FIFTH GRADE STUDENTS' PROCESS OF GENERALIZING DIVISIBILITY RULES

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FIFTH GRADE STUDENTS' PROCESS OF GENERALIZING DIVISIBILITY RULES

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ABSTRACT

FIFTH GRADE STUDENTS' PROCESS OF GENERALIZING DIVISIBILITY RULES

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The purpose of the study was to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. It was ensured that the students conceptualize the divisibility rules with the help of critical questions of a more knowledgeable person through Vygotsky' ZPD.

Three fifth grade students who were studying at private schools in Ankara were selected by using the purposive sampling. Case study design was used to interpret the data. Data were collected in the summer semester of 2023-2024 through activity sheets and clinical interviews. Each student was given three activity sheets during clinical interviews. First, the students were given activity sheets on divisibility by 2, then by 3, and finally by 6 during the three clinical interviews.

The findings of the study demonstrated that the students were able to generalize the divisibility rules conceptually. More specifically, students could reach a generalization about for a number to be divisible by 2 the ones digit of the number needs to be even. Additionally, students could reach a generalization that for a number to be divisible by 3, the sum of digits needs to be multiple of 3. Also, they could generalize that for a number to be divisible by 6, it needs to be divisible by 2 and 3. They could reach these generalizations conceptually even though they did not know the rules before during the clinical interviews.

Keywords: Mathematics Education, Middle School Students, Division, Divisibility Rules, Zone of Proximal Development

ÖZ

BEŞİNCİ SINIF ÖĞRENCİLERİNİN BÖLÜNEBİLME KURALLARINI GENELLEME SÜRECİ

Özcan, Sena
Yüksek Lisans, Matematik Eğitimi, Matematik ve Fen Bilimleri Eğitimi
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Çalışmanın amacı, beşinci sınıf öğrencilerinin 2, 3 ve 6 ile bölünebilme kurallarını, konuyla ilgili herhangi bir ön bilgiye sahip olmadan, klinik görüşmeler sırasında gerektiğinde sunulan rehberlikle sanal bir manipülatif kullanarak nasıl genelleştirdiklerini analiz etmektir. Öğrencilerin, Vygotsky'nin Yakınsal Gelişim Alanı (ZPD) aracılığıyla, daha bilgili bir kişinin kritik sorularıyla bölünebilme kurallarını kavramsallaştırmaları sağlanmıştır.

Amaçlı örneklem yöntemiyle, Ankara'daki özel okullarda eğitim gören üç beşinci sınıf öğrencisi seçilmiştir. Verilerin yorumlanması için durum çalışması deseni kullanılmıştır. Veriler, 2023-2024 yaz döneminde etkinlik kâğıtları ve klinik görüşmeler aracılığıyla toplanmıştır. Klinik görüşmeler sırasında her öğrenciye üç etkinlik kâğıdı verilmiştir. Önce öğrencilere 2'ye, ardından 3'e ve son olarak 6'ya bölünebilme ile ilgili etkinlik kâğıtları verilmiştir.

Çalışmanın bulguları, öğrencilerin 2, 3 ve 6'nın bölünebilme kurallarını kavramsal olarak genelleyebildiklerini göstermiştir. Daha spesifik olarak, öğrenciler bir sayının 2 ile bölünebilmesi için birler basamağındaki rakamın çift olması gerektiği yönünde bir genellemeye ulaşmışlardır. Ayrıca, bir sayının 3 ile bölünebilmesi için rakamlarının toplamının 3’ün katı olması gerektiğini de genelleyebilmişlerdir. Bunun yanı sıra, bir sayının 6 ile bölünebilmesi için hem 2’ye hem de 3’e bölünebilmesi gerektiğini genellemesine ulaşabilmişlerdir. Gerekli rehberlik sağlandığında, bu genellemeleri kavramsal olarak, kuralları önceden bilmemelerine rağmen yapabilmişlerdir.

Anahtar Kelimeler: Matematik Eğitimi, Ortaokul Öğrencileri, Bölünme, Bölünebilme Kuralları, Yakınsak Gelişim Alanı

To My Family

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ABBREVIATIONS

METU	Middle East Technical University
MoNE	Ministry of National Education
ZPD	Zone of Proximal Development

CHAPTER 1

INTRODUCTION

Divisibility rules are used to determine whether a given number can be divided by another without performing long division (Laudano & Donatiello, 2019; Peretti, 2015; Potgieter & Blignaut, 2018; Zazkis & Campbell, 1996). Zazkis and Campbell (1996) state that the division algorithm always provides a solution for dividing one number by another but implementing the division algorithm can sometimes be time-consuming and tiresome. Moreover, although a simple calculator can quickly check the divisibility of a given number by another, it is valuable to understand and teach the logic behind the divisibility rules as they are rooted in good mathematics (Ganzell, 2017). In particular, divisibility rules present great examples for better understanding the principles of number theory, forming new assumptions, and discovering connections between subjects (Szetela, 1980).

Moreover, based on the relevant literature, it can be concluded that understanding the logic behind the division and divisibility rules is crucial for other topics in mathematics (Ganzell, 2017; Harrell & Slavens, 2009; Posamentier, 2003; Szetela, 1980; Young-Loveridge & Mills, 2012). To illustrate, there is a connection between divisibility rules and the discovery and generalization of patterns; therefore, understanding the divisibility rules are significant (Kennedy, 1971). Likewise, Siemon et al. (2012) state that understanding division enables students to develop multiplicative thinking, a big mathematical idea. In other words, multiplicative thinking, which is at the heart of the multiplicative structure, involves recognizing the relationships between multiplication and division (Purnomo et al., 2022), and understanding the logic behind the divisibility rules enable students to develop their multiplicative thinking (Young-Loveridge & Mills, 2012). They show that working with the proof of divisibility rules enables students to develop a deeper understanding of division and multiplication. Therefore, division and divisibility rules are linked to the development of a child's multiplicative thinking, which is essential for the other areas of mathematics.

Based on the literature, we could deduce that divisibility rules are crucial in mathematics, serving as the foundation for other mathematical subjects. However, the literature shows that divisibility rules are usually taught procedurally without giving any attention to the meaning of the rules, which are conceptual. To state differently, upon reviewing the relevant literature, most students do not understand the reasoning behind these rules since divisibility rules are often viewed as a memorization task for students (Posamantier, 2003). Therefore, the conceptual background behind the divisibility rules is unfamiliar to most students (Patodi, 2021). Since "divisibility rules provide an interesting window into the nature of numbers and properties" (Posamentier, 2003, p.53), considering divisibility rules as a memorized subject might cause students to have difficulties in other areas of mathematics, such as multiplication, division, fraction, ratio (Young-Loveridge & Mills, 2012). In addition, since division is a fundamental arithmetic operation in mathematics, memorizing the divisibility rules might hinder students from effectively understanding other mathematical concepts. Therefore, the performance of students who do not have a deep understanding of division may be negatively affected in mathematics, from basic arithmetic to advanced algebra.

For these reasons, we conclude that students might need more support to learn the rationale behind the divisibility rules in a conceptual way. For guiding students toward a conceptual understanding of the divisibility rules, using the zone of proximal development might be helpful because students learn the concepts best in their zone of proximal development (Devi, 2019). More specifically, the zone of proximal development is a concept introduced by Vygotsky that highlights the distinction between what an individual can achieve independently and what they can accomplish with the guidance and support of someone more knowledgeable (Vygotsky, 1978). The zone of proximal development is the central aspect of Vygotsky's social constructivism theory.

Teachers cannot transfer knowledge to students directly; students need to construct knowledge, such as discovering new rules (Bada, 2015). Constructing new knowledge is more straightforward when students are in their zone of proximal development. Since the assumption of help is one of the pathways of the zone of proximal development that emphasizes the positive impact of the knowledgeable person on the child (Chaiklin, 2003), this way emphasizes the interaction between adults or teachers and children. Thus, by considering the student needs, the present study aimed to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews

1.1. The Purpose of the Study and Research Questions

The study aims to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews.

To achieve the aim of the study, the research questions that were addressed are as follows:

1. How do the fifth-grade students generalize the divisibility rules through the clinical interviews with a more knowledgeable person with the help of virtual manipulative?
2. What difficulties or misconceptions do students encounter during the generalization process?

1.2. Significance of the Study

Several researchers agreed that understanding the reason behind the divisibility rules enables students to better understand other areas of mathematics (Ganzell, 2017; Harrell & Slavens, 2009; Posamentier, 2003; Szetela, 1980; Young-Loveridge & Mills, 2012). More specifically, understanding the logic behind the divisibility rules provides excellent examples for a better understanding of number theory properties, developing the concept of place value, common factor, and least common multiple (Szetela, 1980; Harrell & Slavens, 2009). Also, the logic behind the rules enables students to develop their multiplicative thinking, which is a big idea of mathematics (Siemon et al., 2012), and it is a fundamental mathematical concept crucial for developing various mathematical skills and understandings. Since divisibility rules are linked to division, and division is linked to multiplicative thinking, when students generalize the divisibility rules conceptually, this might develop students' multiplicative thinking (Young-Loveridge & Mills, 2012). From this point of view, since this issue is crucial for students to understand topics such as place value better, finding common factors, finding the smallest common multiple, and division of fractions, allowing students to reach generalization of divisibility rules conceptually might enable them to make conceptual sense of other areas of mathematics.

According to the literature, students tend to see the divisibility rules as a subject to memorize (Posamentier, 2003). Based on the literature review, it can be concluded that students generally learn the divisibility rules in a procedural rather than conceptual way. To illustrate, Zazkis and Campbell (1996) argue that there is a risk in treating divisibility rules as a simple routine calculation of digits in a number. Patodi (2021) also states that most students need to learn the conceptual background behind the divisibility rules. Posamentier (2003) suggests that instead of simply memorizing rules, divisibility rules should be taught to students enjoyably. In light of this suggestion, the present study has been designed so students can understand and conceptualize divisibility rules. By understanding the reasoning behind the divisibility rules 2, 3, and 6, students might learn the subject conceptually, and they do not need to memorize the rules without understanding.

Furthermore, several researchers argue that most teachers and pre-service teachers also need to understand the rationale behind divisibility rules. For instance, Patodi (2021) noted that the reason why students do not know the conceptual background of the rules may be due to the fact that many teachers do not know the proof of divisibility rules. Similarly, Zazkis and Campbell (1996) conclude that many pre-service teachers who participated in their study either incorrectly applied or overgeneralized the divisibility rules. Thus, the findings of this study might be an excellent example of how teachers can teach the divisibility rules conceptually to the students by using activities in a technology-enhanced environment. In other words, this study might be an implication for teachers to teach the divisibility rules to students conceptually instead of just giving students a set of rules to memorize.

The existing literature mainly focuses on teachers' or pre-service teachers' understanding of divisibility rules rather than students' understanding. From this point of view, the present study's findings might have a valuable contribution to the literature on how students conceptualize divisibility rules since this study is focused on students' understanding. Thus, the study's focus on students' understanding has the potential to significantly advance the literature, given the limited number of studies conducted on students in this area.

1.3. Definitions of Important Terms

The essential terms which are used in this study are given with their meanings below:

Divisibility Rules: Divisibility rules are a short way of deciding whether a given integer n is divisible by a fixed integer p without performing the division algorithm (Zazkis & Campbell, 1996). Generally, calculating the digits of a given number helps us understand the divisibility. For instance, the divisibility rule by 2 is as follows: "Numbers with 0, 2, 4, 6, or 8 digits in the one's digit (even numbers) can be divided by two without remainder" (MoNE, 2018a, p.32). The divisibility rule by three is as follows: "Numbers whose sum of digits in their digits is a multiple of 3 are divisible by three without remainder" (MoNE, 2018a, p.32). The divisibility rule by six is as follows: "Numbers divisible by both 2 and 3 without remainder are divisible by six without remainder" (MoNE, 2018a, p.33).

Generalization: "The concept of generalization is most commonly understood as a duality between going from particular to general and seeing the particular through the general" (Dumitrascu, 2017). In this study, the generalization means reaching a general expression of the divisibility rules by 2, 3, and 6.

Social Constructivism: Social constructivism is one of the learning theories that indicate the influence of social interaction on cognitive development. This theory assumes that understanding, significance, and meaning are developed in social interaction with other people (Amineh & Asl, 2015; Kozulin, 1990). In this study, relation between social constructivism and zone of proximal development is used.

Zone of Proximal Development: Psychologist Lev Vygotsky defined the concept of the zone of proximal development as follows: "The distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers." (Vygotsky, 1978, p. 86). In this study, critical questions were asked using the zone of proximal development theory to prevent students' possible misconceptions and to learn divisibility rules conceptually.

Virtual manipulatives: "Virtual manipulatives are internet or computer software-based simulations of physical manipulatives" (Bouck & Flanagan, 2010, p 187). In this study, base-ten blocks were used as a virtual manipulative.

CHAPTER 2

LITERATURE REVIEW

The study aims to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. The relevant literature review is presented in this chapter. Based on the study's aims, the literature review section is categorized under the following headings: the division and divisibility rules, studies on divisibility rules, social constructivism, and Vygotsky's zone of proximal development.

2.1. Division and Divisibility Rules

Division is a fundamental arithmetic operation in mathematics and plays an essential role in various mathematical contexts, from basic arithmetic to advanced algebra. Children's ability to divide is the most significant determinant of later arithmetic development (Ellis et al., 2018). Conceptually understanding the division of whole numbers is crucial for mathematical concepts such as algebra, rational numbers, and proportion. More specifically, Young-Loveridge and Mills (2012) emphasized that working with division in whole numbers is an essential background for working with rational numbers.

Division is crucial because it is the basis of other mathematical fields. For instance, it is associated with multiplication; that is, division or multiplication is used depending on the problem structure. Recognizing the relationship between multiplication and division, understanding when division is required and how it relates to multiplication, and developing efficient mental strategies are defined as multiplicative thinking, which is a big idea (Siemon et al., 2012). The term multiplicative thinking refers to a specific type of thinking used to solve various problems involving multiplication and division (Young-Loveridge & Mills, 2012).

Zazkis and Campbell (1996) stated that a division algorithm can always determine the division of one number by another number. They emphasized that implementing the division algorithm can sometimes be tedious and time-consuming. The divisibility rules allow one to infer divisibility without performing division, even if the quotients or remainders are unknown. Some researchers define divisibility rules as a short way of deciding whether a given integer n is divisible by a fixed integer p without carrying out the division algorithm (Laudano & Donatiello, 2020; Peretti, 2015; Potgieter & Blignaut, 2018; Zazkis & Campbell, 1996). To understand divisibility, it is usually necessary to calculate the digits of a given number simply without performing the division (Peretti, 2015). Similarly, Koether (1973) pointed out that divisibility is a topic of interest in number theory. In addition, some studies have shown that understanding the logic behind a divisibility rule increases students' development of multiplicative thinking (Young-Loveridge & Mills, 2012). In other words, understanding the reason behind a divisibility rule is crucial for students to understand multiplication, division, and, therefore, multiplicative thinking (Siemon et al., 2012).

However, the research studies indicate that students have difficulties with divisibility rules. Some studies conclude that students' insufficient practice of exercises on divisibility rules resulted in rapid forgetting (Yünkül, 2006). Additionally, the benefits of quickly dividing one number by another, also known as divisibility rules, are often overlooked (Harrel & Slavens, 2009). Moreover, Posamentier (2003) states that students often see arithmetic as a subject to be memorized, and therefore, arithmetic is a burden for them. Similarly, Patodi (2021) indicates that most students do not know the conceptual background behind the divisibility rules. He explains that the reason for this situation is that the proof of the divisibility rules is not known by many teachers. Even if teachers know the reason behind the division rules, they hesitate to explain it to students because it requires complex algebra, and these reasons result in students not knowing the conceptual background behind the division rules (Patodi, 2021). Moreover, other studies conclude that most of the preservice teachers also do not know the logic behind the divisibility rules, as well. To illustrate, Zazkis and Campbell (1996) conclude that many teachers overgeneralize and misapply the divisibility rules when asked to check divisibility without giving them the divisibility rule.

2.2. Studies on the Divisibility Rules

When the relevant literature is examined, it is seen that there are studies on divisibility rules, such as studies on discovering the divisibility rules with the help of groupable materials, studies on finding a general divisibility test for prime and non-prime numbers, and studies on teachers' or preservice teachers' understanding of the reason behind the division and divisibility rules. Therefore, the studies concentrating on these aspects are presented in this section.

Young-Loveridge and Mills (2012) observe mathematics teachers when they are teaching the divisibility rule 9 to seventh and eighth-grade students. They believe that teaching the logic behind the divisibility by 9 would provide an opportunity to develop students' multiplicative thinking. During the intervention, teachers used grouped materials (e.g., translucent plastic boxes and plastic beans) to prove the rule of divisibility by 9, and students were expected to construct the requested numbers by using plastic beans and translucent plastic boxes in such a way that every ten beans would fit into one plastic box. For instance, when the number 135 was asked, students preferred to use a hundred beans (ten plastic boxes), thirty beans (three plastic boxes), and five single beans, as seen in Figure 2.1.

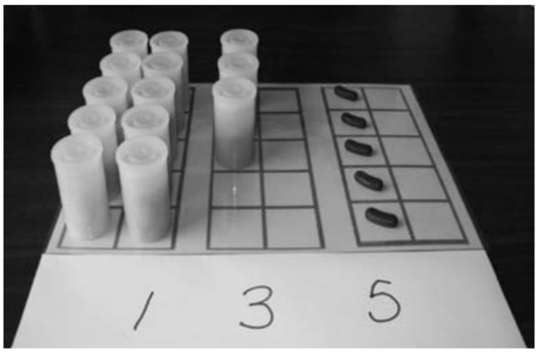

Figure 2. 1. Representation of the number 135 (Young-Loveridge & Mills, 2012)

During the lesson, the teachers asked the students how many groups of 9 were in the number 135. Students noticed a leftover bean from every plastic box, and they took the remainder of the beans to the top of the plastic boxes, as seen in Figure 2.2. The study concluded that students could realize the leftovers from each plastic box and combine the leftover beans with the single beans. Also, they could calculate how many groups of 9 beans were.

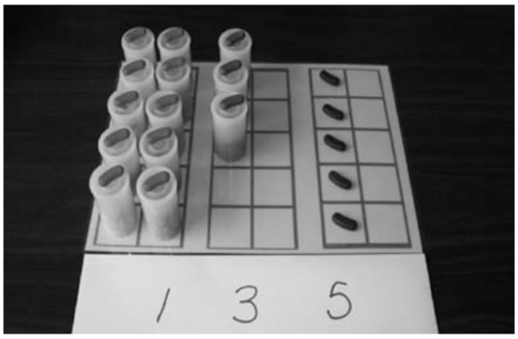

Figure 2. 2. The leftover beans of the number 135 (Young-Loveridge & Mills, 2012)

At the end of the intervention, students could see that the leftover ones corresponded to the digits of a number, which is why the rule of divisibility was 9. This study concludes that the divisibility rule by 9 provides an opportunity to enhance students' multiplicative thinking (Young-Loveridge & Mills, 2012). The researchers suggest that the divisibility rules of other numbers can be explored to develop students' thinking rather than memorizing a rule. Therefore, division and understanding the logic behind the divisibility rules are essential for developing a student's multiplicative thinking.

Similarly, Harrell and Slavens (2009) investigated using base ten diagrams to teach divisibility rules by 2, 3, 4, 5, 6, 8, and 9 to sixth-grade students. In this investigation, students already knew the divisibility rules, and this investigation aims to show them why the divisibility tests work with the help of the base ten blocks, activity sheets, calculators, and pencils. Figure 2.3. below is an example of the activity sheet and ten base diagrams used during the investigation.

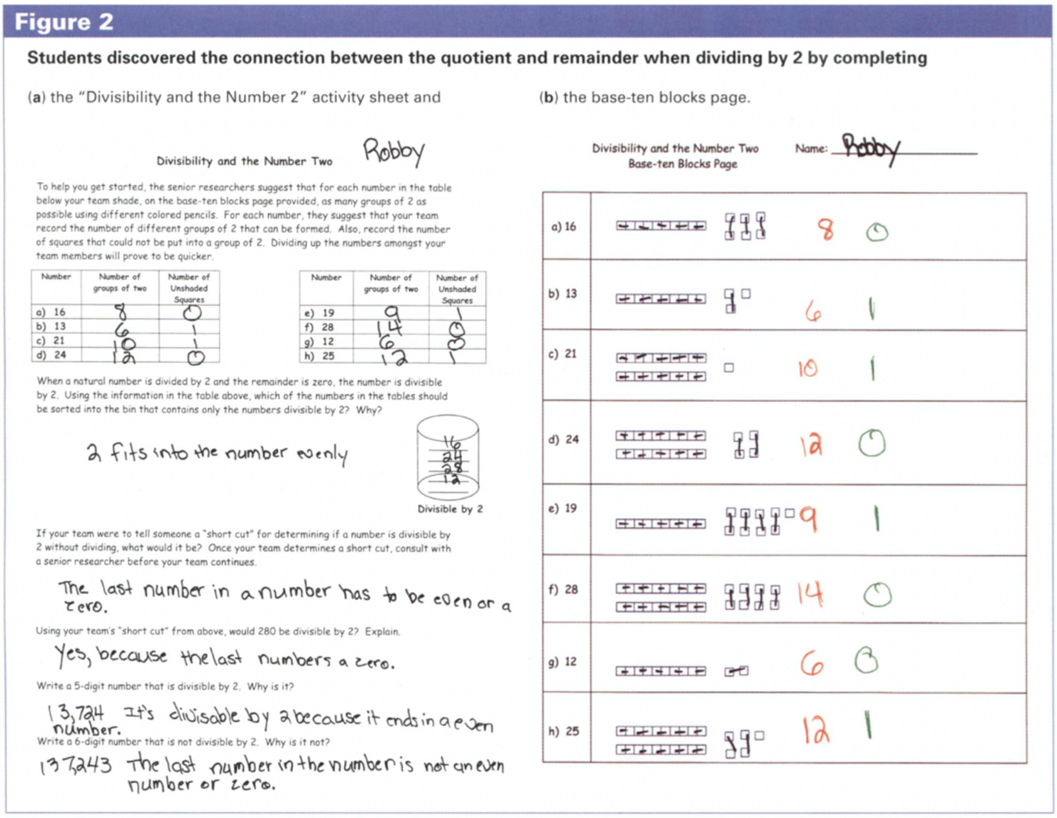
Figure 2

Students discovered the connection between the quotient and remainder when dividing by 2 by completing

(**a**) the "Divisibility and the Number 2" activity sheet and

Divisibility and the Number Two — Robby

To help you get started, the senior researchers suggest that for each number in the table below your team shade, on the base-ten blocks page provided, as many groups of 2 as possible using different colored pencils. For each number, they suggest that your team record the number of different groups of 2 that can be formed. Also, record the number of squares that could not be put into a group of 2. Dividing up the numbers amongst your team members will prove to be quicker.

Number	Number of groups of two	Number of Unshaded Squares
a) 16	8	0
b) 13	6	1
c) 21	10	1
d) 24	12	0

Number	Number of groups of two	Number of Unshaded Squares
e) 19	9	1
f) 28	14	0
g) 12	6	0
h) 25	12	1

When a natural number is divided by 2 and the remainder is zero, the number is divisible by 2. Using the information in the table above, which of the numbers in the tables should be sorted into the bin that contains only the numbers divisible by 2? Why?

2 fits into the number evenly

16, 24, 28, 12 — Divisible by 2

If your team were to tell someone a "short cut" for determining if a number is divisible by 2 without dividing, what would it be? Once your team determines a short cut, consult with a senior researcher before your team continues.

The last number in a number has to be even or a zero.

Using your team's "short cut" from above, would 280 be divisible by 2? Explain.

Yes, because the last numbers a zero.

Write a 5-digit number that is divisible by 2. Why is it?

13,724 It's divisable by 2 because it ends in a even number.

Write a 6-digit number that is not divisible by 2. Why is it not?

137,243 The last number in the number is not an even number or zero.

(**b**) the base-ten blocks page.

Divisibility and the Number Two
Base-ten Blocks Page — Name: Robby

a) 16	8	0
b) 13	6	1
c) 21	10	1
d) 24	12	0
e) 19	9	1
f) 28	14	0
g) 12	6	0
h) 25	12	1

Figure 2. 3. The activity sheet taken from the investigation of Harrell and Slaven (Harrell & Slaven, 2009, p.373)

The activity sheet is shown in Figure 2.3. poses some questions to the students. In the activity sheet, two-digit numbers are given in the table, and there is also a representation of these numbers using base ten blocks. The students were expected to color the groups of 2 from the representation of these numbers using base ten blocks. For instance, in the number 16, the student showed the numbers of the groups of 2 on the activity sheet and showed that there were eight groups. In the number 16, all the boxes could be grouped in pairs, and all the squares were colored. Therefore, the student wrote zero for the number of squares not colored on the activity sheet. In short, the numbers in orange in the b section of the activity sheet show the number of groups of 2, and the numbers in green show the number of unpainted squares left after the coloring. The students were then informed that the remainder must be zero to divide by two, and questions were asked about whether the students found any shortcuts to make faster decisions. Then, students were asked to write five-digit numbers divisible by two and six-digit numbers not divisible by two, and questions were asked about how they chose these numbers. According to Harrell and Slaven (2009), these investigations encourage students to problem-solve and reason, and they require communication skills and making connections between various mathematical concepts and principles. The article aimed to allow students to apply the divisibility tests of the numbers 2, 3, 4, 5, 6, 8, and 9. While the students were performing the divisibility test, they were asked questions that allowed them to find the measurement interpretation of division, i.e., the number of the groups on base ten diagrams. At the end of the investigation, they concluded that using the base ten diagrams can facilitate students' understanding of why several divisibility tests work.

Similar to the investigation above, there was an article about using base-ten blocks to prove why divisibility rules work. Bennent and Nelson (2002) illustrated how base-ten pieces can make divisibility rules more explicit and encourage the exploration of traditional divisibility tests and students' divisibility tests. In other words, they examined how the base-ten pieces could be used to decide the divisibility of a number to another number. They used the measurement concept of division while checking the divisibility as in the investigation of Harrell and Slavens (2009) and Young-Loveridge and Mills (2012). Figure 2.4. is taken from the article as an example.

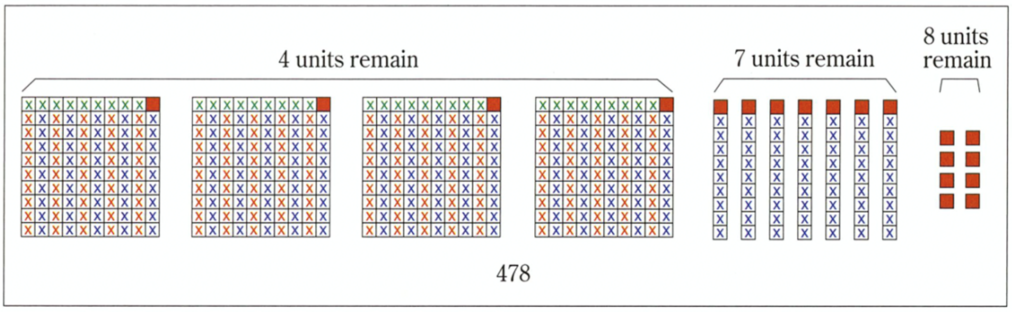

Figure 2. 4. The reason of the divisibility rule by nine taken from the article (Bennent & Nelson, 2002, p.462)

Figure 2.4. represents 478 with base ten blocks using 4 hundreds, 7 tens, and 8 ones. In this representation, each nine-group square on the hundreds and tens blocks is marked with red, green, and blue colors. That is, this representation illustrated the groups of 9 in base-ten blocks, and the remainder of the squares on the base-ten blocks are colored in red. After that, the total number of the remainder (i.e., the red squares) is calculated. This process aims to understand why the divisibility rule by 9 occurs and why the sum of the digits is calculated. Bennett and Nelson (2002) concluded that determining the divisibility starts with initially stating the situation by using base-ten pieces, then stating the situation by using the digits of the number, and finally reaching generalizations by using the variables. The researchers used the measurement concept of division, but they recommended that sometimes the sharing concept of division could be more straightforward.

According to the structure of the problem, there are two primitive division models, namely partitive and quotative division (Fischbein et al., 1985). In the partitive division, also known as fair sharing, the total number of groups is given, and the numbers in each group are asked. On the other hand, in the quotative (measurement) division, the number of objects in each group is given, and the total number of groups is asked. The literature typically explains division rules in terms of measurement division. Unlike the previous studies, this study uses the fair sharing meaning of division, as creating groups of equal size for young children helps them develop a deeper understanding of mathematical concepts. More specifically, sharing a collection of objects or dividing a pie equally among friends are fundamental concepts of fair sharing (Wilson et al., 2012). In this study, students will share a collection of objects equally among the groups with the help of the virtual base-ten block.

Moreover, plenty of research has been done to find a method for checking the divisibility of one number by another (Koether, 1973). Researchers mainly focused on developing divisibility tests to determine the prime numbers' divisibility (Szetela, 1980). For instance, Szatela (1980) indicated that the reason for discovering a divisibility test for prime numbers is that the divisibility rule of a prime number is hard to remember. The article discussed the general divisibility test discovered by one of the students in the classroom. The mentioned divisibility test is used to determine the rule for divisibility by 7 and is presented as an exciting breakthrough in number theory. The researcher also showed that the rules can be extended to various integers and can be used to determine whether these numbers are divisible by several numbers at once by emphasizing the importance of these rules in motivating students to look for patterns in number theory and to formulate hypotheses. Also, the researcher indicated some limitations to the rule, such as the multiplication of large numbers, which is complex to follow (Szetela, 1980).

Similarly, there are studies about the divisibility tests aimed at investigating the discoveries made by students. To illustrate, it is concluded that some students found a divisibility test by 4 and 8 better than the rules in the books (Bennett &Nelson, 2002; Dessart et al., 1999). The students develop a divisibility test: "When you double the tens digit of a two-digit number and add the ones digit, if the sum is divisible by 8, then so is the original number. For a three-digit number, take the hundreds digit with the tens digit, double them, and add the ones digit." For instance, according to the article, the number 96 is divisible by 8 since nine times two is eighteen, and eighteen plus six is twenty-four, which is divisible by eight. However, Bennett and Nelson (2002) pointed out that although the divisibility rule tests developed by students seem to be correct, students can only numerically prove this. As a solution, they suggested using base-ten blocks to generalize the divisibility tests and arrive at a general statement rather than relying solely on numerical proofs.

Different from the studies mentioned above, there are also studies on the teachers' and preservice teachers' understanding of the divisibility rules. To illustrate, Zazkis and Campbell (1996) conducted a study with preservice teachers. They aimed to analyze the relation of divisibility with division, multiplication, prime and composite numbers, factorization, and divisibility rules, prime decomposition. The research methodology involved conducting clinical interviews with 21 preservice teachers. The clinical interviews investigate the participants' depth of understanding and problem-solving strategies related to divisibility and multiplicative structures. The findings suggest that preservice teachers exhibit predominantly procedural understandings, focusing on computational aspects rather than a firm grasp of the underlying conceptual framework. The study emphasizes the importance of equipping preservice teachers with a deep conceptual understanding of mathematical concepts. Their current level of understanding may need to be revised for effective teaching. This view underlines the need for teacher education programs to focus on developing conceptual knowledge and procedural skills (Zazkis & Campbell, 1996).

Similarly, Patodi (2021) states that the reason why the rules are not known by the students is that the teachers do not know them. He explained that the reason for this situation is that the proof of the divisibility rules is not known by many teachers. Even if teachers know the reason behind the division rules, they hesitate to explain it to students because it requires complex algebra, and these reasons result in students not knowing the conceptual background behind the division rules (Patodi, 2021).

Based on the literature review, there are not many studies that focus on students' exploration of divisibility rules. The existing literature suggests that students generally lack a conceptual understanding of these rules, and similarly, preservice teachers also struggle with the logic behind them. Therefore, it was recognized that students might require additional support in understanding division rules conceptually. Accordingly, this study was developed to offer the necessary assistance to students as they work on grasping division rules, utilizing Vygotsky's zone of proximal development.

2.3. Constructivism and Social Constructivism

Constructivism is fundamentally a learning theory that emphasizes the learning process of learners (Fosnot & Perry, 2005) and indicates that students construct a new understanding using their existing understanding. Constructivism represents a perspective that emphases the active construction of knowledge over the passive acquisition of knowledge from external sources. (Applefield et al., 2001). Similarly, constructivism is defined as: "Rather than behaviors or skills as the goal of instruction, cognitive development, and deep understanding are the foci; rather than stages being the result of maturation, they are understood as constructions of active learner reorganization. Rather than viewing learning as a linear process, it is understood to be complex and fundamentally nonlinear" (Fosnot & Perry, 2005, p.25). From this perspective, it can be concluded that learners are not passively but actively involved in constructing their knowledge.

Many articles in the literature discuss how students construct knowledge from their experiences in their minds. To illustrate, Hoover (1996) stated that two main aspects encompass constructed knowledge. The first one is that the students construct a new understanding using their existing knowledge; that is, the prior experiences of a learner affect the construction of new knowledge. The second one is that during the learning process, a learner is not passive but active; that is, it can be concluded that they actively construct their new knowledge on existing knowledge in constructivist learning environments. While doing so, students can adapt to the information they learn in the learning process if it does not match their existing understanding (Hoover, 1996). Furthermore, Bada (2015) emphasized that the active involvement of students in the learning process enables them to learn more, which is a benefit of constructivism. They also noted that constructivism focuses on teaching students how to think and understand rather than simply memorizing information and that lasting learning is achieved through this emphasis on thinking and understanding. Additionally, teachers' role in constructivist learning environments is to guide learners as they actively build their knowledge rather than simply lecturing (Hoover, 1996). Therefore, constructivism promotes a learning environment where students can actively construct their knowledge while guided by teachers. Powel and Kalina (2009) mentioned that constructivism is currently being discussed in many schools as the best learning method for students. To create a learning environment suitable for constructivism, teachers need to know where the student's existing knowledge lies and provide guidance when necessary.

Furthermore, constructivism in education offers a holistic approach that effectively addresses cognitive and social aspects. Two main trends can be detected in mathematics education research in the last decade (Cobb, 1994). The two main trends are cognitive and social constructivism; they follow the ideas of theorists such as Jean Piaget and Lev Vygotsky, emphasizing the importance of social interaction and cognitive development in learning. In other words, the mind is the central part of learning according to cognitive constructivists, although social interaction is the central part of learning according to social constructivists (Fosnot & Perry, 2005). Although the central parts of cognitive and social constructivism differ, the central aspect is similar: Students construct their knowledge in the learning process (Schcolnik et al., 2006). There are similarities between Vygotsky's sociocultural theory and Piaget's cognitive theory (Powel & Kalina, 2009). The most significant similarity of the theories is that both emphasize that students play an active role in constructing their knowledge. Nonetheless, there are also differences between these two theories. The reason for the divergence between the theories lies in the fact that Piaget's cognitive theory is about students developing their knowledge individually, whereas Vygotsky's theory is about students developing their knowledge in diverse forms of social interaction rather than individual efforts (Lourenço, 2012; Schreiber & Valle, 2013; Powel & Kalina, 2009). In other words, social constructivism is characterized by students' social interactions combined with a personal critical thinking process (Powel & Kalina, 2009). Such social interaction can involve using tools such as an abacus or pencil and symbols such as mathematical formulas and language.

2.4. Vygotsky’s Zone of Proximal Development

The zone of proximal development is the central aspect of Vygotsky's social constructivism theory. Psychologist Lev Vygotsky defined the concept of the zone of proximal development as follows: "The distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). In other words, it is a concept introduced by Vygotsky that emphasizes the difference between what a learner can achieve independently and what he or she can achieve with the guidance and help of a more informed person (Vygotsky, 1978). The ZPD shows the influence between two people with different proficiency levels, such that the student with less proficiency has become an independent learner (Chaiklin, 2003). More specifically, the zone of proximal development is the area between what a person can do on their own and what they can do with the help of someone more knowledgeable and competent than that person, and it is in this area that Vygotsky (1978) claims that learning is most effective (Danish et al., 2011; Devi,2019).

According to Lev Vygotsky (1978), children are not all at the same mental level, even if they are the same age. Vygotsky (1978) indicated that children's learning and development were closely connected from birth. In children starting school, writing, reading, and arithmetic operations were taught to the child, considering the child's developmental levels. Vygotsky (1978) also claimed that their potential developmental level should be considered in addition to the child's current developmental level. He stated that learning should be compatible with the child's developmental level and potential development. Vygotsky (1978) theorized that a child's development encompasses both the actual level and the zone of proximal development. The actual level refers to what children can achieve independently. At the same time, the zone of proximal development includes skills that are still developing and require the help of a more experienced individual. The zone of proximal development aims to challenge students with more advanced ideas and emphasizes the role of interaction with others and tools in supporting learning and development (Danish et al., 2011).

Moreover, Wells (1999) stated the difference between the modes of interaction in the zone of proximal development, adult-child, and child-child, as suggested by Vygotsky. It was noted that these modes are not entirely independent of each other. In addition, he stated that there are two forms of adult-child interactions: work in the ZPD of the individual child (individual assistance) and work in the larger ZPD (communal assistance). In other words, adults can help students in two ways. In the form of a larger ZPD, assistance occurs during the whole-class discussions. In the form of individual assistance, in contrast, assistance occurs during the individual's zone of proximal development. Wells (1999) indicated that, in both, the teacher's role is shaped by the needs and situations of the students. Teachers' role is to maintain children's understanding by observing their actions on a specific task. The role of the teacher is responsive. He stated that although teachers have a plan in mind at the beginning of the lesson, this plan can change according to the answers given during the lesson. So, adjusting the plan by considering the students' readiness and needs is vital. Therefore, the zone of proximal development is a concept that reveals a child's independence in the concepts that are unfamiliar to them.

2.5. Summary of Literature Review

Reviewing the relevant literature has shown that divisibility is an essential concept in mathematics since learning it enable students to comprehend other topics in mathematics and also develop students multiplicative thinking which is a big mathematical idea (Laudano & Donatiello, 2019; Peretti, 2015; Potgieter &

Blignaut, 2018; Siemon et al., 2012; Young-Loveridge & Mills, 2012; Zazkis & Campbell, 1996). When dividing one number by another, the determination can be made using a division algorithm. However, the implementation of division algorithm might sometimes be time-consuming and boring for students. (Zazkis and Campbell, 1996). Thus, divisibility rules are a short way for deciding divisibility of one number to another (Laudano & Donatiello, 2020; Peretti, 2015; Potgieter & Blignaut, 2018; Zazkis & Campbell, 1996). However, upon the reviewing the relevant literature, it is seen that students have some difficulties with divisibility rules and most of students do not learn divisibility rules in a conceptual way (Patodi, 2021; Posamentier, 2003). Most students know procedurally the checking of the last digit of a number for deciding the divisibility by 2, 5, or 10. However, only some students know how to check the sum of the digits of a number to decide the divisibility by 3 or 9 or check the last two digits of a number for divisibility by 4. Also, for the divisibility by 7 or 13, it is not easy to find anyone other than a mathematician who knows the divisibility tests of these numbers (Ganzell, 2017). Thus, it is essential to learn divisibility rules conceptually and overcome difficulties of students to understand other areas of mathematics.

There are numerous articles about the divisibility rules shows that using groupable materials helps students to understand the reason behind the divisibility rules (Bennett & Nelson, 2002; Harrell & Slavens, 2009; Young-Loveridge & Mills, 2012) and finding a general divisibility tests for prime numbers (Koether, 1973; Szatela, 1980), and understanding of preservice teachers (Zazkis & Campbell, 1996). When the reviewing the relevant literature, we conclude that for understanding divisibility rules conceptually rather than procedurally, students might be more support. Researchers suggests that students learn better in their proximal development zone (Danish et al., 2011; Devi, 2019). Thus, the aim of the study was to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews.

CHAPTER 3

METHOD

In this study fifth grade students' process of generalizing the divisibility rules were examined. This chapter aims to explain the methodology used in the present study. Therefore, this chapter consists of the design of the study, sampling and participants of study, data collection tools, data collection procedure, the pilot study, data analysis procedure, trustworthiness, researcher role, ethical consideration and limitations of the study.

3.1. Research Design

The qualitative research method was used for the present study to examine fifth-grade students' process of generalizing the divisibility rules of 2, 3 and 6. Fraenkel and Wallen (2006) defined qualitative research as follows: "A researcher might wish to know more than just "to what extent" or "how well" something is done. He or she might wish to obtain a more complete picture, for example, of what goes on in a particular classroom or school" (p.425). Qualitative research is related to research examining the nature of relationships, activities, situations, or materials (Fraenkel & Wallen, 2016). Qualitative researchers focus on how people construct their worlds and the interpretations of their experiences and try to make sense of them (Merriam & Tisdell, 2015). Furthermore, Creswell (2007) stated that qualitative research aims to examine the problems of individuals or groups, and it requires data collection and interpretation of the data in a natural environment.

In this study, I was a part of the study as a researcher interested in analyzing the students' process of generalizing the divisibility rules of 2, 3, and 6. As Creswell indicated, I plan to examine the students' thoughts in natural settings in detail. More specifically, I focused on the interpretations of three students in their natural environment. In addition, as Merriam and Tisdell stated, I plan to obtain a more complete picture of the students' thinking as they conceptualize the divisibility rules. In other words, I conducted a qualitative study to examine the students' thoughts while they generalized the divisibility rules of 2, 3, and 6 more deeply.

Furthermore, the literature has shown that there are five approaches to qualitative research designs, Creswell (2007) named these approaches as narrative research, phenomenological research, grounded theory research, ethnographic research, and case study research. A qualitative case study was used in the present study. The characteristics of the case study are explained in detail below.

3.1.1. Case Study

Frankel and Wallen (2006) described a case study as a detailed analysis of an individual, a few individuals, a class, a school, an event, or an activity. Furthermore, Merriam and Tisdell (2015) defined a case study as an in-depth examination and description of a system bounded by time and space. This bounded system could involve a single person, a program, a group, or an institution (Creswell, 2007; Merriam & Tisdell, 2015).

In addition, Yin (2003, 2008) indicated that there was different case studies based on the number of units of analysis and the number of cases. Single-case and multiple-case designs were associated with the number of cases, while embedded-case and holistic-case designs were associated with the number of units of analysis. Moreover, Yin (2003, 2008) distinguished between single-case and multiple-case designs as multiple-case studies include comparing cases. The model of the single-case embedded design is given below in Figure 3.1.

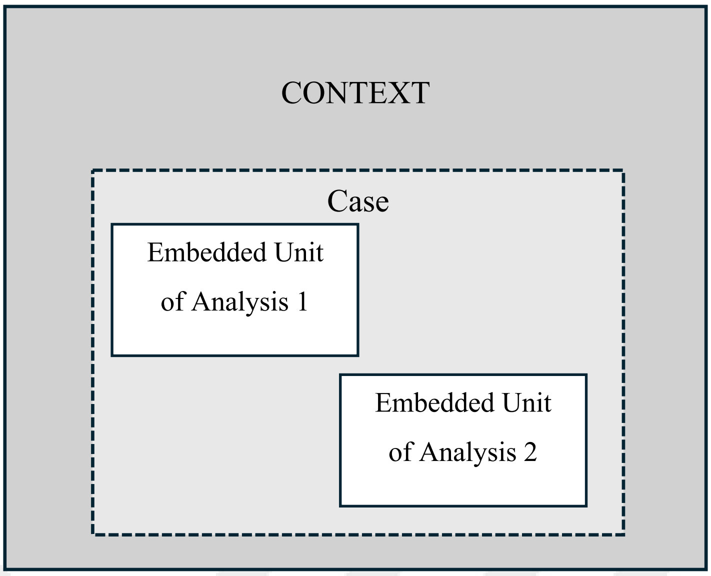

Figure 3. 1. Single-case embedded design (Yin, 2003, p. 40)

In the present study, the case study research was appropriate since it aims to deeply examine the generalization process of the divisibility rules of 2, 3, and 6 in a bounded system. The case was three fifth-grade students, and the case was bounded with Turkish Education System and fifth grade curriculum. The research design of the present study was single-embedded. This is because the case involved three fifth-grade students, and the generalization process of the divisibility rule of 2, 3, and 6 were the units of analysis, respectively. Moreover, the current study did not aim to compare the three fifth-grade students with each other. For these reasons, the single-embedded design was appropriate for the present study. The model of this study with respect to single-case embedded design is given in the Figure 3.2. below.

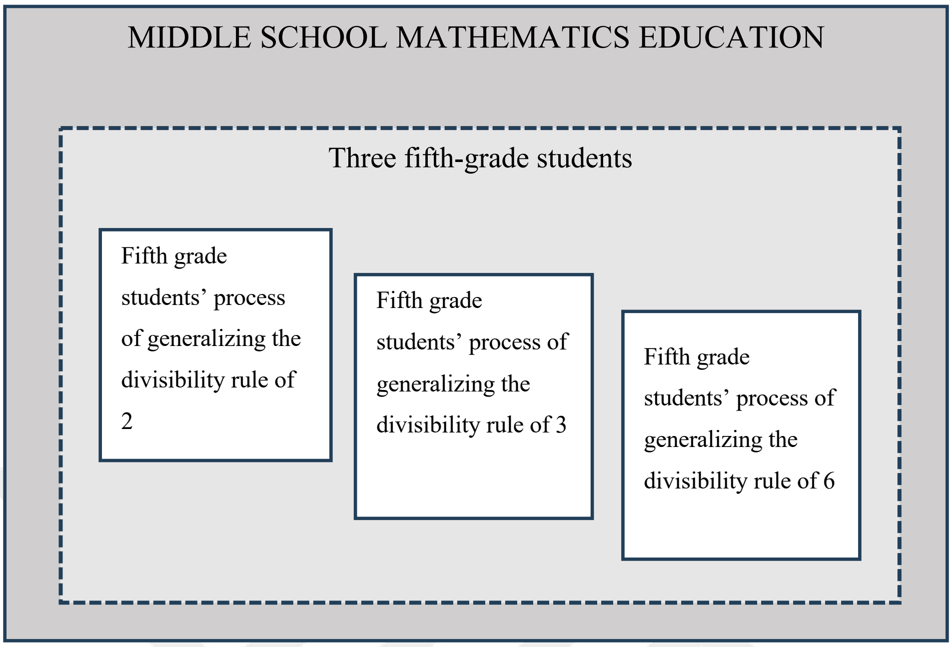

Figure 3. 2. Single-case embedded (three units of analysis) design

3.2. Sampling and the Selection of the Participants

Merriam and Tisdell (2015) present two basic types of sampling: probability and nonprobability sampling. Probability sampling helps generalize the results of the sample to the population. However, in qualitative studies, the researcher's aim is not to generalize the study's findings to the population. Merriam and Tisdell (2015) state that sample selection in qualitative studies is generally non-random, purposeful, and small. For this reason, they state that nonprobability sampling can be used for qualitative studies. Purposive sampling is the most helpful method of nonprobability sampling. Thus, purposive sampling was used to select participants for the present study by taking two criteria into consideration. The first criterion is easy access to participants. The participants had to be selected considering their accessibility since the researcher spent considerable time on data collection. The second criterion is that participants be students who completed the fifth grade and just started the sixth grade. In Turkish curriculum, divisibility rules are taught in the sixth grade (MoNE, 2018a). Since the research aimed to investigate students' process of generalizing divisibility rules, it was determined that students who previously had not learned the rules of divisibility would be studied. In other words, while selecting the participants, they were asked whether they knew the divisibility rules before or not. Three fifth-grade students who did not know the divisibility rules were selected to participate in the study based on those criteria. Some characteristics of the participants are displayed in Table 3.1. below. This table provides information about the gender, grade levels, types of school, and mathematics grade average of the participants. There was one boy and two girls, all studying at private schools in Ankara. All three participants expressed a fondness for mathematics and had high grade point averages in mathematics.

Table 3. 1. Characteristics of Participants

Participant Students	**Gender**	**Grade Level**	**Types of School**	**Mathematics Grade Average**
Elif	Girl	5^{th} grade	Private	95
Doğan	Boy	5^{th} grade	Private	99
Nur	Girl	5^{th} grade	Private	96

Three different clinical interviews were conducted with each participant separately for data collection. During clinical interviews, activity sheets were given to the students. Detailed information about the data collection tools is given below.

3.3. Data Collection Tools

The present research study aimed to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. In order to achieve the aim of the study, data were collected via one-to-one clinical interviews conducted separately with each participant. The technique for collecting data was clinical interviews. In other words, an activity sheet was completed during one-to-one clinical interviews. Firstly, the activity sheet on divisibility by 2 was completed. Then, the activity sheet about divisibility by 3 was completed. Finally, the activity sheet on divisibility by 6 was completed. Each participant was interviewed separately, and completing the activity sheets was the subject of three separate clinical interviews. Organization of the clinical interviews are displayed in Table 3.2. below.

Table 3. 2. Organization of Clinical Interviews

Participants	**Organization of Clinical Interviews**
Elif	Interview 1- Activity Sheet on divisibility by 2 Interview 2- Activity Sheet on divisibility by 3 Interview 3- Activity Sheet on divisibility by 6
Doğan	Interview 1- Activity Sheet on divisibility by 2 Interview 2- Activity Sheet on divisibility by 3 Interview 3- Activity Sheet on divisibility by 6
Nur	Interview 1- Activity Sheet on divisibility by 2 Interview 2- Activity Sheet on divisibility by 3 Interview 3- Activity Sheet on divisibility by 6

The students were given an activity sheet and a tablet during the clinical interviews. A virtual manipulative had previously been opened on the tablet. The students were expected to construct the numbers written on the activity sheet on the tablet simultaneously during the clinical interviews. The following sections provide detailed information about the virtual manipulative, activity sheets, and clinical interviews, respectively.

In the current study, the participants performed the activities using virtual base-ten blocks. The interface of the virtual manipulative is shown in the following figure.

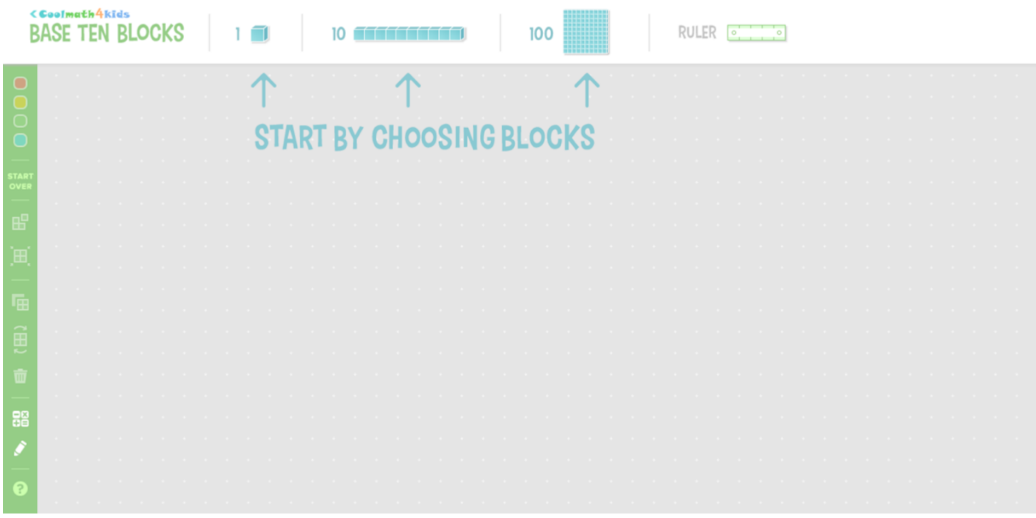

Figure 3. 3. Interface of the virtual manipulative

Using the virtual manipulative shown in Figure 3.3., the students could construct the requested numbers using base-ten blocks on the tablet. The students could use ones, tens, or hundreds blocks as many times as needed. As mentioned before, the students were given a tablet and a pen during the clinical interviews. This virtual manipulative was opened on the tablet, and the participants were expected to simultaneously create the numbers written on the activity sheet using virtual base-ten blocks. Additionally, the virtual manipulative had several features. For instance, if the student receives more blocks than needed to construct the requested number, he/she can delete the extra blocks by pressing the delete button on the left side of the screen. Another useful feature was a button that allowed students to trade when necessary. The button could be used to trade a ten block into ten ones and a hundreds block into ten tens blocks.

As mentioned before, the data were collected separately via clinical interviews with each participant. Moreover, the activity sheets were completed during the one-to-one clinical interviews. There were three different activity sheets, i.e., the first activity sheet was about divisibility by 2, the second activity sheet was about divisibility by 3, and the third activity sheet was about divisibility by 6. The most common feature of these three activity sheets was that the numbers were given in a specific order: first, one-digit, then two-digit, and finally, three-digit numbers were asked of the students. Another common feature was that the students were expected to write a generalization sentence at the end of each activity sheet. Therefore, for example, at the end of the activity about divisibility by 2, the students were expected to write whether they reached any generalization about the divisibility rule of 2. The specific details and the contents of the activity sheets for the research study are described below.

3.3.1. Activity Sheet on Divisibility by 2

The activity sheet on divisibility by 2 included five parts. The numbers in these parts were organized in a way that there was a sequence between numbers such as one-digit, two-digit, and three-digit numbers. In other words, Part 1 consists of one-digit numbers, Part 2 consists of two-digit numbers, and Part 3 consists of three-digit numbers. The researcher aimed to analyze how the students generalize the divisibility rule of 2 in a study planned in this sequence.

Parts 1, 2, and 3 have three subparts, which will be explained below. The first subpart was related to enabling the students to construct the given number by using base ten blocks simultaneously during the clinical interviews. The second subpart was related to using fair sharing, sharing the given number of objects equally between two children, and showing the sharing process using base-ten blocks. The critical step for this part was that the participants needed to pay attention to sharing the base-ten blocks in a way that preserved the integrity of the block structure. The third subpart was related to interpreting the divisibility of the given number. The three subparts are displayed in Figure 3.4. below.

-Can you construct the number ... using base-ten blocks?

-How do you share ... objects between 2 children in such a way that each child gets an equal number of objects?

-Do you think the number ... is divisible by 2? Why?

Figure 3. 4. The subparts

In other words, Parts 1, 2, and 3 included the three subparts mentioned above. In-depth information regarding the numbers in Part 1, Part 2, and Part 3 is as follows.

Part 1 included *one-digit even* and *one-digit odd* consecutive numbers. Part 1 aimed to analyze whether the students could interpret the divisibility of one-digit even and one-digit odd numbers. In addition, the aim of Part 1 was to encourage the students to use ones for constructing the numbers using base-ten blocks. Part 1 and the three subparts of the questions can be seen in Figure 3.5.

Part 1:

- Can you construct the number 8 using base ten blocks?
- How do you share 8 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 8 is divisible by 2? Why?

- Can you construct the number 9 using base ten blocks?
- How do you share 9 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 9 is divisible by 2? Why?

Figure 3. 5. Example of the numbers in Part 1

Part 2 included two-digit numbers. There were questions about two-digit number that are multiples of ten, i.e., the unit place of the number is zero. Also, there were questions about two-digit numbers that are not multiples of ten, i.e., the unit place of the number is different from zero. The aim of giving numbers that are multiples of ten and are not multiples of ten was to analyze whether the students could interpret the structure of base ten blocks (tens and ones)—in other words, giving two-digit numbers that are multiples of ten aimed to encourage the students to use tens blocks while constructing the given number using base-ten blocks. On the other hand, the aim of giving two-digit numbers that are not multiples of ten engaged the students to use ones and tens blocks. This aimed to analyze whether the students could see that the tens block is always divisible by two regardless of how many there are. In other words, the tens block is divisible by two because it consists of ten ones.

The researcher developed the sequences of the two-digit numbers as follows: there were two-digit numbers are multiples of ten with odd digits in the tens place and two-digit even numbers that are not multiples of ten with odd digits in the tens place. Moreover, there were two-digit numbers that are multiples of ten with even digits in the tens place and two-digit odd numbers that are not multiples of ten with even digits in the tens place.

These numbers were asked to see how the students interpreted the even or odd digit in different place values. It will be analyzed to see if it is clear that having an odd digit in the ones place of a number is not the same as having an odd digit in the tens place. Also, a two-digit number that is a multiple of ten with an even digit in the tens place and a two-digit number that is a multiple of ten with an odd digit in the tens place are divisible by two. However, the ones digit of a two-digit number needs to be even in order to be divisible by two. This question was asked to analyze whether the students could make these connections. Detailed information about the numbers is given below.

Part 2:

- Can you construct the number 10 using base ten blocks?
- How do you share 10 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 10 is divisible by 2? Why?

- Can you construct the number 12 using base ten blocks?
- How do you share 12 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 12 is divisible by 2? Why?

Figure 3. 6. Example of the numbers in Part 2

Figure 3.6. shows the example of two-digit numbers that are multiples of ten (e.g., 10) and not multiples of ten (e.g., 12). The two-digit number that is a multiple of ten has an odd digit in the tens place. The number that is not a multiple of ten consists of an odd digit in the tens place and an even digit in the ones place (see Figure 3.6.) In other words, the two-digit number which is not a multiple of ten is an even number with an odd digit in the tens place. Then, the numbers in the figure below were asked (see Figure 3.7.)

- Can you construct the number 20 using base ten blocks?
- How do you share 20 objects between o 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 20 is divisible by 2? Why?

- Can you construct the number 25 using base ten blocks?
- How do you share 25 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 25 is divisible by 2? Why?

Figure 3. 7. Example of the numbers in Part 2

Figure 3.7. shows the example of two-digit numbers that are multiples of ten (e.g., 20) and not multiples of ten (e.g., 25). The two-digit number which is a multiple of ten has an even digit in the tens place. The number that is not a multiple of ten consists of an even digit in the tens place and an odd digit in the ones place. In other words, the two-digit number, which is not a multiple of ten is an odd number with an even digit in the tens place.

The numbers in Figures 3.6. and 3.7. show the combinations of the digits in the ones and tens place of a two-digit number. In other words, a two-digit number with an odd digit in the ones place and an even digit in the tens place and vice versa is asked of the students. Furthermore, for the two-digit numbers that are multiples of ten, numbers with odd digits in the tens place and even digits in the tens place were asked to analyze whether the students could reach a generalization about the divisibility rule of 2. The aim of asking about two-digit numbers that are multiples of ten with an odd digit in the tens place and two-digit numbers that are multiples of ten with an even digit in the tens place was to analyze whether the students reached a generalization about the fact that the ones digit needs to be even for divisibility by 2.

The following Figure, 3.8., had the same purpose as the numbers in Figure 3.6. Numbers that are multiples of ten and those that are not multiples of ten were given to the students. The numbers that are multiples of ten have an odd digit in the tens place, and the numbers that are not multiples of ten have an odd digit in the tens place and an even digit in the ones place. The aim of asking for this part again is to allow the students to generalize their thinking about two-digit numbers.

- Can you construct the number 50 using base ten blocks?
- How do you share 50 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 50 is divisible by 2? Why?

- Can you construct the number 56 using base ten blocks?
- How do you share 56 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 56 is divisible by 2? Why?

Figure 3. 8. Example of the numbers in Part 2

As mentioned before, some numbers are multiples of ten, while others are not multiples of ten. Thus, the aim was to check whether the students could identify that tens blocks are always divisible by 2, and ones can only be divisible if there is an even number of ones. In other words, Part 2 aimed to see how the students generalize that the tens block is always divisible by 2 regardless of how many there are. Therefore, the students needed to consider the place values of numbers to reach this conclusion. The unit place of a number is crucial for divisibility by 2. The aim of Part 2 was to analyze whether the students reached this generalization for two-digit numbers.

Part 3 included three-digit numbers. There were questions about three-digit numbers that are *multiples of ten*, i.e., the unit place of the number is zero. Also, there were questions about three-digit numbers that are *not multiples of ten*, i.e., the unit place of the number is different from zero. The aim of Part 3 was similar to the aim of Part 2. Part 2 and Part 3 differ because Part 2 contains two-digit numbers, and Part 3 contains three-digit numbers. The purpose of Part 3 was to see whether the students realized that the hundreds block is always a multiple of 2 regardless of how many hundreds there are.

The aim of asking about three-digit numbers that are multiples of ten and not multiples of ten was to analyze whether the students could interpret the structure of base ten blocks (hundreds, tens, and ones). In other words, the hundreds block is divisible by 2 since it consists of tens. The structural features of a hundreds and a tens are similar. They are multiples of 2 and thus divisible by 2. A hundreds block consists of ten tens; since ten is a multiple of two, a hundred is also a multiple of 2. No matter how many hundreds there are, they are divisible by 2 for this reason. All numbers that are multiples of ten are also multiples of two, that is, divisible by 2. To establish this connection, the numbers in Part 3 were asked. Similarly, the aim of asking questions about numbers that are not multiple of ten was to see whether the students could make sense of the hundreds, tens, and ones in the number. There were three-digit even numbers with odd digits in the tens and hundreds place and three-digit odd numbers with odd digits in the tens and hundreds place. The students were asked for one hundred, one hundred and one ten, and one hundred, one ten, and one unit. This aimed to make the participants realize the importance of the place values of the numbers. For instance, while asking for the number 111, all the digits of the number have the same number, which is 1. Although the exact number is given in all digits, it is necessary to think about the place values, and this question was included to see if the students would think about it. The aim of asking this was to analyze whether the students reached a generalization that the ones digit needs to be even for divisibility by 2. An example of Part 3 and its three subparts can be seen in the following figure.

Part 3:

- Can you construct the number 100 using base ten blocks?
- How do you share 100 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 100 is divisible by 2? Why?

- Can you construct the number 110 using base ten blocks?
- How do you share 110 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 110 is divisible by 2? Why?

- Can you construct the number 111 using base ten blocks?
- How do you share 111 objects between 2 children in such a way that each child gets an equal number of objects?
- Do you think the number 111 is divisible by 2? Why?

Figure 3. 9. Example of the numbers in Part 3

Furthermore, the three-digit numbers 200 and 300 were asked. The aim of asking about these numbers was to analyze whether the students could identify that there was no difference between three-digit numbers with an even digit in the hundreds place (e.g., 200) and an odd digit in the hundreds place (e.g., 300) when divided by 2. After discussing the divisibility of the given numbers, the participants were asked to give an example of a three-digit number divisible by 2 and asked about how they decided on this number as an example. The aim of asking this question was to see if the students could reach a generalization for the divisibility rule of 2.

The aim of Part 4 was to fill in the table with the numbers interpreted since the beginning of the activity. The students were expected to decide which digits to pay attention to when they decided on divisibility by two. Part 4 aimed to enable the participants to summarize their thinking through the end of the activity. The table can be seen in the following figure.

<u>Part4:</u> Can you fill in the table below.

Numbers	Is this number divisible by 2?	How did you decide on divisibility?
8		
9		
10		
12		
20		
25		
50		
56		
100		
110		
111		
331		
332		

Figure 3. 10. Example of the numbers in Part 4

Part 5 asked the students to write the generalization they had reached at the end of the activity about divisibility by two. The purpose of Part 5 was to see whether the students could make sense of the conditions for divisibility by 2 by considering the role of the ones digit, tens digit and hundreds digit.

Part 5: What generalization can you make about a number that is divisible by 2?

……………………………………………………………………………………………
……………………………………………………………………………………………
……………………………………………………………………………………………
……………………………………………………………………………………………
……………………………………………………………………………………………

Figure 3. 11. Example of the numbers in Part 5

As a result, the organization of the activity sheet on divisibility by 2 can be summarized as follows.

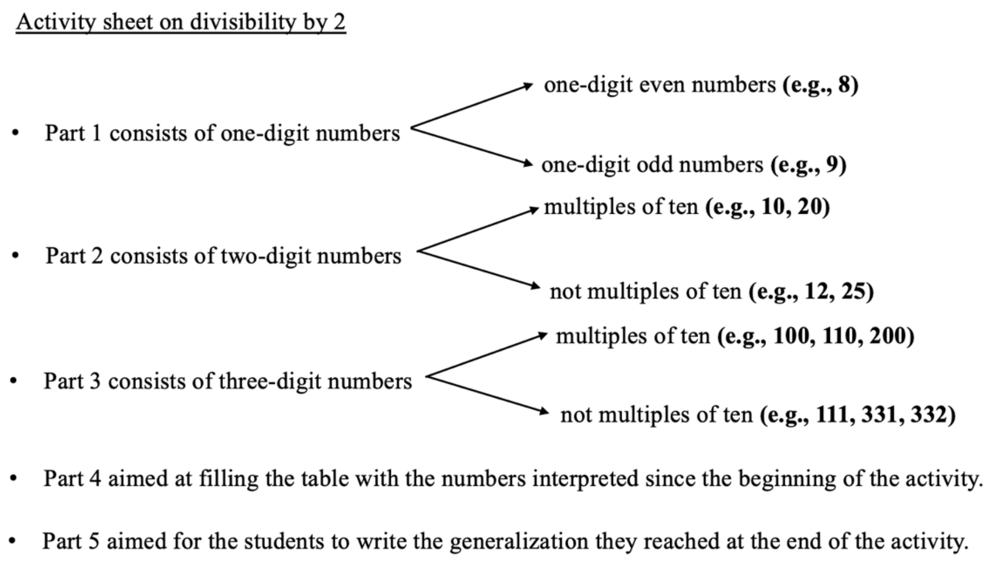

Figure 3. 12. Organization of the activity sheet on divisibility by 2

As can be seen in the Figure 3.12. there was a sequence between the numbers in such a way that first one-digit numbers and then two-and three-digit numbers. The numbers that are multiples and not multiples of ten were given to the students in order to observe possible number combinations.

3.3.2. Activity Sheet on Divisibility by 3

The prepared activity sheets on 2 and 3 had similar sequences. The questions were organized in a way that there is a sequence between numbers such as one-digit, two-digit, and three-digit numbers as in the activity sheet on divisibility by 2. Similarly, the prepared activity sheet on divisibility by 3 included five parts. In other words, Part 1 consists of one-digit numbers, Part 2 consists of two-digit numbers, and Part 3 consists of three-digit numbers. The researcher aimed to analyze how the students generalize the divisibility rule of 3 in a study planned in this sequence. Another similarity in activity sheets 2 and 3 is the similarity of the sub-parts. The difference between the activity sheets 2 and 3 is the numbers that were asked of the students.

On the other hand, the number of parts, the sequence of the numbers, and the subparts were similar. Parts 1, 2, and 3 included the three subparts mentioned above. In-depth information regarding Part 1, Part 2, and Part 3 is as follows.

Part 1 included one-digit numbers. The numbers were chosen based on the remainder after division. That is, numbers with one remainder (e.g., 4), two remainders (e.g., 8), and no remainder (e.g., 3) when divided by 3 were selected. An example of Part 1 can be seen in the following figure.

Part 1:

- Can you construct the number 3 using base ten blocks?
- How can you share 3 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 3 is divisible by 3? What is your final decision?

- Can you construct the number 4 using base ten blocks?
- How can you share 4 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 4 is divisible by 3? What is your final decision?

- Can you construct the number 6 using base ten blocks?
- How can you share 6 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 6 is divisible by 3? What is your final decision?

- Can you construct the number 8 using base ten blocks?
- How can you share 8 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 8 is divisible by 3? What is your final decision?

Figure 3. 13. Example of the numbers in Part 1

Part 2 included two-digit numbers that are *multiples of ten* and *not multiples of ten*. Some numbers include one ten, two tens, and three tens, as seen in the following figure.

Part 2:

- Can you construct the number 10 using base ten blocks?
- How can you share 10 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 10 is divisible by 3? What is your final decision?

- Can you construct the number 20 using base ten blocks?
- How can you share 20 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 20 is divisible by 3? What is your final decision?

- Can you construct the number 30 using base ten blocks?
- How can you share 30 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 30 is divisible by 3? What is your final decision?

Figure 3. 14. Example of the numbers in Part 2

The aim of Part 2 was to see that there would be one remainder unit from each of the ten blocks. The numbers in these questions were two-digit numbers that are multiples of ten, i.e., the ones digit was zero. In order to see how divisibility was affected by the unity, the following questions involved two-digit numbers that are not multiple of ten. The numbers were chosen based on the remainder after division. That is, numbers with one remainder (e.g., 31), two remainders (e.g., 32), and no remainder (e.g., 33) when divided by 3 were selected. An example of Part 2 can be seen in the following figure.

- Can you construct the number 31 using base ten blocks?
- How can you share 31 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 31 is divisible by 3? What is your final decision?

- Can you construct the number 32 using base ten blocks?
- How can you share 32 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 32 is divisible by 3? What is your final decision?

- Can you construct the number 33 using base ten blocks?
- How can you share 33 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 33 is divisible by 3? What is your final decision?

Figure 3. 15. Example of the numbers in Part 2

After constructing and trying to share the given number of objects among three children, the student was asked to give an example of a two-digit number divisible by 3. This aimed to analyze whether the students reached a generalization about the divisibility rule of 3.

Part 3 included three-digit numbers that are multiples of ten and not multiples of ten. The numbers included one hundred, two hundred, and three hundred, as seen in the following figure.

Part 3:

- Can you construct the number 100 using base ten blocks?
- How can you share 100 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 100 is divisible by 3? What is your final decision?

- Can you construct the number 200 using base ten blocks?
- How can you share 200 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 200 is divisible by 3? What is your final decision?

- Can you construct the number 300 using base ten blocks?
- How can you share 300 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 300 is divisible by 3? What is your final decision?

Figure 3. 16. Example of the numbers in Part 3

The aim of Part 3 was to see that there would be one remainder unit from each hundred block. The numbers in Part 3 were three-digit numbers that are multiples of ten, i.e. the ones digit was zero. In order to see how divisibility by 3 was affected by the tens and ones place, the following questions were asked. The numbers included three hundred and a one, three hundred and a ten, as seen in the following figure.

- Can you construct the number 301 using base ten blocks?
- How can you share 301 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 301 is divisible by 3? What is your final decision?

- Can you construct the number 310 using base ten blocks?
- How can you share 310 objects among 3 children in such a way that each child gets an equal number of objects?
- Do you think the number 310 is divisible by 3? What is your final decision?

Figure 3. 17. Example of the numbers in Part 3

These parts aimed to analyze whether the students could identify the place values of the numbers and identify the remainders for each block. To sum up, Parts 1, 2, and 3 aimed to analyze whether the students could reach the summation of the remainder. After discussing the divisibility of the given numbers, the participants were asked to give an example of a three-digit number divisible by 3. This aimed to analyze whether the students reached a generalization about the divisibility rule of 3. In addition, the aim of Part 4 was to fill in the table with the numbers interpreted since the beginning of the activity. The students were expected to write the condition of being divisible by 3. Finally, the aim of Part 5 was to write the divisibility rule of 3 at the end of the activity. The participants were asked to generalize what is required for divisibility by 3.

Part 5: What generalization can you make about a number divisible by 3?
....
....
....
....
....
....

Figure 3. 18. Example of the Part 5

As a result, the organization of the activity sheet on divisibility by 3 can be summarized as follows.

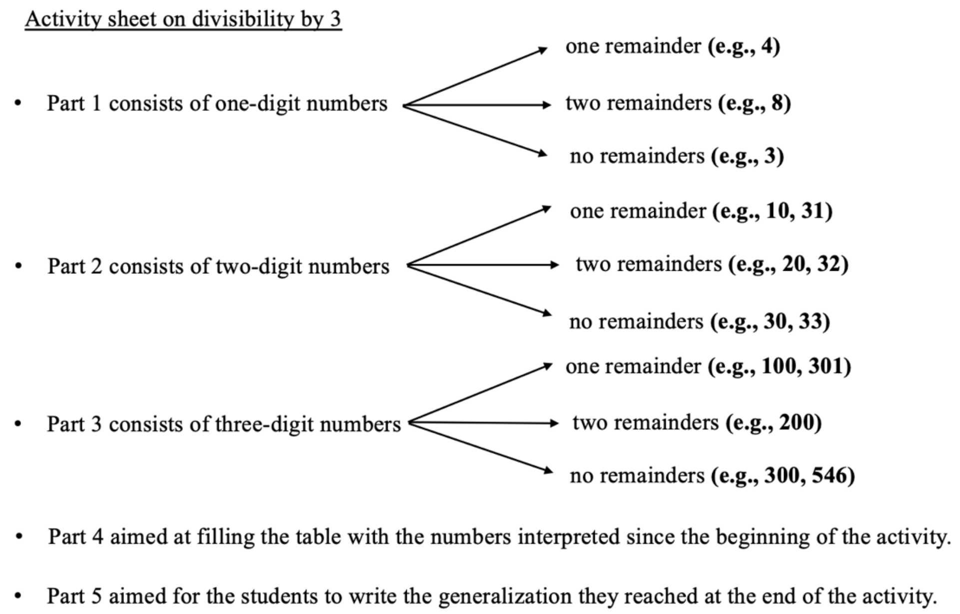

Figure 3. 19. Organization of the activity sheet on divisibility by 3

As can be seen in Figure 3.19., students were given numbers in a sequence. That is, students were given first one-digit numbers, then two- and three-digit numbers, respectively. They were able to have a chance to analyze the number combinations.

3.3.3. Activity Sheet on Divisibility by 6

The prepared activity sheet on divisibility by 6 included four parts. Part 1 consisted of two subparts to see whether the participants could remember the activities about divisibility by 2 and 3 that they had completed before. The participants could use virtual manipulatives as they did in the previous interviews. Also, the parts involved hundreds charts. The participants were expected to color the hundreds charts according to the conditions. The aim of giving hundreds charts was to analyze the students' ability to see that a number that is a multiple of 2 and a multiple of 2 is also a multiple of 6. The first subpart involved questions to remind the students of the activity of divisibility by 2. The first subpart of Part 1 can be seen in the following figure.

Part 1:

- How do we decide whether a given number is divisible by 2 (without division)?

- Is the number 12 divisible by 2? Can you show whether it is divisible using base ten blocks?

- In the hundreds chart below, can you color the boxes with numbers divisible by 2 pinks?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 3. 20. Example of the questions in the first subpart of Part 1

The second subpart involved questions to remind the students of the activity of divisibility by 3. The aim of the parts is similar. As in the first subpart, the participants were expected to color the numbers in the hundreds table that are multiples of three. After the coloring process, the numbers written in some boxes were expected to be colored in both colors. Therefore, at the beginning of the activity, the participants were asked about the common features of the numbers painted in these two colors. The second subpart of Part 1 can be seen in the following figure.

- How do we decide whether a number is divisible by 3 (without division)?
- Is the number 12 divisible by 3? Can you show this using base ten blocks?
- Can you color the boxes with numbers divisible by 3 in the hundreds chart above green?
- After coloring, can you pay attention to the boxes that are both pink and green? What do you think the numbers in these boxes have in common?

Figure 3. 21. Example of the questions in the second subpart of Part 1

Moreover, Part 2 included four tables. The numbers in the first table are divisible by 2 but not by 3 (See Figure 3.22.) The numbers in the second table are divisible by 3 but not by 2 (See Figure 3.22.) The numbers in the third table are not divisible by 2 or 3 (See Figure 3.23.) The numbers in the fourth table are divisible by 2 and 3 (See Figure 3.23.) The participants were expected to fill out the table on divisibility by 2 and 3. The aim of Part 2 was to analyze whether the students could identify the relationship between the multiples of 2 and 3, respectively. The following figure shows the tables in Part 2.

Part 2: Can you fill in the tables below?

Table 1

Numbers	Are numbers shared equally by two people?	Are numbers shared equally by three people?	Are numbers shared equally by six people?
2			
4			
8			
10			
20			
40			
52			
104			

Table 2

Numbers	Are numbers shared equally by two people?	Are numbers shared equally by three people?	Are numbers shared equally by six people?
3			
9			
15			
21			
27			
33			
39			
45			
111			

Figure 3. 22. The example of Table 1 and Table 2 in Part 2

Table 3

Numbers	Are numbers shared equally by two people?	Are numbers shared equally by three people?	Are numbers shared equally by six people?
5			
7			
11			
13			
17			
25			
47			

Table 4

Numbers	Are numbers shared equally by two people?	Are numbers shared equally by three people?	Are numbers shared equally by six people?
6			
12			
18			
24			
30			
36			
42			

Figure 3. 23. The example of Table 3 and Table 4 in Part 2

In addition to these, Part 3 consisted of open-ended questions. It was designed to ask why a number is divisible by 6. For this purpose, the participants were asked to give an example of a number that is odd and divisible by 3 and a number that is both even and not divisible by 3. They had to decide whether the number in the given example was divisible by 6. Part 3 aimed to analyze whether the students reached a generalization about the divisibility of 6. That is, the aim was to analyze whether the students reached a generalization that a number that is even and not divisible by 3 cannot be divisible by 6. Furthermore, Part 3 aimed to examine the student's ability to generalize by asking questions to make them question that the smallest common multiple of 2 and 3 is 6. An example of Part 3 can be observed in the following figure.

Part 3:

- What does it mean when an object is shared equally between 2 and 3 people?
- What does it mean to share equally to 6 people?
- Can an odd number that can be shared by 3 people be shared equally by 6 people? Can you explain with an example?
- Can an even number that can be shared by 3 people be shared equally by 6 people? Can you explain by giving an example?
- What is the smallest number that is divisible by both 2 and 3? Can you find this number from the table? What colors is it painted in?

Figure 3. 24. Example of the questions in Part 3

Lastly, part 4 asked the participants to conclude the divisibility rule of 6. The participants needed to generalize the divisibility rule of six.

Part 4: What is necessary for a number to be shared equally between 6 people? Can you express this in sentences?
………………………………………………………………………………………………
………………………………………………………………………………………………
………………………………………………………………………………………………
………………………………………………………………………………………………
………………………………………………………………………………………………

Figure 3. 25. Example of Part 4

In conclusion, the organization of the activity sheet on divisibility by 6 can be summarized as follows.

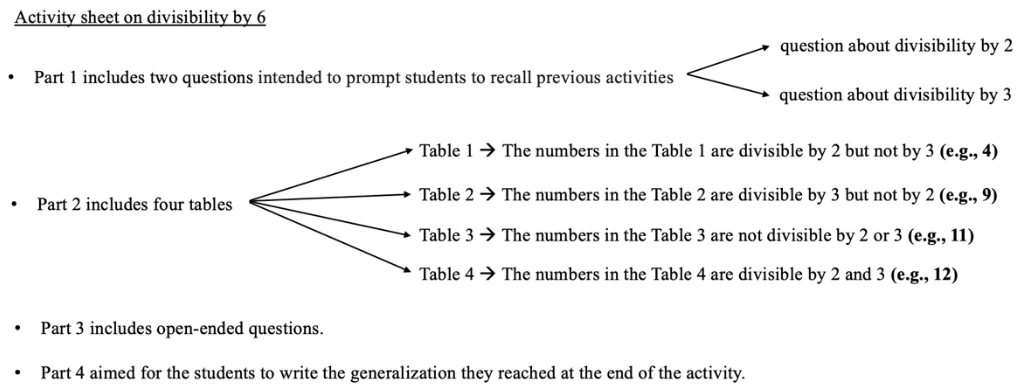

Figure 3. 26. Organization of the activity sheet on divisibility by 6

As can be seen from the above figure, students were given all possibilities for the number combinations, that are numbers that are divisible by 2 but not by 3, numbers that are divisible by 3 but not by 2, numbers that are divisible by 2 and 3, and numbers that are not divisible by 2 or 3.

3.3.4. Clinical Interviews

This part of the chapter describes the features of the clinical interviews and the process of conducting them in the current study.

Clinical interviews can be used to analyze students' different ways of thinking, and conducting clinical interviews allows researchers to obtain ideas about how students construct their understanding, their thought processes and their level of cognitive functioning (Ginsburg, 1997). Moreover, Hunting (1997) noted that any mathematical concept can enable students to talk about their thinking and enable teachers to understand students' mathematical thinking. Also, he stated that any task with manipulative components provides an opportunity to analyze students' thought with their verbal explanations and comments. In other words, it is possible to engage students to use manipulatives to observe their thinking process (Hunting, 1997). Groth et al. (2006) suggested that the interviewer should formulate appropriate follow-up questions when the child responds to the interview task during a clinical interview. Similarly, Hunting (1997) suggested to formulate follow-up questions to elicit students' thoughts during clinical interviews.

As Hunting suggested, the researcher asked questions to probe the students' thoughts during the clinical interviews. Some examples of these questions were, "Can you tell me what you think? Can you say out loud what you are doing? Why do you think so? How did you figure it out?"

The data were gathered from three students through one-to-one clinical interviews. There were three participants in the current study; each participant received an activity sheet separately throughout the clinical interviews. All participants were given an activity sheet related to the rule of divisibility by 2 in the first interview. After that, all participants were given an activity sheet related to the rule of divisibility by 3 in the second interview. Finally, an activity sheet related to the rule of divisibility by 6 was completed during the last clinical interview. The sequence of activity sheets used the clinical interviews was activity sheets on 2, 3, and 6, respectively. The students used virtual base-ten blocks throughout these interviews to engage with and work through the questions in the activity sheets.

3.4. Data Collection Procedure

The study aimed to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. Data were gathered from three fifth-grade students who were accessible for the researcher through one-to-one clinical interviews. During the clinical interviews, the students were given virtual base-ten blocks opened on a tablet, a tablet pencil, and an activity sheet. In the first interview, an activity sheet on divisibility by 2 was completed. The second interview involved an activity sheet on divisibility by 3. Lastly, an activity sheet on divisibility by 6 was completed in the third interview. In other words, three clinical interviews were conducted with each participant separately. During the clinical interviews, the students read the questions on the activity sheet loudly and constructed the numbers on the questions using virtual base-ten blocks simultaneously. While doing so, the researcher posed follow-up questions such as: Can you explain your thoughts? Can you give an example? Please show me what you think using base-ten blocks, etc. The clinical interviews conducted with each student lasted approximately 50 minutes and were conducted at the students' home to ensure their comfort. The students were videotaped during the clinical interviews, the students and their parents were asked to give permission to do so before the interviews.

3.5. Pilot Study

Pilot studies are essential in qualitative research as they allow researchers to test the applicability of their approach before conducting a more extensive study (Leon et al., 2011). Pilot studies can be used in qualitative research to explore teachers' or students' perspectives and can also provide valuable ideas that can guide instructional design (Eldridge et al., 2016). Thus, a pilot study was crucial before conducting the main study. The pilot study was conducted to ensure that some points in the data collection process were added. After developing the data collection instrument, the researcher conducted the pilot study with two students. One of the students was a fifth-grade student, and the other one was a sixth-grade student. The divisibility rules are taught to sixth-grade students in the Turkish curriculum. This aimed to see the differences between instruments for those who had learned the divisibility rules and those who had not. However, since the sixth-grade students had learned the topic but forgotten it, they had to rediscover it. After the pilot study, some changes were made to the data collection instruments. For example, after conducting the pilot study, numbers that are multiples of ten that have an odd/even digit in the tens place and numbers that are not multiples of ten that have an even digit in the tens place and odd digit in the ones place, and numbers that are not multiples of ten that have an odd digit in the tens place and an even digit in the ones place were added to the activity sheet on divisibility by 2. This aimed to analyzed to see if it is clear that having an odd digit in the ones place of a number is not the same as having an odd digit in the tens place.

Moreover, as mentioned at the beginning of the methodology chapter, there was a sequence between the questions. One-digit, two-digit, and three-digit numbers were asked, respectively. At this point, the pilot study helped the researcher see the importance of the sequence between the questions. For this reason, a pilot study is vital for researchers to know the order in which the questions are asked. Moreover, thanks to the pilot study, the researcher could see the time needed to collect the data. Since the data were collected through interviews, it was essential to arrange the time. After the pilot study, the researcher understood how many minutes were needed for three interviews.

3.6. Data Analysis

The data were collected during the one-to-one clinical interviews by giving the participants activity sheets. There were three participants and three different activity sheets. Three different interviews were conducted with each participant. The first interview was about divisibility by 2, the second was about divisibility by 3, and the last one was about divisibility by 6. That is, nine interviews were conducted in total. For data analysis, the researcher transcribed the video recordings of these nine interviews. The participants' responses and the screenshots of the solutions on the tablet were examined in detail using. MAXQDA was used for the coding process. Some arrangements were made according to the students' answers for the generalization process.

Analyzing of this study includes three dimensions for analysis the first research question which are the process of generalizing the divisibility rule of 2, the process of generalizing the divisibility rule of 3, and the process of generalizing the divisibility rule of 6. For the second research question, the misconceptions and difficulties that students encountered during the generalization process of students.

Firstly, key ideas of the process of generalizing the divisibility rules of 2 and 3 were described under the three headings: *one-digit numbers*, *multiples of ten*, and *not multiples of ten*. Moreover, key ideas of the process of generalizing the divisibility rule of 6 were described under the two headings: *numbers that are not* *multiples of six* and *numbers that are multiples of six.* In other words, the solutions describing the students' generalizations were defined under these headings.

Secondly, misconceptions and difficulties that students encountered during the generalization process was coded. The codes for the process of generalizing the divisibility rules and the misconceptions and difficulties are given as follows. The misconceptions and difficulties that students encountered during the generalization process was coded. The difficulties and misconceptions encountered by the participants can be categorized under two main categories: *technology-based difficulties* and *concept-based difficulties.* Under the category of *concept-based difficulties,* the data were coded as *overgeneralizing the divisibility rule of 2 to the divisibility rule of 3*, *misinterpretations of odd digit numbers, inability to identify which digits require an even number for equal sharing*, and *confusion about the place value and number value.*

The findings of the study will be presented by using these headings and subheadings in detail in the findings chapter.

3.7. Trustworthiness

"Validity refers to the appropriateness, meaningfulness and usefulness of the inferences a researcher makes. Reliability refers to the consistency of scores or answers from one administration of an instrument to another, and from one set of items to another" (Fraenkel & Wallen, 2006, p. 147). In addition, Merriam and Tisdell (2015) stated that validity and reliability are concerns that should be considered in the conceptualization of a study, regardless of the type of research, from data collection to the analysis process, that is, in every aspect of the research. These terms, validity and reliability, are used for quantitative studies. Qualitative studies have different terminologies for checking the ethical considerations that will be explained. Lincoln and Guba (1985) defined trustworthiness as how and by what arguments a researcher can persuade readers that the findings of his/her study are worth considering.

To ensure trustworthiness, it is stated that there are four valuable questions that researchers should ask themselves. These questions are related to credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985), which are the correspondents of internal validation, external validation, reliability and objectivity (Creswell, 2007).

3.7.1. Credibility

Merriam and Tisdell (2015) explained the term credibility as extent to which the findings of a study are in line with the reality and to establish the credibility triangulation is the well-known technique. According to Merriam and Tisdell (2015), the aim of triangulation is as follows: "Triangulation- whether you use more than one collection method, multiple sources of data, multiple investigators, or multiple theories- is a powerful strategy for increasing the credibility of your research" (p. 245).

Researchers have suggested that there are four types of triangulations in general, namely: data triangulation, investigator triangulation, theory triangulation, and methodological triangulation (Denzin & Lincoln, 2005; Merriam & Tisdell, 2015; Patton, 2002). In the current study, investigator triangulation and data triangulation were used to ensure the credibility. Firstly, data were collected from three students, rather than one, which established data triangulation. Secondly, the researcher-supervisor collaboration was achieved. The researcher always contacts with supervisor while preparing materials and coding and analyzing data to enhance credibility. In addition, the researcher and the co-coder coded the data of the study. A doctoral student in Mathematics Education coded the data as a co-coder to ensure interrater reliability. The researcher's coding and the co-coder's coding were compared to see the differences and similarities between the codes. Afterwards, by using formula suggested by Miles and Huberman (1994), the interrater reliability was calculated approximately 92%. The differences were discussed, and some adjustments were made by consulting the advisor regularly.

Furthermore, Creswell (2007) suggested that researchers need to use at least two strategies to ensure the validation of any study. Strategies for establishing validation include prolonged engagement, triangulation, peer review, member checking, thick description, etc. In the present study, as Creswell recommended, two strategies are used to ensure credibility: triangulation and member checking. Merriam and Tisdell (2015) indicated that member checking means participant verification to get feedback from interviewees. In the present study, the researcher asked the participants to confirm their responses during the clinical interviews. In this way, the researcher had the chance to understand whether the participants' answers were interpreted correctly or not.

3.7.2. Dependability

Fraenkel and Wallen (2006) defined the reliability as: "Reliability refers to the consistency of scores or answers from one administration of an instrument to another, and from one set of items to another" (p. 147). Lincoln and Guba (1985) suggested using the term dependability for qualitative research studies. Merriam (2015) indicated that this term is related to whether the results are consistent when the study is repeated in other time. To ensure dependability, Merriam and Tisdell (2015) suggested using some strategies: triangulation, peer examination, investigator's position, and the audit trail. As mentioned before, investigator triangulation and data triangulation were used, which also ensured dependability.

3.7.3. Transferability

Transferability refers to the generalizability of a research study (Merriam, 2015; Lincoln & Guba, 1985). However, the aim of qualitative research studies is not to generalize the sample results to the entire population. For this reason, Merriam and Tisdell (2015) suggested providing thick descriptions and using sufficient data. In the present study, there were three different students, and three different interviews conducted with each of them separately, ensuring that there were sufficient data.

3.7.4. Confirmability

Confirmability corresponds to objectivity in quantitate research. As mentioned before, triangulation was used to establish credibility of the present study. According to Shenton (2004) using triangulation can also reduce effect of investigator bias. Lincoln and Guba (1985) proposed strategies to reduce researcher bias in order to ensure confirmability. One of these strategies is triangulation, which reduces the effects of researcher bias. Another strategy to reduce the effects of researcher bias is to explain the methodology of the research study in detail. Additionally, clarifying the roles of the researchers also serves as another strategy. Therefore, in this study, confirmability through triangulation was achieved by providing a detailed description of the study's methodology and explaining the researchers' roles.

3.8. The Researcher's Role

In the current study, first of all, the researcher explained the aim of the study and the process of data collection to the students and parents. The researcher had not previously met any of the students until the time of the study. During the clinical interviews, to make sure that students completed activity sheets without feeling pressure, a friendly environment has been tried to be created. For instance, the researcher had a talk with students before starting the clinical interviews to make them comfortable.

Furthermore, during the clinical interviews, the researcher asked students some critical questions to challenge them. When they had some difficulties during the process of generalizing divisibility rules, the researcher asked them some guiding questions to prevent difficulties. Moreover, time of the clinical interviews was organized according to the suitability of the students, and it was made clear at the beginning of the interviews that there were no time restrictions when solving the activities so that they did not feel pressured.

Lastly, the researcher decided on her thesis topic by being influenced by the "Developing Strategic Competence in Mathematics" course she took during her master's degree.

3.9. Ethical Considerations

The aim of the present study is to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. Firstly, permission was obtained from the Middle East Technical University Human Subjects Ethics Committee and the head of the elementary mathematics education program to conduct the study.

After the necessary permissions were obtained, pilot studies were conducted to ensure the study's trustworthiness. After that, the primary data were collected in the summer semester of the 2023-2024 academic year. The students and parents were informed about the present study's aims and procedures. Necessary permissions were obtained from the students' parents for collecting data. During the interviews, the participants were videotaped with permission, and it was clearly stated to the participants that these audio and video recordings would not be shared with anyone.

Furthermore, it was stated that no one other than the researcher and the advisor would be allowed to watch the videotapes. Three different clinical interviews were planned with each participant. The duration of the clinical interviews varied according to the activity sheet. In order to avoid distraction during the interviews with the participants, calm and quiet places were preferred. Therefore, the interviews were conducted at the participants' homes with the parents.

3.10. Assumptions and Limitations of Study

The main assumptions and limitations of the present study are discussed in this part of the chapter. It was assumed that all students would be able to construct the numbers in the activity sheets using the software on the tablet during the clinical interviews. Furthermore, it was assumed that students would answer the questions on the activity sheets carefully, thoughtfully, and honestly during the one-to-one clinical interviews.

The limitation of this study could be the context of the study. More specifically, the present study aimed to analyze the conceptualization process of fifth-grade students regarding the divisibility rules of 2, 3 and 6. Thus, the present study focused only on three divisibility rules, which were related to 2, 3 and 6. The other divisibility rules were not involved in the present study.

CHAPTER 4

FINDINGS

The findings of the current study are presented in two sections based on the research questions of the study. The purpose of the study was to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. Thus, the first section of this chapter presents the findings obtained by examining the students' answers in the activity sheets. The first part has three sub-sections: the process of generalizing the divisibility rules of 2, 3, and 6, respectively. Moreover, the second research question of the study was to investigate the possible misconceptions or difficulties encountered by fifth-grade students during the process of generalizing the divisibility rules of 2, 3 and 6, respectively. Therefore, this chapter's second section presents the difficulties or misconceptions students encounter during the generalization process.

4.1. The Process of Generalizing the Divisibility Rules

This study aimed to analyze how fifth-grade students generalize the divisibility rules of 2, 3, and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. Three activity sheets were prepared for this purpose, as explained in the methodology chapter. The prepared activity sheets were about divisibility by 2, 3 and 6, respectively. As mentioned, there were three participants in the present study, and clinical interviews were conducted with each student individually. In the following sections, the generalization process of students is explained for divisibility by 2, 3, and 6, respectively.

4.1.1. The Process of Generalizing the Divisibility Rule of 2

An activity sheet on divisibility by 2 was given to the students during the clinical interview. The students were also given a tablet and a tablet pen to use the virtual manipulative simultaneously. They were asked to construct the numbers in the activity sheets using a virtual manipulative. Then, the students tried to share the given number of objects between two children in a way that each person would receive an equal number of objects. The findings were further grouped under the headings of *one-digit numbers*, *multiples of ten*, and *not multiples of ten*. In other words, the students performed fair sharing first with one-digit even and one-digit odd numbers and then with two and three-digit numbers that are multiples of ten and not multiples of ten, and their generalization process for the divisibility rule of 2 was examined. Under the heading of *one-digit numbers*, the subheadings are *one-digit even numbers* and *one-digit odd numbers*. The subheadings of *multiples of ten* are *two-digit numbers* and *three-digit numbers*. Similarly, the subheadings of *not multiples of ten* are *two-digit numbers* and *three-digit numbers*.

Divisibility of one-digit numbers by two

Considering one-digit numbers, the students were given one-digit even and one-digit odd numbers, respectively. The students tried to share the given one-digit numbers between two children in a way that each would receive an equal number of objects. The students' solutions for one-digit even numbers and the interview excerpts will be presented. After that, the students' solutions for one-digit odd numbers will be presented in detail.

One-digit even numbers

All three participants were able to share a one-digit even number between two children and ensure that each child received an equal number of objects. For instance, when asked about the number 8, the students were able to share eight objects by giving each child four objects. To exemplify, one of the participants, Doğan, constructed the following figure using the virtual manipulative on the tablet. The upper part of Figure 4.1. is the first form that the student constructed using base-ten blocks, and lower part of Figure 4.1. is the last form showing how the student shared eight objects between two children. The interview excerpt is as follows:

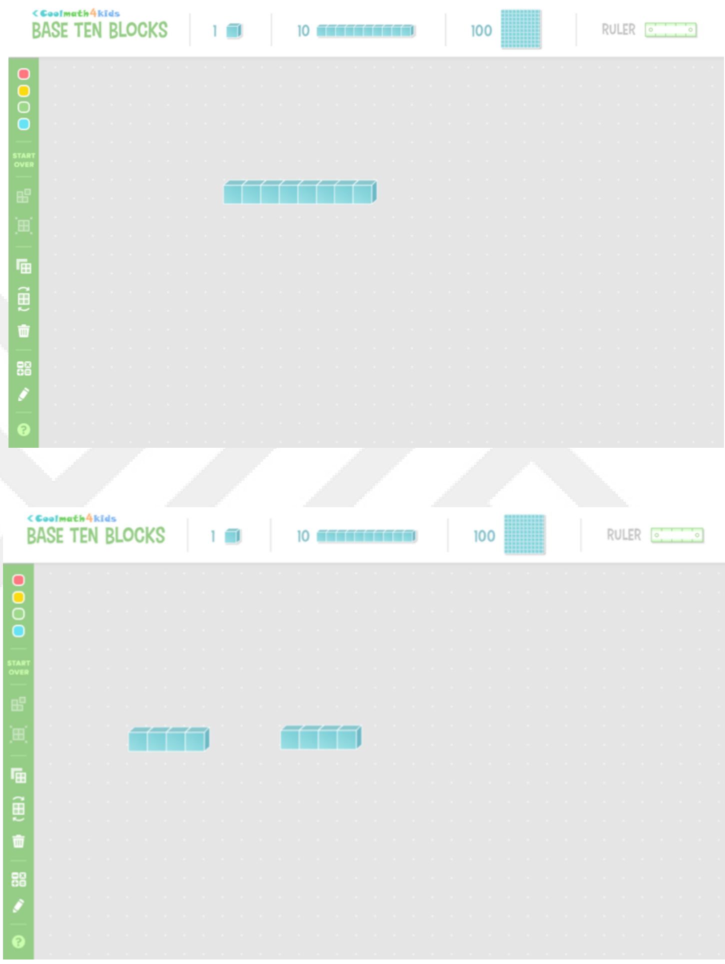

Figure 4. 1. Doğan’s solution for the number 8 on the virtual manipulative

Researcher: Can you show me how do you share eight objects between two children?
Doğan: Each of them took four. Furthermore, there are no objects left.
Researcher: Okay, can you say two children share the 8 objects?
Doğan: Yes.
Researcher: Why is that?
Doğan: Because it is a multiple of two.

The student first constructed the number using eight ones, as seen the upper part of Figure 4.1. Then, he divided the eight ones into two sides of the tablet screen while sharing eight objects between two children, as seen the lower part of Figure 4.1. After that, he decided that the number eight was sharable by two children, and there were no remainders.

In light of the interviews and figures, all three students realized that a one-digit even number is a multiple of two. Thus, they reached the generalization that one-digit even numbers can be sharable and divisible by 2 through the fair sharing approach.

One-digit odd numbers

After working with one-digit even numbers, the students were given one-digit odd numbers. For instance, when the number 9 was given, all three participants were able to share a one-digit odd number between two children so that each child received an equal number of objects, and they realized that there was one remainder. The upper part of Figure 4.2. is the first form that the student constructed using base-ten blocks, and lower part of Figure 4.2. is the last form showing how the student shared nine objects between two children, leaving one remainder.

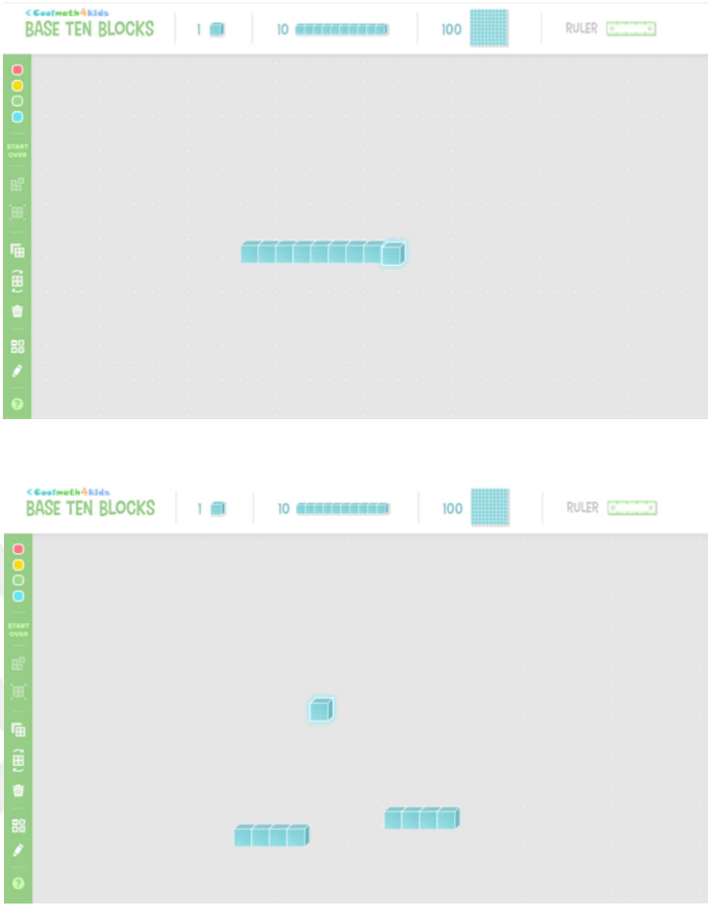

Figure 4. 2. Elif's solution for the number 9 on the virtual manipulative

The student, Elif, first constructed the given number using nine ones using the virtual manipulative, as seen the upper part of Figure 4.2. Also, the lower part of Figure 4.2. shows the student sharing the nine objects between two children, taking four objects on the right side of the screen and the other four objects on the left side of the screen, leaving one object in the middle. Therefore, all students were able to conclude that there was a remainder unit. Two students tried to cut the remaining unit in half. For instance, one of the participants, Elif, stated that she could cut the remaining unit in half with scissors but could not do this as it was necessary not to disturb the integrity. An excerpt from the interview is as follows:

> Elif: To share equally between two children, they need to be equal. I need to give equal blocks to both children, but since the number 9 is not divisible by 2, I keep one of them. The other one is a division with a

remainder, logically. I divide the number by 2 and keep the remainder, just like dividing by 8. I can only divide the remaining one if I have scissors.

Researcher: Then, what can you say about the divisibility?

Elif: 9 is not divisible by 2.

As can be seen from the quotations above, all students were able to identify that while sharing the odd number of objects between two children, there was a remainder. However, while ensuring fair sharing, the students could not divide the remainder in a way that would not disrupt the integrity. In other words, when the odd numbers were divided equally by 2, it was impossible to divide the remainder in a way that would not disrupt the integrity of the whole.

To conclude, the students made a fair sharing while trying to share the odd number of objects between two children *and realized that there was one unity left. Since the remainder could not be divided by 2 in a way that would not disrupt the integrity, they generalized that one-digit odd numbers cannot be divisible by 2.*

Divisibility of numbers that are multiples of ten by two

During the clinical interviews, the students were given virtual manipulatives to construct one-, two-, and three-digit numbers, respectively. Two- and three-digit numbers that are multiples of ten (i.e., the number is zero in the ones place) were given. More specifically, the two-digit numbers that are multiples of ten had even and odd digits in the tens place, e.g., 10, 20, 50. Similarly, the three-digit numbers that are multiples of ten had even and odd digits in the hundreds or tens place, e.g., 100, 110, 200. The data analysis for two-digit numbers and three-digit numbers that are multiples of ten are given in detail below.

Two-digit numbers

While constructing two-digit numbers by using virtual base-ten block, all students realized that a two-digit number that is a multiple of ten could be shared by two, regardless of whether the digit in the tens place is odd or even. They claimed that every two-digit number that is a multiple of ten is also a multiple of two due to the decimal structure of the base-ten block system. In other words, the students were able to claim that a number that is multiple of ten is even and therefore sharable by two. The excerpts from the interviews of two students are as follows.

> Doğan: Tens are, that is, every ten is a multiple of two. In other words, since all its multiples are multiples of two, at the same time, all of them are even numbers. And when you add one to them, you get an odd number.
>
> Elif: Two children can share ten blocks regardless of how many there are.
>
> Nur: Tens can always be shared by two.

While making this generalization, the students developed methods for sharing two-digit numbers that are multiples of ten between two children for numbers with even digits in the tens place and odd digits in the tens place, separately. For instance, for numbers with even digits in the tens place (e.g., 20), they shared the tens blocks between two children without the need to trade the tens for ten ones. On the other hand, the students traded tens for ten ones for numbers with odd digits in the tens place (e.g., 50). Detailed information is given below.

For instance, when the number 20 was asked, the student, Nur, constructed this number using two tens, as seen the upper part of Figure 4.3. Then, she dragged one tens block to the right side of the screen and the other tens block to the left side of the screen, as shown in the lower part of Figure 4.3.

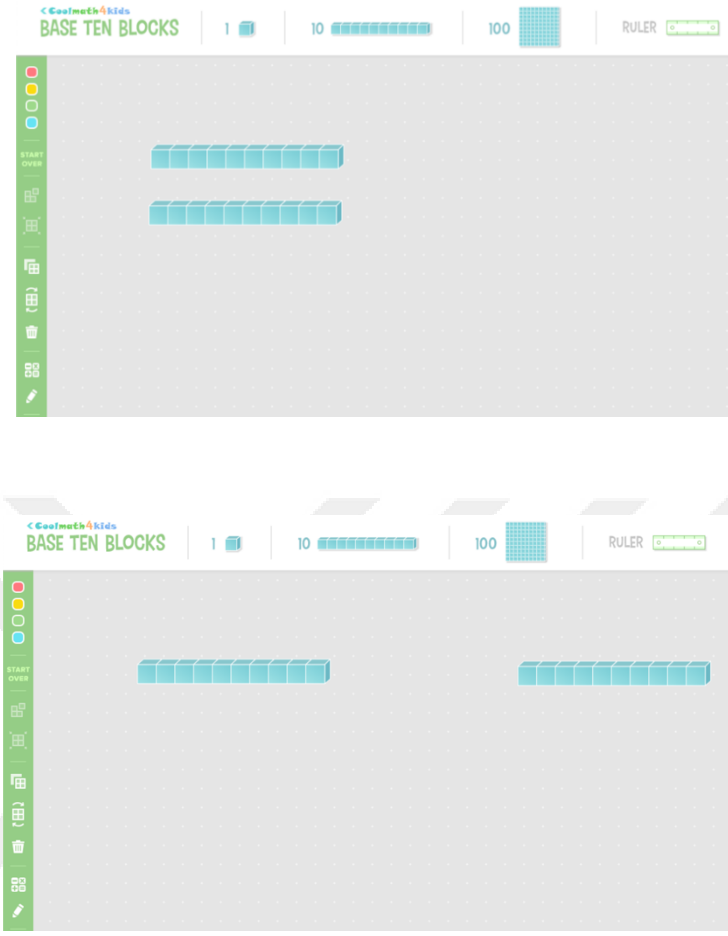

Figure 4. 3. Nur’s solution for the number 20 on the virtual manipulative

To be more specific, the students did not need to trade ten blocks for ten ones, as shown in Figure 4.3. while sharing two tens blocks between two children.

Furthermore, when the number 50 was asked, the students first took five tens blocks to construct the number on the virtual manipulative. Then, they gave each child two tens blocks and traded the remaining tens blocks for ten ones. After trading, they gave each child five ones. Thus, the result of the division was twenty-five. The following figure is a screenshot of the image created by the participant who tried to share the number 50 between two children. The upper part of Figure 4.4. is the first form that the student made on the virtual manipulative, and the lower part of Figure 4.4. is the last form of the solution where the student tried to share the fifty objects between two children equally.

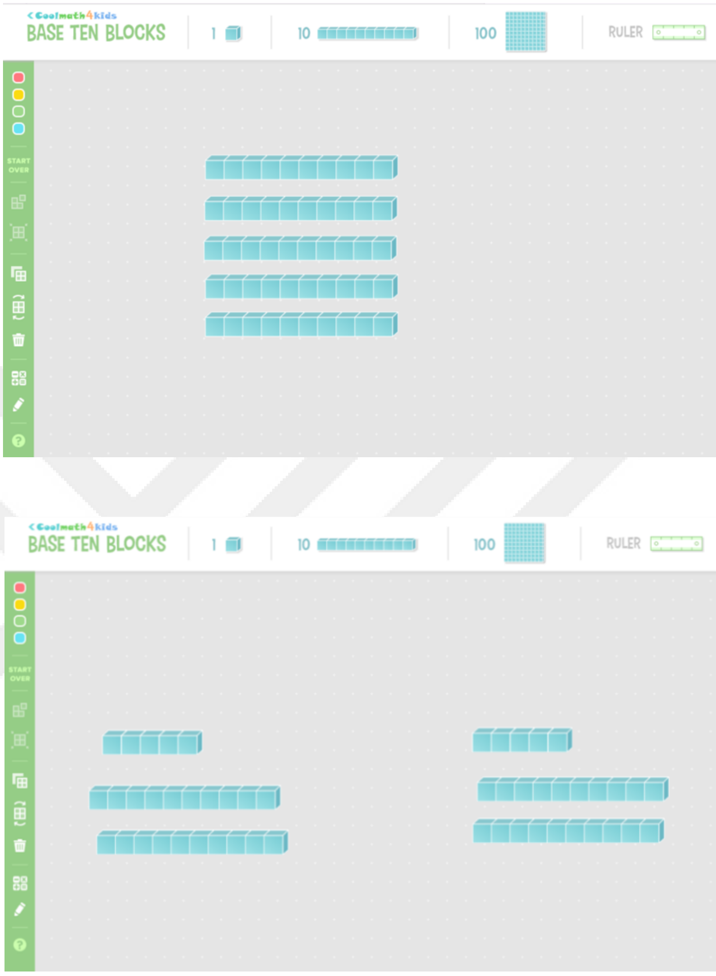

Figure 4. 4. Doğan's solution for the number 50 on the virtual manipulative

All students were able to successfully share the two-digit number, which is a multiple of ten, between two children equally. When the students' solutions were analyzed, it was observed that they used two methods. Numbers with even digits in the tens place were shared by using base-ten blocks without the need to trade them, and for numbers with odd digits in the tens place, the students traded the remaining tens blocks due to the whole structure of the tens blocks.

To sum up, the students were able to generalize that two-digit numbers that are multiples of ten can be shared by two children *regardless of whether the digit in the tens place is odd or even. In other words, the students realized that tens are structurally composed of ten unities and, therefore, can be divided by 2.*

Three-digit numbers

The students were able to investigate the connection between hundreds and tens. They realized that tens are always structurally multiples of two; therefore, they made a similar interpretation about hundreds. An excerpt from the interview on that particular point is as follows:

> Researcher: What do you think about the hundreds place of the number?
>
> Doğan: So, this is like tens. Let me show you that even if there are a million hundreds, they will still be divisible by 2.
>
> Researcher: Why is that?
>
> Doğan: Because it is a multiple of two. More precisely, it consists of tens. Since tens are made of twos, they can be divided by 2.

Based on the examples, the students realized that hundreds are made up of tens, and since tens are already precisely divisible by 2, hundreds can be divided by 2 in the same way. While making sense of the relationship between hundreds and tens, the students developed methods for sharing three-digit numbers that are multiples of ten between two children for both numbers with even digits in the hundreds place and those with odd digits in the hundreds place. For instance, for numbers with even digits in the hundreds place (e.g., 200), they shared the hundreds blocks between two children without trading the hundreds for ten tens. Besides, for numbers with odd digits in the hundreds place (e.g., 100), the students choose to trade hundreds for ten tens. Detailed information about the difference is given below.

The students were asked to share the number 100 between two children equally. The following section consists of two different methods used by the students. Firstly, two students, Elif and Nur, used the same strategy for constructing and sharing the number. These students chose to use a hundreds block to construct the number 100.

After constructing the numbers, they traded the hundreds block for ten tens because of the whole structure of the base hundreds block. The solution of two students (Elif and Nur) who choose to use a hundreds block is shown in Figure 4.5. The upper part of Figure 4.5. is about constructing the number 100 by using a hundreds block. While using a hundreds block, due to the base-ten block structure, the students needed to trade one hundreds block for ten tens blocks. Therefore, the lower part of Figure 4.5. shows the sharing of ten tens, taking five tens on the left and right sides of the screen. As seen in the lower part of Figure 4.5., the students five base-ten blocks on the left and right sides of the screen and concluded that the number 100 is divisible by 2 since there was no remainder left.

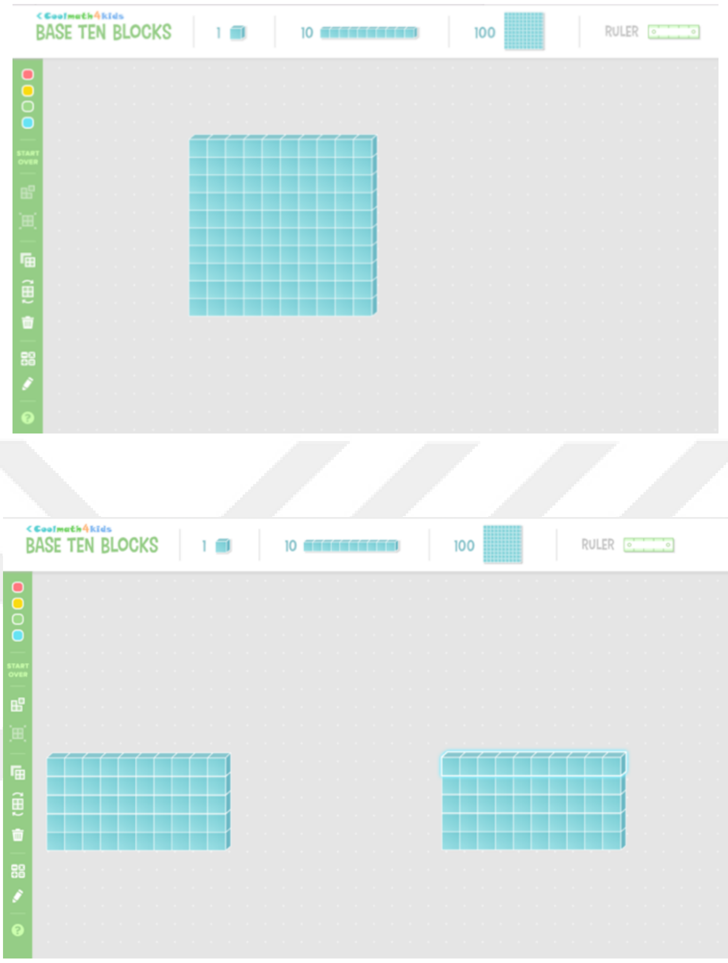

Figure 4. 5. Elif's and Nur's solutions for the number 100 on the virtual manipulative

Secondly, one of the students, Doğan, chose to use ten tens blocks to form the number 100 instead of using a hundreds block. Doğan was able to allocate the ten blocks without needing to trade; he stated that it would be easier to use ten tens blocks rather than using a hundreds block. After taking ten tens blocks as seen in upper part of Figure 4.6., he gave each group five tens blocks as seen in lower part of Figure 4.6.

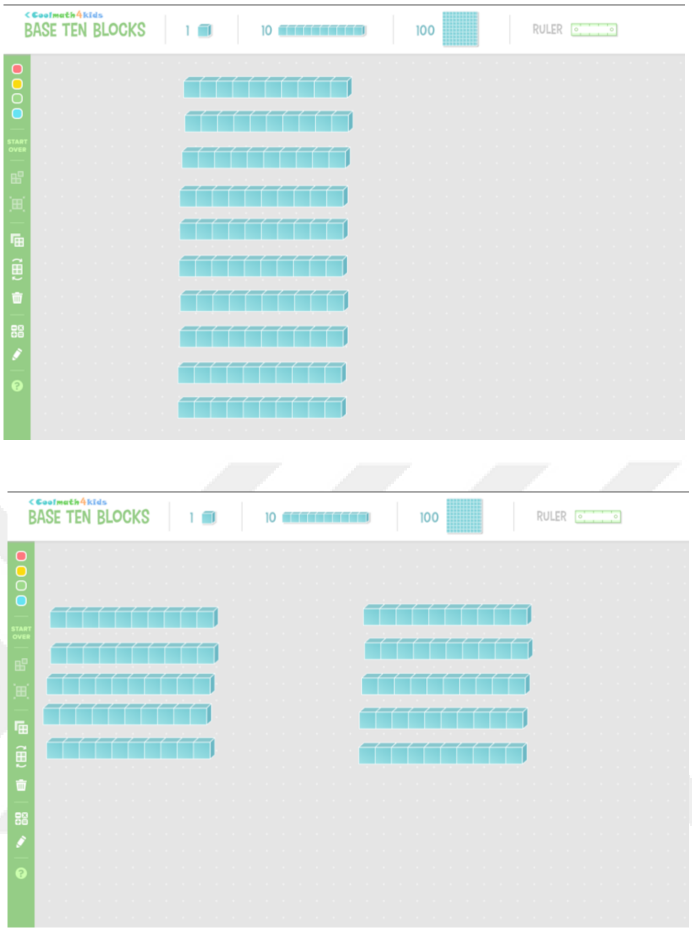

Figure 4. 6. Doğan’s solutions for the number 100 on the virtual manipulative

As a result, in both methods, the students reached the same result, that is they realized that the number 100 is divisible by 2 since there was no remainder left.

Furthermore, when the number 200 was asked to be shared equally between two children, the students were able to share this number by 2. To illustrate, one of the students, Nur, took two hundreds blocks, as shown in the upper part of Figure 4.7. After constructing the number using two hundreds blocks, the student dragged a hundreds block to the left and right sides of the screen, as observed in lower part of Figure 4.7.

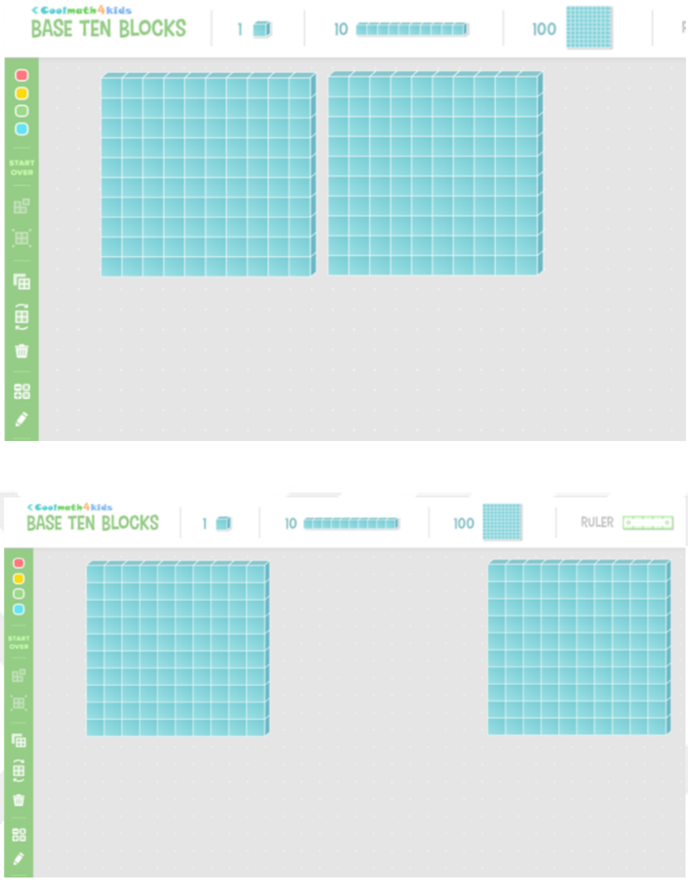

Figure 4. 7. Nur’s solution for the number 200 on the virtual manipulative

To conclude, the students reached a generalization for three-digit numbers that are multiples of ten are sharable by 2 regardless of whether the digit in the hundreds place is even or odd. They were able to share three-digit numbers that are multiples of ten between two groups equally.

Moreover, when the number 110 was asked, all students were able to decide that the number 110 is divisible by 2. They expressed the given number using base-ten language and constructed the number by taking a hundreds and tens block. They traded the hundreds block for ten tens blocks to share the base-ten blocks. After that, they also traded the ten blocks for ten ones to share the ten blocks since they could not be separated due to the block structure. To illustrate, Figure 4.8. consists of Elif’ solution on the virtual manipulative and the interview excerpt is as follow.

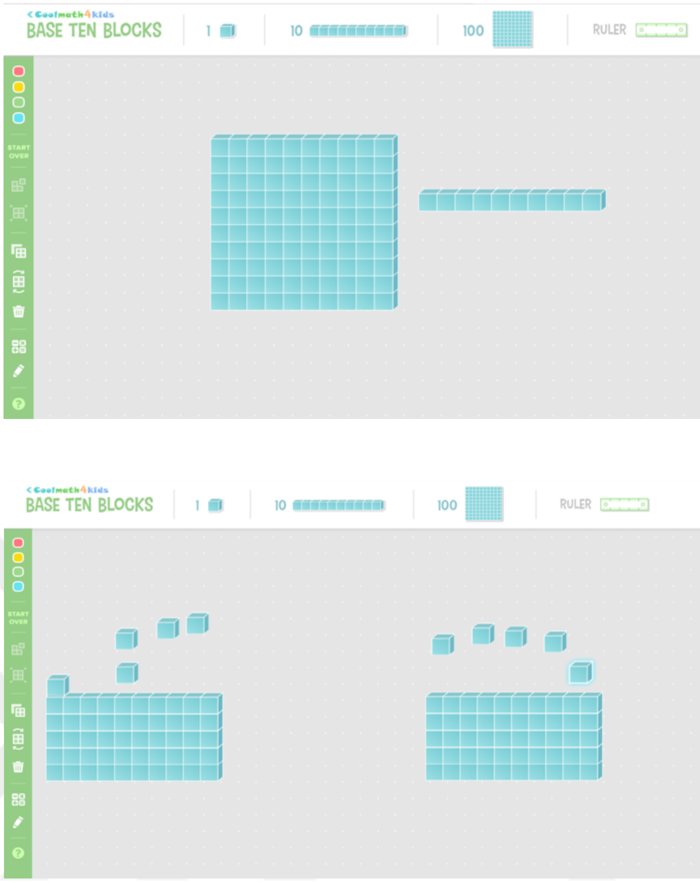

Figure 4. 8. Elif's solution for the number 110 on the virtual manipulative

> Researcher: How did you share, show me?
>
> Elif: I shared the hundred between two children by giving them five tens. Then, I shared the ten blocks between two children, giving each one five ones.

As seen from the figures and excerpts, the student explicitly explained how she tried to share the given number between two groups. Besides, as the numbers that are multiples of ten is even numbers, students realized the connection between the even numbers and the number zero in the ones digit of a number. The example from the interview excerpt about the numbers that are multiples of ten is as follows:

"… because zero is an even number, and the number has a zero at the end of it. This is also true for 100. For example, the numbers 200 or 300 are all divisible by 2. The numbers 10, 20, 30, or 40 are all divisible by because their endings are even numbers."

To sum up, the students were able to see the structure of base tens and hundreds blocks in given two- or three-digit numbers that are multiples of ten by using the virtual manipulative. They realized that a tens block consists of ten ones and a hundreds block consists of ten tens or a hundred ones and are therefore shareable by two children. *They reached a generalization that two- or three-digit numbers that are multiples of ten are consists of tens. Since tens blocks are sharable by 2, they generalize hundreds blocks are also sharable by 2 as hundreds consists of ten tens. Also, they generalize that the two- or three-digit numbers that are multiples of ten are even numbers as they sharable by 2.*

Divisibility of numbers that are not multiples of ten by 2

The students were given numbers that are not multiples of ten, that is, numbers different from those with zero in the units place. The results obtained by the participants for two-digit numbers and three-digit numbers that are not multiples of ten are given in detail below, respectively.

Two-digit numbers

The students were given two-digit even numbers with an odd digit in the tens place. All three students were able to share the given number of objects between two children. When asked about the number 12, the students constructed the number using the virtual manipulative on the tablet. For the number 12, they took one ten and two ones. One tens blocks could not be decomposed because of its decimal structure. Therefore, they traded one ten for ten unities. They were able to share 12 units between two groups, giving each group six unities. Nur's solution for the number 12 is given in Figure 4.9. as an example. The upper part of Figure 4.9. is the first form the student made on the virtual manipulative, and the lower one is the last form in which the student shared 12 objects between two children. Also, the interview excerpt is as follows:

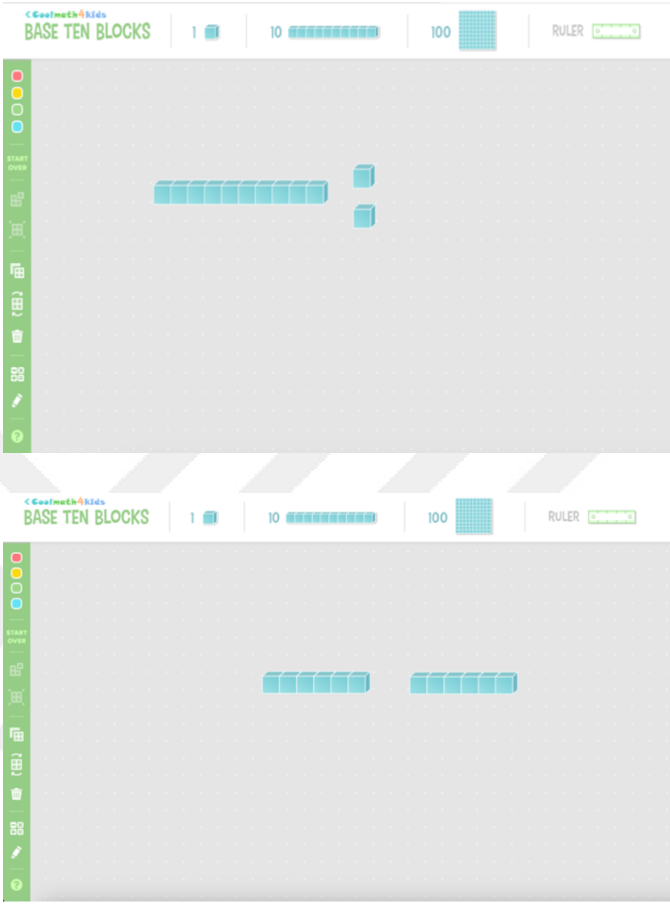

Figure 4. 9. Nur's solution for the number 12 on the virtual manipulative

> Researcher: How do you share 12 objects between two children so that each child gets an equal number of objects?
>
> Nur: We can share them equally.
>
> Researcher: How did you figure that out?
>
> Nur: Because 12, I don't know how to say, but since we divide it between two children, for example, if we give six ones to one child and six ones to another child, it will be equal by dividing one ten into ten ones. So, I gave each child 6 ones.

As can be seen from the interview excerpt, the participant first constructed the number 12 by using one ten and two ones. Then, she chose to trade one ten for ten ones. After trading, she was able to share the number between two children equally.

Furthermore, the students were asked to form 56 as a two-digit even number with an odd digit in the tens place and share the number equally between two children. The students were able to construct and divide the number by 2. One of the participants, Doğan, constructed the following figure using the virtual manipulative on the tablet. The upper part of Figure 4.10. is the first form the student made on the virtual manipulative, and lower part of Figure 4.10. is the last form in which the student shared 56 objects between two children.

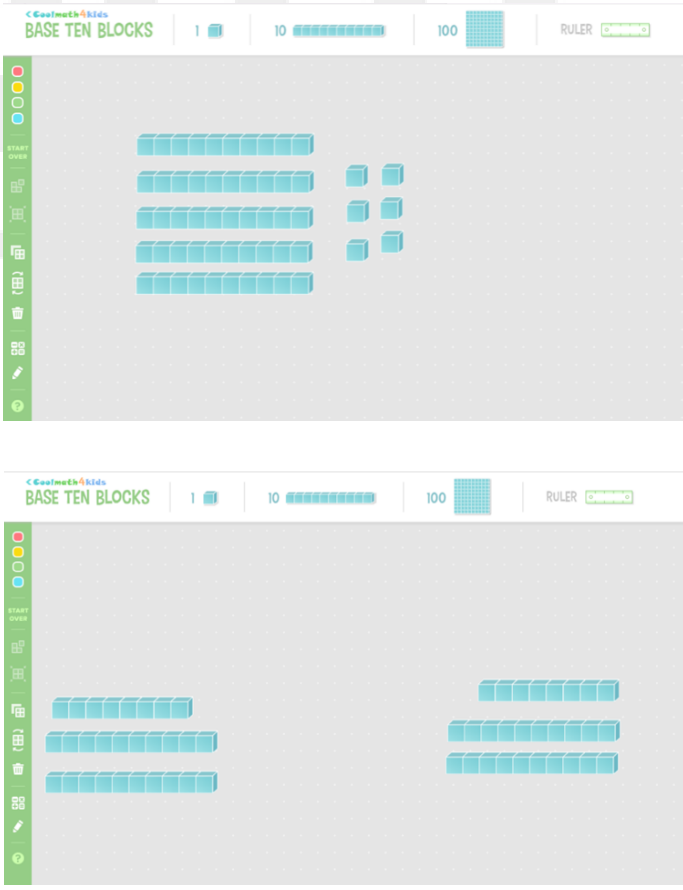

Figure 4. 10. Doğan’s solution for the number 56 on the virtual manipulative

Doğan, one of the participants, first expressed the number as five tens and six ones using base-ten language. He then gave four tens to two children, each receiving two tens. He then traded the remaining ten blocks for ten ones and gave the ten ones to two children, each of them receiving five tens. He gave the six units in the ones place of the number so that each child received three units. As a result, the student was able to divide 56 equally between two children in such each child received 28 objects, and there was no leftover unity. At this stage, the participant benefited from the expression of the number in the base-ten system, the representation of the number with base-ten blocks, and the trading of tens for ones. The excerpt from the interview explained why it is shareable as follows:

> Doğan: That is because 6 is an even number. The tens digit of the number does not affect the sharing of the number because I can break the tens no matter how many there are.

As can be understood from the figures and interviews, the students were able to make sense of the differences between the ones and tens. Base-ten language of numbers was necessary to understand the reason behind the divisibility rule of 2. The students realized that no matter how many tens blocks there were (i.e., whether there were as many odd numbers as even numbers), they could be divided by 2 in different ways.

In the light of the one-to-one interviews and the figures, the students reached the generalization that a two-digit even number can be shared by two children *regardless of whether the numbers in tens place is odd or even. They were able to generalize divisibility is dependent on the ones place not tens place of a number. When the ones digit of a two-digit number is even, this indicates that the number is divisible by 2.*

Moreover, the students were given two-digit odd numbers with an even digit in the tens place. All three students were able to identify that while sharing two-digit odd numbers (e.g. 25), there was remainder unity. One of the participants, Doğan, created the following figure using the virtual manipulative on the tablet. The upper part of Figure 4.11. is the first form the student made on the virtual manipulative, and lower part of Figure 4.11. is the last form in which the student shared 25 objects between two children, with a leftover one. Also, the interview excerpt is as follows:

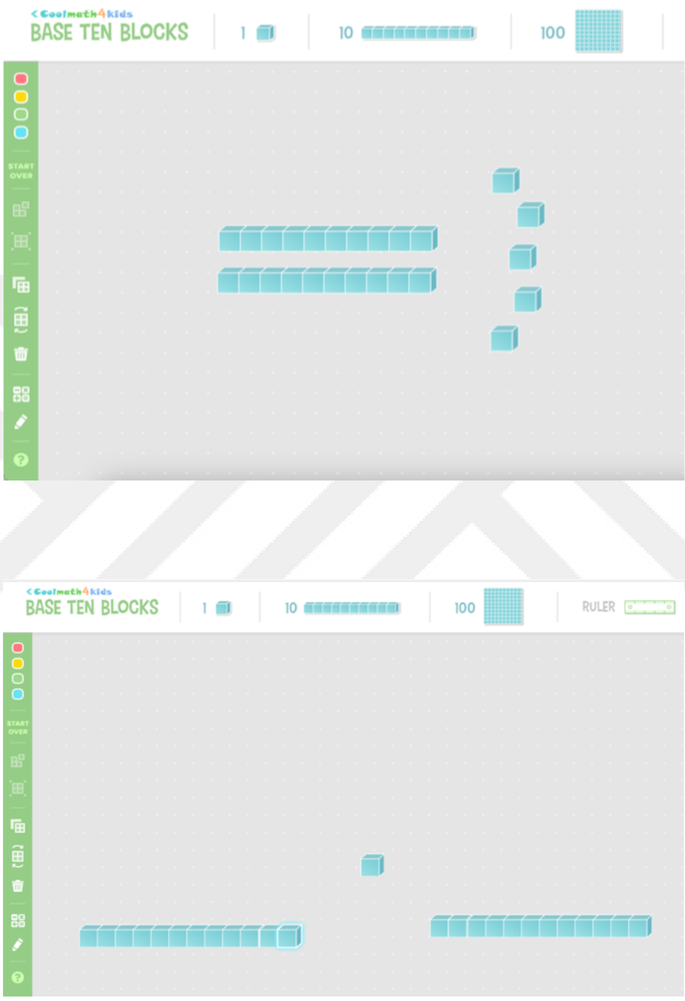

Figure 4. 11. Doğan’s solution for the number 25 on the virtual manipulative

> Doğan: Each child took 12 and there was a remainder unity. So, the number 25 is not divisible by 2.

In conclusion, the students used the virtual manipulative to construct the number 25, using two tens and five ones on the tablet. Then, they divided the two tens by 2, taking a ten block on the right side of the screen and the other tens block on the left side of the screen. Afterward, they shared the five ones so that two were on the right side and two were on the left side of the screen, leaving one unit in the middle of the screen. As seen from the figures and excerpts from interviews, the participants were able to correctly interpret the divisibility of the two-digit odd number with an even digit in the tens place. As it can be understood from the answer given by Doğan, they understand that when the number in the ones place of a two-digit number is a multiple of two, the number can be divided by 2.

In light of the one-to-one interviews and the figures, the students reached the generalization that a two-digit odd number cannot be shared by two children *equally because of the remainder. They were able to generalize that the ones digit of a two-digit number needs to be even to be sharable by two* children *equally.*

Three-digit numbers

Considering the three-digit numbers, three students were able to divide a number by 2 in the correct way. As mentioned, the students formed the numbers 100 and 110 and tried to share these numbers between two children equally. After that, they were asked to construct the number 111 and tried to share this number of objects between two children. All students were able to conclude that there was a remainder unity at the end of the sharing process. The excerpts from the interviews of participants are as follows:

> Researcher: How do you share 111 objects between two children so that each child gets an equal number of objects?
>
> Doğan: There will be one remainder.
>
> Researcher: How did you understand this?
>
> Doğan: Because 111 is an odd number, that is, even plus one equals an odd number, it could not be divided (Even plus odd numbers refers to 100+1=111).
>
> Researcher: Now, why do you think we couldn't share 111, but we could share 110 and 100 objects between two children?

Nur: Well, because 10 is an even number, and 100 is an even number, but 1 is not an even number. 1 is an odd number, so it is not divisible.

As can be understood, the students were able to decide by looking at the ones place of the given number to check the divisibility by 2. That is, it can be concluded that students reach such generalization during the clinical interview. Based on Nur's answer, it can be said that she decides divisibility by looking at the evenness and oddness of the number. According to her, since the number consists of an odd digit in the ones place, this number is odd and not divisible by 2.

Then, the question was asked for constructing and identifying the divisibility of the number 331. All three students were able to recognize that there was a remainder. To illustrate, the following figure was constructed by Elif using the virtual manipulative. The upper part of Figure 4.12. consists of base-ten blocks including three hundreds, three tens, and one unity for constructing the number 331. The lower part of Figure 4.12. shows that student share these base-ten blocks between two children by moving the blocks to the right and left sides of the screen.

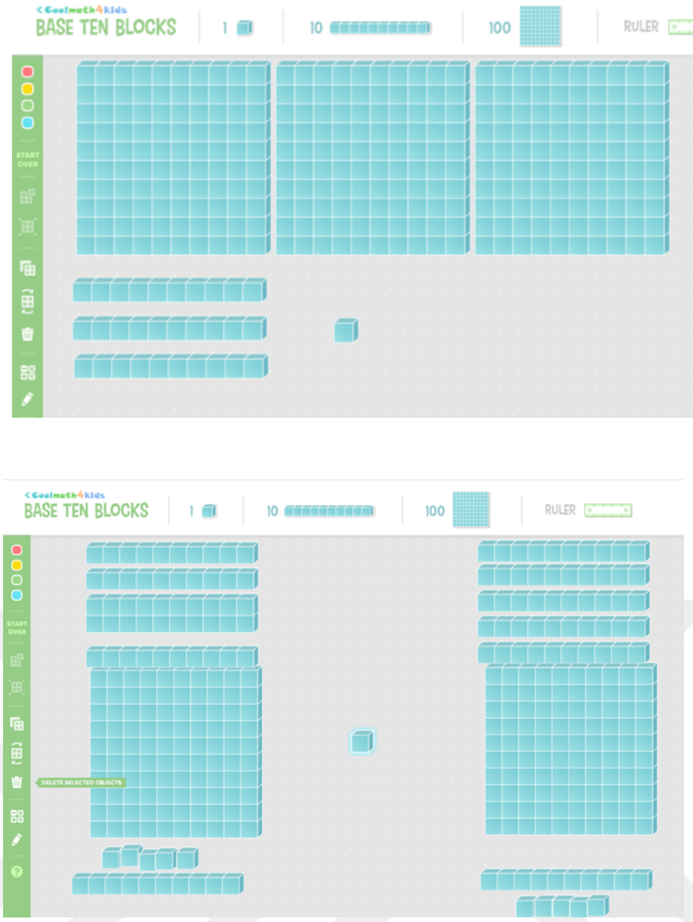

Figure 4. 12. Elif's solution for the number 331 on the virtual manipulative

The student shared two hundreds between two children by giving each child one hundreds block and traded the remaining one hundreds block for ten tens. In this way, she split the number into five tens. In the same way as the three tens, she first shared two tens and then traded the remaining one ten for ten ones. After trading, she gave each child five ones. After sharing the base hundreds and tens blocks, there was a unity left. However, she could not divide one unity in a way that preserves its integrity. During the interview Elif explained the process as follows:

> "At first, I tried to divide 331 separately, but I realized that it would be difficult, so I first divided 200 of 300 by a hundred for two children. Then, I subtracted 200 from 300 and found 100. I divided it as 50 50 for two children, so I got 300. Then, I divided 30 by 2. I found 10. I divided it as 10 10 for two children again. Then, I divided the last ten into two fives. In the end, I had a remainder."

To conclude, it was observed that the students decided the divisibility of a number by analyzing the place values of the numbers. They reached a generalization that three-digit odd numbers cannot be divisible by 2 because the ones digit of the number is odd.

When the students were given the number 332, all students decided on the divisibility by looking at the ones digit of the number. Based on the interviews and the solution made by using the virtual manipulative, it can be seen that the students made decisions by looking at the place values instead of memorizing a rule when deciding on the divisibility of a number. In this number containing three hundreds or three tens, the students discovered a way by themselves, such as separating two hundreds and trading the remaining one hundred block. The following Figure 4.13. was constructed by the student Nur using the virtual manipulative. The upper part of Figure 4.13. consists of base-ten blocks including three hundreds, three tens, and two ones for constructing the number 332, and the lower part of Figure 4.13. shows that the student shares the base-ten blocks between two children by moving the blocks to the right and left sides of the screen, with no remainder left.

Figure 4. 13. Nur's solution for the number 332 on the virtual manipulative

To sum up, the students reached a generalization about three-digit even numbers that are not multiples of ten, which can be shared by two children *equally. The students identified the remainder of the units. If there was no remainder left, then they were able to claim that the number was divisible by 2. For three-digit numbers, the students made a generalization by looking at the ones place of a number to decide on the divisibility. If there are even digits in the ones place, then they made a generalization that there would be no remainders, so the number is divisible by two.*

Furthermore, in the generalization process, the students realized the relationship between consecutive numbers. Two of the students realized the relationship between even and odd consecutive numbers. One of the students claimed that two consecutive numbers cannot be divided by 2 and explained this in the following words.

> "In fact, in consecutive numbers, the even number is divisible by two. In other words, it is never a number that can be divided by 2. That is, the two cannot overlap. Because this is done by adding one to even numbers, they become odd numbers."

For example, the student who was able to divide the number 12 by 2 claimed that the number 13 could not be separated and that there would be one remainder. Establishing the connection of consecutive numbers and the divisibility of these consecutive numbers helps conceptualize the divisibility by 2. The following excerpt from the interview showed that the student realized the relationship of consecutive numbers. The student claimed that tens are even numbers and that the consecutive number is odd when you add one to an even number.

> "They, that is, every ten is a multiple of two. In other words, since all numbers that are multiples of ten are multiples of two, at the same time, all of them are even numbers. Because when you add one to them, you get an odd number."

To sum up, during the clinical interviews, all three students improved and were able to generalize about divisibility by 2. They discovered that they were able to decide whether any given number could be divided by 2 by considering the ones digit of the number. In other words, if the ones digit is even, the number can be divided by 2 and vice versa. The critical point here is not to memorize the rule of divisibility by 2 but to make sense of why that rule exists. At the end of the one-to-one interview on divisibility by 2, three students concluded that the ones digit needs to be even for a given number to be divisible by 2. They expressed the generalization of the divisibility rule by 2 in the following words.

> Researcher: Can you generalize? Can you write it there in your own words?
>
> Nur: When we shared it between two children, only one was left, so I looked at the ones digit. When we divided between two children, we looked at the ones digit because some could not be shared equally.

Researcher: Why didn’t we look at the tens and hundreds digits?

Nur: Because the tens and hundreds digits did not have much effect on whether the number was even or odd, I only checked the ones digit because only the ones digit had an impact.

Researcher: We can understand division without implementing division algorithm, right? From which digit can we understand?

Doğan: Unity.

Researcher: From tens and hundreds?

Doğan: We cannot understand.

Researcher: How can you generalize the rule of division of 2?

Elif: The ones digit needs to be divided by 2.

Researcher: Why did not we look at the tens and hundreds place?

Elif: Because they are divided in half anyway.

4.1.2. The Process of Generalizing the Divisibility Rule of 3

One of the aims of the current study was to analyze how fifth-grade students generalize the divisibility rule of 3. To achieve the aim of the study, the students were given an activity sheet on divisibility by 3 during the clinical interviews. As mentioned in the previous chapters, the students were expected to answer the questions on the activity sheet by using virtual base-ten blocks simultaneously. Firstly, the students were expected to construct the given numbers on the questions by using the virtual manipulative. Then, the students were expected to share the numbers given in the questions among three children equally. In the methodology chapter, detailed information on the questions was given. The following part includes the findings of the students’ answers, headings, and subheadings for analyzing the generalization process of the divisibility rule of 3 in detail.

During the clinical interviews, the students tried to share the given number of objects among three children so that each would receive an equal number of objects, as mentioned. The findings were further grouped under the headings of *one-digit numbers, multiples of ten, not multiples of ten.* In other words, the students performed fair sharing first with one-digit even and one-digit odd numbers and then with two-and three-digit numbers that are multiples of ten and not multiples of ten, and their generalization process for the divisibility rule of 3 was examined. Moreover, while trying to share the given numbers among three groups equally, the students identified that there are three possibilities for the remainders, namely one remainder, two remainders, and no remainders. Thus, under the headings, there were subheadings which are *one remainder*, *two remainders*, and *no remainders,* details of which are given below.

Divisibility of one-digit numbers by 3

Considering the one-digit numbers, there were numbers with one remainder, two remainders, and no remainders after being shared equally by three groups. In other words, during the one-to-one clinical interviews, the students were given one-digit numbers with one remainder, two remainders, and no remainders after being shared equally by three children.

One remainder

All of the students were able to identify that there was a remainder of the given one-digit number after trying to share among three children equally. For the number 4, the students shared four ones among three children and realized that there was a remainder unit after the sharing. For instance, one of the students, Doğan, first constructed the number using four ones, as seen in Figure 4.14. Then, he dragged one object to the left, right and top sides of the tablet screen while sharing four objects among three children. Thus, a remainder unit was seen in the middle of the screen, as in the lower part of Figure 4.14. Figure is a screenshot of the students' answers using the virtual manipulative on the tablet, and the interview excerpts are as follows:

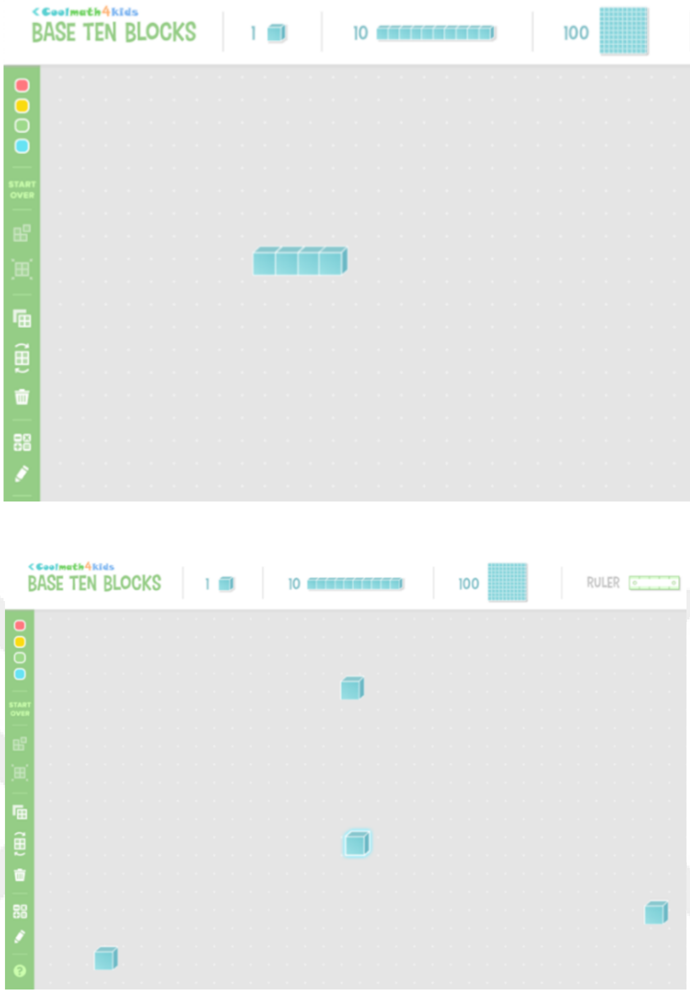

Figure 4. 14. Doğan's solution for the number 4 on the virtual manipulative

> Researcher: Can you construct the number 4 using base-ten blocks? How do you share these four objects equally among three children?
>
> Doğan: One of them is left over.
>
> Researcher: So, what does this left-over object mean?
>
> Doğan: Four cannot be divided by 3.

All of the students were able to see the remainder unity. *In light of the interviews, all three students were able to form the given one-digit number using base-ten blocks and share the number among three* children. *They successfully identified the one-digit number that had one remainder after the sharing.*

Two remainders

Furthermore, the students were given one-digit numbers with two remainders after being shared equally by three groups. For instance, the students were asked to share the number 8 among three children so that everyone would get an equal number of objects. The screenshot of the image created by the student using the virtual manipulative on the tablet is given in Figure 4.15. The upper part of Figure 4.15. shows the construction of the number 8 by using eight ones. The lower part of Figure 4.15. shows the students trying to share the eight units among three children, taking two objects on the right, left, and top sides of the screen and leaving two remainders, as seen in the middle of the screen.

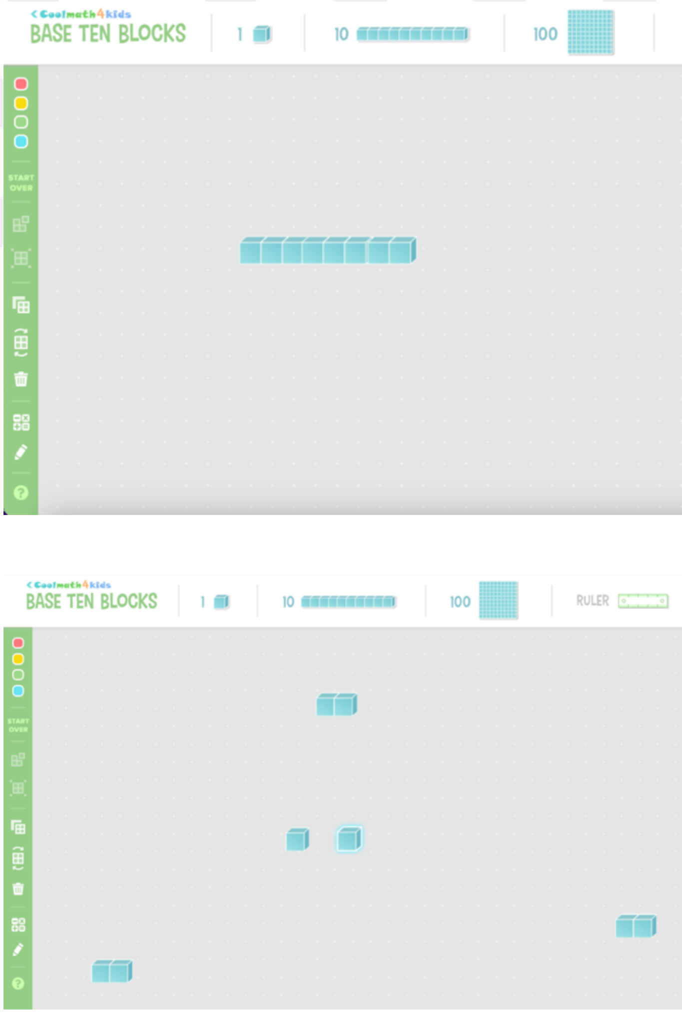

Figure 4. 15. Nur's solution for the number 8 on the virtual manipulative

Researcher: Can you construct the number 8 using base-ten blocks? Can you share the number 8 among three children equally?

Nur: Two ones are left in the middle.

Researcher: So, is the number 8 sharable by 3?

Nur: No. There are two left in the middle.

All students were able to conclude that there were two remainder unities. In light of the interviews, the students were able to form the given one-digit number and share the number among three children*. They successfully identified the one-digit number that had two remainders after the sharing.*

No remainders

Moreover, the students were given one-digit numbers with no remainders after sharing among three groups equally. For instance, the students were asked to share the number 6 among three children so that everyone gets an equal number of objects. Figure 4.16. shows the student's solution using virtual base-ten blocks. Upper part of Figure 4.16. shows formation of the number 6 by using six ones. Lower part of Figure 4.16. demonstrates the students trying to share six units among three children, taking two objects on the right, left, and top sides of the screen, leaving no remainders.

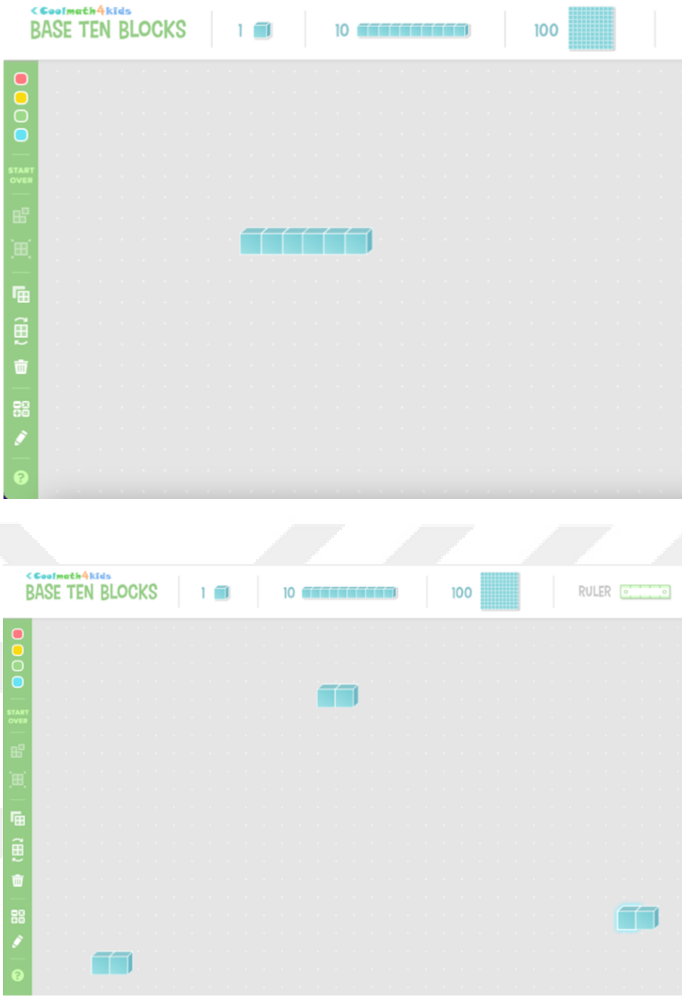

Figure 4. 16. Elif's solution for the number 6 on the virtual manipulative

Researcher: Can you construct the number 6 using base-ten blocks? Can you share the number 6 among three children?

Elif: The first group... the second group... the third group... I divided 6 by 3 since multiples of 3 are already divisible by 3, i.e., 6 is a multiple of 3, so I divided 6 by 3. There were 2 objects for each child.

Researcher: Okay. So, is the number 6 sharable among 3 children equally?

Elif: It was shared because there were no remainders.

All students were able to conclude that there were no remaining units. They were able to identify the number that had no leftovers after the sharing. Furthermore, *they generalized that one-digit numbers that are multiples of three can be sharable and divisible by 3 through the fair sharing approach.*

As mentioned earlier, the students were given two and three-digit numbers that are multiples of ten and not multiples of ten. The findings of two-digit and three-digit numbers that are multiples of ten will be presented in detail.

Multiples of ten

There was a sequence between the given numbers, that is, first one-digit, then two-digit, and finally three-digit numbers were given. Two and three-digit numbers that are multiples of ten (i.e., the number is zero in the ones place) were presented. This part of the findings represents two-digit numbers that are multiples of ten such as 10, 20, 30 i.e., numbers that have zero in the unit place. Similarly, this part of the findings represents three-digit numbers that are multiples of ten such as 100, 200, 300, i.e., numbers with zero in the units place. The remainder possibilities (i.e., one remainder, two remainders, and no remainders) are given in detail below for two- and three-digit numbers that are multiples of ten, respectively.

One remainder

The students were asked to share the number 10 equally among three children. Three students identified that there was a remainder after sharing ten blocks among three groups. When asked about the number 10, the students noticed that one unit block was left over after sharing ten ones among three children. The following figure is a screenshot created by students using the virtual manipulative. The upper part of Figure 4.17. shows the construction of the number 10 using one tens block. After taking one tens block, the students decide to trade the tens block for ten ones to share the ones. The lower part of Figure 4.17. shows the sharing of ten ones, taking three ones on the right, left, and top sides of the screen and leaving one remainder in the middle.

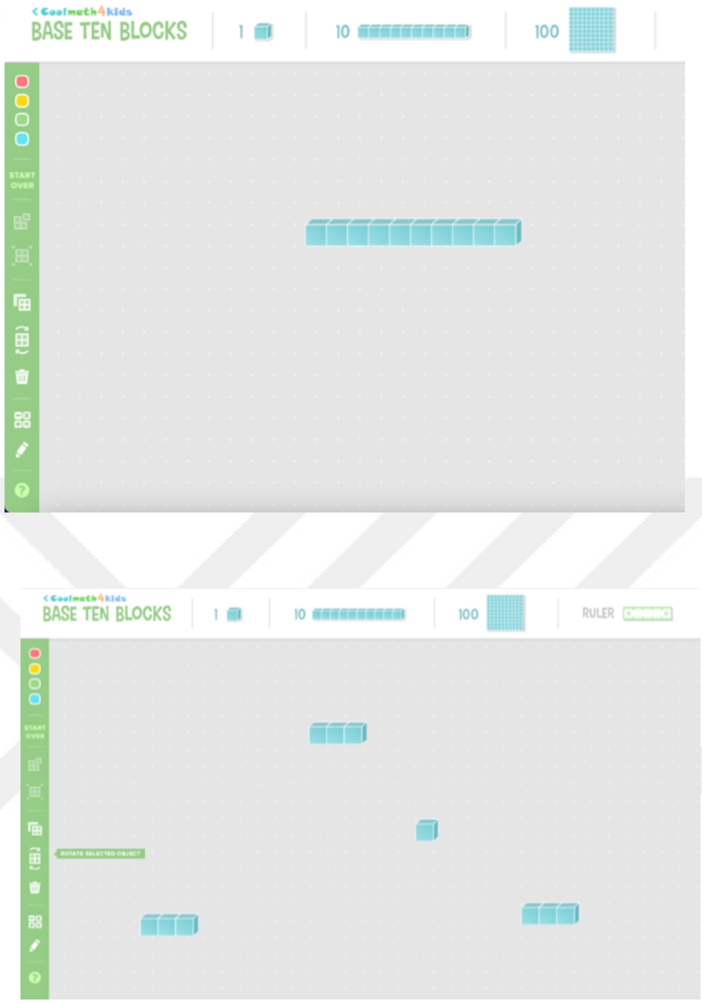

Figure 4. 17. Elif's solution for the number 10 on the virtual manipulative

Researcher: Can you construct the number 10 using base-ten blocks? Can you share the number 10 among three children equally?

Elif: Firstly, since the tens block is a whole, I traded it for ten ones.

Researcher: Why did you feel the need to trade one ten for ten ones?

Elif: If I divide it into units, I can share it equally among three children, but since the tens is a whole, I cannot, so I have to divide it.

Researcher: Okay. Then, what did you do? Can you tell me?

Elif: Yes. It could not be divided. There is one leftover.

All three students were able to trade the ten blocks for ten ones to share the ones. They identified that it was impossible to share ten blocks due to the block structure. After trading, they were able to share the unities among three groups, as can be seen in Figure 4.17. *To conclude, all students were able to generalize that a ten cannot be divisible by 3; there is a leftover after the sharing.*

Furthermore, the students were asked to share the number 100 among three children equally. They observed that one unit block was left over after sharing the number 100 among three children equally. Two of the students used the same strategy for constructing and sharing the number. The following part consists of two different strategies used by the students.

The first strategy used by Dogan is described below. He decided to use ten tens blocks for forming the number 100. Doğan was able to allocate the tens blocks without the need to trade. He stated that it would be easier to use ten tens blocks rather than using a hundreds block. After giving each group three ten blocks, he realized that there was one tens block left, and he used it to trade. He traded a tens block for ten ones because of the whole structure of a tens block. After trading, three ones were given to each group, and there was a remainder which can be seen on the right of the screen. Doğan's solution for the number 100 is shown in the following figure.

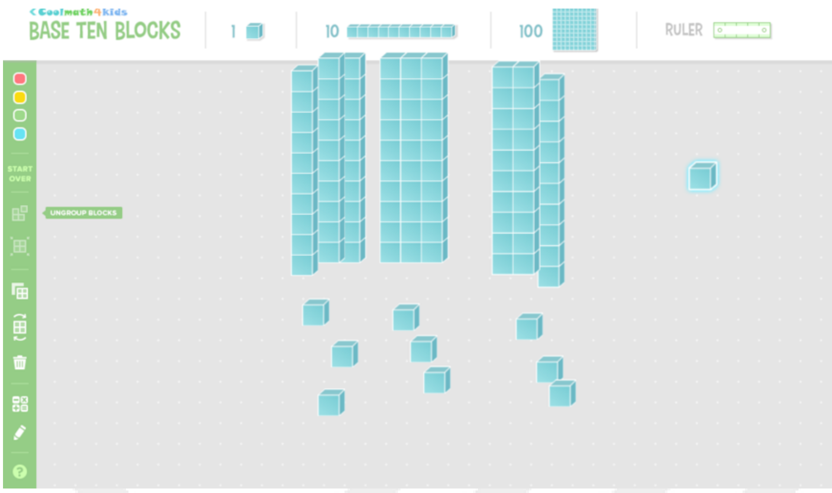

Figure 4. 18. Doğan’s solution for the number 100 on the virtual manipulative

Two students, Elif and Nur, decided to use hundreds blocks for forming the number 100. These students need to trade because of the whole structure of base hundreds blocks. The solution of two students who decided to use a hundreds blocks is shown in Figure 4.19. The first part of Figure 4.19. shows the construction of the number 100 by using a hundreds block. While using a hundreds block, due to the base-ten block structure, the students decided to trade one based hundreds block for ten tens blocks. Thus, the second part of Figure 4.19. shows the sharing of ten tens, taking three tens on the top, bottom and center of the screen and leaving one remainder tens block. The students traded the remainder tens block for ten ones to share these ones. Therefore, they used trading twice for sharing the number 100 (first one is for trading a hundred for ten tens, second one is for trading ten for ten ones). After trading, the students gave each group three ones and observed that there was a remainder unit. In either way, three students were able to identify that while sharing a hundreds block among three children in an equal way, there was one remainder as can be seen in the third part of Figure 4.19.

Figure 4. 19. Elif’s and Nur’s solution for the number 100 on the virtual manipulative

To conclude, although the students chose different ways of sharing, they reached the same result. They were able to generalize that a hundred cannot be divisible by 3 because there is one remainder unit after the sharing. More specifically, they could see that three children take 33 objects and concluded that 100 could not be divisible by 3 because there was a remainder.

Similarly, when the number 310 was asked about, all students were able to identify that there was one remainder left after sharing 310 objects among three children equally. The student constructed the following figure for creating the number 310 by using three hundreds and one tens block.

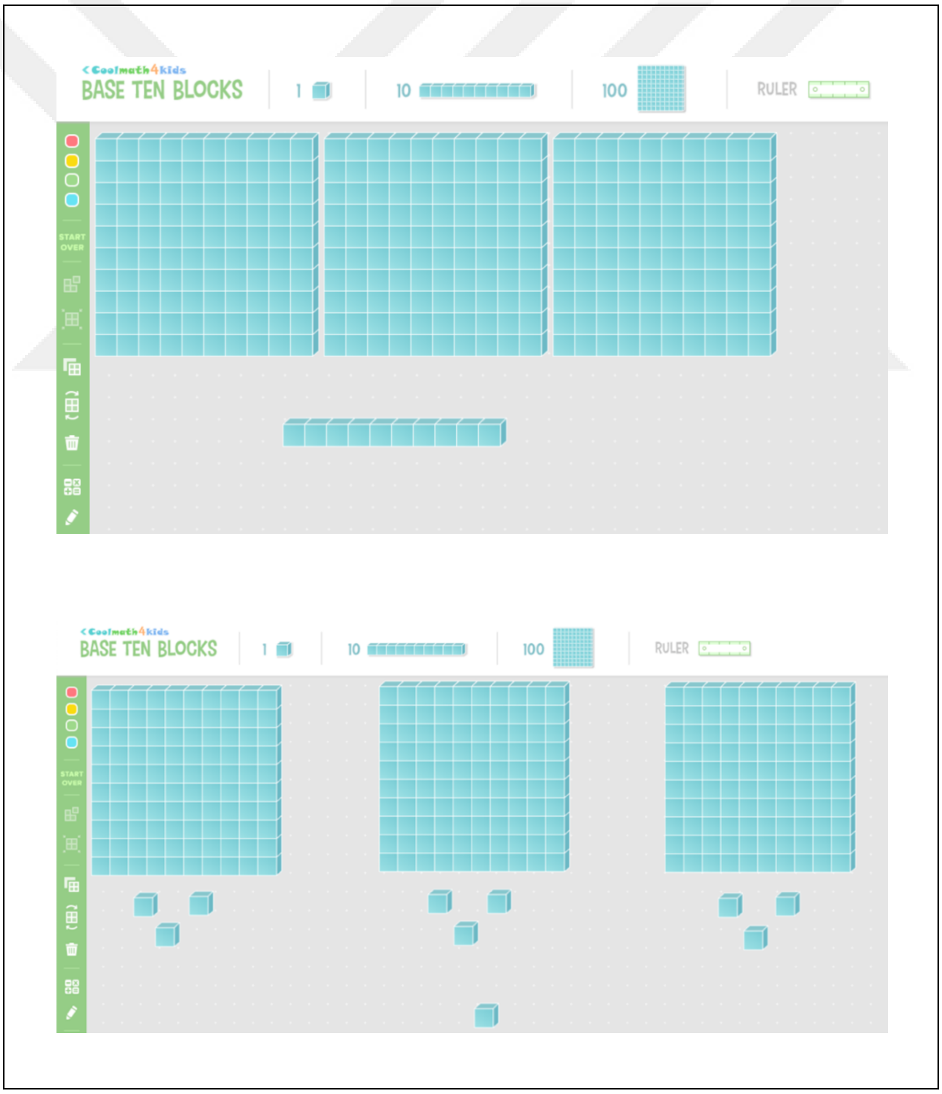

Figure 4. 20. Elif's solution for the number 310 on the virtual manipulative

Researcher: Can you construct the number 310 using base-ten blocks? Can you share the number 310 among three children equally?

Elif: 310 is not sharable.

Researcher How did you realize? Can you explain?

Elif: Because there will be one leftover.

Researcher: How many remainders are there?

Elif: There will be 3 remainders from the hundreds blocks. Also, there is one remainder from tens block. There is no unity. 3+1=4. 4 is not divisible by 3.

The students reached a generalization about the summation of the digits of a number.

Two remainders

Three students reached a generalization that there was a remainder after sharing the tens blocks among three children, as mentioned. When the number 20 was asked about (i.e., there were two tens blocks), three students noticed that there was a remainder unit in each block. The following figure is a screenshot created by Elif using the virtual manipulative. The upper part of Figure 4.21. shows the construction of the number 20 using two tens blocks. The lower part of Figure 4.21. shows that the blocks were traded for ten units separately. There were three groups that got six blocks, leaving two remaining, as seen lower part of Figure 4.21.

Figure 4. 21. Elif's solution for the number 20 on the virtual manipulative

Researcher: How can you share 20 objects among three children so that each one gets an equal number of objects?

Elif: Since there are two base-ten blocks this time, adding them together or multiplying one of the blocks by 2 will give 20. I will trade tens blocks for ten ones and do it in an order.

There's one more.

...

> There are two more.
>
> Researcher: Okay. Can you explain where the leftover ones come from?
>
> Elif: One came from the first base-ten blocks, and the other came from the second base-ten blocks.
>
> Researcher: Okay, so what can you say about the number? Is this number sharable among three children?
>
> Elif: There are two remainders, so the number is not divisible by three.

Another student, Doğan, made the following comment after obtaining three groups of six ones in the same way as the solution above and realizing that there were two leftover ones:

> "... So, three groups take six, but two are left. To share equally, I need one more unity."

As can be seen from Figure 4.21. and the interview excerpts, the students were able to identify the remainder after sharing the numbers among three children equally. *To conclude, all students were able to generalize that a ten cannot be divisible by 3. When there were two tens, they were able to see that there was a remainder in each of the blocks. Therefore, they were able to sum the remainders, giving two remainders total.*

Furthermore, the students were asked to share the number 200 among three children equally. Three students reached a generalization that there was a remainder unit after sharing a hundreds block among three children, as mentioned. When the number 200 was asked about (i.e., there were two hundreds blocks), three students were able to notice that there was a remainder unit for each hundreds block, separately. There were two different solutions for sharing the number 200. The following figure is a screenshot created by Elif and Nur using the virtual manipulative. The students first constructed 200 by taking two hundreds blocks, as seen in Figure 4.22.

Figure 4. 22. Elif's and Nur's solution for the number 200 on the virtual manipulative

The students decided to share two base hundreds blocks separately among groups of three. They first traded one hundreds block for ten tens blocks and then shared these tens blocks among three groups, with each group getting three tens. Then, they dragged the remaining tens blocks to the other corner of the screen, as seen in the following figure.

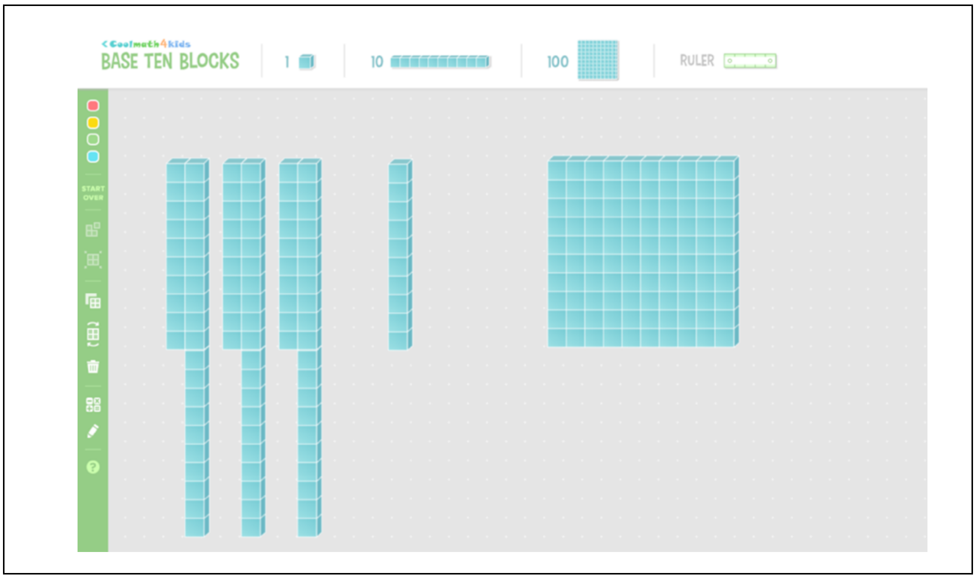

Figure 4. 23. Elif's and Nur's solution for the number 200 on the virtual manipulative

When they moved to the other hundreds block, they followed the same steps. They dragged the remaining tens block to the other corner of the screen.

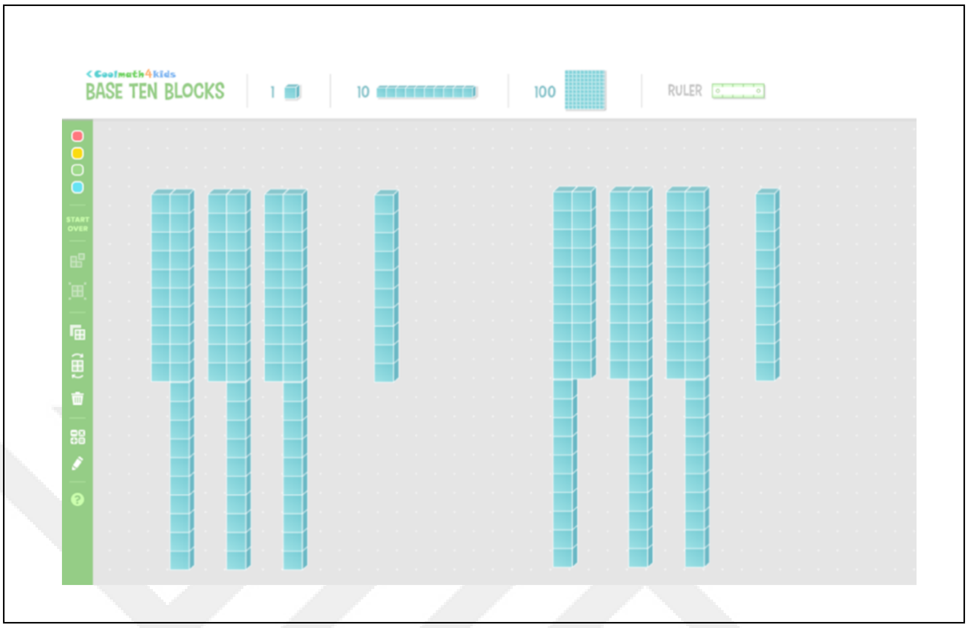

Figure 4. 24. Elif's and Nur's solution for the number 200 on the virtual manipulative

Since they tried to share the given base-ten blocks among three children equally, they tried to get three groups by combining the tens in the right and left corners of the screen. Afterward, they decided to trade the remaining two tens blocks (each of the remaining tens blocks was left over from the hundreds block separately) for ten ones and then shared these ten ones so that each group would receive three ones. As a result, they obtained two remaining ones. This process will be shown in Figure 4.25. The remaining two unities were seen on the left corner of the tablet's screen.

Figure 4. 25. Elif's and Nur's solution for the number 200 on the virtual manipulative

The excerpt from the interview in which the student explained the sharing processes is as follows:

> Elif: I will now divide the number 200 by 3. First, I divided the number 100 by 3 again, and there was a remainder from tens. Now, I divide the other hundred by 3. I will have a remainder from ten again. I will also divide the tens. Then I added them again. There are two remainder ones.

The other student, Doğan, reached the same result but used ten tens blocks directly instead of a hundreds block while forming two hundreds blocks. He gave three tens blocks to three groups. Then, he repeated the same process. Each of the three groups had six tens blocks. Afterward, there were two remaining tens blocks. Then, he traded these tens blocks for ten ones separately. After trading, he gave each of the three groups three ones, and he also realized that there were two remaining ones.

To conclude, all students were able to generalize that a hundred cannot be divisible by 3; there is one remainder after the sharing. When there were two hundreds, they were able to see that there was a remainder for each of the blocks.

Therefore, they were able to sum the remainders, resulting in two remainders in total.

In addition, for the number 320, Elif was able to answer the question without performing division. She explained how she had decided on the divisibility of the number during the interview as follows: There will be three leftovers from three hundreds, two leftovers from two tens, and no ones. 2+3=5. Five is not divisible by 3. Thus, the number 320 is not divisible by 3.

> Researcher: Can you construct the number 320 using base-ten blocks? Can you share the number 320 among three children equally?
>
> Elif: No. 320 is not divisible by 3.
>
> Researcher: How did you decide?
>
> Elif: There will be 3 remainders from the hundreds blocks, and 2 remainders from two tens blocks. There is no unity. 2+3=5. Five is not divisible by 3.

In short, they were able to identify the remainder units from each base-ten block, add these remainder unities, and redistribute them equally among three children.

No remainders

When the number 30 was asked about, all students were able to identify that there was no remainder unit. There was a student who realized the summation of the remainders was zero. That is, she was aware that there is a remainder after sharing a tens block among three. Since there were three base-ten blocks, the number of the remainders was three, and she realized that she could share these three remainders among three groups. She explained her sharing process as follows:

> "Thirty can be sharable by three. Because at first, when I tried to share the base-ten block among three, I got a remainder. Since all tens blocks are the same, if there is one remainder in all of them and those remainders are added, I get 3 remainders. Since these 3 remainders can be shared among three, the number thirty can be divided by 3."

The following figure is a screenshot of the student's solution while trying to share thirty ones among three groups equally.

Figure 4. 26. Elif's solution for the number 30 on the virtual manipulative

As can be observed in Figure 4.26., she split the screen into three groups. She gave each group ten unities with no leftovers. While sharing the number 30, another student commented on the process. He was able to identify the remaining situations. Doğan realized that the number of remaining units after division by three would be two at most. Because when there were three or more remaining units, these units could be shared among the groups again. Therefore, the student emphasized that the remainder could be 1, 2 or 0. The student explained his thoughts as follows:

> "I think we got a situation here. There is one remainder from a tens block. Two more remainders in 20, no more remainder in 30. So, in this case, if we go in order, it has to increase by 1, 2, 0. Thus, there is no remainder while sharing 30 among three children."

In addition, Nur gave commented on the numbers that are multiples of ten as follows:

> "For example, there is a ten, and we can divide ten into ten ones and share it equally, but there is only one leftover. That's why ten, for example,

cannot be divided by 3. Twenty also cannot be divided for that reason, but 30 can be divided because, for example, there is one leftover from the tens, so when there is one left over from all three, there are three, and we can share those three again."

To conclude, the students were able to sum up the remaining unities from each base-ten block. Furthermore, they were able to share the total number of remainders among three children *again. Thus, the students were able to make a generalization that if the total number of the remainders is three, then these remainders can be sharable again among three* children.

Furthermore, the students were asked to share the number 300 among three children equally. All students reached a generalization that there is one remainder from each hundreds block. Since there are three hundreds blocks, the total number of remainders is three, which is sharable again among three children. The relevant interview excerpt is as follows:

Researcher: How can you share 300 objects among three children equally?

Elif: There would be three leftovers.

Researcher: What did you do with those leftovers?

Elif: We shared the three leftovers among three again; that is, we divided them. In the end, there were no leftovers.

In conclusion, the students were able to identify the remainder from each base-ten block separately. They reached a generalization that there is a remainder after sharing the hundreds among three. Similarly, there was a remainder after sharing the tens among three. Also, they were able to add the remainders and then share the remainders again for a two- and three-digit numbers that are multiples of ten.

Numbers that are not multiples of ten

This part of the findings represents two-digit and three-digit numbers that are not multiples of ten, i.e., the numbers with no zero in the units place. The remainder cases (i.e. one remainder, two remainders and no remainder) are given in detail below for two- and three-digit numbers that are not multiples of ten.

The students were given two-digit numbers, which were not multiples of ten, that is, numbers different from those with zero in the units place (31, 32, 33). The results obtained by the participants for two-digit numbers that are not multiples of ten are given in detail below. For the number 31, the students were able to form the number by using base-ten blocks with the help of the virtual manipulative. One of the students constructed the following figure.

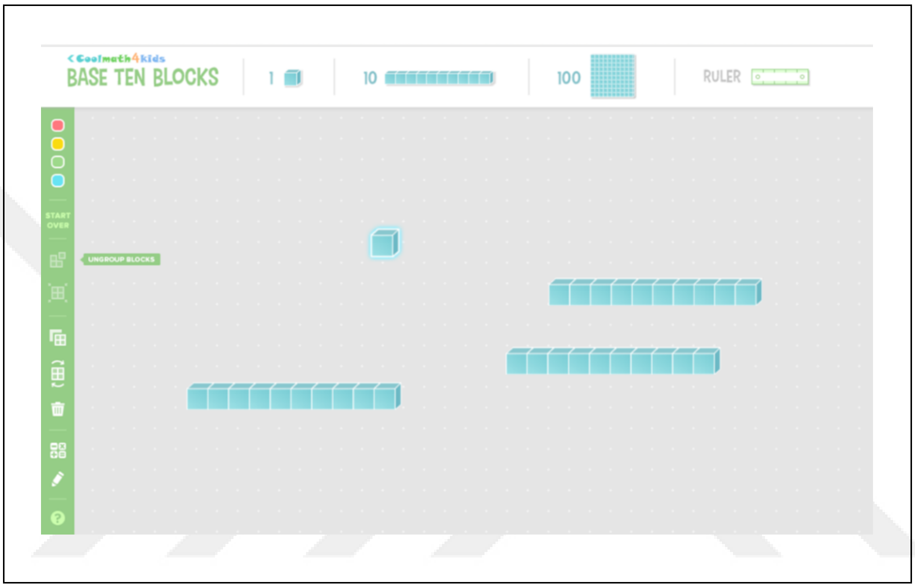

Figure 4. 27. Nur's solution for the number 31 on the virtual manipulative

When constructing the number 31, there were two students who used three tens blocks and one unity, as can be seen in Figure 4.27. There was a student who chose to use thirty-one units. Either way, the students divided the number 31 into three groups with ten unions in each group and found that there was one unity left. The numbers 31, 32, and 33 were asked of the students in this order. All students were able to understand the remainder. They identified that there was one remainder after sharing the number 31, two remainders after sharing the number 32, and there were no remainders after sharing the number 33.

When the number 301 was asked about, all students were able to identify that there was one remainder left after sharing 301 objects among three children equally. The following figure is constructed by the student for cons the number 301 using three hundreds and one ones block.

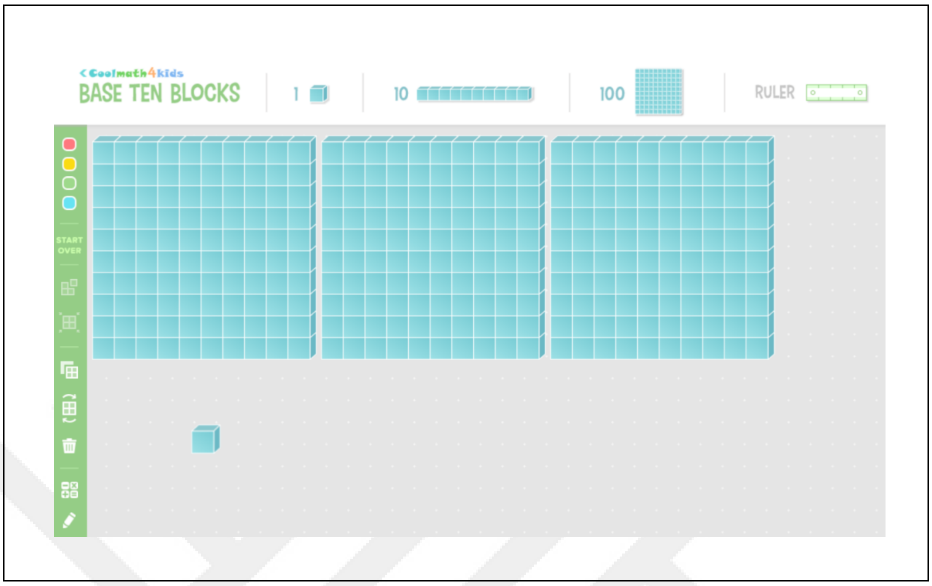

Figure 4. 28. Nur's solution for the number 301 on the virtual manipulative

The students shared three hundreds blocks among three children, with one left. The researcher them encouraged to consider the remainder blocks from each hundreds block. The interview excerpt is given as follows:

> Researcher: Can you construct the number 301 using base-ten blocks? Can you share this number among three children equally?
>
> Nur: We could not share it equally.
>
> Researcher: Why not?
>
> Nur: Because there would be a total of four left, but we shared among three, but there would still be one left.

As can be seen from the interview excerpt, it can be said that the students reached the generalization that a hundreds block gives a remainder when divided by 3, and a tens block gives a remainder when divided by 3 in the same way. They were able to sum the remainders from the hundreds, tens, and ones block. If the summation of the leftovers is larger than three, they choose to share the leftover ones again.

Towards the end of the intervention, the students were asked to give an example of a three-digit number that is divisible by 3 and to share this number using base-ten blocks. For instance, Nur gave the number 123 as an example of a number that is divisible by three. The student first constructed the number using one hundreds, two tens, and three ones. After creating the number, she explained why the number 123 can be divisible by 3 as follows:

> "If there were 123, there would be one left over from this (this refers to the hundreds block), one left over from this (this refers to the tens block), one left over from this (this refers to the tens block), and three ones. Since the total number of ones is six, we can divide it."

Towards the end of the clinical interview, when the number 546 was asked to the students, all students were able to decide by looking at the summation of the digits. If the summation of the digits is a multiple of three, they concluded that the number is divisible by 3. Nur was able to say that this number was divisible by 3 even without the need to construct the number on the virtual manipulative. The interview excerpt is as follows:

> Researcher: Can you construct the number 546 using base-ten blocks? Can you share the number 546 among three children equally?
>
> Nur: It can be shared among three children.
>
> Researcher: Why and how did you understand?
>
> Nur: Because of the remainders. For example, there are 5 hundreds so there are 5 remainders. There are 4 remainders from tens blocks, and there are 6 of these ones left. When we add them up, we get 15. But 15 is divisible because it is divisible by 3.
>
> Researcher: Can you show this by using base-ten blocks.

When students were asked to construct the number by using base-ten blocks, since the number contained five hundreds, four tens and six ones, the students decided to show the remainder block from each base-ten block. For instance, one of the students, Doğan, placed the remaining ones from the blocks on the right corner of the screen. The first part of Figure 4.29. shows that the student formed the number

546 by taking five hundreds, four tens and six ones. In the second part of Figure 4.29., the student took give unity blocks coming from five hundreds, showing them in green color and moving them to the right side of the screen (One unity was left from each hundred. Since there were five hundreds, there were five unities in total). In the third part of Figure 4.29., the student showed four leftover unity blocks from four tens in red color and moved these unities to the right corner of the screen (one unity was left over from each ten. Since there were four tens, he got four leftover unities in total). Lastly, the student dragged six unions from the ones digit of the number to the right side of the screen. Therefore, he got 5+4+6=15 unions in total. Then, since 15 is a multiple of three, he concluded that 546 is divisible by 3.

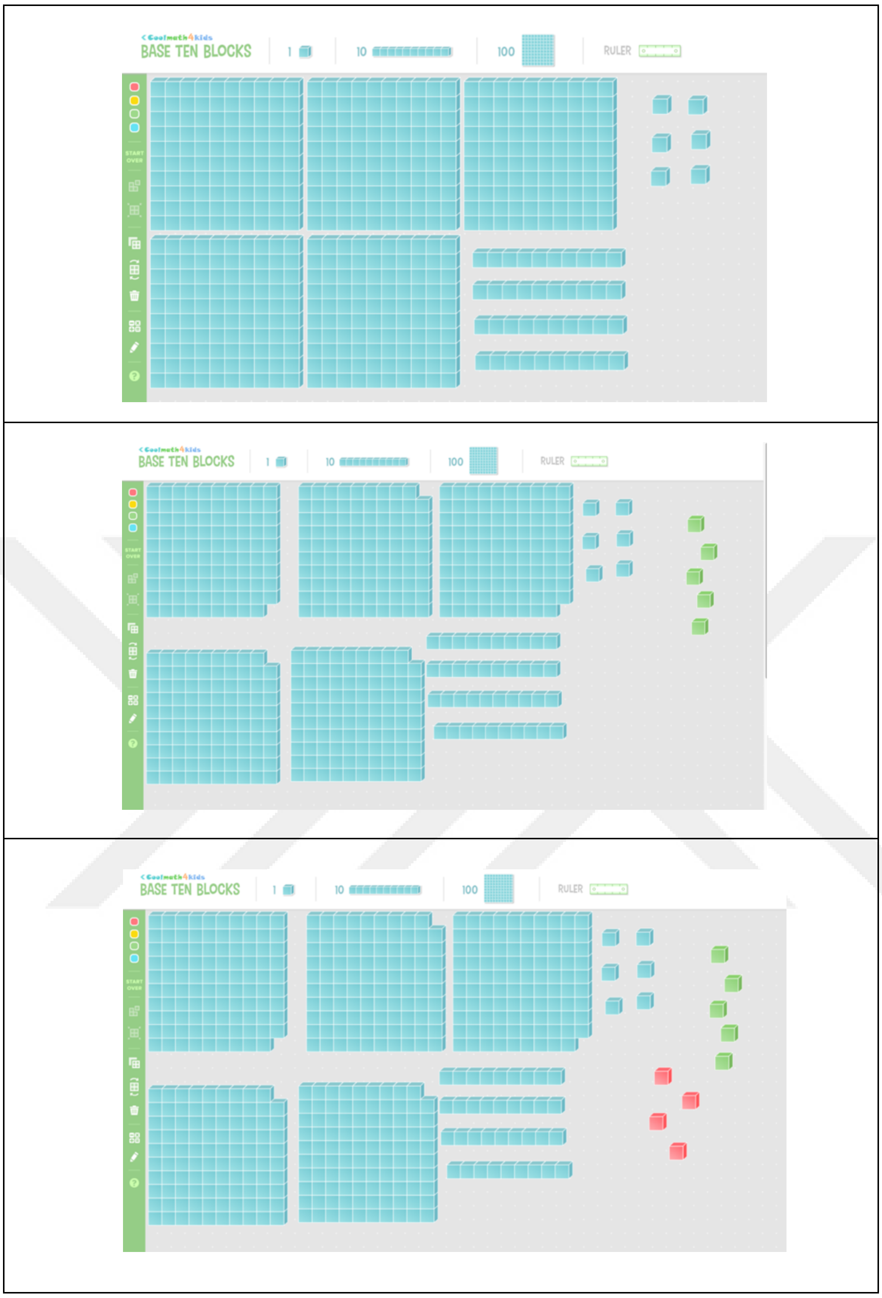

Figure 4. 29. Doğan's solution for the number 546 on the virtual manipulative

All three participants reached the divisibility rule of 3 at the end of the activity. Thanks to the virtual manipulative used, they were able to comprehend the logic behind the rule of divisibility of 3 by forming the numbers using base-ten blocks and using the equal division interpretation of division.

The generalization made by the first participant, Elif, is as follows: "*When we divide all the digits by 3 in order according to their place values, and in fact, when we add the numbers in all digits, if the number is divisible by 3, that number is divisible by 3."*

The second participant, Doğan, showed the digits in the number and expressed it as follows: *"We added them all and divided by 3."*

The third participant, Nur's conclusion for the division of numbers by three is as follows: *"There is a three-digit number, this remainder coming from the hundreds digit, this remainder coming from the tens digit, and these remainders can be combined with the number in the ones digit. We can find the total number of remainders by adding them. We can decide whether the number is a multiple of three or not."*

4.1.3. The Process of Generalizing the Divisibility Rule of 6

One of the aims of the current study was to analyze how fifth-grade students generalize the divisibility rule of 6. To achieve the aim of the study, the students were given an activity sheet on divisibility by 6 during the clinical interviews. As mentioned in the previous chapters, the students were given an activity sheet related to 6 at the end of the process. The students solved the activity sheet about divisibility by 2 in the first interview, the activity sheet about divisibility by 3 in the second interview, and the interview sheet about 6 in the last interview.

The heading and subheadings are explained in detail. The students performed under the following headings: *numbers that are not multiples of six* and *numbers that are multiples of six.* Under the heading of numbers *that are not multiples of six*, the names of the subcategories are *even numbers that are not divisible by 3, odd numbers that are divisible by 3*, and *odd numbers that are not divisible by 3.* In addition, under the heading of *numbers that are multiples of six*, the name of the subcategory is *even numbers that are divisible by* 3.

Numbers that are not multiples of six

The students were given tables that consisted of numbers that are multiples of 6 and not multiples of 6. There were four tables, namely Table 1, Table 2, Table 3, and Table 4, respectively. The students were expected to think about the divisibility of the numbers in the table by 2, 3, and 6. Numbers that are not multiples of 6 are given detail in the following order: *even numbers that are not divisible by 3, odd numbers that are divisible by 3,* and *odd numbers that are not divisible by 3.*

Even numbers that are not divisible by 3

Table 1 consists of even numbers, i.e., numbers that are divisible by 2, but not multiples of 3 and not divisible by 3. All students were able to accurately determine the divisibility of the given numbers by 2, 3, and 6. The students were able to apply the rules learned in the previous activities while checking the divisibility of the numbers in the table by 2 and 3. Excerpts from the interview are as follows:

> Researcher: Is the number 52 divisible by 2? Can you tell me how you decided?
>
> Elif: Okay, the last digit is an even number that can be shared. 52 is divisible by 2.
>
> Researcher: Well, is the number 52 divisible by 3?
>
> Elif: 2 + 5 equals to 7. Therefore, I cannot share equally. For this reason, 52 is not divisible by 3.

When deciding whether 52 is divisible by 6, she considered the numbers that are multiples of six. If it is a number that she can say while counting six by six, she concluded that it is a multiple of 6.

For the number 104, the students were able to demonstrate that this number is divisible by 2 but not by 3 and 6. For example, the following interview excerpt is given for the student, Doğan.

> Researcher: So, can you share the number 104 between two children?
>
> Doğan: Yes.

Researcher: Can you share the number 104 among three children?

Doğan: There will be one more from these hundreds blocks. There will be five more, so it cannot be shared.

Researcher: Okey, can you share the number 104 among six children?

Doğan: If we say 120 and subtract 12, we get 108, and if we subtract 6 more, we get 102, no.

When the participant thinks of numbers that are multiples of 6, he thinks of a number that he knows is a multiple of 6. If the number to be decided is larger than the number he knows, he adds six until he obtains the number; if the number to be decided is smaller, he can rhythmically subtract backward.

The following comments are about the interpretation on the participants about the numbers in Table 1.

Elif: They are all even numbers, so these numbers are divisible by 2. The numbers written here could not be divided by 3 and 6.

Doğan: All of them are even numbers, they could not be divided by 3 and 6.

Nur: They are generally divisible by 2 but not by 3 or 6.

In summary, all three students were able to generalize that even numbers that are not multiples of 3 cannot be divided by 6.

Odd numbers that are divisible by 3

Table 2 includes odd numbers, i.e., numbers that are not divisible by two but are multiples of 3 and divisible by 3. All students were able to correctly determine the divisibility of the given numbers by 2, 3, and 6. The students were able to apply the rules learned in the previous activities while checking the divisibility of the numbers in the table by 2 and 3. Excerpts from the interview for the number 39 are as follows:

Researcher: Can you share the number 39 between two children?

Nur: Since 9 is an odd digit, the number 39 is odd and cannot be divided by 2.

Researcher: Can 39 objects be divided equally among three children?

Nur: Yes, it can.

Researcher: How do you decide?

Nur: We can understand it by adding the digits of the number. 3+9=12. Since 12 is a multiple of 3, it can be divided by 3.

Researcher: How about six children?

Nur: No.

Researcher: How did you understand?

Nur: Well, actually, there is 9 here. 9 is an odd number. So, the number 39 is odd. That's why it can't be divided by 6.

For the number 111, the students were able to state that this number is not divisible by 2 but are divisible by 3 and 6. Below is the excerpt of the interview conducted by the student.

Researcher: Can you share the number 111 between two children?

Elif: No. Because the number has the digit 1 in the ones digit, so it is an odd number.

Researcher: Well, can you share the number 104 among three children?

Elif: It is divisible by 3 because when you sum the digits of the number, it makes three, so it is divisible.

Researcher: Can it be divided by 6? What do you think?

Elif: No, it is not divisible by 6.

Researcher: How did you decide?

Elif: Because 6 times 20 is 120, and 120 minutes 111 equals to 9. Therefore, 9 is not divisible by 6.

Furthermore, the following comments are about the interpretation on the participants about the numbers in Table 2.

Elif: When all the digits are added, they are multiples of 3, and they cannot be divided by 2 and 6.

Doğan: None of them are even. Because they are not multiples of 2, they are numbers divisible by 3 but not by 6.

Nur: All these numbers are odd numbers; they can be divided by 3 but not by 6.

In summary, all three students were able to generalize that odd numbers that are multiples of 3 cannot be divided by 6.

Odd numbers that are not divisible by 3

Table 3 consists of odd numbers, i.e., numbers that are not divisible by 2, multiples of 3 or divisible by 3. All students were able to accurately determine the divisibility of the given numbers by 2, 3, and 6. The students were able to apply the rules learned in the previous activities while checking the divisibility of the numbers in the table by 2 and 3. All students were able to conclude that the numbers in Table 3 are not divisible by 2, 3, and 6. Doğan's interpretation of Table 3 is given as follows:

Researcher: Are the numbers written here even?

Doğan: No.

Researcher: Can you share these numbers among three children?

Doğan: No.

Researcher: Can you divide it by 6?

Doğan: No. If it cannot be divided by 3, it cannot be divided by 6 because 3 times 2 equals 6. In this case, some multiples of 3 can be divided by 6. Or rather, every multiple of three can be divided by 6.

Researcher: Every multiple or some multiples? Can you think of an example?

Doğan: Some multiples. When we count by thirds, all of them must be even here. All multiples are included in this one, too.

Researcher: In which one?

Doğan: In six.

Researcher: Is there an odd number in multiples of six?

Doğan: Let's look at 30. 6, 12, 18, 24, not 30. That's why it is not odd. Even multiples of 3 are divided by 6; odd multiples are not.

Researcher: Why is that?

Doğan: Because 6 is even, and all its multiples are even, so there are no odd multiples.

To sum up, the students were able to reach the generalization that even multiples of 3 are divisible by 6. The following comments are about the interpretation on the participants about the numbers in Table 3.

Elif: The common feature of Table 3 is that no number is divisible by 6.

Doğan: Not divided by any of them (referring to 2, 3 and 6)

Nur: None of them can be divided by 2, 3 or 6.

Multiples of six

As mentioned, the students were asked to color the boxes containing numbers that are multiples of 2 and multiples of 3. They were asked to color the numbers in the hundreds table that are multiples of 2 (i.e., the numbers divisible by 2) in pink, and the numbers that are multiples of 3 (i.e., the numbers divisible by 3) in green. In this case, some boxes were painted in both pink and green. When the students were asked what features of the numbers were painted in pink and green colors in the hundreds table, one of the students, Nur, explained as follows:

"Both of them can be divided by 3 and 2. Generally, they are numbers in rhythmic counting; for example, in other words, they are numbers that are the rhythmic counting of 2 and 3."

The participant was able to realize that the numbers that are multiples of 6 are common multiples of 2 and 3. She also stated that there is a situation that is like a pattern. In addition, Nur commented on the number that are multiples of 6 because in two, for example, when counting by twos, the numbers are even. When counting by 3, some are odd numbers, and others are even numbers. Therefore, both could have overlapped with each other. She initially generalized that to be a number divisible by 6, the number needs to be divisible by 2, i.e., it needs to be even. The numbers given in Table 4 are multiples of 6; that is, they are multiples of 2 and 3. The students were able to generalize that a number that is a multiple of 6 is also a multiple of 2 and a multiple of 3. After filling in the table, the students made the following comments.

> Elif: The numbers in the table are all divisible by 6. Is it divisible by 2? Is it divisible by 3? I found that all numbers that are divisible by 6 are divisible by 2 and 3.
>
> Researcher: What is required for 6 children to share a given number of objects equally?
>
> Elif: What is required? The number has to be divided by both 2 and 3. To be divided by 2, the number needs to be even. To be divided by 3, the sum of the digits of the number needs to be a multiple of 3. I think six is a multiple of 3, so it is divisible.
>
> Researcher: Well, can I ask you something? Why 2 and 3, why not 5, why not 4?
>
> Elif: Because 5 and 4 cannot be divided by 6, but 2 and 3 can.
>
> Researcher: What is the importance of these numbers?
>
> Elif: These numbers are divisible by 6.

Furthermore, the followings are the comments made by the students on the common features of the numbers in Table 4.

> Elif: They are all divisible by 6. Is it divisible by 2. Oh, I found that all numbers that are divisible by 6 are divisible by 2 and 3.
>
> Doğan: All these numbers are odd numbers; they could be divided by 3 but not by 6. It is divided by all of them because if it is divided by 6, six is already a multiple of these two. (He is talking about 2 and 3)
>
> Nur: All of them, that is, all the numbers written here were equally divisible by 2, 3 and 6.

As can be seen in the interview excerpts, all of the students were able to generalize that in order for a number to be divisible by 6, it needs to be divisible by 2 and divisible by 3.

4.2. Difficulties/Misconceptions that Students Encounter During the Generalization Process

The aim of the second question of the present study was to identify the misconceptions and difficulties that students encounter during the conceptualization of the divisibility rules of 2, 3 and 6. The difficulties and misconceptions encountered by the participants can be categorized under two main headings. The first aspect was about the technology-based difficulties, and the second aspect was about the concept-based difficulties. Detailed information is given below.

4.2.1. Difficulties Based on Technology

As mentioned before, the students simultaneously constructed the numbers given during the interviews on the online manipulative. Although technology-based difficulties are not very common, the students had difficulty fitting some of the larger numbers into the screen. For instance, during the second clinical interview on the activity sheet about divisibility by 3, the students were asked to construct the number 546. The students wanted to use 5 hundreds, 4 tens, and 6 ones on the virtual manipulative that was open on the tablet screen. Since there were a large number of base-ten blocks, the students had difficulty in allocating them. In other words, they were going to allocate the base blocks on three sides of the screen and try to share them equally, but they could not do this because the screen was small. Instead, they showed the tens left over from each base block in different colors. This coloring process can be seen in the following figure.

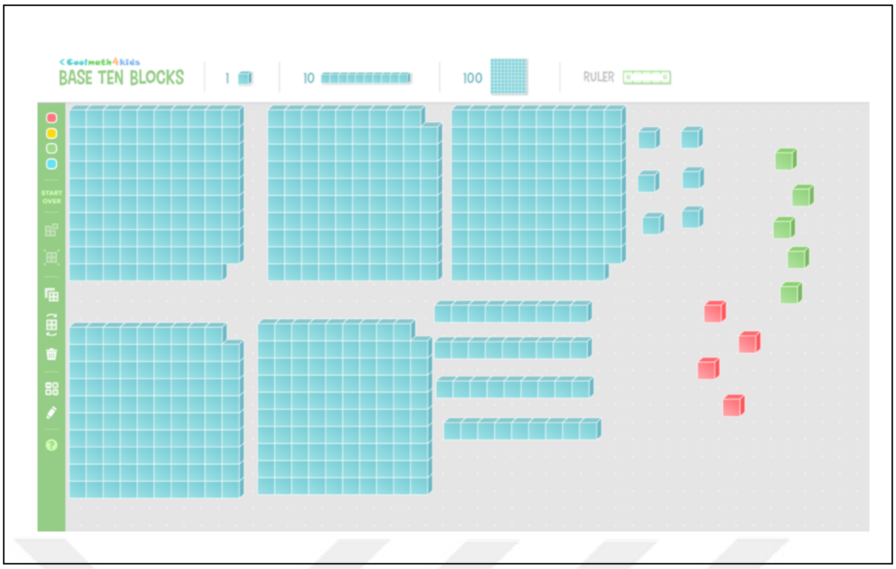

Figure 4. 30. Constructing the number 546 on the virtual manipulative

The coloring process of the remainder ones is displayed in Figure 4.30. This difficulties were overcome with this coloring process.

4.2.2. Difficulties Based on Concept

During the generalization process of the divisibility rules, the students had some difficulties/misconceptions. Those difficulties and misconceptions are categorized under the headings: *overgeneralizing the divisibility rule of 2 to the divisibility rule of 3*, *misinterpretations of odd digit numbers, inability to identify which digits require an even number for equal sharing*, and *confusion about the place value and number value*. Detailed information is given below.

Overgeneralizing the Divisibility Rule of 2 to the Divisibility Rule of 3

The participants were first given an activity sheet about divisibility by 2. After finishing the activity sheet about the rule of divisibility by2, the participants were able to move on to the activity sheet about divisibility by 3. Two of the three participants generalized the divisibility rule of 2 while completing the activity sheet about divisibility by 3. Two students overgeneralized the divisibility rule of 2. The following excerpt is taken from the clinical interview conducted during the activity sheet on 3. The following excerpt is an example of the overgeneralization of the divisibility rule of 2 to the divisibility rule of 3, taken from the interview conducted with one of the participants, Nur:

When the participant was asked whether she could reach any generalization about the divisibility rule of 3, she answered as follows:

> Researcher: Did you reach any generalization about the rule of divisibility by 3?
>
> Nur: We can understand it by looking at the ones or tens digits. For example, the digits of the number need to be a multiple of 3.
>
> Researcher: Is the number 22 divisible by3?
>
> Nur: I think it is not divisible.
>
> Researcher: Why not?
>
> Nur: Because when I look at the ones digit, there is the number 2, but 2 is not divisible by 3.

The participant thought that the number written in the ones place should be a multiple of 3, as in divisibility by 2. Such situations were coded as "*Overgeneralizing the divisibility rule of 2 to the divisibility rule of 3."* In another question, the same participant generalized the divisibility rule of 2 to the divisibility rule of 3. The example from the excerpt is as follows:

> Researcher: Then, can you give me a number consisting of seven tens that are divisible by 3?
>
> Nur: The number can consist of six ones, i.e., 76.
>
> Researcher: Can you show me this number by using base-ten blocks? Why did you say the number 76?
>
> Nur: Because 6 is a multiple of 3. I don't know.

The following quotation is an example of the overgeneralization of the divisibility rule of 2 to the divisibility rule of 3 taken from the interview transcript of Elif: *"It makes sense because, like divisibility by 2, the ones digit is a number that is divisible by3, so it makes sense again."*

As can be seen from the excerpts, it can be seen that these two participants needed to check the ones digit. In the activity sheet related to divisibility by 2, the participants concluded that for any number to be divided by 2, the digit in the ones place must be even, that is, a multiple of 2. As in the rule of divisibility by 2, the participants thought that for the number to be divisible by 3, the digit in the ones place should be a multiple of 3.

In order to address this issue, the researcher instructed the students to provide a number that is divisible by 3 and demonstrate the number and the sharing process using virtual base-ten blocks. Thanks to the virtual base-ten blocks, the students were able to observe that there were remainders in the tens place or hundreds place because 10 or 100 is not divisible by 3. By visualizing the numbers on the virtual manipulative, the students could see nine ones and one remainder from a tens block, and similarly, ninety-nine ones and one remainder from a hundreds block. As a result of constructing the numbers and attempting to distribute the objects among children on the virtual manipulative, the students were able to comprehend the contrast between the divisibility rules of 2 and 3. In essence, while in the divisibility rule of 2, tens or hundreds blocks can be divided regardless of how many tens or hundreds there are, in the divisibility rule of 3, this generalization will not hold true.

Misinterpretation of odd digit numbers

This part of the findings represents the difficulties encountered in two-digit even numbers with an odd digit in the tens place and three-digit even numbers with an odd digit in the tens place during the clinical interview about divisibility by 2. In the activity sheet about divisibility by 2, there was a student who had a misinterpretation about two-digit even numbers with an odd digit in the tens place. Nur thought that a two-digit even number with an odd digit in the tens place was not sharable by 2 with no remainders. The excerpt from the interview is as follows:

> Researcher: How can you share the number 40 between two children equally?

Nur: It can be sharable by giving each child two tens.

Researcher: How about if the number has five tens?

Nur: No. It cannot be sharable.

Researcher: Why do you think it cannot be shared?

Nur: Because there is one more ten, it cannot be shared.

The participant then formed five tens using base-ten blocks. She took five tens and shared two of the tens with one child and the other two with the other child to share equally between the two children. She realized that she could share the remaining ten. She could trade that ten into ten ones and share the tens five by five. Thanks to the regrouped base-ten blocks, the participant could trade tens for ten ones. Initially, the participant could not divide a ten due to the base-ten block structure. Similarly, when the participant formed three-digit numbers by using base-ten blocks and looked at their divisibility, she thought that a three-digit even number with an odd digit in the hundreds place could not be divided by 2.

Nur: I could not divide 700 objects equally between two children.

Researcher: Why do you think so?

Nur: Because 7 is an odd number.

Researcher: Can you make the number 700 using base-ten blocks?

Using the technological tool, the participant generated the number using the base-ten blocks (hundreds). She took seven hundreds blocks. Then, she tried to form the number 700 and share these blocks between two groups equally. The excerpt from the interview is as follows:

> Nur: I am sharing it between two children equally. We can share it by giving each group three hundreds, but only one hundred is left over. We can trade the hundred for ten tens. I was able to divide it. Then, I was able to share seven hundreds.

Nur initially had similar ideas about two-digit and three-digit numbers. The two-digit even number with an odd digit in the tens place and the three-digit even number with an odd digit in the hundreds place was deliberately chosen for the observed student's connection of evenness and divisibility by 2.

Inability to identify the value of a digit for equal sharing

In the activity sheet about the divisibility by 2, there was a student who was unable to identify which digits of a number need to be even to be sharable between two groups equally. This participant, Nur, thought that all digits in the number should be divisible by 2. In other words, she was not sure which digit should be divided by 2 for equal sharing. Examples of the participant's statements during the interview are provided as follows:

> Researcher: Can you give an example of a two-digit number that can be divisible by 2?
>
> Nur: 44.
>
> Researcher: Why did you choose 44?
>
> Nur: Because it is made up of even things. What else could it be, for example? It could have been 26.
>
> Researcher: For a two-digit number to be divided by 2, should there be an even number in both the tens and ones place?
>
> Nur: No, one of them should be even.
>
> Researcher: Which one should be even? Does it matter?
>
> Nur: No, I do not think so; it is enough the number has an even digit.

The student overcame this confusion because the numbers in the activity sheet were given as combinations of even and odd digits in different places. In other words, in two-digit numbers, both odd digits in the ones place and even digits in the tens place, and vice versa, were given. When the student tried to form and allocate these numbers using virtual manipulatives, the confusion she experienced was eliminated.

Confusion about the place value and number value

Two students were confused and needed clarification on the place value of the number and the number value. The difficulty experienced by the participants was related to confusion about the difference between the place values and the number values of the digits. While discovering the divisibility rule by 3, they hesitated whether to sum the digit or the number values of the number.

In summary, some misconceptions and difficulties were encountered during the conceptualization process. Despite these misconceptions and difficulties students generalize divisibility rules conceptually. It can be said that they overcame these difficulties by constructing numbers on the virtual base-ten block.

CHAPTER 5

DISCUSSION, IMPLICATIONS AND RECOMMANDATIONS

The current study aimed to analyze how fifth-grade students generalize the divisibility rules of 2, 3 and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. This chapter presents the discussion, implications for educational practices and recommendations for further research studies.

5.1. Discussion

The findings of the current study are discussed in two main sections based on the research questions. To be more specific, the first section discusses fifth-grade students' process of generalizing the divisibility rules of 2, 3, and 6. In the second section, the difficulties and misconceptions encountered by the students during the generalization process are discussed. Also, the findings are compared and contrasted with previous research studies in the literature.

5.1.1. Students' Process of Generalizing the Divisibility Rules

The current study aimed to analyze how fifth-grade students generalize the divisibility rules of 2, 3 and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. In this respect, the students completed activity sheets on divisibility by 2, 3, and 6 during three clinical interviews. The findings of this study revealed that the students were generally able to generalize the divisibility rules conceptually with no prior knowledge of them. Below is detailed information about the generalization of the divisibility rules of 2, 3, and 6, respectively.

As mentioned in the previous chapters, the Turkish curriculum does not focus on students' ability to discover divisibility rules. Instead, it focuses on providing the rules directly. Students encounter divisibility rules for the first time in 6th grade. When the Turkish curriculum is examined, it is seen that the following objective is included related to divisibility rules: "M.6.1.2.2. Explains and uses the rules of division by 2, 3, 4, 5, 6, 9, and 10 without a remainder" (MoNE, 2018a). Similarly, mathematics textbooks also focus on memorization of the rules by introducing them at the beginning of the topic. Despite the fact that divisibility rules are provided to be memorized in the curriculum and mathematics textbooks, the students in this study were able to learn them conceptually. In other words, the participants of this study were able to generalize the divisibility rules conceptually through clinical interviews instead of memorizing them.

The findings of the study were inconsistent with the ideas of some researchers who suggested that students consider divisibility rules as a memorization task (Posamentier, 2003). On the other hand, the findings were consistent with several research studies in the literature indicating that using groupable materials or base-ten blocks deepens students' understanding of divisibility rules (Bennet & Nelson, 2002; Harrel & Slavens, 2009; Young-Loveridge & Mills, 2012). There might be several reasons underlying this finding. The reason behind the findings of the present study' will be discussed in detail below in terms of the use of a virtual manipulative in clinical interviews, the sequence of activities and the ZPD.

Firstly, during the clinical interviews, the students simultaneously answered the questions on the activity sheets using virtual base-ten blocks, as mentioned. One possible explanation for their ability to reach a generalization might be that using virtual manipulatives may have engaged the students in a conceptual generalization of divisibility rules. The students had the opportunity to see the base-ten representation of the numbers. For instance, when asked about the number 56, they used five tens and six ones to construct the number. In other words, they engaged with the place value of numbers instead of their number values. They were able to make sense of the number 56 as a quantity, not a singular number, and visualize it in groups of tens and ones. By using a virtual manipulative, they were able to see the number of objects in each group and any remaining ones. This allowed them to visually comprehend the numbers and attempt to distribute the objects equally among the given number of children. They were also able to discuss divisibility using the virtual manipulative, focusing on the place values of numbers and attempting to divide the objects by the given numbers while visualizing the remainders. Therefore, the findings suggest that using virtual manipulatives might help students conceptualize divisibility rules instead of simply memorizing them.

In this study, virtual manipulatives were used instead of concrete ones to help students conceptualize divisibility rules. The reason behind the use of virtual manipulatives is that virtual manipulatives are available, time-saving, and motivating (Schackow, 2006-2007). In the present study, it was easy to use virtual manipulatives since they were available on the tablet, giving the students the opportunity to use them. Besides, it was easy to use virtual manipulatives for some larger numbers. For example, the activity sheet involved constructing the number 546. Constructing this number using concrete manipulatives might have been more complicated than using virtual manipulatives since the students could simply click on the virtual manipulatives without spending too much time. Also, they were motivated to use technological devices, such as tablets and tablet pens. As Schackow (2006-2007) mentioned, "Middle school students may find working on a computer with virtual manipulatives more desirable than using concrete manipulatives that they might view as childish" (p.10). Similarly, the students stated that it was fun to use virtual base-ten blocks during the clinical interviews. Hence, it can be concluded that the use of virtual manipulatives in this study might have made it easier for the students to generalize the divisibility rules of 2, 3, and 6 as using them motivates students and makes it easier to see several examples at once, even with larger numbers.

Secondly, another possible reason behind the findings of the study might be the order in which the activity sheets were given and the sequence in which the numbers were used in the activity sheets. The generalization process may have been affected by these factors. The following section includes the discussion of the sequence of the numbers used in the activity sheets and then the order in which the activity sheets were given. The first point to address is the sequence of the numbers on the activity sheet on divisibility by 2. As mentioned, there was a sequence among the numbers written on the activity sheets: first, one-digit, then two-digit, and finally three-digit numbers. To simplify, the students were asked about one-digit odd and even numbers on the activity sheet on divisibility by 2. After that, the students were asked about two-digit and three-digit numbers that are multiples of ten and not multiples of ten. The sequence of two-digit numbers was as follows: there were two-digit numbers that were multiples of ten with an odd digit in the tens place and two-digit even numbers that were not multiples of ten with an odd digit in the tens place. Moreover, there were two-digit numbers that are multiples of ten with an even digit in the tens place and two-digit odd numbers that are not multiples of ten with an even digit in the tens place. The combination of the numbers might have helped the students see all possibilities of numbers, allowing them to generalize the divisibility rule. In other words, the students were able to construct the numbers with all possibilities. Constructing the numbers using virtual manipulatives might have helped them reach a generalization. For instance, the students constructed the number 20 using two tens blocks and the number 25 using two tens and five ones (Part 2 of the activity sheet on divisibility by 2). 20 is a two-digit number that is a multiple of ten with an even digit in the tens place, and 25 is a two-digit odd number that is not a multiple of ten with an even digit in the tens place. The students were able to see that both numbers had the same number in the tens place and different number in the ones place. After constructing the numbers and trying to share these blocks between two children, they were able to argue that having an odd digit in the ones place of a number is not the same as having an odd digit in the tens place. On the other hand, having an odd digit in the tens place or an even digit in the tens place yields the same result for sharing equally. This connection was discovered by thinking about all possibilities of the numbers. Moreover, as mentioned, they were asked about two-digit numbers that are multiples of ten with an odd digit in the tens place and two-digit even numbers that are not multiples of ten with an odd digit in the tens place. To illustrate, the students were asked to construct the numbers 50 and 56 (Part 2 of the activity sheet on divisibility by 2). While working on these numbers, upon observing the even and odd digits in the tens place, the students realized that the number in the tens place does not affect divisibility by two. They were able to share the number equally between two children regardless of whether there was an odd or even number in the tens digit. They were able to generalize that for a number to be divisible by 2, they need to check the ones digit. There was also a sequence of three-digit numbers. There were three-digit numbers that are multiples of ten and not multiples of ten. There were three-digit even numbers with odd digits in the tens and hundreds place and three-digit odd numbers with odd digits in the tens and hundreds place. These number combinations aimed to reveal if the students could identify that there was no difference between three-digit numbers with an even digit in the hundreds place (e.g., 200) and those with an odd digit in the hundreds place (e.g., 300) when divided by 2. In light of this, all students were able to identify that the number in the hundreds place of a number does not affect divisibility by two when they had a chance the opportunity to observe the even and odd digits in the hundreds place. They also could see the connection between hundreds blocks and tens blocks. A hundreds block is divisible by 2 since it consists of tens. The structural features of hundreds and tens are similar. They are multiples of 2 and thus divisible by 2. A hundreds block consists of ten tens; since ten is a multiple of 2, a hundreds block is also a multiple of 2. No matter how many hundreds there are, they are divisible by 2 for this reason. The possible reason behind the students being able to reach this generalization is that they were given the numbers in a sequence. At first, the students analyzed one-digit numbers. Then, they examined two-digit numbers and reached a generalization on the tens digit. Then, while examining three-digit numbers, they were able to reach a generalization on the hundreds digit.

The sequence of the numbers on the activity sheets on divisibility by 2 and 3 was parallel. Similarly, the sequence in which the numbers were presented on the activity sheet on divisibility by 3 might have influenced the students' conceptual understanding. On the sheet, the students were first asked about one-digit numbers, then two-digit numbers, and finally three-digit numbers, similar to the activity sheet on divisibility by 2. In each case, the students were asked to identify the numbers with one remainder (e.g., 4), two remainders (e.g., 8), or no remainders (e.g., 3) when divided by 3. The same pattern was followed for numbers that are multiples of ten and not multiples of ten. For example, the students were asked to identify the numbers with one remainder (e.g., 10, 100), two remainders (e.g., 20, 200), or no remainders (e.g., 30, 300). This sequence might have helped the students understand the three possible remainder scenarios: one remainder, two remainders, or no remainders. It allowed them to generalize that sharing a tens or hundreds block among three children results in one remainder, sharing two tens or hundreds blocks results in two remainders, and sharing three tens or hundreds blocks results in three remainders, which then can be shared among three children again. By analyzing all possible remainder scenarios, the students reached a generalization. Similarly, the students were asked for numbers that are multiples of ten and not multiples of ten, as mentioned. Being asked about numbers that are not multiples of ten (e.g., 31, 301) might have enabled them to realize the remainder of the base-ten representation of the numbers. For the number 31, the students realized that there were three remainders from the tens place, and one from the ones place, so the total number of remainders was 4. After reaching the total number of remainders, they tried to share the remainders among three children again, and in the end, there was one remainder. Therefore, seeing all possibilities of the numbers might have enabled the students to reach a generalization about divisibility rules. In other words, the students realized that there was a remainder from a tens block for two-digit numbers. Similarly, for three-digit numbers, the students realized that there was a remainder from a hundreds block. They successfully found the summation of the remainders to determine divisibility. This finding was consistent with the research in the literature, which concluded that students understood the reason behind divisibility by nine by using groupable materials (Young-Loveridge & Mills, 2012). In their study, the students were taught divisibility by nine with the help of groupable materials (e.g., translucent plastic boxes and plastic beans) instead of providing them with a rule to memorize. The students were expected to construct the given numbers using translucent plastic boxes and plastic beans in a way that every ten beans would fit into one plastic box. While doing so, the students noticed that there was a leftover bean from each box and then found the summation of the leftover beans. At the end of the intervention, the students understood that the leftover ones corresponded to the digits of a number, indicating the divisibility rule of 9. Similarly, in the current study, the students were able to reach the generalization that remainders from each place value correspond to the number values of a number, and the summation of the digits is used to determine divisibility by 3.

As for the sequence of the numbers on the activity sheet on divisibility by 6, it is known that for a number to be divisible by 6, it should be divisible by both 2 and 3. It might be necessary to analyze all possibilities of the numbers to make this generalization. To illustrate, in the activity sheet on divisibility by 6, the students were asked about number combinations such as even numbers that are divisible by 3, even numbers that are not divisible by 3, odd numbers that are divisible by 3, and odd numbers that are not divisible by 3. In addition, the students were asked to provide examples of situations where it is not possible to generalize the divisibility properties of numbers. To illustrate, they were asked whether an odd number that is divisible by three is divisible by six, and they were expected to provide examples.

As mentioned above, the activity sheets were given sequentially throughout the clinical interviews: divisibility by 2, divisibility by 3, and divisibility by 6, respectively. The activity related to divisibility by three included a sequence parallel to the sequence used in the activity related to divisibility by 2. Also, the sequence in the activity related to divisibility by six was established by combining the sequences on the activity sheets on divisibility by 2 and 3. Beyond the sequences for two, three and six, the intersequence, that is, the relationship between two, three and six, may have supported the conceptual generalization. In other words, besides the sequences of two, three, and six, there might have been an underlying relationship or pattern between these numbers. After generalizing the divisibility rules of 2 and 3, the students were also able to generalize the divisibility rule of 6 because they realized the connection between the numbers 2, 3, and 6. The relationship among these numbers might have been established because of the sequence of the activity sheets. Therefore, it can be concluded that the sequence of both activity sheets and the numbers on the activity sheets could have helped the students reach a conceptual generalization of the divisibility rules of 2, 3, and 6.

Another reason behind the findings of the study may be related to the fact that the students were asked critical questions when needed during the clinical interviews. As mentioned, we utilized Vygotsky's zone of proximal development theory to guide the research. We conducted clinical interviews with a fifth-grade student with the aim of ensuring that the student was learning within their zone of proximal development. To state differently, since students learn best in their zone of proximal development and learning is most effective there (Danish et al., 2011; Devi,2019), the researcher, as a more knowledgeable person, asked some critical questions to prevent possible misconceptions and to enable the students to reach generalizations. For instance, during the second clinical interview about the divisibility rule of 3, the students realized the leftover ones from a tens block after trying to share a tens block among three children equally. When they tried to share two tens blocks among three children, they realized that there were two remaining ones. At that point, the researcher asked the students questions to prompt their understanding of remainders and how they relate to the divisibility rule. The researcher aimed to guide the students towards realizing that if the number of remainders from each digit of a number is a multiple of three, then the remainder can be shared among three children equally. The researcher also asked the students to explain the concept of remainders for two- and three-digit numbers to test their understanding. In this sequence, the researcher asked the students: "How many tens can be sharable among three children? How can you figure it out?" This question aimed to reveal if the students could identify if the number of remainders was a multiple of three, which then could be shared among three children equally. In other words, the aim of asking these questions was to help the students realize that the reason of including the digits in the divisibility rule of 3 was related to the number of remainders from each digit of a number. That is, these questions were asked to encourage less proficient students to become independent learners in line with the definition of the ZPD (Chaiklin, 2003). Moreover, the researcher also asked them about the meaning of the remainder. "What does it mean if there is one remainder after sharing the number among three children equally?" Also, for two- and three-digit numbers, the researcher asked them the following questions to check if they were aware of what they were doing: "Can you explain where the remaining ones come from?" "What does it mean if there are two remainders or three remainders?" These questions were asked to engage the students in thinking about remainders, and if the number of remainders is a multiple of three, then the remainders can be shared among three children again. In other words, these questions were designed to encourage the students to think critically about remainders and their relation to the divisibility rule of 3.

In this context, the researcher might have contributed to the students' conceptual understanding by asking them critical questions that they needed. Also, asking critical questions may have prevented them from being confused by misconceptions and enabled them to generalize the divisibility rules successfully.

5.1.2. Miconceptions and Difficulties that the Students Encountered During the Generalization Process

The aim of the second question of the present study was to identify the misconceptions and difficulties that students encounter during the conceptualization of the divisibility rules of 2, 3, and 6. The findings of the current study revealed that the difficulties or misconceptions that encountered by students during the generalization process can be categorized as follows: technology-based and concept-based.

Although the virtual manipulative on the tablet is easy to use, fitting some of the larger numbers on the screen might be challenging for students. For instance, constructing the number 546 on the activity sheet about divisibility by 3 was complex for the students since there were too many base-ten blocks, and the students had difficulty arranging them on the tablet screen. In other words, they tried to move 5 hundreds, 4 tens and 6 ones blocks to three sides of the screen and share them equally, but they could not do this because the screen of the tablet was not big enough. Thanks to the sequence in which the numbers were shown in the activity, the students managed to generalize the divisibility rule of 3. They wanted to show the remainder ones from each base-ten block using a different color. After painting 5 remainder ones from 5 hundreds blocks, 4 remainders from 4 tens blocks and 6 ones, they allocated the ones. Following the coloring process, they found the total number of the remainders and tried to share them among three children equally. As mentioned, the sequence of the numbers might have helped the students learn the divisibility rules conceptually and overcome some of the difficulties. Thus, it can be concluded that use of technology might have led to this difficulty.

Furthermore, the findings of the study demonstrated that the students encountered some concept-based difficulties. To illustrate, two of the students overgeneralized the divisibility rule of 2 to that of 3. They tried to determine the divisibility of a number by 3 by looking at the ones place of the relevant number as if they were determining the divisibility of the number by 2. This overgeneralization of divisibility rules might be due to the sequence of clinical interviews. That is, the students were asked to generalize the divisibility rule of 2 in the first clinical interview, and then they worked on the activity sheet on divisibility by 3 in the second clinical interview. They might have been inclined to use the generalization for divisibility by 2 for divisibility by 3. This finding might be considered consistent with the previous study, which reported that pre-service teachers incorrectly applied or overgeneralized the divisibility rules when specific divisibility rules were not provided to them (Zazkis & Campbell, 1996). Considering that even pre-service teachers have difficulties with divisibility rules, it is not surprising that students might also struggle with them. To overcome this difficulty, the researcher asked the students to provide a number that is divisible by 3 and show the number and the sharing process using virtual base-ten blocks. Thanks to the virtual base-ten blocks, the students realized that tens place or hundreds place had remainders because 10 or 100 is not divisible by 3. Visualizing the numbers on the virtual manipulative allowed the students to see 9 ones and one remainder from a tens block and similarly 99 ones and 1 remainder from a hundreds block. Thus, constructing the numbers and trying to share the objects among children on the virtual manipulative allowed the students to realize the difference between the divisibility rules of 2 and 3. In other words, in the divisibility rule of 2, tens or hundreds blocks can be divided no matter how many tens or hundreds there are, while in the divisibility rule of 3, this generalization will not be valid. Briefly, it can be concluded that during the process of generalizing the divisibility rules of 2 and 3, students might experience some concept-based difficulties such as overgeneralizing the divisibility rule of 2 to that of 3.

Another difficulty was that two students confused the concepts of the place value and the number value of a number. In the second clinical interview, the students tried to generalize the divisibility rule of 3. While doing so, two students were unsure whether the place value or the number value of a number should be summed to determine divisibility by 3. It can be attributed the students' lack of attention to the difference between these terms. One of the students stated that she had difficulty remembering the difference between these terms and this was due to the fact that she had seen these terms before. To avoid such confusion, the authors of mathematics textbooks may remind these terms to students. Also, illustrating the meaning of the place value and number value using base-ten blocks might help students understand the difference between them.

The other difficulty experienced during the clinical interviews was related to a student's misinterpretation of numbers with odd digits. Specifically, she had difficulty with two-digit even numbers with an odd digit in the tens place and three-digit even numbers with an odd digit in the tens place during the clinical interview about divisibility by 2. She thought that a two-digit even number with an odd digit in the tens place was not divisible by 2 because of the odd digit. It may be due to the fact that the student focused on the number value of the digits in the number instead of the digit value. To overcome this difficulty, the students were asked to construct the numbers (in this case, 70) on the virtual manipulative. After constructing the number using 7 tens, the student tried to share these objects between two children. She was able to divide them equally. As mentioned above, using a virtual manipulative was motivating and provided an opportunity to perform visualization. Therefore, using a virtual manipulative might help students learn divisibility rules conceptually and overcome some difficulties.

5.2. Implication for Educational Practices

This study analyzes fifth-grade students' process of generalizing the divisibility rule of 2, 3, and 6. In light of the findings, the present study has several implications for textbook authors, teachers, and teacher educators.

To begin with, when mathematics textbooks are analyzed, it can be seen that the rules are presented at the beginning of the subject. The rules are followed by a numerical example related to the rule (MoNE, 2018a). To illustrate, in a mathematics textbook, the divisibility rule of 2 is as follows: "Numbers with 0, 2, 4, 6, or 8 in the ones digit (even numbers) can be divided by 2 without a remainder" (MoNE, 2018a, p.32). The divisibility rule of 3 is as follows: "Numbers whose sum of digits is a multiple of 3 are divisible by 3 without a remainder" (MoNE, 2018a, p.32). The divisibility rule of 6 is as follows: "Numbers divisible by both 2 and 3 without a remainder are divisible by 6 without a remainder" (MoNE, 2018a, p.33). The rules are presented at the beginning of the subject, and students are not provided with the knowledge to discover the rules. The book provides examples of 38 for the divisibility rule of 2, 48 for the divisibility rule of 3, and 96 for the divisibility rule of 6 (MoNE, 2018a). The findings of the present study may be potentially beneficial for textbook authors as it teaches generalization rather than giving rules. In this study, all students who had no prior knowledge of the subject managed to generalize the divisibility rules of 2, 3, and 6 successfully with the help of a virtual manipulative, which is a virtual base-ten blocks, with guidance offered when needed. With the help of virtual base-ten blocks, the students were able to conceptualize the divisibility rules of 2, 3, and 6 although they did not know them before. Therefore, the findings of the study could be an example for textbook authors in terms of explaining the divisibility rules to students in a conceptual way. More specifially, textbook authors could use the drawings of base-ten blocks or the pictures of groupable materials to illustrate the fair sharing of objects rather than providing students with the rules directly. In other words, the present study suggests that students can reach generalizations when they analyze the base-ten representations of numbers and consider numbers as objects and try to share them among children equally. Thus, textbook authors could use visualizations of base-ten blocks. Furthermore, they could provide students with the links and instructions of the manipulative and ask them to consruct the numbers using these tools. Additionally, textbook authors could use number combinations (even and odd) and the visualization of the numbers on the textbook. In other words, they could use two- or three-digit even and odd numbers instead of giving only one number as an example for students.

Moreover, the findings of the present study might provide valuable implications for teachers. When they learn the reason behind divisibility rules, they might teach their students the rules in a conceptual way. Teachers might use classroom activity sheets and manipulatives or groupable materials when teaching divisibility rules instead of simply providing students with the rules. They might utilize the activity sheets used in this study and also use virtual or concrete manipulatives in their classes. In fact, they can have students work in groups to discuss the activities. As mentioned previously, the findings of the present study revealed that students might encounter some concept-based difficulties during the generalization process. Therefore, teachers might ask them to provide an example for each step and explain their reasoning. Also, to overcome difficulties, teachers could use manipulatives, whether virtual or concrete, to teach divisibility in a conceptual way.

Lastly, the findings of the current study might have implications for teacher educators. Thanks to the activity sheets and virtual base-ten blocks used in this study, the students were able to generalize the divisibility rules in a conceptual way. Similar activity sheets can be used in methods courses when pre-service teachers learn about the teaching of division and divisibility rules. When pre-service teachers become familiar with such activities, they might use similar ones in their classrooms. Therefore, when they become teachers, they can educate students on how to conceptualize divisibility rules instead of memorizing them.

5.3. Recommandations for Further Research Studies

The current study analyzed how fifth-grade students generalize the divisibility rules of 2, 3 and 6, with no prior knowledge on the subject, using a virtual manipulative with guidance offered when necessary during clinical interviews. Some recommendations for further research studies are provided in this part.

To begin with, the present study is limited to the divisibility rules of 2, 3, and 6. Similar studies could be conducted to analyze students' process of generalizing the divisibility rules of other numbers. To illustrate, similar studies might focus on the process of generalizing the divisibility rule of 9. There are similarities between divisibility by 3 and divisibility by 9, so the activity sheet on divisibility by 3 could be improved for divisibility by 9. Also, there are similarities between divisibility by 5 and divisibility by 10, which might be the focus of further studies. Similar studies could be conducted by using concrete manipulatives to generalize divisibility rules.

In addition, when the new Turkish curriculum is examined, it can be seen that the new program aims to teach divisibility rules more conceptually. The new curriculum includes the following objective related to divisibility rules:

"MAT.6.1.2. To be able to make inferences about the criteria for the divisibility of a natural number by 2, 3, 4, 5, 6, 9 and 10.

> *a) Makes assumptions about the divisibility criteria of a natural number by 2, 3, 4, 5, 6, 9 and 10 by considering its multiples or place values.*
>
> *b) Determines generalizations by examining the multiples and place values of 2, 3, 4, 4, 5, 6, 9 and 10.*
>
> *c) Tests whether generalizations meet assumptions using examples.*
>
> *ç) Proposes a proposition about the criteria for the exact division of a natural number by 2, 3, 4, 5, 6, 9 and 10.*
>
> *d) Evaluates the usefulness of the criteria for the exact division of a natural number by 2, 3, 4, 5, 6, 9 and 10 in different situations."* (MoNE, 2024)

Based on the new curriculum, teachers are strongly recommended to use manipulatives when teaching divisibility rules to generalize them conceptually. Therefore, teachers could be used the present study for teaching divisibility rules in a conceptual way by examining the new objectives.

Furthermore, the participants of the present study were three fifth-grade students from private schools in Ankara, so this study is limited to students from private schools. Similar studies could be conducted with students from public schools to reach a generalization for students from both private and public schools. In other words, public school students' process of generalizing divisibility rules can be examined, which can reveal the effects of different school types on the process of generalization.

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APPENDICES

A. Permission from the Ethical Committee at METU

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07 NİSAN 2023

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof. Dr. Mine Işıksal Bostan

Danışmanlığını yürüttüğünüz Sena ÖZCAN'ın **"Ortaokul 5. Sınıf Öğrencilerinin Bölünebilme Kurallarını Anlamlandırması"** başlıklı araştırmanız İnsan Araştırmaları Etik Kurulu tarafından uygun görülerek **0208-ODTUİAEK-2023** protokol numarası ile onaylanmıştır

Bilgilerinize saygılarımla sunarım.

Prof. Dr. Ş. Halil TURAN
Başkan

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Üye

Doç. Dr. Ali Emre Turgut
Üye

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