

ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

**COOPERATIVE CONTROL OF MULTI-AGENT SYSTEM
UNDER TIME DELAY**

Ph.D. THESIS

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Department of Mechatronic Engineering

Mechatronic Engineering Programme

SEPTEMBER 2023

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ÇOK AJANLI SİSTEMLERİN ZAMAN GECİKMESİ ALTINDA
EŞ ZAMANLI KONTROLÜ

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To the 100th anniversary of the Republic of Türkiye,



FOREWORD

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TABLE OF CONTENTS

	<u>Page</u>
FOREWORD.....	ix
TABLE OF CONTENTS.....	xi
ABBREVIATIONS	xiii
SYMBOLS.....	xv
LIST OF TABLES	xvii
LIST OF FIGURES	xix
SUMMARY	xxiii
ÖZET	xxv
1. INTRODUCTION	1
1.1 Purpose of Thesis	3
1.2 Literature Review	4
2. PRELIMINARIES.....	9
2.1 Graph Theory.....	9
2.2 Matrix Theory.....	13
2.3 Introduction to Time Delay Systems.....	19
2.4 Cluster Treatment of Characteristic Roots	22
2.4.1 Review of the CTCR methodologies.....	23
2.4.2 CTCR method for cooperative control systems	28
2.5 Problem Formulation.....	30
2.5.1 Vehicular platooning systems	30
2.5.2 Formation of unmanned aerial vehicles.....	35
2.5.3 Modeling approach of multi-agent systems	37
3. CONTROL OF MAS WITH RETARDED TIME-DELAY.....	49
3.1 Purpose	49
3.2 Leader Agent-Based Constant Headway Policy	50
3.2.1 Stability analysis for nominal case	50
3.2.2 Stability analysis for time delay case	77
3.3 Follower Agent-Based Constant Time Headway Policy	111
3.3.1 Stability analysis for nominal case	111
3.3.2 Stability analysis for time delay case	123
4. MULTI-AGENT CONTROL WITH NEUTRAL TIME-DELAY	135
4.1 Purpose	135
4.2 Stability Analysis for Nominal Cases.....	136
4.3 Stability Analysis for Time Delay Cases.....	150
5. A REAL TIME CASE STUDY.....	157
5.1 Purpose	157
5.2 Problem Formulation for Real Time Case Study	157

5.2.1 Stability analysis with CTCR method via Bezout resultant matrix.....	161
5.3 Simulation Studies on MATLAB/Simulink Environment.....	166
5.4 Real Time Simulation Studies on Industrial Computers	169
6. CONCLUSIONS AND RECOMMENDATIONS.....	175
6.1 Conclusions	175
6.2 Recommendations	177
REFERENCES.....	179
CURRICULUM VITAE.....	191



ABBREVIATIONS

ACC	: Adaptive Cruise Control
ACE	: Auxiliary Characteristic Equation
AHS	: Automated Highway Systems
CC	: Cruise Control
CD	: Constant Distance
CE	: Characteristic Equation
CAS	: Control Augmentation System
CTH	: Constant Time Headway
CACC	: Cooperative Adaptive Cruise Control
CTCR	: Cluster Treatment of Characteristic Roots
IP	: Intellectual Property
LMI	: Linear Matrix Inequality
LTI	: Linear time-invariant
ITS	: Intelligent Transformation System
LTIS	: Linear Time-Invariant System
MAS	: Multi Agent System
MPD	: Mixed Predecessor Following
MTD	: Multiple Time Delay
NTDS	: Neutral Time Delay System
PD	: Proportional Derivative
PF	: Predecessor Following
PID	: Proportional - Integral - Derivative
RT	: Root Tendency
RTDS	: Retarded Time Delay System
SAS	: Stability Augmentation System
SPTF	: Single Predecessor Two Following
TDS	: Time Delay System
TPF	: Two Predecessor Following
TPLF	: Two Predecessor Leader Following
UAV	: Unmanned Aerial Vehicle
V2I	: Vehicle to Infrastructure
V2V	: Vehicle to Vehicle



SYMBOLS

$\mathcal{G}$	: Graph operator
$\mathcal{V}$	: Node operator
$\mathcal{E}$	: Edge operator
v_i	: Node (agent)
$\mathcal{N}$	: Neighbourhood set
$\mathcal{A}$	: Adjacency matrix
$\mathcal{B}$	: Incident matrix
$\mathcal{D}$	: Degree matrix
$\mathcal{L}$	: Laplacian matrix
$\mathcal{L}_p, \mathcal{D}_p$	: Information topology matrix
$\otimes$	: Kronecker product
I, I_n	: Identity matrix
τ	: Time-delay
ω_c	: Crossing frequency
RT	: Root tendency
F_d	: Equivalent air resistance force
ρ	: Specific mass of air
V_{wind}	: Velocity of the wind
η_i	: Throttle input
V_i	: Ground speed
θ_i	: Euler pitch angle
ϕ_i	: Euler roll angle
ψ_i	: Euler yaw angle
λ	: Eigenvalue
h	: Time headway
d	: Safe distance
k_p	: Position controller gain, Proportional controller gain
k_v	: Velocity controller gain
k_a	: Acceleration controller gain
k_i	: Integrator controller gain
k_d	: Derivative controller gain
p_i	: Position of i^{th} vehicle
v_i	: Velocity of i^{th} vehicle
a_i	: Acceleration of i^{th} vehicle
N	: Number of follower vehicle
u	: Control signal
e_p	: Position error
e_v	: Velocity error
e_a	: Acceleration error
J	: Jordan form



LIST OF TABLES

	<u>Page</u>
Table 3.1 : System parameters used in simulation studies	55
Table 3.2 : System parameters used in simulation studies	69
Table 3.3 : Initial States of UAVs.....	69
Table 3.4 : System parameters used in simulation studies	107
Table 3.5 : Initial States of UAVs.....	107
Table 4.1 : System parameters used in simulation studies	149
Table 5.1 : System and controller parameters used in simulation.....	167



LIST OF FIGURES

	<u>Page</u>
Figure 1.1 : Collective motion of animals: a. Birds flying in V formation, b. Birds flocking, c. Herd of wildebeest fleeing danger, d. A school of fish responding as a single organism.	1
Figure 1.2 : Collective motion of systems: a. Aircraft formation flying, b. Platooning system operating on highway, c. UAV mission formation flying for mission, d. NASA's Starling mission.	2
Figure 2.1 : Representation of delay model [1].....	19
Figure 2.2 : Upper and lower level control architecture.....	31
Figure 2.3 : Longitudinal forces acting on a vehicle.....	33
Figure 2.4 : Forces acting on a UAV	36
Figure 2.5 : Vehicle platooning system with a virtual leader.....	38
Figure 2.6 : Vehicle platooning system with a virtual leader.....	38
Figure 2.7 : Platooning information flow topologies [2].....	39
Figure 3.1 : Layout of a platoon for Example 3.1.....	54
Figure 3.2 : Positions and position errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$	56
Figure 3.3 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$	56
Figure 3.4 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 0.4$ and $k_a = 2$	57
Figure 3.5 : Information flow topologies for Example 3.2.	57
Figure 3.6 : Positions and position errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$	58
Figure 3.7 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$	58
Figure 3.8 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 0.756$ and $k_a = 2$	59
Figure 3.9 : Information flow topologies for Example 3.3.	64
Figure 3.10 : Positions and pos. errors for $k_p = 0.2740$, $k_v = 0.9773$ and $k_a = 1.0682$	65
Figure 3.11 : Velocity and velocity errors for $k_p = 0.2740$, $k_v = 0.9773$ and $k_a = 1.0682$	65
Figure 3.12 : Graph of triangular formation structure.....	66
Figure 3.13 : Information flow topologies for Example 3.4.	69
Figure 3.14 : Altitude and velocity for $k_p = 1$ and $k_v = 2$	70
Figure 3.15 : Flight path angle and heading angle for $k_p = 1$ and $k_v = 2$	71
Figure 3.16 : Position errors and velocity errors for $k_p = 1$ and $k_v = 2$	71
Figure 3.17 : UAV 3D trajectory for $k_p = 1$ and $k_v = 2$	72
Figure 3.18 : Information flow topologies for Example 3.5.	74
Figure 3.19 : Flight path angle and altitude for $k_p = 0.3845$ and $k_v = 5.2542$	75
Figure 3.20 : Flight path angle and heading angle for $k_p = 2.9139$ and $k_v = 27.1661$	76
Figure 3.21 : Flight path angle and heading angle for $k_p = 2.9139$ and $k_v = 27.1661$	76

Figure 3.22: Position errors and velocity errors for $k_p = 2.9139$ and $k_v = 27.1661$.	76
Figure 3.23: Stability boundaries for $k_v = 4$ and $k_a = 2$.	82
Figure 3.24: Positions and position errors for $\tau = 0.06$.	83
Figure 3.25: Velocity and velocity errors for $\tau = 0.06$.	83
Figure 3.26: Positions and position errors for $\tau = 0.18$ s.	83
Figure 3.27: Velocity and velocity errors for $\tau = 0.18$ s.	84
Figure 3.28: Velocity and velocity errors for $\tau = 0.2$ s.	84
Figure 3.29: Stability boundaries for $k_v = 4$ and $k_a = 2$.	85
Figure 3.30: Positions and position errors for $\tau = 0.04$ s.	85
Figure 3.31: Velocity and velocity errors for Example $\tau = 0.04$ s.	86
Figure 3.32: Positions and position errors for $\tau = 0.0754$ s.	86
Figure 3.33: Velocity and velocity errors for $\tau = 0.0754$ s.	86
Figure 3.34: Velocities for $\tau = 0.08$ s.	87
Figure 3.35: Stability boundaries for $k_v = 0.9773$ and $k_a = 1.0682$.	89
Figure 3.36: Positions and position errors for $\tau = 0.15$ s.	89
Figure 3.37: Velocity and velocity errors $\tau = 0.15$ s.	90
Figure 3.38: Positions and position errors for $\tau = 0.182$ s.	90
Figure 3.39: Velocity and velocity errors for $\tau = 0.182$ s.	90
Figure 3.40: Velocities for Example $\tau = 0.185$ s.	91
Figure 3.41: Stability boundary for $k_v = 27.1661$.	94
Figure 3.42: Altitude and velocity for $\tau = 0.01$.	95
Figure 3.43: Flight path angle and heading angle for $\tau = 0.01$.	95
Figure 3.44: Position errors and velocity errors for $\tau = 0.01$.	95
Figure 3.45: Flight path angle and heading angle for $\tau = 0.022$ s.	96
Figure 3.46: Altitude and velocity for $\tau = 0.022$.	96
Figure 3.47: Stability boundaries for $k_p = 0.2740$, $k_v = 0.9773$ and $k_a = 1.0682$.	103
Figure 3.48: Positions and position errors for $\tau_1 = 0.05$ s and $\tau_2 = 0.5$ s.	104
Figure 3.49: Velocity and velocity errors for $\tau_1 = 0.05$ s and $\tau_2 = 0.5$ s.	104
Figure 3.50: Positions and position errors for $\tau_1 = 0.144$ s and $\tau_2 = 0.5$ s.	104
Figure 3.51: Velocity and velocity errors for $\tau_1 = 0.144$ s and $\tau_2 = 0.5$ s.	105
Figure 3.52: Velocity and velocity errors for $\tau_1 = 0.16$ s and $\tau_2 = 0.5$ s.	105
Figure 3.53: Information flow topologies for Example 3.11.	106
Figure 3.54: Stability boundaries for $k_p = 6.2806$, $k_v = 25.4619$.	108
Figure 3.55: Altitude and velocity for $\tau = 0.01$.	108
Figure 3.56: Flight path angle and heading angle for $\tau = 0.01$.	109
Figure 3.57: Position errors and velocity errors for $\tau = 0.01$.	109
Figure 3.58: UAV 3D trajectory for $k_p = 1$ and $k_v = 2$.	110
Figure 3.59: Flight path angle and heading angle for $\tau = 0.01$.	110
Figure 3.60: Flight path angle and heading angle for $\tau = 0.01$.	110
Figure 3.61: Positions and position errors for $k_p = 2$, $k_v = 2$ and $k_a = 0.8$.	114
Figure 3.62: Velocity and velocity errors for $k_p = 2$, $k_v = 2$ and $k_a = 0.8$.	114
Figure 3.63: Velocity and velocity errors for $k_p = 2$, $k_v = -0.4444$ and $k_a = 0.8$.	115
Figure 3.64: Positions and position errors for $k_p = 2$, $k_v = 6$ and $k_a = 0.5$.	123
Figure 3.65: Velocity and velocity errors for $k_p = 2$, $k_v = 6$ and $k_a = 0.5$.	124
Figure 3.66: Stability boundary for $k_v = 27.1661$.	127

Figure 3.67: Positions and position errors for $\tau = 0.06$	128
Figure 3.68: Velocity and velocity errors for $\tau = 0.06$	128
Figure 3.69: Velocity and velocity errors for Example $\tau = 0.0788$ s.	128
Figure 3.70: Velocities Example $\tau = 0.08$ s.....	129
Figure 3.71: Stability boundary for $k_p = 2$, $k_v = 6$ and $k_a = 0.5$	132
Figure 3.72: Positions and position errors for $\tau_1 = 0.03s$ and $\tau_2 = 0.1s$	133
Figure 3.73: Velocity and velocity errors for $\tau_1 = 0.03s$ and $\tau_2 = 0.1s$	133
Figure 3.74: Velocity and velocity errors for $\tau_1 = 0.03s$ and $\tau_2 = 0.517s$	134
Figure 3.75: Velocities $\tau_1 = 0.03s$ and $\tau_2 = 0.53s$	134
Figure 4.1 : Information flow topologies for Example 5.1.	148
Figure 4.2 : Positions and position errors for Example 4.1 ($\tau = 0s$)	149
Figure 4.3 : Velocity and velocity errors for Example 4.1 ($\tau = 0s$).....	149
Figure 4.4 : Information flow topologies for Example 4.2.	152
Figure 4.5 : Stability boundary for $k_d = 0.02$, $k_i = 0.02$	153
Figure 4.6 : Positions and position errors ($\tau = 0.01s$).	154
Figure 4.7 : Velocity and velocity errors ($\tau = 0.01s$).....	154
Figure 4.8 : Positions and position errors ($\tau = 0.05s$).	154
Figure 4.9 : Velocity and velocity errors($\tau = 0.055s$).....	155
Figure 5.1 : Vehicle platoon system with a virtual leader.	160
Figure 5.2 : Information flow topology of the platoon system.	166
Figure 5.3 : Stability boundary for the platoon system exposed to multiple time-delay.....	167
Figure 5.4 : Velocity and position curves for $\tau_1 = 0.1$ and $\tau_2 = 0.2$	168
Figure 5.5 : Velocity and position curves for $\tau_1 = 0.191$ and $\tau_2 = 0.3$	168
Figure 5.6 : Velocity and position curves for $\tau_1 = 0.2$ and $\tau_2 = 0.7$	169
Figure 5.7 : Test environment data connection.	170
Figure 5.8 : Logix5000 controllers produced and consumed tags	171
Figure 5.9 : The basic schematic diagram of the industrial controller program ...	172
Figure 5.10: Real time data capture	172
Figure 5.11: Real time simulation points on the $\tau_1 - \tau_2$ plane.	172
Figure 5.12: Real time velocity and position curves for $\tau_1 = 0.1$ and $\tau_2 = 0.2$	173
Figure 5.13: Real time velocity and position curves for $\tau_1 = 0.162$ and $\tau_2 = 0.6$. 173	
Figure 5.14: Real time velocity and position curves for $\tau_1 = 0.165$ and $\tau_2 = 0.6$. 173	
Figure 5.15: Real time velocity and position curves for $\tau_1 = 0.2$ and $\tau_2 = 0.7$	174



COOPERATIVE CONTROL OF MULTI-AGENT SYSTEM UNDER TIME DELAY

SUMMARY

In this Ph.D. dissertation, multi-agent systems are studied in detail using two of the most common examples in practice, which are vehicle platooning systems and formation control of unmanned aerial vehicles. For a better understanding of the study, some basic information such as graph theory, matrix theory, and time-delayed systems are given. Then, the "Cluster Treatment of Characteristic Roots" paradigm, which forms the backbone of the study, is explained, and the existing methods in the literature have been explained. In this study, a new Bezout Resultant matrix-based CTCR method has been proposed, and the steps of the algorithm are explained via simulation examples in detail. The main advantage of the proposed method is that it provides computational convenience for the time-delayed systems in which the degree of characteristic equation is relatively large and not decomposed into factors in obtaining the stability posture of the system in terms of time delay.

First, the distributed controller algorithm is selected as the state feedback controller. The closed-loop system matrix is constructed for the cases with and without time delay. The controller coefficients that make the system stable are obtained by using the Routh table and Lyapunov-based methods for the case where the time delay is neglected. However, in the presence of delay, the system is converted into retarded time delay system, and the stability posture is obtained with CTCR methods for single and multiple time delays. Moreover, the formation geometry between vehicles is considered as constant policy and constant headway policy. For constant policy, the characteristic equation of the system for delayless and single time delay case, is decomposed into factors, which makes the stability analysis easier. But, this case is not possible for the characteristic equation involved multiple time delay, which direct us to utilize Bezout Resultant matrix-based CTCR method. For constant time headway policy, it is seen that, the characteristic equation cannot decomposed into factors for any cases. So, the sufficient condition is derived for determining the stability of multi-agent system for delay-free case with converted the system matrix to block companion form and block Schwarz form.

Then, a PID controller based distributed controller protocol is proposed. The cooperative control problem of multi-agent system with distributed PID controller is converted into an asymptotic stability problem through matrix and state transformations in the absense of time delay. Finally, a Lyapunov function is created and the controller parameters are choosen with the help of linear matrix inequality. In the presence of time delay, the closed-loop system is converted into a neutral-type time delay system. And, the stability posture of the multi-agent system is obtained with the help of Kronecker multiplication and elementary transformation based CTCR method.

Finally, all the theoretical studies and simulation results are evaluated with a real-time experimental study. An industrial controller based real time simulation for the platoon

system with five connected vehicle including a virtual leader is proposed. The constant time headway policy is selected to modeled the desired inter-vehicle distance and the vehicle dynamic states-based distributed control strategy is used to converge to their desired velocities and inter-vehicle distances. Then the multi-agent platooning control problem is converted into LTI system stability analysis problem. The delay-based stability analysis is studied by means of Bezout Resultant matrix-based CTCR method. Numerical simulations are provided to verify the validity of the proposed method. The real-time experiments are carried out on industrial computers to show the applicability of the proposed method in real time systems.

The study concluded by evaluating the results and recommendations.



ÇOK AJANLI SİSTEMLERİN ZAMAN GECİKMESİ ALTINDA EŞ ZAMANLI KONTROLÜ

ÖZET

Bu doktora çalışmasında, çok ajanlı sistemler uygulamada en çok karşılaşılan iki örnek model kullanılarak detaylı olarak incelenmiştir. Son yıllarda ulaşım sorunları sosyal, çevresel ve ekonomik problemlerden dolayı tüm dünyada büyük ilgi görmektedir. Akıllı ulaşım sistemleri, ulaşım endüstrisine güvenlik, verimlilik ve tam otomasyon getirecek popüler konulardan biridir. Araç müfreze konsepti, bağlı araçların yerleşik sensörler, araçtan araca (V2V) ve araçtan altyapıya (V2I) iletişim kullanarak ortak kontrolünü amaçlayan umut vaat eden teknolojilerden biridir. Bağlantılı araçların ana amaçlarından biri, istenen referans hızı korurken sabit araçlar arası mesafe ile dizideki araçların boylamsal hareketlerini koordine etmektir. Bu teknikte, her araç, yerleşik sensörler ve/veya V2V iletişimleri aracılığıyla komşu bilgileri alır. Bu bilgiler kullanılarak gerekli kontrol sinyalleri üretilir. Uyarlanabilir hız sabitleyici (ACC) ve gelişmiş versiyonu işbirlikçi uyarlanabilir hız sabitleyici (CACC) biçimi, araç takım oluşturma sistemlerinde iyi bilinen metodolojilerdir. Birbirleriyle V2V veya V2I kullanarak haberleşen bağlantılı araçlar analiz edilirken, sistemin doğal yapısından kaynaklanan gecikme, paket kayıpları, bağlantı problemleri gibi etkenlerin iletişimin sınırlandırılması kaçınılmazdır. Bu gibi durumlarda, zaman gecikmesi takım oluşturma sisteminin performansını düşürebileceğinden ve hatta araç dizisinin kararsızlığına neden olabileceğinden, sistem performansını değerlendirmek için gecikmeye bağlı kararlılık koşulları detaylıca incelenmelidir. Bu bilgiler ışığında seçilen ilk örnek model araç müfreze sistemleridir (vehicular platooning systems). Bu sistemler çok ajanlı sistemler açısından zaman gecikmesi olmadığı durumda, giriş ve iletişim gecikmeleri varlığında gecikmeye bağlı kararlılık koşulları elde edilerek incelenmiştir. Seçilen ikinci örnek model ise insansız hava araçlarının eş zamanlı kontrol problemidir. Çoklu insansız hava araçları (İHA) sistemlerinin oluşum kontrolü, hedef arama ve yer belirleme, gözetleme ve keşif, radyasyon tespiti ve kontur haritalama gibi hem sivil hem de askeri alanlardaki büyük ölçüde potansiyel uygulamaları nedeniyle büyük ilgi çekmiştir.

İkinci bölümünde, çalışmanın geri kalanının daha iyi anlaşılması için bazı temel bilgilerin verilmesi amaçlanmıştır. Çok ajanlı sistemlerin modellenmesinde kullanılan çizgi kuramı (graph theory) ile ilgili temel bilgiler verildikten sonra doğrusal cebir ve sistem teorisi hakkında kullanılan yaklaşımlar tanıtılmıştır. Daha sonra zaman gecikmeli sistemlere giriş yapılmış ve çalışmanın omurgasını oluşturan "Cluster Treatment of Characteristic Roots" paradigmasının oluşturulmasında bu zamana kadar literatürde var olan yöntemler anlatılmıştır. Bu çalışmada bu metodun oluşturulmasında Bezout matrisi tabanlı yeni bir yaklaşım önerilmiş ve bu yaklaşımın çok ajanlı sistemlere nasıl uygulanacağı anlatılmıştır. Bu bölümün sonunda uygulama alanı olarak seçilen araç müfreze sistemlerinin ve insansız hava araçlarının oluşum uçuşlarının çok ajanlı sistemler açısından ajan modellerinin nasıl oluşturulduğu

açıklanmıştır. Her iki sistem için de (araçların boylamsal modeli ve insansız hava araçlarının basitleştirilmiş modeli) literatürde edinilen bilgiler ve çalışmanın ilerleyişindeki çıkarımlar sonucu doğrusal modellerin yeterli olacağı görülmüş ve doğrusal olmayan modeller geribeslemeli doğrusallaştırma (feedback linearization) metodu ile doğrusallaştırılmıştır.

Üçüncü bölümde, ajan dinemikleri elde edilen sistemlere durum geri beslemeli temelli ayırık kontrolcü yapıları önerilmiştir ve analiz, araçlar arasındaki mesafenin değişmediği (constant policy) ve sistemdeki ajanların hızlarına göre değiştiği (time headway policy) olmak üzere iki farklı yaklaşım altında değerlendirilmiştir. Öncelikle zaman gecikmesi ihmal edilmiş ve Laplacian matrisinin yapısına göre sistemi kararlı yapan kontrolcü katsayıları araştırılmıştır. Laplacian matrisinin özdeğerlerinin gerçek olduğu durumda tüm sistemi ifade eden karakteristik denklem faktörlerine ayrıştırıldığı için sistemi kararlı yapan kazançların sınırları Routh tablosu yöntemi ile elde edilmiştir. Laplacian matrisinin özdeğerlerinin complex olduğu durumda ise sistemi kararlı yapan kontrolcü kazançları doğrusal matrix eşitsizliği (LMI) çözümleri ile elde edilmiştir. Sistemde tek bir zaman gecikmesi olduğu durumda karakteristik denklemin faktörlerine ayrıştığı durumlarda her bir faktörün zaman gecikmesine bağlı kararlılık sınırları elde edilmiş ve bunların kesişimi tüm sistemin kararlılık sınırını oluşturmuştur. Her bir faktörün zaman gecikmesi altında kararlılık sınırı elde ederken literatürde var olan CTCR yöntemleri kullanılmış ve sonuç olarak neden yeni bir yaklaşım önerme ihtiyacı duyulduğu anlatılmıştır. Sistemde hem giriş hem de iletişim gecikmesi olduğu durumda ise (multiple time delay) karakteristik denklem faktörlerine ayrışmadığı için önerilen Bezout matrisi tabanlı CTCR metodu kullanılmış ve zaman gecikmelerine bağlı kararlılık sınırı oluşturulmuştur. Bu bölümde elde edilen zaman gecikmeli sistemler literatürde geciktirilmiş zaman gecikmesi (retarded-time delay) yapısındadır.

Dördüncü bölümde sisteme oransal-integral-türevsel (PID) kontrolcü yapısı önerilmiştir. Bu kontrolcü yapısının seçilmesinin temel nedeni literatürde ve gerçek uygulamalarda en sık karşılaşılan yapılardan biri olmasıdır. Bu durumda en genel yaklaşımla Laplacian matrisinin özdeğerlerinin karmaşık olduğu düşünülüp analiz yapılmıştır. Sistemin kararlılık analizi zaman gecikmesinin ihmal edildiği ve tek bir zaman gecikmesi olduğu durumlar için ayrı ayrı incelenmiştir. Zaman gecikmesinin ihmal edildiği durumda eşzamanlı kontrol problemi bazı matris dönüşümleri yardımıyla doğrusal kararlılık problemine dönüştürülmüştür. Bir Lyapunov fonksiyonu oluşturularak, denetleyici protokolünün kazanç matrisleri doğrusal matris eşitsizliğine yardımıyla elde edilmiştir. Ancak zaman gecikmesinin varlığında sistemin kararlılığını etkileyen baskın unsur gecikme değeridir. Stabilite duruşu ise zaman gecikmesine dayalı stabilite analiz yöntemi yardımıyla araştırılmalıdır. Önerilen PID tabanlı kooperatif kontrol probleminde tek bir zaman gecikmesi ele alınmıştır. Ayrıca, kapalı çevrim sistemin yapısı, nötral tip bir zaman geciktirme sistemini temsil eder. Ayrıca, zaman gecikmesi altında sistemin kararlı duruşunu elde etmek için Kronecker çarpma ve temel dönüşüm tabanlı CTCR yöntemi kullanılmıştır.

Beşinci bölümde elde edilen teorik çalışmalar ve simülasyon sonuçları değerlendirilerek gerçek zamanlı uygulama yapılmıştır. Bu bölümde, araç konvoy sistemi için endüstriyel denetleyici tabanlı gerçek zamanlı bir simülasyon önerilmiştir. Bu gerçek zamanlı simülasyon sistemi, önerilen yöntemin doğruluğunu ve güvenilirliğini doğrulamak için bir platform sağlar. Burada, her araç modeli ve önerilen kontrol protokolü bir endüstriyel kontrolörde kodlanmıştır. Denetleyiciler birbirleriyle

Ethernet/IP ağı aracılığıyla iletişim kurar. EtherNet/IP ağı, standart Ethernet iletişim ürünlerini kullanarak gerçek zamanlı kontrol, usta/çırak (master/slave) mesajlaşma, bilgi alışverişi ve genel mesajlaşma sağlar. Bu gerçek zamanlı simülasyon için Allen Bradley endüstriyel otomasyon ürünleri kullanılmıştır.

Son bölümde çalışmanın sonuçları, literatüre katkıları özetlenmiş ve gelecek çalışmalar için önerilere yer verilmiştir.





1. INTRODUCTION

Cooperative control of multi-agent systems is an emerging multidisciplinary engineering field. Observations and descriptions of collective behavior in nature, such as schools of fish, birds' flocks, and insects' swarms, initially inspired the concept of multi-agent cooperation [3]. Although each individual has their tendencies and movements, their collective action makes them seem like a single entity reacting to external events. These behaviors provide an advantage to the herd in migration, seeking food, and avoiding dangers, so the study and application of such behaviors have attracted the attention of researchers in various fields. Figure 1.1 shows examples of the collective motion of animals [4, 5].

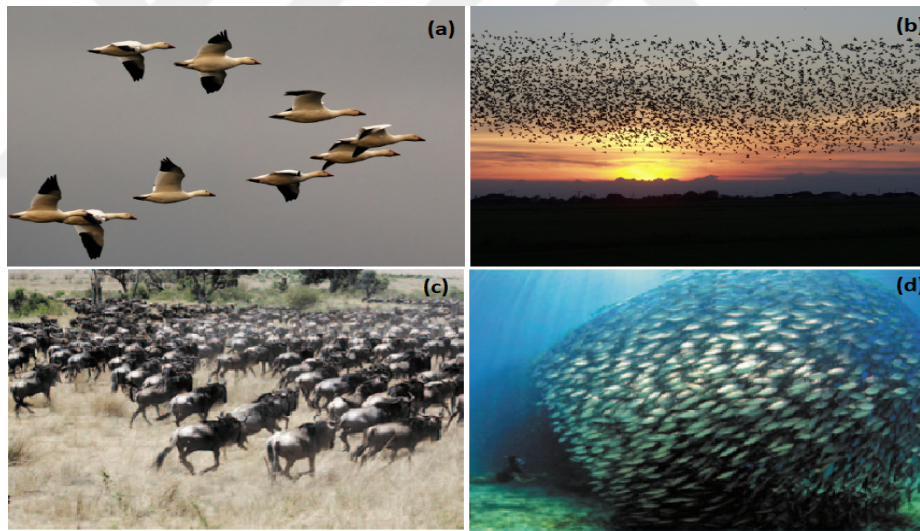


Figure 1.1 : Collective motion of animals: a. Birds flying in V formation, b. Birds flocking, c. Herd of wildebeest fleeing danger, d. A school of fish responding as a single organism.

It has been observed that these behavior patterns can be applied to many real-world problems where agents can exchange data with each other. Thus, it has been witnessed that the cooperative control of a multi-agent system approach has been applied to more and more research topics. Some of them are sensor networks, formation control of unmanned ground or aerial vehicles, traffic control systems, vehicular platooning

systems, smart grids, automated highway systems, and satellite formation. Figure 1.2 shows examples of cooperative control of real-time applications [6–8].

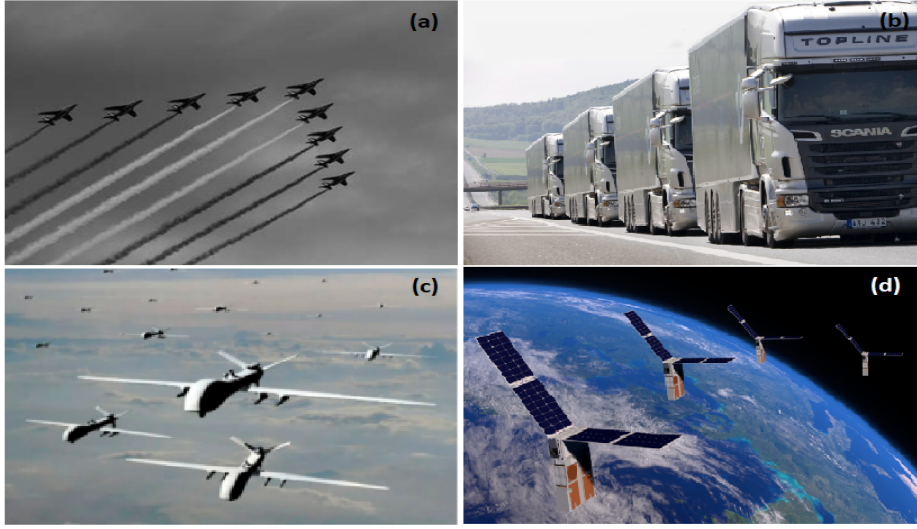


Figure 1.2 : Collective motion of systems: a. Aircraft formation flying, b. Platooning system operating on highway, c. UAV mission formation flying for mission, d. NASA's Starling mission.

Generally, the cooperative control of multi-agent systems means that agents agree on their mutual status by exchanging information. As a result, agents should be able to reach a common trajectory based on consensus protocol by exchanging data with each other under a fixed or variable data flow topology. The cooperative control of multi-agent systems problem decomposes into four interrelated sub-components: agent model, information flow topology, formation geometry, and controller structure. For example, in a formation flight of an unmanned aerial system, each unmanned aerial vehicle (UAV) is an individual agent. So, each agent may be modeled with differential equations. The system is called homogeneous if each agent has the same dynamic; otherwise, it is heterogeneous. Besides, the agent model may be linear or nonlinear behavior depending on the nature of the differential equation. The information flow topology describes the interconnections among agents using a graph. If the information flow topology does not change with time, it is named fixed; otherwise, it is a switched topology. Moreover, the topology may be categorized as directed or undirected according to the flow direction. The formation geometry describes the desired shape or headway between agents. The formation may be constant or variable regarding the states of agents or time. The last one is controller architectures, which regulate the agent inputs to obtain the desired behavior. Three different controller structure options,

centralized, distributed, and decentralized, are proposed in the literature for developing cooperative controllers of multi-agent systems. In the centralized controller, all the agents communicate with a central unit and each other. Thus, each agent shares common knowledge with the others. In distributed control, some agents do not share their knowledge with others. Also, the central unit may need to send the information to each agent. In decentralized control, the agent communicates with the central unit; however, there is no communication with each other.

The most critical criterion in the design of a multi-agent system is internal stability, as in any control system. A multi-agent system with linear time-invariant (LTI) dynamics is internally stable if all real parts of the roots of the closed-loop system are left half of the complex plane. So, the internal stability depends on the four interrelated sub-components of multi-agent systems. In other words, internal stability is affected not only by agent dynamics and controller gains but also by formation geometry and information flow topology. Since cooperative control of multi-agent systems relies on communication between agents to achieve desired behavior, network-induced problems must be addressed. These problems can generally degrade the system's performance or even destroy stability. The most known network imperfections are quantization errors, packet dropouts, input and communication time delay, time-varying sampling/transmission intervals, and communication constraints. Time delay-based network imperfections are fundamental and critical for multi-agent systems.

1.1 Purpose of Thesis

This dissertation describes the analysis and design process for cooperative control of multi-agent systems, focusing on the effects of time delays in communication and input channels. Platooning systems and formation flight of unmanned aerial vehicles, two real-world examples of cooperative control problems, are used for simulation studies to verify the theoretical analysis.

The following topics are studied in this dissertation in detail: The distributed strategies based on state feedback and PID controller are studied. The cooperative control problem is solved effectively through linear dynamic models. The stability analysis is conducted for the nominal case, which means there is no time delay in the system. The

effect of different information flow topologies corresponding to Laplacian matrices' real and complex eigenvalues on the stability analysis has been investigated. The analysis is conducted for the constant spacing policy and extended for a constant time-headway policy. Then, the stability analysis is extended for time delay cases. The Cluster Treatment of Characteristic Roots (CTCR) methodology is utilized to obtain the stability boundary in case of a single time delay. Also, the methodology is enhanced for obtaining the stability boundary for multiple time-delayed multi-agent systems via Bezout's resultant theory. Simulation studies of determined real-time problems are conducted to support all the theoretical outputs. In the end, the stability and control of the platooning system are analyzed for homogeneous vehicles in the presence of multiple time delays with its experimental validation with an industrial controller-based real-time simulation.

1.2 Literature Review

Many exciting and inspiring results have been obtained for consensus problems after the 80s. In [9], a three-rule-based which are collision avoidance, velocity matching, and flock centering algorithm is proposed to model the collective behavior of flocking agents. In [10], a novel model of particle systems in discrete time is proposed, and all particles are driven with an identical speed. Their heading is decided by their local average direction of movement. The theoretical explanation for the consensus behavior of the vicsek's model given in [10], is presented and extended with the help of graph and matrix theories in [11, 12]. The authors of [13] established a new consensus algorithm for multi-agent systems using both fixed and dynamic topologies. They also defined a new type of consensus algorithm known as average consensus. In this algorithm, the agents reach a consensus on the initial states' average for the particular feature of interest. They are also investigated undirected interconnection graphs and time-delay by Lyapunov-based approach. In [14], several fundamental properties of compositions of directed graphs which help explain how a consensus is achieved under various conditions, have been established in a dynamically changing environment. The conclusions on finite-time consensus problems for a group of interacting agents have been established in [15]. In addition, the theoretical findings have been applied to sensor networks, namely to compute the maximum-likelihood estimate of unknown

parameters. The consensus problem for multi-agent systems with directed topologies and nonlinear dynamics are also investigated in [16, 17].

Nearly most of all earlier publications on cooperative control of multi-agent systems with time delays have relied on Lyapunov-Krasovskii, Razhumikin-based approaches or generalised Nyquist criterion for stability analysis. In [18], the consensus problem of second-order discrete-time multi-agent systems is investigated with nonuniform time delays and dynamically changing communication topologies. They perform a model transformation and turn the system into an equivalent undelayed one whose system matrix is stochastic. And, the adequate conditions are given under which all agents reach a consensus based on this derived equivalent system. The paper given in [19], the leaderless and leader-following consensus algorithms with communication and input delays in first-order kinematics and second-order dynamics under a directed network topology are examined. The stability or boundlessness conditions are analyzed using time-domain and frequency-domain approaches. In [20], the delay bound is obtained for the consensus of a group of linear dynamic agents with a uniform communication delay. By using the Lyapunov-Krasovskii method, stability conditions which contain the feedback gain and delay conditions, are obtained for systems over fixed and switching topologies. In [21], the formation control is investigated for a network of second-order dynamic agents with heterogeneous communication delays. Based on the frequency-domain analysis and matrix theory, sufficient conditions for the multi-agent systems are obtained.

The methods given above provide sufficient conditions for delays in achieving asymptotic stability. They produce very conservative results, leading to stability bounds for minimal delays. Moreover, these methods require the solution of complex linear matrix inequalities. And they are always imprecise, conservative, and cumbersome to implement. A stability paradigm which is called cluster treatment of characteristic Roots (CTCR) is also employed as a different way to find the stability boundary of a multi-agent system exposed to the time delay. This method provides a practical and numerically efficient procedure for assessing the stability of linear time-invariant (LTI) systems with multiple rationally independent delays as given in [22, 23]. The benefit of this method is that it creates precise, exhaustive and clear stability regions in the time delay domain. CTCR methodology consists of

two primary steps: the first one requires the determination of the possible imaginary roots crossings of the characteristic equation and the second one decides the direction of that crossing roots by evaluating the root tendency as explain in [24]. Some of the earlier investigations of this method mostly focus on consensus control of multi agent systems under single or multiple time delays in [25–27]. In [28], the consensus problem of a general class of second-order multi-agent systems with distributed delays is investigated. A spectral equivalence of such a system is created, and using CTCR and Dixon resultant theory, the exact upper bound of time delay is determined. In [29], a unique frequency-domain technique is provided to determine the exact frequency range and stability characteristic of the time-delay system. By sweeping the resulting range, a complete set of stability-switching curves are determined, with the use of the CTCR paradigm for stability analysis.

In parallel with the development of multi-agent systems, the Intelligent transformation system (ITS) has also attracted significant interest worldwide due to bringing safety, efficiency, and total automation to the transportation industry problems [30]. The vehicle platooning concept is one of the promising ITS technologies, which aims for the cooperative control of connected vehicles using onboard sensors, vehicle-to-vehicle (V2V) and vehicle-toinfrastructure (V2I) communications [31,32]. One of the main purposes of connected vehicles is to coordinate the longitudinal motions of the vehicles in the string with constant inter-vehicle distance while maintaining the desired reference velocity. In this technique, each vehicle obtains neighbouring information via on-board sensors and/or V2V communications. The necessary control signals are produced using this information as given in [33]. Adaptive cruise controller (ACC) and its extended form of cooperative adaptive cruise controller (CACC) are well-known methodologies in vehicle platooning systems as in [34]. The history of platooning controller dates back to the end of the 1980s [35]. Since then, many essential topics have been examined. In [36], the cooperative control of platooning system problem is investigated from the perspective of networked control. A great number of researchers concentrate on the longitudinal dynamics of vehicular platoon since these approaches are highly sufficient to examine the system in every respect [37,38]. However, lateral dynamics is rarely analyzed, particularly for platoon merging and lane changing [39]. A platooning system is defined to be homogeneous if all

vehicles have identical dynamics, otherwise it is heterogeneous. Most of the papers in the literature report homogeneous system approaches to simplify the analysis, although the heterogeneous one may also be used to represent different conditions [40,41]. The development of sensors and communication technologies enable vehicles to exchange information over the V2V and V2I technologies. Therefore, the studies which use many advanced topologies such as two-predecessor-following (TPF), two predecessor-leader following (TPLF) and mixed predecessor following (MPF) are started to appear in the literature [42, 43]. There are two major policies which define the formation between consecutive vehicles in the platooning system: constant distance (CD) policy and constant time-headway (CTH) policy. In CD policy, the desired distance between two adjacent vehicles is constant, however in the CTH policy, desired inter-vehicle spacing varies with the vehicle's velocity [44, 45]. There are some remarkable policies to increase the performance of the system such as nonlinear policies, delay-based policies and the combination of CD and CTH policies [46, 47]. Several approaches have been considered for the control of the platooning system in the literature. One of the well-known studies is a distributed linear feedback controller which basically uses the neighboring information that is obtained via flow topology to track desired formation geometry [48]. Some advanced control methods are also implemented for vehicular platooning to achieve better performance. A distributed robust control method is studied for heterogeneous vehicular platoons to balance the performance of robust stability and disturbance attenuation in [49]. A model predictive control design procedure for vehicle platoons, accommodating both string stability and constraints satisfaction, is proposed in [50]. The limitation of communication and network uncertainties, such as delay, packet drops due to the natural structure of the system are inevitable when analyzing connected vehicles that communicate with each other either using V2V or V2I. In such cases, delay-dependent stability conditions are obtained to evaluate system performance as time delay may degrade the performance of the platooning system and even cause the instability of the vehicle string. In recent years, plenty of prominent studies have been conducted related to the time delay phenomena in the connected vehicular system. A decentralized and distributed platoon control strategy based on the solution of a high-order consensus problem for a network of dynamical agents in the presence of heterogeneous time varying delay is formed using Lyapunov-Razumikhin theorem in [51, 52]. A novel

proportional-integral-derivative (PID) controller structure is addressed to cancel out time-varying homogeneous parameter uncertainties and time-varying communication delay in [53]. The asymptotic stability of the closed-loop system is achieved by using the Lyapunov-Krasovskii method and linear matrix inequality (LMI).

The multi-agent system modeling approach is also utilized for the formation flying of unmanned aerial vehicles in recent years. In [54], it is introduced an extension of a consensus algorithm for systems modeled by second-order dynamics. And the proposed algorithm is applied to the formation control problem of UAV systems by appropriately choosing the information states on which consensus is reached. In [55], a consensus-based feedback linearization strategy is suggested to maintain a specific time-varying geometric configuration for the formation flight of UAV systems. In the proposed strategy, leaders are not required in the team, and only local neighbor-to-neighbor information exchange is needed. In [56], formation control analysis and design problems for unmanned aerial vehicle (UAV) swarm systems to achieve time-varying formations are investigated. and necessary and sufficient conditions are obtained for the stability of the system which contains a spanning tree in the fixed topology. In [57], it is investigated the distributed finite-time fault-tolerant containment control problem for multiple unmanned aerial vehicles (multi-UAVs) in the presence of actuator faults and input saturation. In [58], The formation control of fixed-wing UAV swarms with five different types of motion constraints and unknown disturbances is examined. The motion constraints have been addressed by introducing the rotating formation errors in the Frenet-Serret frame and the saturation function. The disturbances have been adaptively estimated by employing a finite-time disturbance observer. In [59], the time-varying formation control problem for the second-order discrete-time multi-agent systems is investigated. both the non-uniform communication time-delays and switching topology are taken into account.

2. PRELIMINARIES

This section will provide background knowledge for the rest of this dissertation. Some related information on graph theory is introduced, and the associated matrices used to model multi-agent systems, adjacency, degree, pinning, and Laplacian, are defined. The basic information on linear algebra is then given, and some important definitions of the control system theory are reminded. Afterward, the time-delay system theory is summarized to understand the "Cluster Treatment of Characteristic Roots" paradigm, which is the backbone of the study. First, the time delay concept in control systems is introduced, and some preliminary information about the time-delay systems is explained. Then, the CTCR method is presented with the proposed novelties used in the study. It should be noted that the discussed problem arises due to delays in LTI systems modeled by delayed differential equations. In other words, input and communication time delay cases are investigated for the system under single and multiple time delays.

At the end of this chapter, the problem cases are introduced. Two different case study is examined: vehicular platooning systems and the formation of unmanned aerial vehicles. These cases are used to evaluate theoretical techniques in this dissertation's context. Finally, the methodology of modeling multi-agent systems is introduced.

2.1 Graph Theory

Networked dynamic systems have consisted of dynamic systems known as agents or nodes, information flow topology used to interact with the agents, formation geometry, which depicts the desired goal, and distributed controller, which implements each agent's local feedback controller. The general feature of these subsystems can be achieved by using graph theory. Graphs that are an abstract representation are used to define the interconnection of agents with each other. A graph is defined as follows;

Definition 2.1 A regular graph is denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = (v_1, v_2, \dots, v_n)$ is a finite nonempty set of nodes representing all the agents, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ where

$\mathcal{E} = (e_1, e_2, \dots, e_n)$ is a set of pairs of nodes, called edges representing the connections between each pair of the agents in the graph.

The graph may be classified in terms of information flow that is *directed*, which represents unidirectional flow, and *undirected*, which means bidirectional flow. The directed graph is also referred as digraph. The pair of nodes (v_i, v_j) represents the edge and in a directed graph, all edges have a unique direction. In other words, the edge starts at vertex i and terminates at vertex j , which means agent i sent information to agent j . However, the edges have no orientation, so the information flow may go either direction along the edges for the undirectional graph, meaning that (v_i, v_j) is the same as (v_j, v_i) .

Here some of the basic description used in graph theory are given;

- **Subgraph:** A subgraph of a graph $\mathcal{G}$ is a graph whose node set and edge set are subset of those $\mathcal{G}$.
- **Adjacent:** Two nodes are called adjacent if there is an edge between them.
- **Incident:** An edge and a node are incident with one another if the node is contained in the edge.
- **Neighbourhood:** The neighbourhood of the node v_i is the set of all nodes which are adjacent to v_i and is denoted by $N(v_i)$.

$$\mathcal{N}(v_i) = \{v_j \in \mathcal{V} | (v_i, v_j) \in \mathcal{E}\}$$

- **Degree:** The degree of a node in a graph is the number of edges incident with the vertex. For a directed graph, this term is defined as the number of edges leaving or entering the relevant node.
- **Loop:** A loop (also called a self-loop) is an edge that connects a node to itself.
- **Walk:** A walk consists of an alternating sequence of vertices and edges consecutive elements of which are incident begins and ends with a node. A walk is called closed if the starting and ending vertices are identical, open otherwise.
- **Path:** It is a walk without repeated nodes. A directed path from node v_i to v_j is a sequence of edges of from $(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \dots, (v_{i_k}, v_j)$ in the directed graph with distinct node $(v_l \in \mathcal{V})$ for $l = 1, 2, \dots, k$.

- **Root:** A node is called a root if it has a directed path to every other node.
- **Connected graph:** An undirected graph is called connected if there is a path between every distinct pair of nodes. For a digraph, the distinction is made between a graph being strongly or weakly connected. A digraph is called strongly connected if there exist a path between any distinct pair of nodes. A digraph is weakly connected if there is a path between every two nodes in the underlying undirected graph.
- **Cycle:** A closed walk with no repeated vertices except at the initial and terminal vertices is called a cycle.
- **Tree:** A tree is a connected graph without cycles.
- **Spanning Tree:** A spanning tree of a connected graph $\mathcal{G}$ is a tree containing all the vertices of $\mathcal{G}$.
- **Directed Spanning Tree:** It is a directed tree which contains all nodes of the graph and has only one root.

Control applications of graph theory for multi-agent systems make use of the matrices associated with the graph. Properties of graph can be reflected in the algebraic analysis of these matrices. A couple of essential matrices are introduced below.

Definition 2.2 *Adjacency matrix:* $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ of the graph $\mathcal{G}$ is defined as;

$$\mathcal{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix} \quad (2.1)$$

where $a_{i,j}$ represents the edge of (v_i, v_j) , which describes the communication between agent i and agent j , and,

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases} \quad (2.2)$$

Also, it is assumed that there is no self loop, i.e. $a_{ii} = 0, \forall i \in N$.

Definition 2.3 *Incident matrix:* $\mathcal{B} = [b_{ij}] \in \mathbb{R}^{N \times N}$ of the graph $\mathcal{G}$ is defined as;

$$b_{ij} = \begin{cases} 1 & \text{if } e_j = (v_i, v_k) \text{ for some } k \\ 0 & \text{otherwise} \end{cases} \quad (2.3)$$

Definition 2.4 Degree matrix: $\mathcal{D} = \text{diag}\{d_{ii}\} \in \mathbb{R}^{N \times N}$ is a diagonal matrix, where d_{ii} defines the number of vehicles that i^{th} vehicle communicates with, given as;

$$d_{ii} = \sum_{k=1}^n a_{ik} \quad (2.4)$$

Definition 2.5 Laplacian matrix: $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ which plays an important role for the dynamic of MAS is defined as;

$$\mathcal{L} = \begin{bmatrix} \sum_{j \in \mathcal{N}_1} a_{1,j} & -a_{1,2} & \dots & -a_{1,n} \\ -a_{2,1} & \sum_{j \in \mathcal{N}_2} a_{2,j} & \dots & -a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n,1} & -a_{n,2} & \dots & \sum_{j \in \mathcal{N}_n} a_{n,j} \end{bmatrix} \quad (2.5)$$

This matrix is constructed from the Adjacency matrix and Degree matrix as follows;

$$\mathcal{L} = \mathcal{D} - \mathcal{A} \quad (2.6)$$

The Laplacian matrix $\mathcal{L}$ plays an essential role in the cooperative controller design. In brief, certain conditions need to be declared to accommodate a general understanding of the Laplacian matrix $\mathcal{L}$ both directed and undirected graphs [60].

- (i) $\mathcal{L}$ is symmetric for an undirected graph. However, it is not necessary for digraph.
- (ii) The sum of each row of $\mathcal{L}$ is zero for both directed and undirected graphs. Therefore 0 is the eigenvalues of $\mathcal{L}$ with the corresponding eigenvector of $[1, \dots, 1]^T$, and $n \times 1$ column vector of ones.
- (iii) $\mathcal{L}$ is diagonally dominant which means that the diagonal elements of $\mathcal{L}$ is non-negative.
- (iv) 0 is a simple eigenvalue of $\mathcal{L}$ for undirected graph, if the undirected graph is connected. Whereas, 0 is a simple eigenvalue of $\mathcal{L}$ if the directed graph is strongly connected.
- (v) (iv)] Gersgorin circle criterion states that all of $\mathcal{L}$'s non-zero eigenvalues are positive ($\mathcal{L}$ is positive semidefinite) for an undirected graph. However, for a directed graph, all of the non-zero eigenvalues of $\mathcal{L}$ have positive real parts.

Definition 2.6 *Pinning matrix*: $\mathcal{P} = \text{diag}\{p_{ii}\} \in \mathbb{R}^{N \times N}$ is a diagonal matrix which defines the nodes who communicate with the (virtual) leader for a leader following topology is defined as follows;

$$p_{ii} = \begin{cases} 1 & \text{if } v_i \text{ communicate with } v_0 \\ 0 & \text{otherwise} \end{cases} \quad (2.7)$$

Definition 2.7 *Information topology matrix*: Additional information topology matrices were needed to formulate the MAS for different controller structures throughout the study.

$$\begin{aligned} \mathcal{L}_p &= \mathcal{L} + \mathcal{P} \\ \mathcal{D}_p &= \mathcal{D} + \mathcal{P} \end{aligned} \quad (2.8)$$

Example 2.1: consider the following directed graph given in Figure 2.2. Then, its Adjacency matrix, Incident matrix, Degree matrix and Laplacian matrix can be given as follows;

$$\begin{aligned} \mathcal{A} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \mathcal{D} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \end{aligned}$$

2.2 Matrix Theory

Definition 2.8 *Kronecker Product*: The Kronecker product of matrices of $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is defined as [4];

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix},$$

The properties of Kronecker product are listed below;

$$(i) \quad (A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

$$(ii) (A \otimes B) + (A \otimes C) = (A \otimes (B + C))$$

$$(iii) (A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

$$(iv) (A + B) \otimes C = (A \otimes C) + (B \otimes C)$$

$$(v) (iv)] (A \otimes B)^T = A^T \otimes B^T$$

(vi) If $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ are both positive definite (positive semi-definite), so $A \otimes B$.

Definition 2.9 Kronecker Summation: The Kronecker sum of two square matrices of $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{n \times n}$ is defined as;

$$A \oplus B = A \otimes I_m + I_n \otimes B$$

Definition 2.10 Gershgorin's Circle Criterion [5]: Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, all eigenvalues of A in the union of Gershgorin circles are given by;

$$G(A) = \bigcup_{i=1}^n \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j=1, j \neq i}^n |a_{i,j}|\}, 1 \leq i \leq n$$

Therefore, the i -th disk in the Gershgorin circle criterion is drawn with a center at the diagonal element a_{ii} and with a radius equal to the i -th absolute row sum with the diagonal element deleted, that is $\sum_{j=1, j \neq i}^n |a_{i,j}|$. All the eigenvalues of A lie in the region formed by the union of all the n discs. The information given above is references [61].

Definition 2.11 Let $A \in \mathbb{R}^{m \times n}$. Then rank A is defined as

$$\text{rank}(A) = \dim(\mathbb{R}(A))$$

Since $\text{rank}(A) = \text{rank}(A^*)$, then $\text{rank}(A)$ equals the maximum number of independent rows or columns.

Definition 2.12 A matrix $A \in \mathbb{R}^{n \times n}$ is diagonal if $a_{ij} = 0$ for $i \neq j$.

Definition 2.13 A block diagonal matrix $A \in \mathbb{C}^{n \times n}$ is of the form;

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk} \end{bmatrix} = \text{diag}(A_{11}, A_{22}, \dots, A_{kk})$$

in which $A_{ii} \in \mathbb{C}^{n_i \times n_i}, i = 1, \dots, k, \sum_{i=1}^k n_i = n$, and all blocks above and below the block diagonal are zero blocks.

Definition 2.14 A matrix $A \in \mathbb{C}^{n \times n}$ is unitary if $A^* A = A A^* = I$. A matrix $A \in \mathbb{R}^{n \times n}$ is orthogonal if $A^T A = A A^T = I$.

Definition 2.15 A matrix $A \in \mathbb{C}^{n \times n}$ is Hermitian if $A^* A = A$. A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if $A^T = A$.

Definition 2.16 A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is

- positive definite is and only if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^{n \times n}$ and it is written as $A > 0$.
- positive semidefinite is and only if $x^T A x \geq 0$ for all nonzero $x \in \mathbb{R}^{n \times n}$ and it is written as $A \geq 0$.
- negative definite if $-A$ is positive definite and it is written as $A < 0$.
- negative semidefinite if $-A$ is positive semidefinite and it is written as $A \leq 0$.

Definition 2.17 Eigenvalues and Eigenvectors [60]: For a square matrix $A \in \mathbb{C}^{n \times n}$, the eigenvalues of A are defined as the solution of the equation

$$\det(\lambda I - A)$$

where I is defined as

$$I = \text{diag}(1) \in \mathbb{R}^{n \times n}$$

for the set of all eigenvalues $\{\lambda_i\}_{i=1}^n$ if there is any non-zero vector $v_r \in \mathbb{C}^m$ satisfying;

$$A v_r = \lambda_i v_r$$

then the non-zero vector v_r is called the right eigenvectors of A .

Similarly, if a non-zero vector $v_l \in \mathbb{C}^m$ satisfy

$$v_l^T A = \lambda_i v_l^T$$

then the non-zero vector v_l is called the left eigenvector of A .

Definition 2.18 [62] The system, $\dot{X} = AX$ ($A \in R^{N \times N}$) is quadratically stable if there exist a positive definite symmetric matrix $P > 0$, such that;

$$AP + PA^T < 0 \quad (2.9)$$

Definition 2.19 *Diagonally dominant matrix* : [63] Let $A = [a_{ij}]$, ($i, j = 1, 2, \dots, n$) be a complex matrix of order n . Inclusion regions for the eigenvalues of A are closely connected with conditions which guarantee that A is non-singular. The most classical results of this type are the Gersgorin theorem and Levy-Desplanques theorem. Let,

$$R_i = |a_{i1}| + \dots + |a_{i,i-1}| + |a_{i,i+1}| + \dots + |a_{in}|, \quad (i = 1, 2, \dots, n)$$

denote the sum of the module of the off-diagonal elements in row i of A . The matrix A is called *diagonally dominant* provided

$$|a_{ii}| > R_i, \quad (i = 1, 2, \dots, n)$$

Notice that a diagonally dominant matrix can have no zeros on its main diagonal. If the matrix A is diagonally dominant, then $\det(A) \neq 0$.

Definition 2.20 *Matrix polynomial* [64]: A matrix polynomials may be defined as a set of linear, time-invariant, ordinary differential equations as follows;

$$\begin{aligned} P(X) &= \sum_{i=1}^{n+1} A_i \frac{d^{i-1}}{dt^{i-1}} X(t) = 0, \quad A_{n+1} = I \\ &= A_1 X(t) + A_2 \frac{d}{dt} X(t) + A_3 \frac{d^2}{dt^2} X(t) + \dots + A_{n-1} \frac{d^n}{dt^n} X(t) = 0 \quad (2.10) \\ \frac{d^{i-1}}{dt^{i-1}} X(0) &= [\alpha_i] \quad i = 0, 1, \dots, n \end{aligned}$$

where $X(t)$ is the n -dimensional state of the system. The A_i are $n \times n$ real constant matrices. The matrix I is an identity matrix.

Definition 2.21 *Block companion matrix* [65]: Let the corresponding state equation of the characteristic polynomial given in Eq. (2.10) is defined as

$$\dot{X} = AX, \quad X(0) = 0 \quad (2.11)$$

Then the block companion matrix A of the characteristic polynomial is written as;

$$A = \begin{bmatrix} 0 & I & 0 & 0 & \dots & 0 \\ 0 & 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & I \\ -A_1 & -A_2 & -A_3 & -A_4 & \dots & A_n \end{bmatrix} \quad (2.12)$$

The A is the matrix in *block companion* form. Where, the dimensions of the matrix, A , each block element in the A , and the state vector x are $(n \times m) \times (n \times m)$, $m \times m$ and $(n \times m) \times 1$, respectively.

Definition 2.22 *Skew-symmetric matrix* A square matrix A is said to be skew-symmetric if $a_{ij} = -a_{ji}$ for all i and j . In other words, it is said that matrix A is said to be skew-symmetric if transpose of matrix is A is equal to negative of matrix A ($A^T = -A$).

- The main diagonal element in the skew-symmetric matrix are zero.
- The any square matrix A with real number entries, $A + A^T$ is a symmetric matrix, $A - A^T$ is a skew-symmetric matrix.
- The eigenvalues of skew-symmetric matrix is purely imaginary or zero.

Definition 2.23 *Block Schwarz matrix* [66]: Let, $A = [A_{ij}] \in \mathcal{R}$, where

$$S_{ij} = \begin{cases} I_n & \text{if } j - i = 1 \\ -S_k & \text{if } i - j = 1 \\ -S_n & \text{if } i = j = n \\ 0 & \text{otherwise} \end{cases}, \quad k = 1, 2, \dots, n \quad (2.13)$$

The matrix

$$S = \begin{bmatrix} 0 & I & 0 & 0 & \dots & 0 & 0 \\ -S_1 & 0 & I & 0 & \dots & 0 & 0 \\ 0 & -S_2 & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & I \\ 0 & 0 & 0 & 0 & \dots & S_{n-1} & S_n \end{bmatrix} \quad (2.14)$$

is named the block Schwarz matrix. The dimension of each block element in the matrix S is $m \times m$.

Definition 2.24 *Matrix Routh algorithm* [65]: Eq. (2.11) can be transform into the block Schwarz form by using the following linear transformation K .

$$\begin{aligned} Y &= KX \\ \dot{Y} &= KAK^{-1}Y \\ &= SY \end{aligned} \quad (2.15)$$

where

$$K = \begin{bmatrix} I & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ C_{n-1,1}^{-1}C_{n-2,2} & . & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & . & 0 & I & 0 & 0 & 0 & 0 & 0 \\ C_{n-3,1}^{-1}C_{n-3,3} & . & C_{6,1}^{-1}C_{6,2} & 0 & I & 0 & 0 & 0 & 0 \\ 0 & . & 0 & C_{5,1}^{-1}C_{5,2} & 0 & I & 0 & 0 & 0 \\ C_{n-5,1}^{-1}C_{n-5,4} & . & C_{4,1}^{-1}C_{4,3} & 0 & C_{4,1}^{-1}C_{4,2} & 0 & I & 0 & 0 \\ 0 & . & 0 & C_{3,1}^{-1}C_{3,3} & 0 & C_{3,1}^{-1}C_{3,2} & 0 & I & 0 \\ . & . & C_{2,1}^{-1}C_{2,4} & 0 & C_{2,1}^{-1}C_{2,3} & 0 & C_{2,1}^{-1}C_{2,2} & 0 & I \end{bmatrix} \quad (2.16)$$

The dimension of each block element in the matrix K is $m \times m$. The linear transformation matrix K is constructed by following the approach which is given in [65]. The block elements $C_{i,j}$ may be obtained from *matrix Routh algorithm* and *matrix Routh array*. The steps are given below;

- Define l as follow;

$$l = \begin{cases} (n/2) + 1 & \text{if } n \text{ is an even number} \\ n + 1/2 & \text{otherwise} \end{cases} \quad (2.17)$$

- Define double subscripted block elements $C_{1,j}$ and $C_{2,2}$ as follows;

$$\begin{aligned} C_{1,j} &= A_{n+3-2j}, & j &= 1, 2, 3, \dots, l \\ C_{2,j} &= A_{n+2-2j} & j &= 1, 2, 3, \dots, l \\ C_{1,1} &= I \end{aligned} \quad (2.18)$$

where A_i are the $m \times m$ real constant matrices given in Eq. (2.12).

- Create the Routh array and the matrix Routh algorithm as follows;

$$\begin{aligned} H_1 &= C_{1,1}C_{2,1}^{-1} & C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & \dots \\ H_2 &= C_{2,1}C_{3,1}^{-1} & C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & \dots \\ H_3 &= C_{3,1}C_{4,1}^{-1} & C_{3,1} & C_{3,2} & C_{3,3} & . & \dots \\ H_4 &= C_{4,1}C_{5,1}^{-1} & C_{4,1} & C_{4,2} & C_{4,3} & . & \dots \\ H_5 &= C_{5,1}C_{6,1}^{-1} & C_{5,1} & C_{5,2} & . & . & \dots \\ H_6 &= C_{6,1}C_{7,1}^{-1} & C_{6,1} & C_{6,2} & . & . & \dots \\ & & C_{7,1} & . & . & . & \dots \\ & & . & . & . & . & \dots \\ H_n &= C_{n,1}C_{n+1,1}^{-1} & C_{n,1} & . & . & . & \dots \\ & & C_{n+1,1} & . & . & . & \dots \end{aligned} \quad (2.19)$$

where

$$\begin{aligned} C_{i,j} &= C_{i-2,j+1} - H_{i-2}C_{i-1,j+1}, & j &= 1, 2, \dots, & i &= 3, 4, \dots \\ H_i &= C_{i,1}(C_{i+1,1})^{-1} & i &= 1, 2, \dots, n \\ \det(C_{i+1,1}) &\neq 0 \end{aligned} \quad (2.20)$$

The matrices H_i are called the matrix quotients.

Definition 2.25 *Stability of a matrix polynomial [65]:* The multivariable system, whose characteristic polynomial has the form of shown in Eq. (2.10) is asymptotically stable if the matrix quotients H_i are real symmetric positive definite matrices.

2.3 Introduction to Time Delay Systems

Many control systems exhibit the behaviour of time delay in the state, the control input, or the measurements. Aircraft control systems, process control systems, and communication networks may be given significant examples of systems that have a time delay. Time delay is observed in control systems since taking and processing information, and creating and executing the decision take time. This delay may evaluated as input delay or state delay. Besides, actuators, sensors, and other network elements are involved in delays named communication, transportation, or measurement delay, mainly in feedback lines. The effect of delay on a signal is shown in Figure 2.1. A general form of delayed differential-difference equation is described as;

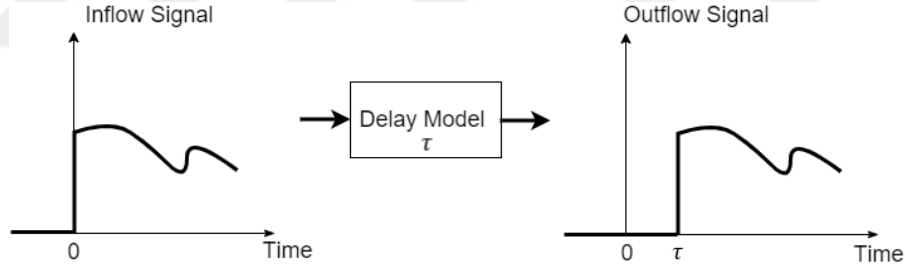


Figure 2.1 : Representation of delay model [1]

$$\dot{x}(t) - \sum_{k=1}^m A_k \dot{x}(t - \tau_k) = A_0 x(t) + \sum_{k=1}^m B_k x(t - \tau_k) \quad (2.21)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_k \in \mathbb{R}^{n \times n}$, $A_0 \in \mathbb{R}^{n \times n}$ and $B_k \in \mathbb{R}^{n \times n}$ are real constant matrices, $\tau = (\tau_1, \dots, \tau_m) \in \mathbb{R}^{m+}$ represents the time delay vector and m denotes the number of delays in τ which are positive and rationally independent from each other. If the matrix of $A_k = 0$, then the equation of Eq. (2.21) is called as *retarded type* LTI time delay system (LTI-RTDS), however, if $A_k \neq 0$, then it is called as *neutral type* LTI time delay system (LTI-NTDS). In LTI-RTDS, the derivative of state variable ($\dot{x}(t)$) at time t depends only on the current ($x(t)$) and past values ($x(t - \tau_k)$) of the state variables. Whereas in LTI-NTDS, $\dot{x}(t)$ at time t not only depends on $x(t)$ and

$x(t - \tau_k)$ but also $\dot{x}(t - \tau_k)$. It should be remarked that the two sides of Eq. (2.21) need not contain an equal number of delays.

So, the general characteristic equation of Eq. (2.21) is written as follows;

$$CE(s, \tau) = \det[s(I - \sum_{k=1}^m A_k e^{-\tau_k s}) - A_0 - \sum_{k=1}^m B_k e^{-\tau_k s}] = 0 \quad (2.22)$$

Eq. (2.22) is known as quasi-polynomial that has infinity number of roots over the complex plane $\mathbb{C}$ due to the presence of the exponential terms of $e^{-\tau_k s}$. It is well known from the control theory, a system is asymptotically stable if and only if all the roots of the characteristic equation are on the left half of the $\mathbb{C}$. As the system have many infinite roots, it is impossible to track all of them. Thusly, the question of stability translates into some conditions on τ to guarantee that all of these infinitely many characteristic roots lie on the stable left half of the complex plane. Even so, if the system is retarded type, the *root continuity argument*, which means that for any positive value of the delay, the position of the poles varies continuously with respect to delay, helps us to investigate the stability of the system. However, this property does not hold for LTI-NTDS due to small variation in time delay τ imparts indefinitely large root variations. During the transition of components of delay vector τ from 0 to 0^+ , the root continuity argument may collapse and infinity many bounded characteristic roots may appear in $\mathbb{C}^+$. Though, such a root discontinuity never occurs anywhere else in the $\tau \in R^+$ domain. If it occurs, even if the system is stable for $\tau = 0$, this condition which is known as *small-delay phenomenon* or *strong stability condition* precludes the stability in $\tau \in R^+$. So before performing with a neutral-type time-delay system, one may guarantee the root continuity in the transition of τ from 0 to 0^+ [67]. For this, the kernel difference operator of the system which is defined in Eq. (2.23) should be stable.

$$K(x) = x(t) - \sum_{k=1}^m A_k x(t - \tau_k) \quad (2.23)$$

That is, the roots of characteristic equation given in Eq. (2.23) on the left-half of complex plane. This condition directly translates into another one on the spectral radius of $\mathbb{C}$, $\rho(\mathbb{C}) : K(x)$ is stable if and only if $\rho(\mathbb{C}) < 1$.

$$CE_D(s, \tau) = \det[(I - \sum_{k=1}^m A_k e^{-\tau_k s})] = 0 \quad (2.24)$$

The necessary condition for this feature which is proven in [68] states that the LTI-NTDS is stabilizable on the condition that the kernel difference operator Eq. (2.23)

is stable. Alter all , if kernel difference operator of an LTI-NTDS is stable, the dynamic is regarded as τ -*stabilizable* [69]. It should be pointed out that τ -*stabilizability* does not guarantee that the system will be stable for the values of $\tau \in R^+$. Conversely, if τ -*stabilizable* of the system is not satisfied any of the values of $\tau \in R^+$ can not grant stability even if the system is stable for $\tau = 0$. That is to say, τ -*stabilizability* is a necessary but not sufficient condition for for a stabilizing of LTI-NTDS system.

In this study, the discussed cooperative control problems are converted to the widely accepted structures of LTI-NTDS and LTI-RTDS as follows;

- A general form of single-delay LTI-NTDS

$$\dot{x}_i(t) - A_1 \dot{x}_i(t - \tau) = A_0 x_i(t) + B_1 x_i(t - \tau) \quad (2.25)$$

- A general form of single-delay LTI-RTDS

$$\dot{x}_i(t) = A_0 x_i(t) + B_1 x_i(t - \tau) \quad (2.26)$$

- A general form of multiple-delay LTI-RTDS

$$\dot{x}_i(t) = A_0 x_i(t) + B_1 x_i(t - \tau_1) + B_2 x_i(t - \tau_2) \quad (2.27)$$

Here, i represent the number of agents in the cooperative system. It should be noted that due to the nature of the multi-agent systems discussed, and the controller forms are chosen, the overall structure is converted to a single delayed ($m = 1$) neutral time-delay system as well as a single($m = 1$) and multiple($m = 2$) time-delayed systems.

The determination of stability analysis of time-delay systems is widely studied in the literature. However, the method of *Cluster Treatment of Characteristic Roots*, *CTCR* who brought a practical and numerically efficient procedure for the problem becomes prominent. This procedure provides a non-conservative and complete stability picture in the region of uncertain delays, $\tau \in R^+$. Even if it is initially introduced for retarded time-delay systems, the applicability it for a neutral time-delay system is proved [67]. The CTCR framework is briefly explained in the next section.

2.4 Cluster Treatment of Characteristic Roots

The CTCR method is a mathematical tool that solves the stability problem of the time-delay system precisely and explicitly. The time-delay system has infinitely many characteristic roots due to the exponential term, and it is impracticable to track all the roots for all the variation of delay to assess the stability. It is also known that the stability posture of the system only changes when the system presents a conjugated pair of purely imaginary characteristic roots $s \pm j\omega_c$. Thus, the main idea behind the CTCR method is the *clustering* of all possible imaginary crossing roots (or frequencies) in the delay domain, which shows that the solution of the time-delay system is a continuous function of the delay. The basic concept in the CTCR paradigm is introduced in the following propositions without proof [70].

Proposition 1. *Imaginary Spectra:* The complete set of crossing frequencies of LTI-TDS system for delay vector $\tau \in R^+$ is denoted as; follows;

$$\begin{aligned}\Omega &= \{\omega_c \mid CE(s = j\omega, \tau) = 0, \tau \in R^+, \omega_c \in R\} \\ \Omega &= \{\omega_c \mid \langle \tau \omega_c \rangle, \tau \in R^+, \omega_c \in R\}\end{aligned}\tag{2.28}$$

where $\langle \tau \omega_c \rangle$ indicates that for a $\tau \in R^+$, there exist an imaginary root, $j\omega_c$ of Eq. (2.22). With this notation the stability switching boundaries are classified into two classes.

Proposition 2. *Kernel Hypersurfaces, $\mathcal{P}_0^{DS}$:* The curves that consists of all the points $\tau \in R^+$ exhaustively, which cause an imaginary root and satisfy the constraint $0 < \tau_k \omega_c < 2\pi, (k = 1, 2, \dots, m)$ are called the *kernel hypercurves*. The points on the curve contain all the smallest delay composition which correspond to all possible imaginary roots. By its definition, the kernel hypercurves formation is unique for a given characteristic equation of Eq. (2.22). The characteristic equation of Eq. (2.22) possess imaginary roots along countably infinite number of hypercurves. Mathematical representation of this set is shown as follows;

$$\xi_0^{DS} = \{\tau : (\tau_1, \tau_2, \dots, \tau_m) \mid CE(j\omega_c) = 0, \omega_c \in R^+, 0 \leq \tau \omega_c < 2\pi\}\tag{2.29}$$

Proposition 3. Offspring Hypersurfaces, $\mathcal{O}^{DS}$: The hypercurves obtained from the kernel hypercurves by the following point-wise nonlinear transformation.

$$\{\langle \tau_1 \pm \frac{2\pi}{\omega_c} j_1, \tau_2 \pm \frac{2\pi}{\omega_c} j_2, \dots, \tau_m \pm \frac{2\pi}{\omega_c} j_m \rangle | j_1, j_2, \dots, j_m \in N\} \quad (2.30)$$

This definition simply utilizes the fact that any point on the kernel hypersurface will result in $\inf_m(m\text{-dimensional infinity})$ offspring, which correspond to the infinite dimensionality of the characteristic roots of a transcendental equation.

Proposition 4. Root Tendency - RT: For an imaginary characteristic root, $j\omega$ of Eq. (2.22) caused by any of the infinitely many grid points in $\tau \in R^+$ domain defined by expression Eq. (2.30). So, the *Root Tendency*, $RT|_{s=j\omega}^{\tau_k}$ remains invariant (-1 for **stablizing** and $+1$ for **destablizing**) so long as the grid points on different offspring hyperspaces are obtained keeping all $p-1$ delays $\tau_j = j = 1, 2, \dots, m$, fixed. The *Root Tendency* is defined as follows;

$$RT|_{s=j\omega_c}^{\tau_k} = \text{sgn}[RE(S_{\tau_j}^s | s = j\omega_c)] \quad (2.31)$$

2.4.1 Review of the CTCR methodologies

Therewith, the starting and crucial point for the stability analysis of LTI-TDS is to the determination of all imaginary roots completely and accurately. In literature, different approaches are observed to find the imaginary roots of a TDS, but the listed below is actively used in this study.

- a) Rekasius substitution
- b) Matrix pencil, Kronecker sum method
- c) Half-angle tangent substitution
- d) Kronecker multiplication and elementary transformation

These methods are briefly explained as follows;

(a) Rekasius substitution

This method was firstly introduced in [71] and utilized in most studies. It may be applied both neutral and retarded type time delay systems. In this substitution the

exponential terms are replaced with a rational polynomial as follows;

$$e^{-\tau_k s} = \frac{1 - T_k s}{1 + T_k s}, T_k \in R, k = 1, 2, \dots, m \quad (2.32)$$

This representation becomes exact for $s = \pm j\omega$, with the following relation between τ_k and T_k ;

$$\tau_k = \frac{2}{\omega} [\tan^{-1}(T_k \omega) \pm j\pi], j = 1, 2, \dots \quad (2.33)$$

This exact substitution transforms the original characteristic equation of $CE(s, \tau) = 0$ given by Eq. (2.22) into a new form $CE(s, T) = 0$ where $T = (T_1, T_2, \dots, T_m) \in R_m$, which is a fractional polynomial type.

$$CE(s, T)_T = CE(s, T) \prod_{k=1}^L (1 + T_k s)^{c_k} = \sum_{k=0}^m a_k(T) s_k \quad (2.34)$$

where $c_k = \text{rank}(B_k \leq n)$ is the commensurate degree for τ_k and $m = n + \sum_{k=1}^L \leq (L+1)n$. This substitution only holds for crossing frequencies when the original system has poles on the imaginary axis. In other words, the question is to determine all $T \in R$ values, which cause imaginary roots of $s = i\omega$. The most basic method to achieve this is to forming Routh array with the parameter of T . Then, using the relation given Eq. (2.33), the time delay values are obtained.

(b) Matrix pencil, Kronecker sum method

This method is presented in [72, 73] and used in literature such as [22, 25]. A generic form of single delay LTI-RTDS given in Eq. (2.26) is evaluated to understand the process in an easy way. The characteristic equation of Eq. (2.26) has the form;

$$CE(s, z) = \det(sI - A_0 - B_1 z) = 0, z = e^{-\tau s} \quad (2.35)$$

Here, finding the roots of Eq. (2.35) is equivalent to finding the eigenvalues of the matrix of;

$$M_1 = A_0 + B_1 z \quad (2.36)$$

As A_0 and B_1 are real-valued matrices, the solution of Eq. (2.35) come in complex conjugate terms. This is to say that, Eq. (2.35) is also satisfied by the complex conjugate of s and z ;

$$CE(s^*, z^*) = \det(s^* I - A_0 - B_1 z^*) = 0 \quad (2.37)$$

Similarly, the roots of this equation is also equivalent of the eigenvalues of the following matrix;

$$M_2 = A_0 + B_1 z^* \quad (2.38)$$

Here, for $s = j\omega$, $z = e^{-j\omega\tau}$, and $z^* = z^{-1}$. Since, it is interested in finding the purely imaginary characteristic roots of the system, it is concluded that if $s = j\omega$ is a root of Eq. (2.35), then $s^* = -j\omega$ is also root of Eq. (2.37). Analogically, if there is a common solution to Eq. (2.35) and Eq. (2.37), then $s = j\omega$ is an eigenvalue of M_1 while M_2 has $s^* = -j\omega$ as an eigenvalue. Frankly, the sum of these two eigenvalues is zero, which means that the *Kronecker Summation* of $M_1 \oplus M_2$ has a zero eigenvalue. And its determinant must be zero. So, by using this deduction the *auxiliary characteristic equation* is defined for the system given Eq. (2.26).

$$CE_{ACE}(z) = \det[M_1 \oplus M_2] = \det[(A_0 + B_1 z) \oplus (A_0 + B_1 z^{-1})] = 0 \quad (2.39)$$

Note that, Eq. (2.39) is an algebraic equation in z , and may be solved by using numerical tools. The detail analysis of this equation may be found in [22]. After solve the Eq. (2.39) with a proper numerical tool, then the solution which have unity magnitude, $|z| = 1$ is extracted. This roots are used in Eq. (2.35) to find the values of ω , and form $z = e^{-j\omega\tau}$ the values of the delay are obtained. However, one of the disadvantage of this method is Eq. (2.39) yield to a polynomial in z of degree $2n^2$ for $n = m$ and to solve this equation for cooperative systems with scores of agent may be increase computational complexity.

(c) Half-angle tangent substitution

To find the complex set of frequencies of Ω given in Eq. (2.28) using this method in the 2-D domain of delays, for example τ_1, τ_2 without loss of generality while all remaining delays $\tau_3, \tau_4, \dots, \tau_m$ are fixed, for the root $s = j\omega$ the *Euler's formula* is used for transcendental terms as;

$$e^{-j\omega\tau_k} = \cos(v_k) - j\sin(v_k), \quad v_k = \tau_k \omega, \quad k = 1, 2. \quad (2.40)$$

and express them in term of a single parameter, the half-angle tangent:

$$\begin{aligned} \cos(v_k) &= \frac{1 - z_k^2}{1 + z_k^2}, \quad \sin(v_k) = \frac{2z_k}{1 + z_k^2} \\ z_k &= \tan \frac{v_k}{2}, \quad k = 1, 2. \end{aligned} \quad (2.41)$$

Remark that, the inverse transformation from z_k to τ_k is multi-values as;

$$\tau_k = \frac{2 \tan^{-1}(z_k)}{\omega} + \frac{2r\pi}{\omega} \quad 0 \leq \tan^{-1}(\cdot) \leq \pi, \quad k = 1, 2, \dots, r = 0, 1, 2, \dots \quad (2.42)$$

Substituting Eq. (2.40) into Eq. (2.23) for an imaginary root $s = j\omega$, a new characteristic equation of $CE(s, z, e^{-j\omega\tau_3}, \dots, e^{-j\omega\tau_m}) = 0$ where $z = (z_1, z_2) \in R^2$. Following this domain transformation from τ to $z, \tau_3, \tau_4, \dots$ and multiplying with the least common multiple of the denominators of $CE(s, z, e^{-j\omega\tau_3}, \dots, e^{-j\omega\tau_m}) = 0$, a modified characteristic equation is obtained as follows;

$$CE_{HTS}(s, z, e^{-j\omega\tau_3}, \dots, e^{-j\omega\tau_m}) = CE(s, z, e^{-j\omega\tau_3}, \dots, e^{-j\omega\tau_m}) \prod_{k=1}^2 (1 + z_k^2)^{c_k} = 0 \quad (2.43)$$

where $c_k = \text{rank}(B_k) \leq n$ is the commensurate degree of τ_k . This operation simplifies the problem of determining the complete set of imaginary roots of Eq. (2.40) in the space of $z \in R^2$. If the fixed delay arguments are suppressed, Eq. (2.40) is restated as;

$$CE_{HTS}(j\omega, z_1, z_2) = CE_{HTS}(s, z, e^{-j\omega\tau_3}, \dots, e^{-j\omega\tau_m}) = 0 \quad (2.44)$$

Then, the complete set of purely imaginary eigenvalues of Eq. (2.44) as $\bar{\Omega} \subset \Omega$. So, if there is a solution of $\omega \in R^+$, it should make both the real and imaginary parts of Eq. (2.44) should be zero, simultaneously.

$$\begin{aligned} \text{Re}[CE_{HTS}(j\omega, z_1, z_2)] &= 0 \\ \text{Im}[CE_{HTS}(j\omega, z_1, z_2)] &= 0 \end{aligned} \quad (2.45)$$

(d) Kronecker multiplication and elementary transformation

The elementary transformation converts a matrix to a vector form and back again. This method was firstly used for time-delay systems in [74]. Two important property of this transformation is explained in below;

- The first property basically convert a $n \times n$ matrix to a $n^2 \times 1$ vector and is symbolized as $\xi : \mathcal{C}^{n \times n} \rightarrow \mathcal{C}^{n^2 \times 1}$ The transformation may be shown as follows;

$$M^{(n \times n)} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}_{(n \times n)} \iff (\xi M)^{(n^2 \times 1)} = \begin{bmatrix} m_1^T \\ m_2^T \\ \vdots \\ m_n^T \end{bmatrix}_{(n^2 \times 1)} \quad (2.46)$$

- Another important property of ξ is the multiplication of three $n \times n$ matrices of P_1, P_2, P_3 converts into a Kronecker product of dimension $n^2 \times n^2$, and is symbolized as $\xi : \mathcal{C}^{n \times n} \rightarrow \mathcal{C}^{n^2 \times 1}$. The transformation is given as follows;

$$\xi(P_1 P_2 P_3) = (P_1 \otimes P_3^T) \xi P_2 \quad (2.47)$$

These two property is used in solving the pure imaginary roots of time-delayed system given in Eq. (2.25) and Eq. (2.26). It should be remarked that, this method may be applied for single-delay LTI-N/RTDS systems. Without loss of generality, the rest of the method is explained for Eq. (2.25) due to retarded time- delay system is a subset of neutral time delay systems. So, if the solution of the form $x(t) = e^{st}v$ is suggested for Eq. (2.25), where $(s, v^{n \times 1})$ in an eigenvalue-eigenvector pair, and is substituted in *Laplace* transform of Eq. (2.25), then the following equation is obtained;

$$(s(I + A_1 e^{-\tau s}) - A_0 - B_1 e^{-\tau s})v = 0 \quad (2.48)$$

Eq. (2.48) may also written as follows;

$$(sI - A_0)v = -e^{-\tau s}(sA_1 - B_1)v \quad (2.49)$$

It is known from linear algebra that if $s = j\omega$ is a root of Eq. (2.49) so is its conjugate $s^* = -j\omega$. Therefore, the conjugated complex equation of Eq. (2.49) is written as;

$$v^*(sI + A_0^T) = -e^{-\tau s}v^*(sA_1^T + B_1^T) \quad (2.50)$$

If Eq. (2.49) and Eq. (2.50) are multiplied side-by-side, one get

$$(sI - A_0)V(sI + A_0^T) = (sA_1 - B_1)V(sA_1^T + B_1^T) \quad (2.51)$$

with $V^{n \times n} = vv^*$. This equation is exactly in the form to be transformed by using ξ defined in Eq. (2.44). It returns;

$$\begin{aligned} ((sI - A_0) \otimes (sI + A_0) - (sA_1 - B_1) \otimes (sA_1^T + B_1^T)) \xi V &= 0 \\ \lambda(s) \xi V &= 0 \end{aligned} \quad (2.52)$$

For non-trivial solution of $\xi V \neq 0$, the only way to satisfy Eq. (2.48) is to set;

$$\det(\lambda(s)) = 0 \quad (2.53)$$

Hence, it can be concluded that the desired imaginary roots are determined by solving the polynomial of Eq. (2.50). It is also extracted from the elementary transformation properties that $|\lambda(s)| = |sE_0 - J_0|$ where E_0 and J_0 is defined as;

$$E_0 = \begin{bmatrix} I \otimes I & A_1 \otimes I \\ I \otimes A_1 & I \otimes I \end{bmatrix}, J_0 = \begin{bmatrix} A_0 \otimes I & B_1 \otimes I \\ -I \otimes B_1 & -I \otimes A_0 \end{bmatrix} \quad (2.54)$$

Computation with retarded time delay systems is particularly straightforward, since $A_1 = 0$ and $E_0 = I_0$, and thus all imaginary eigenvalues of the delay equation are also eigenvalue of the matrix J_0 . Also, noting that, $\lambda(s)$ converts a second degree matrix polynomial in s as follows;

$$\lambda(s) = G_0 s^2 + G_1 s + G_2 \quad (2.55)$$

where;

$$G_0 = I \otimes I$$

$$G_1 = I \otimes A_0 - A_0 \otimes I$$

$$G_2 = B_1 \otimes B_1 - A_0 \otimes A_0$$

The matrix polynomial of Eq. (2.52) may be linearized on the fact that G_0 is an invertible matrix. So, the linearized form of Eq. (2.52) is expressed as;

$$F = \begin{bmatrix} 0 & T_0 \\ -T_2 & -T_1 \end{bmatrix} \quad (2.56)$$

with $T_0 = I \otimes I$, $T_1 = G_0^{-1} G_1$, $T_2 = G_0^{-1} G_2$, where the roots of $\det(\lambda(s)) =$ are the eigenvalues of F , which means $\det(\lambda(s)) = \det(sI - F) = 0$.

2.4.2 CTCR method for cooperative control systems

The method explained in the previous subsection is also valid for cooperative control systems. However, the deployment of the process is computationally intensive as it is a numerical method, so it may not be suited for the problem of a higher-order multi-agent systems. The overall dynamic model of cooperative control of multi-agent systems has its individual agent dynamics and the dynamics of its communication behaviour, stem from information flow topology. So, if the number of the agents in the system is increased, the degree of the overall system becomes larger. But, if the characteristic equation of the system is block diagonalized, it is easily analyzed with one of the methods given in Section 3.3.1. The diagonalizable methodology was introduced by

[26]. On the contrary, if it is not, some additional modification is needed to solve the following situations.

- For Rekasius substitution, investigation of the stability region of the characteristic equation with the Routh array method becomes more complex since the table is constituted with the parameter of T and the dynamic of the system is higher.
- In the Matrix pen and Kronecker sum method, solving the equation of Eq. (2.39) with a numerical tool may not be possible because the polynomial degree is increased while creating the auxiliary characteristic equation.
- For Half-angle tangent substitution, solving the real and imaginary equation given in Eq. (2.45) to obtaining common roots increase computational complexity for the system with a number of agents.
- For Kronecker multiplication and elementary transformation, the problem is similar with the others given below. The increase the number of the agent in the cooperative system yields the rise in the dimension of the matrix of E_0 and J_0 given in Eq. (2.54).

It should be noted that the complexity is basically originated from the following situations; the dynamics of the individual agent (if the dynamics of the agent is higher due to its innate nature), the types of information flow topology (bi-directional or unidirectional), and the number of time delays in the system (input delay, state delay, communication delay). And using CTCR method, the investigation of the stability region of the cooperative control system under time delay is converted to the problem of solving the determinant of some higher-order polynomial equation. A practical procedure that deploys Bezout resultant methodology to create the necessary and sufficient conditions to have common roots for polynomial pairs is proposed in this study. The Bezout resultant method may be used to solve a pair of polynomial equations to obtain nontrivial common solutions [75]. In the following, it is explained how to constructed the Bezout matrix of two polynomials [76]. Consider the two polynomials

$$\begin{aligned} Q_1(s, z) &= \sum_{i=0}^n f_i(z) s^i = 0 \\ Q_2(s, z) &= \sum_{i=0}^n g_i(z) s^i = 0 \end{aligned} \tag{2.57}$$

Let write the Bezout matrix of $Q_1(s, z)$ and $Q_2(s, z)$ as follows;.

$$B(Q_1(s, z), Q_2(s, z), s) = b_{i,j=1,\dots,n} \quad (2.58)$$

where;

$$b_{n-i+1,j} = \text{Coeff}\left(\left(\sum_{r=i}^n g_{n-r}s^{n-r}\right)\left(\sum_{r=i}^i f_{n-r+1}s^{i-r}\right) - \left(\sum_{r=i}^n f_{n-r}s^{n-r}\right)\left(\sum_{r=i}^i g_{n-r+1}s^{i-r}\right), s, n-j\right), \quad \text{for } i, j = 1, \dots, n$$

The Bezout matrix given in Eq. (2.58) is singular for common roots of polynomial equations for $Q_1(s, z)$ and $Q_2(s, z)$.

2.5 Problem Formulation

In the last of this chapter, the problem cases are introduced. After the mathematical models of case studies are given, the modeling approach of multi-agent systems is defined. The theoretical outputs are evaluated in two case studies: vehicular platooning systems and the formation of unmanned aerial vehicles.

2.5.1 Vehicular platooning systems

Platooning systems are based on the longitudinal control of the vehicle. Longitudinal control management is responsible for maintaining the speed, acceleration, and distance between consecutive cars on the lane. Regarding longitudinal control of autonomous vehicles, the controller structure is divided into upper-level and lower-level controls, as shown in Figure 2.2. The upper-level controller determines the desired acceleration or deceleration to maintain a predefined distance between vehicles and ensures the stability of the platoon. The lower one generates the throttle and brake actuator inputs to track the desired acceleration specified by the upper-level control [77, 78].

In this dissertation, only upper-level controller is considered and it is assumed that the lower-level controller has been already designed.

Cruise Control (CC) is one of the primary longitudinal controllers. It aims to keep the speed set by the driver by changing the throttle and brake pedal positions. In a dangerous situation, the driver manually disengages the system by taking control of

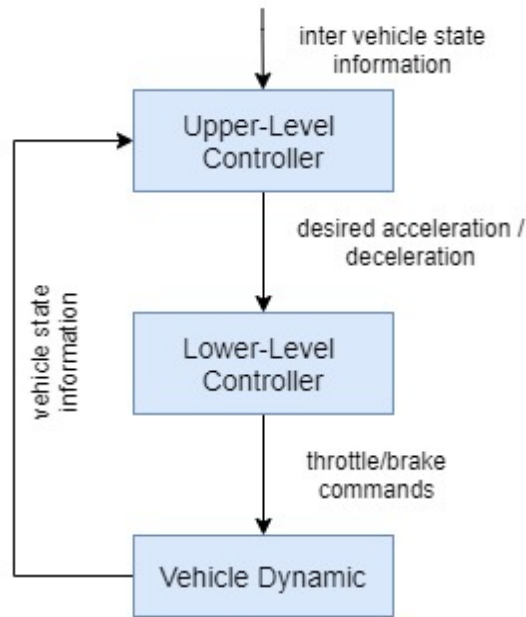


Figure 2.2 : Upper and lower level control architecture

the brake pedal. Adaptive Cruise Control (ACC) is an extended version of the CC system. Its main difference from CC is that it adjusts the vehicle speed to keep the distance between the vehicles by controlling the throttle or brake. The vehicle with ACC is equipped with range sensors like Radar, which can measure the preceding vehicle's relative distance and velocity. The ACC system works in speed control and distance control mode. The transition between those two modes occurs automatically without driver intervention [79]. *Cooperative Adaptive Cruise Control*(CACC) system, an improved version of ACC, use information exchanged by vehicles through wireless and onboard sensors. Such an improvement decreases the headway distance between two consecutive vehicles, improving the road network's traffic throughput. Another significant advantage of CACC systems is the fact that it improves string stability, which in turn enhances safety and throughput, as well as decrease fuel consumption and aerodynamic drag forces. *Platooning systems* is a technique where highway traffic is organized into groups of close-following vehicles. It was designed initially for *Automated Highway Systems* (AHS). Platooning systems have always relied on vehicle-to-vehicle (V2V) communication technology to help compensate for sensor delays inherent in radars and lidars. The improvement in wireless communication and vehicle control technology make platooning feasible for partially automated vehicles, such as CACC vehicles. Moreover, most automated vehicle control projects discussing platooning have emphasized close coupling between vehicles to maintain a constant

distance from consecutive vehicles. However, following short gaps between vehicles is likely to require special lanes and both speed control and steering control in vehicles. Dedicated lanes for platooning systems enhance safety as the interaction of the platoon vehicles with other traffic participants will be minimized. Nevertheless, it is not feasible to dedicate a line for platoon vehicles if considering today's traffic infrastructure or building new lines for the short term. At this point, CACC is an intermediate step for a vision of platooning operations on both long-haul and short-haul freight corridors. At this point, CACC is an intermediate step for a vision of platooning operations on both long-haul and short-haul freight corridors. With CACC, only vehicle speed control will be automated, using V2V communication to supplement forward sensors. However, the drivers will still be responsible for actively steering the vehicle, lane-keeping, and monitoring roadway and traffic conditions. Additionally, rather than the very closely coupled, constant distance control strategy, CACC relies on a more loosely coupled, constant-time-gap following strategy whereby the distance between the vehicles varies with speed [80].

In summary, the longitudinal motion of partially-automated platooning systems in which only the speed of the vehicles is controlled with a kind of CACC system by using V2V communication and onboard sensors will be investigated in this study.

The longitudinal dynamic model of a vehicle shows nonlinear behavior due to some nonlinearities in power-train systems, which consist of engine, driveline, brake system, and aerodynamic drag. However, the simplified model, which is more frequently used in platoon control problems, is created with the following reasonable assumptions;

- (a) Each vehicle is equipped with an onboard sensor to measure its distance from the preceding vehicle and a wireless communication device to send/receive information to/from other vehicles.
- (b) The platoon is assumed to be homogeneous, and the longitudinal dynamics of each vehicle are composed of an engine, a drive line, the rolling resistance, and the gravitational force.
- (c) The longitudinal tire slip is negligible, and the power-train dynamics are transformed into a first-order inertial model.

- (d) The vehicle body is considered to be rigid and symmetric.
- (e) The influence of pitch and yaw motions is ignored.
- (f) The driving and braking forces are controllable inputs.

The longitudinal forces acting on a vehicle is shown in Figure 2.3.

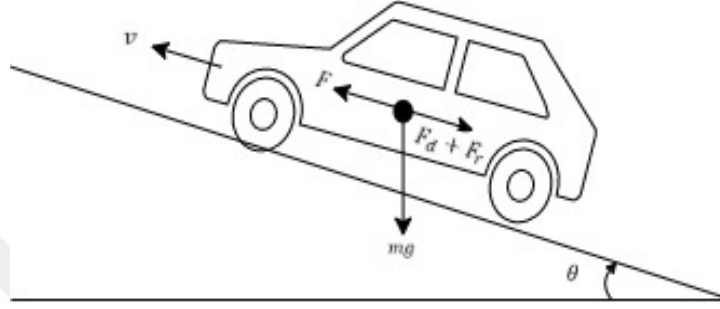


Figure 2.3 : Longitudinal forces acting on a vehicle

A simple model for longitudinal dynamic of i – th vehicle is

$$m_i a_i = F_i - m_i g \sin \theta - F_{d_i} - F_{r_i} \quad (2.59)$$

where a_i is vehicle's longitudinal acceleration, m_i denotes the i – th vehicle's mass, g denotes the acceleration of gravity, θ denotes the angle between the road surface and a horizontal plane, F_{d_i} is the force due to the air resistance, F_{r_i} denotes the mechanical drag which is constant and F_i denotes the force produced by the i – th vehicle's engine. The equivalent air resistance force on a vehicle can be represented as

$$F_{d_i} = \frac{\rho A_i C_{d_i}}{2} (v + V_{wind})^2 \text{sgn}(v + V_{wind}) \quad (2.60)$$

where v_i is vehicle's longitudinal velocity, ρ denotes the specific mass of air, A_i denotes the cross-sectional area of the i – th vehicle, C_{d_i} denotes the i – th vehicle's drag coefficient, and V_{wind} denotes the velocity of the wind gust. The force produced by the engine can be modeled as

$$\begin{aligned} F_i &= m_i \xi \\ \dot{\xi} &= -\frac{\xi}{\gamma_i} + \frac{\eta_i}{m_i \gamma_i} \end{aligned} \quad (2.61)$$

where, γ_i denotes i – th vehicle's engine time constant, η_i denotes the throttle input to the i – th vehicle's engine. Then, if the Eq. (2.60) and Eq. (2.61) are substituted into

Eq. (2.59) then it is obtained as;

$$m_i a_i = m_i \xi - m_i g \sin \theta - \frac{\rho A_i C_{d_i}}{2} (v + V_{wind})^2 \text{sgn}(v + V_{wind}) - F_{r_i} \quad (2.62)$$

To simplify the longitudinal dynamic model given in Eq. (2.62), it is assumed that the road surface is horizontal ($\theta = 0$), there is no wind gust ($V_{wind} = 0$) and the vehicles are assumed to travel in the same direction at all times ($\text{sgn}(v) = 1$), then $K_{d_i} = \frac{\rho A_i C_{d_i}}{2}$. Finally, the dynamics of the simplified longitudinal model are described by two nonlinear differential equations as follows;

$$m_i a_i = m_i \xi - K_{d_i} v^2 - F_{r_i} \quad (2.63)$$

$$\dot{\xi} = -\frac{\xi}{\gamma_i} + \frac{\eta_i}{m_i \gamma_i} \quad (2.64)$$

Then, the exact linearization method is used to linearize the input-output behavior of vehicle. From Eq. (2.63) it is obtained;

$$\xi = a_i + \frac{K_{d_i}}{m_i} v^2 + \frac{F_{r_i}}{m_i} \quad (2.65)$$

Substituting Eq. (2.65) into Eq. (2.64) yields

$$\dot{\xi} = -\frac{1}{\gamma_i} [a_i + \frac{K_{d_i}}{m_i} v^2 + \frac{F_{r_i}}{m_i}] + \frac{\eta_i}{m_i \gamma_i} \quad (2.66)$$

And, after both sides of Eq. (2.64) are differentiated with respect to time and Eq. (2.66) is substituted the following expression is obtained [81];

$$\dot{a}_i = -\frac{2K_{d_i}}{m_i} v_i a_i - \frac{1}{\gamma_i} [a_i + \frac{K_{d_i}}{m_i} v_i^2 + \frac{F_{r_i}}{m_i}] + \frac{\eta_i}{m_i \gamma_i} \quad (2.67)$$

The longitudinal dynamics of the i -th vehicle are simplified, but still nonlinear. They may also described as;

$$\begin{aligned} \dot{p}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= a_i(t) \\ \dot{a}_i(t) &= f_i(v_i, a_i) + g_i(v_i) \eta_i \end{aligned} \quad (2.68)$$

where p_i, v_i and a_i denote the position, velocity and acceleration of i -th vehicle, respectively. η_i is the engine input and the function of $f(.,.)$ and $g(.,.)$ are given by;

$$\begin{aligned} f_i(v_i, a_i) &= -\frac{2K_{d_i}}{m_i} v_i a_i - \frac{1}{\gamma_i} [a_i + \frac{K_{d_i}}{m_i} v_i^2 + \frac{F_{r_i}}{m_i}] \\ g_i(v_i) &= \frac{1}{m_i \gamma_i} \end{aligned} \quad (2.69)$$

Assume that all system parameters are a priori known. And, to linearise the $i - th$ vehicle nonlinear longitudinal dynamic, an exogenous input signal u_i , which is related to the $i - th$ vehicle throttle input η_i may be created as follows;

$$\eta_i = m_i u_i + K_{di} v_i^2 + F_{r_i} + 2K_d \gamma_i v_i a_i \quad (2.70)$$

The input transformation is globally well defined since it was previously assumed that $m_i > 0$. After adopting this input transformation the longitudinal vehicle model is given by [82], [83], [84];

$$\begin{aligned} \dot{p}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= a_i(t) \\ \dot{a}_i(t) &= -\frac{1}{\gamma_i} a_i + \frac{1}{\gamma_i} u_i \end{aligned} \quad (2.71)$$

Therefore, the state (A_i) and input (B_i) matrices of each vehicle i are written as;

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\gamma_i} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\gamma_i} \end{bmatrix} \quad (2.72)$$

2.5.2 Formation of unmanned aerial vehicles

Point-mass aircraft model is used to describe the dynamic behavior of formation of flying unmanned aerial vehicles (UAVs). The inner loop control structures, known as the stability augmentation system (SAS) and control augmentation system (CAS), as well as the attitude autopilot controllers of each UAV, are assumed to be designed.

The nonlinear point-mass model is created with the following assumptions;

- (a) The aircraft thrust is directed along the velocity vector.
- (b) The aircraft always performs coordinated maneuvers.
- (c) The earth is flat and the fuel expenditure is negligible, which means the center of mass is time-invariant.
- (d) The angle of attack is assumed to be small enough to be taken as zero, and the flight path angle and the pitch angle are equal to each other.
- (e) The angle of sideslip is assumed to be small enough to be taken as zero, and the heading angle and the yaw angle are equal to each other.

- (f) The inner loop systems (SAS, CAS, attitude autopilots, and engine systems) are assumed to be modeled as first-order dynamical systems.
- (g) The command signals are reference roll angle, pitch angle and velocity values.
- (h) The related variables are defined with respect to inertial coordinate frame (x_i, y_i, z_i) .

The forces acting on the UAV and the related variables are shown in Figure 2.4 [4].

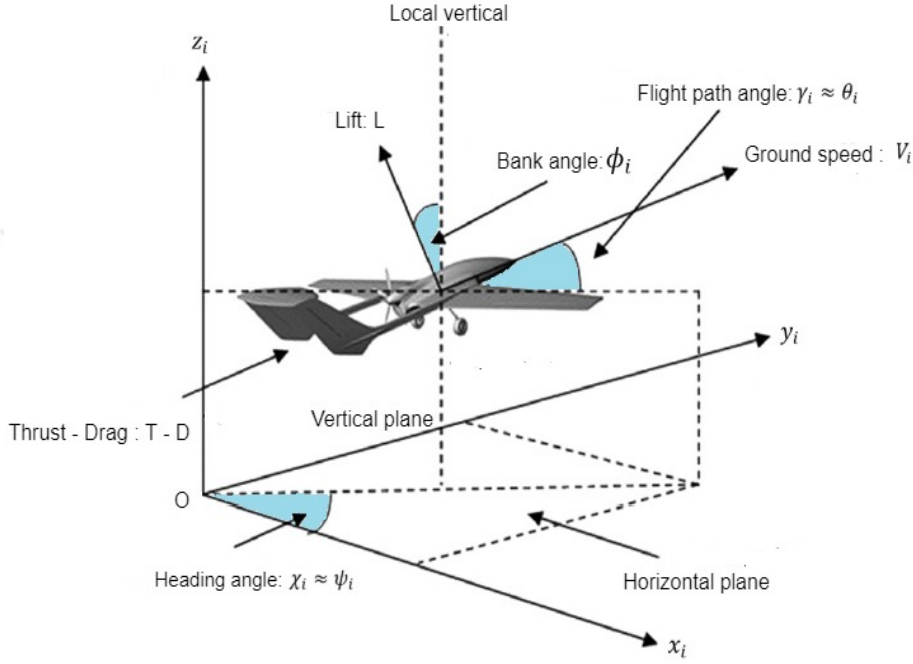


Figure 2.4 : Forces acting on a UAV

Under these assumptions, the simplified kinematic and dynamic UAV equations of motion can be described as follows;

$$\begin{aligned}
 \dot{x}_i &= V_i \cos \theta_i \cos \psi_i \\
 \dot{y}_i &= V_i \cos \theta_i \sin \psi_i \\
 \dot{z}_i &= V_i \sin \theta_i \\
 \dot{\psi}_i &= \frac{g}{V_i} \tan \phi_i \cos \theta_i \\
 \dot{\psi}_i &= \frac{1}{\tau_\psi} (\psi_{r_i} - \psi_i) \\
 \dot{\theta}_i &= \frac{1}{\tau_\theta} (\theta_{r_i} - \theta_i) \\
 \dot{V}_i &= \frac{1}{\tau_v} (V_{r_i} - V_i)
 \end{aligned} \tag{2.73}$$

where x_i is the down-range displacement, y_i is the cross range displacement, z_i is the altitude, V_i is ground speed, θ_i is the Euler pitch angle (flight path angle), ψ_i is the

Euler yaw angle (heading angle), ϕ_i is the Euler roll angle (bank angle), g is the gravity constant, ψ_{r_i} , θ_{r_i} and V_{r_i} are the command values of heading angle, theta angle and velocity of aircraft, τ_ψ , τ_θ and τ_v are the time constant of the controlled system. In here, the system states can be selected as $X_i = [x_i, y_i, z_i, \psi_i, \theta_i, v_i]^T$, $i = 1, 2, \dots, n$ and the control input vector $U_i = [\psi_{r_i}, \theta_{r_i}, V_{r_i}]^T$, respectively.

The nonlinear UAV model can be transformed into a linear time-invariant form by feedback linearization. And, to linearise the i -th UAV nonlinear dynamic, the exogenous signal $u_i = [u_{x_i} \ u_{y_i} \ u_{z_i}]^T \in R^3$, which is related to actual control variables $U_i = [\psi_{r_i}, \theta_{r_i}, V_{r_i}]^T$, may be created as follows;

$$\begin{aligned}\psi_{r_i} &= \tau_\psi(-u_{x_i} \sin \psi_i + u_{y_i} \cos \psi_i) / (v_i \cos \theta_i) + \psi_i \\ \theta_{r_i} &= \tau_\theta(-u_{x_i} \cos \psi_i \sin \theta_i - u_{y_i} \sin \psi_i \sin \theta_i + u_{z_i} \cos \theta_i) / v_i + \theta_i \\ V_{r_i} &= \tau_v(u_{x_i} \cos \psi_i \cos \theta_i + u_{y_i} \sin \psi_i \cos \theta_i + u_{z_i} \sin \theta_i) + V_i\end{aligned}\quad (2.74)$$

Then, the double integrator model can be obtained by differentiating x_i , y_i , and z_i with respect to time twice and substitute of $\dot{\psi}_i$, $\dot{\theta}_i$ and $\dot{V}_i$. Let define $p_i = [x_i \ y_i \ z_i]^T \in R^3$ then the double integrator model;

$$\begin{aligned}\dot{p}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= u_i(t)\end{aligned}\quad (2.75)$$

where, u_{x_i} , u_{y_i} and u_{z_i} are the new control variables in double-integrator point-mass model. Therefore, the state (A_i) and input (B_i) matrices of each UAV i are written as;

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\quad (2.76)$$

2.5.3 Modeling approach of multi-agent systems

The cooperative control of a multi-agent system problem may be decomposed into four sub-components [36]. This decomposition gives a unified four-component structure to analyze, design, and synthesize the multi-agent systems. These components include node dynamics, information flow topology, formation geometry, and distributed controllers.

- **Node Dynamics:** The node dynamic is basically describe the behavior of each node which means the mathematical model of node. The node dynamic of each vehicle in the platooning system is given in Eq. (2.71), and each UAV in the formation is given in Eq. (2.75). Besides node dynamics, homogeneity is crucial in analyzing

the whole system. A multi-agent system is said to be *homogeneous* if all the node dynamics are thought identical; otherwise, it is called *heterogeneous*. This study assumes that all the node dynamics in multi-agent systems are homogeneous.

The structure of node dynamics in a vehicular system is defined as follows: A platoon of connected vehicles indexed by 0 and 1,2,...,N, respectively. So, the vehicular platooning system may be considered a collection of nodes representing the longitudinal dynamic of each vehicle in the platoon.

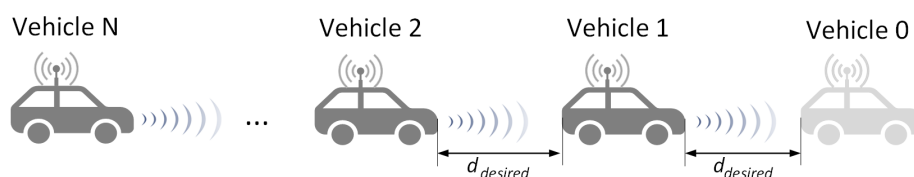


Figure 2.5 : Vehicle platooning system with a virtual leader

A similar structure may be evaluated for the formation of an aircraft system. In the formation, one leader aircraft can be virtual or not, and n followers. According to the route planner, the leader is maneuvered to follow the desired trajectory, and each follower aircraft is instructed to monitor its intended position. One of the exemplary representative formation structures is given in the figure.

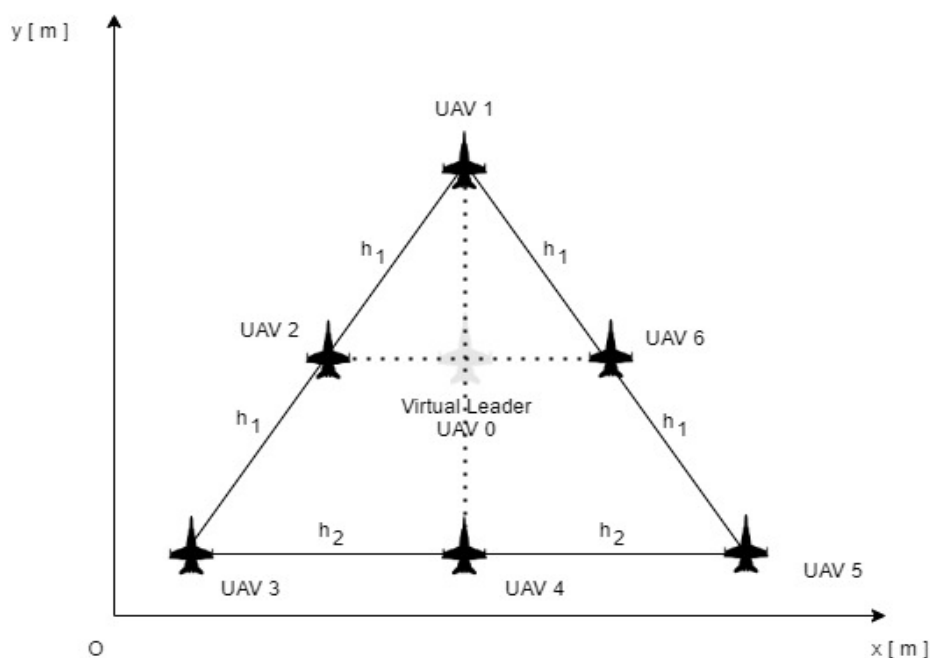


Figure 2.6 : Vehicle platooning system with a virtual leader

Remark 2.1: It is noted that the leader vehicle moves at a constant velocity, since it is necessary to achieve the goal [85]. Therefore, the dynamic of the leader vehicle

(it is thought as a virtual leader) is formulated as follows;

$$\begin{aligned}\dot{p}_0(t) &= v_0(t) \\ \dot{v}_0(t) &= 0\end{aligned}\tag{2.77}$$

- **Information Flow Topologies:**

The information flow topology basically describes the data flow among the nodes. So, it has a significant impact on the stability of multi-agent system. According to time, the information flow topologies may be divided into two types which are fixed and switched. In fixed topology, the data flow is constant; however, it is time-varying for switched topology. Moreover, the information flow topology may be categorized as *unidirectional (directed)* and *bidirectional (undirected)*. In unidirectional flow topology, i^{th} node (or agent) gets the information from its predecessor nodes and sends the reference to the follower nodes, but not get any feedback from the followers. However, in bidirectional flow topology, i^{th} nodes may get information both from its predecessor and follower nodes.

Typical information flow topologies used in the multi-agent systems are shown in Fig. 2.7.

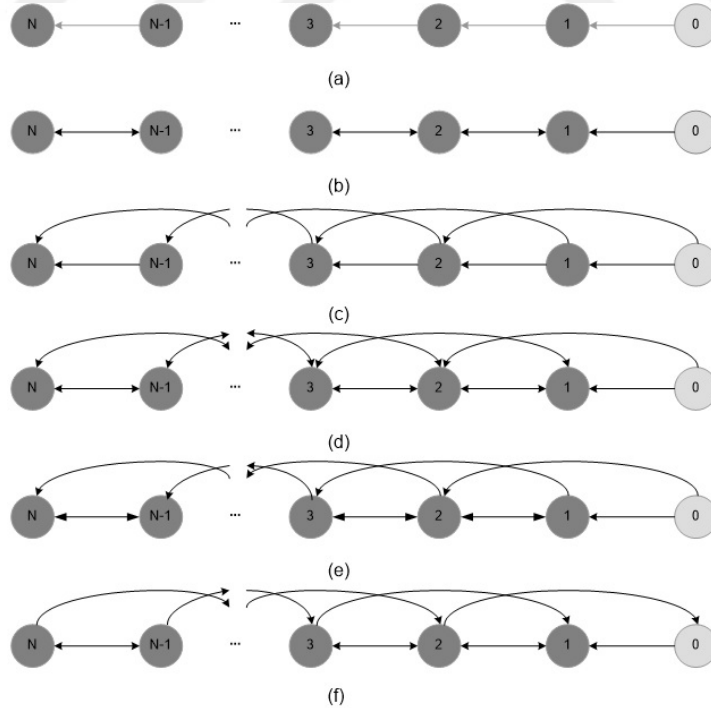


Figure 2.7 : Platoon information flow topologies [2]

They are called as (a) predecessor following (PF), (b) bidirectional predecessor following (BPF), (c) two predecessor following (TPF), (d) two bidirectional

predecessor following (TBPF), (e) two predecessor single following (TPSF), (f) single predecessor two following (SPTF) ([2]). Even if these topologies primarily apply to the vehicular platooning system, they are also used in the formation flight of unmanned aerial systems with minor modifications.

As shown in Fig. 2.7, (a) and (c) are unidirectional, besides the others are bidirectional type flow topologies. For unidirectional topology, if one or more of the following vehicles are exposed to any disturbances (braking/stopping condition or communication loss), the platoon may split and the main goal of the system fails. The unidirectional topologies can significantly simplify the theoretical analysis of platoon control, while the bidirectional types improve safety but increase the computational complexity. The dynamic effect of information flow topology is added to the whole system by using the *Adjacency Matrix*, *Pinning Matrix*, *Laplacian Matrix* which are defined in Section 2.1. The information topology matrices of the flows given in 2.7 is defined below to understand the content of the study in detail.

- (o) *Pinning Matrices*: All the pinning matrices of the given topologies are diagonal due to its natural definition. The Pinning matrix of PF (a), BPF (b) and SPTF (f) is equals since just 1st vehicle takes information from the leader vehicle.

$$\mathcal{P}_{PF} = \mathcal{P}_{BPF} = \mathcal{P}_{SPTF} = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \quad (2.78)$$

And, the rest of three topology has the same pinning matrix structure given below;

$$\mathcal{P}_{BPF} = \mathcal{P}_{TPF} = \mathcal{P}_{TPSF} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \quad (2.79)$$

- (o) *Adjacency Matrices and Laplacian Matrices*: It is known that, if the topology is unidirectional, then the adjacency matrix (and Laplacian) is always lower-triangular (a, c). Besides, if the vehicle has the same number of predecessors and followers, its adjacency matrix (and Laplacian) is always symmetric (b, d). The adjacency and Laplacian matrices of topologies are given as below;

$$\mathcal{A}_{PF} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}, \mathcal{L}_{PF} = \begin{bmatrix} 0 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix} \quad (2.80)$$

$$\mathcal{A}_{BPF} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}, \mathcal{L}_{BPF} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 1 \end{bmatrix} \quad (2.81)$$

$$\mathcal{A}_{TPF} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 1 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & 0 \end{bmatrix}, \mathcal{L}_{TPF} = \begin{bmatrix} 0 & & & & \\ -1 & 1 & & & \\ -1 & -1 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -1 & 2 \end{bmatrix} \quad (2.82)$$

$$\mathcal{A}_{TBPF} = \begin{bmatrix} 0 & 1 & 1 & & & \\ 1 & 0 & 1 & 1 & & \\ 1 & 1 & 0 & 1 & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & 1 & 1 & 0 & 1 & 1 \\ & & & 1 & 1 & 0 & 1 \\ & & & & 1 & 1 & 0 \end{bmatrix}, \quad (2.83)$$

$$\mathcal{L}_{TBPF} = \begin{bmatrix} 2 & -1 & -1 & & & \\ -1 & 3 & -1 & -1 & & \\ -1 & -1 & 4 & -1 & -1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & -1 & -1 & 4 & -1 & -1 \\ & & & -1 & -1 & 3 & -1 \\ & & & & -1 & -1 & 2 \end{bmatrix}$$

$$\mathcal{A}_{TPSF} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ 1 & 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots & \ddots \\ & & 1 & 1 & 0 & 1 \\ & & & 1 & 1 & 0 \end{bmatrix}, \quad (2.84)$$

$$\mathcal{L}_{TPSF} = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ -1 & -1 & 3 & -1 & \\ & \ddots & \ddots & \ddots & \ddots \\ & & -1 & -1 & 3 & -1 \\ & & & -1 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned}
\mathcal{A}_{SPTF} &= \begin{bmatrix} 0 & 1 & 1 & & \\ 1 & 0 & 1 & \ddots & \\ & 1 & 0 & \ddots & 1 \\ & & 1 & \ddots & 1 & 1 \\ & & & \ddots & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}, \\
\mathcal{L}_{SPTF} &= \begin{bmatrix} 2 & -1 & -1 & & \\ -1 & 3 & -1 & \ddots & \\ & -1 & 3 & \ddots & -1 \\ & & -1 & \ddots & -1 & -1 \\ & & & \ddots & 3 & -1 \\ & & & & -1 & 1 \end{bmatrix}
\end{aligned} \tag{2.85}$$

It should be noted that the topology used over the study is fixed which means all the topology-related matrices are not time-varying.

- **Formation Geometry:**

The formation between vehicles is defined with desired inter-agent distances which named as *range policy*. There is extensive literature on formation geometry where different range policies have been applied. Indeed, all range policies have distinct goals depending on the available sensors and communication techniques. The most prevalent range policies in platooning systems are *Constant Distance Policy* and *Constant Time Headway Policy*. Still, most of the range policies used in platooning systems found in the literature are summarized in the following.g [86].

- (a) *Constant Policy*: It was used in the first platooning applications. The main purpose of this range policy is to keep the distance of two consecutive vehicles constant. So, the desired reference range between i^{th} and $i - 1^{th}$ vehicles is defined as follows;

$$d_{i,i-1} = d_0, \quad i = 1, 2, \dots, N \tag{2.86}$$

where d_0 is a fixed gap. Therefore, if d_0 is short enough, the fuel consumption is reduced due to aerodynamic drag reduction. By doing so, a very high traffic capacity can be achieved.

- (b) *Constant Time Headway Policy*: It shows a dynamic behavior that is the most similar to the human driver. The desired inter-vehicle distance varies

with vehicle's velocity, but it has limits on achievable traffic capacity. It is formulated as follows;

$$d_{i,i-1} = h_i v_i + d_0, \quad i = 1, 2, \dots, N, \quad h_i > 0, d_0 > 0 \quad (2.87)$$

where h_i is the time headway and d_0 is the standstill gap. This range policy is considerably preferred for ACC/CACC systems, and may also be applied to the platooning control systems.

- (c) *Safety Distance Policy*: In such a range policy, the distance gap is derived from an estimate of the stopping distance required in the worst braking scenario. It is formulated as;

$$d_{i,i-1} = h_1(v_i^2 - v_{i-1}^2) + h_2 v_i + h_3, \quad i = 1, 2, \dots, N, \quad h_1, h_2, h_3 > 0 \quad (2.88)$$

where h_1 , h_2 and h_3 denote three positive constant, which may be calculated from the human reaction time, vehicle full acceleration, deceleration time and maximum allowable jerk [87].

- (d) *Quadratic Policy* : It is generally used for a particular case: the follower vehicle has less deceleration capabilities than the leading one, and even higher inter distances must be kept to avoid collisions. It proposed to maintain a spacing gap that is a quadratic function of the speed as follows [88];

$$d_{i,i-1} = h_1 + h_2 v_i + h_3 v_i^2, \quad i = 1, 2, \dots, N, \quad h_1, h_2, h_3 > 0 \quad (2.89)$$

where h_1 , h_2 and h_3 denote three positive constant.

- (e) *Adaptive Policy*: This policy is proposed for the platooning system with multiple predecessor vehicles. It is basically used the advantages of classical leader-predecessor following and predecessor following architectures are combined without their disadvantages. The proposed range policy is summarized as follows;

$$D_{i,0}(t) = D_{i-1,0}(t) + D_{i,i-1}(t), \quad i = 1, 2, \dots, N \quad (2.90)$$

Here, the inter-vehicle distance is designed between vehicle i and vehicle 0 (leader vehicle). $D_{i,i-1}$ is the free choose and $D_{i-1,0}(t)$ is the spacing policy of the preceding vehicle with respect to leader but unknown. So, this term is

named as *virtual spacing policy* and it reflect the behaviour of the preceding platoon and is computed on the board of the vehicle i based on the available measurements [89], [47].

- (f) *Delay-based Policy*: This policy ensures that all vehicles in the platoon track the same velocity profile in the spatial domain, which offers advantages as road properties such as hills, bends, or road speed limits are specified in this domain. So the delay-based spacing policy is defined for the i^{th} vehicle in the platoon;

$$d_{ref,i}(t) = s_{i-1}(t - \Delta t) \quad i = 1, 2, \dots, N, \quad \Delta t > 0 \quad (2.91)$$

In this spacing policy, the vehicle tracks the time-delay version of the trajectory of its predecessor. When this spacing policy is perfectly tracked, each vehicle in the platoon achieves the same velocity on the same point on the road [90], [46].

- (g) *Variable Time Headway Policy*: The main focus of this range policy is to improve ACC's ability and adaptability to coordinate various goals in a complex and changing driving environment, especially the safety under deceleration. It may be evaluated as an extended version constant time headway policy. The variable time headway policy is formulated as follows [91], [92];

$$d_{ref,f}(t) = v_f(t_0 - c_v(v_l - v_f) - c_a a_l) + d_0 \quad (2.92)$$

where v_f is the speed of ACC equipped vehicle, v_l and a_l are the speed and acceleration of preceding vehicle, d_0 is the minimum safe distance, c_v and c_a are the weight coefficient of relative velocity and acceleration, respectively.

- (h) *Nonlinear Policy*: Besides the policies summarized above, nonlinear policies have gained attention in recent years due to their more flexible strategy. In this policy, the desired inter-vehicle distance is a nonlinear function of vehicles velocity.

In this study, constant policy and constant time headway policy have been considered since they are the most preferred range policies to the platooning systems and, more convenient to the defined problem.

In terms of unmanned aerial vehicles, the formation can be divided into leader following (leader-wingman), virtual leader (virtual structure), behavioral approaches [93]. Leader-follower approaches set a specific vehicle as a leader and the rest of vehicles as followers, which chase the leader's path and maintain formation [94]. This approach has been utilized extensively in various applications due to its simplicity and viability. The disadvantage of the leader-follower method is that the leader is less robust than the followers. Thus, when the leader fails, the entire formation will become headless, resulting in formation failure. The virtual structure method defines the formation shape as a virtual rigid body. The desired position of other agents is defined with the help of the virtual rigid body. The motion trajectory of a virtual structure is the reference trajectory of formation. The formation can be formed as long as the agents track the corresponding reference points on the virtual rigid body. There is no actual leader in the virtual structure method, which reduces the impact of the single point failure problem of the leader-follower formation method [95]. The behaviour-based method refers to a set of basic behaviors of agents that are defined in advance, such as avoiding collision, avoiding obstacles, searching targets, formation keeping, trajectory tracking, etc. Then the control amount of the agents. can be obtained by calculating the weighted average of these basic behaviors. The difficulty of this method is the design of different basic behaviors and the determination of the weights of basic behaviors [96].

The leader-follower, behavior-based, and virtual structure methods have their advantages and disadvantages; however, this research examines the virtual structure-based formation for application in formation control of unmanned aerial vehicles.

- ***Distributed Controller:***

The distributed controller is basically a kind of feedback controller which uses the neighbouring information to achieve a certain objective. The neighbouring information is determined in accordance with the information flow topology which is used in communication. So the controller objective that provides vehicles to maintain the desired distance while travelling at the same velocity is formulated for

i^{th} vehicle as follows;

$$\begin{aligned}\lim_{t \rightarrow \infty} \|p_i(t) - p_0(t) + d_{i,0}\| &= 0 \\ \lim_{t \rightarrow \infty} \|v_i(t) - v_0(t)\| &= 0 \\ \lim_{t \rightarrow \infty} \|a_i(t) - a_0(t)\| &= 0\end{aligned}\tag{2.93}$$

The controller problem is defined as virtual leader-follower approach, which means a virtual leader is assigned to the multi-agent system and remaining vehicles are the followers. In this approach, the leader follows the desired trajectory while the followers track the position of the leader to achieve the objective which is defined in Eq. (2.93). The advantage of this method is the reduced tracking errors and can be analysed using standard control methods. Another benefit is that only the leader is responsible for planning trajectories and followers must follow the coordinates of the leader. In terms of disadvantage, a leader's fault can penalise the whole formation.

These all four sub-components which are node dynamic, formation geometry, information flow topology, and distributed controller affect the closed-loop system matrix of multi-agent dynamics; this separates the problem from the classical controller design. However, the last two components have a more dominant influence on the stability of the system. Basically, the definition of internal stability of a multi-agent system may be clarified as follows;

Definition 4.1 *Multi-agent system stability:* A multi-agent system with linear time-invariant dynamics is stable if the closed-loop system has eigenvalues with strictly negative real part, which means that in the open left half of the complex plane [43].

Therefore, the stability of a multi-agent systems depends not only on the distributed controller but also on the information flow topologies. So, the structure of $\mathcal{L}_p$ widely penetrates the stability posture of the system, especially when the time delay is ignored. Thus, the eigenvalues of $\mathcal{L}_p$ should be investigated first. It is know that $\mathcal{L}_p = \mathcal{L} + \mathcal{P}$ is a square real matrix. But, the eigenvalues of $\mathcal{L}_p$ can be real or complex as the selected information flow topology. It is known from linear algebra that a real-valued symmetric matrix or a triangular matrix has real eigenvalues. Except for these, it is not easy to determine whether the eigenvalues are real or complex for an arbitrary real matrix. Hence the delay-free stability analysis is investigated in terms of the real and

complex eigenvalues of $\mathcal{L}_p$ separately. For the first four information flow topologies which is given in 2.7, the eigenvalues of $\mathcal{L}_p$ is real since the structure of $\mathcal{L}_p$ is either symmetric or triangular.

In the following sections of the dissertation, the multi-agent control problems are converted to the general form of single-delay LTI-RTDS/NTDS and multiple-delay LTI-RTDS structures. A general state-feedback controllers and PD-PID type controllers are evaluated in the following sections. For all defined problems below, delay-free stability conditions are obtained first, then the time-delayed stability conditions are derived. And the results are supported with the simulations which are conducted with MATLAB/Simulink environment.





3. CONTROL OF MAS WITH RETARDED TIME-DELAY

3.1 Purpose

Time delay is unavoidable in multi-agent systems due to each vehicle (or agent) has its actuator and sensors, and communication channels. Specifically, parasitic delays and lags originate from the natural characteristics of actuators and sensors in the mechanical and control systems of the vehicle. With the increase of the level of vehicle autonomy, the number of time delay sources rises.

This section presents a stability analysis of multi-agent systems without and with time delay in detail. The controller structure is chosen as a *state-feedback controller*. The system presents a linear time-invariant system when the time delay is ignored and the stability analysis may be achieved using the classical stability analysis method like Routh-Hurwitz. However, in the presence of time delay, these methods are no longer viable because the characteristic equation is not finite-dimensional. Thus, in this study, the stability analysis is performed using the CTCR paradigm, and stability regions are obtained in the delay space. One of the significant contributions of this study is the implementation of CTCR methodology with a novel approach to the problem. In the single-time delay case, one approach presents that the characteristic equation of this type of system may be transformed easily into a product of a set of factors that have the same degree as the dynamics of the individual vehicles [26]. Although this method reduces the problem's complexity, applying it to the multi-agent system whose characteristic equation can not decompose into factors is impossible. Other approaches are using the Rekasius transformation to achieve a finite-dimensional polynomial or creating an auxiliary characteristic equation (ACE) for the system by extended Kronecker summation method, which is also utilized for multiple time-delay systems. However, these approaches require significant computational effort, mainly when the number of vehicles is high. The solution to this computational problem may also be achieved by using matrix-based methods with better performance and

satisfactory numerical stability, especially in the case of large sets of polynomials. In this work, the Bezout matrix, which is a particular square matrix associated with two polynomials is used for the utilization of the CTCR methodology to the multi-agent system.

In the rest of this section, all these methods are investigated in the problems of vehicular platooning and the formation of unmanned aerial systems. Firstly the stability analysis is conducted for delay-free cases, which is named nominal case, then for time delay case.

3.2 Leader Agent-Based Constant Headway Policy

The following constant time headway policy which defined in terms of leader vehicle velocity is proposed as the formation geometry of the system as follows;

$$d_{i,0}(t) = lh_{i,0}v_0 + ld_{i,0}, \quad i = 1, 2, \dots, N, \quad h_{i,0} > 0, d_{i,0} > 0 \quad (3.1)$$

where v_0 is the leader vehicle velocity, l is the number of vehicles between i^{th} and leader vehicle, $h_{i,0}$ is the time headway and $d_{i,0}$ is the standstill gap. The reference leader motion is defined in Eq. (2.77). The leader position and velocity are thought of as virtual reference signals which are defined by the motion planner and they have not a real dynamic relationship between them. The advantage of the proposed policy in Eq. (3.1) is that it simplifies the problem while taking some advantage of the general time headway policy given in Eq. (2.87).

3.2.1 Stability analysis for nominal case

In this sub-chapter, all the analyses are conducted with the help of the vehicular platooning system model since it has a third-order model first. Then, these analyses are evaluated for the formation control of the unmanned aerial vehicle system model.

To achieve consensus on the goal in Eq. (2.93), the following distributed control strategy is proposed for the platooning system.

$$\begin{aligned}
u_i(t) = & - \sum_{j=1}^N a_{ij} \left[k_p \left(p_i(t) - p_j(t) + d_{i,j} \right) + k_v \left(v_i(t) - v_j(t) \right) + k_a \left(a_i(t) \right. \right. \\
& \left. \left. - a_j(t) \right) \right] - p_{ii} \left[k_p \left(p_i(t) - p_0(t) + d_{i,0} \right) + k_v \left(v_i(t) - v_0(t) \right) + \right. \\
& \left. k_a \left(a_i(t) - a_0(t) \right) \right]
\end{aligned} \tag{3.2}$$

where $p_i(t)$, $v_i(t)$ and $a_i(t)$ are the absolute position, velocity and acceleration of i^{th} vehicle, respectively. k_p , k_v and k_a are the position, velocity and acceleration control gains. a_{ij} and p_{ii} is the corresponding element of Adjacency matrix and Pinning matrix which are defined in the previous chapter, respectively. N is the number of vehicles in the platoon for $i = 1, 2, \dots, N$, and the leader vehicle is indexed as 0. $d_{i,j}$ and $d_{i,0}$ are the inter-vehicle distance between i^{th} and j^{th} vehicles and i^{th} and 0^{th} vehicles, respectively. The position, velocity, and acceleration tracking errors between i^{th} and 0^{th} vehicle are defined as follows to facilitate the analysis and the rest of the equations are formulated in terms of the tracking errors.

$$\begin{aligned}
e_{p_i}(t) &= p_i(t) - p_0(t) + d_{i,0} \\
e_{v_i}(t) &= v_i(t) - v_0(t) \\
e_{a_i}(t) &= a_i(t) - a_0(t)
\end{aligned} \tag{3.3}$$

Then the dynamic of tracking errors are obtained by using Eq. (2.71), (3.3) and (3.1);

$$\begin{aligned}
\dot{e}_{p_i}(t) &= e_{v_i}(t) \\
\dot{e}_{v_i}(t) &= e_{a_i}(t) \\
\dot{e}_{a_i}(t) &= -\frac{1}{\gamma_i} e_{a_i}(t) + \frac{1}{\gamma_i} u_i(t)
\end{aligned} \tag{3.4}$$

The control signal is written in terms of tracking errors as follows.;

$$\begin{aligned}
u_i(t) = & - \sum_{j=1}^N a_{ij} \left[k_p \left(e_{p_i}(t) - e_{p_j}(t) \right) + k_v \left(e_{v_i}(t) - e_{v_j}(t) \right) + \right. \\
& \left. k_a \left(e_{a_i}(t) - e_{a_j}(t) \right) \right] - p_{ii} \left[k_p e_{p_i}(t) + k_v e_{v_i}(t) + k_a e_{a_i}(t) \right]
\end{aligned} \tag{3.5}$$

The following vector definition is used to facilitate the notation;

$$\begin{aligned}
E(t) &= [e_{p_1}(t) \ e_{v_1}(t) \ e_{a_1}(t) \ e_{p_2}(t) \ e_{v_2}(t) \ e_{a_2}(t) \ \dots \ e_{p_N}(t) \ e_{v_N}(t) \ e_{a_N}(t)]^T \\
U(t) &= [u_1(t) \ u_2(t) \ \dots \ u_N(t)]^T \\
K &= [k_p \ k_v \ k_a]
\end{aligned} \tag{3.6}$$

Finally, if the control signal given in Eq. (3.5) is substituted into Eq. (3.4) and redesigned with the help of Eq. (3.6), the closed-loop state-space model for N vehicle may be acquired as in Eq. (3.7) by means of Kronecker product.

$$\dot{E}(t) = \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK) \right) E(t) \quad (3.7)$$

Hence, the overall closed-loop system matrix of delay-free system is;

$$A_c = (I_N \otimes A) - (\mathcal{L}_p \otimes BK) \quad (3.8)$$

where;

$$I_N \otimes A = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & \mathbf{0}_{3 \times 3} & & \mathbf{0}_{3 \times 3} \\ 0 & 0 & -\frac{1}{\gamma} & & & \\ & \mathbf{0}_{3 \times 3} & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \\ & & & 0 & 0 & -\frac{1}{\gamma} \\ & \dots & & \dots & \ddots & \dots \\ & \mathbf{0}_{3 \times 3} & & \mathbf{0}_{3 \times 3} & & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\gamma} \end{bmatrix} \end{bmatrix},$$

and

$$\mathcal{L}_p \otimes BK = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{k_p}{\gamma} l_{p11} & \frac{k_v}{\gamma} l_{p11} & \frac{k_a}{\gamma} l_{p11} & \frac{k_p}{\gamma} l_{p12} & \frac{k_v}{\gamma} l_{p12} & \frac{k_a}{\gamma} l_{p12} & & \frac{k_p}{\gamma} l_{p1N} & \frac{k_v}{\gamma} l_{p1N} & \frac{k_a}{\gamma} l_{p1N} \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & & & \\ \frac{k_p}{\gamma} l_{p21} & \frac{k_v}{\gamma} l_{p21} & \frac{k_a}{\gamma} l_{p21} & \frac{k_p}{\gamma} l_{p22} & \frac{k_v}{\gamma} l_{p22} & \frac{k_a}{\gamma} l_{p22} & & \frac{k_p}{\gamma} l_{p2N} & \frac{k_v}{\gamma} l_{p2N} & \frac{k_a}{\gamma} l_{p2N} \\ & \vdots & & & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ \frac{k_p}{\gamma} l_{pN1} & \frac{k_v}{\gamma} l_{pN1} & \frac{k_a}{\gamma} l_{pN1} & \frac{k_p}{\gamma} l_{pN2} & \frac{k_v}{\gamma} l_{pN2} & \frac{k_a}{\gamma} l_{pN2} & & \frac{k_p}{\gamma} l_{pNN} & \frac{k_v}{\gamma} l_{pNN} & \frac{k_a}{\gamma} l_{pNN} \end{bmatrix}$$

where $l_{p(\cdot)}$ is the corresponding element of $\mathcal{L}_p = \mathcal{L} + \mathcal{P}$ matrix.

Hence, the stability analysis of the given problem may be investigated in terms of the structure of L_p matrix, which is also explained in [97], [43] and [2] in detail. Here, a short summary is given in order to integrity of the topic.

Theorem 3.1 : [5] Consider the homogeneous multi-agent system whose dynamic model given in Eq. (3.7). Let the information topology matrix $\mathcal{L}_p$ be symmetric or triangular and have only real eigenvalues including multiplicity; $\lambda_i, i = 1, 2, \dots, N$, Then, the multi-agent system is stable if and only if the following N matrices are Hurwitz.

$$A - \lambda_i BK \quad (i = 1, 2, \dots, N) \quad (3.9)$$

Proof: Define the transformation M such that $J_p = M^{-1} \mathcal{L}_p M$ is upper-triangular with the eigenvalues $\lambda_i \quad (i = 1, 2, \dots, N)$ of $\mathcal{L}_p$ on the diagonal. Then the Jordan form of A_c which is given in Eq. (3.8) may be obtained as follows;

$$\begin{aligned} J_{A_c} &= (M^{-1} \otimes I_r) [(I_N \otimes A) - \mathcal{L}_p \otimes BK] (M \otimes I_r) \\ &= (M^{-1} I_N \otimes I_r A) - M^{-1} \mathcal{L}_p \otimes I_r BK (M \otimes I_r) \\ &= (M^{-1} I_N M \otimes I_r A I_r) - M^{-1} \mathcal{L}_p M \otimes I_r BK I_r \\ &= (I_N \otimes A) - (J_p \otimes BK) \\ &= \text{diag}(J_{A_{c_1}}, J_{A_{c_2}}, \dots, J_{A_{c_N}}) \end{aligned} \quad (3.10)$$

where J_{A_c} is the Jordan form of A_c and $J_{A_{c_i}}$ is the Jordan block of A_c . Let define a new state vector as $F(t)$ such that $F(t) = (M^{-1} \otimes I)E(t)$. Then the transformed system is in block triangular form with diagonal blocks of the form

$$\dot{F}_i(t) = (A - \lambda_i BK) F_i(t) \quad (3.11)$$

Hence, stability of Eq. (3.7) and Eq. (3.11) are equivalent, since state space transformation does not change the eigenvalues.

Remark 4.1 : It should be noted that the Theorem 3.1 is valid for all the multi-agent systems with a dynamic as Eq. (3.7) and information topology matrix with real-eigenvalues.

Therefore, the following lemma may be written for the platooning system whose $\mathcal{L}_p$ matrix has real eigenvalues.

Lemma 3.1 : [97], [98] Consider the homogeneous platooning system using a leader velocity-dependent CTH policy as in Eq. (3.1) and a linear distributed controller as in Eq. (3.2) with the closed-loop dynamics in Eq. (3.7). If the information topology matrix $\mathcal{L}_p$ has only real eigenvalue, then the stability of the platoon is guaranteed if

and only if the following conditions are held.

$$\begin{cases} k_p > 0 \\ k_v > \frac{k_p \gamma}{\lambda_i k_a + 1} \\ k_a > -\frac{1}{\lambda_i} \end{cases} \quad (3.12)$$

Proof: The characteristic equation of the system given in Eq. (3.7) may also be written with the help of Theorem 3.1 as follows;

$$\begin{aligned} Q(s) &= \det(sI_{3N} - A_c) = \prod_{i=1}^N \det(sI_3 - A + \lambda_i BK) \\ &= \prod_{i=1}^N s^3 + \frac{\lambda_i k_a + 1}{\gamma} s^2 + \frac{\lambda_i k_v}{\gamma} s + \frac{\lambda_i k_p}{\gamma}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (3.13)$$

Therefore the stability of Eq. (3.13) is examined using the Routh-Hurwitz stability criterion which constituted with the following Routh table;

$$\begin{array}{ccc} s^3 & 1 & \frac{\lambda_i k_v}{\gamma} \\ s^2 & \frac{\lambda_i k_a + 1}{\gamma} & \frac{\lambda_i k_p}{\gamma} \\ s^1 & \frac{\lambda_i (k_v - k_p \gamma + k_a k_v \lambda)}{\gamma (\lambda_i k_a + 1)} & \\ s^0 & \frac{\lambda_i k_p}{\gamma} & \end{array} \quad (3.14)$$

Given the fact that $\gamma > 0$ and $\lambda_i > 0, i = 1, 2, \dots, N$, Eq. (3.7) is asymptotically stable if the and only if Eq. (3.12) is satisfied. Thus the proof is completed.

Simulation Example 3.1: The following simulation study is conducted with the help of MATLAB/Simulink environment to demonstrate the results of Lemma 3.1. Consider a platooning with 1 leader and 5 followers. The TPF topology is selected which shown in Figure 3.1 to exchange information between vehicles. The $\mathcal{L}_p$ matrix of the system is also given in Eq. (3.15) which is a lower triangular form.

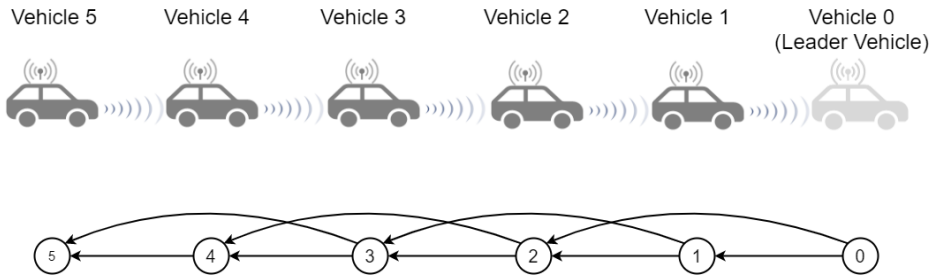


Figure 3.1 : Layout of a platoon for Example 3.1.

$$\mathcal{L}_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \quad (3.15)$$

The eigenvalues of $\mathcal{L}_p$ matrix are found as follows;

$$\mathcal{L}_p : \lambda_1 = 1, \quad \lambda_{2,3,4,5} = 2$$

Besides, the platooning system is considered homogeneous, which means that the power-train time constant is equal for each vehicle. So the system parameters used in simulation studies are given in Table 3.1.

Table 3.1 : System parameters used in simulation studies

Parameter	Description	Value	Unit
N	Number of following vehicles	5	-
γ	Power-train time constant	0.5	[s]
d	Safe distance	10	[m]
v_0	Velocity of virtual leader	20	[m/s]
h	Time headway	0.5	[s]

The characteristic equation of the system may be written by using Eq. (3.13) as follows;

$$\begin{aligned} Q(s) &= \det(sI_3 - A + \lambda_i BK) \\ &= (s^3 + 2(k_a + 1)s^2 + 2k_v s + 2k_p)(s^3 + 2(2k_a + 1)s^2 + 4k_v s + 4k_p)^4 \end{aligned} \quad (3.16)$$

Since the characteristic equation is decomposed into factors, the Routh Table method may be easily applied to each factor. And the stability boundary of the whole system is obtained by intersecting the stability region of each factor. Therefore, if the system parameters are selected as in Table 3.1, the conditions of controller gains are founded with the help of Eq. (3.12) as follows;

$$K := \begin{cases} k_p > 0 \\ k_v > \frac{0.5k_p}{k_a+1} \\ k_a > -0.5 \end{cases} \quad (3.17)$$

The speed profile of the leading vehicle is given in Eq. (3.18).

$$v_0 = \begin{cases} 20 \text{ m/s} & \text{if } t \leq 20 \\ 1.5t-10 \text{ m/s} & \text{if } 20 < t \leq 40 \\ 60 \text{ m/s} & \text{if } 40 < t \leq 70 \\ 120-t \text{ m/s} & \text{if } 70 < t \leq 80 \\ 40 \text{ m/s} & \text{if } 80 < t < 80 \end{cases} \quad (3.18)$$

Then, positions, position tracking errors, velocities, and velocity tracking errors for the simulation parameters given in Table 3.1 are shown in Figure 3.2 and 3.3, for the control parameters ($k_p = 2.4$, $k_v = 4$ and $k_a = 2$) in the stable region, respectively.

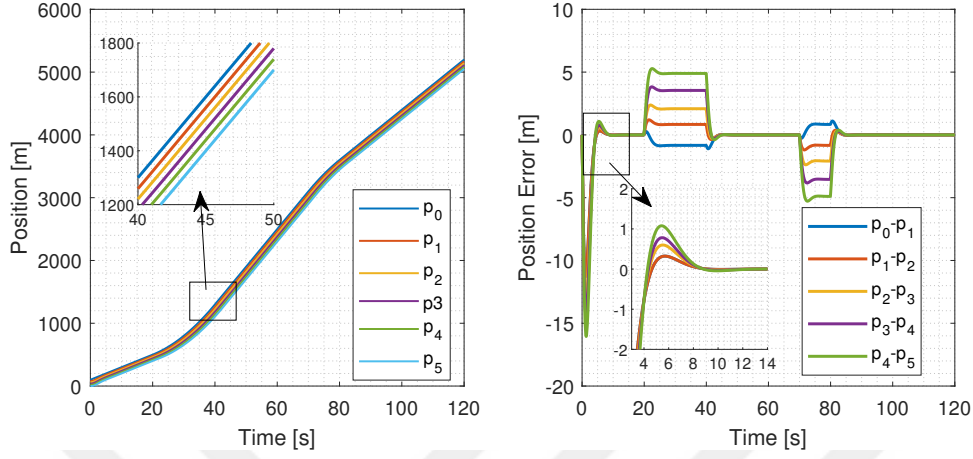


Figure 3.2 : Positions and position errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$.

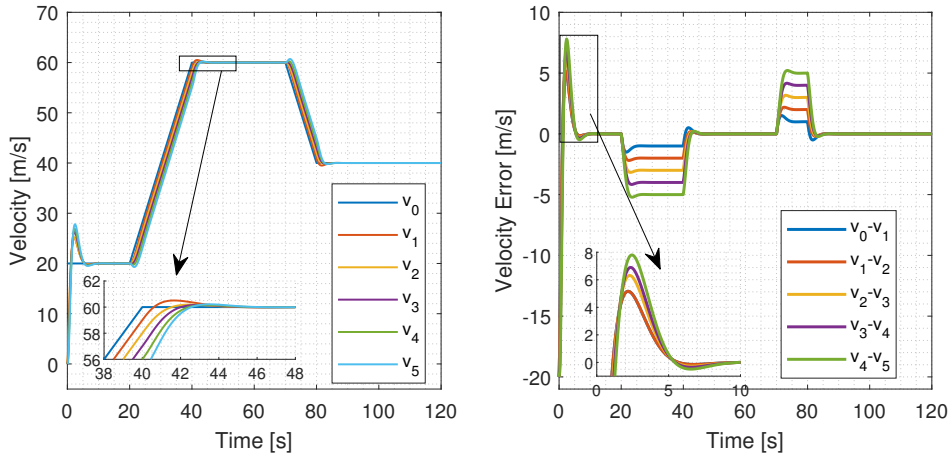


Figure 3.3 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$.

In Figure 3.4, the velocity controller parameter is selected on the stability boundary as $k_v = 0.4$, and the system displays the marginal stability behavior.

Simulation Example 3.2: The following simulation study is conducted with the help of MATLAB/Simulink environment to demonstrate the results of Lemma 3.1. Consider a platooning with 1 leader and 5 followers. The TBPf topology is selected which shown in Figure 3.5 to exchange information between vehicles. The $\mathcal{L}_p$ matrix of the system is also given in Eq. (3.15) which is a symmetric form.

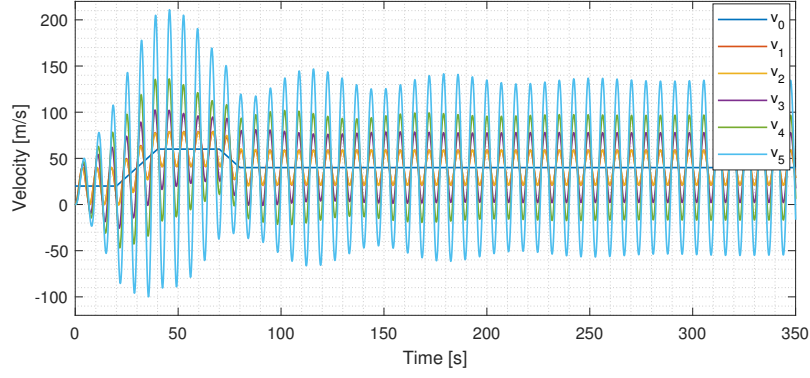


Figure 3.4 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 0.4$ and $k_a = 2$.

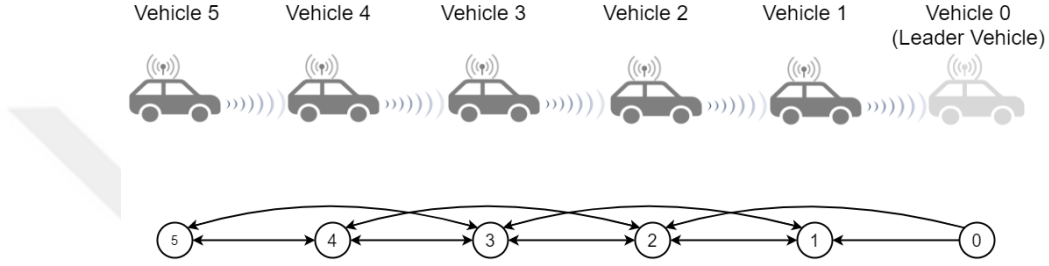


Figure 3.5 : Information flow topologies for Example 3.2.

$$\mathcal{L}_p = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \quad (3.19)$$

The eigenvalues of $\mathcal{L}_p$ matrices are found as follows;

$$\mathcal{L}_p : \lambda_1 = 0.294, \quad \lambda_2 = 2.132, \quad \lambda_3 = 3.39, \quad \lambda_4 = 5, \quad \lambda_5 = 5.184$$

The system parameters in Table 3.1 are used and the speed profile of the leader vehicle is considered as in Eq. (3.18). Then the characteristic equation of the system is written as in Eq. (3.20).

$$\begin{aligned} Q(s) &= \det(sI_3 - A + \lambda_i BK) \\ &= (s^3 + (0.587k_a + 2)s^2 + 0.587k_v s + 0.587k_p) \\ &\quad (s^3 + (4.264k_a + 2)s^2 + 4.264k_v s + 4.264k_p) \\ &\quad (s^3 + (6.78k_a + 2)s^2 + 6.78k_v s + 6.78k_p) \\ &\quad (s^3 + (10k_a + 2)s^2 + 10k_v s + 10k_p) \\ &\quad (s^3 + (10.368k_a + 2)s^2 + 10.368k_v s + 10.368k_p) \end{aligned} \quad (3.20)$$

The characteristic equation is decomposed into factors due to the symmetric structure of $\mathcal{L}_p$ matrix. The intersection of the stability of all factors gives the stability region

of $Q(s)$ which is found as Eq. (3.21).

$$K := \begin{cases} k_p > 0 \\ k_v > \frac{(1+0.2935k_a)k_p}{2+(1.174+0.172k_a)k_a} \\ k_a > 0 \end{cases} \quad (3.21)$$

The results are shown in Figure 3.6, 3.7 and 3.8.

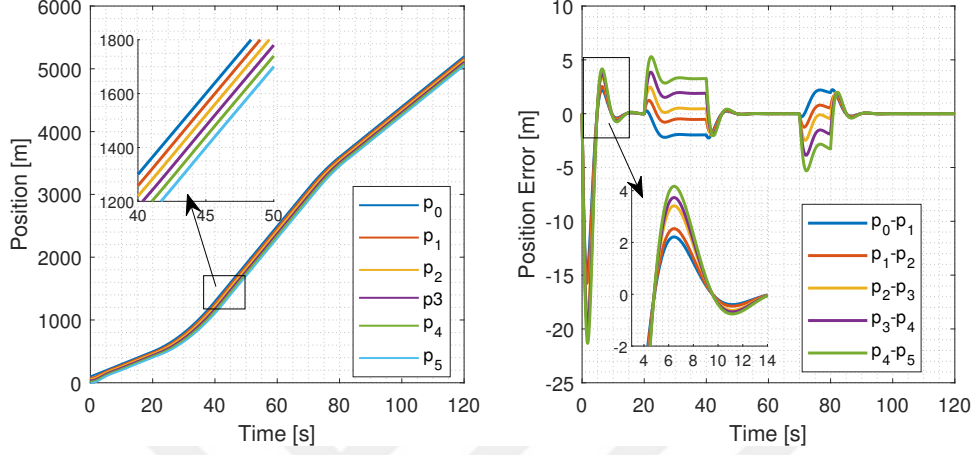


Figure 3.6 : Positions and position errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$.

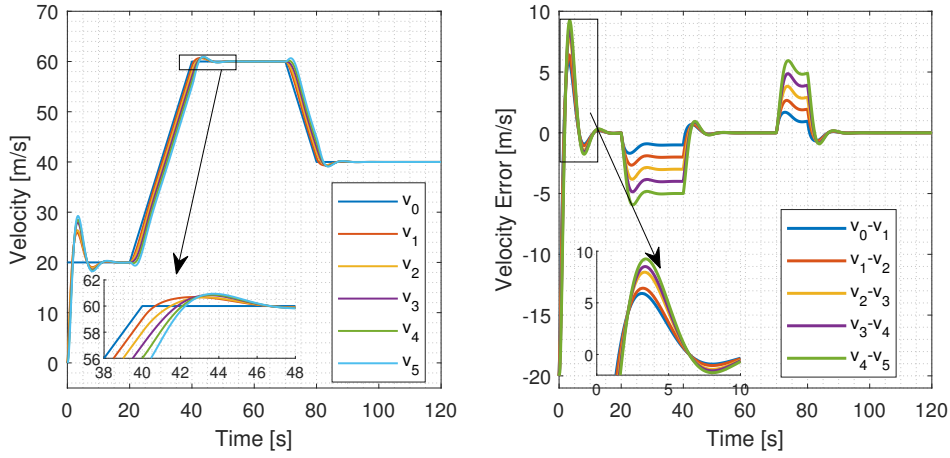


Figure 3.7 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 4$ and $k_a = 2$.

Figure 3.6 and 3.7 show the positions, position tracking errors, velocities, and velocity tracking errors for the simulation parameters given in Table 3.1, respectively. The controller gains are selected in the stable region as $k_p = 2.4$, $k_v = 4$ and $k_a = 2$.

In Figure 3.4, the velocity controller parameter is selected on the stability boundary as $k_v = 0.756$, and the system displays the marginal stability behavior as expected.

Moreover, if the information topology matrix $\mathcal{L}_p$ has also complex eigenvalue, then the Theorem 3.1 may be expanded as follows;

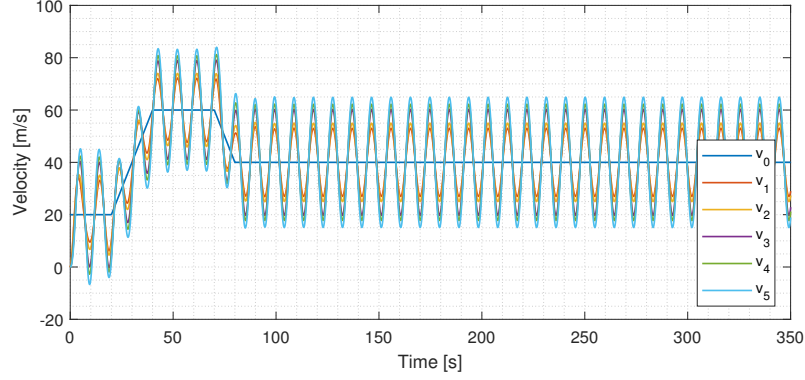


Figure 3.8 : Velocity and velocity errors for $k_p = 2.4$, $k_v = 0.756$ and $k_a = 2$.

Theorem 3.2: [97], [2] Consider the homogeneous multi-agent system whose dynamic model given in Eq. (3.7). Let the information topology matrix $\mathcal{L}_p$ is an arbitrary square matrix, $\lambda_i, i = 1, 2, \dots, q$ are the real eigenvalues of $\mathcal{L}_p$ including multiplicity, and $\lambda_i = \sigma_i + j\omega_i, \lambda_i^* = \sigma_i - j\omega_i, i = q+1, q+2, \dots, q+r, (q+r = N)$ are the complex conjugate eigenvalues of $\mathcal{L}_p$ including multiplicity. Then, the multi-agent system is stable if and only if the following matrices are Hurwitz.

$$\begin{cases} A - \lambda_i BK, & i = 1, 2, \dots, q \\ I_2 \otimes A - \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes BK, & i = q+1, \dots, q+r \end{cases} \quad (3.22)$$

Proof: Let define the transformation M such that $J_p = M^{-1} \mathcal{L}_p M, M \in \mathbb{R}$, such that.

$$J_p = \begin{bmatrix} J_i & & & & \\ & \ddots & & & \\ & & J_q & & \\ & & & J_{q+1} & \\ & & & & \ddots \\ & & & & & J_{q+r} \end{bmatrix} \quad (3.23)$$

It is basically seen that the J_p matrix can be divided into two sub-blocks where the first one $J_i, i = 1, 2, \dots, q$ is the Jordan canonical form of $\mathcal{L}_p$ corresponding to the real eigenvalues, which shown as follows;

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad i = 1, 2, \dots, q \quad (3.24)$$

And the second sub-block corresponding to complex eigenvalue of $\mathcal{L}_p$ which have the following form;

$$\begin{cases} J_i = \begin{bmatrix} S_i & I_2 & & \\ & S_i & \ddots & \\ & & \ddots & I_2 \\ & & & S_i \end{bmatrix} \\ S_i = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \end{cases}, \quad i = q+1, \dots, q+r \quad (3.25)$$

where $\lambda_i = \sigma_i + j\omega_i$, $i = q+1, q+2, \dots, q+r$.

Thus far, it is just stated that an arbitrary square matrix is converted into a canonical form without changing eigenvalues. Then, the next step is that rearranging the closed-loop system matrix of $A_c = (I_N \otimes A) - (\mathcal{L}_p \otimes BK)$. A non-singular transformation matrix M is defined to convert the $\mathcal{L}_p$ into Jordan canonical form of J_p as $M^{-1}\mathcal{L}_pM = J_p$. Then, J_{A_c} , the Jordan form of A_c may also be obtained as in Eq. (3.10). So, the closed-loops system matrix A_c is now converted into a block diagonal J_{A_c} matrix. And it can also be divided into tow sub blocks which correspond to the real and complex eigenvalues of $\mathcal{L}_p$ as follows, respectively;

$$\begin{bmatrix} \Lambda_i & BK & & \\ & \Lambda_i & \ddots & \\ & & \ddots & BK \\ & & & \Lambda_i \end{bmatrix} \quad (3.26)$$

where

$$\begin{cases} \Lambda_i = A - \lambda_i BK \\ \lambda_i \in R \end{cases} \quad i = 1, \dots, q+1$$

and

$$\begin{bmatrix} \Gamma_l & I_2 \otimes BK & & \\ & \Gamma_l & \ddots & \\ & & \ddots & I_2 \otimes BK \\ & & & \Gamma_l \end{bmatrix} \quad (3.27)$$

where

$$\begin{cases} \Gamma_i = I_2 \otimes A - \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes BK \\ \lambda_i = \sigma_i + j\omega_i, \end{cases} \quad i = q+1, \dots, q+r$$

Then, the stability property of A_c matrices under the complex eigenvalues of $\mathcal{L}_p$ is equivalent the stability property of Λ_i and Γ_l matrices. So the condition Eq. (3.22) is

hold if and only if

$$\begin{cases} \Lambda_i & \text{is Hurwitz,} & i = 1, 2, \dots, p \\ \Gamma_i & \text{is Hurwitz,} & i = q + 1, \dots, q + r \end{cases} \quad (3.28)$$

Thus the proof is completed.

Remark 4.2: It should be noted that the Theorem 3.2 is valid for all the multi-agent systems with a dynamic as Eq. (3.7) and information topology matrix has both real and complex eigenvalues.

Theorem 3.2 shows us that, the characteristic equation of the system Eq. (3.7) may be written as a product of the characteristic equations of Λ_i and Γ_i matrices shown below;

$$\begin{aligned} Q(s) &= \det(sI_{3N} - A_c) = \det(sI_{3N} - J_{A_c}) \\ &= \det(sI_{3N} - (I_N \otimes A) + (J_i \otimes BK)) \\ &= \prod_{i=1}^q \det(sI_3 - A + \lambda_i BK) \prod_{i=q+1}^r \det(sI_6 - (I_2 \otimes A) + (S_i \otimes BK)) \end{aligned} \quad (3.29)$$

After all, the characteristic equation of the system Eq. (3.7) is now decomposed into a set of third-order and sixth-order systems. But, even if Theorem 3.1 is valid for the first part (left part, third-order part) of Eq. (3.29), by using Routh tabulation method, it is not easy to obtain a stability condition depending on the controller gains for the sixth-order system. The second part is decomposed to its sub-factors, however, all these sub-factors have complex coefficients. And, it is known that all the coefficients of any characteristic equation should be real to apply the Routh Table method. So, a method based on the *Riccatti inequality* may be used to design a state feedback controller K . The following lemma may be written for the platooning system whose $\mathcal{L}_p$ matrix has both real and complex eigenvalues.

Lemma 3.2 : [99] [100] Consider the homogeneous platooning system using a leader velocity-dependent CTH policy as in Eq. (3.1) and a linear distributed controller as in Eq. (3.2) with the closed-loop dynamics in Eq. (3.7). Suppose the information topology matrix $\mathcal{L}_p$ has both only real and complex eigenvalues. Let $\lambda_i, i = 1, 2, \dots, q$ be the real eigenvalues of $\mathcal{L}_p$ including multiplicity, and $\lambda_i = \sigma_i + j\omega_i, \lambda_i^* = \sigma_i - j\omega_i, i = q + 1, q + 2, \dots, q + r$ be the complex conjugate eigenvalues of $\mathcal{L}_p$. Then, if there exist a positive scalar v and positive symmetric matrix $P > 0$, such that

$$0 < v \leq \min_{1 \leq i \leq q+r} \left(\frac{\lambda_i(\mathcal{L}_p) + \lambda_i^*(\mathcal{L}_p)}{2} \right), \quad i = 1, 2, \dots, N \quad (3.30)$$

$$AP + PA^T - vBB^T < 0 \quad (3.31)$$

then the stability of platooning system Eq. (3.7) can be guaranteed, if the controller gain is constructed as;

$$K = \frac{B^T P^{-1}}{2} \quad (3.32)$$

Proof: The information topology matrix $\mathcal{L}_p$ is a diagonally dominant matrix by definition. The magnitude of the diagonal entry is greater than or equal to the sum of the magnitudes of the off-diagonal entries in its row.

$$|l_{p_{ii}}| \geq \sum_{j \neq i}^N |l_{p_{ij}}|, \quad (i = 1, 2, \dots, N) \quad (3.33)$$

where $l_{p_{ij}}$, $1 \leq i, j \leq N$ are the corresponding element of the matrix $\mathcal{L}_p$. It guaranteed that $l_{p_{ii}} > 0$ since the definition of $\mathcal{L}_p$. Due to $\mathcal{L}_p$ is a square matrix, the Gersgorian disk theorem may be applied as follows;

$$\begin{aligned} |z - l_{p_{ii}}| &\leq \sum_{j \neq i}^N |l_{p_{ij}}| \\ l_{p_{ii}} - \sum_{j \neq i}^N |l_{p_{ij}}| &\leq z \leq l_{p_{ii}} + \sum_{j \neq i}^N |l_{p_{ij}}| \end{aligned} \quad (3.34)$$

Then, the left part of second inequality given in Eq. (3.34) may be written with the help of Eq. (3.33) as follows;

$$z \geq l_{p_{ii}} - \sum_{j \neq i}^N |l_{p_{ij}}| \geq 0 \quad (3.35)$$

Therefore, all the Gersgorian disk of $\mathcal{L}_p$ are located in the closed right half-plane. And all the eigenvalue of $\mathcal{L}_p$ have positive real parts the following inequality may be written.

$$\frac{\lambda_i + \lambda_i^*}{2} > 0 \quad (i = 1, 2, \dots, q + r) \quad (3.36)$$

Eq. (3.36) simply shows that there exist a positive v which satisfies Eq. (3.30).

In the next, to demonstrate the control gain given in Eq. (3.32) stabilizes the system, it should be shown that Eq. (3.31) is valid both real and complex eigenvalues of $\mathcal{L}_p$. For real eigenvalues of $\mathcal{L}_p$ the following inequality may be written;

$$(A - \lambda_i BK)P + P(A - \lambda_i BK)^T < 0 \quad (i = 1, 2, \dots, q) \quad (3.37)$$

By replacing Eq. (3.32) in Eq. (3.37), it may be written;

$$AP + PA^T - \lambda_i BB^T < 0 \quad (i = 1, 2, \dots, q) \quad (3.38)$$

Since inequality Eq. (3.37) holds, one can easily say that;

$$AP + PA^T - vBB^T < 0 \quad (i = 1, 2, \dots, q) \quad (3.39)$$

This results show that, Eq. (3.31) is valid for the system whose $\mathcal{L}_p$ has real eigenvalues.

Similarly, for complex eigenvalues of $\mathcal{L}_p$ the inequality given below may be written;

$$\left[I_2 \otimes A - \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes BK \right] (I_2 \otimes P) + (I_2 \otimes P) \left[I_2 \otimes A - \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes BK \right]^T < 0$$

(3.40)

By replacing Eq. (3.32) in Eq. (3.40), then;

$$I_2 \otimes AP + I_2 \otimes PA^T - \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes \frac{BB^T}{2} - \begin{bmatrix} \sigma_i & -\omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes \frac{BB^T}{2} < 0$$

(3.41)

By employing the properties of Kronecker product given in Section 2, Eq. (3.41) is written as follows:

$$\begin{aligned} I_2 \otimes (AP + PA^T) - \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \otimes \frac{BB^T}{2} &< 0 \\ I_2 \otimes (AP + PA^T) - \sigma_i I_2 \otimes BB^T &< 0 \\ I_2 \otimes [AP + PA^T - \sigma_i BB^T] &< 0, (i = q, q + 1, \dots, r) \end{aligned} \quad (3.42)$$

To hold Eq. (3.42), it is easily seen that the following inequality should be pertinent.

$$AP + PA^T - vBB^T < 0 \quad (i = q, q + 1, \dots, r) \quad (3.43)$$

Thus, the proof is completed.

Simulation Example 3.3: This numerical simulation is presented for the analysis of a platoon system with $\mathcal{L}_p$ of the information topology matrix consisting of both real and complex eigenvalues. The information flow topology between vehicles is considered as TPSF which is shown in Figure 3.9 and the corresponding $\mathcal{L}_p$ matrix is given in Eq. (3.44).

$$\mathcal{L}_p = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \quad (3.44)$$

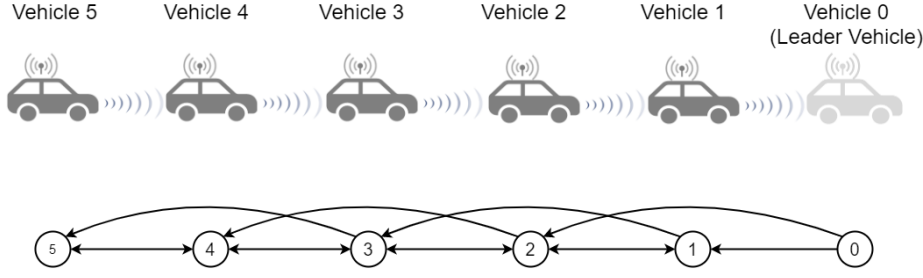


Figure 3.9 : Information flow topologies for Example 3.3.

The eigenvalues of $\mathcal{L}_p$ is found as follows;

$$\mathcal{L}_p : \lambda_1 = 0.603, \quad \lambda_2 = 1.427, \quad \lambda_3 = 2.816,$$

$$\lambda_4 = 4.0765 - 0.533j, \quad \lambda_5 = 4.0765 + 0.533j$$

The system parameters in Table 3.1 are used and the speed profile of the leader vehicle is considered as in Eq. (3.18), which is the same with Example 3.1 and 3.2. According to Theorem 3.2, the system is stable if both of the matrices given Eq. (3.22) are Hurwitz stable and the designed system should also hold the conditions given in Lemma 3.2. And, the characteristic equation for real and complex eigenvalues are written as in Eq. (3.45).

$$\begin{aligned} Q(s)_{real} &= \det(sI_3 - A + \lambda_i BK) \quad i = 1, 2, 3 \\ &= (s^3 + (1.21k_a + 2)s^2 + 1.21k_v s + 1.21k_p) \\ &\quad (s^3 + (2.855k_a + 2)s^2 + 2.855k_v s + 2.855k_p) \\ &\quad (s^3 + (5.632k_a + 2)s^2 + 5.632k_v s + 5.632k_p) \\ Q(s)_{complex} &= \det(sI_6 - (I_2 \otimes A) + (S_i \otimes BK)) \quad i = 4, 5 \\ &= s^6 + (16.31k_a + 4)s^5 + (4 + 32.61k_a + 67.61k_a^2 + 16.31k_v)s^4 \\ &\quad + (16.31k_p + 32.61k_v + 135.22k_a k_v)s^3 \\ &\quad + (32.61k_p + 135.218k_a k_p + 67.61k_v^2)s^2 \\ &\quad + 135.218k_p k_v s + 67.61k_p^2 \end{aligned} \quad (3.45)$$

The design procedure given in Lemma 3.2 is followed to find the state feedback controller gain K . Thus v is set as in Eq. (3.46) according to the inequality given in Eq. (3.30). Then, the *Riccati inequality* given in Eq. (3.31) is solved with the help of the MATLAB/CVX Toolbox [101]. And P is obtained as in Eq. (3.47).

$$v = 0.6 \quad (3.46)$$

$$P = \begin{bmatrix} 9.9201 & -2.2441 & -0.4914 \\ -2.2441 & 1.7489 & -1.0244 \\ -0.4914 & -1.0244 & 1.9995 \end{bmatrix} \quad (3.47)$$

Finally, the static feedback is obtained as follows by means of Eq. (3.32) as in Eq. (3.48).

$$K = \frac{B^T P^{-1}}{2} = [0.2740 \quad 0.9773 \quad 1.0682] \quad (3.48)$$

The results are shown in Figure 3.10, 3.11;

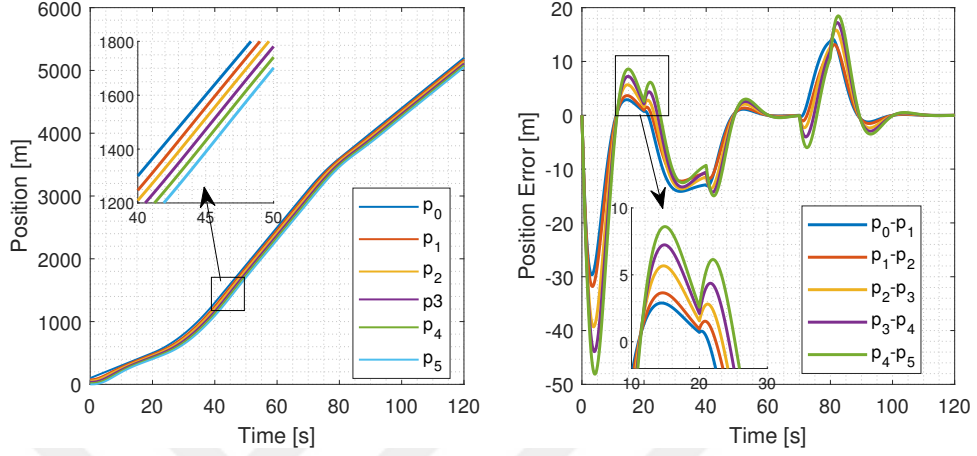


Figure 3.10 : Positions and pos. errors for $k_p = 0.2740$, $k_v = 0.9773$ and $k_a = 1.0682$.

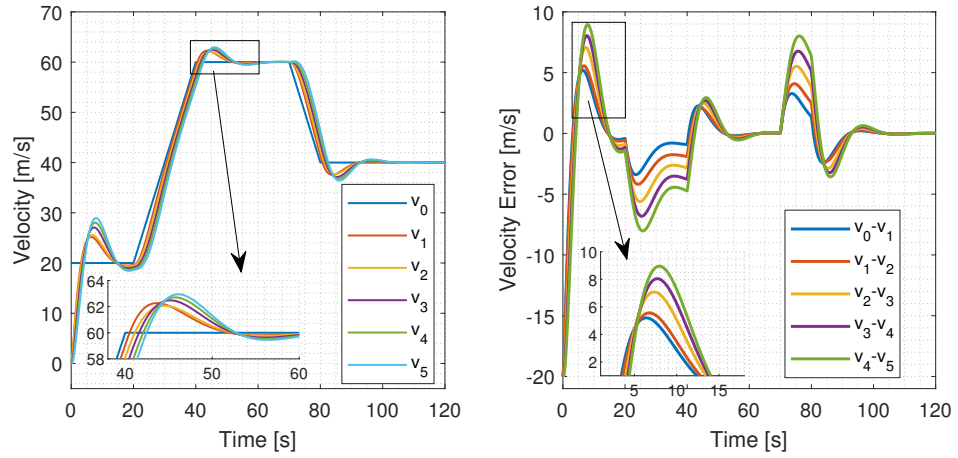


Figure 3.11 : Velocity and velocity errors for $k_p = 0.2740$, $k_v = 0.9773$ and $k_a = 1.0682$.

After that, Figure 3.10 and 3.11 show the positions, position tracking errors, velocities, and velocity tracking errors for the simulation parameters given in Table 3.1, respectively. The controller gains found as $k_p = 0.2740$, $k_v = 0.9773$ and $k_a = 1.0682$. It should be noted that Lemma 3.2 always provides controller gains that stabilize the system, however, it does not guarantee the convergence speed of spacing errors. This topic is out of the scope of this study and is examined in [2] in detail.

In the following, the findings deduced from Theorem 3.1, 3.2 Lemma 3.1, 3.2 for vehicular platooning systems are discussed for the formation control of UAV systems.

Then, the following distributed controller algorithm is proposed for the formation control of UAV system;

$$u_i(t) = - \sum_{j=1}^N a_{ij} \left[k_p \left(p_i(t) - p_j(t) + d_{i,j} \right) + k_v \left(v_i(t) - v_j(t) \right) \right] - p_{ii} \left[k_p \left(p_i(t) - p_0(t) + d_{i,0} \right) + k_v \left(v_i(t) - v_0(t) \right) \right] \quad (3.49)$$

where $p_i = [x_i \ y_i \ z_i]^T \in R^3$ is the position state, $v_i = [v_{x_i} \ v_{y_i} \ v_{z_i}]^T \in R^3$ is the velocity state, $u_i(t)$ solves the formation control problem is the states of the system satisfy the controller objective given in Eq. (2.93). A model transformation is proposed to analyze the formation control of the UAV system. For this purpose, a virtual structure-based formation is introduced with the virtual leader at the center of the formation. A triangular formation is taken as an example for convenient and easy to understand the formation problem, shown in Figure 3.12.

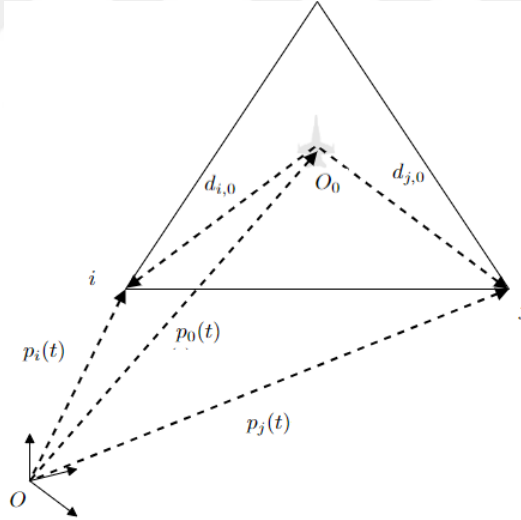


Figure 3.12 : Graph of triangular formation structure.

In Figure 3.12, O is the origin of Cartesian coordinates, O_0 is the formation center, $p_i(t)$ and $p_j(t)$ are positions of UAV i and j in plane coordinate system, respectively, and $p_0(t)$ is the virtual leader position. The distance between UAV i and j and the virtual leader are $d_{i,0}$, $d_{j,0}$, respectively. The control protocol given in Eq. (3.49) may be transformed into;

$$u_i(t) = - \sum_{j=1}^N a_{ij} \left[k_p \left((p_i(t) - p_0(t) + d_{i,0}) - (p_j(t) - p_0(t) + d_{j,0}) \right) + k_v \left(v_i(t) - v_j(t) \right) \right] - p_{ii} \left[k_p \left(p_i(t) - p_0(t) + d_{i,0} \right) + k_v \left(v_i(t) - v_0(t) \right) \right] \quad (3.50)$$

The position and velocity tracking errors between i^{th} and 0^{th} vehicle are defined as in Eq. (3.3), and the formation geometry between UAVs is considered as constant policy as in Eq. (2.86). The reference leader motion is defined in Eq. (2.77). The leader position and velocity are thought of as virtual reference signals which are defined by the motion planner and they have not a real dynamic relationship between them.

Then the dynamic of tracking errors are obtained by using Eq. (2.75), Eq. (3.3) and Eq. (2.86);

$$\begin{aligned} \dot{e}_{p_i}(t) &= e_{v_i}(t) \\ \dot{e}_{v_i}(t) &= u_i(t) \end{aligned} \quad (3.51)$$

The control signal is written in terms of tracking errors as follows.;

$$u_i(t) = - \sum_{j=1}^N a_{ij} \left[k_p \left(e_{p_i}(t) - e_{p_j}(t) \right) + k_v \left(e_{v_i}(t) - e_{v_j}(t) \right) \right] - p_{ii} \left[k_p e_{p_i}(t) + k_v e_{v_i}(t) \right] \quad (3.52)$$

The following vector definition is used to facilitate the notation;

$$\begin{aligned} E(t) &= [e_{p_1}(t) \ e_{v_1}(t) \ e_{p_2}(t) \ e_{v_2}(t) \ \dots \ e_{p_N}(t) \ e_{v_N}(t)]^T; \\ U(t) &= [u_1(t) \ u_2(t) \ \dots \ u_N(t)]^T \\ K &= [k_p \ k_v] \end{aligned} \quad (3.53)$$

Consequently, if the control signal given in Eq. (3.52) is substituted into Eq. (3.51) and redesigned with the help of Eq. (3.53), the closed-loop state-space model for N UAV may be acquired as in Eq. (3.54) by means of Kronecker product.

$$\dot{E}(t) = \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK) \right) E(t) \quad (3.54)$$

where A and B matrices are defined in Eq. (2.76). Therefore, the overall closed-loop system matrix of delay-free system is;

$$A_c = (I_N \otimes A) - (\mathcal{L}_p \otimes BK) \quad (3.55)$$

It is strictly noted that the closed-loop structure formation of UAV systems given in Eq. (3.54) is similar to closed-loop structure vehicular platooning systems Eq. (3.7).

Therefore, Theorem 3.1 is valid if the information flow topology matrix has only real eigenvalues, and Theorem 3.2 is valid if the information flow topology matrix has both real and complex eigenvalues for the formation of UAV systems.

Finally, the following lemma may be proposed to obtain the controller gains for $\mathcal{L}_p$ has real eigenvalue that stabilizes the UAV systems formation control closed-loop system.

Lemma 3.3: Consider the homogeneous formation control of UAV system using a constant policy, and a linear distributed controller as in Eq. (3.49) with the closed-loop dynamics in Eq. (3.54). If the information topology matrix $\mathcal{L}_p$ has only real eigenvalue, then the stability of the platoon is guaranteed if and only if the following conditions are held.

$$\begin{cases} k_p > 0 \\ k_v > 0 \end{cases} \quad (3.56)$$

Proof: The characteristic equation of the system given in Eq. (3.54) may be written with the help of Theorem 3.1 as follows;

$$\begin{aligned} Q(s) &= \det(sI_{2N} - A_c) = \prod_{i=1}^N \det(sI_2 - A + \lambda_i BK) \\ &= \prod_{i=1}^N s^2 + \lambda_i k_v s + \lambda_i k_p, \quad i = 1, 2, \dots, N. \end{aligned} \quad (3.57)$$

Therefore the stability of Eq. (3.13) is examined using the Routh-Hurwitz stability criterion which constituted with the following Routh table;

$$\begin{array}{lcl} s^2 : & 1 & \lambda k_p \\ s^1 : & \lambda k_v & 0 \\ s^0 : & \lambda k_p & \end{array} \quad (3.58)$$

Given the fact that $\lambda_i > 0, i = 1, 2, \dots, N$, Eq. (3.58) is asymptotically stable if the and only if Eq. (3.56) is satisfied. Thus the proof is completed.

Simulation Example 3.4: This numerical simulation is presented for the analysis of formation control of the UAV system with the information topology matrix $\mathcal{L}_p$ consisting of only real eigenvalues. The information flow topology between UAVs is shown in Figure 3.13, and the corresponding $\mathcal{L}_p$ matrix is given in Eq. (3.59).

$$\mathcal{L}_p = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 \end{bmatrix} \quad (3.59)$$

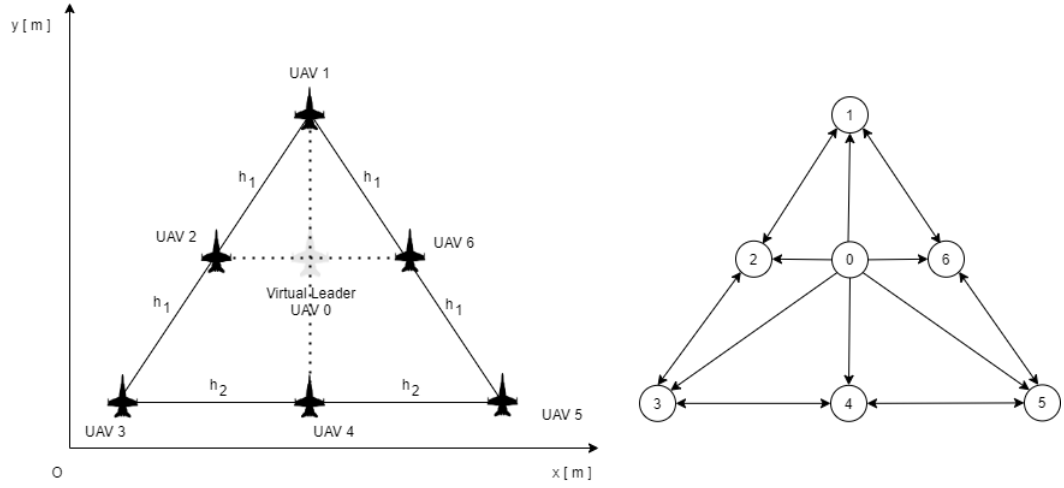


Figure 3.13 : Information flow topologies for Example 3.4.

The eigenvalues of $\mathcal{L}_p$ is found as follows;

$$\mathcal{L}_p : \lambda_1 = 1, \quad \lambda_{2,3} = 2, \quad \lambda_{4,5} = 4, \lambda_6 = 5$$

The formation control system is considered as homogeneous. The system parameters and initial states of UAVs used in simulation are given in Table 3.2 and in Table 3.3, respectively.

Table 3.2 : System parameters used in simulation studies

Parameter	Description	Value	Unit
N	Number of following UAVs	6	-
h_1, h_2	Constant distances	650 , 500	[m]
τ_v	Velocity time constant	500	[sec]
τ_θ	Flight path angle time constant	50	[sec]
τ_ψ	Heading angle time constant	50	[sec]

Table 3.3 : Initial States of UAVs

$x_i[m]$	$y_i[m]$	$z_i[m]$	$v_i[mPs]$	$\psi_i[deg]$	$\theta_i[deg]$
UAV_0	707.7	707.7	40	30	5
UAV_1	1021.9	1021.9	65	15	3
UAV_2	600.95	954.45	35	10	4
UAV_3	0	707	25	-10	8
UAV_4	353.5	353.5	30	0	9
UAV_5	707	0	30	25	10
UAV_6	954.45	600.95	45	5	6

The characteristic equation of the system may be written by using Eq. (3.57) as follows;

$$\begin{aligned} Q(s)_{real} &= \prod_{i=1}^N \det(sI_2 - A + \lambda_i BK) \quad i = 1, 2, \dots, 6 \\ &= (k_p + k_v s + s^2)(2k_p + 2k_v s + s^2)^2(4k_p + 4k_v s + s^2)^2 \\ &\quad (5k_p + 5k_v s + s^2) \end{aligned} \quad (3.60)$$

The reference velocity, heading angle and flight path angle profile of virtual UAV (UAV_0) is determined as follows;

$$\begin{aligned} v_{ref} &= 50 \text{ m/s}, \quad \psi_{ref} = 45 \text{ deg} \\ \theta_{ref} &= \begin{cases} 0 \text{ deg} & \text{if } t \leq 40 \\ 10 \text{ deg} & \text{if } 40 < t \leq 80 \\ 0 \text{ deg} & \text{if } 80 < t \leq 120 \\ -5 \text{ deg} & \text{if } 120 < t \leq 150 \\ 0 \text{ deg} & \text{if } t \leq 200 \end{cases} \end{aligned} \quad (3.61)$$

The formation task is climbing, level flight, and gliding as given in Eq. (3.132). Figure 3.14, Figure 3.15, Figure 3.16 and Figure 3.17 are shown the altitude and velocity, flight path angle and heading angle, position and velocity errors and 3D trajectory of formation flying, respectively.

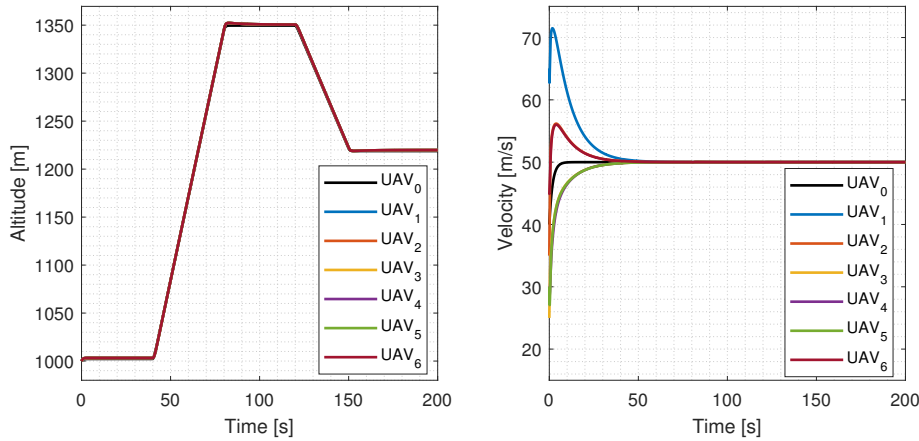


Figure 3.14 : Altitude and velocity for $k_p = 1$ and $k_v = 2$.

The multi-UAV system may complete the desired trajectory of virtual UAV with the reference velocity, heading angle, flight path angle and maintain the desired formation during the flight. The controller parameters are $k_p = 0.4$ and $k_v = 4$ for stability. Since the distances between the UAVs are relatively large and the speed differences are slight, a small value is assigned according to the speed controller gain to reduce the effect of the position controller gain on the system response. All the position and velocity errors converge to zero, which means the controller objective given in Eq. (2.93) is achieved.

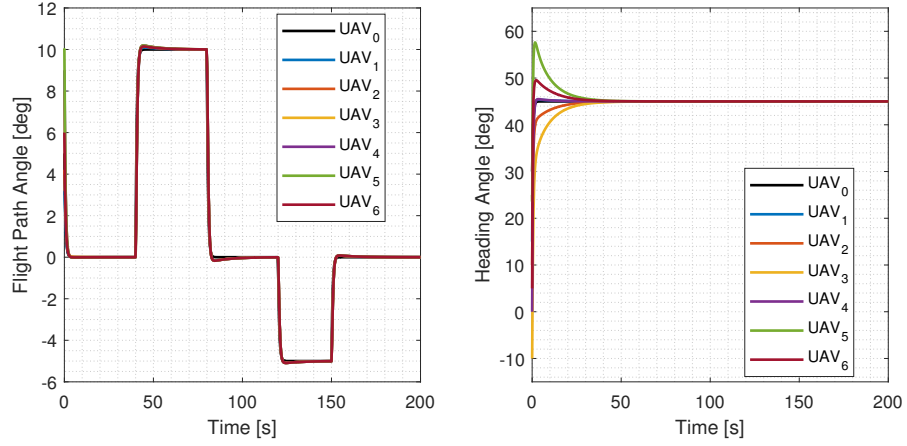


Figure 3.15 : Flight path angle and heading angle for $k_p = 1$ and $k_v = 2$.

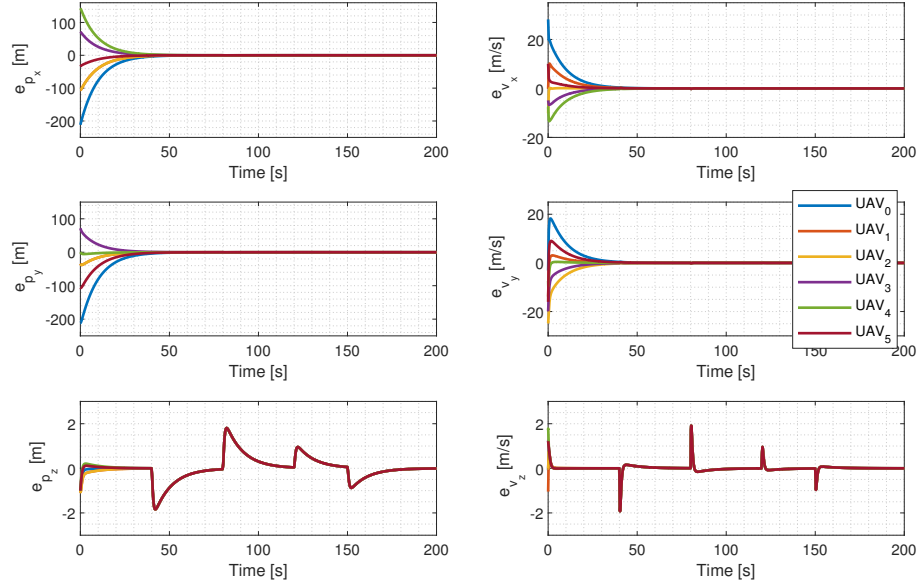


Figure 3.16 : Position errors and velocity errors for $k_p = 1$ and $k_v = 2$.

Theorem 3.2 is applied if the information topology matrix $\mathcal{L}_p$ has a complex eigenvalue for the UAV generation flight. The controller gains found with the help of Lemma 3.2 provides stability; however, there is no guarantee for the convergence speed of the reference trajectory. A long convergence time may significantly affect the performance of formation flying. To overcome this drawback, the following lemma may be proposed;

Lemma 3.4: [2] Consider the homogeneous formation control of UAV system and a linear distributed controller as in Eq. (3.49) with the closed loop dynamic in Eq. (3.54). Suppose the information topology matrix $\mathcal{L}_p$ has both real and complex eigenvalues.

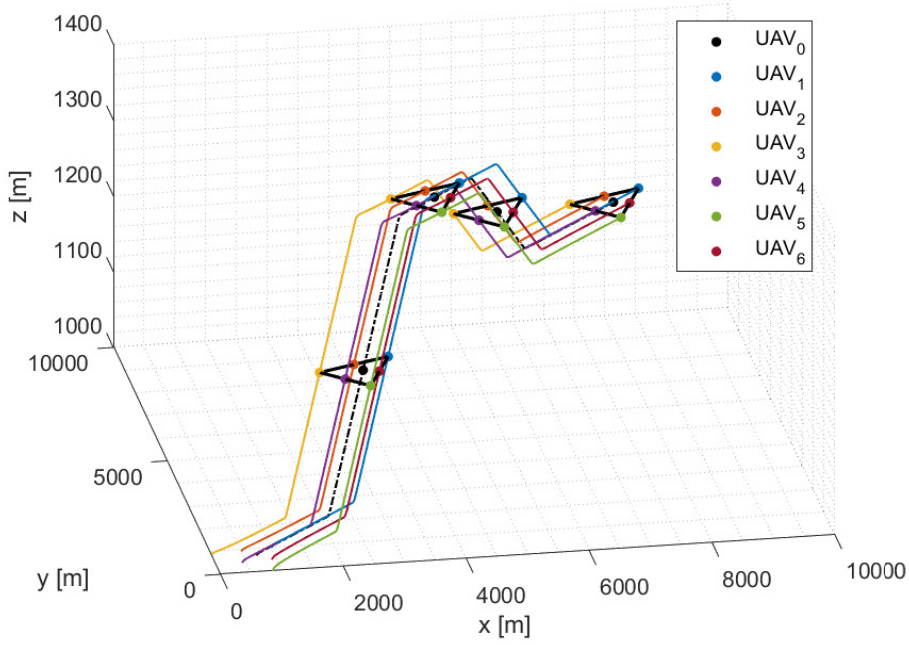


Figure 3.17 : UAV 3D trajectory for $k_p = 1$ and $k_v = 2$.

Let $\lambda_i, i = 1, 2, \dots, q$ be the real eigenvalues of $\mathcal{L}_p$ including multiplicity, and $\lambda_i = \sigma_i + j\omega_i, \lambda_i^* = \sigma_i - j\omega_i, i = q+1, q+2, \dots, q+r$ be the complex conjugate eigenvalues of $\mathcal{L}_p$. Let v be a positive scalar defined in Lemma 3.2. Then, if there exist a positive scalar δ and symmetric matrix $P > 0$, such that

$$AP + PA^T - vBB^T + 2\delta P < 0 \quad (3.62)$$

then the errors converge to zero exponentially faster than $e^{-\delta t}$ with the controller gains defined by Eq. (3.32).

Proof: [2] For the controller gains defined by Eq. (3.32), two conditions below may be confirmed, similar to Lemma 3.2.

For real eigenvalues of $\mathcal{L}_p$ the following inequality may be written;

$$\begin{aligned} (A - \lambda_i BK + \delta I_2)P + [(A - \lambda_i BK + \delta I_2)P]^T &< 0 \\ AP - \lambda_i B \frac{B^T P^{-1}}{2} P + \delta P + PA^T - \lambda_i P \frac{P^{-1} B}{2} B^T + \delta P &< 0 \\ AP + PA^T - \lambda_i BB^T + 2\delta P &\leq AP + PA^T - vBB^T + 2\delta P < 0 \end{aligned} \quad (3.63)$$

And, the complex eigenvalues of $\mathcal{L}_p$;

$$\begin{aligned}
& ((I_2 \otimes A) - \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes (BK) + \delta I_4)(I_2 \otimes P) + \\
& (I_2 \otimes P)[((I_2 \otimes A) \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes (BK) + \delta I_4)]^T < 0 \\
& (I_2 \otimes A)(I_2 \otimes P) - (\begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes (BK))(I_2 \otimes P) + \delta(I_2 \otimes P) + \\
& (I_2 \otimes P)(I_2 \otimes A^T) - (I_2 \otimes P)(\begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}^T \otimes (BK)^T) + (I_2 \otimes P)\delta < 0 \\
& (I_2 \otimes AP) - (\begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix} \otimes BKP) + \delta(I_2 \otimes P) + (I_2 \otimes PA^T) \\
& - \begin{bmatrix} \sigma_i & -\omega_i \\ \omega_i & \sigma_i \end{bmatrix} \otimes PK^T B^T + \delta(I_2 \otimes P) < 0 \\
& (I_2 \otimes (AP + PA^T)) - \sigma I_2 \otimes BB^T + 2\delta(I_2 \otimes P) < 0 \\
& I_2 \otimes (AP + PA^T - \sigma BB^T + 2\delta P) \leq I_2 \otimes (AP + PA^T - v BB^T + 2\delta P) < 0
\end{aligned} \tag{3.64}$$

If these inequalities are satisfied, it is clear that the following system should be defined and the system matrix $\bar{A}_c$ is Hurwitz.

$$\begin{aligned}
\dot{\tilde{X}} &= \bar{A}_c \tilde{X} \\
\bar{A}_c &= (I_N \otimes A - \mathcal{L}_p \otimes (BK) + \delta I_{2N}) \in S^{2N \times 2N}
\end{aligned} \tag{3.65}$$

Let define the Lyapunov function as follows;

$$\begin{aligned}
V &= \tilde{X}^T (I_N \otimes P) \tilde{X} > 0 \\
\dot{V} &= \dot{\tilde{X}}^T (I_N \otimes P) \tilde{X} + \tilde{X}^T (I_N \otimes P) \dot{\tilde{X}} < 0 \\
&= \tilde{X}^T \bar{A}_c^T (I_N \otimes P) \tilde{X} + \tilde{X}^T (I_N \otimes P) \bar{A}_c \tilde{X} < 0 \\
&= \tilde{X}^T (\bar{A}_c^T (I_N \otimes P) + (I_N \otimes P) \bar{A}_c) \tilde{X} < 0 \\
&= \tilde{X}^T \left((I_N \otimes A - \mathcal{L}_p \otimes (BK) + \delta I_{2N})(I_N \otimes P) + \right. \\
&\quad \left. (I_N \otimes P)(I_N \otimes A^T - \mathcal{L}_p^T \otimes (BK)^T + \delta I_{2N}) \right) \tilde{X} < 0
\end{aligned} \tag{3.66}$$

And, the following inequality is obtained;

$$\begin{aligned}
\dot{V} &= \tilde{X}^T \left((I_N \otimes A - \mathcal{L}_p \otimes (BK))(I_N \otimes P) + \right. \\
&\quad \left. (I_N \otimes P)(I_N \otimes A^T - \mathcal{L}_p^T \otimes (BK)^T) \right) \tilde{X} < -2\delta \tilde{X}^T (I_N \otimes P) \tilde{X} = -2\delta V \\
\dot{V} &< -2\delta V
\end{aligned} \tag{3.67}$$

It is basically obtained from Eq. (3.67) that $V(t) < e^{-2\delta t} V(0)$. Also;

$$\begin{aligned}
\lambda_{\min}(P) \|\tilde{X}\|_2^2 &\leq V(t) < e^{-2\delta t} V(0) \leq \lambda_{\max}(P) \|\tilde{X}(0)\|_2^2 \\
\|\tilde{X}\|_2 &< e^{-\delta t} \|\tilde{X}(0)\|_2 \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}
\end{aligned} \tag{3.68}$$

Since the state variable $\tilde{X}$ is formed of position and velocity errors, the condition Eq. (3.68) points that the errors converge to zero exponentially with speed no slower than $e^{-2\delta t}$.

Simulation Example 3.5: This numerical simulation is presented for the analysis of formation control of the UAV system with the information topology matrix $\mathcal{L}_p$ consisting of both real and complex eigenvalues. The information flow topology between UAVs is shown in Figure 3.18, and the corresponding $\mathcal{L}_p$ matrix is given in Eq. (3.69).

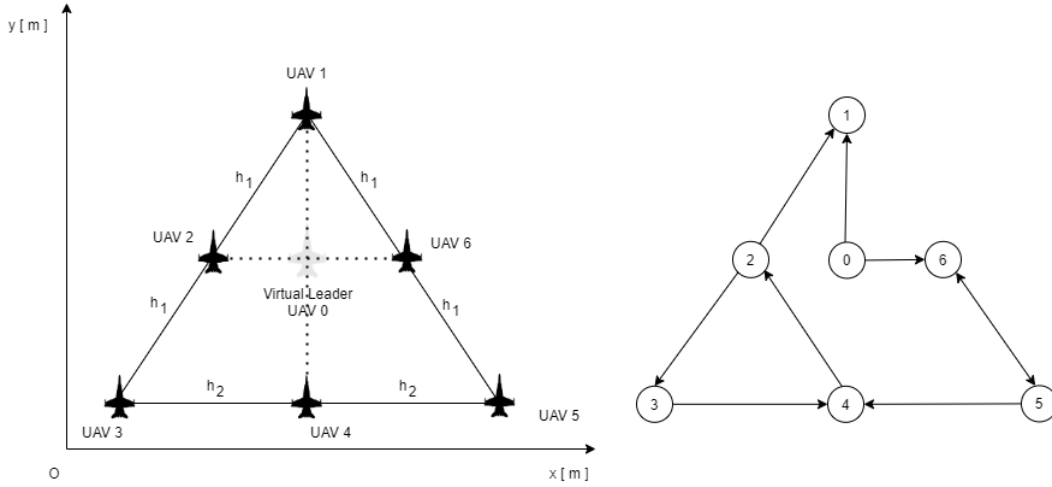


Figure 3.18 : Information flow topologies for Example 3.5.

$$\mathcal{L}_p = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \quad (3.69)$$

The eigenvalues of $\mathcal{L}_p$ is found as follows;

$$\mathcal{L}_p : \lambda_1 = 0.2451, \quad \lambda_2 = 0.3820, \quad \lambda_3 = 2, \quad \lambda_4 = 2.6180$$

$$\lambda_{5,6} = 1.8774 \mp 0.7449i,$$

The formation control system is considered as homogeneous. The system parameters and initial states of UAVs used in simulation are given in Table 3.2 and in Table 3.3, respectively.

The characteristic equation of the system may be written by using Eq. (3.57) as follows;

$$\begin{aligned} Q(s)_{real} &= \det(sI_2 - A + \lambda_i BK) \quad i = 1, 2, 3 \\ &= (s^2 + 0.2451k_v s + 0.2451k_p)(s^2 + 0.3820k_v s + 0.3820k_p) \\ &\quad (s^2 + 2k_v s + 2k_p)(s^2 + 2.6180k_v s + 2.6180k_p) \end{aligned} \quad (3.70)$$

$$\begin{aligned} Q(s)_{complex} &= \det(sI_4 - (I_2 \otimes A) + (S_i \otimes BK)) \quad i = 4, 5 \\ &= s^4 + (3.75488s^3 + (4.0796k_v^2 + 3.75488k_p + \\ &\quad 8.15919k_p k_v)s + 4.0796k_p^2 \end{aligned}$$

The v is set as $v = 0.02$ according to the inequality given in Eq. (3.30). Then, the *Riccati inequality* given in Eq. (3.31) is solved with the help of the MATLAB/CVX Toolbox. And P and K are obtained as in Eq. (3.71).

$$\begin{aligned} P &= \begin{bmatrix} 19.8202 & -1.4505 \\ -1.4505 & 0.2013 \end{bmatrix}, \\ K &= \frac{B^T P^{-1}}{2} = \begin{bmatrix} 0.3845 & 5.2542 \end{bmatrix} \end{aligned} \quad (3.71)$$

Moreover, If the *Riccati inequality* given in Eq. (3.62) is solved for $v = 0.02$, $\delta = 0.08$ P and K are obtained as in Eq. (3.72).

$$\begin{aligned} P &= \begin{bmatrix} 6.4998 & -0.6972 \\ -0.6972 & 0.0932 \end{bmatrix}, \\ K &= \frac{B^T P^{-1}}{2} = \begin{bmatrix} 2.9139 & 27.1661 \end{bmatrix} \end{aligned} \quad (3.72)$$

In case of the reference trajectory is selected as Eq. (3.132), the flight path angle and altitude results are obtained for the controller gains given in Eq. (3.71) as follows;

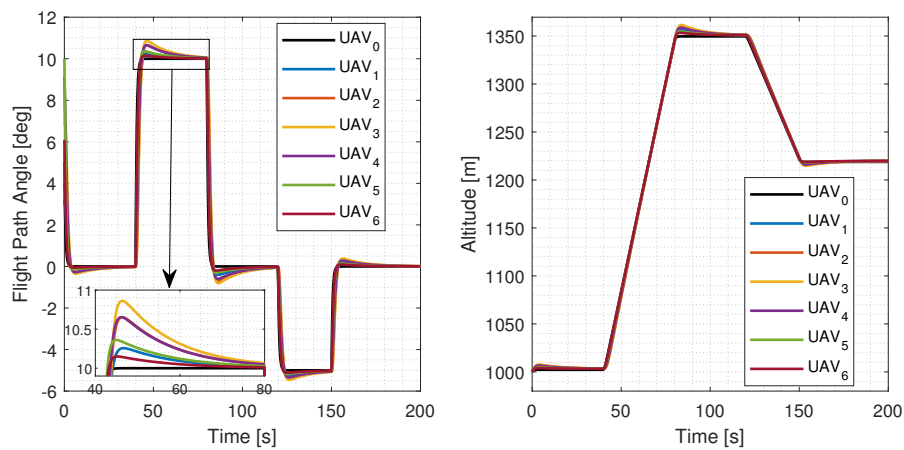


Figure 3.19 : Flight path angle and altitude for $k_p = 0.3845$ and $k_v = 5.2542$.

For the controller gains given in Eq. (3.72), Figure 3.14, Figure 3.15, Figure 3.16 and Figure 3.17 are shown the altitude and velocity, flight path angle and heading angle, position and velocity errors of formation flying, respectively.

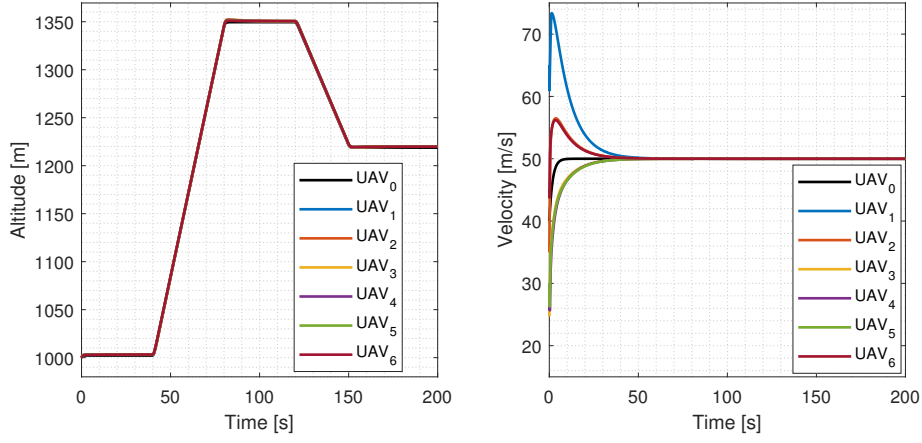


Figure 3.20 : Flight path angle and heading angle for $k_p = 2.9139$ and $k_v = 27.1661$.

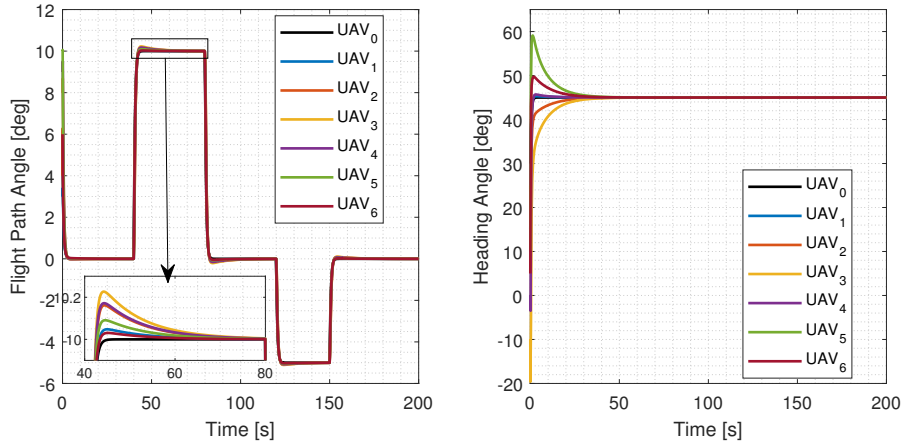


Figure 3.21 : Flight path angle and heading angle for $k_p = 2.9139$ and $k_v = 27.1661$.

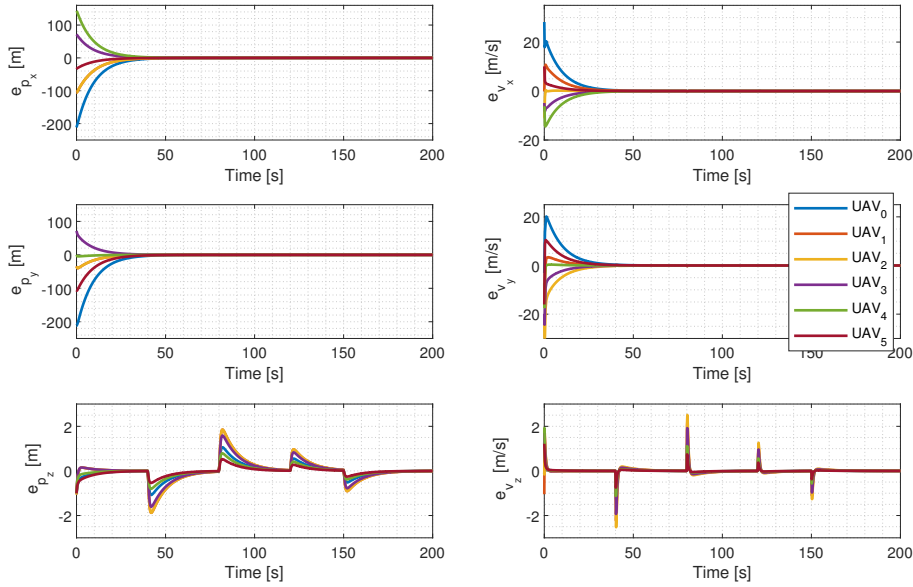


Figure 3.22 : Position errors and velocity errors for $k_p = 2.9139$ and $k_v = 27.1661$.

The controller gains obtained with the help of eq provide a better converge speed, as seen in the flight path angle and altitude results. Also, the virtual leader UAV(UAV_0) communicates with UAV_1 and UAV_6 . So, the other UAVs access the trajectory relatively. For this reason, the performance of UAVs is different from each other. All the position and velocity errors converge to zero, which means the controller objective given in Eq. (2.93) is achieved.

3.2.2 Stability analysis for time delay case

This section creates stability conditions for the system under single and multiple time delays. Similar to the previous section, the analysis is conducted for platooning system model first, then formation control of the unmanned aerial vehicle system model.

The control signal is written in terms of tracking errors in the presence of communication time delay for platooning system;

$$u_i(t - \tau) = - \sum_{j=1}^N \left[a_{ij} \left(k_p (e_{p_i}(t - \tau) - e_{p_j}(t - \tau)) + k_v (e_{v_i}(t - \tau) - e_{v_j}(t - \tau)) + k_a (e_{a_i}(t - \tau) - e_{a_j}(t - \tau)) \right) \right] - p_{ii} \left[k_p e_{p_i}(t - \tau) + k_v e_{v_i}(t - \tau) + k_a e_{a_i}(t - \tau) \right] \quad (3.73)$$

Then, the closed-loop system is obtained by substituting Eq. (3.73) into Eq. (3.4) a more compact form as follows;

$$\dot{E}(t) = (I_N \otimes A)E(t) - (\mathcal{L}_p \otimes BK)E(t - \tau) \quad (3.74)$$

The characteristic equation of the system given Eq. (3.74) is $3N$ -degree quasi-polynomial since each internal dynamic of vehicles is 3, and the number of vehicles in the platoon is N . Increasing the number of agents heightens the degree of these equations swiftly. This makes the carry-out CTCR methods onerous to obtain an exact determination of stability posture. To avoid this difficulty, one can decompose the characteristic equation into factors. This method is named factorization property and is similar to the method explained in the previous section.

Theorem 3.3: [26], [102] Consider the homogeneous multi-agent system whose dynamic model given in Eq. (3.74) under communication time delay. Let the information topology matrix $\mathcal{L}_p$ is an arbitrary square matrix, $\lambda_i, i = 1, 2, \dots, q$ are

the real eigenvalues of $\mathcal{L}_p$ including multiplicity, and $\lambda_i = \sigma_i + j\omega_i$, $\lambda_i^* = \sigma_i - j\omega_i$, $i = q+1, q+2, \dots, q+r$, ($q+r = N$) are the complex conjugate eigenvalues of $\mathcal{L}_p$ including multiplicity. The characteristic equation of Eq. (3.74) may be decomposed as a set of third-order and sixth-order factors as follows;

$$\begin{aligned} Q(s, \tau) &= \det(sI_{3N} - (I_N \otimes A) + (\mathcal{L}_p \otimes BK)e^{-\tau s}) \\ &= \det(sI_{3N} - (I_N \otimes A) + (J_i \otimes BK)e^{-\tau s}) \\ &= \prod_{i=1}^q \det(sI_3 - A + \lambda_i B K e^{-\tau s}) \prod_{i=q+1}^r \det(sI_6 - (I_2 \otimes A) + (S_i \otimes BK)e^{-\tau s}) \end{aligned} \quad (3.75)$$

Proof: The transformation of $J_p = M^{-1} \mathcal{L}_p M$, $M \in R$ which is defined in the proof of Theorem 3.2 is also conducted here. The structure of J_p is the same as Eq. (3.23). Then, a state transformation $E(t) = (M \otimes I_3) \Upsilon(t)$ is applied to the Eq. (3.74).

$$\begin{aligned} (M \otimes I_3) \dot{\Upsilon}(t) &= (I_N \otimes A)(M \otimes I_3) \Upsilon(t) - (\mathcal{L}_p \otimes BK)(M \otimes I_3) \Upsilon(t - \tau) \\ \dot{\Upsilon}(t) &= (M^{-1} \otimes I_3)(I_N \otimes A)(M \otimes I_3) \Upsilon(t) - (M^{-1} \otimes I_3) \\ &\quad (\mathcal{L}_p \otimes BK)(M \otimes I_3) \Upsilon(t - \tau) \\ &= (M^{-1} I_N M \otimes I_3 A I_3) \Upsilon(t) - (M^{-1} \mathcal{L}_p M \otimes I_3 B K I_3) \Upsilon(t - \tau) \end{aligned} \quad (3.76)$$

Then Eq. (3.76) becomes;

$$\dot{\Upsilon}(t) = (I_N \otimes A) \Upsilon(t) - (J_p \otimes BK) \Upsilon(t - \tau) \quad (3.77)$$

Since I_N and J_p are diagonal and block diagonal, Eq. (3.77) is block-diagonalized; thus it can be represented as a set of $q+r$ dynamically decoupled subsystems as;

$$\begin{aligned} \dot{\Upsilon}_i(t) &= A \Upsilon_i(t) - \lambda_i B K \Upsilon_i(t - \tau), \quad i = 1, 2, \dots, q \\ \dot{\Upsilon}_i(t) &= (I_2 \otimes A) \Upsilon_i(t) + (S_i \otimes BK) \Upsilon_i(t - \tau), \quad i = q+1, q+2, \dots, q+r \end{aligned} \quad (3.78)$$

Therefore, the characteristic equations of Eq. (3.74) becomes a product of $q+r$ individual subsystems as follows;

$$Q(s, \tau) = \prod_{i=1}^q \det(sI_3 - A + \lambda_i B K e^{-\tau s}) \prod_{i=q+1}^r \det(sI_6 - (I_2 \otimes A) + (S_i \otimes BK)e^{-\tau s}) \quad (3.79)$$

where S_i is defined in Eq. (3.25), the left and right products correspond to the real and complex eigenvalues of $\mathcal{L}_p$, respectively.

Thus the proof is completed.

This factorization property simplifies the problem considerably, by transforming it from $3N$ -order system into a set of third-order and sixth-order systems. A similar

transformation is also applied for second-order systems under different distributed controllers in [102] and [103] for multiple time delays. Notice that, the stability problem of the system is now reduced to obtaining the specific eigenvalues of $\mathcal{L}_p$ and superposing the stability outlook of each characteristic equation in Eq. (3.79) to obtain the ensemble stability region for the system. The finalized stability region of whole system in the parametric space are the found from the common intersection of the stability region of each factor. Hence, the stability analysis of each factor may be performed with CTCR methodology explained in Section 2 as shown in the following examples.

Simulation Example 3.6: In this numerical simulation, the problem explained in Example 3.1 is evaluated in case of time delay. So, the stability of the closed-loop platooning system not only bounded of the controller gains and the information flow topologies but also dependent on time delay. Then, the characteristic equations given in Eq. (3.16) is converted to the following form when the system is exposed to the single time delay.

$$\begin{aligned} Q(s, \tau) &= \det(sI_3 - A + \lambda_i B K e^{-\tau s}) \\ &= \left((s^2(s+2+2k_a e^{-\tau s}) + 2(k_v s + k_p) e^{-\tau s}) \right) \\ &\quad \left((s^2(s+2+4k_a e^{-\tau s}) + 4(k_v s + k_p) e^{-\tau s}) \right)^4 \end{aligned} \quad (3.80)$$

Any more, the CTCR paradigm which is explained in Section 2 in detail, may be used to obtain the stability posture of the system. All the methods given in Section 2.4.1 will be used in the examples in this study. In this example, one of the most basic topologies used in platooning systems is chosen. The information topology matrix has 5 eigenvalues, however, four of them are equal to each other. Therefore, even if the characteristic equation of the system decomposes into 5 factors, it is investigated just two of them given in Eq. (3.80). Clearly, the delay term results in infinitely many finite characteristic roots for each factor. So, the system given in Eq. (3.80) is asymptotically stable is and only if all the characteristic roots are on the left half of the complex s plane. The CTCR paradigm enables investigating all these characteristic roots in an exact and explicit manner. This paradigm is based on the D-subdivision method (continuity argument). This argument states that the solutions of each factor of Eq. (3.80) are a continuous function of the delay. Consequently, a change in the system's stability due to the delay is possible only when, for a given value of

the delay, the system displays a conjugated pair of purely imaginary characteristic roots ($s = j\omega$). Therefore, the delay space is divided into pockets where the number of unstable characteristic roots of the system remains constant. Also, these purely imaginary roots of are periodic with respect to the delay.

The first step of the CTCR methodology is to find all the points τ values that cause a pair of imaginary characteristic roots of each factor at $s = j\omega_c$, and satisfy the constraint $0 < \omega_c < 2\pi$. It is easily found with the help of magnitude and phase equations of each factors. The procedures are also given in [26] and [104]. The second step uses the invariance property of the root crossing tendency at these imaginary roots as the delay increases. The transformation of $s = j\omega$ is applied for each factor, to determine the complex set of imaginary characteristic roots of $Q(s, \tau)$, Then, the magnitude and phases of each factors must be equal when the suggested root is substituted in place.

Then, the general factor form of Eq. (3.73) for $\mathcal{L}_p$ which has only real-eigenvalue is given below;

$$s^3 + \frac{1}{\gamma}s^2 + \frac{\lambda_i k_a}{\gamma}e^{-\tau s}s^2 + \frac{\lambda_i k_v}{\gamma}e^{-\tau s}s + \frac{\lambda_i k_p}{\gamma}e^{-\tau s} \quad (3.81)$$

Eq. (3.81) may be rewritten as;

$$s^3 + \frac{1}{\gamma}s^2 = -(k_a s^2 + k_v s + k_p) \frac{\lambda_i}{\gamma} e^{-\tau s} \quad (3.82)$$

If Eq. (3.82) has an imaginary roots $s = \omega j$, then the magnitude and phase of two sides must be equal while the imaginary root is substituted in place. So, the magnitude equation is written as follows;

$$\omega^6 + \frac{1 - k_a^2 \lambda_i^2}{\gamma^2} \omega^4 + \frac{2k_a k_p \lambda_i^2 - k_v^2 \lambda_i^2}{\gamma^2} \omega^2 - \frac{k_p^2 \lambda_i^2}{\gamma^2} = 0 \quad (3.83)$$

Since the positive values of ω is needed, Eq. (3.83) is reduced with a substitution of $\omega = \omega^2$. So, Eq. (3.83) becomes;

$$\omega^3 + a\omega^2 + b\omega + c = 0 \quad (3.84)$$

where;

$$a = \frac{1 - k_a^2 \lambda_i^2}{\gamma^2}, \quad b = \frac{2k_a k_p \lambda_i^2 - k_v^2 \lambda_i^2}{\gamma^2}, \quad c = -\frac{k_p^2 \lambda_i^2}{\gamma^2}$$

It is known that the imaginary roots $s = \omega j$ of Eq. (3.82) may arise is the cubic equation given in Eq. (3.84) has positive real roots. One way is to determine the number of

positive real roots of a polynomial is to utilize Descartes' rule of sign. It tells us that the number of positive real roots in a polynomial functions the same or less than by an even numbers of changes in the sign of the coefficients. Since $c < 0$, one of the following sign changes should happen for the system to have at least one positive real roots on the condition that $k_p > 0$, $k_v > 0$ and $k_a > 0$;

$$\begin{aligned}
\text{Condition 1: } & a > 0 \quad \wedge \quad b > 0 \\
& k_a \lambda_i < 1 \quad \& \quad k_v^2 < 2k_a k_p \\
\text{Condition 2: } & a > 0 \quad \wedge \quad b < 0 \\
& k_a \lambda_i > 1 \quad \& \quad k_v^2 < 2k_a k_p \\
\text{Condition 3: } & a < 0 \quad \wedge \quad b > 0 \\
& k_a \lambda_i < 1 \quad \& \quad k_v^2 > 2k_a k_p \\
\text{Condition 4: } & a < 0 \quad \wedge \quad b < 0 \\
& k_a \lambda_i > 1 \quad \& \quad k_v^2 > 2k_a k_p
\end{aligned} \tag{3.85}$$

On the other hand, the phase equality condition of Eq. (3.82) is given as;

$$\arctan(\omega \gamma) - \arctan\left(\frac{k_v \omega}{k_p - k_a \omega^2}\right) = -\tau \omega \tag{3.86}$$

Then, Eq. (3.86) produces the infinitely many equidistant time delay value which generate the only imaginary crossing as the delay reaches the values;

$$\tau_{c,k} = -\frac{1}{\omega} \arctan\left(\frac{\omega(k_a \gamma \omega^2 - k_p \gamma + k_v)}{\omega^2(k_a - k_v \gamma) - k_p}\right) + k \frac{2\pi}{\omega}, \quad k = 0, 1, \dots \tag{3.87}$$

Then the root tendency which is explain in Section 2 - Proposition 4, is used to determine the direction of the roots;

$$RT = \text{sgn} \left[\text{Re} \left(\frac{ds}{d\tau} \right) \Big|_{s=\omega_{c,j}, \tau=\tau_c} \right] \tag{3.88}$$

So, by the help of the following partial derivative formula;

$$\frac{\partial F}{\partial x} \frac{\partial x}{\partial \tau} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial \tau} = 0 \tag{3.89}$$

the $\frac{ds}{d\tau}$ is found for this particular case;

$$\frac{ds}{d\tau} = \frac{\lambda_i s (k_a s^2 + K_v s_p^k)}{3s^2 + \frac{2}{\gamma} s + \frac{\lambda_i}{\gamma} [(2k_a s + k_v) - \tau (k_a s^2 + k_v s + k_p)] e^{-\tau s}} \tag{3.90}$$

It is concluded that, at the imaginary root of each factor of Eq. (3.80), which defined by the real solution of Eq. (3.83) and the corresponding time delays of Eq. (3.87),

the root tendency is always destabilizing ($RT = +1$). It means that the only imaginary roots always moves to the right half of complex plane. Then the stability margin for k_p and τ for the system parameters given in Table 3.1 and $k_v = 4$ and $k_a = 2$ is obtained. For the given controller parameters, Eq. (3.83) should provide Condition 2 for k_p is between 0 and 4 and provide Condition 4 for $k_p > 4$. The stability margin is shown in Figure 3.23.

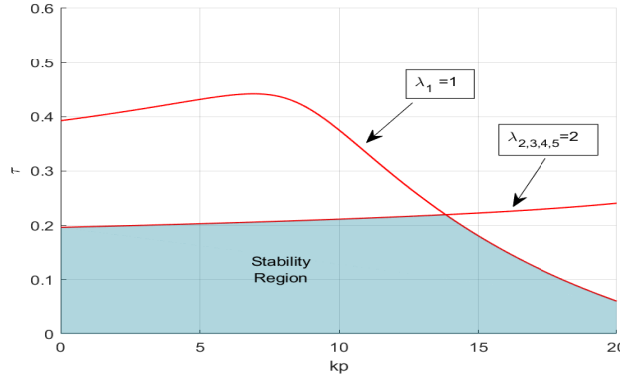


Figure 3.23 : Stability boundaries for $k_v = 4$ and $k_a = 2$.

In the Figure 3.23, the shaded regions indicate the parameters that yield no characteristic roots on the right half of the complex-plane. At any point on the boundary of the stability zone, the characteristic equation has purely imaginary roots which makes the system marginally stable. This stability map is validated by simulations for some selected test points. The test point are selected as $k_p = 2.4$, $\tau = 0.06$ s which is on the stable region ($\tau = 0.199$ s at the stability boundary for $k_p = 2.4$), and the simulation result for positions, position tracking errors and the velocity, velocity tracking errors are given in Figure 3.24 and 3.25, respectively.

On the other hand, the test point is selected as $k_p = 2.4$ and $\tau = 0.18$ s, which is in the stable region but very close to the stability boundary. Consequently, the effect of time delay is visible directly on the simulation results given in the Figure 3.26 and 3.27.

Then, another set of parameters are selected at $k_p = 2.4$ and $\tau = 2$ s which is on the unstable region. The figure in Figure 3.28, the velocities of vehicles are indicated and the system is unstable which conveys that at least one pair of roots of the closed-loop system is on the right half of the complex plane.

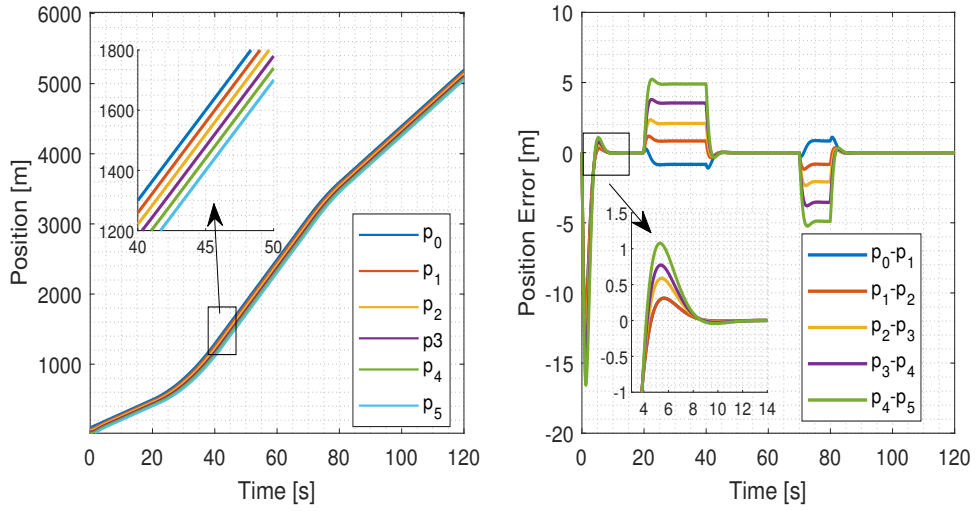


Figure 3.24 : Positions and position errors for $\tau = 0.06$.

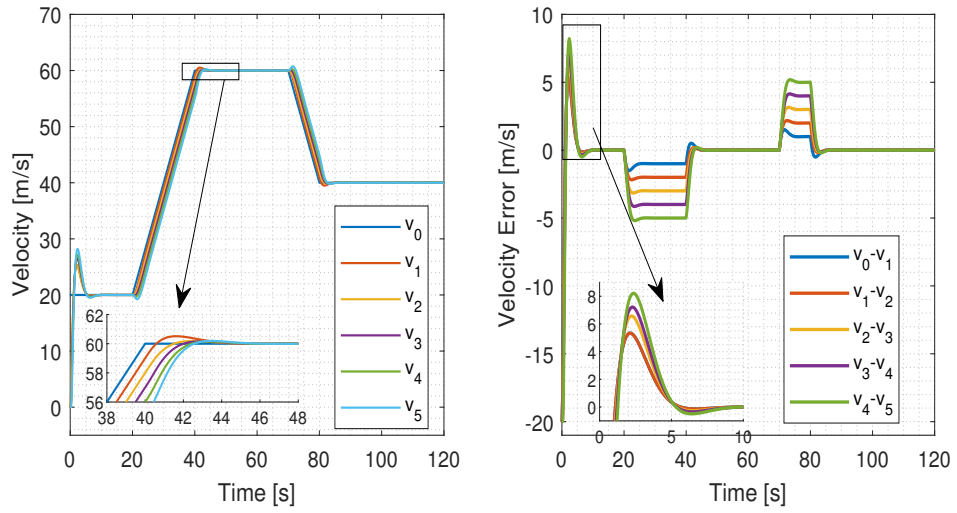


Figure 3.25 : Velocity and velocity errors for $\tau = 0.06$.

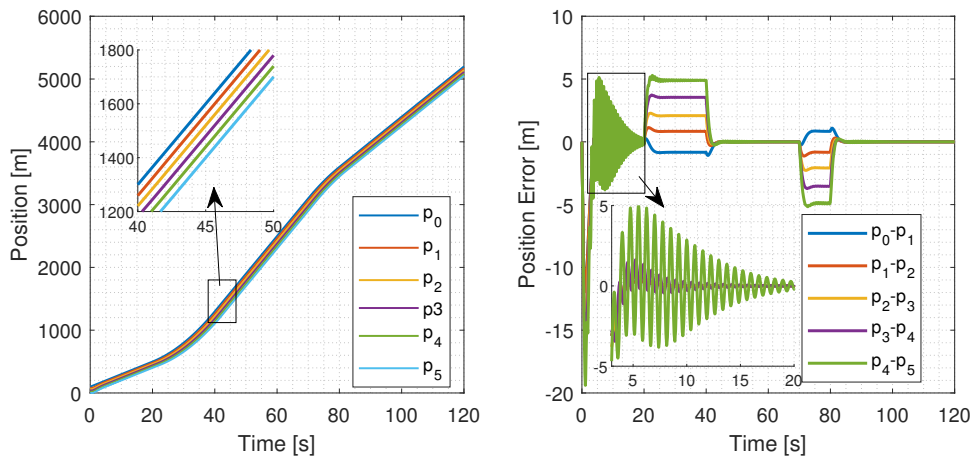


Figure 3.26 : Positions and position errors for $\tau = 0.18$ s.

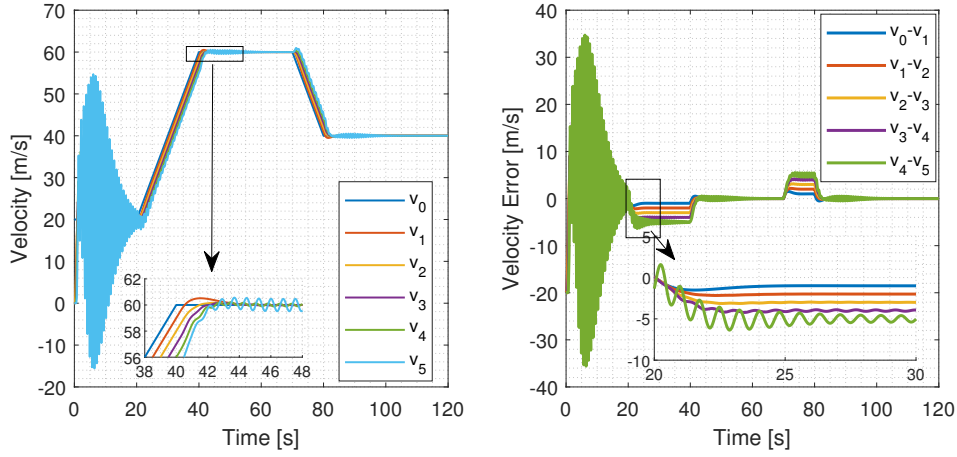


Figure 3.27 : Velocity and velocity errors for $\tau = 0.18$ s.

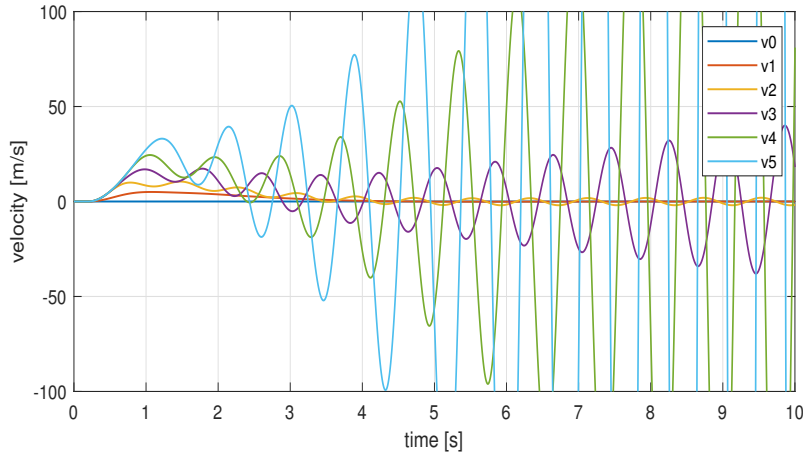


Figure 3.28 : Velocity and velocity errors for $\tau = 0.2$ s.

Simulation Example 3.7: In this numerical simulation, the problem explained in Example 3.2 is evaluated in case of time delay. $\mathcal{L}_p$ is symmetric; all the eigenvalues are real and different from each other. The characteristic equations given in Eq. (3.20) is converted to the following form when the time delay is considered.

$$\begin{aligned}
 Q(s, \tau) &= \det(sI_3 - A + \lambda_i B K e^{-\tau s}) \\
 &= \left((s^2(s+2+0.587k_a e^{-\tau s}) + 0.587(k_v s + k_p) e^{-\tau s}) \right) \\
 &\quad \left((s^2(s+2+4.264k_a e^{-\tau s}) + 4.264(k_v s + k_p) e^{-\tau s}) \right) \\
 &\quad \left((s^2(s+2+6.78k_a e^{-\tau s}) + 6.78(k_v s + k_p) e^{-\tau s}) \right) \\
 &\quad \left((s^2(s+2+10k_a e^{-\tau s}) + 10(k_v s + k_p) e^{-\tau s}) \right) \\
 &\quad \left((s^2(s+2+10.368k_a e^{-\tau s}) + 10.368(k_v s + k_p) e^{-\tau s}) \right)
 \end{aligned} \tag{3.91}$$

And, if the same procedure which is given in Example 3.5, for the system parameters given in Table 3.1 and for the controller gains of $k_v = 4$ and $k_a = 2$, is followed, the stability region for $k_p - \tau$ is found as in Figure 3.29.

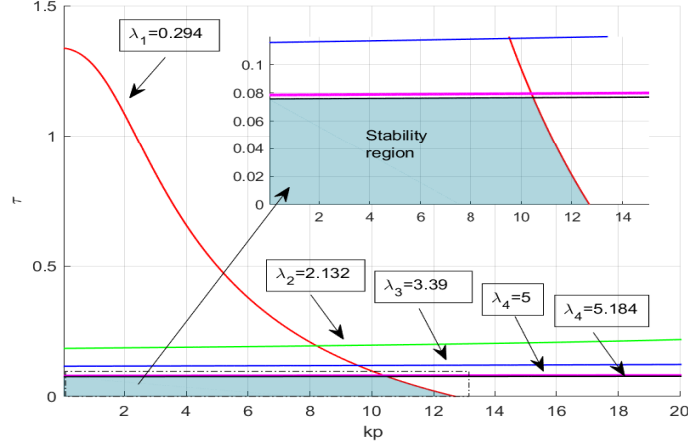


Figure 3.29 : Stability boundaries for $k_v = 4$ and $k_a = 2$.

Then, some of the set of parameters are selected for investigation of stability switching and simulations are conducted for the considered information flow topology. In first set, parameters $k_p = 2.4$ and $\tau = 0.04$ s which are on the stable region are tested. The results given in Figure 3.31 and 3.31 reveal that the system is stable, as expected.

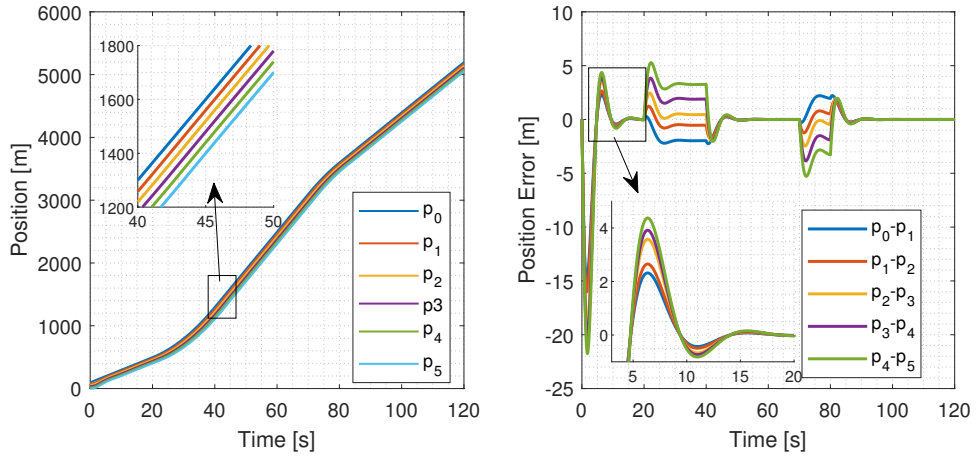


Figure 3.30 : Positions and position errors for $\tau = 0.04$ s.

In the second one, $k_p = 2.4$ kept constant and time delay is selected as $\tau = 0.0754$ s which is which in the stable region but close to the stability boundary. The results are shown in Figure 3.32 and 3.33. For the last test point, $k_p = 2.4$ is still kept constant, $\tau = 0.08$ s is selected in unstable region. In Figure 3.34, it is seen that the system is unstable.

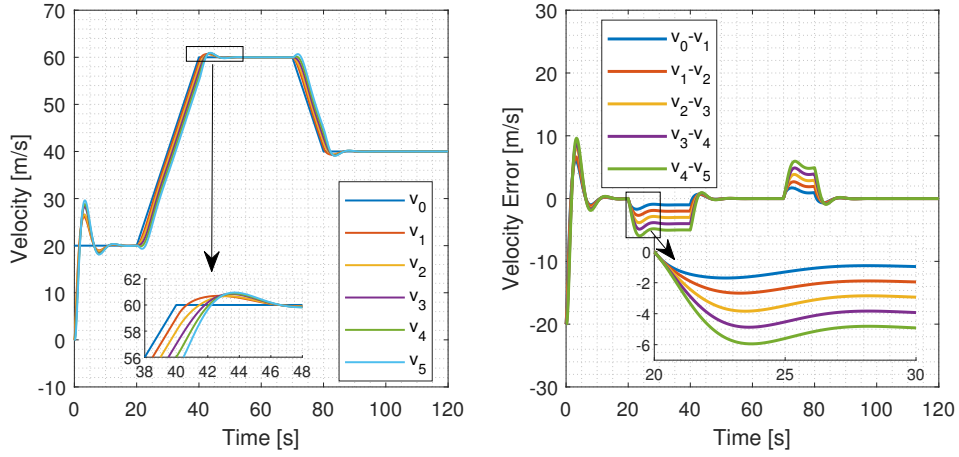


Figure 3.31 : Velocity and velocity errors for Example $\tau = 0.04$ s.

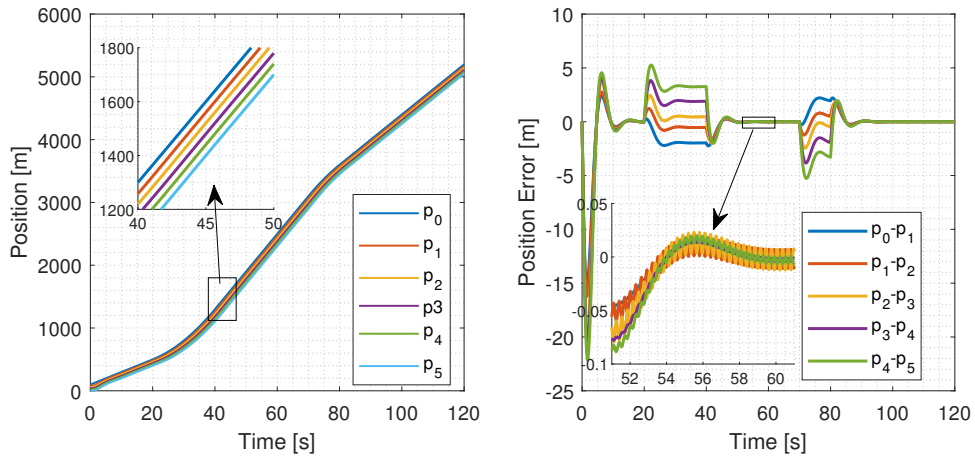


Figure 3.32 : Positions and position errors for $\tau = 0.0754$ s.

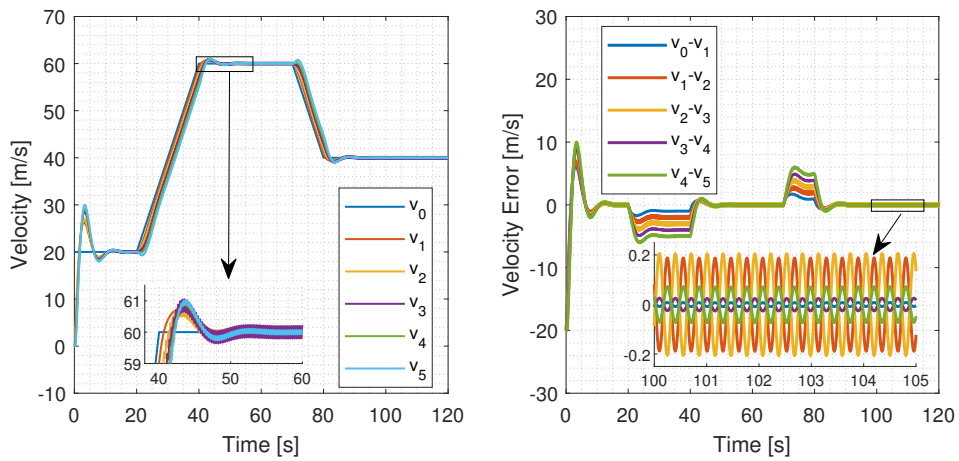


Figure 3.33 : Velocity and velocity errors for $\tau = 0.0754$ s.

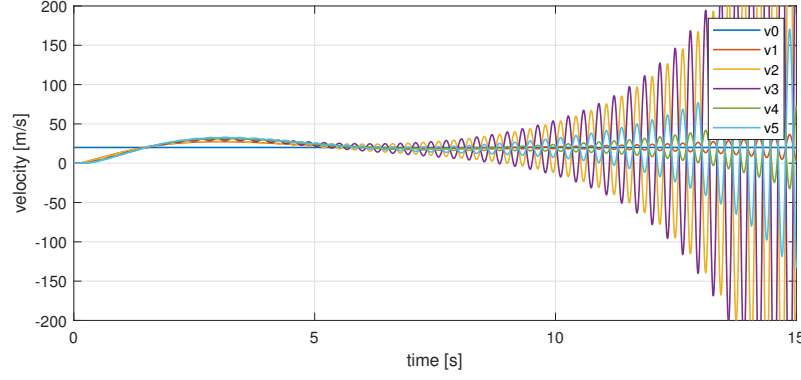


Figure 3.34 : Velocities for $\tau = 0.08$ s.

Simulation Example 3.8: In this numerical simulation, the problem explained in Example 3.3 is evaluated in case of time delay. $\mathcal{L}_p$ has both both real and complex eigenvalues. The characteristic equations given in Eq. (3.45) is converted to the following form when the time delay is considered.

$$\begin{aligned}
 Q(s, \tau)_{real} &= \det(sI_3 - A + \lambda_i B K e^{-\tau s}) \quad i = 1, 2, 3 \\
 &= (s^3 + (1.21k_a e^{-\tau s} + 2)s^2 + 1.21k_v e^{-\tau s} s + 1.21k_p e^{-\tau s}) \\
 &\quad (s^3 + (2.855k_a e^{-\tau s} + 2)s^2 + 2.855k_v s + 2.855k_p e^{-\tau s}) \\
 &\quad (s^3 + (5.632k_a e^{-\tau s} + 2)s^2 + 5.632k_v e^{-\tau s} s + 5.632k_p e^{-\tau s}) \\
 Q(s, \tau)_{complex} &= \det(sI_6 - (I_2 \otimes A) + (S_i \otimes B K) e^{-\tau s}) \quad i = 4, 5 \quad (3.92) \\
 &= (s^3 + ((8.15307 - 1.0661j)k_a e^{-\tau s} + 2)s^2 + (8.15307 \\
 &\quad - 1.0661j)k_v e^{-\tau s} s + (8.15307 - 1.0661j)k_p e^{-\tau s}) \\
 &\quad (s^3 + ((8.15307 + 1.0661j)k_a e^{-\tau s} + 2)s^2 + (8.15307 \\
 &\quad + 1.0661j)k_v e^{-\tau s} s + (8.15307 + 1.0661j)k_p e^{-\tau s})
 \end{aligned}$$

The procedure which is explained in Examples 4.6 and 4.7 is inefficient to obtain the stability boundary of the system in case of time delay since $\mathcal{L}_p$ matrix has complex eigenvalues. The reason is that the condition set given in Eq. (3.85) in Example 3.6 is obtained on the condition that the eigenvalues of $\mathcal{L}_p$ matrix are real. One of the CTCR methodologies which are given in Section 2 may be used to overcome the difficulties. Because of the ease of its implementation for higher-order systems, the *Matrix pencil*, *Kronecker summation* method introduced by [22] is used in this example to find the stability regions of the system. Although the methodology is explained in Section 2, a detailed analysis is given also here. The real and complex part of the characteristic

equation in Eq. (3.92) may be written in a generic form as in Eq. (3.93).

$$CE(s, z) = \det(sI - A_0 - B_1 z) \quad (3.93)$$

where;

$$z = e^{-\tau s}$$

$$\text{for real part: } A_0 = A, \quad B_1 = -\lambda_r BK$$

$$\text{for complex part: } A_0 = I_2 \otimes A, \quad B_1 = -(S_i \otimes BK)$$

Similarly, the complex conjugate characteristic equation of the system;

$$CE(s, z) = \det(s^* I - A_0 - B_1 z^*) \quad (3.94)$$

It is known that, finding the roots of Eq. (3.93) and Eq. (3.94) are equivalent of the following two matrix, respectively.

$$\begin{aligned} M_1 &= A_0 + B_1 z \\ M_2 &= A_0 + B_1 z^* \end{aligned} \quad (3.95)$$

As finding the kernel and offspring set is the main purpose, the purely imaginary roots of the system should be obtained. Therefore, we focus on the solutions to Eq. (3.93) for $s = \omega j$, and $z = e^{-\tau \omega j}$. Then, $s^* = -\omega j$ and $z^* = z^{-1}$. So, the *Auxiliary Characteristic Equation*, (ACE) for each factor;

$$CE_{ACE}(z) = \det[M_1 \oplus M_2] = \det[(A_0 + B_1 z) \oplus (A_0 + B_1 z^{-1})] = 0 \quad (3.96)$$

Notice that, Eq. (3.96) is algebraic equation in z for each factor, which is solved by *Maple* that is a symbolic and numerical computational environment. For this example, there are 5 different ACE, three of them is corresponded to real eigenvalues of $\mathcal{L}_p$ and the others for complex eigenvalues of $\mathcal{L}_p$. Then, Eq. (3.96) is solved for z and the unit norm are extracted. These extracted results are used in Eq. (3.93) to find the values of ω , and from $z = e^{-\tau \omega j}$ the values of delay are obtained. When finding the delays, one must always find the minimum positive value, to make sure it corresponds to a kernel point. The root tendencies are found from their definition in Eq. (3.88). This completes the construction of the kernel. Thus, for the system parameters given in 3.1 and for the controller gains of $k_v = 0.9773$ and $k_a = 1.0682$, which is same as delay-free case in Example 3.3, the stability region for $k_p - \tau$ is obtained as in Figure 3.35.

In Figure Eq. (??), the shaded region indicate the parameters that yield no characteristic root on the right half of the complex plane. The selected test points are used to

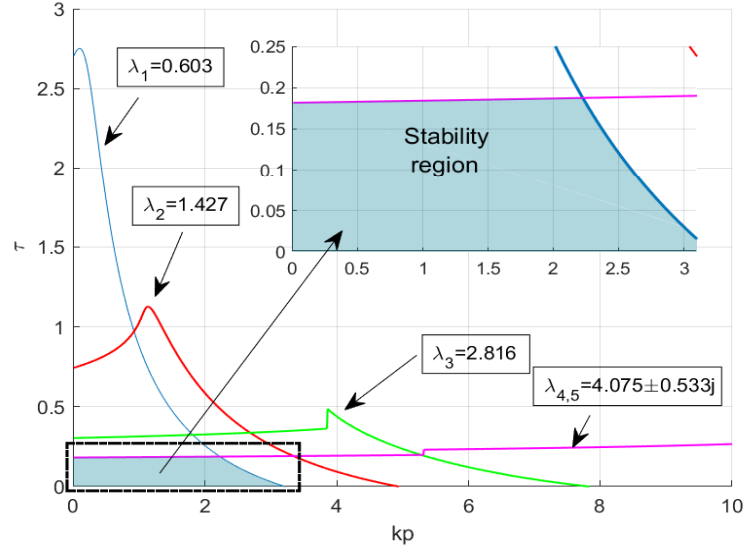


Figure 3.35 : Stability boundaries for $k_v = 0.9773$ and $k_a = 1.0682$.

investigate to validation of the stability region found in Figure 3.35. The first test points are chosen as $k_p = 0.2$ and $\tau = 0.15$ s, which is in the shaded region. The results are given in Figure ?? and ??, respectively and shown that the system is stable as expected.

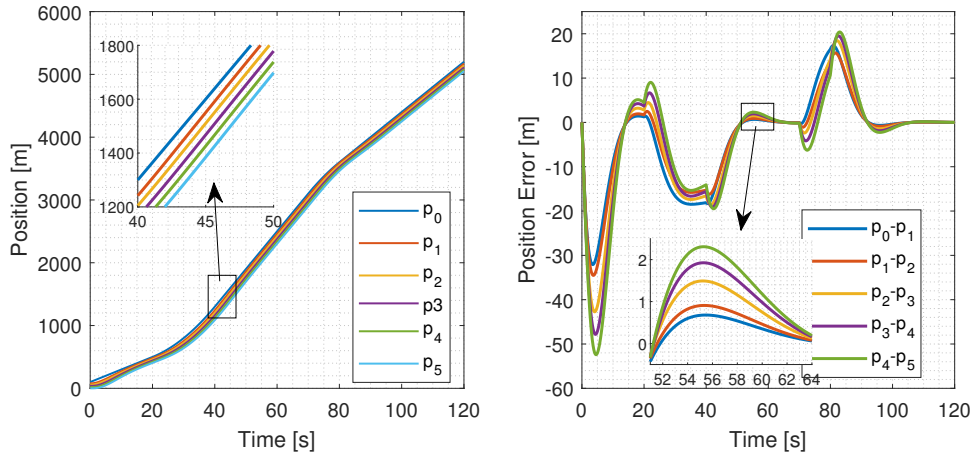


Figure 3.36 : Positions and position errors for $\tau = 0.15$ s.

On the other hand, the test points are selected as $k_p = 0.2$ and $\tau = 0.1819$ s, which is in the stable region but very close to the stability boundary ($\tau = 0.182$ s at the stability boundary for $k_p = 0.2$). Consequently, the effect of the time delay is visible directly on the simulation results given in Figure 3.38 and 3.39.

Finally, the test points are selected as $k_p = 0.2$ and $\tau = 0.185$ s, which is in unstable region. And, it is seen in the Figure 3.40 that the system is unstable, which conveys that

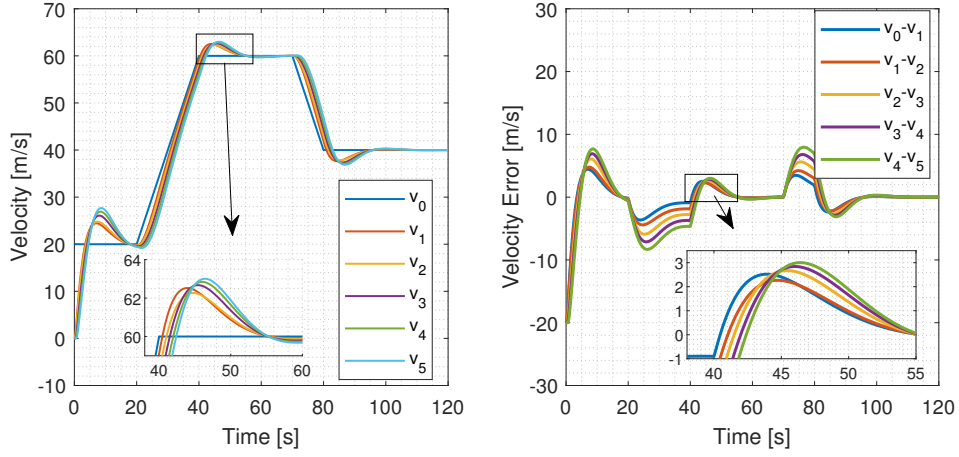


Figure 3.37 : Velocity and velocity errors $\tau = 0.15$ s.

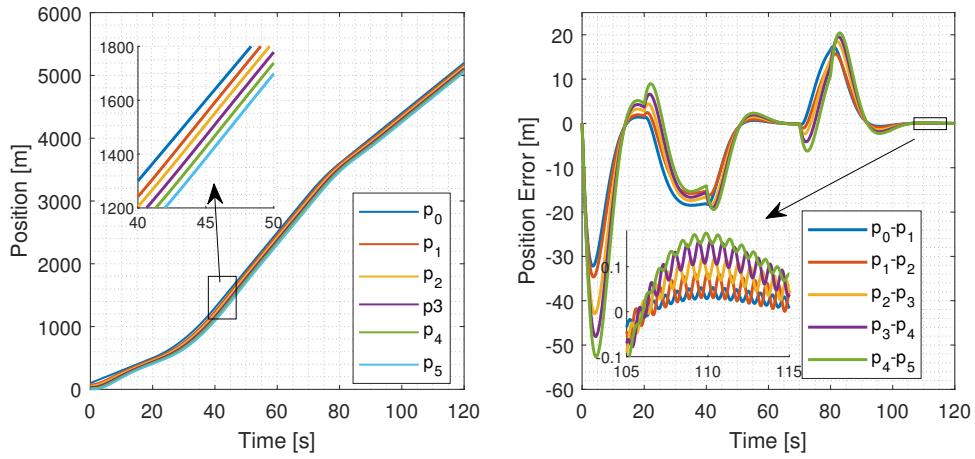


Figure 3.38 : Positions and position errors for $\tau = 0.182$ s.

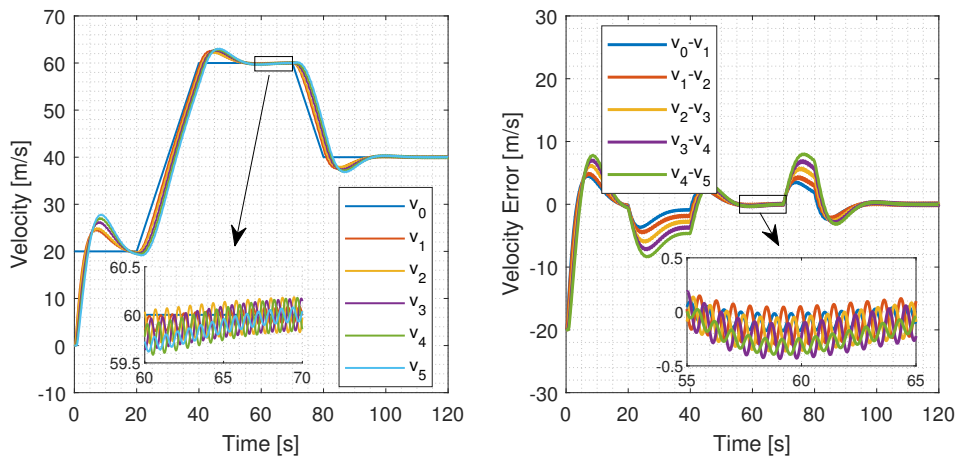


Figure 3.39 : Velocity and velocity errors for $\tau = 0.182$ s.

at least one pair of roots of the closed-loop system is on the right half of the complex plane.

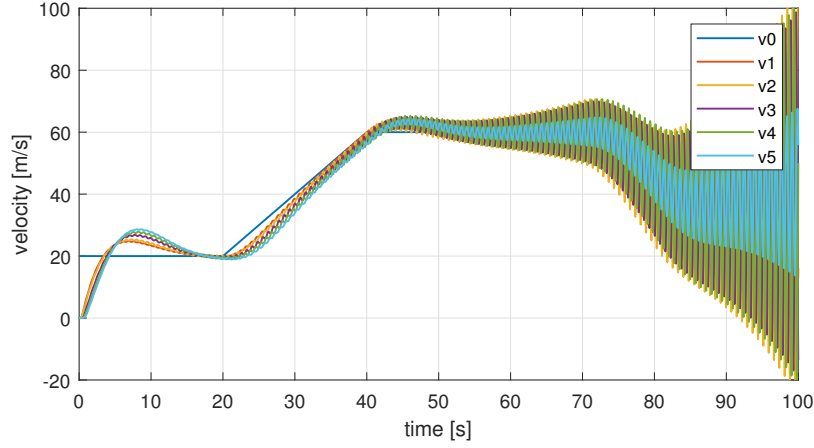


Figure 3.40 : Velocities for Example $\tau = 0.185$ s.

The rest of this section examines the stability conditions for the formation flight of UAV systems under single-time delay. The control signal is written in terms of tracking errors in the presence of time delay as follows;

$$u_i(t - \tau) = - \sum_{j=1}^N a_{ij} \left[k_p \left(e_{p_i}(t - \tau) - e_{p_j}(t - \tau) \right) + k_v \left(e_{v_i}(t - \tau) - e_{v_j}(t - \tau) \right) \right] - p_{ii} \left[k_p e_{p_i}(t - \tau) + k_v e_{v_i}(t - \tau) \right] \quad (3.97)$$

Then, the closed-loop system is obtained by substituting Eq. (3.97) into Eq. (3.51) a more compact form as follows;

$$\dot{E}(t) = (I_N \otimes A)E(t) - (\mathcal{L}_p \otimes BK)E(t - \tau) \quad (3.98)$$

where A and B matrices are defined in Eq. (2.76). It is seen that the structure of Eq. (3.98) is similar to Eq. (3.74). However, the matrix dimensions are different. Therefore, Theorem 3.3 may also be applicable for the formation flight system under time delay.

$$\begin{aligned} Q(s, \tau) &= \det(sI_{2N} - (I_N \otimes A) + (\mathcal{L}_p \otimes BK)e^{-\tau s}) \\ &= \det(sI_{2N} - (I_N \otimes A) + (J_i \otimes BK)e^{-\tau s}) \\ &= \prod_{i=1}^q \det(sI_2 - A + \lambda_i B K e^{-\tau s}) \prod_{i=q+1}^r \det(sI_4 - (I_2 \otimes A) + (S_i \otimes BK)e^{-\tau s}) \end{aligned} \quad (3.99)$$

In the following example, how CTCR methodology is applied to the formation flight under the time delay of the UAVs system is examined in detail.

Simulation Example 3.9: In this numerical simulation, the problem explained in Example 3.5 is evaluated in case of time delay. $\mathcal{L}_p$ has both both real and complex eigenvalues. The characteristic equations given in Eq. (3.70) is converted to the following form when the time delay is considered.

$$\begin{aligned}
Q(s)_{real} &= \det(sI_2 - A + \lambda_i BK) \quad i = 1, 2, 3, 4 \\
&= (s^2 + 0.2451k_v s e^{-\tau s} + 0.2451k_p e^{-\tau s}) \\
&\quad (s^2 + 0.3820k_v s e^{-\tau s} + 0.3820k_p e^{-\tau s}) \\
&\quad (s^2 + 2k_v s e^{-\tau s} + 2k_p e^{-\tau s}) \\
&\quad (s^2 + 2.6180k_v e^{-\tau s} s + 2.6180k_p e^{-\tau s}) \quad (3.100)
\end{aligned}$$

$$\begin{aligned}
Q(s)_{complex} &= \det(sI_4 - (I_2 \otimes A) + (S_i \otimes BK)) \quad i = 5, 6 \\
&= s^2 + (1.8774 + 0.7449j)k_v s e^{-\tau s} + (1.8774 + 0.7449j)k_p e^{-\tau s} \\
&\quad s^2 + (1.8774 - 0.7449j)k_v s e^{-\tau s} + (1.8774 - 0.7449j)k_p e^{-\tau s} \\
Q(s) &= Q(s)_{real} Q(s)_{complex}
\end{aligned}$$

All the quasi-polynomials in Eq. (3.100) have infinitely many characteristic roots, and all of them should be in the left half of the complex plane. This example applies a new proposed CTCR algorithm based on the Bezout resultant to each factor. First, the method is explained in a general class of a single-time delay system. Then, it is applied one of the factors of Eq. (3.100). The characteristic equation of a general form of the single time-delay system may be written as in Eq. (3.101)

$$Q(s, \tau) = \det(sI - A - B e^{-\tau s}) \quad (3.101)$$

The first step in the CTCR method is finding the kernel and offspring spaces of the system. To this end, the exhaustive determination of the finite number of crossing frequencies is needed. It is achieved by finding the solution of $s = j\omega$, which corresponds to the imaginary axis crossing of the roots of the characteristic equation. Then, the characteristic equation is rearranged with the transform of $z = e^{-\tau s}$ as follows;

$$Q_1(s, z) = \det(sI - A - Bz) \quad (3.102)$$

As the system matrices A , B_1 and B_2 pose real elements, then the roots of characteristic equation may only be either real or complex conjugate pairs. Thus, the complex conjugates of s and z also satisfy the equation. Therefore, the characteristic equation

may also be written for $s^* = -s = -j\omega$ and $z^* = z^{-1}$ as follows:

$$Q_2(s^*, z^*) = \det(s^* I - A - Bz^*) \quad (3.103)$$

Consequently, the problem of finding the kernel and offspring curves of the system is transformed into obtaining the common purely imaginary roots of two polynomials Q_1 and Q_2 . Then, the two polynomials Q_1 and Q_2 can be rewritten as a polynomial of s .

$$\begin{aligned} Q_1(s, z) &= \sum_{i=0}^n f_i(z) s^i = 0 \\ Q_2(s, z) &= \sum_{i=0}^n g_i(z) s^i = 0 \end{aligned} \quad (3.104)$$

The necessary condition for Eq. (3.104) have a common root, the following Bezout resultant matrix should be singular.

$$B(Q_1(s, z), Q_2(s, z), s) = b_{i,j=1,\dots,n} \quad (3.105)$$

where;

$$\begin{aligned} b_{n-i+1,j} &= \text{Coeff} \left(\left(\sum_{r=i}^n g_{n-r} s^{n-r} \right) \left(\sum_{r=i}^i f_{n-r+1} s^{i-r} \right) - \right. \\ &\quad \left. \left(\sum_{r=i}^n f_{n-r} s^{n-r} \right) \left(\sum_{r=i}^i g_{n-r+1} s^{i-r} \right), s, n-j \right), \quad \text{for } i, j = 1, \dots, n \end{aligned} \quad (3.106)$$

In Eq. (3.106), the command *Coeff* gives the coefficients of s^{n-j} . That is,

$$\det(B) = D(z) = 0 \quad (3.107)$$

Then, from the solution of Eq. (3.107), extract those with $|z| = 1$, replace these values of z in Eq. (3.102), and solve for s . The roots of Eq. (3.102) reveals the crossing frequencies for $s = \omega j$. Finally, the respective delay points are found as follows;

$$\tau_{jk} = \frac{\arg(z_j) \pm 2k\pi}{\omega} \quad (3.108)$$

The following results are obtained if these operations are applied to the first factor, $(s^2 + 0.2451k_v s e^{-\tau s} + 0.2451k_p e^{-\tau s})$, in the Eq. (3.100) for $k_v = 27.1661$. Then, $Q_1(s, z)$ and $Q_2(s, z)$ are written as follows;

$$\begin{aligned} Q_1(s, z) &= s^2 + 54.3322zs + 2zk_p \\ Q_2(s, z) &= zs^2 - 54.3322s + 2k_p \end{aligned} \quad (3.109)$$

The related Bezout resultant matrix;

$$B = \begin{bmatrix} 2k_p - 2z^2k_p & 217.3288zk_p \\ -54.3322 - 54.3322z^2 & 2k_p - 2z^2k_p \end{bmatrix} \quad (3.110)$$

And the determinant of Bezout resultant matrix Eq. (3.110)

$$D(z) = 4k_p^2z^4 + 11807.95183k_pz^3 - 8k_p^2z^2 + 11807.95183k_pz + 4k_p^2 = 0 \quad (3.111)$$

Specifically, the value of k_p is increased from 0 to 10 with an iterative step of 0.01 and for each of these values the Eq. (3.111) is solved. Those roots with unity magnitude form the corresponding z values. Substitute z into Eq. (3.109) and solve for $s = \omega j$, the imaginary crossing frequency of the system. Lastly, the delay points are found with Eq. (3.108). Then, increase k_p and repeat the steps.

The stability boundary region given in Figure 3.41 is obtained when these steps are applied to each factor of Eq. (3.100).

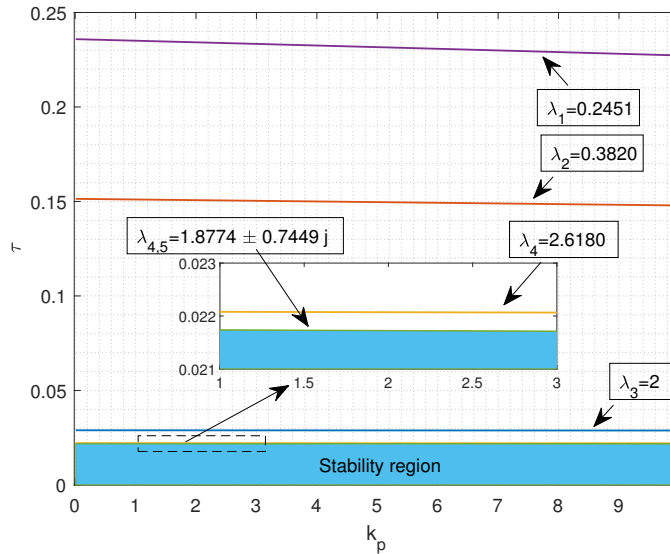


Figure 3.41 : Stability boundary for $k_v = 27.1661$.

In Figure 3.41, the shaded region indicate the parameters that yield no characteristic root on the right half of the complex plane. The selected test points are used to investigate to validation of the stability region found in Figure 3.41. The first test points are chosen as $k_p = 3$ and $\tau = 0.01$ s, which is in the shaded region. The results are given in Figure 3.42, 3.43 and 3.44, respectively and shown that the system is stable as expected.

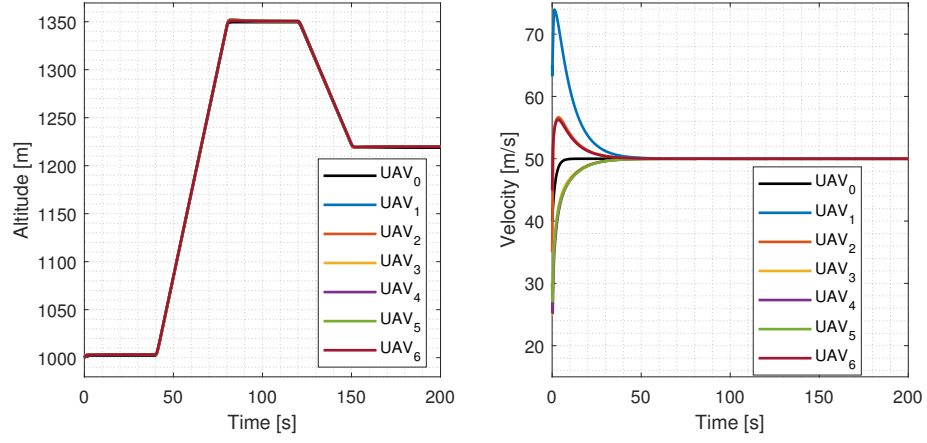


Figure 3.42 : Altitude and velocity for $\tau = 0.01$.

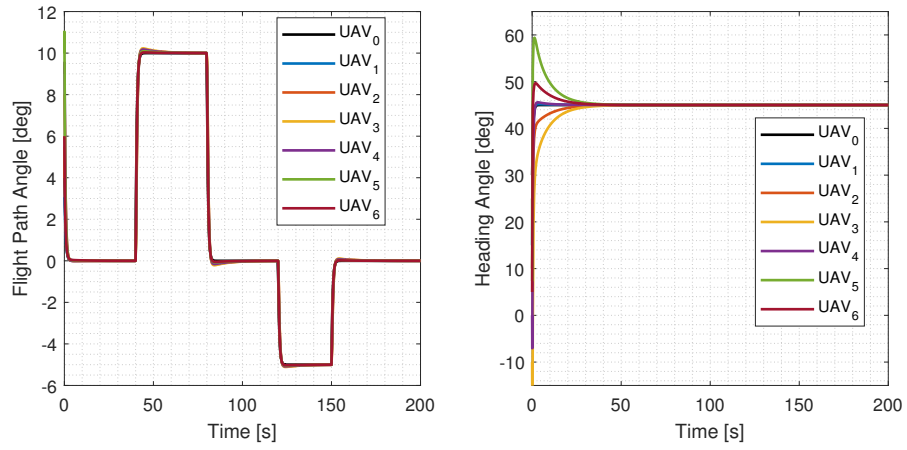


Figure 3.43 : Flight path angle and heading angle for $\tau = 0.01$.

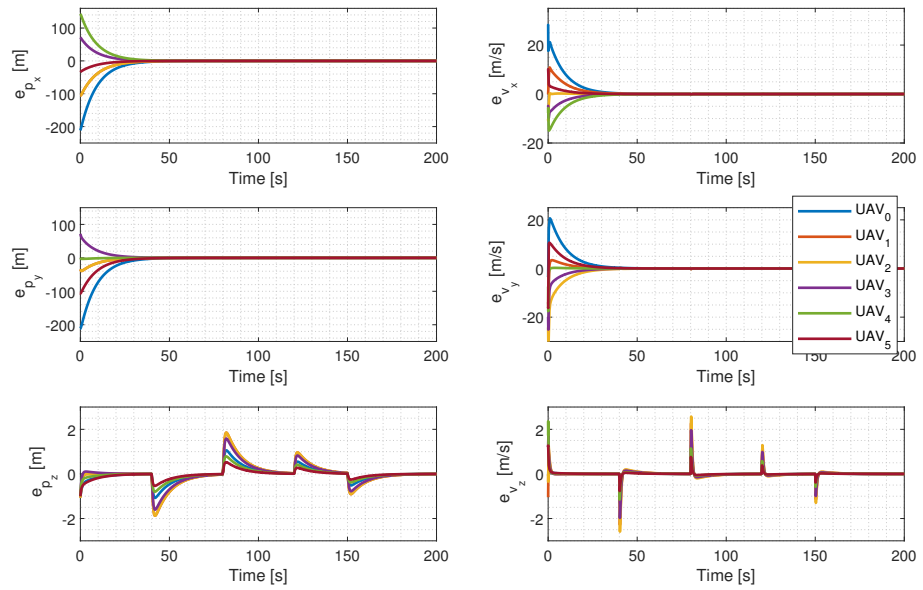


Figure 3.44 : Position errors and velocity errors for $\tau = 0.01$.

On the other hand, the test points are selected as $k_p = 3$ and $\tau = 0.017$ s, which is on the stability boundary. Consequently, the effect of the time delay is visible directly on the simulation results of flight path angle and heading angles in Figure 3.45.

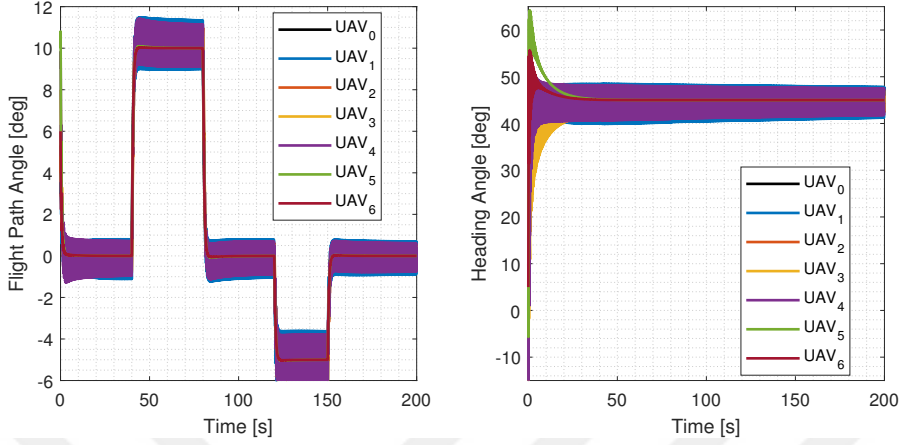


Figure 3.45 : Flight path angle and heading angle for $\tau = 0.022$ s.

The last test point is selected on an unstable region as $\tau=0.022$, and the simulation results of altitude and velocity given in Figure 3.45 show that the system is unstable.

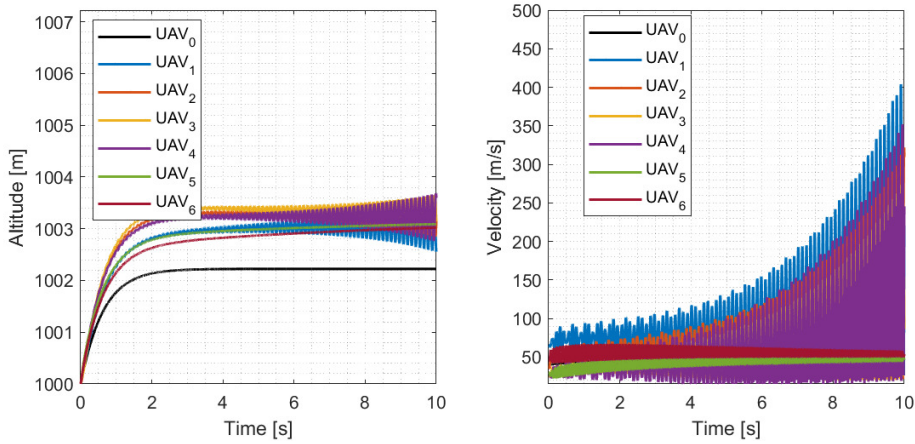


Figure 3.46 : Altitude and velocity for $\tau = 0.022$.

When the system is exposed to both input and communication time delays, the control signal of the platooning system is written in the following form as given in Eq. (3.112);

$$u_i(t - \tau) = - \sum_{j=1}^N \left[a_{ij} \left(k_p (e_{p_i}(t - \tau_1) - e_{p_j}(t - \tau_2)) + k_v (e_{v_i}(t - \tau_1) - e_{v_j}(t - \tau_2)) + k_a (e_{a_i}(t - \tau_1) - e_{a_j}(t - \tau_2)) \right) \right] - p_{ii} \left[k_p e_{p_i}(t - \tau_1) + k_v e_{v_i}(t - \tau_1) + k_a e_{a_i}(t - \tau_1) \right] \quad (3.112)$$

The closed-loop system formulation of the system under multiple time delay is obtained by substituting Eq. (3.112) into Eq. (3.4). Then, the compact form is written with the help of the Kronecker product as follows;

$$\dot{E}(t) = (I_N \otimes A)E(t) - (\mathcal{D}_p \otimes BK)E(t - \tau_1) - (\mathcal{A} \otimes BK)E(t - \tau_2) \quad (3.113)$$

In here, the definition of A and B matrices are given in Eq. (2.72), $E(t)$ and K are given in Eq. (3.6). Moreover $\mathcal{A}$ and $\mathcal{D}_p$ are Adjacency and Information Topology Matrices which are given in Section 2.

As seen from the structure of Eq. (3.113), is not an easy task to factorize the characteristic equation into sub-factors as the delay parts of the equation can not converted into diagonal form. So, the characteristic equation should be evaluate as a whole, which makes the analysis is cumbersome. If the degree of individual vehicle dynamic characteristic equation is r ($r = 3$), and the platoon has N follower vehicles, the characteristic equation of the whole platoon becomes $N * r$. This means that each vehicles adds three poles to the system. When the characteristic equation is evaluated as a whole, getting rid of the exponential terms simplifies the analysis. To achieve this, an exact relation which transform the transcendental characteristic equation ($Q(s, \tau)$) to a quasi polynomials ($Q(s, q)$) by replacing the exponential term caused by the delays may be used. It is known from the Section 2 that $s = \omega j$ solutions of $Q(s, \tau) = 0$ and $Q(s, q) = 0$ are identical for some τ and q .

Remark 4.3 : The stability analysis of this kind of multiple time-delay systems are generally obtained by Lyapunov-Razumikhin and Lyapunov-Krasowski theorems [105, 106], and the solution of this method generally bases on the LMIs. Bezout resultant matrix based CTCR method which used in this study is an alternative way to obtain the stability boundaries of the the system for uncertain delays. This methodology is a practical and numerically efficient procedure for analyzing robustness of the systems against uncertain delays. On the other hand, it should be indicated that, when the number of vehicles in the platoon increases, this method requires more computational power since the sizes of the closed-loop matrices increases drastically. However this drawback may be eliminated with using high performance computers (super computer) or with some modification on algorithm.

Applying the CTCR paradigm to multiple time-delay systems is explained below.

The characteristic equation of Eq. (3.113) is given in the form;

$$Q(s, \tau_1, \tau_2) = \det(sI - A_1 - B_1 e^{-s\tau_1} - B_2 e^{-s\tau_2}) = 0 \quad (3.114)$$

where;

$$A_1 = I_N \otimes A$$

$$B_1 = \mathcal{D}_p \otimes BK$$

$$B_2 = \mathcal{A} \otimes BK$$

The first step is finding the kernel and offspring spaces of the system. The solution at $s = \omega j$ which corresponds to the imaginary axis crossings of the roots of the characteristic equation should be obtained. It may possible to rewrite the characteristic equation by substituting $z_1 = e^{-s\tau_1}$ and $z_2 = e^{-s\tau_2}$ to reduce the complexity.

$$Q_1(s, z_1, z_2) = \det(sI - A_1 - B_1 z_1 - B_2 z_2) = 0 \quad (3.115)$$

As the system matrices A_1 , B_1 and B_2 pose real elements, then the roots of characteristic equation may only be either real or complex conjugate pairs. Thus, the complex conjugates of s , z_1 and z_2 also satisfy the equation. Therefore, the characteristic equation may also be written as follows;

$$Q_2(s^*, z_1^*, z_2^*) = \det(s^*I - A_1 - B_1 z_1^* - B_2 z_2^*) = 0 \quad (3.116)$$

Where $s^* = -\omega j$, $z_1^* = z_1^{-1}$ and $z_2^* = z_2^{-1}$. As a consequence, the problem of finding kernel and offspring of the system is transformed into obtaining the common roots of two polynomials Q_1 and Q_2 . In other words, if there is an imaginary solution at $s = \omega j$ of Eq. (3.114), both Q_1 Eq. (3.115) and Q_2 Eq. (3.116) have a common root.

To find the resultant of Eq. (3.115) and Eq. (3.116) Bezout resultant theory which is explained in Section 2 may be utilized. Bezout matrix is created with the help of the coefficients of Q_1 and Q_2 . And, it is singular for a common root of Q_1 and Q_2 . The algorithm to find the delay space of the characteristic equation in Eq. (3.114) is given in detail with the following steps;

Step 1: Write the characteristic equation of the time-delay system as Eq. (3.114). And create Q_1 and Q_2 forms by substituting $z_i = e^{-j\omega\tau_i}$. In here, the Q_1 and Q_2 may be written one of the following structures.

$$Q_1(s, z_1, z_2) = \sum_{i=1}^{r*N} f_i(z_1, z_2) s^i, \quad Q_2(s, z_1, z_2) = \sum_{i=1}^{r*N} g_i(z_1, z_2) s^i \quad (3.117)$$

$$Q_1(s, z_1, z_2) = \sum_{i=1}^N f_i(s, z_2) z_1^i, \quad Q_2(s, z_1, z_2) = \sum_{i=1}^N g_i(s, z_2) z_1^i \quad (3.118)$$

$$Q_1(s, z_1, z_2) = \sum_{i=1}^N f_i(s, z_1) z_2^i, \quad Q_2(s, z_1, z_2) = \sum_{i=1}^N g_i(s, z_1) z_2^i \quad (3.119)$$

Step 2: Create the Bezout resultant matrix B_Q , of Q_1 and Q_2 which contains their coefficient parameters. The element of Bezout resultant matrix is constituted with the help of Eq. (3.106)

$$B(Q_1(s, z_1, z_2), Q_2(s, z_1, z_2)) = b_{i,j=1,\dots,n} \quad (3.120)$$

Step 3: The determinant of Bezout resultant matrix $D = \det(B) = 0$ is created according to selection of Q_1 and Q_2 given in Eq. (3.117), Eq. (3.118) and Eq. (3.119). Since the determinant is also a polynomial structure, it can be written in two ways for each different choice of Q_1 and Q_2 as follows.

For Eq. (3.117);

$$(a) \quad D(Q_1, Q_2) = \sum_{i=1}^M c_i(z_1) z_2^i, \quad (b) \quad D(Q_1, Q_2) = \sum_{i=1}^M c_i(z_2) z_1^i \quad (3.121)$$

For Eq. (3.118);

$$(a) \quad D(Q_1, Q_2) = \sum_{i=1}^M c_i(s) z_2^i, \quad (b) \quad D(Q_1, Q_2) = \sum_{i=1}^M c_i(z_2) s^i \quad (3.122)$$

For Eq. (3.119);

$$(a) \quad D(Q_1, Q_2) = \sum_{i=1}^M c_i(s) z_1^i, \quad (b) \quad D(Q_1, Q_2) = \sum_{i=1}^M c_i(z_1) s^i \quad (3.123)$$

Step 4: If the determinant of Bezout matrix is created with the first form given in Eq. (3.121);

4.1: Select z_1 using a secondary parameter, θ , as $z_1 = e^{j\theta}$, $\theta \in [0, 2\pi]$; start with $\theta = 0$.

4.2: With this value of z_1 , solve the roots of Eq. (3.121)-(a). Those roots with unity magnitude forms the corresponding z_2 values.

4.3: Substitute z_1 and z_2 into Eq. (3.115) and solve ω ($s = \omega j$), the imaginary spectrum of the system.

4.4: Then increase θ by a desired resolution, $\delta\theta$, and repeat steps 4.1 – 4.3.

4.5: As θ reaches 2π a complete set of imaginary spectra and the corresponding kernel points $(\tau_1, \tau_2) \in \wp Ker$ are obtained using Eq. (3.124).

$$\tau_{jk} = \frac{\arg(z_j) \pm 2k\pi}{\omega}, \quad j = 1, 2, \quad k = 0, 1, 2, \dots \quad (3.124)$$

Similar steps may be followed in case of the determinant of Bezout matrix is created as Eq. (3.121)-b.

Step 5: If the determinant of Bezout matrix is created with the first form given in Eq. (3.122);

5.1: Select $s = \omega_c j$ using a secondary parameter, ω_c , as $\omega_c \in [0, \omega_{c_{max}}]$; start with $\omega = 0$.

5.2: With the value of s , solve the roots of Eq. (3.122)-(a). Those roots unity magnitude forms the corresponding z_2 values.

5.3: Substitute s and z_2 into Eq. (3.115) and solve it. Those roots unity magnitude forms the corresponding z_2 values.

5.4: Then increase ω_c by a desired resolution, $\delta\omega$, and repeat steps 5.1 – 5.3.

5.5: As ω_c reaches $\omega_{c_{max}}$ a complete set of imaginary spectra and the corresponding kernel points $(\tau_1, \tau_2) \in \wp Ker$ are obtained using Eq. (3.124).

Similar steps may be followed in case of the determinant of Bezout matrix is created as Eq. (3.123)-a.

Step 6: If the determinant of Bezout matrix is created with the second form given in Eq. (3.123);

6.1: Select z_1 using a secondary parameter, θ , as $z_1 = e^{j\theta}$, $\theta \in [0, 2\pi]$; start with $\theta = 0$.

6.2: With this value of z_1 , solve the roots of Eq. (3.121)-(a). Those roots with purely imaginary value form the corresponding s value.

6.3: Substitute s and z_1 into Eq. (3.115) and solve it. Those roots unity magnitude forms the corresponding z_1 values.

6.4: Then increase θ by a desired resolution, $\delta\theta$, and repeat steps 6.1 – 6.3.

6.5: As θ reaches 2π a complete set of imaginary spectra and the corresponding kernel points $(\tau_1, \tau_2) \in \wp Ker$ are obtained using Eq. (3.124).

In the following example, the Bezout-resultant-based CTCR methodology explained in above is applied to an example.

Simulation Example 3.10: In this numerical simulation, the problem given in Example 3.3 is evaluated in case of multiple time delay. The Adjacency matrix $\mathcal{A}$ and information topology matrix $\mathcal{D}_p = \mathcal{D} + \mathcal{P}$ which take part in Eq. (3.113), are given as;

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{D}_p = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad (3.125)$$

The system parameters given in 3.1 are utilized and the leader speed profile given in Eq. (3.18). The controller parameters are selected as follows which is found in Eq. (3.48) for delay-free case.

$$K = [kp \quad kv \quad ka] = [0.2740 \quad 0.9773 \quad 1.0682] \quad (3.126)$$

The characteristic equation of the system is obtained for $z_1 = e^{-s\tau_1}$ and $z_2 = e^{-s\tau_2}$ as follows;

$$\begin{aligned} Q_1(s, z_1, z_2) &= \det(sI_{15} - (I_N \otimes A) - (\mathcal{D}_p \otimes BK)z_1 - (\mathcal{A} \otimes BK)z_2 \\ &= s^{15} + (10.0 + 27.7732z_1)s^{14} + (40.0 + 305.8017z_1^2 - 18.2568z_2^2 + \\ &\quad 247.5954z_1)s^{13} + (-292.5290z_1z_2^2 + 80.0 + 1667.4154z_1^3 - 29.2529z_2^3 \\ &\quad + 2394.3685z_1^2 - 142.9473z_2^2 + 876.9592z_1)s^{12} + (-291.6475z_1z_2^3 \\ &\quad - 1541.5655z_1^2z_2^2 + 4499.7048z_1^4 + 62.4959z_2^4 + 11246.2347z_1^3 \\ &\quad - 197.3023z_2^3 + 7439.8209z_1^2 - 444.1684z_2^2 + 1555.5696z_1 - \\ &\quad 1973.0236z_1z_2^2 + 80.0)s^{11} + (311.5378z_1z_2^4 - 2670.3248z_1^3z_2^2 \\ &\quad - 712.0866z_1^2z_2^3 + 89.0108z_2^5 + 4806.5847z_1^5 + 25466.5939z_1^4 \\ &\quad + 353.7026z_2^4 + 30446.1845z_1^3 - 534.1435z_2^3 - 703.3879z_2^2 + \\ &\quad 11781.7485z_1^2 - 1650.6125z_1z_2^3 - 8724.6664z_1^2z_2^2 - \\ &\quad 5341.4358z_1z_2^2 + 32.0 + 1428.4608z_1)s^{10} + (407.1816z_2^5 + \\ &\quad 21987.8077z_1^5 + 60150.0072z_1^4 + 835.4167z_2^4 - 768.6314z_2^3 + \\ &\quad 43811.9918z_1^3 + 1425.1356z_1z_2^4 - 12215.4487z_1^3z_2^2 \\ &\quad - 3257.4530z_1^2z_2^3 - 3898.6115z_1z_2^3 - 20606.9469z_1^2z_2^2 - \end{aligned} \quad (3.127)$$

$$\begin{aligned}
& 7686.3143 z_1 z_2^2 - 615.6405 z_2^2 + 10311.9793 z_1^2 + 634.5248 z_1) s^9 \\
& + (859.2230 z_2^5 + 46398.0435 z_1^5 + 1123.4294 z_2^4 + 80886.9183 z_1^4 \\
& + 3007.28060 z_1 z_2^4 - 25776.6908 z_1^3 z_2^2 - 6873.7842 z_1^2 z_2^3 - \\
& 27711.2590 z_1^2 z_2^2 - 5242.6706 z_1 z_2^3 - 662.8628 z_2^3 + 37783.1801 z_1^3 - \\
& 307.2174 z_2^2 + 5145.8926 z_1^2 + 113.984 z_1 - 6628.6281 z_1 z_2^2) s^8 + \\
& (1099.4401 z_2^5 + 59369.7678 z_1^5 - 8795.5211 z_1^2 z_2^3 - 32983.2043 z_1^3 z_2^2 \\
& + 3848.0405 z_1 z_2^4 - 23787.5209 z_1^2 z_2^2 + 964.3589 z_2^4 + 69433.8449 z_1^4 \\
& - 358.1196 z_2^3 + 20412.8209 z_1^3 + 1389.6859 z_1^2 - 82.9663 z_2^2 \\
& - 4500.3418 z_1 z_2^3 - 3581.1966 z_1 z_2^2) s^7 + (943.7330 z_2^5 + \\
& 50961.58482 z_1^5 - 7549.8644 z_1^2 z_2^3 - 28311.9915 z_1^3 z_2^2 + \\
& 3303.0656 z_1 z_2^4 + 553.2106 z_2^4 + 39831.1641 z_1^4 + 6845.2057 z_1^3 \\
& - 120.0913 z_2^3 + 160.9629 z_1^2 - 9.6097 z_2^2 - 2581.6495 z_1 z_2^3 - \\
& 13645.8617 z_1^2 z_2^2 - 1200.9132 z_1 z_2^2) s^6 + (567.5035 z_2^5 + \\
& 30645.1921 z_1^5 - 4540.0284 z_1^2 z_2^3 - 17025.1067 z_1^3 z_2^2 \\
& + 1986.2624 z_1 z_2^4 + 213.3714 z_2^4 + 15362.7413 z_1^4 + \\
& 1317.0346 z_1^3 - 23.1058 z_2^3 - 995.7332 z_1 z_2^3 - 5263.16138 z_1^2 z_2^2 \\
& - 231.0587 z_1 z_2^2) s^5 + (242.0734 z_2^5 + 13071.9661 z_1^5 \\
& - 1936.5875 z_1^2 z_2^3 - 7262.20341 z_1^3 z_2^2 + 847.2570 z_1 z_2^4 + \\
& 3859.2472 z_1^4 + 53.6006 z_2^4 + 112.5635 z_1^3 - 1.9747 z_2^3 - 250.1363 z_1 z_2^3 \\
& - 1322.1495 z_1^2 z_2^2 - 19.7479 z_1 z_2^2) s^4 + (72.3381 z_2^5 + 3906.2616 z_1^5 \\
& - 578.7054 z_1^2 z_2^3 - 2170.1453 z_1^3 z_2^2 + 253.1836 z_1 z_2^4 + 575.3111 z_1^4 \\
& + 7.9904 z_2^4 - 37.2886 z_1 z_2^3 - 197.0973 z_1^2 z_2^2) s^3 + \\
& (14.5010 z_2^5 + 783.0574 z_1^5 - 116.0085 z_1^2 z_2^3 - 435.0319 z_1^3 z_2^2 \\
& + 50.7537 z_1 z_2^4 + 38.9588 z_1^4 + 0.541 z_2^4 - 2.5251 z_1 z_2^3 \\
& - 13.3470 z_1^2 z_2^2) s^2 + (1.7627 z_2^5 + 95.1861 z_1^5 - 14.1016 z_1^2 z_2^3 \\
& - 52.8812 z_1^3 z_2^2 + 6.1694 z_1 z_2^4) s + 0.09884 z_2^5 + 5.3373 z_1^5 \\
& + 0.3459 z_1 z_2^4 - 2.9652 z_1^3 z_2^2 - 0.7907 z_1^2 z_2^3
\end{aligned}$$

Since the platooning system consists of $N = 5$ follower vehicle and each vehicle comprises $r = 3$ dynamics, the characteristic equation has $N * r = 15$ orders. Moreover

the coefficients includes two different time delays, so it is not possible to simplify the characteristic equation by factorization. The stability problem of the entire system should be tackled. The Bezout resultant based CTCR method which is explained in above is applied to the system the stability delay space for (τ_1, τ_2) are obtained as in Figure 3.47.

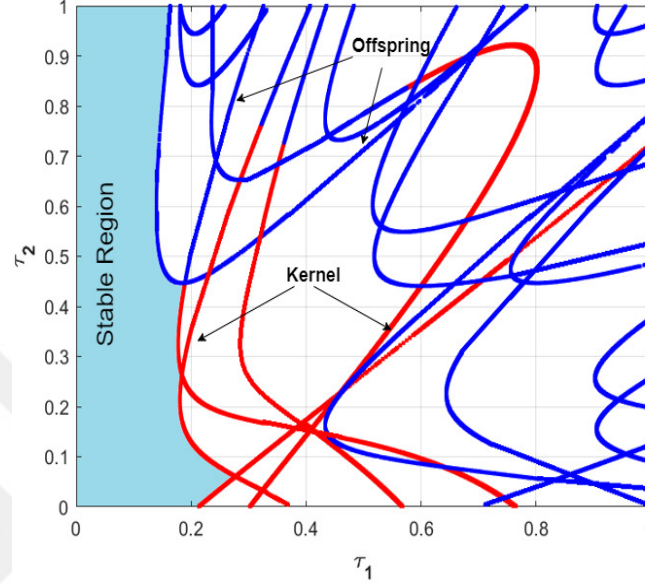


Figure 3.47 : Stability boundaries for $k_p = 0.2740$, $k_v = 0.9773$ and $k_a = 1.0682$.

One point which should be emphasized is that when the system is exposed to multiple time delays the stability map becomes more complicated as it compares with the single time delay case. The shaded region in Figure 3.47 shows the stability space. The kernels are depicted thick (red) and offsprings are thin (blue) in Figure 3.47, respectively. Time delay-induced instability occurs when time delay pairs are out of the shaded region, which is simulated and described in the following simulations. The test point, $\tau_1 = 0.05$ and $\tau_2 = 0.5$ is selected on the shaded region to exhibit the stability of the system. The results are displayed for position and position errors in Figure 3.48 and velocity and velocity errors in Figure 3.49.

Then, the second test point is selected on the stability boundary as $\tau_1 = 0.144s$ and $\tau_2 = 0.5s$ to display the marginal stability behavior of the system. The simulation results are depicted for position/position errors and velocity/velocity errors in Figure 3.50 and 3.51, respectively.

Finally, the test point is selected in unstable region as $\tau_1 = 0.16s$ and $\tau_2 = 0.5s$. As expected, the closed-loop system given in Eq. (3.114) is unstable, which verifies at

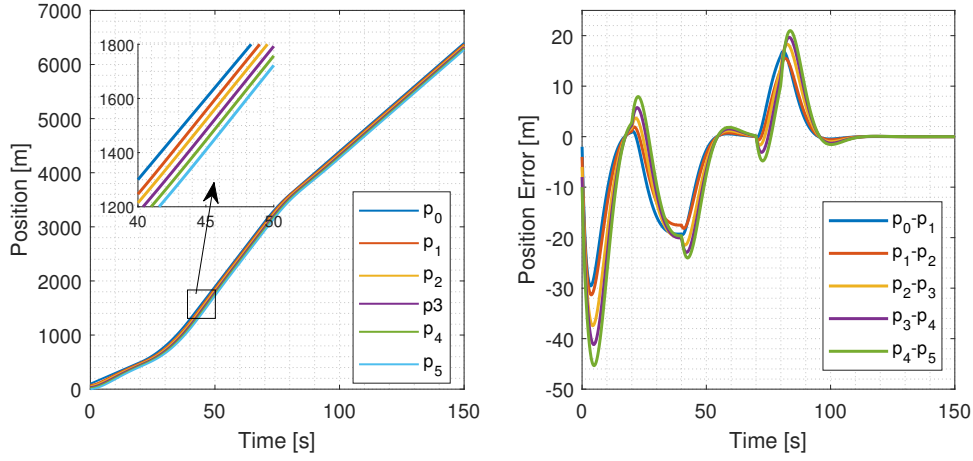


Figure 3.48 : Positions and position errors for $\tau_1 = 0.05s$ and $\tau_2 = 0.5s$.

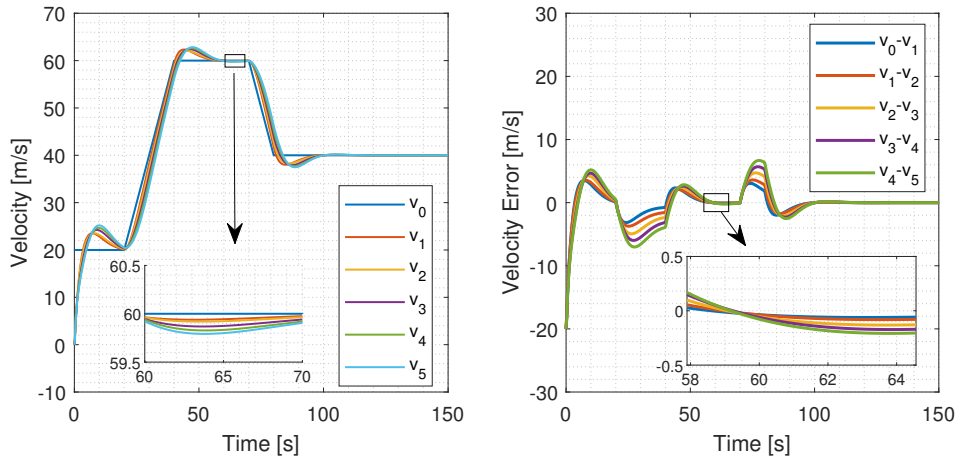


Figure 3.49 : Velocity and velocity errors for $\tau_1 = 0.05s$ and $\tau_2 = 0.5s$.

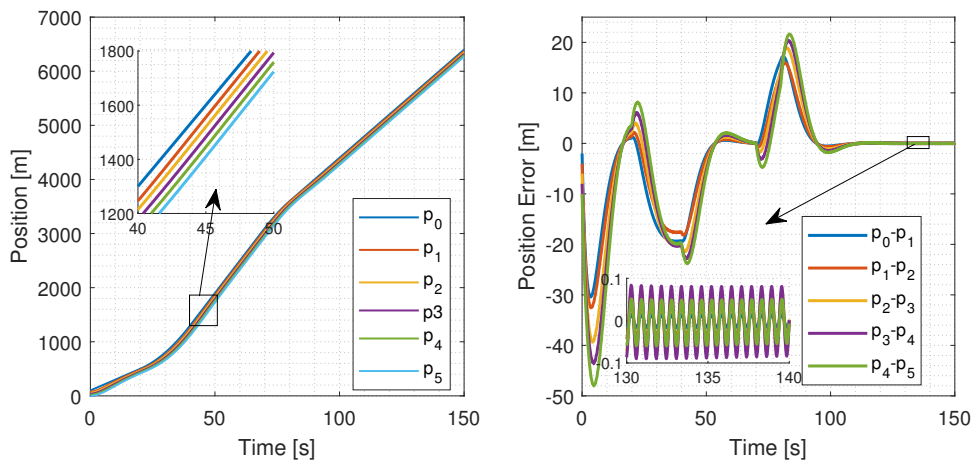


Figure 3.50 : Positions and position errors for $\tau_1 = 0.144s$ and $\tau_2 = 0.5s$.

least one pair of the roots of characteristic equation of the system lies on the right half of the complex plane. Figure 3.52 displays unstable behavior of the system.

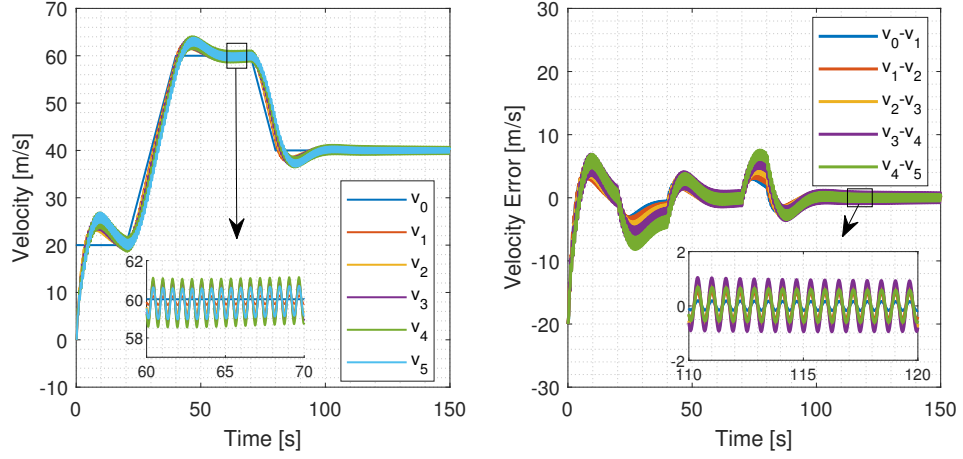


Figure 3.51 : Velocity and velocity errors for $\tau_1 = 0.144s$ and $\tau_2 = 0.5s$.

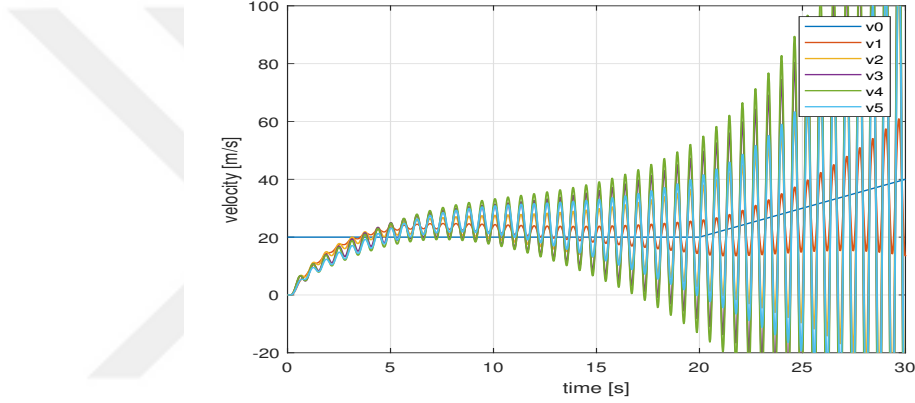


Figure 3.52 : Velocity and velocity errors for $\tau_1 = 0.16s$ and $\tau_2 = 0.5s$

The rest of this section examines the stability conditions for the formation flight of UAV systems under multiple-time delay. The control signal is written in terms of tracking errors in the presence of input and communication delay as follows;

$$u_i(t - \tau) = - \sum_{j=1}^N a_{ij} \left[k_p \left(e_{p_i}(t - \tau_1) - e_{p_j}(t - \tau_2) \right) + k_v \left(e_{v_i}(t - \tau_1) - e_{v_j}(t - \tau_2) \right) \right] - p_{ii} \left[k_p e_{p_i}(t - \tau_1) + k_v e_{v_i}(t - \tau_1) \right] \quad (3.128)$$

Then, Eq. (3.128) is substituted into Eq. (3.51), and the closed loop system is obtained in terms of Kronecker product.

$$\dot{E}(t) = (I_N \otimes A)E(t) - (\mathcal{D}_p \otimes BK)E(t - \tau_1) - (\mathcal{A} \otimes BK)E(t - \tau_2) \quad (3.129)$$

where A and B matrices are defined in Eq. (2.76). $E(t)$ and K are given in Eq. (3.53). Moreover $\mathcal{A}$ and $\mathcal{D}_p$ are Adjacency and Information Topology Matrices which are given in Section 2. It is seen that the closed-loop formulation for formation of UAV

systems given in Eq. (3.129) is similar to Eq. (3.113). However, the matrix dimensions are different. In the following example, the stability boundary for delay space is obtained via the Bezout resultant based CTCR methodology which is explained above.

Simulation Example 3.11: The numerical simulation is presented for the analysis of formation control of the UAV system under multiple time delay. The information flow topology between UAVs is shown in Figure 3.53.

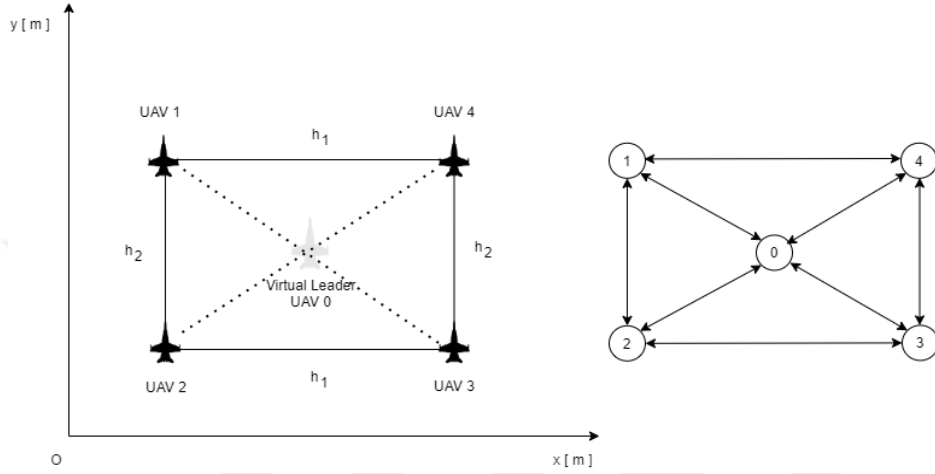


Figure 3.53 : Information flow topologies for Example 3.11.

The Adjacency matrix $\mathcal{A}$ and information topology matrix $\mathcal{D}_p = \mathcal{D} + \mathcal{P}$ are given as;

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad \mathcal{D}_p = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad (3.130)$$

The formation control system is assumed as homogeneous. The system parameters and initial states of UAVs used in simulation are given in Table 3.4 and in Table 3.5, respectively. The controller parameters which makes the closed-loop system stable for delay-free case are selected as follows;

$$K = [kp \quad kv] = [6.2806 \quad 25.4619] \quad (3.131)$$

Table 3.4 : System parameters used in simulation studies

Parameter	Description	Value	Unit
N	Number of following UAVs	4	-
h_1, h_2	Constant distances	700 , 600	[m]
τ_v	Velocity time constant	1.8	[sec]
τ_θ	Flight path angle time constant	0.6	[sec]
τ_ψ	Heading angle time constant	0.5	[sec]

Table 3.5 : Initial States of UAVs

$x_i[m]$	$y_i[m]$	$z_i[m]$	$v_i[mPs]$	$\psi_i[deg]$	$\theta_i[deg]$
UAV_0	350	350	40	30	5
UAV_1	0	700	65	-10	3
UAV_2	0	0	55	10	4
UAV_3	700	0	30	15	6
UAV_4	700	700	35	-15	-5

The reference velocity, flight path angle and heading angle profile of virtual UAV (UAV_0) is determined as follows;

$$v_{ref} = 50 \text{ m/s}, \quad \theta_{ref} = 5 \text{ deg}$$

$$\psi_{ref} = \begin{cases} 0 \text{ deg} & \text{if } t \leq 40 \\ 10 \text{ deg} & \text{if } 40 < t \leq 80 \\ 0 \text{ deg} & \text{if } 80 < t \leq 120 \\ -5 \text{ deg} & \text{if } 120 < t \leq 150 \\ 0 \text{ deg} & \text{if } t \leq 200 \end{cases} \quad (3.132)$$

The characteristic equation of the system is obtained for for $z_1 = e^{-s\tau_1}$ and $z_2 = e^{-s\tau_2}$ as follows;

$$\begin{aligned}
Q_1(s, z_1, z_2) &= \det(sI_{15} - (I_N \otimes A) - (\mathcal{D}_p \otimes BK)z_1 - (\mathcal{A} \otimes BK)z_2 \\
&= s^8 + 305.543z_1s^7 + (35008.651z_1^2 - 2593.233z_2^2 + 75.367z_1)s^6 + \\
&(1782773.541z_1^3 + 17270.928z_1^2 - 1279.328z_2^2 - 396171.898z_1z_2^2) \\
&s^5 + (34044601.22z_1^4 + 1319252.0z_1^3 - 15130933.88z_1^2z_2^2 - \\
&293167.1112z_1z_2^2 + 2130.08z_1^2 - 157.783z_2^2)s^4 + (33590662.51z_1^4 - \\
&14929183.34z_1^2z_2^2 - 72314.531z_1z_2^2 + 325415.3898z_1^3)s^3 + \\
&(12428541.17z_1^4 - 5523796.076z_1^2z_2^2 - 5945.859z_1z_2^2 + \\
&26756.367z_1^3)s^2 + (2043803.897z_1^4 - 908357.2876z_1^2z_2^2)s - \\
&56015.348z_1^2z_2^2 + 126034.5335z_1^4
\end{aligned} \quad (3.133)$$

Since the formation of UAVs system consist of $N = 4$ follower UAV and each UAV has $r = 3$ dynamics, the closed-loop characteristic equation has $N * r = 8$ orders. The

stability maps depending on the delays are obtained utilizing CTCR with the Bezout matrix approach and given in Figure 3.54. The kernel is derived and offsprings are generated with the methodology. The stable region is determined using root tendencies of each curve. The kernels are depicted thick (red) and offsprings are thin (blue).

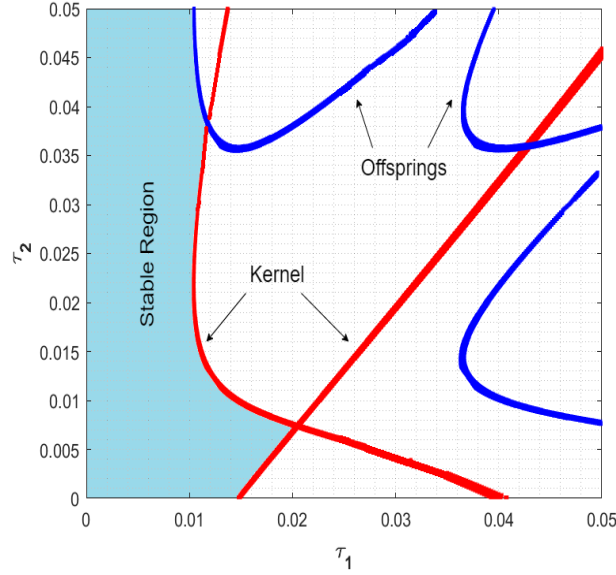


Figure 3.54 : Stability boundaries for $k_p = 6.2806$, $k_v = 25.4619$.

Time delay-induced instability occurs when time delay pairs are out of the shaded region, which is simulated and described in the following simulations. The test point, $\tau_1 = 0.01$ and $\tau_2 = 0.01$ is selected on the shaded region to exhibit the stability of the system. The results for altitude and velocity, flight path angle and heading angle, positions and velocity errors are displayed in Figure 3.55, 3.56 and 3.57, respectively. Moreover, the 3D trajectory of formation of UAVs are shown in Figure 3.58.

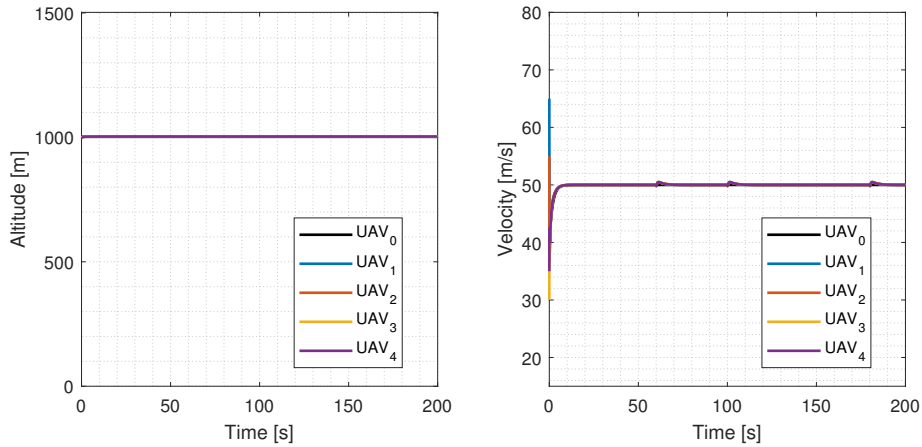


Figure 3.55 : Altitude and velocity for $\tau = 0.01$.

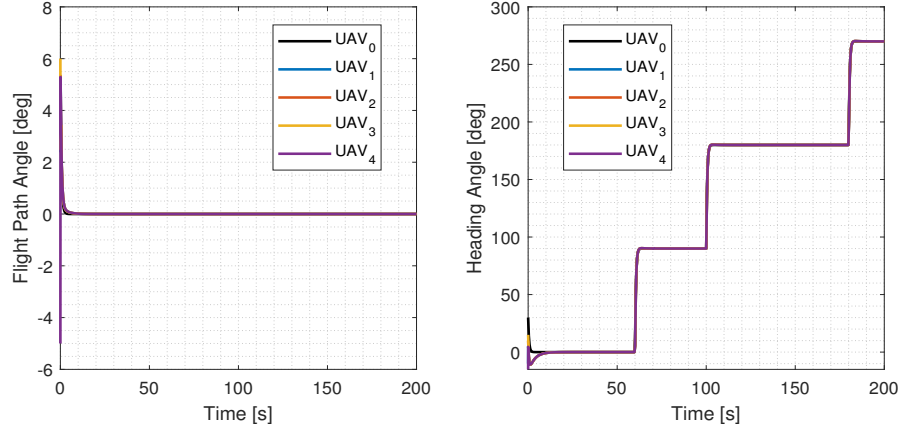


Figure 3.56 : Flight path angle and heading angle for $\tau = 0.01$.

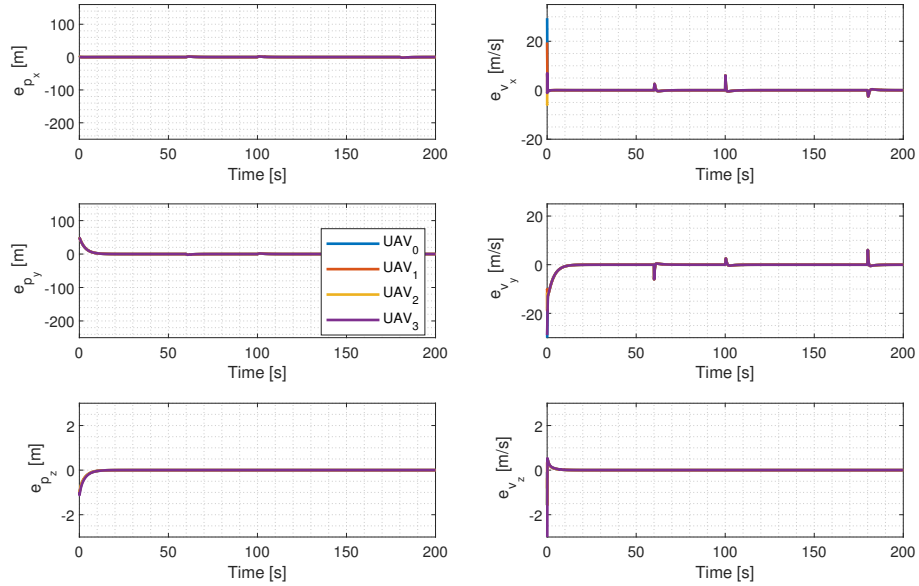


Figure 3.57 : Position errors and velocity errors for $\tau = 0.01$.

The test points are selected as $\tau_1 = 0.01$ and $\tau_2 = 0.01916$ which are on the stability boundary, the flight path angle and heading angle are obtained as in Figure 3.59. The results show that the system display the marginal stability behaviour. Then, the test point is selected as $\tau_1 = 0.01$ and $\tau_2 = 0.02$, which is in the unstable region. As expected, the closed-loop system is unstable, which verifies at least one pair of the roots of characteristic equation of the system lies on the right half of the complex plane.

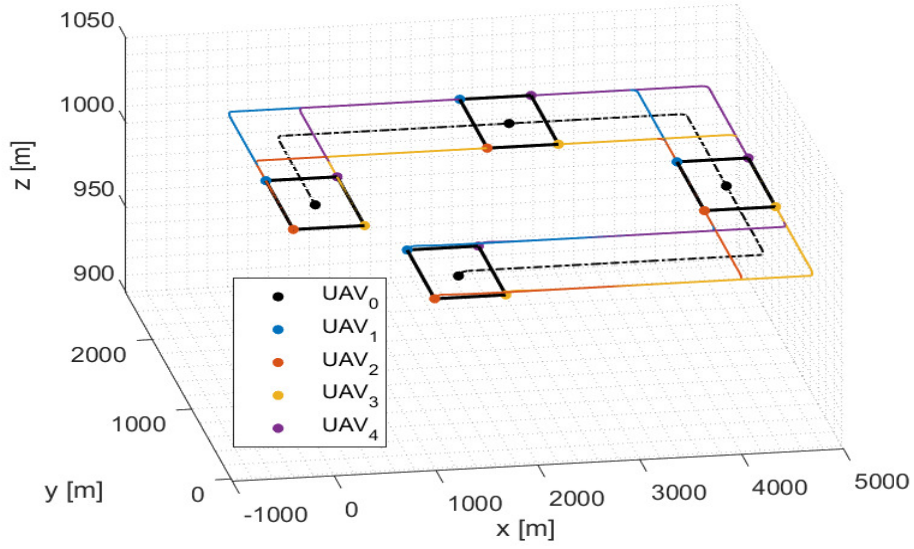


Figure 3.58 : UAV 3D trajectory for $k_p = 1$ and $k_v = 2$.

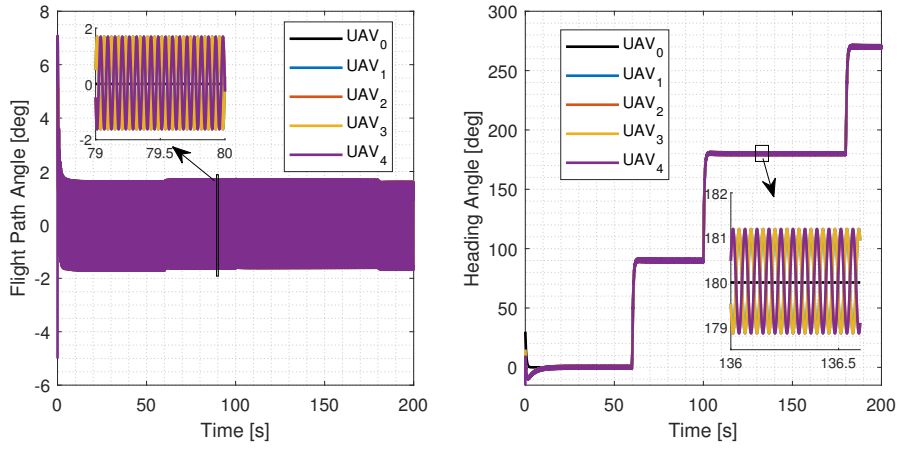


Figure 3.59 : Flight path angle and heading angle for $\tau = 0.01$.

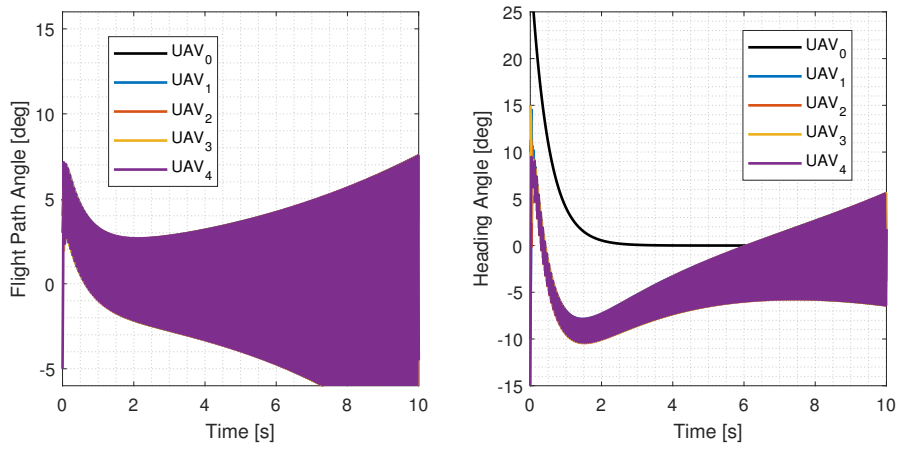


Figure 3.60 : Flight path angle and heading angle for $\tau = 0.01$.

3.3 Follower Agent-Based Constant Time Headway Policy

The following constant time headway policy is used to formulate follower velocity-dependent constant time-headway policy [48];

$$d_{i,0} = \sum_{k=1}^i (h_k v_k + d_k), \quad i = 1, 2, \dots, N, \quad k = 1, 2, \dots, i, \quad h_k > 0, d_k > 0. \quad (3.134)$$

where h_k is the time headway and d_k is the standstill gap. The main advantage of the time headway policy given in Eq. (3.134) is that it is more similar to human behavior since it makes the desired distances proportional to follower agent speeds. It also contributes to safety and driving comfort, as well as improve string stability.

Similar to the previous analysis, first, the vehicular platooning system model is studied. Then, these analyses are evaluated for the formation control of the unmanned aerial vehicle system model.

3.3.1 Stability analysis for nominal case

In the rest of this subsection, the stability analysis of the system with follower-dependent constant time-headway policy is analyzed by ignoring time delay. Then, the dynamic of tracing errors are obtained with the help of Eq. (2.71), Eq. (3.3) and Eq. (3.134);

$$\begin{aligned} \dot{e}_{p_i}(t) &= e_{v_i}(t) + \sum_{k=1}^i h_k e_{a_k}(t) \\ \dot{e}_{v_i}(t) &= e_{a_i}(t) \\ \dot{e}_{a_i}(t) &= -\frac{1}{\gamma_i} e_{a_i}(t) + \frac{1}{\gamma_i} u_i(t) \end{aligned} \quad (3.135)$$

The controller signal is written as in Eq. (3.5) regarding position, velocity, and acceleration errors. Unlike Eq. (3.7), the closed-loop of the platooning system under follower velocity-dependent CTH policy can not be written using the Kronecker product since the structure of CTH policy given in Eq. (3.134). However, the following vector definition may be used to facilitate the notation.

$$\begin{aligned} e_p &= [e_{p_1}, e_{p_2}, \dots, e_{p_N}]^T \\ e_v &= [e_{v_1}, e_{v_2}, \dots, e_{v_N}]^T \\ e_a &= [e_{a_1}, e_{a_2}, \dots, e_{a_N}]^T \\ u &= [u_1, u_2, \dots, u_N]^T \end{aligned} \quad (3.136)$$

Then, the system without delay is an LTI vehicle platooning system, and the closed-loop model of delay-free error dynamics is obtained as follows [48];

$$\dot{E}(t) = A_c E(t) \quad (3.137)$$

And the closed-loop system matrix of a delay-free system is;

$$A_c = \begin{bmatrix} 0_{N \times N} & I_{N \times N} & H \\ 0_{N \times N} & 0_{N \times N} & I_{N \times N} \\ -\Lambda K_p \mathcal{L}_p & -\Lambda K_v \mathcal{L}_p & -(\Lambda + \Lambda K_a \mathcal{L}_p) \end{bmatrix} \quad (3.138)$$

where;

$$\begin{aligned} E(t) &= [e_p^T, e_v^T, e_a^T]^T \\ K_\alpha &= \text{diag}[k_\alpha, k_\alpha, \dots, k_\alpha] \in \mathbb{R}^{N \times N}, \quad \alpha \in \{p, v, a\} \\ I_N &= \text{diag}[1, 1, \dots, 1] \in \mathbb{R}^{N \times N} \\ \mathcal{L}_p &= \mathcal{L} + \mathcal{P} \end{aligned}$$

And H is the time headway matrix and Λ is the time constant matrix which are defined as follows;

$$H = \begin{bmatrix} h_1 & 0 & \dots & 0 \\ h_1 & h_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & \dots & h_N \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \frac{1}{\gamma_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\gamma_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\gamma_N} \end{bmatrix} \quad (3.139)$$

It is known that, the stability condition is not only depend on distributed controller, but also the structure of $\mathcal{L}_p$ and H (or h). Then, it is also explained in above, if the structure of $\mathcal{L}_p$, which depend on the information flow topology, is triangular or symmetric, then the eigenvalues of $\mathcal{L}_p$ is real. The characteristic equation may be decomposed into its factors whose coefficients are real. It means that the stability posture of the whole system may be obtained by investigated each of the factors via Routh Tabulation method. Hence, it is possible to extend of Lemma 3.1 for follower velocity-dependent CTH policy as follows;

Lemma 3.5 : [48], Consider the homogeneous platooning system using a follower velocity-dependent CTH policy as in Eq. (3.134) and a linear distributed controller as in Eq. (3.5) with the closed-loop dynamics in Eq. (3.137). If the information topology matrix $\mathcal{L}_p$ has lower-triangular, then the stability of the platoon is guaranteed if and only if the following conditions are held for $k_p > 0$, $\lambda_i > 0$, $h > 0$, $\gamma > 0$.

$$\begin{cases} k_p > 0 \\ k_v > k_p \left(\frac{\gamma}{\lambda_i k_a + 1} - h_i \right) \\ k_a > -\frac{1}{\lambda_i} \end{cases} \quad (3.140)$$

Proof : The lower-triangular structure of the $\mathcal{L}_p$ matrix means that the vehicles just get information from their predecessors. And, the λ_i eigenvalue, which is also the corresponding diagonal element of $\mathcal{L}_p$, gives the number(s) of predecessor vehicle(s) of i -th vehicle. Then, characteristic equation of the system given in Eq. (3.137) may also be written as follows;

$$\begin{aligned} Q(s) &= \det(sI_{3N} - A_c) = \det(sI_3 - (A - \lambda_i BK)) \\ &= \prod_{i=1}^N s^3 + \frac{\lambda_i k_a + 1}{\gamma} s^2 + \frac{\lambda_i(k_v + h k_p)}{\gamma} s + \frac{\lambda_i k_p}{\gamma}, \quad i = 1, 2, \dots, N. \end{aligned} \quad (3.141)$$

Therefore the stability of Eq. (3.141) is examined using the Routh-Hurwitz stability criterion which constituted with the following Routh table;

$$\begin{array}{ccc} s^3 & 1 & \frac{\lambda_i(k_v + h k_p)}{\gamma} \\ s^2 & \frac{\lambda_i k_a + 1}{\gamma} & \frac{\lambda_i k_p}{\gamma} \\ s^1 & \frac{\lambda_i((\lambda_i k_a + 1)(k_v + h k_p) - k_p \gamma)}{\gamma(\lambda_i k_a + 1)} & \\ s^0 & \frac{\lambda_i k_p}{\gamma} & \end{array} \quad (3.142)$$

Given the fact that $\gamma > 0$ and $\lambda_i > 0, i = 1, 2, \dots, N$, Eq. (3.141) is asymptotically stable if the and only if Eq. (3.140) is satisfied. Thus, the proof is completed.

Simulation Example 3.12: The simulation study given in *Simulation Example 3.1* is also conducted for follower based constant time headway policy to demonstrate the results of Lemma 3.5. The platooning system has 1 leader and 5 follower. The TPF topology is shown in Figure 3.1. The $\mathcal{L}_p$ matrix of the topology is given in Eq. (3.15), which is a lower triangular form. The platooning system is homogeneous, the system parameters is given in Table 3.1.

The characteristic equation of the system is written by using Eq. (3.141) as follows;

$$\begin{aligned} Q(s)_{TPF} &= \det(sI_{3N} - A_c) \\ &= (s^3 + 2(k_a + 1)s^2 + (2k_v + k_p)s + 2k_p) \\ &\quad (s^3 + 2(2k_a + 1)s^2 + 2(2k_v + k_p)s + 4k_p)^4 \end{aligned} \quad (3.143)$$

The characteristic equations are decomposed into factors due to the lower-triangular structure of $\mathcal{L}_p$ and H matrices as seen from Eq. (3.15) and Eq. (3.139). So, the Routh table method may be easily applied to each factor. The stability boundary of the systems for TPF topology is obtained by using Lemma 3.5 for $h = 0.5, \gamma = 0.5$ as

follows;

$$K_{TPF} := \begin{cases} k_p > 0 \\ k_v > k_p \left(\frac{0.5}{\lambda_i k_a + 1} - 0.5 \right) \\ k_a > -\frac{1}{\lambda_i} \end{cases} \quad (3.144)$$

If the speed profile given in Eq. (3.18) to the leader vehicle. The controller parameters are selected as $k_p = 2$, $k_a = 0.8$, the stability condition of $k_v = 2$ is obtained as $k_v > 0.4444$. Then, positions, position tracking errors, velocities, and velocity tracking errors are shown in Figure 3.61 and 3.62, for the controller parameters $k_p = 2$, $k_v = 2$, $k_a = 0.8$, in the stable region, respectively.

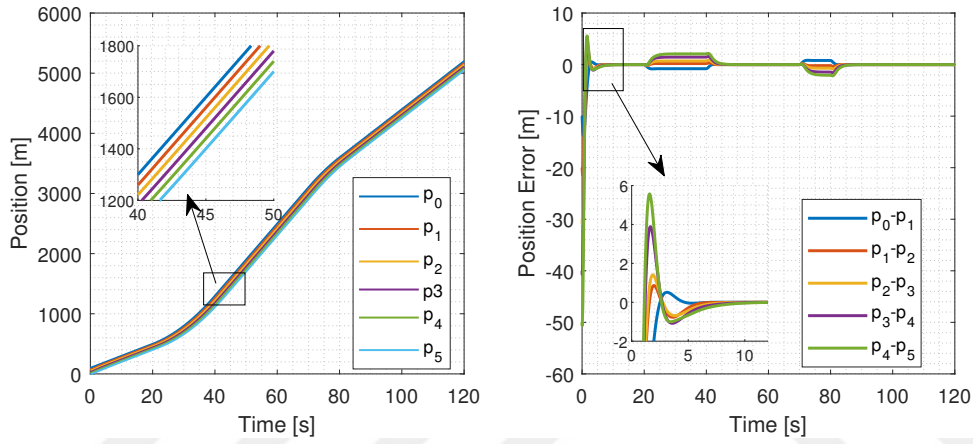


Figure 3.61 : Positions and position errors for $k_p = 2$, $k_v = 2$ and $k_a = 0.8$.

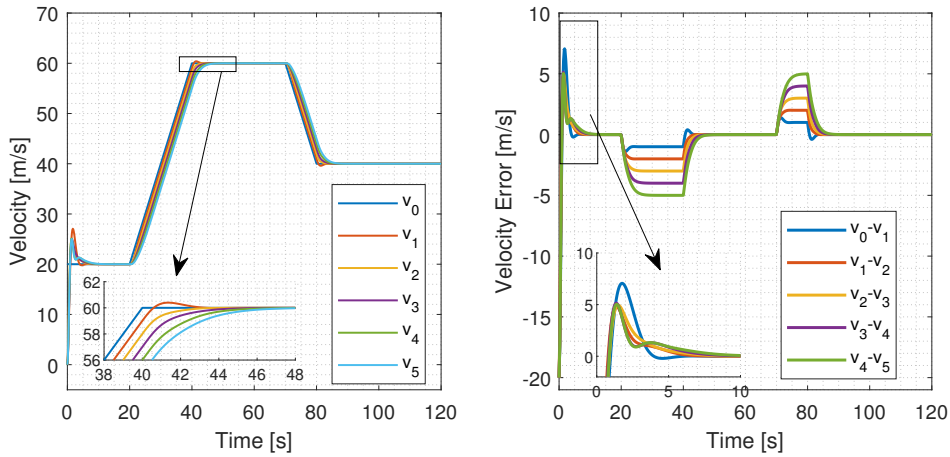


Figure 3.62 : Velocity and velocity errors for $k_p = 2$, $k_v = 2$ and $k_a = 0.8$.

In Figure 3.63, the velocity controller parameter is selected on the stability boundary as $k_v = -0.4444$, and the system displays the marginal stability behavior.

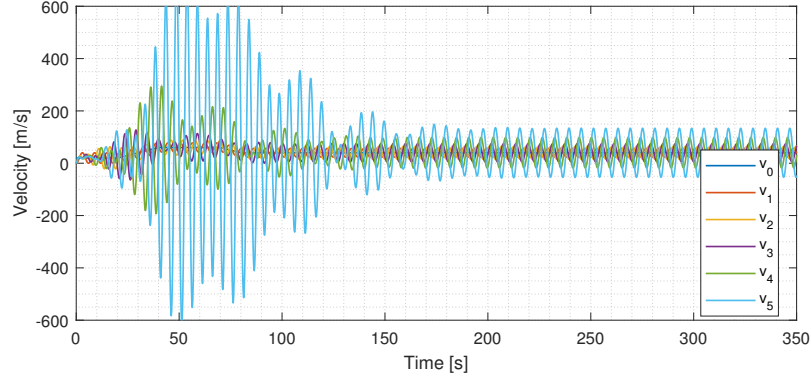


Figure 3.63 : Velocity and velocity errors for $k_p = 2$, $k_v = -0.4444$ and $k_a = 0.8$

However, if the structure of $\mathcal{L}_p$ is not lower-triangular, which means the vehicle gets information from both its predecessor and follower, creating the stability condition is not easy like given in Lemma 3.5. As the number of vehicles in the platooning system increases and the information flow topology becomes more complex, it may not be possible to factorize the closed-loop characteristic equation. The case which is the structure of $\mathcal{L}_p$ symmetric is examined in [107, 108]. A similar procedure is followed here with the help of matrix eigenvalue perturbation theory and block matrix polynomials.

For any arbitrary $\mathcal{L}_p$, the closed-loop system matrix given in Eq. (3.138) may be converted into *block companion* form, which plays a fundamental role in stability analysis. Then, by using the matrix eigenvalue perturbation theory, the sufficient conditions are obtained to guarantee the stability of the system.

So, the state matrix A_c given in Eq. (3.138) is converted into the block companion matrix using Definition 2.20 and 2.21 as follows;

$$B_c = \begin{bmatrix} 0 & I_N & 0 \\ 0 & 0 & I_N \\ -\Lambda K_p \mathcal{L}_p & -(\Lambda K_p \mathcal{L}_p H + \Lambda K_v \mathcal{L}_p) & -(\Lambda + \Lambda K_a \mathcal{L}_p) \end{bmatrix} \quad (3.145)$$

It is known that, A_c and B_c have the same eigenvalues, which means that obtaining the stability condition of B_c is also equivalent to the stability condition for A_c . Then, let write $\mathcal{L}_p$ matrix as follows;

$$\begin{aligned} \mathcal{L}_{p\alpha} &= \frac{1}{2}(\mathcal{L}_p + \mathcal{L}_p^T) \\ \mathcal{L}_{p\beta} &= \frac{1}{2}(\mathcal{L}_p - \mathcal{L}_p^T) \end{aligned} \quad (3.146)$$

where $\mathcal{L}_{p\alpha}$ is symmetric and $\mathcal{L}_{p\beta}$ is skew-symmetric matrix. And, let write B_c matrix in a matrix perturbation form as follows;

$$B_c = B_\alpha + B_\beta + B_\delta \quad (3.147)$$

where the B_α is block symmetric, B_β is block skew-symmetric and B_δ contains the parts of time headway matrix H and time constant Λ as follows;

$$B_\alpha = \begin{bmatrix} 0 & I_N & 0 \\ 0 & 0 & I_N \\ -\Lambda K_p \mathcal{L}_{p\alpha} & -\Lambda K_v \mathcal{L}_{p\alpha} & -\Lambda K_a \mathcal{L}_{p\alpha} \end{bmatrix} \quad (3.148)$$

and

$$B_\beta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Lambda K_p \mathcal{L}_{p\beta} & -\Lambda K_v \mathcal{L}_{p\beta} & -\Lambda K_a \mathcal{L}_{p\beta} \end{bmatrix} \quad (3.149)$$

and

$$B_\delta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\Lambda K_p \mathcal{L}_{p\alpha} H & -\Lambda \end{bmatrix} \quad (3.150)$$

It is seen that the dominant part of B_c matrix is B_α . B_β and B_δ matrices are the perturbations over the matrix B_α . Then, if all the eigenvalues of B_α have strictly negative real parts, then there must exist B_β and B_δ matrices that all eigenvalues of B_c have strictly negative real parts. It is reminded that if $\mathcal{L}_p$ is symmetric, B_β is a zero matrix. If not, since the B_β structure is block skew-symmetric, the eigenvalues of B_β matrix are zero and/or purely imaginary. Also, the eigenvalues of B_δ is just connect to Λ matrix.

Since the structure of B_α is block-companion form, the matrix Routh algorithm may be utilized to investigate the stability condition. Then the following Lemma may be written.

Lemma 3.6 : Consider the homogeneous platooning system using the distributed controller with the closed-loop system matrix A_c which is given in Eq. (3.138). Let the information topology matrix of $\mathcal{L}_p$ is general structure. Let A_c is written in block companion form as B_c , which is written as Eq. (3.145), and B_c is written in a matrix perturbation form as $B_c = B_\alpha + B_\beta + B_\delta$. All eigenvalues of B_α have negative real parts, if the following condition is holds;

$$\left(\Delta K_v - J_\alpha^{-1} K_a^{-1} K_p \right)^{-1} > 0 \quad (3.151)$$

Proof : The stability conditions are extracted by using matrix Routh array method, which is given in Section 2. So, after the matrix Routh algorithm given in Eq. (2.19) and Eq. (2.20) is performed for the B_0 the matrix Routh array is obtained as follows;

$$\begin{array}{llll} C_{11} & = & I_N & C_{12} = \Delta K_v \mathcal{L}_{p\alpha} & C_{13} = 0 \\ C_{21} & = & \Delta K_a \mathcal{L}_{p\alpha} & C_{22} = \Delta K_p \mathcal{L}_{p\alpha} & C_{23} = 0 \\ C_{31} & = & C_{12} - H_1 C_{22} & C_{32} = 0 & C_{33} = 0 \\ C_{41} & = & \Delta K_p \mathcal{L}_{p\alpha} & C_{42} = 0 & C_{43} = 0 \end{array} \quad (3.152)$$

Moreover, the matrix quotients may be expressed as follows;

$$\begin{aligned} H_1 &= C_{11} C_{21}^{-1} \\ &= \left(\Lambda K_a \mathcal{L}_{p\alpha} \right)^{-1} = M J_\alpha^{-1} K_a^{-1} \Lambda^{-1} M^{-1} \\ H_2 &= C_{21} C_{31}^{-1} \\ &= \Lambda K_a \mathcal{L}_{p\alpha} \left(\Lambda K_v \mathcal{L}_{p\alpha} - \left(\Lambda K_a \mathcal{L}_{p\alpha} \right)^{-1} \Lambda K_p \mathcal{L}_{p\alpha} \right)^{-1} \\ &= M \Lambda K_a \left(\Lambda K_v - J_\alpha^{-1} K_a^{-1} K_p \right)^{-1} M^{-1} \\ H_3 &= C_{31} C_{41}^{-1} \\ &= \left(\Lambda K_v \mathcal{L}_p - \left(\Lambda (I_N + K_a \mathcal{L}_p) \right)^{-1} \Lambda K_p \mathcal{L}_p \right) (\Lambda K_p \mathcal{L}_p)^{-1} \\ &= M \Lambda K_a \left(\Lambda K_v - J_\alpha^{-1} K_a^{-1} K_p \right)^{-1} M^{-1} \end{aligned} \quad (3.153)$$

So, if H_1 , H_2 and H_3 are real symmetric, positive definite matrices then Eq. (3.151) holds. The proof is completed.

The stability conditions given in Eq. (3.151) just contain the B_α matrix. The stability condition for B_c (or A_c) matrix may be obtained by the help of matrix perturbation theory over the B_α matrix. Then, the following Lemma may be written to guarantee stability of B_c (or A_c) matrix;

Lemma 3.7 : Consider the homogeneous platooning system using the distributed controller with the closed-loop system matrix A_c which is given in Eq. (3.138). Let the information topology matrix of $\mathcal{L}_p$ is general structure. Let A_c is written in block companion form as B_c , which is written as Eq. (3.145), and B_c is written in a matrix perturbation form as $B_c = B_\alpha + B_\beta + B_\delta$. It is known that all eigenvalues of B_α have negative real parts. In this condition, a stable B_c (or A_c) may always be obtained.

Proof : The proof of this lemma may easily shown with the help of Lyapunov stability theorem as follows. Let assume that the matrix B_c and B_α is stable. Then the following

inequalities may be written.

$$\begin{aligned}
B_c &= B_\alpha + B_\beta + B_\delta \\
P(B_\alpha + B_\beta + B_\delta) + (B_\alpha + B_\beta + B_\delta)^T P &< 0 \\
PB_\alpha + P(B_\beta + B_\delta) + B_\alpha^T P + (B_\beta + B_\delta)^T P &< 0 \\
\left(PB_\alpha + B_\alpha^T P \right) + \left(P(B_\beta + B_\delta) + (B_\beta + B_\delta)^T P \right) &< 0
\end{aligned} \tag{3.154}$$

Since B_α is stable, the first part of inequality should always satisfy the following condition;

$$PB_\alpha + B_\alpha^T P < 0 \tag{3.155}$$

So, the structure of B_β and B_δ and the selected parameters which makes B_α stable, allow the following condition to be satisfied.

$$P(B_\beta + B_\delta) + (B_\beta + B_\delta)^T P < 0 \tag{3.156}$$

The proof is completed.

Simulation Example 3.13: The simulation study given in *Simulation Example 3.3* is also conducted for follower based constant time headway policy to demonstrate the results of Lemma 3.6 and Lemma 3.7. The platooning system has 1 leader and 5 followers. The information flow topology is shown in Figure 3.9. The $\mathcal{L}_p$ matrix of the topology consists of both real and complex eigenvalues and is given in Eq. (3.44). The platooning system is homogeneous, the system parameters is given in Table 3.1.

The characteristic equation of the system is written as follows;

$$\begin{aligned}
Q_1(s) &= \det(sI_{15} - A_c) \\
&= s^{15} + 32(10.3125 + 0.8126k_a)s^{14} + 32(1.25 + 6.5k_a + 7.875k_a^2 + \\
&0.28125k_p + 0.8125k_v)s^{13} + 32(2.5 + 19.5k_a + 47.25k_a^2 + 34.5k_a^3 + \\
&3.0625k_p + 5.8125k_a k_p + 6.5k_v + 15.75k_a k_v)s^{12} + 32(2.5 + 26k_a + \\
&94.5k_a^2 + 138k_a^3 + 65.5k_a^4 + 13.25k_p + 50.625k_a k_p + 41.25k_a^2 k_p + \\
&1.125k_p^2 + 19.5k_v + 94.5k_a k_v + 103.5k_a^2 k_v + 5.8125k_p k_v + 7.875k_v^2)s^{11} + \\
&32(1 + 13k_a + 63k_a^2 + 138k_a^3 + 131k_a^4 + 41k_a^5 + 28.5k_p + 164.25k_a k_p + \\
&268.5k_a^2 k_p + 115.25k_a^3 k_p + 12.5625k_p^2 + 17.0625k_a k_p^2 + 26k_v + 189k_a k_v + \\
&414k_a^2 k_v + 262k_a^3 k_v + 50.625k_p k_v + 82.5k_a k_p k_v + 47.25k_v^2 + 103.5k_a k_v^2)s^{10}
\end{aligned} \tag{3.157}$$

$$\begin{aligned}
& 32(30.5k_p + 235.5k_a k_p + 579.k_a^2 k_p + 492.5k_a^3 k_p + 102.5k_a^4 k_p + 56.25k_p^2 + \\
& 150.75k_a k_p^2 + 77.75k_a^2 k_p^2 + 2.40625k_p^3 + 13.k_v + 126.k_a k_v + 414.k_a^2 k_v + \\
& 524k_a^3 k_v + 205.k_a^4 k_v + 164.25k_p k_v + 537k_a k_p k_v + 345.75k_a^2 k_p k_v + \\
& 17.0625k_p^2 k_v + 94.5k_v^2 + 414k_a k_v^2 + 393k_a^2 k_v^2 + 41.25k_p k_v^2 + 34.5k_v^3)s^9 + \\
& 32(13k_p + 126k_a k_p + 414k_a^2 k_p + 524k_a^3 k_p + 205k_a^4 k_p + 126k_p^2 + \\
& 501.75k_a k_p^2 + 501.25k_a^2 k_p^2 + 102.5k_a^3 k_p^2 + 26.6875k_p^3 + 23.625k_a k_p^3 + \\
& 235.5k_p k_v + 1158k_a k_p k_v + 1477.5k_a^2 k_p k_v + 410k_a^3 k_p k_v + 150.75k_p^2 k_v + \\
& 155.5k_a k_p^2 k_v + 63k_v^2 + 414k_a k_v^2 + 786k_a^2 k_v^2 + 410k_a^3 k_v^2 + 268.5k_p k_v^2 + \\
& 345.75k_a k_p k_v^2 + 138.k_v^3 + 262k_a k_v^3)s^8 + 32(141k_p^2 + 744k_a k_p^2 + 1084.5k_a^2 k_p^2 + \\
& 410k_a^3 k_p^2 + 119.125k_p^3 + 202.75k_a k_p^3 + 51.25k_a^2 k_p^3 + 2.71875k_p^4 + 126k_p k_v + \\
& 828k_a k_p k_v + 1572k_a^2 k_p k_v + 820k_a^3 k_p k_v + 501.75k_p^2 k_v + 1002.5k_a k_p^2 k_v + \\
& 307.5k_a^2 k_p^2 k_v + 23.625k_p^3 k_v + 579k_p k_v^2 + 1477.5k_a k_p k_v^2 + 615k_a^2 k_p k_v^2 + \\
& 77.75k_p^2 k_v^2 + 138k_v^3 + 524k_a k_v^3 + 410k_a^2 k_v^3 + 115.25k_p k_v^3 + 65.5k_v^4)s^7 + \\
& 32(63k_p^2 + 414k_a k_p^2 + 786k_a^2 k_p^2 + 410k_a^3 k_p^2 + 267.75k_p^3 + 656.75k_a k_p^3 + \\
& 307.5k_a^2 k_p^3 + 29.0625k_p^4 + 12.8125k_a k_p^4 + 744k_p^2 k_v + 2169k_a k_p^2 k_v + \\
& 1230k_a^2 k_p^2 k_v + 202.75k_p^3 k_v + 102.5k_a k_p^3 k_v + 414k_p k_v^2 + 1572k_a k_p k_v^2 + \\
& 1230k_a^2 k_p k_v^2 + 501.25k_p^2 k_v^2 + 307.5k_a k_p^2 k_v^2 + 492.5k_p k_v^3 + 410k_a k_p k_v^3 + \\
& 131.k_v^4 + 205k_a k_v^4)s^6 + 32(303k_p^3 + 953.5k_a k_p^3 + 615k_a^2 k_p^3 + 125k_p^4 + \\
& 102.5k_a k_p^4 + 1.28125k_p^5 + 414k_p^2 k_v + 1572k_a k_p^2 k_v + 1230k_a^2 k_p^2 k_v + \\
& 656.75k_p^3 k_v + 615k_a k_p^3 k_v + 12.8125k_p^4 k_v + 1084.5k_p^2 k_v^2 + 1230k_a k_p^2 k_v^2 + \\
& 51.25k_p^3 k_v^2 + 524k_p k_v^3 + 820k_a k_p k_v^3 + 102.5k_p^2 k_v^3 + 102.5k_p k_v^4 + 41.k_v^5)s^5 + \\
& 32(138k_p^3 + 524k_a k_p^3 + 410k_a^2 k_p^3 + 270.75k_p^4 + 307.5k_a k_p^4 + 12.8125k_p^5 + \\
& 953.5k_p^3 k_v + 1230k_a k_p^3 k_v + 102.5k_p^4 k_v + 786k_p^2 k_v^2 + 1230k_a k_p^2 k_v^2 + \\
& 307.5k_p^3 k_v^2 + 410k_p^2 k_v^3 + 205k_p k_v^4)s^4 + 32(296k_p^4 + 410k_a k_p^4 + \\
& 51.25k_p^5 + 524k_p^3 k_v + 820k_a k_p^3 k_v + 307.5k_p^4 k_v + 615k_p^3 k_v^2 + 410k_p^2 k_v^3)s^3 + \\
& 32(131k_p^4 + 205k_a k_p^4 + 102.5k_p^5 + 410k_p^4 k_v + 410k_p^3 k_v^2)s^2 + 32(102.5k_p^5 + \\
& 205k_p^4 k_v)s + 1312k_p^5
\end{aligned}$$

The platooning system is stable if and only if the eigenvalues of the A_c system matrix or the roots of the $Q(s)$ characteristic equation are in the open left half of the complex

plane. Since the structure of $\mathcal{L}_p$ is not lower-triangular and constant time headway policy is utilized, the characteristic equation cannot be simplified by factorization. And the A_c closed-loop system matrix which is given in Eq. (3.138) for may be written as follows;

$$A_c = \begin{bmatrix} 0_{N \times N} & I_{N \times N} & A_{c13} \\ 0_{N \times N} & 0_{N \times N} & I_{N \times N} \\ A_{c31} & A_{c32} & A_{c33} \end{bmatrix} \quad (3.158)$$

where;

$$A_{c13} = H = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix},$$

$$A_{c31} = -\Lambda K_p \mathcal{L}_p = \begin{bmatrix} -4k_p & 2k_p & 0 & 0 & 0 \\ 2k_p & -6k_p & 2k_p & 0 & 0 \\ 2k_p & 2k_p & -6k_p & 2k_p & 0 \\ 0 & 2k_p & 2k_p & -6k_p & 2k_p \\ 0 & 0 & 2k_p & 2k_p & -4k_p \end{bmatrix},$$

$$A_{c32} = -\Lambda K_v \mathcal{L}_p = \begin{bmatrix} -4k_v & 2k_v & 0 & 0 & 0 \\ 2k_v & -6k_v & 2k_v & 0 & 0 \\ 2k_v & 2k_v & -6k_v & 2k_v & 0 \\ 0 & 2k_v & 2k_v & -6k_v & 2k_v \\ 0 & 0 & 2k_v & 2k_v & -4k_v \end{bmatrix},$$

$$A_{c33} = -(\Lambda + \Lambda K_a \mathcal{L}_p) = \begin{bmatrix} -2-4k_a & 2k_a & 0 & 0 & 0 \\ 2k_a & -2-6k_v & 2k_a & 0 & 0 \\ 2k_a & 2k_a & -2-6k_v & 2k_a & 0 \\ 0 & 2k_a & 2k_a & -2-6k_v & 2k_a \\ 0 & 0 & 2k_a & 2k_a & -2-4k_a \end{bmatrix}$$

Let's choose the controller gains $k_p = 2$ and $k_a = 0.5$ to reduce the system's complexity. And let's set the k_v controller gain as the free parameter. As seen in the system's characteristic equation given in Eq. (3.157), the Routh table method is miserable to create the stability region of the system since the degree of the characteristic polynomial is rather high. So, sufficient condition to guarantee the stability of the system is obtained from the stability of matrix polynomial and matrix eigenvalue perturbation theory.

First of all, the A_c system matrix is converted into the block companion matrix, B_c which is given in Eq. (3.159) as follows;

$$B_c = \begin{bmatrix} 0_{N \times N} & I_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & I_{N \times N} \\ B_{c31} & B_{c32} & B_{c33} \end{bmatrix} \quad (3.159)$$

where;

$$B_{c31} = -\Lambda K_p \mathcal{L}_p = \begin{bmatrix} -8 & 4 & 0 & 0 & 0 \\ 4 & -12 & 4 & 0 & 0 \\ 4 & 4 & -12 & 4 & 0 \\ 0 & 4 & 4 & -12 & 4 \\ 0 & 0 & 4 & 4 & -8 \end{bmatrix},$$

$$B_{c32} = -\Lambda K_p \mathcal{L}_p H - \Lambda K_v \mathcal{L}_p = \begin{bmatrix} -2-4k_v & 2+2k_v & 0 & 0 & 0 \\ -2-2k_v & -4-6k_v & 2+2k_v & 0 & 0 \\ 2k_v & -2-2k_v & -4-6k_v & 2+2k_v & 0 \\ 0 & 2k_v & -2-2k_v & -4-6k_v & 2+2k_v \\ 0 & 0 & 2k_v & -2-2k_v & -4-2k_v \end{bmatrix},$$

$$B_{c33} = -(\Lambda + \Lambda K_a \mathcal{L}_p) = \begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -5 & 1 & 0 & 0 \\ 1 & 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & -5 & 1 \\ 0 & 0 & 1 & 1 & -4 \end{bmatrix}$$

Let write $\mathcal{L}_p$ matrix as follows;

$$\mathcal{L}_{p\alpha} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}, \quad (3.160)$$

$$\mathcal{L}_{p\beta} = \begin{bmatrix} 2 & -1 & -0.5 & 0 & 0 \\ -1 & 3 & -1 & -0.5 & 0 \\ -0.5 & -1 & 3 & -1 & -0.5 \\ 0 & -0.5 & -1 & 3 & -1 \\ 0 & 0 & -0.5 & -1 & 2 \end{bmatrix}$$

where $\mathcal{L}_{p\alpha}$ is symmetric and $\mathcal{L}_{p\beta}$ is skew-symmetric matrix. And, let write B_c matrix in a matrix perturbation form as $B_c = B_\alpha + B_\beta + B_\delta$ as follows;

$$B_\alpha = \begin{bmatrix} 0_{N \times N} & I_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & I_{N \times N} \\ B_{\alpha31} & B_{\alpha32} & B_{\alpha33} \end{bmatrix} \quad (3.161)$$

Where;

$$B_{\alpha31} = -\Lambda K_p \mathcal{L}_{p\alpha} = \begin{bmatrix} -8 & 4 & 2 & 0 & 0 \\ 4 & -12 & 4 & 2 & 0 \\ 2 & 4 & -12 & 4 & 2 \\ 0 & 2 & 4 & -12 & 4 \\ 0 & 0 & 2 & 4 & -8 \end{bmatrix}$$

$$\begin{aligned}
B_{\alpha_{32}} &= -\Lambda K_p \mathcal{L}_{p\alpha} = \begin{bmatrix} -4k_v & 2k_v & k_v & 0 & 0 \\ 2k_v & -6k_v & 2k_v & k_v & 0 \\ k_v & 2k_v & -6k_v & 2k_v & k_v \\ 0 & k_v & 2k_v & -6k_v & 2k_v \\ 0 & 0 & k_v & 2k_v & -4k_v \end{bmatrix} \\
B_{\alpha_{33}} &= -\Lambda K_p \mathcal{L}_{p\alpha} = \begin{bmatrix} -2 & 1 & 0.5 & 0 & 0 \\ 1 & -3 & 1 & 0.5 & 0 \\ 0.5 & 1 & -3 & 1 & 0.5 \\ 0 & 0.5 & 1 & -3 & 1 \\ 0 & 0 & 0.5 & 1 & -2 \end{bmatrix} \\
B_\beta &= \begin{bmatrix} 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ B_{\beta_{31}} & B_{\beta_{32}} & B_{\beta_{33}} \end{bmatrix} \tag{3.162}
\end{aligned}$$

Where;

$$\begin{aligned}
B_{\beta_{31}} &= -\Lambda K_p \mathcal{L}_{p\beta} = \begin{bmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 2 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \\
B_{\beta_{32}} &= -\Lambda K_p \mathcal{L}_{p\beta} = \begin{bmatrix} 0 & 0 & -k_v & 0 & 0 \\ 0 & 0 & 0 & -k_v & 0 \\ k_v & 0 & 0 & 0 & -k_v \\ 0 & k_v & 0 & 0 & 0 \\ 0 & 0 & k_v & 0 & 0 \end{bmatrix} \\
B_{\beta_{33}} &= -\Lambda K_p \mathcal{L}_{p\beta} = \begin{bmatrix} 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 \\ 0.5 & 0 & 0 & 0 & -0.5 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \end{bmatrix} \\
B_\beta &= \begin{bmatrix} 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & B_{\delta_{32}} & B_{\delta_{33}} \end{bmatrix} \tag{3.163}
\end{aligned}$$

Where;

$$B_{\delta_{32}} = -\Lambda K_p \mathcal{L}_p = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 \\ -2 & -4 & 2 & 0 & 0 \\ 0 & -2 & -4 & 2 & 0 \\ 0 & 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & -2 & -4 \end{bmatrix}$$

$$B_{\delta_{33}} = -\Lambda = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The stability condition for the matrix of B_α is obtained with the help of Lemma 3.6. After the condition given in Eq. (3.151) are executed via numerical analysis programs (Mathematica or Maple), the following stability condition is obtained;

$$k_v > 5.15372 \quad (3.164)$$

After determining the minimum value of k_v that keeps the eigenvalues of B_{alpha} in the left half plane, the value of k_v will make the system matrix B_c stable may be obtained by a simple search algorithm. It should be stated that, the condition given in Eq. (3.164) is the sufficient for the stability of the whole system.

The simulation is conducted for $k_p = 2$, $k_a = 0.5$ and $k_v = 6$, and results are shown in Figure 3.64 and 3.65, respectively.

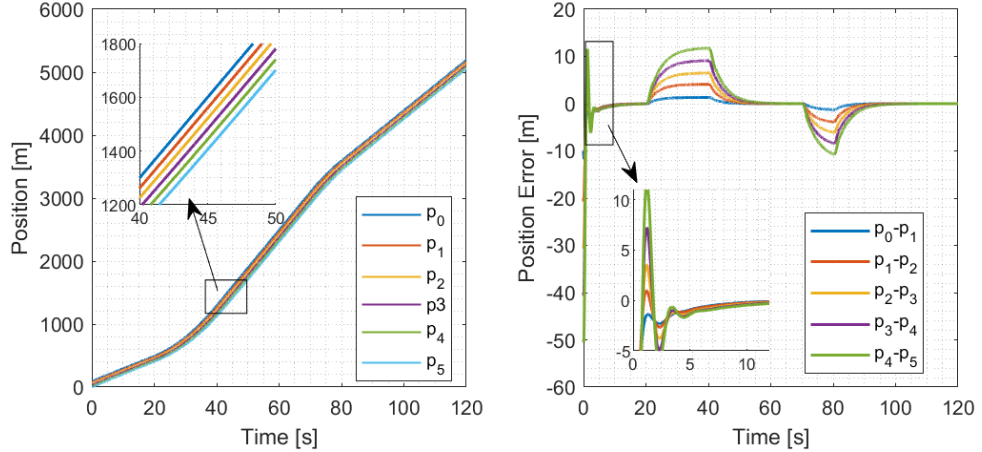


Figure 3.64 : Positions and position errors for $k_p = 2$, $k_v = 6$ and $k_a = 0.5$.

3.3.2 Stability analysis for time delay case

This section creates stability conditions for the system under single and multiple time delays, respectively. If the system is exposed to the controller signal given in Eq. (3.73) for the existence of time delay the general closed-loop formulation for single delay case is obtained as follows.

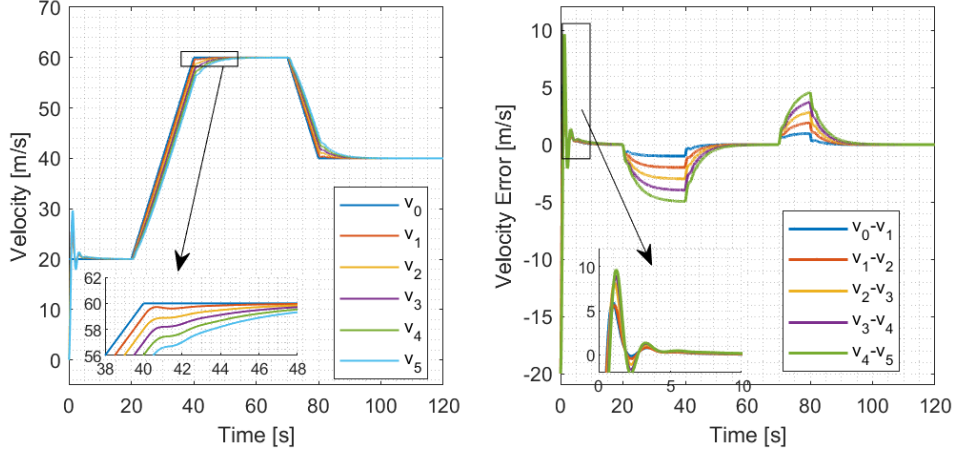


Figure 3.65 : Velocity and velocity errors for $k_p = 2$, $k_v = 6$ and $k_a = 0.5$.

$$\dot{E}(t) = AE(t) + BE(t - \tau) \quad (3.165)$$

where;

$$A = \begin{bmatrix} 0 & I_N & H \\ 0 & 0 & I_N \\ 0 & 0 & -\Lambda \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Lambda K_p \mathcal{L}_p & -\Lambda K_v \mathcal{L}_p & -\Lambda K_a \mathcal{L}_p \end{bmatrix} \quad (3.166)$$

The stability analysis of the system given in Eq. (3.165) may be achieved over the whole characteristic equation rather than the factorization property which is explained in Section 3.2.2, since the system can not be decomposed into its factors. The effect of time headway is frankly seen when the closed-loop systems given in Eq. (3.74) and Eq. (3.165) are compared. So, under the time headway policy given in Eq. (3.134), the closed-loop system given in Eq. (3.165) can not be written in terms of the Kronecker product and not decomposed. The characteristic equation of the system in Eq. (3.165) is written as follows;

$$Q(s, \tau) = \det(sI_{3N} - A - Be^{-\tau s}) \quad (3.167)$$

The stability posture of the system with respect to $(k_p/k_v/k_a, \tau)$ by performing the proposed Bezout resultant matrix-based CTCR method. The method is applied to a single-time delay system in *Simulation Example 3.9* in Section 3.2.2. In that example, the characteristic equation decomposed into factors. The method is applied to each factor, and the intersection of the stability regions is accepted as the stability posture of the system. However, since the characteristic equation given in Eq. (3.167) is not be decomposed into factors, the method should be applied to the whole system. All the

steps of the Bezout resultant matrix-based CTCR method is given in from Eq. (3.101) to Eq. (3.108).

Simulation Example 3.14: In this simulation study, the problem examined in *Simulation Example 3.13* is evaluated under time delay. The platooning system has 1 leader and 5 followers. The information flow topology is shown in Figure 3.9. The $\mathcal{L}_p$ matrix of the topology consists of both real and complex eigenvalues and is given in Eq. (3.44). The platooning system is homogeneous, the system parameters is given in Table 3.1. And, the velocity and acceleration controller gains are selected as $k_v = 6$ and $k_a = 0.5$. Then the stability posture with respect to (k_p, τ) is investigated via Bezout resultant matrix-based CTCR method whose steps is given in from Eq. (3.101) to Eq. (3.108).

The A and B matrices in Eq. (3.167) are written as follows;

$$A = \begin{bmatrix} 0_{N \times N} & I_{N \times N} & A_{c13} \\ 0_{N \times N} & 0_{N \times N} & I_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & A_{33} \end{bmatrix} \quad (3.168)$$

where;

$$A_{13} = H = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix},$$

$$A_{33} = -\Lambda = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \quad (3.169)$$

where;

$$B_{31} = -\Lambda K_p \mathcal{L}_p = \begin{bmatrix} -4k_p & 2k_p & 0 & 0 & 0 \\ 2k_p & -6k_p & 2k_p & 0 & 0 \\ 2k_p & 2k_p & -6k_p & 2k_p & 0 \\ 0 & 2k_p & 2k_p & -6k_p & 2k_p \\ 0 & 0 & 2k_p & 2k_p & -4k_p \end{bmatrix},$$

$$B_{32} = -\Lambda K_v \mathcal{L}_p = \begin{bmatrix} -8 & 4 & 0 & 0 & 0 \\ 4 & -12 & 4 & 0 & 0 \\ 4 & 4 & -12 & 4 & 0 \\ 0 & 4 & 4 & -12 & 4 \\ 0 & 0 & 4 & 4 & -8 \end{bmatrix},$$

$$B_{33} = -\Lambda K_a \mathcal{L}_p = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 1 & 1 & -3 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & 1 & -2 \end{bmatrix},$$

Then characteristic equation of the system, $Q(s, \tau)$ given in Eq. (3.167) may be rearranged with the transform of $z = e^{\tau s}$ as follows;

$$\begin{aligned} Q(s, z, k_p) &= \det(sI_{3N} - A - Bz) \\ &= s^{15} + (13z + 10)s^{14} + (9zk_p + 40 + 63z^2 + 260z)s^{13} + (93z^2k_p + 1560z + \\ &138z^3 + 80 + 1890z^2 + 98zk_p)s^{12} + (36k_p^2z^2 + 330k_pz^3 + 1926k_pz^2 + \\ &424k_pz + 131z^4 + 5520z^3 + 18900z^2 + 4160z + 80)s^{11} + (273k_p^2z^3 + \\ &402k_p^2z^2 + 461k_pz^4 + 10068k_pz^3 + 12348k_pz^2 + 912k_pz + 41z^5 + \\ &6550z^4 + 80040z^3 + 73080z^2 + 5200z + 32)s^{10} + (77k_p^3z^3 + 622k_p^2z^4 + \\ &5688k_p^2z^3 + 1800k_p^2z^2 + 205k_pz^5 + 18566k_pz^4 + 103704k_pz^3 + 35304k_pz^2 + \\ &976k_pz + 2460z^5 + 125760z^4 + 496800z^3 + 120960z^2 + 2496z)s^9 + \\ &(378k_p^3z^4 + 854k_p^3z^3 + 410k_p^2z^5 + 18938k_p^2z^4 + 36972k_p^2z^3 + 4032k_p^2z^2 + \\ &10250k_pz^5 + 272168k_pz^4 + 423792k_pz^3 + 47232k_pz^2 + 416k_pz + \\ &59040z^5 + 1131840z^4 + 1192320z^3 + 72576z^2)s^8 + (87k_p^4z^4 + 410k_p^3z^5 + \\ &7780k_p^3z^4 + 3812k_p^3z^3 + 16400k_p^2z^5 + 194484k_p^2z^4 + 108240k_p^2z^3 + \\ &4512k_p^2z^2 + 196800k_pz^5 + 1723104k_pz^4 + 746496k_pz^3 + 24192k_pz^2 + \\ &708480z^5 + 4527360z^4 + 953856z^3)s^7 + (205k_p^4z^5 + 930k_p^4z^4 + \\ &12300k_p^3z^5 + 49436k_p^3z^4 + 8568k_p^3z^3 + 237800k_p^2z^5 + 791952k_p^2z^4 + \\ &149472k_p^2z^3 + 2016k_p^2z^2 + 1771200k_pz^5 + 4309632k_pz^4 + 476928k_pz^3 + \\ &4250880z^5 + 5432832z^4)s^6 + (41k_p^5z^5 + 4100k_p^4z^5 + 4000k_p^4z^4 + \\ &123000k_p^3z^5 + 141352k_p^3z^4 + 9696k_p^3z^3 + 1476000k_p^2z^5 + 1400256k_p^2z^4 + \\ &79488k_p^2z^3 + 7084800k_pz^5 + 3621888k_pz^4 + 10202112z^5)s^5 + (410k_p^5z^5 + \end{aligned} \quad (3.170)$$

$$\begin{aligned}
& 24600k_p^4z^5 + 8664k_p^4 * z^4 + 475600k_p^3z^5 + 191456k_p^3z^4 + 3542400k_p^2z^5 + \\
& 905472k_p^2z^4 + 8501760k_pz^5)s^4 + (1640k_p^5z^5 + 4416k_p^3z^3 + 65600k_p^4z^5 + \\
& 9472k_p^4z^4 + 787200k_p^3z^5 + 100608k_p^3z^4 + 2833920 + k_p^2z^5)s^3 + \\
& (3280k_p^5z^5 + 82000k_p^4z^5 + 4192k_p^4z^4 + 472320k_p^3z^5)s^2 + (3280k_p^5z^5 + \\
& 39360k_p^4z^5)s + 1312k_p^5z^5
\end{aligned}$$

Then, the stability posture for $k_p - \tau$ is found as in Figure 3.66.

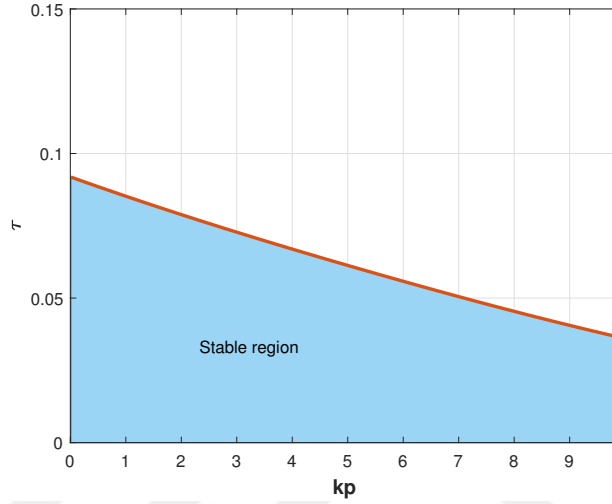


Figure 3.66 : Stability boundary for $k_v = 27.1661$.

In the Figure 3.66, the shaded regions indicate the parameters that yield characteristic roots on the left half of the complex-plane. At any point on the boundary of the stability zone, the characteristic equation has purely imaginary roots, which makes the system marginally stable. This stability map is validated by simulations for some selected test points. The first test point are selected in stable region as $k_p = 2$, and $\tau = 0.06s$ and the simulation results for positions, position tracking errors and the velocity, velocity tracking errors are given in Figure 3.67 and 3.68, respectively.

On the other hand, the test point is selected as $k_p = 2$ and $\tau = 0.0788s$, which is on the stability boundary. Consequently, the system displays marginal stable behavior and the effect of time delay is visible directly on the simulation results given in the Figure 3.69.

Ultimately, the test points are selected as $k_p = 0.2$ and $\tau = 0.08s$ in the unstable region. And it is seen in Figure 3.70 that the system is unstable, which conveys that at least one pair of roots of the closed-loop system is on the right half of the complex plane.

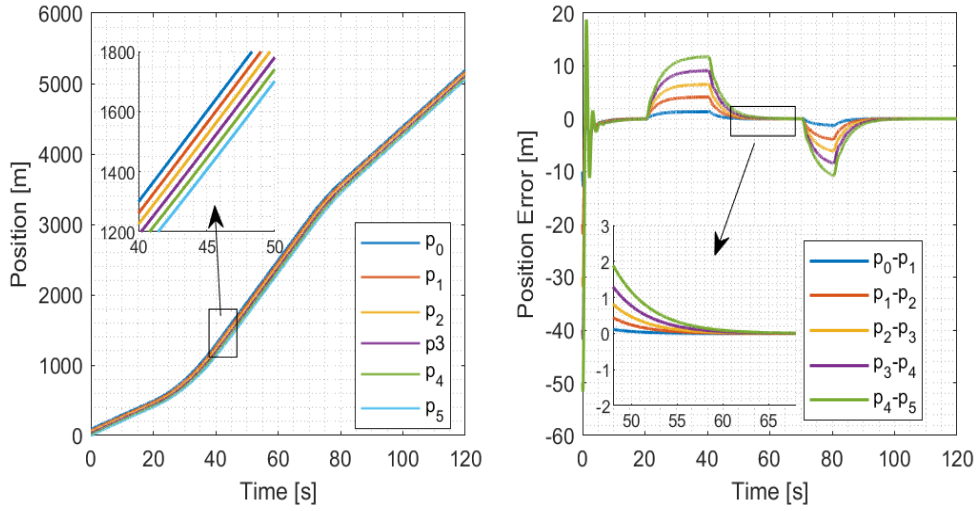


Figure 3.67 : Positions and position errors for $\tau = 0.06$.

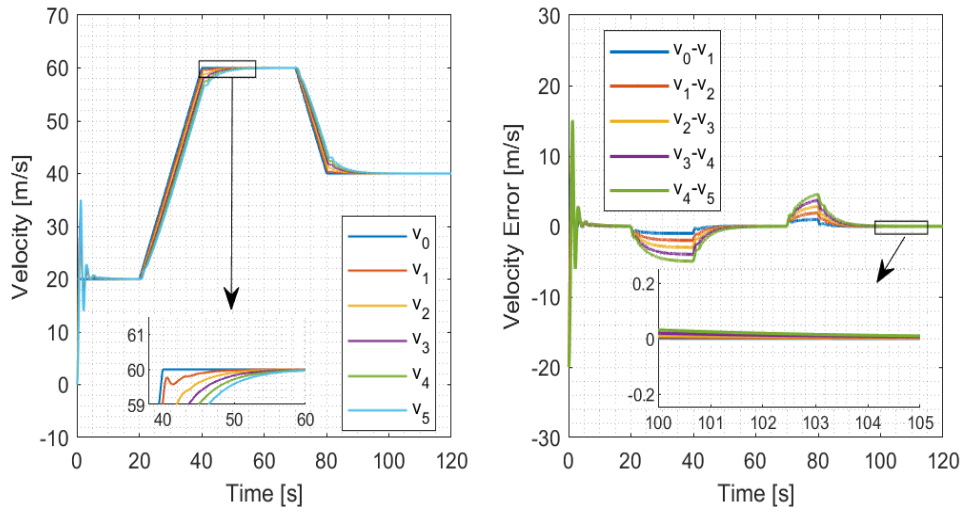


Figure 3.68 : Velocity and velocity errors for $\tau = 0.06$.

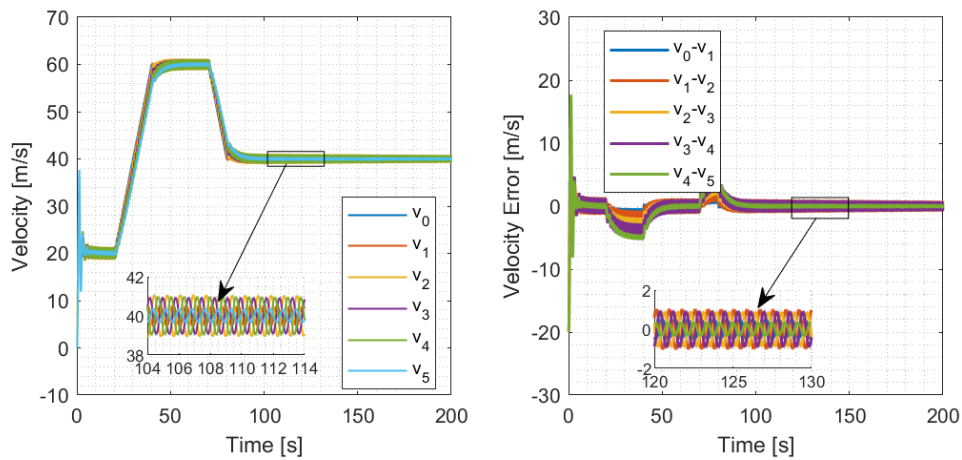


Figure 3.69 : Velocity and velocity errors for Example $\tau = 0.0788$ s.

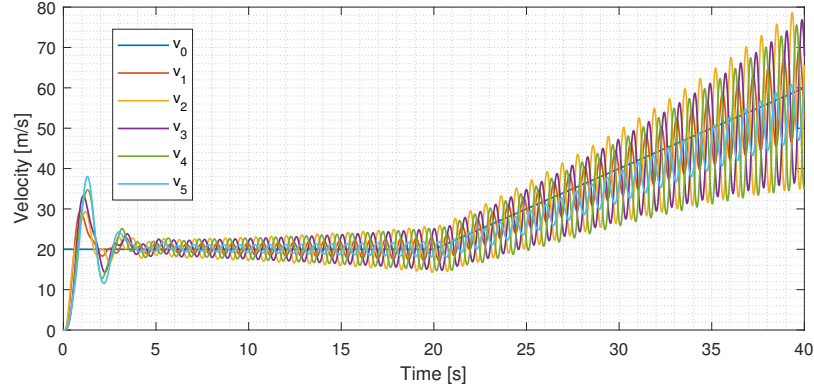


Figure 3.70 : Velocities Example $\tau = 0.08$ s

In the following of this part, the case where the vehicle platooning system is subject to both input and communication delay is examined. The applied controller signal is given in Eq. (3.112). Then, the closed loop formulation of platooning system which exposed to both input and communication time delays may be obtained as in Eq. (??) with the help of tracking errors and vector definition given in Eq. (3.135) and Eq. (3.136), respectively.

$$\dot{E}(t) = AE(t) + B_1E(t - \tau_1) + B_2E(t - \tau_2) \quad (3.171)$$

where;

$$\begin{aligned} A &= \begin{bmatrix} 0 & I_N & H \\ 0 & 0 & I_N \\ 0 & 0 & -\Lambda \end{bmatrix} \\ B_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Lambda K_p \mathcal{D}_p & -\Lambda K_v \mathcal{D}_p & -\Lambda K_a \mathcal{D}_p \end{bmatrix}, \\ B_2 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Lambda K_p \mathcal{A} & \Lambda K_p \mathcal{A} & \Lambda K_p \mathcal{A} \end{bmatrix} \end{aligned} \quad (3.172)$$

Where $\mathcal{D}_p = \mathcal{D} + \mathcal{P}$ is the information topology matrix which is defined in Section 2. It needs to be clearly stated that, the cooperative control problem of connected vehicles under time delay is transformed into a stabilization problem of error dynamics of a platoon tracking system with multiple time delays. The characteristic equation of a closed-loop system is not decomposed into its factors like the Kronecker product-type closed-loop formulation given in Section 3.2.2. And, Bezout's resultant matrix-based CTCR methodology, explained in the previous section, followed to achieve the $\tau_1 - \tau_2$ stability region.

Simulation Example 3.15: The same problem given in *Simulation Example 3.13* is evaluated for both input and communication time delays. The system parameters is given in Table 3.1 and the controller gains are as follows; $k_p = 2$, $k_v = 6$ and $k_a = 0.5$.

The matrices given in Eq. (3.172) are written as follows. The A matrix are the same as in Eq. (3.168). And, B_1 and B_2 matrices are given as follows;

$$B_1 = \begin{bmatrix} 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ B_{131} & B_{132} & B_{133} \end{bmatrix} \quad (3.173)$$

where;

$$B_{131} = -\Lambda K_p \mathcal{D}_p = \begin{bmatrix} -8 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 \\ 0 & 0 & -12 & 0 & 0 \\ 0 & 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix},$$

$$B_{132} = -\Lambda K_v \mathcal{D}_p = \begin{bmatrix} -24 & 0 & 0 & 0 & 0 \\ 0 & -36 & 0 & 0 & 0 \\ 0 & 0 & -36 & 0 & 0 \\ 0 & 0 & 0 & -36 & 0 \\ 0 & 0 & 0 & 0 & -24 \end{bmatrix},$$

$$B_{133} = -\Lambda K_a \mathcal{D}_p = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ 0_{N \times N} & 0_{N \times N} & 0_{N \times N} \\ B_{231} & B_{232} & B_{233} \end{bmatrix} \quad (3.174)$$

where;

$$B_{231} = -\Lambda K_p \mathcal{A} = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 \\ 4 & 4 & 0 & 4 & 0 \\ 0 & 4 & 4 & 0 & 4 \\ 0 & 0 & 4 & 4 & 0 \end{bmatrix},$$

$$B_{232} = -\Lambda K_v \mathcal{A} = \begin{bmatrix} 0 & 12 & 0 & 0 & 0 \\ 12 & 0 & 12 & 0 & 0 \\ 12 & 12 & 0 & 12 & 0 \\ 0 & 12 & 12 & 0 & 12 \\ 0 & 0 & 12 & 12 & 0 \end{bmatrix},$$

$$B_{233} = -\Lambda K_a \mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

The characteristic equation of the system for $z_1 = e^{\tau_1 s}$ and $z_2 = e^{\tau_2 s}$ as follows;

$$\begin{aligned} Q_1(s, z_1, z_2) &= \det(sI_{15} - A - B_1 z_1 - B_2 z_2 \\ &= s^{15} + (13z_1 + 10)s^{14} + (40 - 8z_2 + 67z_1^2 + 286z_1 - 4z_2^2)s^{13} - (-171z_1^3 - \\ &2278z_1^2 + 30z_1z_2^2 + 60z_1z_2 - 1820z_1 + 3z_2^3 + 142z_2^2 + 64z_2 - 80)s^{12} - \\ &(-216z_1^4 - 7866z_1^3 + 74z_1^2z_2^2 + 148z_1^2z_2 - 25728z_1^2 + 14z_1z_2^3 + 1408z_1z_2^2 + \\ &1200z_1z_2 - 5200z_1 - 3z_2^4 + 130z_2^3 + 1632z_2^2 + 192z_2 - 80)s^{11} - (-108z_1^5 - \\ &12528z_1^4 + 60z_1^3z_2^2 + 120z_1^3z_2 - 132012z_1^3 + 16z_1^2z_2^3 + 4324z_1^2z_2^2 + \\ &4736z_1^2z_2 - 112560z_1^2 - 7z_1z_2^4 + 792z_1z_2^3 + 23972z_1z_2^2 + 6000z_1z_2 - \\ &7280z_1 - 2z_2^5 - 180z_2^4 + 2036z_2^3 + 7176z_2^2 + 256z_2 - 32)s^{10} - (-7560z_1^5 - \\ &281664z_1^4 + 4200z_1^3z_2^2 + 5280z_1^3z_2 - 965808z_1^3 + 1120z_1^2z_2^3 + 97840z_1^2z_2^2 + \\ &47360z_1^2z_2 - 225120z_1^2 - 490z_1z_2^4 + 17344z_1z_2^3 + 176888z_1z_2^2 + 12000z_1z_2 - \\ &4576z_1 - 140z_2^5 - 4180z_2^4 + 13984z_2^3 + 14352z_2^2 + 128z_2)s^9 - (-213840z_1^5 - \\ &3030912z_1^4 + 118800z_1^3z_2^2 + 82080z_1^3z_2 - 2927520z_1^3 + 31680z_1^2z_2^3 + \\ &1058336z_1^2z_2^2 + 153920z_1^2z_2 - 205824z_1^2 - 13860z_1z_2^4 + 181840z_1z_2^3 + \\ &537344z_1z_2^2 + 9600z_1z_2 - 832z_1 - 3960z_2^5 - 46296z_2^4 + 41632z_2^3 + \\ &13056z_2^2)s^8 - (-3084480z_1^5 - 15382656z_1^4 + 1713600z_1^3z_2^2 + 513600z_1^3z_2 - \\ &3863232z_1^3 + 456960z_1^2z_2^3 + 5390304z_1^2z_2^2 + 189440z_1^2z_2 - 72896z_1^2 - \\ &199920z_1z_2^4 + 903616z_1z_2^3 + 707552z_1z_2^2 + 1920z_1z_2 - 57120z_2^5 - \\ &240000z_2^4 + 55936z_2^3 + 4544z_2^2)s^7 - (-23302080z_1^5 - 30765312z_1^4 + \\ &12945600z_1^3z_2^2 + 1027200z_1^3z_2 - 2112192z_1^3 + 3452160z_1^2z_2^3 + \\ &10780608z_1^2z_2^2 + 75776z_1^2z_2 - 8576z_1^2 - 1510320z_1z_2^4 + 1807232z_1z_2^3 + \\ &383552z_1z_2^2 - 431520z_2^5 - 480000z_2^4 + 32576z_2^3 + 512z_2^2)s^6 - \\ &(-82518912z_1^5 - 24247296z_1^4 + 45843840z_1^3z_2^2 + 656640z_1^3z_2 - \\ &503424z_1^3 + 12225024z_1^2z_2^3 + 8466688z_1^2z_2^2 + 9472z_1^2z_2 - 5348448z_1z_2^4 + \\ &1454720z_1z_2^3 + 90112z_1z_2^2 - 1528128z_2^5 - 370368z_2^4 + 8320z_2^3)s^5 - \\ &(-93208320z_1^5 - 9013248z_1^4 + 51782400z_1^3z_2^2 + 168960z_1^3z_2 - 43776z_1^3 + \end{aligned} \quad (3.175)$$

$$\begin{aligned}
& 13808640z_1^2z_2^3 + 3130880z_1^2z_2^2 - 6041280z_1z_2^4 + 555008z_1z_2^3 + \\
& 7680z_1z_2^2 - 1726080z_2^5 - 133760z_2^4 + 768z_2^3)s^4 - (-49351680z_1^5 - \\
& 1603584z_1^4 + 27417600z_1^3z_2^2 + 15360z_1^3z_2 + 7311360z_1^2z_2^3 + \\
& 553472z_1^2z_2^2 - 3198720z_1z_2^4 + 101376z_1z_2^3 - 913920z_2^5 - 23040z_2^4)s^3 \\
& (13685760z_1^5 + 110592z_1^4 - 7603200z_1^3z_2^2 - 2027520z_1^2z_2^3 - 37888z_1^2z_2^2 + \\
& 887040z_1z_2^4 - 7168z_1z_2^3 + 253440z_2^5 + 1536z_2^4)s^2 + (1935360z_1^5 - \\
& 1075200z_1^3z_2^2 - 286720z_1^2z_2^3 + 125440z_1z_2^4 + 35840z_2^5)s + 110592z_1^5 - \\
& 61440z_1^3z_2^2 - 16384z_1^2z_2^3 + 7168z_1z_2^4 + 2048z_2^5
\end{aligned}$$

By deploying Bezout's resultant matrix-based CTCR methodology whose steps are starts from Eq. (3.115), considering the controller gains $k_p = 2$, $k_v = 6$ and $k_a = 0.5$, the stability chart of Eq. (3.91) is obtained in the domain of time delays. The kernel points and offsprings curves are depicted in red and in blue, respectively. At each point on those curves the system has a pair of imaginary characteristic roots. By increasing either of the delays at one of those points the number of unstable characteristic roots changes. The shaded region in that plot corresponds to combinations of delays τ_1, τ_2 for which the number of unstable roots is zero. That is, this is the stable operating region. For any delay combination within the shaded area, characteristic equation of the system has not an unstable root.

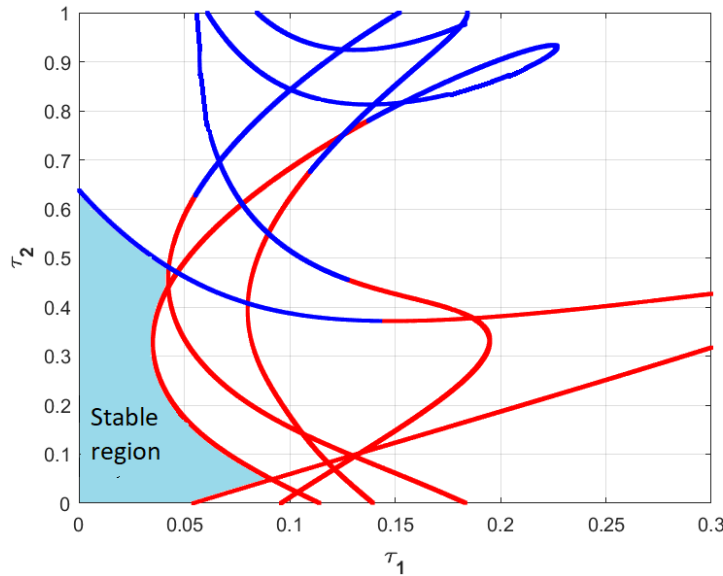


Figure 3.71 : Stability boundary for $k_p = 2$, $k_v = 6$ and $k_a = 0.5$.

The test point, $\tau_1 = 0.03 \text{ s}$ and $\tau_2 = 0.1 \text{ s}$ is selected on the shaded region to exhibit the stability of the system. The results are displayed for position and position errors in Figure 3.72 and velocity and velocity errors in Figure 3.73.

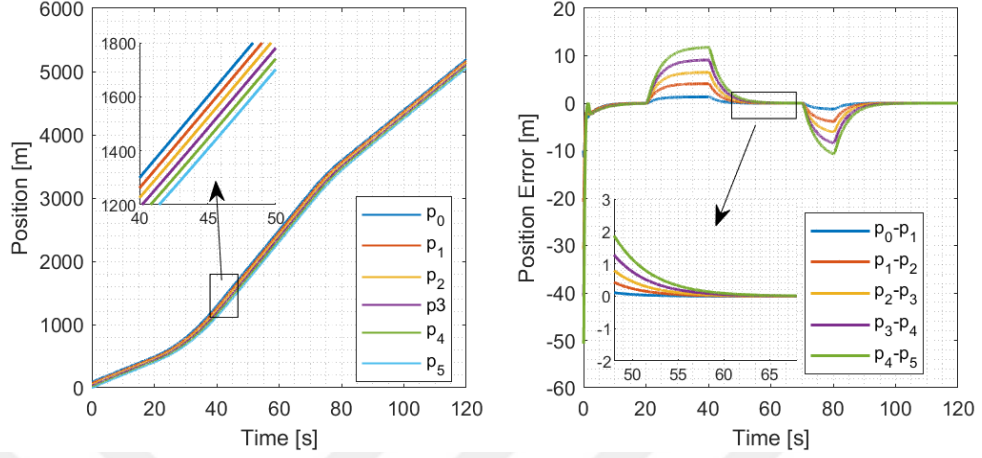


Figure 3.72 : Positions and position errors for $\tau_1 = 0.03s$ and $\tau_2 = 0.1s$.

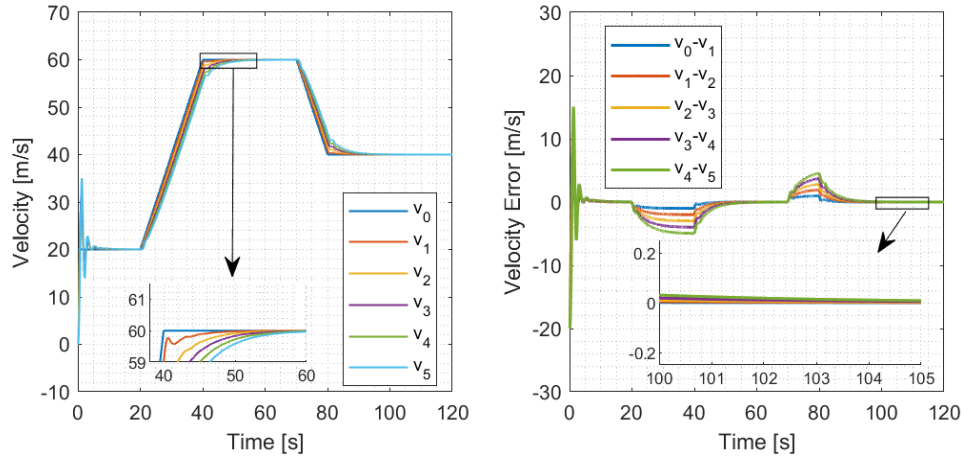


Figure 3.73 : Velocity and velocity errors for $\tau_1 = 0.03s$ and $\tau_2 = 0.1s$.

Then, the test point is selected as $\tau_1 = 0.03 \text{ s}$ and $\tau_2 = 0.517 \text{ s}$, which is on the stability boundary. Consequently, the system displays marginal stable behavior as shown in the Figure 3.74. Finally, the test points are selected as $\tau_1 = 0.03 \text{ s}$ and $\tau_2 = 0.53 \text{ s}$ in the unstable region. And it is seen in Figure 3.75 that the system is unstable, which conveys that at least one pair of roots of the closed-loop system is on the right half of the complex plane.

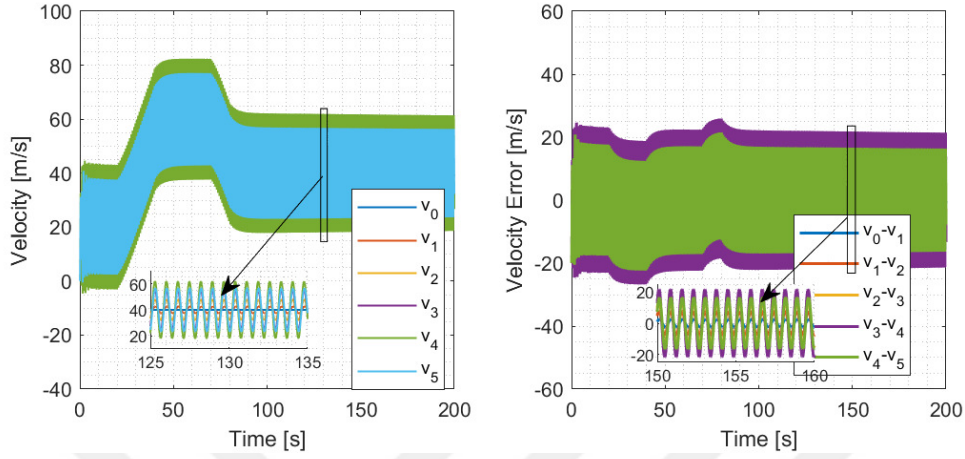


Figure 3.74 : Velocity and velocity errors for $\tau_1 = 0.03s$ and $\tau_2 = 0.517s$.

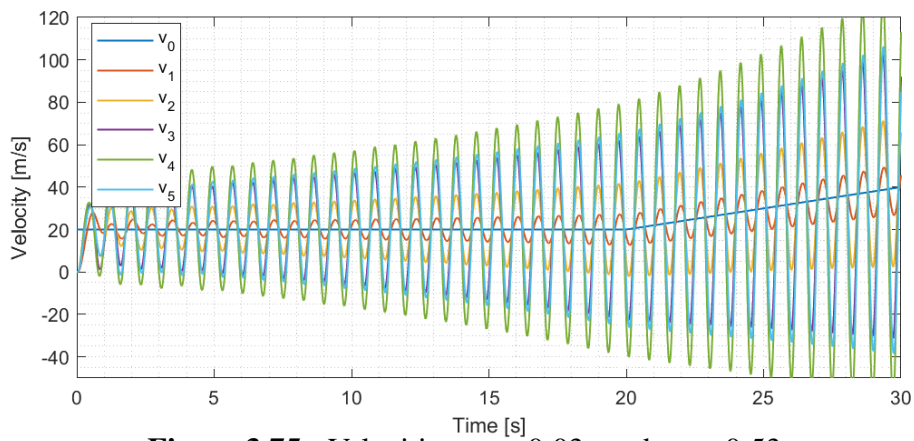


Figure 3.75 : Velocities $\tau_1 = 0.03s$ and $\tau_2 = 0.53s$

4. MULTI-AGENT CONTROL WITH NEUTRAL TIME-DELAY

4.1 Purpose

This section aims to address and solve the platooning control problem via *Proportional-Integral-Derivative* (PID) controller under time delay. The main reason to investigate PID controller is due to its simple structure and easy implementation. Besides, these controllers are the most preferred structures, both industrial applications, and embedded systems. In this part of the thesis, the theoretical studies are examined with the help of platooning system model. In the proposed approach, each vehicle computes its own distributed control signal using the state information received from its neighbors via vehicle-to-vehicle communication links. It should be remarked that the whole platooning system can be thought of as a leader-following consensus problem with the distributed PID controller. Each vehicle takes the state information from its neighbors. The algorithm is basically run as follows; the calculated error, the integral of the error, and the derivative of error are weighted with corresponding controller gains which are k_p , k_i and k_d , respectively. Some similar studies are conducted in the following references [53, 103, 109].

Then, the cooperative control of platooning system problem is converted into an asymptotic stability problem with the help of some transformations. And, a Lyapunov function is constructed and the gain matrices of the controller protocol are assigned based on the linear matrix inequality. However, in the presence of time delay, the delay value is the dominant element that affects the system's stability. And the stability posture should be investigated with the help of the time-delay-based stability analysis method. A single time delay is considered in the proposed PID-based cooperative control problem. Moreover, the structure of the closed-loop system represents a neutral-type time delay system. And, the *Kronecker multiplication and elementary*

transformation based CTCR method is utilized to obtain the stability posture of the system under time delay.

4.2 Stability Analysis for Nominal Cases

This part of the dissertation considers the control problem of vehicular platooning system formation without time delay. The aim is to construct a suitable distributed PID controller, so that the consensus problem can be solved and provide a sufficient performance. The proposed structure of distributed PID controller for the vehicle i is written as in Eq. (4.1).

$$\begin{aligned}
 u_i(t) = & -k_p \left[\sum_{j=1}^N a_{ij} \left(x_i(t) - x_j(t) + \bar{d}_{i,j} \right) + p_{ii} \left(x_i(t) - x_0(t) + \bar{d}_{i,0} \right) \right] \\
 & -k_i \left[\sum_{j=1}^N a_{ij} \left(\int_0^t (x_i(t) - x_j(t) + \bar{d}_{i,j}) dt \right) + p_{ii} \left(\int_0^t (x_i(t) - x_0(t) + \bar{d}_{i,0}) dt \right) \right] \\
 & -k_d \left[\sum_{j=1}^N a_{ij} \left(\dot{x}_i(t) - \dot{x}_j(t) + \dot{\bar{d}}_{i,j} \right) + p_{ii} \left(\dot{x}_i(t) - \dot{x}_0(t) + \dot{\bar{d}}_{i,0} \right) \right]
 \end{aligned} \tag{4.1}$$

where $x_i(t) = [p_i(t), v_i(t), a_i(t)]^T$ are the absolute position, velocity and acceleration of i^{th} vehicle, respectively. k_p , k_i and k_d are the proportional, integral and derivative control gains. a_{ij} and p_{ii} is the corresponding elements of Adjacency matrix and Pinning matrix which are defined in Section 2. N is the number of vehicles in the platoon for $i = 1, 2, \dots, N$, and the leader vehicle is indexed as 0. $\bar{d}_{i,j} = [d_{i,j} \ 0 \ 0]^T$ and $\bar{d}_{i,0} = [d_{i,0} \ 0 \ 0]^T$ are the inter-vehicle distance vectors between i^{th} and j^{th} vehicles and i^{th} and 0^{th} vehicles, respectively. In this study, the inter-vehicle distance is calculated with the constant distance (CD) policy, which means $d_{i,i-1} = d_0$ and $d_{i,i+1} = -d_0$, where d_0 is a constant number.

The position, velocity, and acceleration tracking errors between i^{th} and 0^{th} vehicle are defined as in Eq. (4.2) also given in below for reminder.

$$\begin{aligned}
 e_{p_i}(t) &= p_i(t) - p_0(t) + d_{i,0} \\
 e_{v_i}(t) &= v_i(t) - v_0(t) \\
 e_{a_i}(t) &= a_i(t) - a_0(t)
 \end{aligned} \tag{4.2}$$

Therefore, the controller structure given in Eq. (4.3) may be written in terms of error terms as follows;

$$\begin{aligned}
u_i(t) = & -k_p \left[\sum_{j=1}^N a_{ij} \left((e_{p_i}(t) - e_{p_j}(t)) + (e_{v_i}(t) - e_{v_j}(t)) + (e_{a_i}(t) - e_{a_j}(t)) \right) \right. \\
& + p_{ii} \left(e_{p_i}(t) + e_{v_i}(t) + e_{a_i}(t) \right) \left. \right] - k_i \left[\sum_{j=1}^N a_{ij} \left(\int_0^t (e_{p_i}(t) - e_{p_j}(t)) dt + \int_0^t \right. \right. \\
& \left. \left(e_{v_i}(t) - e_{v_j}(t) \right) dt + \int_0^t (e_{a_i}(t) - e_{a_j}(t)) dt \right) + p_{ii} \int_0^t (e_{p_i}(t) + e_{v_i}(t) + \\
& e_{a_i}(t)) dt \left. \right] - k_d \left[\sum_{j=1}^N a_{ij} \left((\dot{e}_{p_i}(t) - \dot{e}_{p_j}(t)) + (\dot{e}_{v_i}(t) - \dot{e}_{v_j}(t)) + (\dot{e}_{a_i}(t) - \right. \right. \\
& \left. \left. \dot{e}_{a_j}(t)) \right) + p_{ii} (\dot{e}_{p_i}(t) + \dot{e}_{v_i}(t) + \dot{e}_{a_i}(t)) \right]
\end{aligned} \tag{4.3}$$

To facilitate the notation of closed-loop system the vector definition given in Eq. (4.4) is used.

$$\begin{aligned}
e_i(t) &= [e_{p_i}(t) \quad e_{v_i}(t) \quad e_{a_i}(t)]^T \\
E(t) &= [e_1(t) \quad e_2(t) \quad \dots \quad e_N(t)]^T \\
U(t) &= [u_1(t) \quad u_2(t) \quad \dots \quad u_N(t)]^T \\
K_p &= [k_{p_p} \quad k_{p_v} \quad k_{p_a}] \\
K_i &= [k_{i_p} \quad k_{i_v} \quad k_{i_a}] \\
K_d &= [k_{d_p} \quad k_{d_v} \quad k_{d_a}]
\end{aligned} \tag{4.4}$$

It is known that from the previous studies, the structure of information topology matrix, $\mathcal{L}_p$ has a significant effect on the stability posture of the system. The stability analysis is relatively facile compared to the complex case if the eigenvalues $\mathcal{L}_p$ are real. In this study, the eigenvalues of $\mathcal{L}_p$ are considered complex. Then, the first step is to convert the cooperative control of platooning problem into an asymptotic stability problem. The control signal in Eq. (4.3) may be written as in Eq. (4.5) to reduce the complexity.

$$\begin{aligned}
u_i(t) = & -k_p \left[\sum_{j=1}^N a_{ij} (e_i(t) - e_j(t)) + p_{ii} (e_i(t)) \right] - k_i \left[\sum_{j=1}^N a_{ij} \left(\int_0^t (e_i(t) - \right. \right. \\
& \left. \left. e_j(t) dt \right) + p_{ii} \left(\int_0^t (e_i(t)) dt \right) \right] - k_d \left[\sum_{j=1}^N a_{ij} (\dot{e}_i(t) + \dot{e}_j(t)) + p_{ii} (\dot{e}_i(t)) \right]
\end{aligned} \tag{4.5}$$

Then, the control signal obtained in Eq. (4.5) is substituted into Eq. (3.4), and compact form of the error dynamic system is obtained as in Eq. (4.6) with the help of vector

definition given in Eq. (4.4).

$$\begin{aligned}\dot{E}(t) = & \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) E(t) - (\mathcal{L}_p \otimes BK_i) \int_0^t E(t) dt \\ & - (\mathcal{L}_p \otimes BK_d) \dot{E}(t)\end{aligned}\quad (4.6)$$

The differential part of Eq. (4.6) may be moved to the left side and the the closed loop formulation of platooning error dynamic system with distributed PID controller is obtained as in Eq. (4.7);

$$\begin{aligned}\left((I_N \otimes I_r) + (\mathcal{L}_p \otimes BK_d) \right) \dot{E}(t) = & \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) E(t) \\ & - (\mathcal{L}_p \otimes BK_i) \int_0^t E(t) dt\end{aligned}\quad (4.7)$$

To reduce the complexity, let define;

$$\bar{A} = (I_N \otimes I_r) + (\mathcal{L}_p \otimes BK_d) \quad (4.8)$$

On the condition that $\bar{A}$ is always invertible Eq. (4.7) is depicted as in Eq. (4.9).

$$\dot{E}(t) = \bar{A}^{-1} \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) E(t) - \bar{A}^{-1} (\mathcal{L}_p \otimes BK_i) \int_0^t E(t) dt \quad (4.9)$$

Now, under the condition that the error state vector, $E(t)$ has second derivative and all the coefficient matrix are real, then Eq. (4.9) may be written as;

$$\ddot{E}(t) = \bar{A}^{-1} \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) \dot{E}(t) - \bar{A}^{-1} (\mathcal{L}_p \otimes BK_i) E(t) \quad (4.10)$$

Let define the following augmented state variables;

$$\begin{aligned}F(t) &= \dot{E}(t) \\ \dot{F}(t) &= \ddot{E}(t)\end{aligned}\quad (4.11)$$

So, a new closed-loop state space form is conducted as in Eq. (4.12).

$$\begin{bmatrix} \dot{E}(t) \\ \dot{F}(t) \end{bmatrix} = \begin{bmatrix} 0_N \otimes 0_r & I_N \otimes I_r \\ -\bar{A}^{-1} (\mathcal{L}_p \otimes BK_i) & \bar{A}^{-1} \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) \end{bmatrix} \begin{bmatrix} E(t) \\ F(t) \end{bmatrix} \quad (4.12)$$

Where;

$$A_c = \begin{bmatrix} 0_N \otimes 0_r & I_N \otimes I_r \\ -\bar{A}^{-1} (\mathcal{L}_p \otimes BK_i) & \bar{A}^{-1} \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) \end{bmatrix} \quad (4.13)$$

Let remind the following definitions;

Definition 4.1: [110] Let λ_i , $i = 1, 2, \dots, N$ be the eigenvalue of $\mathcal{L}_p$, then all eigenvalues of $\mathcal{L}_p$ have positive real parts, if there exist a directed spanning tree whose

root is the leader vehicle. So, the eigenvalues of $\mathcal{L}_p$ satisfy $Re(\lambda_1) \leq Re(\lambda_2) \leq \dots \leq Re(\lambda_N)$. It means that every follower vehicle is globally reachable from the leader vehicle. Namely, leader vehicle can sent its information to every follower vehicle directly or indirectly.

Definition 4.2: [44] Let the $\mathcal{L}_p$ is an arbitrary square matrix, $\lambda_i, i = 1, 2, \dots, q$ are the real eigenvalues of $\mathcal{L}_p$ including multiplicity, and $\lambda_i = \sigma_i + j\omega_i, \lambda_i^* = \sigma_i - j\omega_i, i = q + 1, q + 2, \dots, q + r$ are the complex conjugate eigenvalues of $\mathcal{L}_p$ including multiplicity. There exist a non-singular matrix $M \in \mathbb{R}^{N \times N}$ such that

$$J_p = M^{-1} \mathcal{L}_p M \quad (4.14)$$

where, J_p is the Jordan normal form of $\mathcal{L}_p$.

According to Definition 4.2 and matrix theory it is possible write $\bar{A}$ given in Eq. (4.8) into Jordan normal form as in Eq. (4.15).

$$\begin{aligned} J_{\bar{A}} &= (M \otimes I_r)^{-1} \left((I_N \otimes I_r) + (\mathcal{L}_p \otimes BK_d) \right) (M \otimes I_r) \\ &= \left(M^{-1} I_N \otimes I_r^{-1} I_r + M^{-1} \mathcal{L}_p \otimes I_r^{-1} BK_d \right) (M \otimes I_r) \\ &= M^{-1} I_N M \otimes I_r^{-1} I_r I_r + M^{-1} \mathcal{L}_p M \otimes I_r^{-1} BK_d I_r \\ &= I_N \otimes I_r + M^{-1} M J_p M^{-1} M \otimes BK_d \\ &= I_N \otimes I_r + J_p \otimes BK_d \\ &= \text{diag}(J_{\bar{A}_1}, J_{\bar{A}_2}, \dots, J_{\bar{A}_N}) \end{aligned} \quad (4.15)$$

where $J_{\bar{A}}$ is the Jordan form of $\bar{A}$ and $J_{\bar{A}_i}$ is the Jordan block of $\bar{A}$. And, the value of Jordan blocks of are written as given in the following definition;

Definition 4.3: Let define a system matrix which is form of $A_c = I_N \otimes A + J \otimes BK$. Where $A \in \mathbb{R}^{r \times r}$, the product of $BK \in \mathbb{R}^{r \times r}$ and $J \in \mathbb{R}^{N \times N}$ is a Jordan form of any matrix whose eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_N$. Then the Jordan blocks of A_c are written as follows;

$$J_{A_c} = I_r + \lambda_i BK \quad (4.16)$$

Therefore, it can be written that that $J_{\bar{A}_i} = I_r + \lambda_i BK_d$, where λ_i is the eigenvalue of $\mathcal{L}_p$. The inverse matrix of $\bar{A}$, which is placed in the closed-loop system matrix as seen in Eq. (4.12) may be obtained as as in Eq. (4.17).

$$\begin{aligned} \bar{A}^{-1} &= \left[(M \otimes I_r) J_{\bar{A}} (M \otimes I_r) \right]^{-1} \\ &= (M \otimes I_r) J_{\bar{A}}^{-1} (M \otimes I_r)^{-1} \end{aligned} \quad (4.17)$$

The definitions and information are utilized to decomposed the closed-loop platooning system given in Eq. (4.12) into some individual subsystem to reduce the analytic difficulty. To this aim the similarity transformations are defined as in Eq. (4.18).

$$\begin{aligned}\tilde{E}(t) &= (M^{-1} \otimes I_r)E(t) \\ \tilde{F}(t) &= (M^{-1} \otimes I_r)F(t)\end{aligned}\tag{4.18}$$

To simplify the mathematical analysis, let define the following states, states transformation matrix and its inverse.

$$\xi(t) = \begin{bmatrix} E(t) \\ F(t) \end{bmatrix}, \quad \tilde{\xi}(t) = \begin{bmatrix} \tilde{E}(t) \\ \tilde{F}(t) \end{bmatrix}\tag{4.19}$$

$$T = \begin{bmatrix} M \otimes I_r & \\ & M \otimes I_r \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} M^{-1} \otimes I_r & \\ & M^{-1} \otimes I_r \end{bmatrix}\tag{4.20}$$

Then, the closed-loop system matrix given in Eq. (4.12) may be written in terms of the new states given in Eq. (4.19) and (4.20)

$$\begin{aligned}\dot{\xi}(t) &= A_c \xi(t) \\ \dot{\tilde{\xi}}(t) &= T^{-1} A_c T \tilde{\xi}(t) \\ \dot{\tilde{\xi}}(t) &= \bar{A}_c \tilde{\xi}(t)\end{aligned}\tag{4.21}$$

where;

$$\begin{aligned}\bar{A}_{c11} &= 0_N \otimes 0_r \\ \bar{A}_{c12} &= I_N \otimes I_r \\ \bar{A}_{c21} &= (M^{-1} \otimes I_r) \bar{A}^{-1} (\mathcal{L}_p \otimes BK_i) (M \otimes I_r) \\ &= (M^{-1} \otimes I_r) (M \otimes I_r) J_{\bar{A}}^{-1} (M^{-1} \otimes I_r) (MJ_p M^{-1} \otimes BK_i) (M \otimes I_r) \\ &= J_{\bar{A}}^{-1} (J_p \otimes BK_i) \\ \bar{A}_{c22} &= (M^{-1} \otimes I_r) \bar{A}^{-1} \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) (M \otimes I_r) \\ &= (M^{-1} \otimes I_r) (M \otimes I_r) J_{\bar{A}}^{-1} (M^{-1} \otimes I_r) \left((I_N \otimes A) - (\mathcal{L}_p \otimes BK_p) \right) (M \otimes I_r) \\ &= J_{\bar{A}}^{-1} \left((M^{-1} I_N \otimes I_r A) - (M^{-1} \mathcal{L}_p \otimes I_r BK_p) \right) (M \otimes I_r) \\ &= J_{\bar{A}}^{-1} \left((M^{-1} I_N M \otimes I_r A I_r) - (M^{-1} M J_p M^{-1} M \otimes I_r BK_p I_r) \right) \\ &= J_{\bar{A}}^{-1} \left(I_N \otimes A - J_p \otimes BK_p \right)\end{aligned}\tag{4.22}$$

Then, $\bar{A}_c$ is obtained with Eq. (4.23).

$$\bar{A}_c = \begin{bmatrix} 0_N \otimes 0_r & I_N \otimes I_r \\ -J_{\bar{A}}^{-1} (J_p \otimes BK_i) & J_{\bar{A}}^{-1} (I_N \otimes A - J_p \otimes BK_p) \end{bmatrix}\tag{4.23}$$

where $J_p = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ is the Jordan normal form of $\mathcal{L}_p$ and $J_{\bar{A}}^{-1} = \text{diag}(J_{\bar{A}_1}^{-1}, J_{\bar{A}_2}^{-1}, \dots, J_{\bar{A}_N}^{-1})$ is the inverse of Jordan normal form of $\bar{A}$. It should be reminded that each state variable may satisfy the statements in Eq. (4.24).

$$\begin{aligned}\dot{\tilde{e}}_i(t) &= \tilde{f}_i(t) \\ \dot{\tilde{f}}_i(t) &= -\lambda_i J_{\bar{A}_i}^{-1} B K_i \tilde{e}_i(t) + J_{\bar{A}_i}^{-1} (A - \lambda_i B K_p) \tilde{f}_i(t)\end{aligned}\quad (4.24)$$

In the next step, the consensus problem of platooning system is converted into an asymptotically stability problem. Then, to analysis the stability problem over the matrix of $\bar{A}_c$, it needs to transform the $\bar{A}_c$ to diagonal or block upper-diagonal form since it is known that the closed-loop system given in Eq. (4.21) is stable if and only if the N subsystems defined taking the diagonal blocks is stable. Actually, the transformation changes the sequence of states as in Eq. (4.25).

$$\tilde{\xi}(t) = \begin{bmatrix} \tilde{E}(t) \\ \tilde{F}(t) \end{bmatrix} = \begin{bmatrix} \tilde{e}_1(t) \\ \tilde{e}_2(t) \\ \tilde{e}_3(t) \\ \vdots \\ \tilde{e}_N(t) \\ \tilde{f}_1(t) \\ \tilde{f}_2(t) \\ \tilde{f}_3(t) \\ \vdots \\ \tilde{f}_N(t) \end{bmatrix} \xrightarrow{T} \begin{bmatrix} \tilde{e}_1(t) \\ \tilde{f}_1(t) \\ \tilde{e}_2(t) \\ \tilde{f}_2(t) \\ \tilde{e}_3(t) \\ \tilde{f}_3(t) \\ \vdots \\ \tilde{e}_N(t) \\ \tilde{f}_N(t) \end{bmatrix} \quad (4.25)$$

To obtain the diagonal compact form of closed-loop system, it needs to define a new state transformation as in Eq. (4.26) which gives the state order as in Eq. (4.25).

$$\begin{aligned}\tilde{\xi}(t) &= T_R \tilde{\Sigma}(t) \\ \tilde{\Sigma}(t) &= T_L \tilde{\xi}(t) \\ T_L &= T_r^{-1}\end{aligned}\quad (4.26)$$

where;

$$\begin{aligned}\tilde{\Sigma}_i(t) &= \begin{bmatrix} \tilde{e}_i^T(t) & \tilde{f}_i^T(t) \end{bmatrix}^T \\ \tilde{\Sigma}(t) &= \begin{bmatrix} \tilde{\Sigma}_1(t) & \tilde{\Sigma}_2(t) & \dots & \tilde{\Sigma}_N(t) \end{bmatrix}^T\end{aligned}\quad (4.27)$$

Thus, the following transformation may be written with the help of Eq. (4.21);

$$\begin{aligned}\tilde{\Sigma}(t) &= T_L \tilde{\xi}(t) \\ \dot{\tilde{\Sigma}}(t) &= T_L \dot{\tilde{\xi}}(t) \\ \dot{\tilde{\Sigma}}(t) &= T_L \bar{A}_c \tilde{\xi}(t) \\ \dot{\tilde{\Sigma}}(t) &= T_L \bar{A}_c T_R \tilde{\Sigma}(t) \\ \dot{\tilde{\Sigma}}(t) &= \bar{A}_T \tilde{\Sigma}(t)\end{aligned}\quad (4.28)$$

where;

$$\begin{aligned}\bar{A}_T &= T_L \bar{A}_c T_R \\ &= \text{diag}\{\bar{A}_{T_1}, \bar{A}_{T_2}, \dots, \bar{A}_{T_N}\}\end{aligned}\quad (4.29)$$

and

$$\bar{A}_{T_i} = \begin{bmatrix} 0_r & I_r \\ -\lambda_i J_{\bar{A}_i}^{-1} B K_i & J_{\bar{A}_i}^{-1} (A - \lambda_i B K_p) \end{bmatrix} \quad (4.30)$$

Through the transformation given Eq. (4.28), the cooperative control of platooning problem is converted into an asymptotic stability problem. So, the following theorem is given;

Theorem 4.1 : Consider the homogeneous platooning system using the distributed PID controller given in Eq. (4.3) with the closed-loop dynamics in Eq. (4.6). Let the information topology matrix $\mathcal{L}_p$ is an arbitrary square matrix, $\lambda_i, i = 1, 2, \dots, q$ are the real eigenvalues and $\lambda_i = \sigma_i + j\omega_i$, $\lambda_i^* = \sigma_i - j\omega_i, i = q + 1, q + 2, \dots, q + r$ are the complex conjugate eigenvalues of $\mathcal{L}_p$. Therefore, $q + r = N$, where N is the length of the platoon. Then, the platooning system achieves the controller objectives given in Eq. (2.93) if and only if $\bar{A}_{T_i}, i = 1, 2, \dots, N$ is Hurwitz.

Proof : The closed-loop platooning system given in Eq. (4.6) is transformed into a new format as given in Eq. (4.21) with the help of the similarity transformation given in Eq. (4.18). Then, the order of states of these new state space form is rearranged and the closed-loop system matrix is converted into block diagonal form. The transformations of system matrix are shown below;

$$A_c \implies \bar{A}_c \implies \bar{A}_T \quad (4.31)$$

And;

$$\begin{aligned}\dot{\tilde{\Sigma}}(t) &= \bar{A}_T \tilde{\Sigma}(t) \\ &= T_L \bar{A}_c T_R \tilde{\Sigma}(t) \\ &= T_L T^{-1} A_c T T_R \tilde{\Sigma}(t) \\ &= (T_L T^{-1} T_R) (T_L A_c T_R) (T_L T T_R) \tilde{\Sigma}(t) \\ &= (T_L T^{-1} T_R) (T_L A_c T_R) (T_L T T_R) \tilde{\Sigma}(t) \\ &= (T_R^{-1} T^{-1} T_L^{-1}) (T_L A_c T_R) (T_L T T_R) \tilde{\Sigma}(t) \\ &= (T_L T T_R)^{-1} (T_L A_c T_R) (T_L T T_R) \tilde{\Sigma}(t)\end{aligned}\quad (4.32)$$

Let define, $J_T = T_L T T_R$ and $A_T = T_L A_c T_R$. Then;

$$\dot{\tilde{\Sigma}}(t) = J_T^{-1} A_T J_T \tilde{\Sigma}(t) \quad (4.33)$$

So, the expression given in Eq. (4.34) is obtained easily;

$$\begin{aligned}\dot{\Sigma}(t) &= A_T \Sigma(t) \\ \Sigma(t) &= \left[\Sigma_1(t) \quad \Sigma_i(t), \dots, \Sigma_N(t) \right]^T \\ \Sigma_i(t) &= \left[e_i^T(t) \quad f_i^T(t) \right]^T, \quad i = 1, 2, \dots, N.\end{aligned}\tag{4.34}$$

The transformation shows that, if the system matrix $\bar{A}_T$ is Hurwitz, which means all its block diagonal matrices, $\bar{A}_{T_i}, i = 1, 2, \dots, N$ are Hurwitz, then the platooning system is internally stable and reaches consensus.

Thus, the proof is completed.

Theorem 4.1 states the necessary and sufficient conditions for the platooning system to satisfy system stability. However, it does not contain any conditions for the gain matrices K_p , K_i , and K_d to guarantee stability. So, in the following a Riccati inequality based algorithm is presented to compute the controller gains. In this study, the $\mathcal{L}_p$ matrix is an arbitrary square matrix, which assumes that it has both real and complex eigenvalues for the general structure. The following definitions is given to understand the rest of the analysis.

Definition 4.4 : Let $C = A + jB$ is an complex matrix where A, B are real matrices and $j^{-1} = -1$. Then the inverse of C is calculated as in Eq. (4.35).

$$C^{-1} = (A + BA^{-1}B)^{-1} - jA^{-1}B(A + BA^{-1}B)^{-1}\tag{4.35}$$

Definition 4.5 : Let $C = A + jB$ is a complex matrix, where A, B are real matrices, it may also be expressed by another square matrix as given in Eq. (4.36). It means that the eigenvalues of C are the same the eigenvalues of $\hat{C}$.

$$\hat{C} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}\tag{4.36}$$

Theorem 4.1 gives that, the stability properties of the closed-loop system given in Eq. (4.12) are equivalent to the stability properties of the N subsystems given in Eq. (4.37).

$$\begin{aligned}\dot{\tilde{\Sigma}}_i(t) &= \bar{A}_{T_i} \tilde{\Sigma}_i(t) \\ \dot{\tilde{\Sigma}}_i(t) &= \begin{bmatrix} 0_r & I_r \\ -\lambda_i J_{\bar{A}_i}^{-1} B K_i & J_{\bar{A}_i}^{-1} (A - \lambda_i B K_p) \end{bmatrix} \tilde{\Sigma}_i(t)\end{aligned}\tag{4.37}$$

in that they have the same eigenvalues.

According to Definition 4.4 and 4.5, the inverse of $J_{\bar{A}_i} = I_r + \lambda_i BK_d$, which is the Jordan block of $J_{\bar{A}}$ is obtain as in Eq. (4.38). In here, $J_{\bar{A}}$ is Jordan normal form of $\bar{A} = (I_N \otimes I_r) + (\mathcal{L}_p \otimes BK_d)$ and $\lambda_i = \sigma_i + j\omega_i$, $i = 1, 2, \dots, N$ is the eigenvalue of $\mathcal{L}_p$.

$$\begin{aligned}
J_{\bar{A}_i}^{-1} &= (I_r + (\sigma_i + j\omega_i)BK_d)^{-1} \\
&= ((I_r + \sigma_i BK_d) + j(\omega_i BK_d))^{-1} \\
&= [(I_r + \sigma_i BK_d) + \omega_i BK_d (I_r + \sigma_i BK_d)^{-1} \omega_i BK_d]^{-1} \\
&\quad - j(I_r + \sigma_i BK_d)^{-1} \omega_i BK_d [(I_r + \sigma_i BK_d) + \omega_i BK_d (I_r + \sigma_i BK_d)^{-1} \omega_i BK_d]^{-1} \\
&= [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} \\
&\quad - j\omega_i (I_r + \sigma_i BK_d)^{-1} BK_d [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1}
\end{aligned} \tag{4.38}$$

And, since it is also a complex matrix, $\bar{A}_{T_i}$ which is the square diagonal blocks of block diagonal matrix $\bar{A}_T$ may be written as in Eq. (4.39).

$$\bar{A}_{T_i} = \bar{A}_{Re} + j\bar{A}_{Im} \tag{4.39}$$

where $\bar{A}_{Re}$ and $\bar{A}_{Im}$ are shown below;

$$\bar{A}_{Re} = \begin{bmatrix} 0_r \\ -\sigma_i [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} BK_i - \omega_i^2 (I_r + \sigma_i BK_d)^{-1} BK_d [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} BK_i \end{bmatrix} \tag{4.40}$$

$$\begin{aligned}
&\begin{bmatrix} I_r \\ [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} (A - \sigma_i BK_d) - \omega_i^2 (I_r + \sigma_i BK_d)^{-1} BK_d [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} BK_p \end{bmatrix} \\
\bar{A}_{Im} &= \begin{bmatrix} 0_r \\ \sigma_i \omega_i (I_r + \sigma_i BK_d)^{-1} BK_d [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} \\ BK_d]^{-1} BK_i - \omega_i [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} \\ BK_i \end{bmatrix} \\
&\begin{bmatrix} 0_r \\ -\omega_i [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} BK_p - \omega_i (I_r + \sigma_i BK_d)^{-1} BK_d [(I_r + \sigma_i BK_d) + \omega_i^2 BK_d (I_r + \sigma_i BK_d)^{-1} BK_d]^{-1} \\ (A - \sigma_i BK_d) \end{bmatrix}
\end{aligned} \tag{4.41}$$

In accordance with Definition 4.5, the subsystem given in Eq. (4.37) may also be written as in Eq. (4.42).

$$\dot{\tilde{\Sigma}}_i(t) = \begin{bmatrix} \bar{A}_{Re} & -\bar{A}_{Im} \\ \bar{A}_{Im} & \bar{A}_{Re} \end{bmatrix} \tilde{\Sigma}_i(t) \tag{4.42}$$

It is known from the control theory and consensus theory, the system given in Eq. (4.42) is stable, if the real parts of the all eigenvalues of $\bar{A}_{T_i}$ are in negative half plane, which means $\bar{A}_{T_i}$ is Hurwitz and the system reach consensus.

Finally, the condition that gain matrices K_p , K_i , and K_d should satisfy to guarantee stabilities is given in the following Lemma.

Lemma 4.1 Consider the homogeneous platooning system using the distributed PID controller given in Eq. (4.3) with the closed-loop dynamics in Eq. (4.6). Let the information topology matrix $\mathcal{L}_p$ is an arbitrary square matrix, $\lambda_i, i = 1, 2, \dots, q$ are the real eigenvalues and $\lambda_i = \sigma_i + j\omega_i$, $\lambda_i^* = \sigma_i - j\omega_i, i = q + 1, q + 2, \dots, q + r$ are the complex conjugate eigenvalues of $\mathcal{L}_p$. Therefore, $q + r = N$, where N is the length of the platoon. Then, the platooning system is internally stable if there exist a positive-definite symmetric matrix $P \in R^{r \times r}$, so that the controller gain matrices K_p , K_i and K_s satisfies the following linear matrix inequality.

$$\Psi_i = \begin{bmatrix} 0_r & \Psi_{12} & 0_r & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & 0_r \\ 0_r & \Psi_{32} & 0_r & \Psi_{34} \\ \Psi_{41} & 0_r & \Psi_{43} & \Psi_{44} \end{bmatrix} < 0, \quad i = 1, 2, \dots, N. \quad (4.43)$$

Proof : For the system given in Eq. (4.43), the following Lyapunov function may be considered;

$$V_i(t) = \tilde{\Sigma}_i^T(t)(I_4 \otimes P)\tilde{\Sigma}_i(t) \quad (4.44)$$

Since P is assumed a positive-definite matrix, so is $I_4 \otimes P$. Then, the time derivative of $V_i(t)$ is obtained and rearranged as in Eq. (4.45);

$$\begin{aligned} \dot{V}_i(t) &= \dot{\tilde{\Sigma}}_i^T(t)(I_4 \otimes P)\tilde{\Sigma}_i(t) + \tilde{\Sigma}_i^T(t)(I_4 \otimes P)\dot{\tilde{\Sigma}}_i(t) \\ &= \tilde{\Sigma}_i^T(t) \begin{bmatrix} \bar{A}_{Re}^T & \bar{A}_{Im}^T \\ -\bar{A}_{Im}^T & \bar{A}_{Re}^T \end{bmatrix} (I_4 \otimes P)\tilde{\Sigma}_i(t) + \tilde{\Sigma}_i^T(t)(I_4 \otimes P) \begin{bmatrix} \bar{A}_{Re} & -\bar{A}_{Im} \\ \bar{A}_{Im} & \bar{A}_{Re} \end{bmatrix} \tilde{\Sigma}_i(t) \\ &= \tilde{\Sigma}_i^T(t) \left(\begin{bmatrix} \bar{A}_{Re}^T & \bar{A}_{Im}^T \\ -\bar{A}_{Im}^T & \bar{A}_{Re}^T \end{bmatrix} (I_4 \otimes P) + (I_4 \otimes P) \begin{bmatrix} \bar{A}_{Re} & -\bar{A}_{Im} \\ \bar{A}_{Im} & \bar{A}_{Re} \end{bmatrix} \right) \tilde{\Sigma}_i(t) \end{aligned} \quad (4.45)$$

If the matrices of $\bar{A}_{Re}$ and $\bar{A}_{Im}$ given in Eq. (4.40) and (4.41) are shortened as,

$$\bar{A}_{Re} = \begin{bmatrix} 0_r & I_r \\ \bar{A}_{Re21} & \bar{A}_{Re22} \end{bmatrix}, \quad \bar{A}_{Im} = \begin{bmatrix} 0_r & 0_r \\ \bar{A}_{Im21} & \bar{A}_{Im22} \end{bmatrix} \quad (4.46)$$

and substituting into Eq. (4.45), then;

$$\begin{aligned}
&= \tilde{\Sigma}_i^T(t) \left(\begin{bmatrix} 0_r & \bar{A}_{Re21}^T & 0_r & \bar{A}_{Im21}^T \\ I_r & \bar{A}_{Re22}^T & 0_r & \bar{A}_{Im22}^T \\ 0_r & -\bar{A}_{Im21}^T & 0_r & \bar{A}_{Re21}^T \\ 0_r & -\bar{A}_{Im22}^T & I_r & \bar{A}_{Re22}^T \end{bmatrix} \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} + \right. \\
&\quad \left. \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \begin{bmatrix} 0_r & I_r & 0_r & 0_r \\ \bar{A}_{Re21} & \bar{A}_{Re22} & -\bar{A}_{Im21} & -\bar{A}_{Im22} \\ 0_r & 0_r & 0_r & I_r \\ \bar{A}_{Im21} & \bar{A}_{Im22} & \bar{A}_{Re21} & \bar{A}_{Re22} \end{bmatrix} \right) \tilde{\Sigma}_i(t) \\
&= \tilde{\Sigma}_i^T(t) \left(\begin{bmatrix} 0_r & \bar{A}_{Re21}^T P & 0_r & \bar{A}_{Im21}^T P \\ P & \bar{A}_{Re22}^T P & 0_r & \bar{A}_{Im22}^T P \\ 0_r & -\bar{A}_{Im21}^T P & 0_r & \bar{A}_{Re21}^T P \\ 0_r & -\bar{A}_{Im22}^T P & P & \bar{A}_{Re22}^T P \end{bmatrix} \begin{bmatrix} 0_r & P & 0_r & 0_r \\ P\bar{A}_{Re21} & P\bar{A}_{Re22} & -P\bar{A}_{Im21} & -P\bar{A}_{Im22} \\ 0_r & 0_r & 0_r & P \\ P\bar{A}_{Im21} & P\bar{A}_{Im22} & P\bar{A}_{Re21} & P\bar{A}_{Re22} \end{bmatrix} \right) \tilde{\Sigma}_i(t) \\
&= \tilde{\Sigma}_i^T(t) \begin{bmatrix} 0_r & \bar{A}_{Re21}^T P + P & 0_r & \bar{A}_{Im21}^T P \\ P + P\bar{A}_{Re21} & \bar{A}_{Re22}^T P + P\bar{A}_{Re22} & -P\bar{A}_{Im21} & \bar{A}_{Im22}^T P - P\bar{A}_{Im22} \\ 0_r & -\bar{A}_{Im21}^T P & 0_r & \bar{A}_{Re21}^T P + P \\ P\bar{A}_{Im21} & -\bar{A}_{Im22}^T P + P\bar{A}_{Im22} & P + P\bar{A}_{Re21} & \bar{A}_{Re22}^T P + P\bar{A}_{Re22} \end{bmatrix} \tilde{\Sigma}_i(t) \\
&= \tilde{\Sigma}_i^T(t) \begin{bmatrix} 0_r & \Psi_{12} & 0_r & \Psi_{14} \\ \Psi_{21} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ 0_r & \Psi_{32} & 0_r & \Psi_{34} \\ \Psi_{41} & \Psi_{42} & \Psi_{43} & \Psi_{44} \end{bmatrix} \tilde{\Sigma}_i(t) \\
&= \tilde{\Sigma}_i^T(t) \Psi_i \tilde{\Sigma}_i(t)
\end{aligned} \tag{4.47}$$

where;

$$\begin{aligned}
\Psi_{12} &= -\sigma_i K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T P - \omega^2 K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T K_d^T B^T \{ (I_r + \sigma_i B K_d)^{-1} \}^T P + P \\
\Psi_{14} &= \sigma_i \omega_i K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T K_d^T B^T \{ (I_r + \sigma_i B K_d)^{-1} \}^T P - \omega K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T P \\
\Psi_{21} &= \Psi_{12}^T = -\sigma_i P [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_i - \omega^2 P (I_r + \sigma_i B K_d)^{-1} [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_i \\
\Psi_{22} &= (A - \sigma_i B K_d)^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T P + \omega^2 K_p^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T K_d^T B^T \{ (I_r + \sigma_i B K_d)^{-1} \}^T P + P [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} (A - \sigma_i B K_d) - \omega^2 P (I_r + \sigma_i B K_d)^{-1} B K_d [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_p
\end{aligned}$$

$$\Psi_{23} = -\sigma_i \omega_i P (I_r + \sigma_i B K_d)^{-1} B K_d [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_i \\ + \omega_i P [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_i$$

$$\Psi_{24} = -\omega_i K_p^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T P - \omega_i (A - \sigma_i B K_d)^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T K_d^T B^T \{ (I_r + \sigma_i B K_d)^{-1} \}^T P \\ + \omega_i P [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_p \\ + \omega_i (I_r + \sigma_i B K_d)^{-1} B K_d [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \\ (A - \sigma_i B K_d)$$

$$\Psi_{32} = \Psi_{23}^T = -\sigma_i \omega_i K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T K_d^T B^T \{ (I_r + \sigma_i B K_d)^{-1} \}^T P \omega_i K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T P \\ B K_d]^{-1} \}^T P$$

$$\Psi_{34} = -\sigma_i K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T P - \omega_i^2 K_i^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T K_d^T B^T \{ (I_r + \sigma_i B K_d)^{-1} \}^T P + P$$

$$\Psi_{41} = \Psi_{14}^T = \sigma_i \omega_i P (I_r + \sigma_i B K_d)^{-1} B K_d [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_i \omega_i [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i P B K_d)^{-1} B K_d]^{-1} B K_i$$

$$\Psi_{42} = \Psi_{24}^T = \omega_i K_p^T B^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T P + \omega_i (A - \sigma_i B K_d)^T \{ [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \}^T K_d^T B^T \{ (I_r + \sigma_i B K_d)^{-1} \}^T P - \omega_i P [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_p \\ - \omega_i (I_r + \sigma_i B K_d)^{-1} B K_d [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} \\ (A - \sigma_i B K_d)$$

$$\Psi_{43} = \Psi_{34}^T = -\sigma_i P [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_i - \omega^2 P (I_r + \sigma_i B K_d)^{-1} B K_d [(I_r + \sigma_i B K_d) + \omega_i^2 B K_d (I_r + \sigma_i B K_d)^{-1} B K_d]^{-1} B K_i + P$$

$$\Psi_{44} = \Psi_{22}$$

On the condition that the inequality given in Lemma 4.1 holds, then $\dot{V}_i(t) < 0$. This implies that, $\lim_{t \rightarrow \infty} \tilde{\Sigma}_i(t) = 0$. Therefore, the closed-loop platooning system is asymptotically stable.

The proof is completed.

For the platooning system with the vehicle dynamics described by Eq. (2.71), the formation geometry defined in Eq. (2.86) and the distributed PID controller given in Eq. (4.3) is considered. Here, the time delay is ignored. The information topology matrix $\mathcal{L}_p$ is not symmetric and may have complex eigenvalues. Then, Lemma 4.1 is used to show the controller gains guarantee the stability of the closed loop system given in Eq. (4.42).

Simulation Example 4.1: The following simulation study is conducted for the platooning system which is controlled with distributed PID structure explained in previous. The information flow topology between vehicles is considered as mixed type which is shown in Figure 4.1.

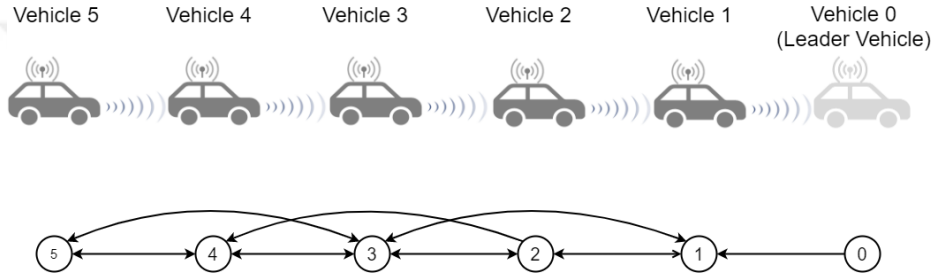


Figure 4.1 : Information flow topologies for Example 5.1.

The $\mathcal{L}_p$ matrix of the system is also given in Eq. (??).

$$\mathcal{L}_p = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix} \quad (4.48)$$

The eigenvalues of $\mathcal{L}_p$ matrix are found as follows;

$$\mathcal{L}_p : \lambda_1 = 0.1747, \quad \lambda_2 = 1.5771, \quad \lambda_3 = 5.0891,$$

$$\lambda_4 = 3.5795 - 0.1461j, \quad \lambda_5 = 3.5795 + 0.1461j$$

The platooning system is considered homogeneous, which means that the power-train time constant is equal for each vehicle. So the system parameters used in simulation studies are given in 4.1 as follows;

Then, if the controller parameters and the positive definite symmetric matrix are chosen as;

$$K_i = [0.1 \quad 0.1 \quad 0.1], \quad K_d = [0.01 \quad 0.01 \quad 0.01], \quad P = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}$$

Table 4.1 : System parameters used in simulation studies

Parameter	Description	Value	Unit
N	Number of following vehicles	5	-
γ	Power-train time constant	0.45	[s]
d	Safe distance	20	[m]
v_0	Velocity of virtual leader	20	[m/s]

Through, solving the inequality given in Eq. (4.43) with the help of the CVX Toolbox for MATLAB environment, then we obtain;

$$K_p = [6.1036 \quad 4.7308 \quad 24629]$$

The positions and relative position errors of the followers are shown in Figure 4.2, and the velocity and velocity errors are given in Figure 4.3. respectively.

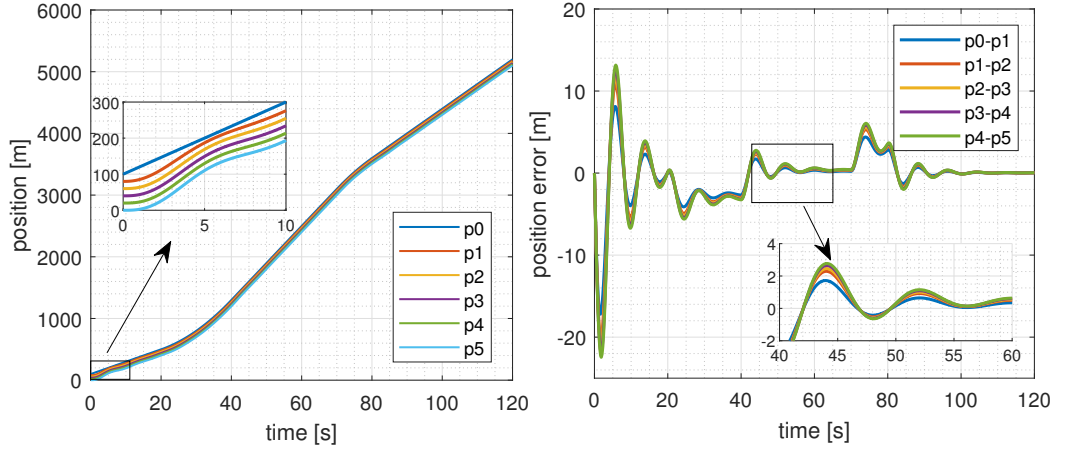


Figure 4.2 : Positions and position errors for Example 4.1 ($\tau = 0s$)

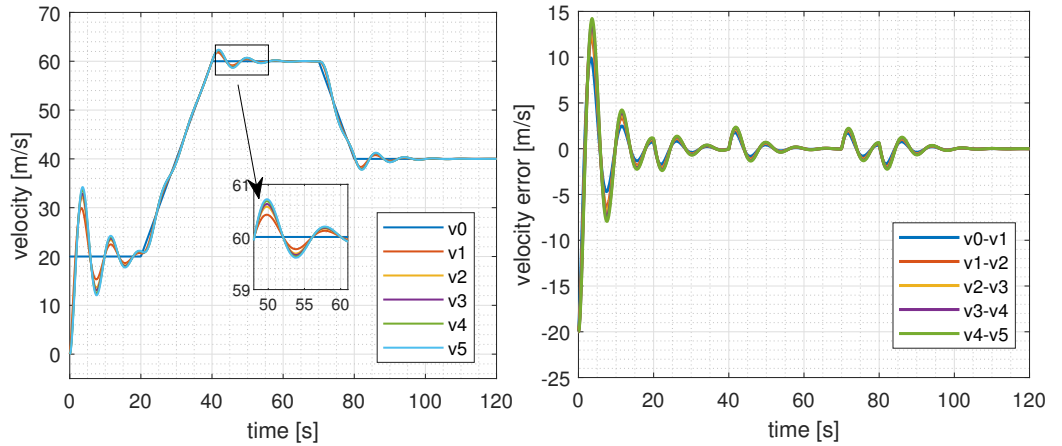


Figure 4.3 : Velocity and velocity errors for Example 4.1 ($\tau = 0s$)

4.3 Stability Analysis for Time Delay Cases

To deal with the problem in the presence of time delay, the controller structure given in Eq. (4.1) is written as in Eq. (4.49) ;

$$\begin{aligned}
 u_i(t - \tau) = & -k_p \left[\sum_{j=1}^N a_{ij} \left(x_i(t - \tau) - x_j(t - \tau) + \bar{d}_{i,j} \right) + p_{ii} \left(x_i(t - \tau) \right. \right. \\
 & \left. \left. - x_0(t - \tau) + \bar{d}_{i,0} \right) \right] - k_i \left[\sum_{j=1}^N a_{ij} \left(\int_0^{t-\tau} (x_i(s) - x_j(s) + \bar{d}_{i,j}) dt \right) \right. \\
 & \left. + p_{ii} \left(\int_0^{t-\tau} (x_i(s) - x_0(s) + \bar{d}_{i,0}) dt \right) \right] - k_d \left[\sum_{j=1}^N a_{ij} \left(\dot{x}_i(t - \tau) \right. \right. \\
 & \left. \left. - \dot{x}_j(t - \tau) + \dot{\bar{d}}_{i,j} \right) + p_{ii} \left(\dot{x}_i(t - \tau) - \dot{x}_0(t - \tau) + \dot{\bar{d}}_{i,0} \right) \right]
 \end{aligned} \tag{4.49}$$

If the position, velocity and acceleration tracking errors between i^{th} and 0^{th} vehicle are defined as in Eq. (3.3), the controller structure given in Eq. (4.49) is written in terms of error terms as in Eq. (4.50).

$$\begin{aligned}
 u_i(t - \tau) = & -k_p \left[\sum_{j=1}^N a_{ij} \left(e_i(t - \tau) - e_j(t - \tau) \right) + p_{ii} \left(e_i(t - \tau) \right) \right] \\
 & - k_i \left[\sum_{j=1}^N a_{ij} \left(\int_0^{t-\tau} (e_i(s) - e_j(s)) ds \right) + p_{ii} \left(\int_0^{t-\tau} (e_i(s)) ds \right) \right] \\
 & - k_d \left[\sum_{j=1}^N a_{ij} \left(\dot{e}_i(t - \tau) - \dot{e}_j(t - \tau) \right) + p_{ii} \left(\dot{e}_i(t - \tau) \right) \right]
 \end{aligned} \tag{4.50}$$

Then, the control signal obtained in Eq. (4.50) is substituted into Eq. (3.4), and compact form of the error dynamic system is obtained as in Eq. (4.51) with the help of vector definition given in Eq. (4.4).

$$\begin{aligned}
 \dot{E}(t) = & (I_N \otimes A)E(t) - (\mathcal{L}_p \otimes BK_p)E(t - \tau) - (\mathcal{L}_p \otimes BK_i) \int_0^{t-\tau} E(s) ds \\
 & - (\mathcal{L}_p \otimes BK_d)\dot{E}(t - \tau)
 \end{aligned} \tag{4.51}$$

To deal with the integrator term the descriptor transformation is applied [53]. Let define;

$$F(t) = \int_0^t E(s) ds, \quad \dot{E}(t) = G(t) \tag{4.52}$$

Then, the descriptor form of closed-loop system equation of error dynamic platooning system is represented as in Eq. (4.53).

$$\begin{aligned}
 \dot{F}(t) = & E(t) \\
 \dot{E}(t) = & (I_N \otimes A)E(t) - (\mathcal{L}_p \otimes BK_p)E(t - \tau) - (\mathcal{L}_p \otimes BK_i)F(t - \tau) \\
 & - (\mathcal{L}_p \otimes BK_d)\dot{E}(t - \tau)
 \end{aligned} \tag{4.53}$$

And, by defining $H(t) = [F(t)^T \ E(t)^T]^T$, the closed-loop form of Eq. (4.53) can be re-written in a more compact form as in Eq. (4.54).

$$\dot{H}(t) + A_0 \dot{H}(t - \tau) = A_1 H(t) + A_2 H(t - \tau) \quad (4.54)$$

where;

$$A_0 = \begin{bmatrix} I_{3r} & 0_{3r} \\ 0_{3r} & -(\mathcal{L}_p \otimes BK_d) \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0_{3r} & I_{3r} \\ 0_{3r} & (I_N \otimes A) \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0_{3r} & 0_{3r} \\ -(\mathcal{L}_p \otimes BK_i) & -(\mathcal{L}_p \otimes BK_p) \end{bmatrix}$$

Now, the stability analysis of time-delay system is simplified dramatically. The CTCR method is employed to calculate the exact stability regions of delays. To found the complete set of purely imaginary roots of time-delay system in Eq. (4.54), the *Kronecker multiplication and elementary transformation* method which is explained in Section 2 is used.

It is known from Section 2, all the imaginary axis eigenvalues of Eq. (4.54) are zeros of $|\Delta(s)|$, and thus also of $|K(s)|$, so that they are generalized eigenvalues of the matrix pair (J_0, E_0) , which means the zeros of $|sE_0 - J_0|$. Where;

$$\Delta(s) = (sI - A_1) \otimes (sI + A_1) - (sA_0 - A_2) \otimes (sA_0 + A_2)$$

$$K(s) = \begin{bmatrix} (sI - A_1) \otimes I & (sA_0 - A_2) \otimes I \\ I \otimes (sA_0 + A_2) & (sI - A_1) \otimes I \end{bmatrix}$$

$$= sE_0 - J_0$$

$$E_0 = \begin{bmatrix} I \otimes I & A_0 \otimes I \\ I \otimes A_0 & I \otimes I \end{bmatrix}$$

$$J_0 = \begin{bmatrix} -I \otimes A_2 & A_2 \otimes I \\ I \otimes A_0 & -I \otimes A_1 \end{bmatrix}$$

So, the algorithm steps are listed below;

Step 1: Find the coefficient matrices of delay differential equation given in Eq. (4.54), which are A_0 , A_1 and A_2 .

Step 2: First, check the eigenvalues of A_0 to show it is an asymptotically stable discrete matrix.

Step 3: Check the system with zero delay is asymptotically stable. Note that with zero delay, the system given in Eq. (5.54) is converted to $(I + A_0)\dot{H}(t) = (A_1 + A_2)H(t)$, and since $(I + A_0)$ is invertible, it is written as in Eq. (4.55).

$$\dot{H}(t) = (I + A_0)^{-1}(A_1 + A_2)H(t) \quad (4.55)$$

Step 4: To determine the smallest time delay for which the system is not asymptotically stable, find all the possible imaginary axis eigenvalues of the system. Thus, form the following matrices given in Eq. (4.56).

$$E_0 = \begin{bmatrix} I \otimes I & A_0 \otimes I \\ I \otimes A_0 & I \otimes I \end{bmatrix}, \quad J_0 = \begin{bmatrix} -I \otimes A_2 & A_2 \otimes I \\ I \otimes A_0 & -I \otimes A_1 \end{bmatrix} \quad (4.56)$$

And, obtain the zeros of $|sE_0 - J_0|$ by determining the generalized eigenvalues of the matrix pair (J_0, E_0) .

Step 5: Extract the roots of $|sE_0 - J_0|$ with purely imaginary values which have all possible eigenvalues of the system given in Eq. (4.54). Let say these candidate sets as Ω .

Step 6: It is known from [111] that, for real ω , the matrix $T = j\omega I - A_1 + e^{j\omega\tau}(j\omega A_0 - A_2)$ is singular if and only if $z = e^{j\omega\tau}$ is a unit magnitude generalized eigenvalue of the matrix pair $(A_1 - j\omega I, j\omega A_0 - A_2)$. Then, check all members of Ω and found the ω values, the corresponding z values which lies on the unit circle.

Step 7: Finally, find the τ value with the following transformation given in Eq. (4.57).

$$\tau_{jk} = \frac{\arg(z_j) \pm 2k\pi}{\omega}, \quad j = 1, 2, \quad k = 0, 1, 2, \dots \quad (4.57)$$

Simulation Example 4.2: The following simulation study is conducted for the platooning system which is controlled with distributed PID structure with time delay explained in Section 4.3. The information flow topology between vehicles is considered as mixed type which is shown in Figure 4.4.

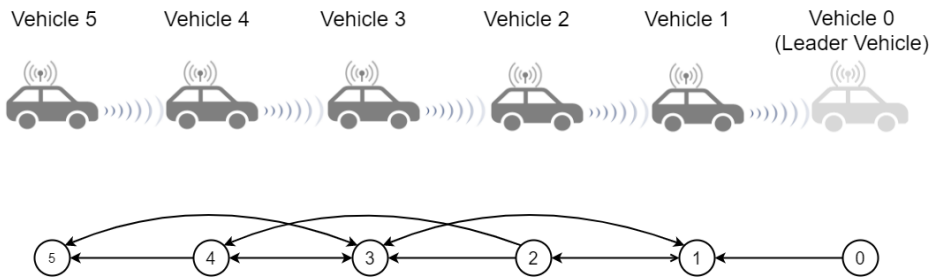


Figure 4.4 : Information flow topologies for Example 4.2.

The $\mathcal{L}_p$ matrix of the system is also given in Eq. (4.58).

$$\mathcal{L}_p = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \quad (4.58)$$

The eigenvalues of $\mathcal{L}_p$ matrix are found as follows;

$$\mathcal{L}_p : \lambda_1 = 0.4174, \quad \lambda_2 = 1, \quad \lambda_3 = 2.4213,$$

$$\lambda_4 = 4.0807 - 0.4127j, \quad \lambda_5 = 4.0807 + 0.4127jj$$

The platooning system is considered homogeneous, which means that the power-train time constant is equal for each vehicle. The system parameters given in Table 4.1 is used. The stability boundary map corresponds to k_p and τ is found as in Figure 5.5 for $k_d = 0.02$, $k_i = 0.02$ and the system parameters given in Table 4.1.

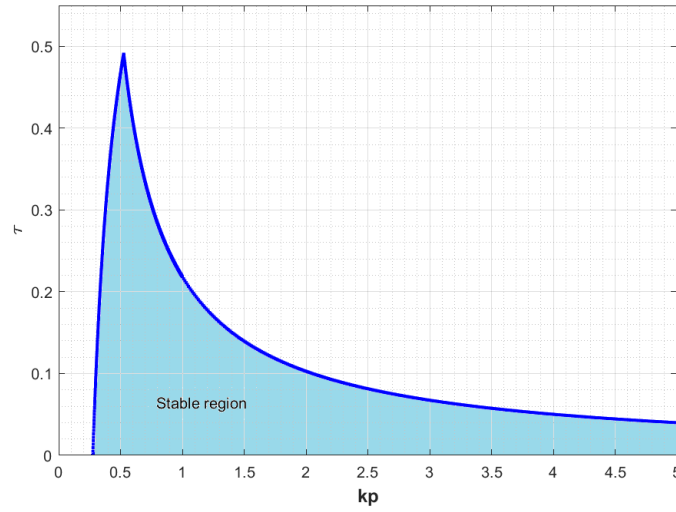


Figure 4.5 : Stability boundary for $k_d = 0.02$, $k_i = 0.02$.

The shaded region in ?? defines the sector of $k_p - \tau$ plane in which all the roots of Figure 4.5 are in the left half of the complex plane. At any point on the boundary of the stability zone, the characteristic equation has purely imaginary roots which makes the system marginally stable. To confirm the exactitude of the delay bound obtained some test points are chosen. For the controller gains as $k_p = 4$, $k_i = 0.02$ and $k_d = 0.02$, the simulation is conducted with delay values smaller than the bound ($\tau = 0.01$ seconds), exactly at the bound ($\tau = 0.05$ seconds) and larger than the bounds ($\tau = 0.055$ seconds). Figure 5.6 and 5.7 are shows the simulation result for positions, position tracking errors, and the velocity, velocity tracking errors for the delay value on the stable region. Figure 5.8 and 5.9 show the behaviors of the closed-loop system for the delays which are on the bound and larger than the bound.

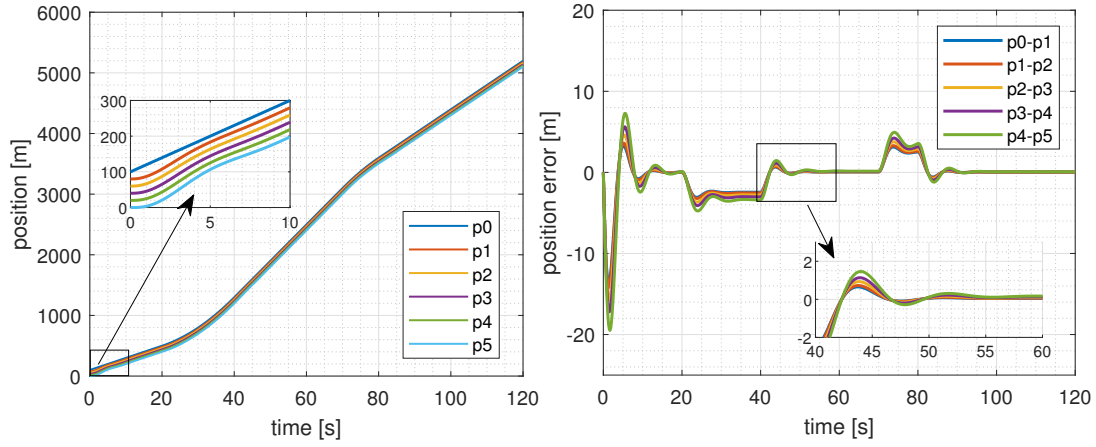


Figure 4.6 : Positions and position errors ($\tau = 0.01s$).

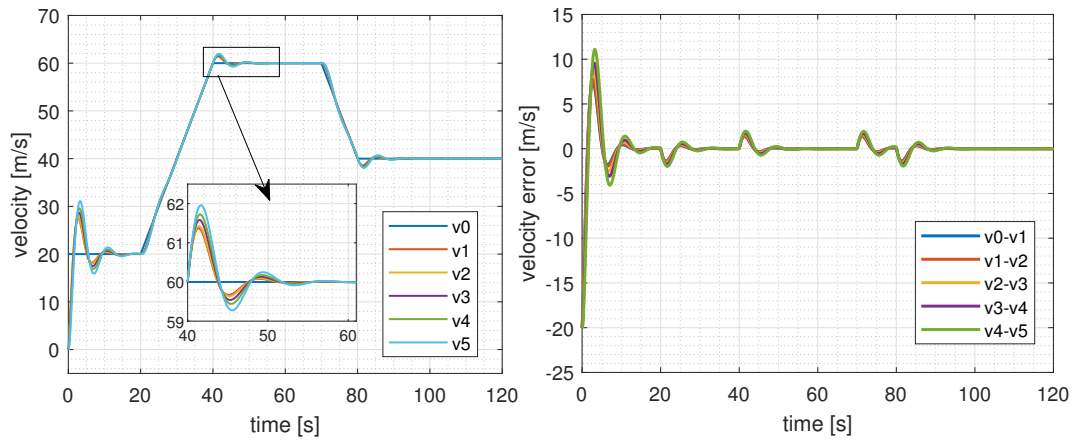


Figure 4.7 : Velocity and velocity errors ($\tau = 0.01s$).

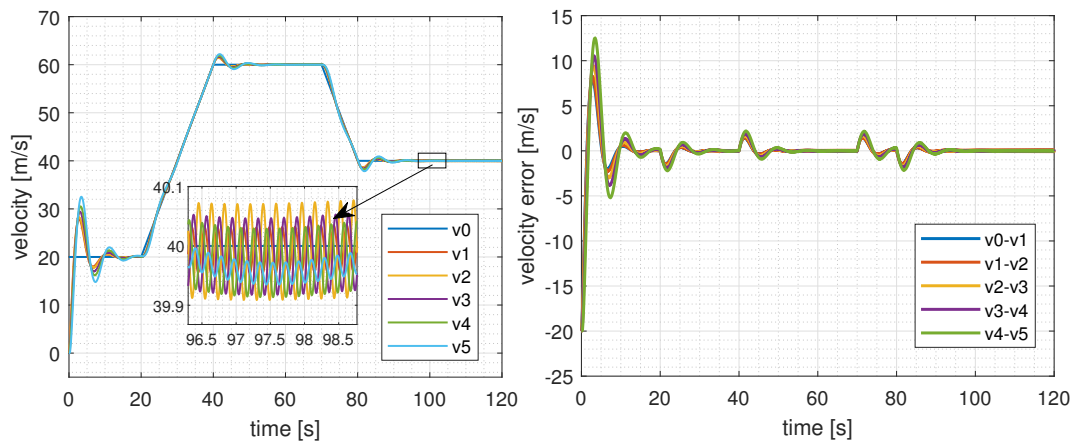


Figure 4.8 : Positions and position errors ($\tau = 0.05s$).

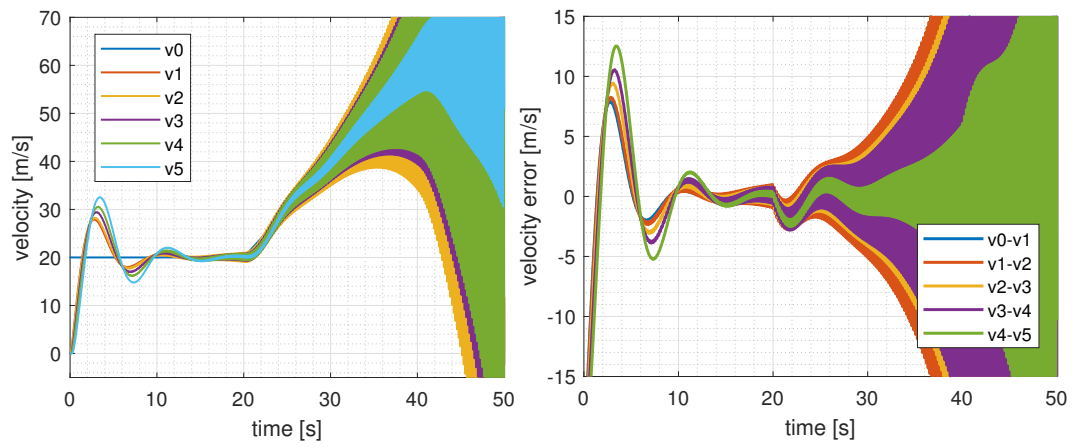


Figure 4.9 : Velocity and velocity errors($\tau = 0.055s$).



5. A REAL TIME CASE STUDY

5.1 Purpose

In this section, the stability and control of platooning system is analyzed for homogeneous vehicles in the presence of multiple time delays with its experimental validation. The addressed problem is developed based on a multi-agent system approach with vehicle dynamics, information flow topology, desired inter-vehicle distances (formation geometry), and distributed control architecture as explained in the previous sections. The closed loop formulation of the system is built based on the vehicle states as opposed to the errors between the states of the vehicles. The stability posture with respect to delays is obtained with the Cluster Treatment of Characteristic Root (CTCR) method based on the Bezout resultant matrix. It is verified that the vehicles are converged to their desired velocities and inter-vehicle distances when they are exposed time delays lying in delay space determined as stable regions. The results of several numerical simulations and real-time experiments exhibit the efficiency of the presented method in a platooning system that has one leader and four following vehicles. In addition, real-time experiments on industrial computers present the usability of the proposed method in real applications.

5.2 Problem Formulation for Real Time Case Study

This section considers control of a platoon system with $N + 1$ connected vehicles, including one leader and N followers that share their information via V2V communication under multiple time delays. The objective of platoon control is to drive all the agents of the group to move at the same speed while maintaining a predefined formation geometry. The longitudinal dynamics of each vehicle ($i = 1, 2, \dots, N$) in the

platoon system is modeled with third order differential equations in Eq. (5.1).

$$\begin{aligned}\dot{p}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= a_i(t) \\ \lambda_i \dot{a}_i(t) + a_i(t) &= u_i(t)\end{aligned}\tag{5.1}$$

where p_i, v_i and a_i represent the position, velocity and acceleration of i^{th} vehicle, respectively while u_i is the control signal. Here, the power train and engine dynamics of each vehicle is transformed into a first order system with a time constant λ_i .

The platoon is assumed to be homogeneous and inertial time constant of each vehicle is the same, i.e. $\lambda_i = \lambda$. It is also assumed that the driving and braking forces are controllable actions for the vehicles. Each vehicle in the platoon utilizes its onboard sensors to measure its acceleration, velocity and position and a deterministic communication device to send/receive information to/from other vehicles.

For the virtual leader of the platoon the dynamic equations given in Eq. (5.2) are valid.

$$\begin{aligned}\dot{p}_0(t) &= v_0(t) \\ \dot{v}_0(t) &= a_0(t) \\ \dot{a}_0(t) &= f(t)\end{aligned}\tag{5.2}$$

Here $f(t)$ represents the desired acceleration change of the platoon system. Note that, the virtual leader is assumed to be uncontrolled, traveling at desired acceleration and its inertial time constant is not defined.

A vehicle platoon system consists of a virtual leader and N followers, indexed by 0 (virtual leader) and $1, 2, \dots, N$, respectively. The information flow topology between followers can be described by graph theory with a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = (v_1, v_2, \dots, v_n)$ is a set of nodes representing all the following vehicles and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges representing the connections between vehicle pairs. An edge denoted by the pair (v_j, v_i) represents a communication link from j^{th} vehicle to i^{th} vehicle. The Adjacency, Degree, Laplacian, Pinning, and Information Topology matrices which are defined in Section 2, are utilized to model the information topology of a vehicle platooning system.

The most used information flow topologies in vehicle platoon systems are predecessor following (PF), bidirectional predecessor following (BPF), two predecessor following (TPF), two bidirectional predecessor following (TBPF), two predecessor single

following (TPSF) and single predecessor two following (SPTF), which are explained in detail in Section 2. Also, see [2] for explanation. Note that, in all of these topologies every follower receives information from its one predecessor.

Spacing policy is the specification of the desired inter-vehicle distance to provide a safety distance between neighboring vehicles traveling in the platoon. Constant distance (CD) policy, where the desired distance is constant and independent of vehicle's velocity, and constant time headway (CTH) policy, where the desired inter-vehicle spacing is proportional to the vehicle velocity, are the best known ones (See Section 2).

In terms of providing safe driving conditions, the CTH policy is more reasonable than constant spacing policy. CTH imitates the normal driving intuition that vehicles should slow down as the inter-vehicle distance decreases [112]. In this case study, the CTH formulation given in Eq. (5.3) is used for the desired distance between i^{th} vehicle and $i - 1^{th}$ vehicle [48].

$$d_{i,i-1} = (h_i v_i + d_i) \quad (5.3)$$

where h_k is the time headway which represent the proportionality constant between desired following distance and speed of the vehicle, and d_k is the standstill gap.

In general, for each vehicle ($i = 1, 2, \dots, N$) in the platoon system the desired distance between i^{th} vehicle and j^{th} vehicle in case of $i > j$ is given as Eq. (5.4);

$$d_{i,j} = \sum_{k=j+1}^i (h_k v_k + d_k) \quad (5.4)$$

and, for $i < j$ is given in Eq. (5.5);

$$d_{i,j} = \sum_{k=i+1}^j (h_k v_k + d_k) \quad (5.5)$$

Then, the desired inter-vehicle distance between i^{th} vehicle and the leader vehicle becomes as in Eq. (5.6)

$$d_{i,0} = \sum_{k=1}^i (h_k v_k + d_k) \quad (5.6)$$

The vehicle platooning system with the CTH spacing policy as in Eq. (5.6) may displayed in Figure 5.1.

The objective of platoon control is to ensure that vehicles in the platoon follow the speed of the leader and to maintain the formation governed by an inter-vehicle spacing

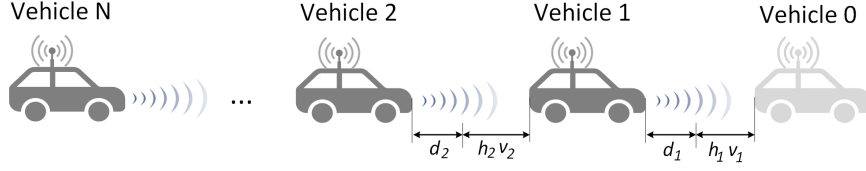


Figure 5.1 : Vehicle platoon system with a virtual leader.

policy. Then, the control objective is formulated as in Eq. (5.7). (It is also given in Eq. (2.93))

$$\begin{aligned}\lim_{t \rightarrow \infty} \|p_i(t) - p_0(t) + d_{i,0}\| &= 0 \\ \lim_{t \rightarrow \infty} \|v_i(t) - v_0(t)\| &= 0 \\ \lim_{t \rightarrow \infty} \|a_i(t) - a_0(t)\| &= 0\end{aligned}\tag{5.7}$$

This objective, where the leader is a global reference, requires an information flow topology where all vehicles receive information from the leader. However, the leader may not be connected to all the followers in practice [48]. Therefore, this control objective should be achieved using local information in a distributed way as in Eq. (5.8).

$$\begin{aligned}\text{For } i > j \quad u_i(t) &= - \sum_{j=1}^n a_{ij} \left[k_{p_i} \left(p_i(t) - p_j(t) + \sum_{k=j+1}^i (h_k v_k(t) + d_k) \right) \right. \\ &\quad \left. + k_{v_i} \left(v_i(t) - v_j(t) \right) + k_{a_i} \left(a_i(t) - a_j(t) \right) \right] \\ &\quad - p_{ii} \left[k_{p_i} \left(p_i(t) - p_0(t) + \sum_{k=1}^i (h_k v_k(t) + d_k) \right) \right. \\ &\quad \left. + k_{v_i} \left(v_i(t) - v_0(t) \right) + k_{a_i} \left(a_i(t) - a_0(t) \right) \right] \\ \text{For } i < j \quad u_i(t) &= - \sum_{j=1}^n a_{ij} \left[k_{p_i} \left(p_i(t) - p_j(t) - \sum_{k=i+1}^j (h_k v_k(t) + d_k) \right) \right. \\ &\quad \left. + k_{v_i} \left(v_i(t) - v_j(t) \right) + k_{a_i} \left(a_i(t) - a_j(t) \right) \right] \\ &\quad - p_{ii} \left[k_{p_i} \left(p_i(t) - p_0(t) + \sum_{k=1}^i (h_k v_k(t) + d_k) \right) \right. \\ &\quad \left. + k_{v_i} \left(v_i(t) - v_0(t) \right) + k_{a_i} \left(a_i(t) - a_0(t) \right) \right]\end{aligned}\tag{5.8}$$

where k_{p_i} , k_{v_i} and k_{a_i} are the position, velocity and acceleration feedback gains, a_{ij} and p_{ii} are the corresponding elements of the adjacency matrix and the pinning matrix, respectively. Here, each vehicle uses spacing errors, velocity errors, and acceleration errors with respect to its communication neighbors to produce the control input.

Control without time delays is an ideal case, however, V2V and V2I wireless communication networks within a platoon experience communication delays and

packet losses caused by many impairments such as channel fading, shadowing, etc. Also, in practice the vehicles suffer from input delays such as measuring delay of sensors, breaking delay and fueling delay, etc. Therefore, these inevitable time delays must be taken into account in the design of effective controllers.

Suppose that each vehicle has an input delay τ_{in} which affects all the state feedback lines, and also each vehicle receives information from its neighbors with a communication delay τ_{com} . Taking these delays into account, a more accurate control protocol is designed as in Eq. (5.9).

$$\begin{aligned}
\text{For } i > j \quad u_i(t) = & - \sum_{j=1}^n a_{ij} \left[k_{p_i} \left(p_i(t - \tau_1) - p_j(t - \tau_2) + h_i v_i(t - \tau_1) + d_i \right. \right. \\
& + \left. \sum_{k=j+1}^{i-1} (h_k v_k(t - \tau_2) + d_k) \right) + k_{v_i} \left(v_i(t - \tau_1) - v_j(t - \tau_2) \right) \\
& + \left. k_{a_i} \left(a_i(t - \tau_1) - a_j(t - \tau_2) \right) \right] - p_{ii} \left[k_{p_i} \left(p_i(t - \tau_1) - p_0(t - \tau_2) \right. \right. \\
& + \left. h_i v_i(t - \tau_1) + d_i + \sum_{k=1}^{i-1} (h_k v_k(t - \tau_2) + d_k) \right) + k_{v_i} \left(v_i(t - \tau_1) \right. \\
& - \left. v_0(t - \tau_2) \right) + \left. k_{a_i} \left(a_i(t - \tau_1) - a_0(t - \tau_2) \right) \right] \\
\text{For } i < j \quad u_i(t) = & - \sum_{j=1}^n a_{ij} \left[k_{p_i} \left(p_i(t - \tau_1) - p_j(t - \tau_2) - \sum_{k=i+1}^j (h_k v_k(t - \tau_2) + d_k) \right) \right. \\
& + \left. k_{v_i} \left(v_i(t - \tau_1) - v_j(t - \tau_2) \right) + k_{a_i} \left(a_i(t - \tau_1) - a_j(t - \tau_2) \right) \right] \\
& - p_{ii} \left[k_{p_i} \left(p_i(t - \tau_1) - p_0(t - \tau_2) + h_i v_i(t - \tau_1) + d_i \right. \right. \\
& + \left. \sum_{k=1}^{i-1} (h_k v_k(t - \tau_2) + d_k) \right) + k_{v_i} \left(v_i(t - \tau_1) - v_0(t - \tau_2) \right) \\
& + \left. k_{a_i} \left(a_i(t - \tau_1) - a_0(t - \tau_2) \right) \right]
\end{aligned} \tag{5.9}$$

where $\tau_1 = \tau_{in}$ and $\tau_2 = \tau_{in} + \tau_{com}$.

Note that in the control signal $u_i(t)$ in Eq. (5.9), the states of i^{th} vehicle appear with delay time τ_1 since they are exposed to only input delays. On the other hand, the states of other vehicles appear with delay time τ_2 since they are exposed to both input and communication delays.

5.2.1 Stability analysis with CTCR method via Bezout resultant matrix

Consider a platoon of $N + 1$ vehicles, which are described by the vehicle dynamics of Eq. (5.1) and Eq. (5.2), with the distributed control structure in Eq. (5.9). The

platoon is assumed to be homogeneous, i.e. $\lambda_i = \lambda, \forall i \in N$ with time constant matrix $\Lambda = \text{diag}[\frac{1}{\lambda}] \in \mathbb{R}^{N \times N}$. The CTH policy is employed with identical time headway and standstill gap, i.e. $h_i = h \forall i \in N$ and $d_i = d \forall i \in N$. It is also assumed that controller gains are identical, i.e. $k_{p_i} = k_p, k_{v_i} = k_v$ and $k_{a_i} = k_a \forall i \in N$.

The states of the platoon can be given in Eq. (5.10).

$$x = [p_0 \ v_0 \ a_0 \ p_1 \ v_1 \ a_1 \ \dots \ p_N \ v_N \ a_N]^T \quad (5.10)$$

with the corresponding as in Eq. (5.11);

$$\dot{x}(t) = Ax(t) + B_1x(t - \tau_1) + B_2x(t - \tau_2) + D_1 + D_2 \quad (5.11)$$

Here, it is crucial to note that system matrices A , B_1 and B_2 are $3(N+1) \times 3(N+1)$ matrices since the states of leader are included in the platoon dynamics. Therefore, $\mathcal{A}$, $\mathcal{D}_p$, $\mathcal{P}$ and Λ matrices should be used in extended forms as follows;

$$\begin{aligned} \mathcal{A}_e &= \begin{bmatrix} 0 & 0 \dots 0 \\ 0 \\ \vdots & \mathcal{A} \\ 0 \end{bmatrix}, & \mathcal{D}_{pe} &= \begin{bmatrix} 0 & 0 \dots 0 \\ 0 \\ \vdots & \mathcal{D}_p \\ 0 \end{bmatrix} \\ \mathcal{P}_e &= \begin{bmatrix} 0 & 0 \dots 0 \\ 0 \\ \vdots & \mathcal{P} \\ 0 \end{bmatrix}, & \Lambda_e &= \begin{bmatrix} 0 & 0 \dots 0 \\ 0 \\ \vdots & \Lambda \\ 0 \end{bmatrix} \end{aligned}$$

Also, two matrices $\mathcal{H}_p$ and $\mathcal{H}_v$, associated with the spacing policy CTH are defined as $\mathcal{H}_p = [h_{p_{ij}}] \in \mathbb{R}^{N \times N}$ and $\mathcal{H}_v = [h_{v_{ij}}] \in \mathbb{R}^{N \times N}$ where;

$$\begin{aligned} h_{p_{ij}} &= \begin{cases} 0 & i \neq j \\ \sum_{k=1}^i (a_{ik}) + p_{ii} & i = j \end{cases} \\ h_{v_{ij}} &= \begin{cases} \sum_{k=1}^{j-1} a_{ik} - \sum_{k=1}^N a_{ik} & i < j \\ 0 & i = j \\ p_{ii} + \sum_{k=1}^N a_{ik} - \sum_{k=j}^N a_{ik} & i > j \end{cases} \end{aligned}$$

The extended forms of $\mathcal{H}_p$ and $\mathcal{H}_v$ become as follows.

$$\mathcal{H}_{pe} = \begin{bmatrix} 0 & 0 \dots 0 \\ 0 \\ \vdots & \mathcal{H}_p \\ 0 \end{bmatrix}, \quad \mathcal{H}_{ve} = \begin{bmatrix} 0 & 0 \dots 0 \\ 0 \\ \vdots & \mathcal{H}_v \\ 0 \end{bmatrix}$$

Then, the system matrices in Eq. (5.11) may be rewritten into a compact form as;

$$A = \begin{bmatrix} 0 & I_{N+1} & 0 \\ 0 & 0 & I_{N+1} \\ 0 & 0 & -\Lambda_e \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Lambda_e \mathcal{D}_{pe} K_p & -\Lambda_e (\mathcal{D}_{pe} K_v + \mathcal{H}_{pe} K_p h) & -\Lambda_e \mathcal{D}_{pe} K_a \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \Lambda_e (\mathcal{P}_e E_m + \mathcal{A}_e) K_p & \Lambda_e (\mathcal{P}_e E_m + \mathcal{A}_e) K_v - \Lambda_e \mathcal{H}_{ve} K_p h & \Lambda_e (\mathcal{P}_e E_m + \mathcal{A}_e) K_a \end{bmatrix}$$

$$D_1 = [0 \quad 0 \quad -d \Lambda_e (\mathcal{H}_{ve} + \mathcal{H}_{pe}) K_p E_m D]^T$$

$$D_2 = [0 \quad 0 \quad [f \ 0 \ 0 \ 0 \ 0]]^T$$

where

$$K_\alpha = \text{diag}[k_\alpha, k_\alpha, \dots, k_\alpha] \in \mathbb{R}^{N+1 \times N+1}, \alpha \in \{p, v, a\},$$

$$I_{N+1} = \text{diag}[1, 1, \dots, 1] \in \mathbb{R}^{N+1 \times N+1},$$

$$E_m = \begin{bmatrix} 0 & 0 \dots 0 \\ 1 & \\ \vdots & 0 \\ 1 & \end{bmatrix}_{(N+1) \times (N+1)}, \quad E_{mD} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{(N+1) \times (N+1)}$$

Here, block matrices 0 in the system matrices A , B_1 and B_2 are $(N+1) \times (N+1)$ matrices, while block matrices 0 in the system matrices D_1 and D_2 are $1 \times (N+1)$ matrices.

Note that, the dynamics of the platoon system in Eq. (5.11) has to be handled according to spacing policy since B_1 , B_2 and D_1 matrices depend on the spacing policy.

The cooperative control problem of platoon system is transformed into a stability problem of linear time-invariant system with multiple time delays by using the expression in Eq. (5.11). A platoon with linear time-invariant dynamics is internally stable if and only if all the eigenvalues of the closed-loop system matrix have negative real parts. However, in the presence of delay, the closed-loop characteristic equation has infinitely-many roots. Therefore, the stability problem translates into some

conditions on τ_1 and τ_2 to guarantee that all of these infinitely many characteristic roots lie on the open left half plane. Since tracking all of these roots is not possible, just to investigate the imaginary axis crossing roots is handy to assess the stability.

In this thesis, the CTCR method is utilized to study the stability analysis of the given system. The method is practical and numerically efficient for analyzing the systems against single or multiple time delays. However, if the number of following vehicles in the platoon increases, this technique needs more computational sources since the sizes of the closed-loop matrices increase dramatically. (See Section 2 and 3 for detail explanation about CTCR). In the following, the algorithm steps given in Section 3 are re-explained briefly to understand the integrity of studies.

All the other constants including controller gains k_p , k_v and k_i must be predefined to study the root behavior with CTCR according to time delays.

The general form of characteristic equation of multiple time-delay system given in Eq. (5.12).

$$\Delta(s; \tau_1, \tau_2) = \det \left[sI - A - B_1 e^{-s\tau_1} - B_2 e^{-s\tau_2} \right] = 0 \quad (5.12)$$

Equation (5.12) is called a quasi-polynomial which is a special class of polynomials due to the exponential terms. The kernel and offspring hyperspaces include all the purely imaginary characteristic roots in the delay space, so creating them is a prominent step of CTCR method. Therefore, the solutions of the characteristic equation Eq. (5.12) with $s = \omega j$ should be focused on.

The characteristic equation may be also written with the transformation of $z_1 = e^{-s\tau_1}$ and $z_2 = e^{-s\tau_2}$ to eliminate the exponential terms as in Eq. (5.13).

$$\Delta_1(s; z_1, z_2) = \det \left[sI - A - B_1 z_1 - B_2 z_2 \right] = 0 \quad (5.13)$$

The complex conjugates of s , z_1 and z_2 also satisfy Eq. (5.13) since A , B_1 and B_2 are all constant matrices in $R^{n \times n}$. Then, the characteristic equation may also be written as in Eq. (5.14).

$$\Delta_2(s^*; z_1^*, z_2^*) = \det \left[s^* I - A - B_1 z_1^* - B_2 z_2^* \right] = 0 \quad (5.14)$$

where $s^* = -j\omega$, $z_1^* = z_1^{-1}$ and $z_2^* = z_2^{-1}$.

It is basically visible that, if a solution $s = \omega j$ exist for Eq. (5.12), it should also satisfy both Eq. (5.13) and (5.14). Then, the problem of finding kernel and offspring space

of the system is transformed into obtaining the common roots of two polynomials of Δ_1 and Δ_2 . In this study, the Bezout resultant method is utilized to find the common roots of Δ_1 and Δ_2 . In literature, there are also some special resultant methods such as Sylvester, Macaulay, Dixon, and Sparse [29, 113] which may also used to find common roots.

The following steps are followed to apply the CTCR method based on Bezout resultant method to the system in Eq. (5.11) with the characteristic equation given in Eq. (5.12).

- (i) Create the Δ_1 and Δ_2 characteristic polynomials with the the form given in Eq. (5.13) and Eq. (5.14) and rearrange them as Eq. (5.15).

$$\begin{aligned}\Delta_1(s; z_1, z_2) &= \sum_{i=0}^{3 \times (N+1)} f_i(z_1, z_2) s^i \\ \Delta_2(s; z_1, z_2) &= \sum_{i=0}^{3 \times (N+1)} g_i(z_1, z_2) s^i\end{aligned}\tag{5.15}$$

- (ii) Construct the Bezout matrix of order $3 \times (N + 1)$ associated with the polynomials Δ_1 and Δ_2 as in Eq. (5.16).

$$B(\Delta_1, \Delta_2) = (b_{i,j})_{i,j=0,\dots,(3 \times (N+1))-1}\tag{5.16}$$

Let $m_{ij} = \min(i, (3 \times (N + 1)) - 1 - j)$ for each $i, j = 0, \dots, (3 \times (N + 1)) - 1$, then the entries $b_{i,j}$ results from;

$$b_{i,j} = \sum_{k=0}^{m_{ij}} (f_{j+k+1} g_{i-k} - f_{i-k} g_{j+k+1})\tag{5.17}$$

- (iii) Obtain the determinant of Bezout matrix as Eq. (5.18);

$$D(z_1, z_2) = \det(B(\Delta_1, \Delta_2)) = 0\tag{5.18}$$

It is explained above that if Eq. (5.13) and (5.14) have a common root, then Eq. (??) is singular. In other expression, the singularity of Eq. (??) is a necessary condition for Eq. (??) and (??) to have a common solution.

- (iv) Select $z_{1(or\ 2)} = e^{j\theta}$, where $\theta \in [0, 2\pi]$. Then, solve Eq. (5.18) in a loop at small intervals by starting $\theta = 0$. And, filter the roots of Eq. (5.18) with unity magnitude which correspond to $z_{2(or\ 1)}$ values.

- (v) Replaces the values of z_1 and z_2 into Eq. (5.13) or (5.14) and solve it for $s = j\omega$ which corresponds to purely imaginary roots of Eq. (5.12).
- (vi) Increase θ by a desired resolution of $\delta\theta$, and repeat step 1 – 5.
- (vii) As θ reaches 2π a complete set of imaginary spectra and the corresponding kernel points τ_1, τ_2 are obtained with using the formula as given in Eq. (5.19).

$$\tau_{jk} = \frac{\arg(z_j) \mp 2k\pi}{\omega}, \quad j = 1, 2. \quad k = 0, 1, 2, \dots \quad (5.19)$$

The offspring points are trivially generated using Eq. (5.19).

5.3 Simulation Studies on MATLAB/Simulink Environment

In this section, a platoon system with five connected vehicle including a virtual leader is simulated via MATLAB/Simulink. The vehicles are communicated with each other according to the information flow topology given in Figure 5.2 with the following Adjacency and Pinning matrices.

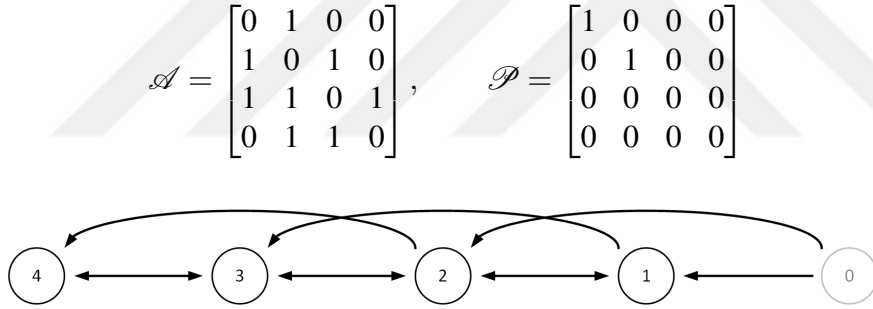


Figure 5.2 : Information flow topology of the platoon system.

The parameters of platoon system with the controller coefficients are given in Table 5.1. Note that, for these parameters, the closed-loop system for the delay-free case ($\tau_1 = \tau_2 = 0$) is stable.

The stability map on the τ_1 and τ_2 plane is obtained for the platoon system using CTCR with Bezout resultant method. First, the kernel curves (red lines) are derived and offsprings (blue lines) are generated as in Figure 5.3. Then, the stable region is determined using root tendency of each curve. As a result, it is determined that the platoon system is stable for all the (τ_1, τ_2) points inside the shaded region in Figure 5.3.

The platoon system is simulated for the velocity profile of the lead vehicle given in Eq. (5.20) to verify the delay bound shown in Figure 5.3. It is also the velocity profile

Table 5.1 : System and controller parameters used in simulation

System and Controller Parameters			
Parameter	Description	Value	Unit
N	Number of following vehicles	4	
λ	Inertial time constant	0.5	[s]
d	Safe distance	10	[m]
h	Time headway	0.5	[s]
τ_1, τ_2	Time delay		[s]
k_p, k_v, k_a	Controller parameters	2, 2, 0.8	

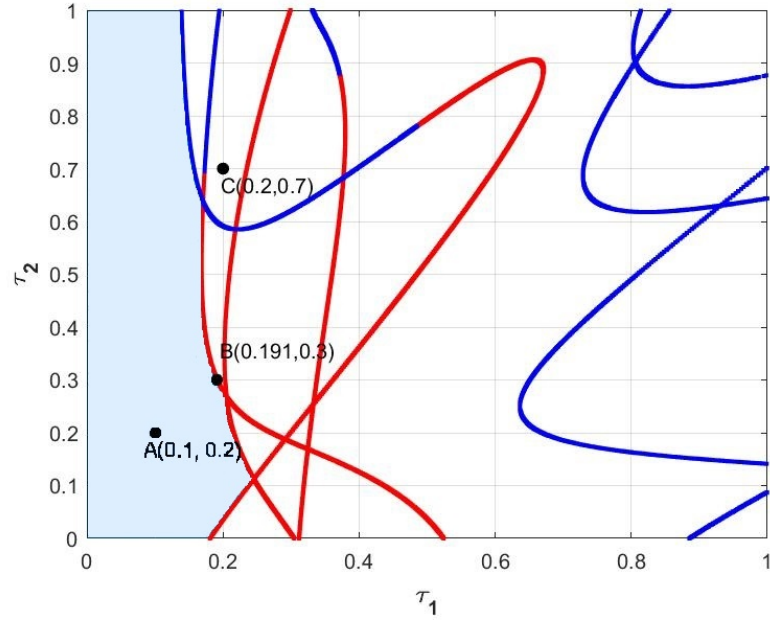


Figure 5.3 : Stability boundary for the platoon system exposed to multiple time-delay.

that followers must follow while keeping the desired spacing. Here, the corresponding acceleration change may be considered as the system input $f(t)$ given in Eq. (5.1). At $t = 0$, the position of lead vehicle is 40m and there is a safety distance of 10m between two consecutive vehicles, i.e. the initial position of 4th follower vehicle is 0 m. The lengths of vehicles are neglected. Initial velocities of all vehicles are set to 0 m/s. Therefore, the initial position and velocity errors are zero.

$$v_0(t) = \begin{cases} 2t \text{ m/s} & t \leq 10 \\ 20 \text{ m/s} & 10 < t \leq 30 \\ 1.5t - 25 \text{ m/s} & 30 < t \leq 50 \\ 50 \text{ m/s} & 50 < t \leq 70 \\ -t + 120 \text{ m/s} & 70 < t \leq 90 \\ 30 \text{ m/s} & t > 90 \end{cases} \quad (5.20)$$

The following figures depict the behavior of the platoon system with different multiple time delays. First, communication and input delays are both set to 0.1 seconds ($\tau_1 = 0.1, \tau_2 = 0.2$), which is shown as point A in Figure 5.3. Second, the simulation is run for the point B ($\tau_1 = 0.191, \tau_2 = 0.3$) which addresses a point exactly at the stability bound. Then, the delays are increased and the pair ($\tau_1 = 0.2, \tau_2 = 0.7$) at point C is chosen which is out of the stable region.

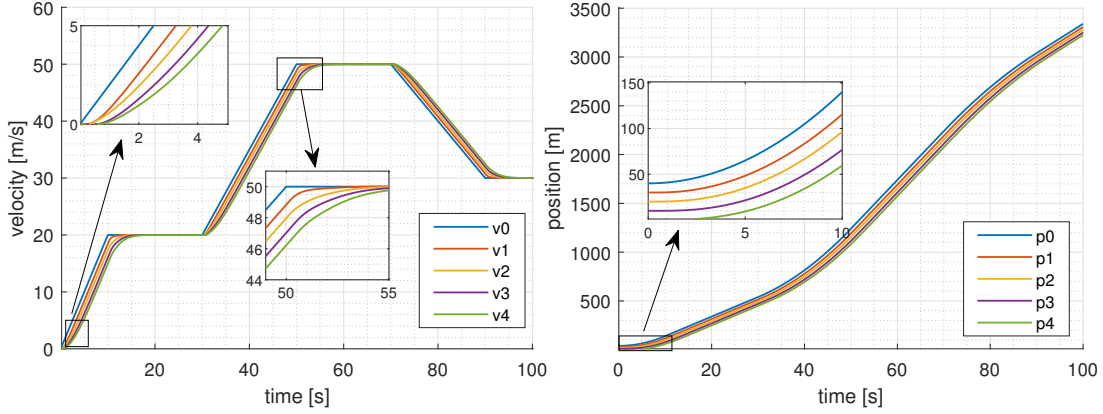


Figure 5.4 : Velocity and position curves for $\tau_1 = 0.1$ and $\tau_2 = 0.2$.

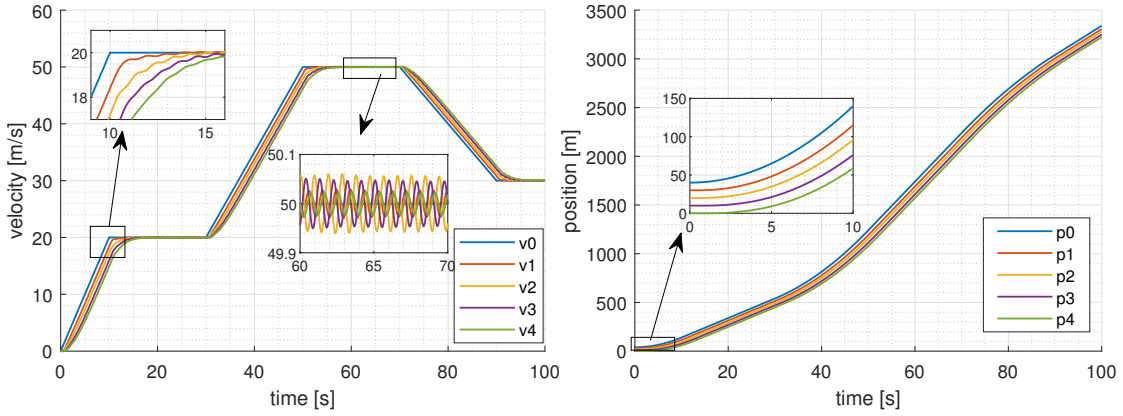


Figure 5.5 : Velocity and position curves for $\tau_1 = 0.191$ and $\tau_2 = 0.3$.

All vehicles achieve the same velocity profile with the leader as visible in Figure 5.4 and 5.5. The platoon system is stable for the time delay pair in the stable region (Figure 5.4). The stable platoon system evolves towards a marginally stable state when the time delays are increased exactly to a point at the stability boundray (Figure 5.5). It is observed that the distributed controller in Eq. (5.9) can stabilize the platoon with appropriate time delays. The platoon system responds by oscillating with increasing amplitude and the system becomes unstable when the delays go beyond the stability

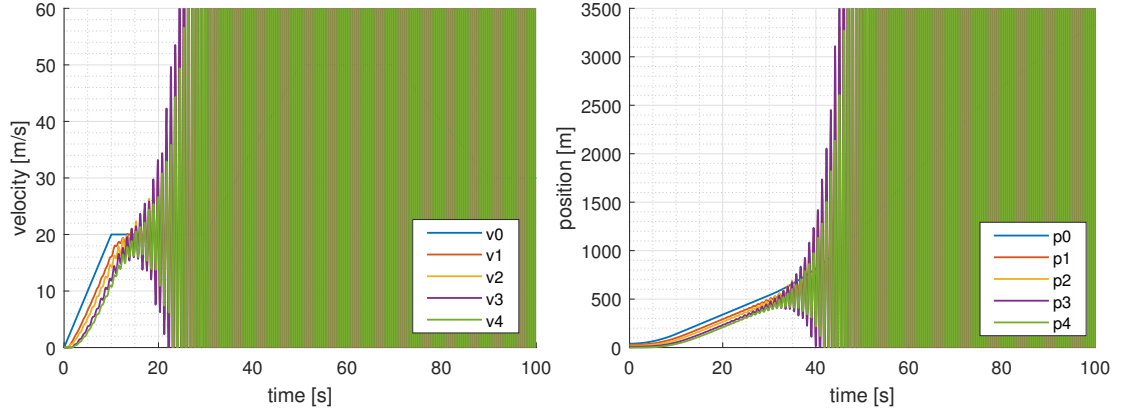


Figure 5.6 : Velocity and position curves for $\tau_1 = 0.2$ and $\tau_2 = 0.7$.

boundary (Figure 5.6). The simulation results verify the stability boundary in Figure 5.3.

Also, the effect of information flow topology may be observed from the first zoomed part in Figure 5.4. According to the information flow topology in Figure 5.2, 2^{nd} and 3^{rd} vehicles receive information directly from the leader while 4^{th} and 5^{th} vehicles receive information directly from the 2^{nd} vehicle. Therefore, vehicles begin to move in pairs and 2^{nd} and 3^{rd} vehicle pair begins to move before other vehicles. In addition, the position difference between consecutive vehicles increases as the velocities increase in accordance with the CTH spacing policy.

5.4 Real Time Simulation Studies on Industrial Computers

In this section, a industrial controller based real time simulation for the previous platoon system is proposed. This real time simulation system provides a platform for verifying accuracy and reliability of proposed method. Here, each vehicle model and its control signal is coded in one industrial controller. The controllers communicate with each other via Ethernet/IP network. EtherNet/IP network enables real time control, I/O messaging, information exchange, and general messaging using standard Ethernet communication products. For this real time simulation Allen Bradley industrial automation products are used. A schematic of the test environment is given in Figure 5.7.

ControlLogix controllers are used to emulate the vehicles, their sensors and control rules. These controllers are based on the Logix5000 operating system which provide a common control engine for all control disciplines. The controllers connect to

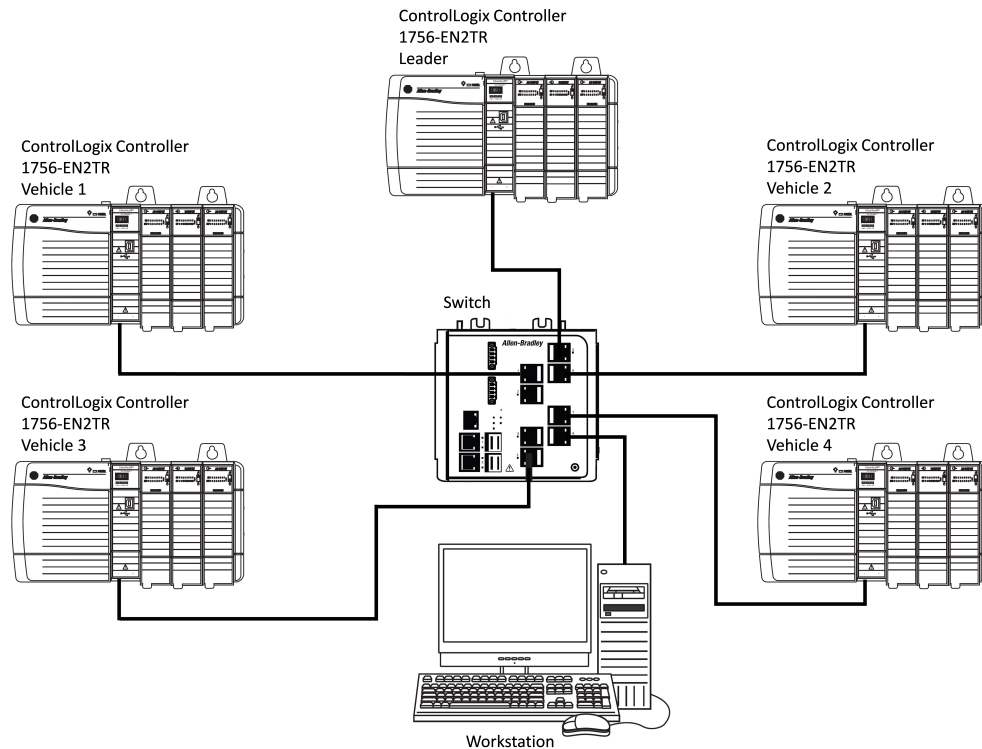


Figure 5.7 : Test environment data connection.

the network through 1756-EN2TR modules. RSLogix 5000/Studio 5000 Design Software is used to configure and program the industrial controllers. It is an integrated development environment that combines elements of design into one standard framework and provides users with access to real-time information. A switch is only used for observing the internal values of controllers and initiating the simulation (dashed blue lines in Figure 5.7). A workstation is employed to configure devices, and upload or download projects to the controllers and then to monitor the whole system.

Controllers communicating with each other according to the flow topology in Figure 5.2 are connected to the same Ethernet/IP network (solid blue lines in Figure 5.7). When controllers are attached to the same network, they allow sharing produced (broadcast) and consumed (receive) tags. By configuring a controller's tag as produced tag, the tag is made available for use by the subscribed controllers. A produced tag sends its data to consumed tags (consumers) and a consumed tag receives the data of a produced tag. Multiple subscribed controllers can simultaneously consume (receive) the same data as in Figure 5.8. This produced and consumed tag framework provides comprehensive data transfer between different controllers on the same network.

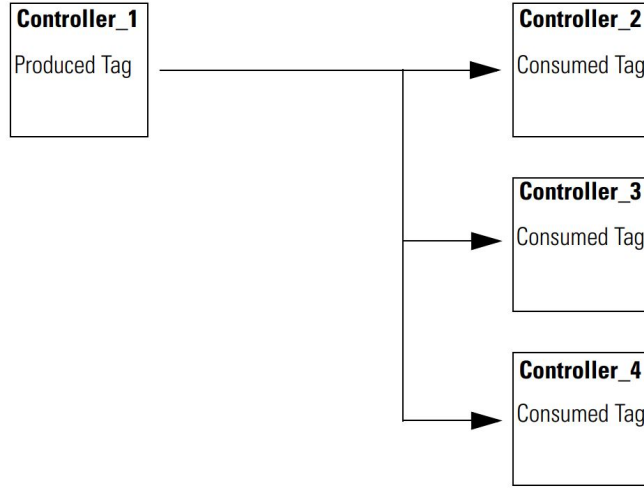


Figure 5.8 : Logix5000 controllers produced and consumed tags

A delay instruction block, which uses a data buffer to store delayed data, is applied to the system states to implement the communication and input delays on the platoon system. The instruction performs a delay of its single input for any desired length of dead time. The basic schematic diagram of an industrial controller program is given in Figure 5.9. For instance, according to the information flow topology in Figure 5.2, controller 2 (second vehicle) consumes leader, first and third vehicle's produced delayed data while it produces its delayed position, velocity and acceleration for other controllers. All the controllers start to run at the same time. They run all the time and repeatedly executes their tasks in every 10 ms.

For the real time simulation, the parameters in the previous section are used. The measurements are acquired in real time via the workstation over the network (dashed blue lines in Figure 5.7) as in Figure 5.10. The results are given in the following figures for different time delays in Figure 5.11.

Figure 5.12 shows the velocity and position behaviors of the platoon under the delay combination of point A ($\tau_1 = 0.1, \tau_2 = 0.2$) which is in the stability region. Then, both delays are increased and the points are approached the stability boundary and the test point B ($\tau_1 = 0.162, \tau_2 = 0.6$) is set for simulation. The velocity and position dynamics show small oscillatory behavior, however still follow the references as shown in Figure 5.13. Thereafter, the test point is brought a little closer to the stability boundary and selected as point C ($\tau_1 = 0.165, \tau_2 = 0.6$). As seen in Figure 5.14, the system shows stable behavior at the beginning, however then it becomes unstable. Here, the points B

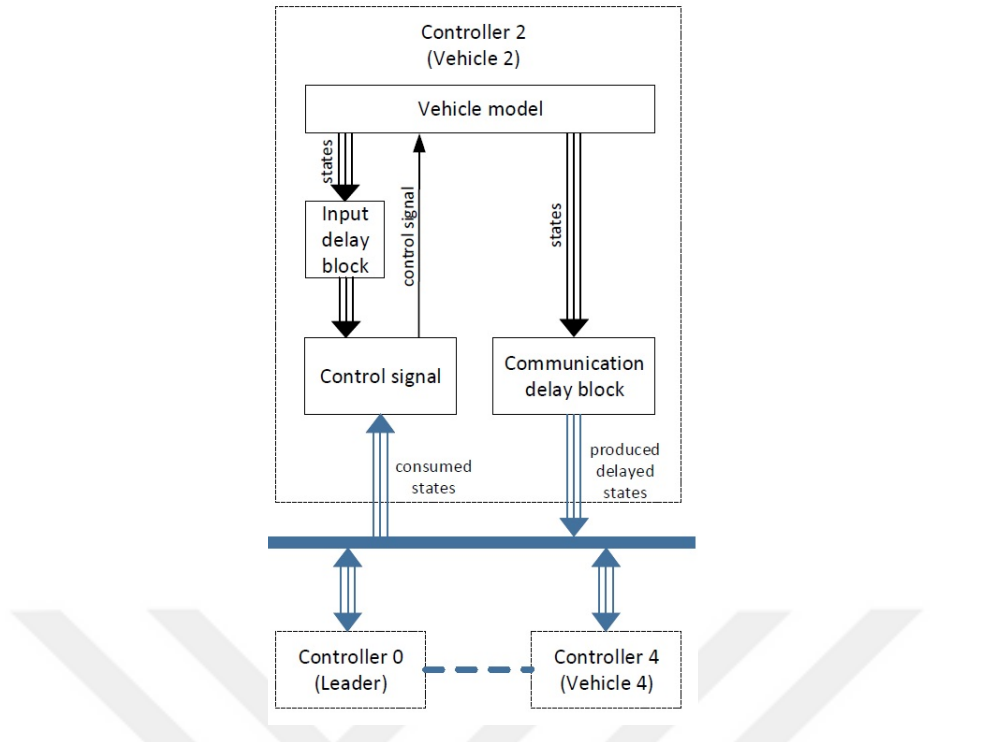


Figure 5.9 : The basic schematic diagram of the industrial controller program

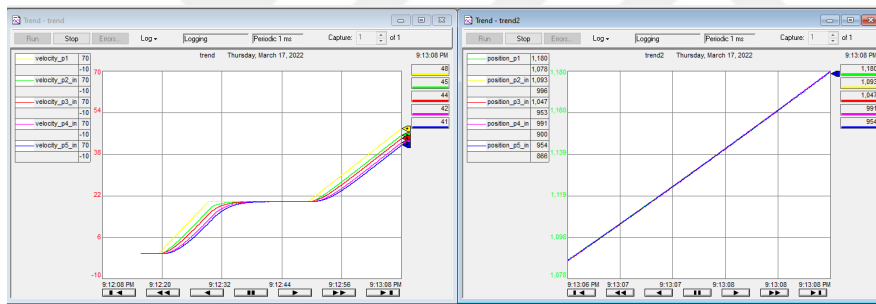


Figure 5.10 : Real time data capture

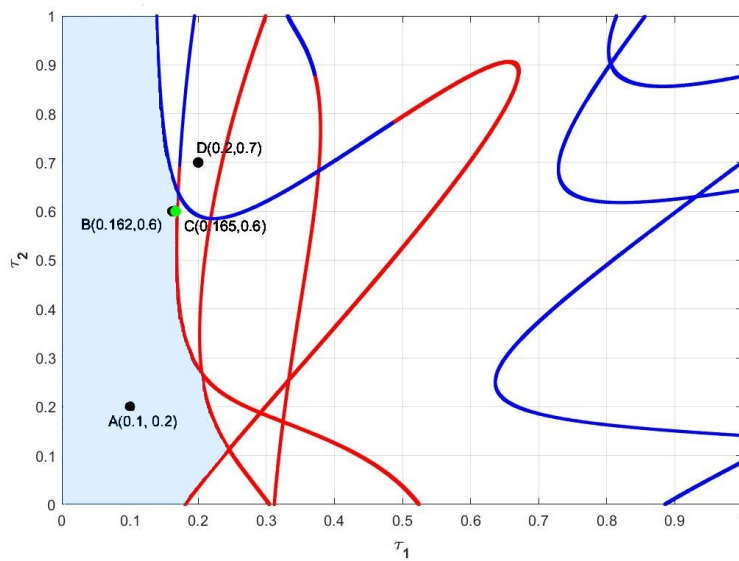


Figure 5.11 : Real time simulation points on the $\tau_1 - \tau_2$ plane.

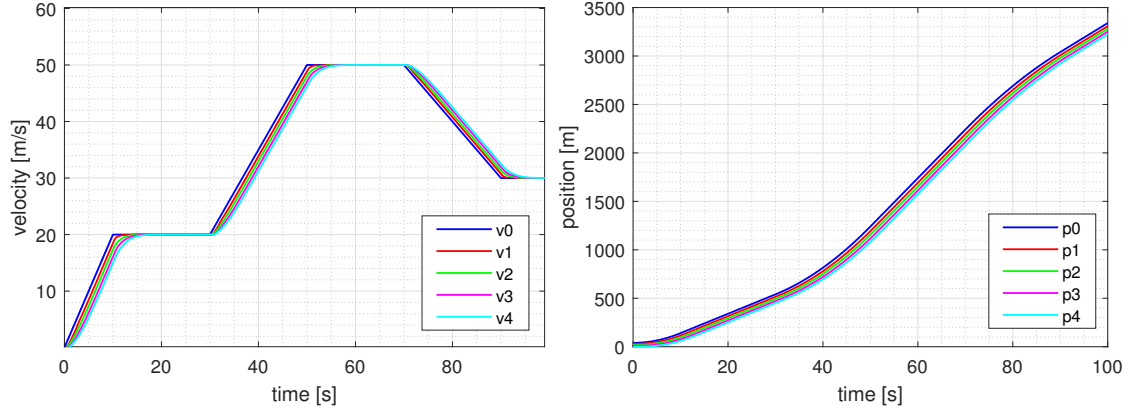


Figure 5.12 : Real time velocity and position curves for $\tau_1 = 0.1$ and $\tau_2 = 0.2$.

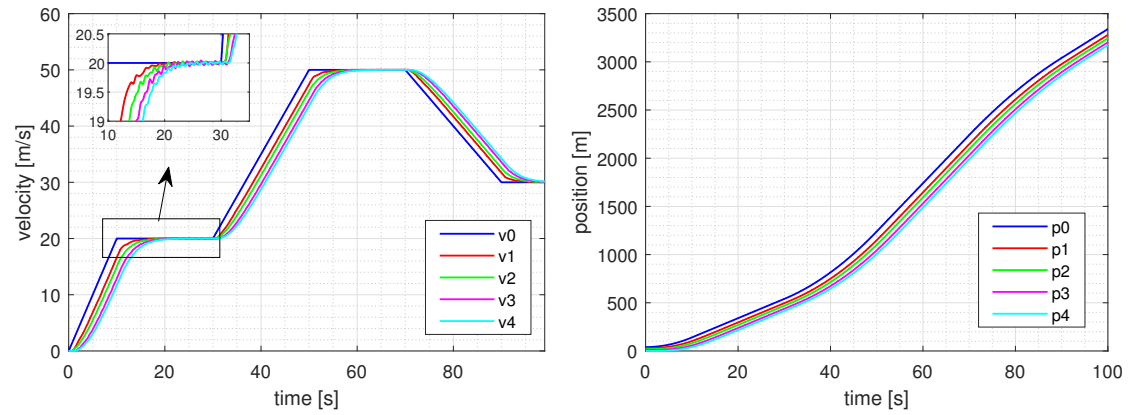


Figure 5.13 : Real time velocity and position curves for $\tau_1 = 0.162$ and $\tau_2 = 0.6$.

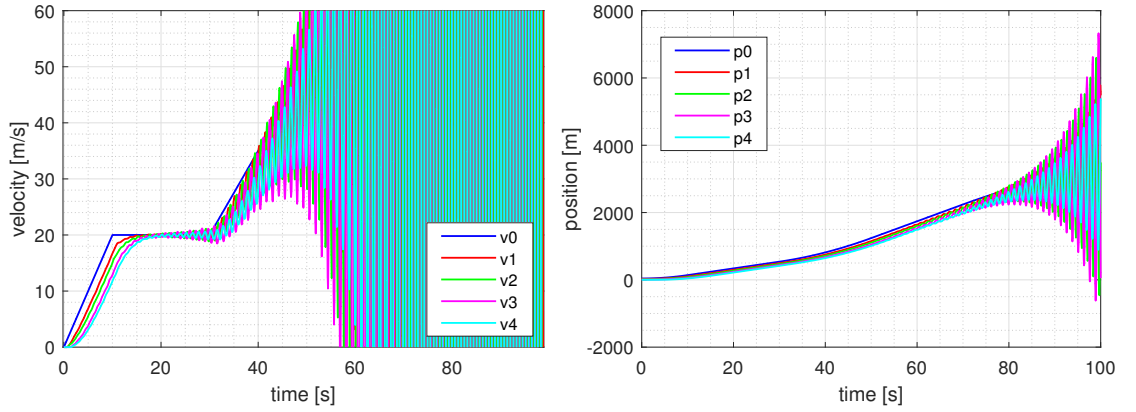


Figure 5.14 : Real time velocity and position curves for $\tau_1 = 0.165$ and $\tau_2 = 0.6$.

and C are close to the computed stability boundary in Figure 5.11. The system presents underdamped behavior at point B however unstable behavior at point C although both points are in the stable region. The observation reveals that since point C is very close to the stability boundary, both industrial computer cycle and communication delays place the platoon system into the marginally stable-unstable region. The dynamics

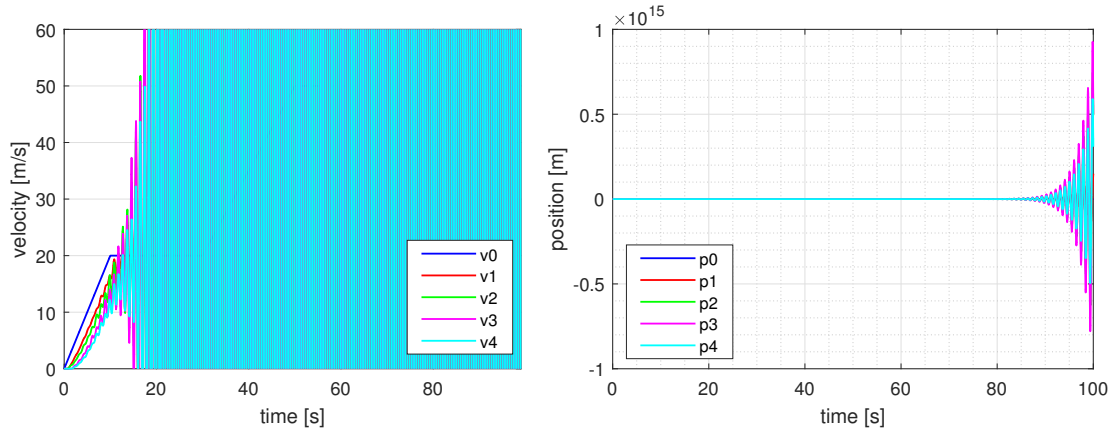


Figure 5.15 : Real time velocity and position curves for $\tau_1 = 0.2$ and $\tau_2 = 0.7$.

become unstable for the test points which is out of the stable region such as point D ($\tau_1 = 0.2, \tau_2 = 0.7$) as seen in Figure 5.15.

In this study, the platooning control problem with multiple time delays is modeled as a multi-agent control problem. The vehicle dynamics are thought of as nodes. Graph theory-based information flow topology is utilized to represent the communication between vehicles. The constant time headway policy is selected to model the desired inter-vehicle distance and the vehicle dynamic states-based distributed control strategy is used to converge to their desired velocities and inter-vehicle distances. Then, the multi-agent platooning control problem is converted into multiple time delayed linear time-invariant system (MTD-LTIS) stability analysis problem. Here, control structure is based on states of the vehicles instead of errors between the states. This provides more realistic inclusion of communication and sensing delays of the vehicles in the platoon. The delay-based stability analysis is conducted utilizing CTCR method via Bezout resultant matrix. Numerical simulations are provided to verify the validity of the proposed method. The real-time experiments are carried out on industrial computers with deterministic communication capabilities to present the applicability of the proposed method in real time systems. The results of the both simulations verify the findings of the stability region computations of the method.

6. CONCLUSIONS AND RECOMMENDATIONS

This chapter summarizes the main contributions of this study and explores possible directions for future research. The main aim of this dissertation is to deal with cooperative control for multi-agent systems (MASs) in the case of with and without time delay. The theoretical studies are applied to vehicle platooning and formation flight control problems. Finally, the results have been reinforced with the real-time application.

6.1 Conclusions

Section 2 explains the theoretical background, which is crucial to understanding the whole study. The vital points of graph theory are presented, and the associated matrices used to create the mathematical model of multi-agent systems are defined. These matrices, which are adjacency, degree, pinning, and Laplacian matrices, play a key component in the stability of the multi-agent system. Then, the information about linear algebra, especially the Kronecker product and Kronecker summation, is offered. After the necessary knowledge is supplied about the system theory, the time delay systems are introduced. The paradigm of the "Cluster Treatments of Characteristic Roots" used to create the stability posture of time delay systems, is described. The different methods to apply the steps of the CTCR paradigm have been presented in detail. A new CTCR method based on the Bezout resultant matrix, which is this study's critical point, is addressed, and the proposed algorithm is presented. These applications are selected because they are proper for the proposed multi-agent system model approach, the time delay plays a vital role in the stability of these systems, and they represent real-time problems.

Section 3 presents the distributed cooperative control problem for vehicular platooning and the formation of UAV systems. Within this approach, the longitudinal vehicle dynamics are modeled as a third-order linear system, and a single UAV is modeled as a double-integrator system. The multi-agent system model comprises the following

components: node dynamics, information flow topology, formation geometry, and distributed controller. Then the stability of multi-agent systems depends not only on the node dynamics and distributed controllers but also on the information flow topologies and formation geometry. In case of time delay is ignored, the information flow may cast fundamental limitations for stability. The section focuses on the influence on the stability of different information flow topologies, which constitute with information flow topology matrix containing real and complex eigenvalues. The distributed linear feedback controllers are proposed for vehicular platooning and the formation of UAV systems for constant policy and constant headway policy. Then, the stability problems are examined against single and multiple-time delays. All the multi-agent time delay systems are assumed as retarded time delay systems. The CTCR methodologies in the literature are used for the multi-agent system whose overall characteristic equation may decompose into factors; otherwise, a new algorithm based on the Bezout resultant theory is proposed to apply the CTCR steps. Due to the increasing computational need as the number of agents (and the order of the system) increases, the proposed algorithm offers an alternative solution when known CTCR methods cannot be directly applied to cooperative control systems.

Section 4 addressed the multi-agent control problem for the proportional-integral-derivative (PID) controller for delay-free and time-delay cases. The information topology matrix has assumed both real and complex eigenvalues. The state transformation eliminates the integrator term, and a new closed-loop state space form is obtained. The necessary and sufficient conditions to satisfy internal stability are created. Finally, a Riccati inequality-based algorithm is presented to compute the controller gains. When the system is exposed to time delay, the multi-agent closed-loop system is converted into a neutral time delay system. Furthermore, the Kronecker multiplication and elementary transformation method is offered to obtain the stability posture. The method was proposed for time-delay systems many years ago; however, it is the first time applied to time-delayed multi-agent systems.

In Section 5, the stability and control of the platooning system are analyzed for homogeneous vehicles in the presence of multiple time delays with its experimental validation. The closed-loop formulation of the system is built based on the vehicle states as opposed to the errors between the states of the vehicles given in Section

3. The CTCR method based on the Bezout resultant matrix obtains the stability posture concerning delays. The results of several numerical simulations and real-time experiments exhibit the efficiency of the presented method in a platooning system with one leader and four following vehicles. In addition, real-time experiments on industrial computers present the usability of the proposed method in real applications.

6.2 Recommendations

In future work, it may be beneficial to prove the string stability of a vehicular platoon. String stability requires the attenuation of spacing errors along the platoon. In this study, the internal stability of the vehicular platooning system is investigated for delay-free and time-delay cases, and string stability phenomena are assumed out of scope.

From a realistic point of view, it would also be beneficial to implement a real wireless communication network protocol for vehicular platooning systems and the formation of UAV systems. Implementing CAN or Mavlink protocols in-vehicle or in-UAV communication may also be beneficial to evaluate the system's performance against time delay.

Using a more complex or nonlinear model for vehicular platooning systems and the formation of UAV systems may be another way to improve the proposed controller algorithms. These complex models contain more detailed dynamic features, and the effects of time delay may be observed in detail.

Time-varying formation problems may be investigated for platooning and UAV systems with switching information flow topologies. In this study, the flow topologies are assumed fixed; however, investigating the switched topology in terms of time delay may show advanced results.

The proposed algorithms should be implemented in real vehicular platooning and the formation of UAV systems. Providing that the systems are also heterogenous structures, they result in more meaningful data.



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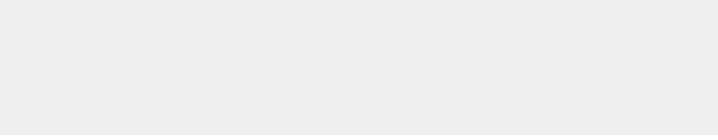
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PUBLICATIONS, PRESENTATIONS AND PATENTS ON THE THESIS:

A. Journal Papers

- **Akkaya, S.,** Akbati, O., Ergenc, A. F. 2021. Stability analysis of connected vehicles with V2V communication and time delays: CTCR method via Bézout's resultant. *Transactions of the Institute of Measurement and Control*, 43(8), 1802-1829. DOI: 10.1177/0142331220981426

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