}(-du)-Ke^{-rT}\phi(\beta_2)\quad \text{for}\quad u=\sigma-y\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\beta_1}xe^{-u^2/2}du-Ke^{-rT}\phi(\beta_2)\quad \text{for}\quad \beta_1=\sigma+\beta_2\\
&= x\phi(\beta_1)-Ke^{-rT}\phi(\beta_2).
\end{aligned}
$$

Therefore, the value of the European call option under the Black-Scholes-Merton setting for any $t\in[0,T]$ with delay effect is;

$$
\begin{aligned}
V(S,t) &= e^{rt}E_Q\left[\left(\tilde{S}(T)-Ke^{-rT},0\right)^+\Big|F_t\right]\\
&= e^{rt}E_Q\left[E_Q\left[\left(\tilde{S}(T)-Ke^{-rT},0\right)^+\Big|F_{T-l}\right]\Big|F_t\right],\text{ by Tower property}\\
&= e^{rt}E_Q\left[E_Q\left[\left(\tilde{S}(T)-Ke^{-rT},0\right)^+\right]\Big|F_t\right]\\
&= e^{rt}E_Q\left[H(x,m,\sigma^2)\mid F_t\right]
\end{aligned}
$$

where

$$
\begin{aligned}
&H(x,m,\sigma^2)=x\phi(\beta_1)-Ke^{-rT}\phi(\beta_2)\\
&x=\tilde{S}(t)\\
&m=-\frac{1}{2}\int_t^T g^2(S(u-b))du\\
&\sigma^2=\int_t^T g^2(S(u-b))du\ \text{so that}\ m=-\sigma^2/2\\
&\beta_1=\frac{\ln\left(\frac{x}{K}\right)+m+rT+\sigma^2}{\sigma}\\
&\beta_2=\frac{\ln\left(\frac{x}{K}\right)+m+rT}{\sigma}=\beta_1-\sigma.
\end{aligned}
$$

Case 1: When $t \in [T-l, T]$; since H is F_{T-l}-measurable and $F_{T-l} \subset F_t$ turn to;

$$\begin{aligned} V(t) &= e^{rt} E_Q \left(H(x, m, \sigma^2) \mid F_t \right) \\ &= e^{rt} H(x, m, \sigma^2) \\ &= e^{rt} \left[\tilde{S}(t) \phi(\beta_1) - K e^{-rT} \phi(\beta_2) \right] \\ &= S(t) \phi(\beta_1) - K e^{-r\tilde{t}} \phi(\beta_2) . \end{aligned}$$

Case 2: when $T > l$ and $t < T - l$, then consider to write $S(T)$ in terms of $S(T-l)$;

$$\tilde{S}(T) = \tilde{S}(T-l) \exp \left[\int_{T-l}^{T} g(S(u-b)) dW(u) - \frac{1}{2} \int_{T-l}^{T} g(S(u-b))^2 du \right]$$

$$V(t) = e^{rt} E_Q \left(H \left(\tilde{S}(T-l), -\frac{1}{2} \int_{T-l}^{T} g(S(u-b))^2 du, \int_{T-l}^{T} g(S(u-b))^2 du \right) \Big| F_t \right)$$

$$H\left(x, m, \sigma^2\right) = x \phi(\beta_1) - K e^{-rT} \phi(\beta_2)$$

$$x = \tilde{S}(T-l)$$

$$m = \frac{-1}{2} \int_{T-l}^{T} g^2(S(u-b)) du$$

$$\sigma^2 = \int_{T-l}^{T} g^2 S(u-b)) du \quad st \quad m = -\sigma^2/2.$$

$$\alpha_1 = \frac{\ln\left(\frac{x}{K}\right) + m + rT + \sigma^2}{\sigma}$$

$$\alpha_2 = \frac{\ln\left(\frac{x}{K}\right) + m + rT}{\sigma} = \beta_1 - \sigma.$$

This completes the proof. □

CURRICULUM VITAE

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