ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

LATE-TIME COSMOLOGICAL ACCELERATION IN $f(R,\phi,X)$ GRAVITY

M.Sc. THESIS

Büşra TERZİ

Department of Physics Engineering

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FEBRUARY 2024

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$f(R,\phi,X)$ TEORİDE GEÇ-ZAMAN İVMELENMESİ

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Büşra TERZİ
(509191103)

Fizik Mühendisliği Anabilim Dalı

Fizik Mühendisliği Programı

Tez Danışmanı: Prof. Dr. Neşe ÖZDEMİR

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Büşra TERZİ, a M.Sc. student of ITU Graduate School student ID 509191103 successfully defended the thesis entitled "LATE-TIME COSMOLOGICAL ACCELERATION IN $f(R,\phi,X)$ GRAVITY", which she prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

Thesis Advisor :	**Prof. Dr. Neşe ÖZDEMİR** İstanbul Technical University	.....
Jury Members :	**Prof. Dr. Abdurrahman Savaş ARAPOĞLU** İstanbul Technical University	.....
	Prof. Dr. Özgür DELİCE Marmara University	.....

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To my beloved family and friends,

FOREWORD

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Büşra TERZİ
(Physicist)

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ABBREVIATIONS

FLRW : Friedmann-Lemaître-Robertson-Walker
ΛCDM : Lambda-Cold Dark Matter
CMB : Cosmic Microwave Background
GUTs : Grand Unified Theories
QFTs : Quantum Field Theories
DE : Dark Energy

SYMBOLS

$(-,+,+,+)$	**:** The signature of the metric
$\mu,\nu,\rho,...$	**:** Spacetime indices
$i,j,k,...$	**:** Spatial indices
∂_μ **or** $_{,\mu}$	**:** Partial derivative
∇_μ **or** $_{;\mu}$	**:** Covariant derivative
$\Box$	**:** d'Alembertian operator
$\dot{}$	**:** Derivative with respect to time
$'$	**:** Derivative with respect to redshift
S	**:** Action
R	**:** Ricci scalar
ϕ	**:** Scalar field
X	**:** Kinetic term
L_m	**:** Matter Lagrangian
κ	**:** Einstein gravitational constant
g	**:** Metric determinant
$g_{\mu\nu}$	**:** Metric tensor
$R_{\mu\nu}$	**:** Ricci tensor
$R^{\rho}_{\mu\rho\nu}$	**:** Riemann tensor
$\Gamma^{\rho}_{\mu\nu}$	**:** Christoffel symbols
$T_{\mu\nu}$	**:** Energy-momentum tensor
$G_{\mu\nu}$	**:** Einstein tensor
t	**:** Cosmic (proper) time
a	**:** Scale factor
H	**:** Hubble parameter
H_0	**:** Hubble constant
u^μ	**:** Four-velocity
ρ	**:** Energy density
p	**:** Pressure
ω	**:** Equation of state
Ω	**:** Density parameter
ρ_c	**:** Critical density
Λ	**:** Cosmological constant
z	**:** Redshift

LATE-TIME COSMOLOGICAL ACCELERATION IN $f(R,\phi,X)$ GRAVITY

SUMMARY

Numerous observational data indicates that the universe has undergone two phases of accelerated expansion. First phase, inflation, is thought to have occurred before the radiation-dominated era. The second phase is the late-time cosmic acceleration. But the mechanism that caused it hasn't been properly understood yet. Since it interacts, matter wouldn't be able to cause it; thus, the concept of dark energy arose.

In the standard model of cosmology, the cosmological constant Λ was considered to be the best candidate for dark energy. Λ corresponds to the vacuum energy in quantum field theories. And based on our understanding of the subject and vacuum fluctuations, we can calculate its density. However, a substantial problem emerged when the expected vacuum energy density was 10^{120} times higher than the observed dark energy density. Consequently, interest in modified gravity theories increased as an alternative solution.

In this thesis, we examine and review the late-time cosmological acceleration for several models in the $f(R,\phi,X)$ type modified gravity.

In the first chapter, we discussed the scientific discovery process that led us to this theory in more detail.

In the second chapter, we introduced the general formalism of the $f(R,\phi,X)$ theory by replacing the Ricci scalar in the Einstein-Hilbert action with the said arbitrary function.

In the third chapter, we examined cosmological applications by considering specific models and reconstructed the gravitational actions that corresponded to them.

And lastly, we discussed our results in the "conclusions" section.

$f(R,\phi,X)$ TEORİDE GEÇ-ZAMAN İVMELENMESİ

ÖZET

Friedmann-Lemaître-Robertson-Walker (FLRW) metriği, homojen ve izotropik bir uzayı tanımlar. Kozmolojinin standart modeli ΛCDM'den söz edildiğinde, düz $(k=0)$ FLRW metriği ile tanımlanan bir evrenden bahsedilmektedir. Homojenlik terimi, evrenin uzayın her noktasında aynı görünmesini; izotropiklik terimi ise, evrenin her yönde aynı görünmesini ifade eder. Burada yalnızca uzaydan bahsedildiği, evrenin zamanda homojen ya da izotropik olmadığı unutulmamalıdır.

ΛCDM modeli evrenin hızlanarak genişlemesi, kozmik mikrodalga arkaplan ışıması (CMB) anizotropilerinin ve evrenin büyük ölçekli yapılarının karakteristikleri, ayrıca hidrojen, döteryum, helyum ve lityum bolluğu gibi gözlemlenen özelliklerini oldukça başarılı bir şekilde açıklamaktadır. Ancak açıklamakta yetersiz kaldığı çeşitli problemler bulunmaktadır.

ΛCDM modeli ile bugün gözlemlediğimiz evrendeki gibi bir düzlük üretebilmek için Hubble sabitinin başlangıç değeri olağanüstü hassasiyetle ayarlanmalıdır; bu "düzlük problemi" olarak adlandırılır. Ayrıca, gözlemsel veriler CMB'nin neredeyse mükemmel şekilde tekdüze ve izotrop olduğunu göstermektedir. Evrenin farklı bölgelerinin, görünüşte hiçbir zaman termal dengeye gelmek için yeterli zamanları olmamış olmasına rağmen, aynı 2.725 K sıcaklığında olması sorunu "ufuk problemi" olarak ifade edilir. Son olarak, büyük birleşik teoriler tarafından öngörülüyor olmalarına rağmen manyetik monopoller, yalıtılmış tek bir manyetik kutbu (manyetik yükü) olan teorik parçacıklar, hala gözlemlenememiştir.

1981 yılında Alan Guth, bu problemleri çözen enflasyon modelini, evrenin ilk aşamalarında meydana gelen ve üstel genişmeleyle sonuçlanan mekanizmayı önerdi. Guth'ın teorisindeki bazı sorunlara, Andrei Linde'nin slow-roll (yavaş yuvarlanma) ve kaotik enflasyon katkılarıyla çözüm sağlandı. Böylece model daha kapsamlı hale getirildi.

1998-1999 yıllarında Tip Ia süpernova gözlemlerinin verileri, evrenin beklenenden daha yüksek bir oranda hızlandığını ortaya çıkardı. Bu daha sonra CMB'nin, evrendeki büyük ölçekli yapıların ve baryon akustik salınımlarının gözlem verileriyle de desteklendi. Geç-zaman ivmelenmesine neden olan bu itici olgu karanlık enerji olarak isimlendirildi.

Standart kozmoloji modelinde bu ivmelenme ile ilişkilendirilen öge kozmolojik sabit Λ'dır ve kuantum alan teorilerindeki vakum enerjisine (boş uzayın enerjisi) karşılık gelmektedir. Ancak evreni ivmelenerek genişleten şeyin kozmolojik sabit Λ olması önemli bir sorunu beraberinde getirmektedir. Çünkü kuantum alan teorisi

ve vakum dalgalanmaları hakkında bildiklerimizi kullanarak hesapladığımız vakum enerji yoğunluğu ile gözlemlenen karanlık enerji yoğunluğu arasında 10^{120} kat fark bulunmaktadır; gözlemlenen değer, beklenen değerden çok daha küçüktür.

Bu sorunu çözmenin bir yolu geometrinin kendisini değiştirmek, genel göreliliği modifiye etmektir. Bu modifikasyona basit bir örnek $f(R)$ teorisidir; burada Einstein-Hilbert eylemindeki Lagrangian, Ricci skalerinin keyfi bir fonksiyonuyla genelleştirilir.

Bu tezde Ricci skaleri R'ye ek olarak keyfi fonksiyona ϕ skaler alanı ve X kinetik terimi eklenmiş; $f(R,\phi,X)$ teorisi incelenmiştir. Çalışmamızda Bahamonde, Böhmer, Lobo ve Sáez-Gómez'in makalesi referans alınmış ve makaledeki sıra izlenmiştir.

İkinci bölümde $f(R,\phi,X)$ teorisinin genel formalizmi ele alınmıştır. Öncelikle eylemin metriğe ve skaler alana göre varyasyonu sıfıra eşitlenerek alan denklemleri; ardından alan denklemleri kullanılarak Friedmann denklemleri elde edilmiştir. Skaler alanın karanlık enerji gibi davrandığı ve Friedmann denklemlerini sağladığı varsayılarak, karanlık enerji yoğunluğu ve basıncı tanımlanmıştır. Bu karanlık enerji bileşenlerinin süreklilik denklemini sağladığı gösterilmiştir. Son olarak, karanlık enerji için hal denklemi ve yoğunluk parametresi tanımlanmıştır.

Üçüncü bölümün ilk aşamasında $f(R,X,\phi) = \gamma(X,\phi)\,R$ seçilerek Brans-Dicke tipi bir eylem tanımlanmıştır. Burada Friedmann denklemleri kullanılarak

$$\frac{d^2\gamma(t)}{dt^2} + 2H\,\frac{d\gamma(t)}{dt} + \left(3H^2 + 2\dot{H}\right)\gamma(t) = 0$$

denklemi elde edilmiştir. Bu denklem de-Sitter, üs-yasası (power-law) ve ΛCDM gibi geç-zaman ivmelenmesini üretebildiği literatürde iyi bilinen çeşitli Hubble parametreleri tanımlanarak çözülmüştür.

Üçüncü bölümün ikinci aşamasında ise $f(R,X,\phi) = \alpha(R) + \gamma(X,\phi)$ seçilerek skaler alanın minimal bağlı olduğu bir başka kütle çekimsel eylem, daha sonra Starobinsky enflasyonu da tanımlanarak incelenmiştir.

Son bölümde ise elde edilen sonuçların değerlendirmesi yapılmıştır.

1. INTRODUCTION

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + sin^2\theta d\phi^2\right)\right] \tag{1.1}$$

describes a homogeneous and isotropic universe. Here, k can take the values $\{-1, 0, +1\}$. And depending on its value, it indicates whether the universe is open (hyperbolic), flat or closed (spherical), respectively.

When we refer to the "ΛCDM model", also known as the "standard model of cosmology", we are talking about an adiabatically expanding radiation-dominated universe described by a flat FLRW metric [1]. And it quite successfully explains the observed features of the universe; such as its accelerating expansion, the characteristics of the cosmic microwave background (CMB) anisotropies, the characteristics of the large-scale structures of the universe, as well as the abundances of hydrogen, deuterium, helium, and lithium [2]. However, there were the remaining flatness, horizon, and magnetic monopole problems.

The universe we observe today appears to be quite flat. To produce such flatness in ΛCDM model, the initial value of the Hubble constant (H_0) must be fine-tuned with exceptional precision. This is known as the flatness problem. Furthermore, the observational data reveals that the CMB is almost perfectly uniform and isotropic. Which raises the question: How can these distant regions be at the same temperature, even though they have never been in causal contact to reach thermal equilibrium? This is referred to as the horizon problem [3]. Lastly, magnetic monopoles are self-explanatory theoretical particles, that were predicted by the grand unified theories (GUTs). But despite the theoretical predictions, they have never been detected in the present-day universe.

As a resolution to these problems, A. Guth suggested the inflationary model. A mechanism that takes place at early stages of the universe and results in an exponential

expansion [1]. Guth's model initially had some problems but became more complete with Linde's slow-roll and chaotic inflation contributions [4,5].

In 1998-1999, observations of Type Ia supernovae revealed that the universe is accelerating at a rate higher than expected [6,7]. This finding was later supported by observations of CMB, large-scale structures, and baryon acoustic oscillations. The repulsive phenomenon causing this late-time cosmic acceleration was named dark energy.

In standard model of cosmology, the element associated with this acceleration is the cosmological constant Λ. And it corresponds to the vacuum energy (energy of the empty space) in quantum field theories (QFTs). However, the observed dark energy density is too small compared to the expected value of vacuum energy density. This remains a major unresolved matter in modern cosmology.

One way to approach this problem is to modify the geometry itself. A simple modification to general relativity is the $f(R)$ gravity; in which the Lagrangian in the Einstein-Hilbert action is generalized by an arbitrary function of the Ricci scalar [8].

In this study, we are going to consider $f(R,\phi,X)$ gravity; where in addition to the Ricci scalar, we introduce a scalar field and a kinetic term to our arbitrary function.

2. $f(R,\phi,X)$ TYPE GRAVITY MODELS

Generalized theories of gravity, such as $f(R)$, $f(R,\phi)$, $f(R,T)$ gravity, are widely studied and their observational constraints are presented in modern cosmology literature [8]–[15]. In this thesis, we consider the gravity model $f(R,\phi,X)$ to understand the cosmological effects of modified gravity on the late-time acceleration and whether it can overcome the shortcomings of ΛCDM cosmologies. In accordance with this purpose, let us consider the general action as follows:

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{2\kappa^2} f(R,\phi,X) + L_m\right]. \tag{2.1}$$

Here, $f(R,\phi,X)$ is an arbitrary function of the Ricci scalar R, the scalar field ϕ and the kinetic term X. The kinetic term is defined by [16]

$$\begin{aligned} X &= -\frac{1}{2}(\nabla\phi)^2, \\ X &= -\frac{1}{2} g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi. \end{aligned} \tag{2.2}$$

On the other hand, $L_m = L_m(g,\psi)$ is the matter Lagrangian. Here, matter is minimally coupled to the metric field and ψ represents all matter fields.

2.1 The Field Equations

The field equations for this action can be obtained by taking its variation with respect to the metric tensor $g_{\mu\nu}$ and the scalar field ϕ. Varying the action with respect to the metric tensor gives us

$$\delta_g S = \int d^4x \left[\frac{f}{2\kappa^2}\,\delta_g\sqrt{-g} + \frac{\sqrt{-g}}{2\kappa^2}\,\frac{\partial f}{\partial R}\,\delta_g R + \frac{\sqrt{-g}}{2\kappa^2}\,\frac{\partial f}{\partial X}\,\delta_g X + \delta_g\left(\sqrt{-g}L_m\right)\right]. \tag{2.3}$$

To solve this equation, first we need to find the variation of the metric determinant g, where

$$g = det\ g_{\mu\nu}. \tag{2.4}$$

For any matrix M, $Tr(lnM) = ln(detM)$. The variation of this identity yields [17]

$$Tr\left(M^{-1}\delta M\right) = \frac{1}{det\ M}\ \delta(det\ M).$$

Applying this to the metric using $M = g_{\mu\nu}$ transformation leads to

$$g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{g}\ \delta g, \tag{2.5}$$

and with this result, one can see that

$$\delta\sqrt{-g} = \frac{\sqrt{-g}}{2}\ g^{\mu\nu}\ \delta g_{\mu\nu}. \tag{2.6}$$

Using the following transformation

$$\begin{gathered} g_{\mu\nu}\ g^{\mu\nu} = \delta^{\mu}_{\mu}\ \rightarrow\ \delta\left(g_{\mu\nu}\ g^{\mu\nu}\right) = \delta\left(\delta^{\mu}_{\mu}\right) = 0, \\ g^{\mu\nu}\delta g_{\mu\nu} = -g_{\mu\nu}\delta g^{\mu\nu}, \end{gathered} \tag{2.7}$$

we obtain

$$\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2} g_{\mu\nu}\delta g^{\mu\nu}. \tag{2.8}$$

Next, we will determine the variation of the Ricci scalar (curvature scalar). The Ricci scalar is defined by [18]:

$$R = g^{\mu\nu}R_{\mu\nu} = g_{\mu\nu}R^{\mu\nu}, \tag{2.9}$$

where $R_{\mu\nu}$ is the Ricci tensor. Hence, the variation of the Ricci scalar can be written as

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{2.10}$$

The Ricci tensor is the contraction of the Riemann tensor $R^{\rho}_{\mu\rho\nu}$ [18], which is given by

$$\begin{aligned} R_{\mu\nu} &= R^{\rho}_{\mu\rho\nu}, \\ R^{\rho}_{\mu\rho\nu} &= \partial_{\rho}\Gamma^{\rho}_{\mu\nu} - \partial_{\nu}\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\rho\mu}. \end{aligned} \tag{2.11}$$

So the variation of the Ricci tensor is

$$\delta R_{\mu\nu} = \left(\partial_{\rho}\delta\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\rho\sigma}\delta\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\rho\mu}\delta\Gamma^{\rho}_{\nu\sigma}\right) - \left(\partial_{\nu}\delta\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\sigma}_{\mu\nu}\delta\Gamma^{\rho}_{\rho\sigma}\right).$$

The variation $\delta\Gamma^{\rho}_{\rho\mu}$ itself is a tensor. Therefore its covariant derivative is [17]

$$\nabla_\nu \delta\Gamma^{\rho}_{\rho\mu} = \partial_\nu \delta\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\nu\sigma}\delta\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\sigma}_{\nu\rho}\delta\Gamma^{\rho}_{\sigma\mu} - \Gamma^{\sigma}_{\mu\nu}\delta\Gamma^{\rho}_{\rho\sigma}. \tag{2.12}$$

This gives us the Palatini Identity:

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^{\rho}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\rho\mu}, \tag{2.13}$$

and the variation of the Ricci scalar (2.9) turns into

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left(\nabla_\rho \delta\Gamma^{\rho}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\rho\mu}\right). \tag{2.14}$$

The variation of the Christoffel symbols can be written as [19]

$$\delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\, g^{\rho\alpha}\left(\nabla_\mu \delta g_{\alpha\nu} + \nabla_\nu \delta g_{\alpha\mu} - \nabla_\alpha \delta g_{\mu\nu}\right), \tag{2.15}$$

therefore

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} g^{\rho\alpha}\left(2\nabla_\rho\nabla_\mu \delta g_{\alpha\nu} - \nabla_\rho\nabla_\alpha \delta g_{\mu\nu} - \nabla_\nu\nabla_\mu \delta g_{\alpha\rho}\right).$$

Using (2.7) we can find

$$g^{\mu\nu}g^{\rho\alpha}\nabla_\rho\nabla_\mu \delta g_{\alpha\nu} = -\underbrace{g_{\alpha\nu}g^{\rho\alpha}}_{\delta^{\rho}_{\nu}}\nabla_\rho\nabla_\mu \delta g^{\mu\nu},$$
$$g^{\mu\nu}g^{\rho\alpha}\nabla_\rho\nabla_\mu \delta g_{\alpha\nu} = -\nabla_\nu\nabla_\mu \delta g^{\mu\nu},$$

and rearrange the equation

$$\delta R = \left(R_{\mu\nu} - \nabla_\nu\nabla_\mu\right)\delta g^{\mu\nu} + \frac{1}{2}\left(g_{\mu\nu}g^{\rho\alpha}\nabla_\rho\nabla_\alpha \delta g^{\mu\nu} + g^{\mu\nu}g_{\rho\alpha}\nabla_\nu\nabla_\mu \delta g^{\rho\alpha}\right),$$
$$\delta R = \left(R_{\mu\nu} - \nabla_\nu\nabla_\mu\right)\delta g^{\mu\nu} + \frac{1}{2}\left(g_{\mu\nu}\nabla^\rho\nabla_\rho \delta g^{\mu\nu} + g_{\rho\alpha}\nabla^\nu\nabla_\nu \delta g^{\rho\alpha}\right).$$

Changing the indices $\nu \to \rho$, $\rho \to \mu$ and $\alpha \to \nu$ gives us

$$\delta R = \left(R_{\mu\nu} - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu} + g_{\mu\nu}\nabla^\rho\nabla_\rho \delta g^{\mu\nu}.$$

The covariant d'Alembertian operator is

$$\Box = \nabla^\rho\nabla_\rho = \nabla_\rho\nabla^\rho. \tag{2.16}$$

Therefore, the variation of the Ricci scalar is obtained

$$\delta R = \left(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu}. \tag{2.17}$$

Lastly, from the equation (2.2) we find the variation of the kinetic term X with respect to metric as

$$\delta X = -\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi. \tag{2.18}$$

Using the equations (2.8), (2.17), (2.18) and denoting

$$F = \frac{\partial f}{\partial R}, \tag{2.19}$$

the variation of the action (2.3) turns into

$$\begin{aligned}\delta S = \int d^4x \Bigg[-\frac{1}{2\kappa^2}\Bigg(-\frac{f}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} + \sqrt{-g}\,F\left(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu} \\ - \frac{\sqrt{-g}}{2}f_{,X}\,\delta g^{\mu\nu}\,\phi_{,\mu}\phi_{,\nu} + \delta\left(\sqrt{-g}L_m\right)\Bigg)\Bigg].\end{aligned} \tag{2.20}$$

Using integration by parts, one gets

$$\begin{aligned}&Fg_{\mu\nu}\Box\delta g^{\mu\nu} = \nabla^\alpha\left[Fg_{\mu\nu}\nabla_\alpha\delta g^{\mu\nu}\right] - \left[\nabla^\alpha Fg_{\mu\nu}\right]\nabla_\alpha\delta g^{\mu\nu},\\ &\left[\nabla^\alpha Fg_{\mu\nu}\right]\nabla_\alpha\delta g^{\mu\nu} = \nabla_\alpha\left[g_{\mu\nu}\nabla^\alpha F\delta g^{\mu\nu}\right] - \delta g^{\mu\nu}\left[g_{\mu\nu}\nabla^\alpha\nabla_\alpha F\right],\\ &Fg_{\mu\nu}\Box\delta g^{\mu\nu} = \nabla^\alpha\left[Fg_{\mu\nu}\nabla_\alpha\delta g^{\mu\nu}\right] - \nabla_\alpha\left[g_{\mu\nu}\nabla^\alpha F\delta g^{\mu\nu}\right] + \delta g^{\mu\nu}\left[g_{\mu\nu}\nabla^\alpha\nabla_\alpha F\right].\end{aligned}$$

The first two terms are total derivatives, resulting in a boundary term that generally vanishes in the action integral. Therefore

$$Fg_{\mu\nu}\Box\delta g^{\mu\nu} = \delta g^{\mu\nu}g_{\mu\nu}\Box F. \tag{2.21}$$

Using the same method, one can prove that

$$F\nabla_\mu\nabla_\nu\delta g^{\mu\nu} = \delta g^{\mu\nu}\nabla_\mu\nabla_\nu F. \tag{2.22}$$

Substituting (2.21) and (2.22) into (2.20) results in

$$\begin{aligned}\delta S = \int d^4x\sqrt{-g}\Bigg[\frac{1}{2\kappa^2}\Bigg(-\frac{1}{2}fg_{\mu\nu}\delta g^{\mu\nu} + \delta g^{\mu\nu}\left(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)F \\ - \frac{1}{2}\delta g^{\mu\nu}f_{,X}\phi_{,\mu}\phi_{,\nu}\Bigg) + \frac{1}{\sqrt{-g}}\delta\left(\sqrt{-g}L_m\right)\Bigg],\end{aligned}$$

$$\begin{aligned}\delta S = \int d^4x\,\frac{\sqrt{-g}}{2}\,\delta g^{\mu\nu}\Bigg[\frac{1}{\kappa^2}\Bigg(-\frac{1}{2}fg_{\mu\nu} + FR_{\mu\nu} + g_{\mu\nu}\Box F - \nabla_\mu\nabla_\nu F - \frac{1}{2}f_{,X}\phi_{,\mu}\phi_{,\nu}\Bigg) \\ + \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}L_m\right)}{\delta g^{\mu\nu}}\Bigg].\end{aligned} \tag{2.23}$$

The matter energy-momentum tensor $T_{\mu\nu}$ and the Einstein tensor $G_{\mu\nu}$ are defined as [17]:

$$T^{(m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\,\frac{\delta\left(\sqrt{-g}L_m\right)}{\delta g^{\mu\nu}}, \tag{2.24}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;\;\rightarrow\;\; R_{\mu\nu} = G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R. \tag{2.25}$$

Using these definitions and keeping in mind that the action remains invariant under variations of the metric $\delta S = 0$ [18], the equation (2.23) becomes

$$\begin{aligned}\delta S = \int d^4x\,\frac{\sqrt{-g}}{2}\,\delta g^{\mu\nu}\Bigg[\frac{1}{\kappa^2}\Bigg(-\frac{1}{2}f g_{\mu\nu} + FG_{\mu\nu} + \frac{1}{2}g_{\mu\nu}FR + g_{\mu\nu}\Box F \\ - \nabla_\mu\nabla_\nu F - \frac{1}{2}f_{,X}\phi_{,\mu}\phi_{,\nu}\Bigg) - T^{(m)}_{\mu\nu}\Bigg] = 0.\end{aligned} \tag{2.26}$$

Hence, varying the action (2.1) with respect to the metric tensor $g_{\mu\nu}$, yields the following field equation

$$FG_{\mu\nu} = \frac{1}{2}\left(f - FR\right)g_{\mu\nu} + \nabla_\mu\nabla_\nu F - g_{\mu\nu}\Box F + \frac{1}{2}f_{,X}\phi_{,\mu}\phi_{,\nu} + \kappa^2 T^{(m)}_{\mu\nu}. \tag{2.27}$$

Now we will vary the action (2.1) with respect to scalar field ϕ as follows

$$\delta_\phi S = \int d^4x\,\frac{\sqrt{-g}}{2\kappa^2}\left[\frac{\partial f}{\partial\phi}\delta_\phi\phi + \frac{\partial f}{\partial X}\delta_\phi X\right]. \tag{2.28}$$

Through the definition of the kinetic term (2.2), its variation with respect to scalar field can be expressed as

$$\begin{aligned} X &= -\frac{1}{2}\left(\nabla_\mu\phi\right)\left(\nabla^\mu\phi\right), \\ \delta X &= -\frac{1}{2}\left[\left(\nabla_\mu\delta\phi\right)\left(\nabla^\mu\phi\right) + \left(\nabla_\mu\phi\right)\left(\nabla^\mu\delta\phi\right)\right], \\ \delta X &= -\left(\nabla_\mu\delta\phi\right)\left(\nabla^\mu\phi\right). \end{aligned} \tag{2.29}$$

Using integration by parts

$$f_{,X}\left(\nabla_\mu\delta\phi\right)\left(\nabla^\mu\phi\right) = \nabla_\mu\left(f_{,X}\nabla^\mu\phi\,\delta\phi\right) - \nabla_\mu\left(f_{,X}\nabla^\mu\phi\right)\delta\phi,$$

and eliminating the total derivative

$$f_{,X}\delta X = f_{,X}\left(\nabla_\mu\delta\phi\right)\left(\nabla^\mu\phi\right) = \nabla_\mu\left(f_{,X}\nabla^\mu\phi\right)\delta\phi, \tag{2.30}$$

one can see that the equation (2.28) turns into

$$\delta S = \int d^4x \, \frac{\sqrt{-g}}{2\kappa^2} \left[f_{,\phi} + \nabla_\mu \left(f_{,X} \nabla^\mu \phi \right) \right] \delta\phi = 0. \tag{2.31}$$

Therefore, variation of the action (2.1) with respect to the scalar field ϕ, yields

$$f_{,\phi} + \nabla_\mu \left(f_{,X} \nabla^\mu \phi \right) = 0. \tag{2.32}$$

2.2 The Friedmann Equations

Now for (2.27), taking into account a flat FLRW metric (1.1) in spherical coordinates

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + r^2 \left(d\theta^2 + sin^2\theta d\varphi^2 \right) \right], \tag{2.33}$$

where a(t) is the scale factor. Therefore the metric and the inverse metric tensors takes the form

$$\begin{aligned} g_{\mu\nu} &= diag\left(-1,\ a^2,\ a^2r^2,\ a^2r^2sin^2\theta\right), \\ g^{\mu\nu} &= diag\left(-1,\ \frac{1}{a^2},\ \frac{1}{a^2r^2},\ \frac{1}{a^2r^2sin^2\theta}\right). \end{aligned} \tag{2.34}$$

Christoffel symbols are given by [18]

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right), \tag{2.35}$$

so for the metric (2.33), the non-zero components of Christoffel symbols are

$$\begin{aligned} \Gamma^0_{11} &= a\dot{a}, & \Gamma^1_{22} &= -r \\ \Gamma^0_{22} &= a\dot{a}r^2, & \Gamma^1_{33} &= -rsin^2\theta \\ \Gamma^0_{33} &= a\dot{a}r^2sin^2\theta, & \Gamma^2_{33} &= -sin\theta\ cos\theta \\ \Gamma^1_{01} &= \frac{\dot{a}}{a}, & \Gamma^3_{03} &= \frac{\dot{a}}{a} \\ \Gamma^2_{02} &= \frac{\dot{a}}{a}, & \Gamma^3_{13} &= \frac{1}{r} \\ \Gamma^2_{12} &= \frac{1}{r}, & \Gamma^3_{23} &= cot\theta. \end{aligned} \tag{2.36}$$

Using the definition of Ricci tensor (2.11), the values of the Christoffel symbols given above, we obtain the following results:

$$\begin{aligned} R_{00} &= -3\,\frac{\ddot{a}}{a}, & R_{22} &= r^2\left(a\ddot{a} + 2\dot{a}^2\right) \\ R_{11} &= a\ddot{a} + 2\dot{a}^2, & R_{33} &= r^2sin^2\theta\left(a\ddot{a} + 2\dot{a}^2\right). \end{aligned} \tag{2.37}$$

So the Ricci scalar (2.9) can be found as

$$R = \frac{6}{a^2}\left(a\ddot{a} + \dot{a}^2\right). \tag{2.38}$$

The Hubble parameter (the rate of expansion) and its derivative with respect to time are

$$H = \frac{\dot{a}}{a} \;\rightarrow\; \dot{H} = \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}. \tag{2.39}$$

Therefore, the Ricci scalar can be expressed in terms of the Hubble parameter as

$$R = 6\left(\dot{H} + 2H^2\right). \tag{2.40}$$

Energy-momentum tensor for a perfect fluid is [18]

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\, g^{\mu\nu}, \tag{2.41}$$

where ρ is the energy density, p is the pressure and four-velocity $u^\mu = (1,\ 0,\ 0,\ 0)$. Therefore:

$$T_{\mu\nu} = diag\left(\rho,\ p\, g_{ii}\right). \tag{2.42}$$

The first Friedmann equation is derived from the 00 components of the field equations (2.27) [18]:

$$FG_{00} = \frac{1}{2}(f - FR)\, g_{00} + \nabla_0\nabla_0 F - g_{00}\nabla_0\nabla^0 F + \frac{1}{2} f_{,X}\phi_{,0}\phi_{,0} + \kappa^2 T_{00}^{(m)},$$
$$F\left(R_{00} - \frac{1}{2}g_{00}R\right) = \frac{1}{2}(FR - f) + \ddot{F} + \nabla_0\nabla^0 F + \frac{1}{2} f_{,X}\left(\frac{\partial\phi}{\partial t}\right)^2 + \kappa^2\rho_m.$$

With the definition of the covariant derivative, one can prove that

$$\Box F = \nabla_\alpha\nabla^\alpha F = \frac{1}{\sqrt{-g}}\,\partial_\alpha\left(\sqrt{-g}\; g^{\alpha\beta}\;\partial_\beta F\right). \tag{2.43}$$

The determinant g (2.4) for FLRW metric (2.34)

$$g = -a^6 r^4 sin^2\theta \;\rightarrow\; \sqrt{-g} = a^3 r^2 sin\theta, \tag{2.44}$$

hence

$$\nabla_0\nabla^0 F = \frac{1}{a^3}\,\frac{\partial}{\partial t}\left(-a^3\dot{F}\right) \;\rightarrow\; \nabla_0\nabla^0 F = -3H\dot{F} - \ddot{F}. \tag{2.45}$$

Using (2.45) and our findings from (2.37) to (2.40), we obtain

$$F\left(-3\frac{\ddot{a}}{a}+\frac{R}{2}\right)=\frac{FR}{2}-\frac{f}{2}+\ddot{F}-3H\dot{F}-\ddot{F}+\frac{1}{2}f_{,X}\dot{\phi}^2+\kappa^2\rho_m,$$
$$-3F\left(\frac{R}{6}-H^2\right)=-\frac{f}{2}-3H\dot{F}+\frac{1}{2}f_{,X}\dot{\phi}^2+\kappa^2\rho_m.$$

Scalar field ϕ is only dependent on time and covariant derivative of scalar field is simply a partial derivative $\nabla_\mu\phi=\partial_\mu\phi$. Therefore

$$X=-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\ \rightarrow\ X=-\frac{1}{2}g^{00}\partial_0\phi\partial_0\phi,$$
$$X=\frac{1}{2}\left(\dot{\phi}\right)^2. \tag{2.46}$$

Thus, we obtain the first Friedmann equation

$$3FH^2=f_{,X}X+\frac{1}{2}\left(FR-f\right)-3H\dot{F}+\kappa^2\rho_m. \tag{2.47}$$

The initial step in deriving the second Friedmann equation is to find the trace of the field equations [18]. With that aim, first we will multiply the field equations (2.27) with $g^{\mu\nu}$:

$$g^{\mu\nu}F\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)=\frac{1}{2}\left(f-FR\right)g^{\mu\nu}g_{\mu\nu}+g^{\mu\nu}\nabla_\mu\nabla_\nu F-g^{\mu\nu}g_{\mu\nu}\Box F$$
$$+\frac{1}{2}g^{\mu\nu}f_{,X}\phi_{,\mu}\phi_{,\nu}+\kappa^2g^{\mu\nu}T^{(m)}_{\mu\nu}.$$

Using the definition of the Ricci scalar (2.9), d'Alembertian operator (2.16) and the equations below

$$g^{\mu\nu}g_{\mu\nu}=4, \qquad\qquad g^{ii}g_{ii}=3 \tag{2.48}$$

as well as the following result for the energy-momentum tensor in FLRW space-time (2.42)

$$g^{\mu\nu}T_{\mu\nu}=g^{00}\rho+pg^{ii}g_{ii},$$
$$T=g^{\mu\nu}T_{\mu\nu}=3p-\rho, \tag{2.49}$$

we can find the trace of the field equations

$$FR-2FR=2f-2FR+\Box F-4\Box F+\frac{1}{2}f_{,X}\phi^{,\nu}\phi_{,\nu}+\kappa^2T,$$
$$FR=2f-3\Box F+\frac{1}{2}f_{,X}\phi^{,\nu}\phi_{,\nu}+\kappa^2T \tag{2.50}$$

If we rearrange this as follows

$$\frac{1}{2}f g_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\left(FR + 3\Box F - \frac{1}{2}f_{,X}\phi^{,\nu}\phi_{,\nu} - \kappa^2 T\right) \tag{2.51}$$

and plug (2.51) into field equations (2.27)

$$\begin{aligned} F\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \frac{1}{4}g_{\mu\nu}\left(FR + 3\Box F - \frac{1}{2}f_{,X}\phi^{,\nu}\phi_{,\nu} - \kappa^2 T\right) - \frac{FR}{2}g_{\mu\nu} + \nabla_\mu\nabla_\nu F \\ - g_{\mu\nu}\Box F + \frac{1}{2}f_{,X}\phi_{,\mu}\phi_{,\nu} + \kappa^2 T^{(m)}_{\mu\nu}, \end{aligned}$$

then find the 00 components

$$\begin{aligned} FR_{00} = \frac{1}{4}g_{00}\left(FR + 3\nabla^0\nabla_0 F - \frac{1}{2}f_{,X}\ g_{00}\ \phi_{,0}\phi_{,0} - \kappa^2 T\right) + \nabla_0\nabla_0 F - g_{00}\nabla^0\nabla_0 F \\ + \frac{1}{2}f_{,X}\phi_{,0}\phi_{,0} + \kappa^2 T^{(m)}_{00}, \end{aligned}$$

$$\begin{aligned} -3F\dot{H} - 3FH^2 &= -\frac{1}{4}\left(6F\dot{H} + 12FH^2 - 3\ddot{F} - 9H\dot{F} + f_{,X}X - \kappa^2 T\right) - 3H\dot{F} \\ &\quad + f_{,X}X + \kappa^2\rho_m, \\ -4F\dot{H} - 4FH^2 &= -2F\dot{H} - 4FH^2 + \ddot{F} + 3H\dot{F} - \frac{1}{3}f_{,X}X + \frac{1}{3}\kappa^2 T - 4H\dot{F} \\ &\quad + \frac{4}{3}f_{,X}X + \frac{4}{3}\kappa^2\rho_m, \\ -2F\dot{H} &= \ddot{F} - H\dot{F} + f_{,X}X + \frac{1}{3}\kappa^2\left(3p_m - \rho_m\right) + \frac{4}{3}\kappa^2\rho_m, \end{aligned}$$

we obtain the second Friedmann equation

$$-2F\dot{H} = \ddot{F} - H\dot{F} + f_{,X}X + \kappa^2\left(\rho_m + p_m\right). \tag{2.52}$$

Now, applying the covariant derivative in the second term on the left hand side of equation (2.32), gives us

$$\nabla_\mu\left(f_{,X}\ g^{\mu\nu}\partial_\nu\phi\right) = \partial_\mu\left(f_{,X}\ g^{\mu\nu}\partial_\nu\phi\right) + \Gamma^\mu_{\alpha\mu}\left(f_{,X}\ g^{\alpha\nu}\partial_\nu\phi\right). \tag{2.53}$$

The only non-zero Christoffel (2.36) and metric (2.34) terms results in

$$\begin{aligned} \nabla_\mu\left(f_{,X}\ g^{\mu\nu}\partial_\nu\phi\right) &= \partial_0\left(f_{,X}\ g^{00}\partial_0\phi\right) + \Gamma^i_{0i}\left(f_{,X}\ g^{00}\partial_0\phi\right), \\ \nabla_\mu\left(f_{,X}\ g^{\mu\nu}\partial_\nu\phi\right) &= -\left(\dot{f_{,X}}\dot{\phi}\right) - 3\frac{\dot{a}}{a}\left(f_{,X}\dot{\phi}\right), \end{aligned} \tag{2.54}$$

and the equation (2.32) turns into

$$f_{,\phi} - \left(\dot{f_{,X}}\dot{\phi}\right) - 3\frac{\dot{a}}{a}\left(f_{,X}\dot{\phi}\right) = 0.$$

Rearranging this using the fact that

$$(a^3 \dot{f_{,X}} \dot{\phi}) = a^3 (\dot{f_{,X}} \dot{\phi}) + 3\frac{\dot{a}}{a}(a^3 f_{,X} \dot{\phi}),$$

leads us to

$$\frac{1}{a^3}(a^3 \dot{f_{,X}} \dot{\phi}) - f_{,X} = 0. \tag{2.55}$$

At this stage, we think that the scalar field behaves like dark energy and assume that it satisfies the following Friedmann equations [16]:

$$3H^2 = \kappa^2 (\rho_m + \rho_{DE}), \tag{2.56}$$

$$-2\dot{H} = \kappa^2 (\rho_m + p_m + \rho_{DE} + p_{DE}). \tag{2.57}$$

Now let us compare these with Friedmann equations (2.47) and (2.52) to define dark energy density and pressure. Substituting (2.56) into (2.47) gives us

$$\begin{aligned} 3FH^2 &= f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} + \left(3H^2 - \kappa^2 \rho_{DE}\right), \\ \kappa^2 \rho_{DE} &= f_{,X}X + \frac{1}{2}(FR - f) - 3H\dot{F} + 3H^2(1 - F). \end{aligned} \tag{2.58}$$

Applying (2.57) into (2.52)

$$-2F\dot{H} = \ddot{F} - H\dot{F} + f_{,X}X + \left(-2\dot{H} - \kappa^2 (\rho_{DE} + p_{DE})\right),$$

and using the equation (2.58), gives the following results

$$\begin{aligned} \kappa^2 p_{DE} &= \ddot{F} - H\dot{F} + f_{,X}X - 2\dot{H}(1 - F) - f_{,X}X - \frac{1}{2}(FR - f) + 3H\dot{F} - 3H^2(1 - F), \\ \kappa^2 p_{DE} &= \ddot{F} + 2H\dot{F} - \frac{1}{2}(FR - f) - \left(2\dot{H} + 3H^2\right)(1 - F). \end{aligned} \tag{2.59}$$

Now, we are going to see if these effective dark energy components satisfy the usual conservation equation. The covariant generalization of the conservation equation of a perfect fluid [18]

$$T^{\mu\nu}_{;\mu} = T^{\mu\nu}_{,\mu} + \Gamma^{\mu}_{\beta\mu} T^{\beta\nu} + \Gamma^{\nu}_{\beta\mu} T^{\mu\beta} = 0. \tag{2.60}$$

When $\nu = 0$

$$\begin{aligned} \nabla_\mu T^{\mu 0} =& T^{\mu 0}_{,\mu} + \Gamma^{\mu}_{\beta\mu} T^{\beta 0} + \Gamma^{0}_{\beta\mu} T^{\mu\beta}, \\ \nabla_0 T^{00} + \nabla_i T^{i0} =& T^{00}_{,0} + T^{i0}_{,i} + \Gamma^{0}_{00} T^{00} + \Gamma^{i}_{0i} T^{00} + \Gamma^{0}_{00} T^{00} + \Gamma^{0}_{0i} T^{i0} \\ &+ \Gamma^{0}_{i0} T^{0i} + \Gamma^{0}_{ii} T^{ii}. \end{aligned}$$

Using the non-zero components of the Christoffel symbols for FLRW metric (2.36) and energy-momentum tensor (2.42) for matter $T^{\mu\nu}_{(m)} = diag\left(\rho_m,\ p_m\, g^{ii}\right)$, the equation becomes

$$\begin{aligned}
\nabla_0 T^{00}_{(m)} &= \partial_0 \rho_m + 3H\rho_m + \Gamma^0_{11} p_m\, g^{11} + \Gamma^0_{22} p_m\, g^{22} + \Gamma^0_{33} p_m\, g^{33}, \\
\nabla_0 T^{00}_{(m)} &= \dot{\rho_m} + 3H\rho_m + a\dot{a}\,\frac{p_m}{a^2} + a\dot{a}r^2\,\frac{p_m}{a^2 r^2} + a\dot{a}r^2 sin^2\theta \frac{p_m}{a^2 r^2 sin^2\theta}, \\
\nabla_0 T^{00}_{(m)} &= \dot{\rho_m} + 3H\rho_m + 3p_m\frac{\dot{a}}{a} = 0, \\
&\dot{\rho_m} + 3H\left(\rho_m + p_m\right) = 0
\end{aligned} \tag{2.61}$$

Turning back to the equations (2.56) and (2.57) to obtain a similar result:

$$\begin{aligned}
\kappa^2\left(\rho_m + \rho_{DE}\right) = 3\,\frac{\dot{a}^2}{a^2} \quad \rightarrow \quad \kappa^2\left(\dot{\rho_m} + \dot{\rho_{DE}}\right) &= 6\left(\frac{\dot{a}\ddot{a}}{a^2} - \frac{\dot{a}^3}{a^3}\right), \\
\kappa^2\left(\dot{\rho_m} + \dot{\rho_{DE}}\right) &= 6H\dot{H}, \\
\dot{H} &= \frac{\kappa^2}{6H}\left(\dot{\rho_m} + \dot{\rho_{DE}}\right).
\end{aligned} \tag{2.62}$$

Insterting (2.62) into (2.57)

$$\begin{aligned}
-\frac{\kappa^2}{3H}\left(\dot{\rho_m} + \dot{\rho_{DE}}\right) &= \kappa^2\left(\rho_m + p_m + \rho_{DE} + p_{DE}\right), \\
\dot{\rho_m} + 3H\left(\rho_m + p_m\right) + \dot{\rho_{DE}} + 3H\left(\rho_{DE} + p_{DE}\right) &= 0,
\end{aligned}$$

results in

$$\dot{\rho_{DE}} + 3H\left(\rho_{DE} + p_{DE}\right) = 0. \tag{2.63}$$

Finally, we will define the dark energy equation of state ω_{DE}. The general formula of the equation of state is [18]

$$\omega = \frac{p}{\rho}. \tag{2.64}$$

Rearranging the equation (2.56)

$$\kappa^2\rho_{DE} = 3H^2 - \kappa^2\rho_m, \tag{2.65}$$

and (2.57)

$$\begin{aligned}
-2\dot{H} &= \kappa^2\left(p_m + p_{DE}\right) + 3H^2, \\
\kappa^2 p_{DE} &= -\left(\kappa^2 p_m + 3H^2 + 2\dot{H}\right).
\end{aligned} \tag{2.66}$$

The ratio of these two equations, (2.66) and (2.65), gives us

$$\omega_{DE} = \frac{p_{DE}}{\rho_{DE}} \quad \rightarrow \quad \omega_{DE} = -\frac{2\dot{H} + 3H^2 + \kappa^2 p_m}{3H^2 - \kappa^2 \rho_m}. \tag{2.67}$$

Since $p_m = \omega\rho_m$ from (2.64), we can also express this equation as

$$\omega_{DE} = -\frac{2\dot{H} + 3H^2 + \kappa^2 \omega\rho_m}{3H^2 - \kappa^2 \rho_m}. \tag{2.68}$$

Density parameter is [18]

$$\Omega = \frac{\rho}{\rho_c}, \tag{2.69}$$

where ρ_c is critical density and

$$\rho_c = \frac{3H^2}{\kappa^2}. \tag{2.70}$$

Now let us define [16]

$$\tilde{\Omega}_m = F\Omega_m = \frac{\kappa^2\rho_m}{3H^2}, \qquad \omega_{eff} = -\left(1 + \frac{2\dot{H}}{3H^2} + \omega\,\tilde{\Omega}_m\right), \tag{2.71}$$

therefore

$$\omega_{DE} = -\frac{\frac{2\dot{H}}{3H^2} + 1 + \omega\frac{\kappa^2\rho_m}{3H^2}}{1 - \frac{\kappa^2\rho_m}{3H^2}} \quad \rightarrow \quad \omega_{DE} = \frac{\omega_{eff}}{1 - \tilde{\Omega}_m}. \tag{2.72}$$

3. LATE-TIME COSMIC ACCELERATION EXAMPLES

The aim of this chapter is to consider some specific forms of the action (2.1) and then reconstruct some well-known cosmological solutions which are capable of reproducing the late-time cosmic acceleration, such as de-Sitter [20,21], power-law [12,22,23] and exact ΛCDM solutions.

3.1 Brans-Dicke Type Models

In 1961, C. Brans and R.H. Dicke introduced a non-minimally coupled scalar-tensor theory with the intent of incorporating Mach's principle into General Relativity [24]. In recent years, Brans-Dicke theories have regained interest due to their cosmological implications, providing an alternative to the existence of a dark energy source [25]. In this section, we choose $f(R,X,\phi)=\gamma(X,\phi)R$ to examine a Brans-Dicke type action

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\gamma(X,\phi)R+L_m\right]. \tag{3.1}$$

Here, the function $\gamma(X,\phi)$ is coupled to the gravitational field through the Ricci scalar R. So the kinetic term is also part of the coupling to the Ricci curvature. First, we vary the action with respect to metric tensor

$$\begin{aligned}\delta S=\int d^4x\Bigg[&\frac{\gamma(X,\phi)R}{2\kappa^2}\delta\sqrt{-g}+\frac{\sqrt{-g}R}{2\kappa^2}\frac{\partial\gamma(X,\phi)}{\partial X}\delta X+\frac{\gamma(X,\phi)\sqrt{-g}}{2\kappa^2}\delta R\\&+\delta\left(\sqrt{-g}L_m\right)\Bigg].\end{aligned} \tag{3.2}$$

Substituting the equations (2.8), (2.17) and (2.18) into the equation (3.2), one can see that

$$\begin{aligned}\delta S=\int d^4x\Bigg[&\frac{\gamma R}{2\kappa^2}\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)+\frac{\sqrt{-g}R}{2\kappa^2}\gamma_{,X}\left(-\frac{1}{2}\delta g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)\\&+\frac{\sqrt{-g}\gamma}{2\kappa^2}\left(\left(R_{\mu\nu}+g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu}\right)+\delta\left(\sqrt{-g}L_m\right)\Bigg].\end{aligned}$$

Here, using integration by parts, like we did for equations (2.21) and (2.22)

$$\gamma g_{\mu\nu}\Box\delta g^{\mu\nu}=\delta g^{\mu\nu}g_{\mu\nu}\Box\gamma,$$
$$\gamma\nabla_\mu\nabla_\nu\delta g^{\mu\nu}=\delta g^{\mu\nu}\nabla_\mu\nabla_\nu\gamma,$$

we obtain the following solution

$$\delta S = \int d^4x \frac{\sqrt{-g}}{2} \delta g^{\mu\nu} \left[\frac{1}{\kappa^2} \left(-\frac{\gamma R}{2} g_{\mu\nu} - \frac{R}{2} \gamma_{,X} \phi_{,\mu} \phi_{,\nu} + \gamma R_{\mu\nu} + g_{\mu\nu} \Box\gamma - \nabla_\mu \nabla_\nu \gamma \right) + \frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g} L_m\right)}{\delta g^{\mu\nu}} \right].$$

Since $\delta S = 0$ and the definitions of $T_{\mu\nu}$ and $G_{\mu\nu}$ are given as (2.24) and (2.25), we obtain the field equations as follows

$$\begin{aligned} -\frac{\gamma R}{2} g_{\mu\nu} - \frac{R}{2} \gamma_{,X} \phi_{,\mu} \phi_{,\nu} - \gamma G_{\mu\nu} + \frac{\gamma R}{2} g_{\mu\nu} + g_{\mu\nu} \Box\gamma - \nabla_\mu \nabla_\nu \gamma &= \kappa^2 T^{(m)}_{\mu\nu}, \\ \gamma G_{\mu\nu} = \frac{R}{2} \gamma_{,X} \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \Box\gamma + \nabla_\mu \nabla_\nu \gamma + \kappa^2 T^{(m)}_{\mu\nu}. \end{aligned} \tag{3.3}$$

The first Friedmann equation is derived from the 00 components of the field equations (3.3)

$$\gamma \left(R_{00} - \frac{1}{2} g_{00} R \right) = \frac{R}{2} \gamma_{,X} \phi_{,0} \phi_{,0} - g_{00} \nabla_0 \nabla^0 \gamma + \nabla_0 \nabla_0 \gamma + \kappa^2 T^{(m)}_{00}$$

Similarly to (2.43), one can see that

$$\begin{aligned} \Box\gamma &= \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \gamma \right), \\ \nabla_0 \nabla^0 \gamma &= -\ddot{\gamma} - 3H\dot{\gamma}. \end{aligned} \tag{3.4}$$

Here, using the equations (2.34), (2.37), (2.38) and (2.42), the first Friedmann equation is obtained as follows

$$\begin{aligned} \gamma \left(-3 \frac{\ddot{a}}{a} + \frac{R}{2} \right) &= \frac{R}{2} \gamma_{,X} \dot{\phi}^2 + \ddot{\gamma} - 3H\dot{\gamma} - \ddot{\gamma} + \kappa^2 \rho_m, \\ -3\gamma\dot{H} - 3\gamma H^2 + \frac{\gamma R}{2} &= \gamma_{,X} XR - 3H\dot{\gamma} + \kappa^2 \rho_m, \\ -3\gamma\dot{H} - 3\gamma H^2 + \frac{\gamma}{2} \left(6\dot{H} + 12H^2 \right) &= \gamma_{,X} XR - 3H\dot{\gamma} + \kappa^2 \rho_m, \\ 3\gamma H^2 &= \gamma_{,X} XR - 3H\dot{\gamma} + \kappa^2 \rho_m. \end{aligned} \tag{3.5}$$

Now for the second Friedmann equation, we are going to multiply the field equations (3.3) with $g^{\mu\nu}$ to find its trace

$$\begin{aligned} \gamma g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) &= \frac{R}{2} \gamma_{,X} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - g^{\mu\nu} g_{\mu\nu} \Box\gamma + g^{\mu\nu} \nabla_\mu \nabla_\nu \gamma + \kappa^2 g^{\mu\nu} T_{\mu\nu}, \\ 2\gamma R &= \gamma R + 3\Box\gamma - \frac{R}{2} \gamma_{,X} \phi^{,\nu} \phi_{,\nu} - \kappa^2 T. \end{aligned} \tag{3.6}$$

If we rearrange this equation

$$\frac{\gamma R}{2}g_{\mu\nu} = \frac{g_{\mu\nu}}{4}\left(\gamma R + 3\Box\gamma - \frac{R}{2}\gamma_{,X}\phi^{,\nu}\phi_{,\nu} - \kappa^2 T\right),$$

and plug it into field equations (3.3), we get

$$\gamma R_{\mu\nu} - \frac{g_{\mu\nu}}{4}\left(\gamma R + 3\Box\gamma - \frac{R}{2}\gamma_{,X}\phi^{,\nu}\phi_{,\nu} - \kappa^2 T\right) = \frac{R}{2}\gamma_{,X}\phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\Box\gamma + \nabla_\mu\nabla_\nu\gamma + \kappa^2 g^{\mu\nu}T^{(m)}_{\mu\nu}.$$

Considering the 00 components of this equation

$$-3\gamma\dot{H} - 3\gamma H^2 + \frac{1}{4}\left(\gamma\left(6\dot{H} + 12H^2\right) - 3\ddot{\gamma} - 9H\dot{\gamma} - \gamma_{,X}XR - \kappa^2 T\right) = \gamma_{,X}XR - \ddot{\gamma} - 3H\dot{\gamma} + \ddot{\gamma} + \kappa^2\rho_m,$$

$$-2\gamma\dot{H} - \ddot{\gamma} - 3H\dot{\gamma} - \frac{XR}{3}\gamma_{,X} - \frac{\kappa^2}{3}T = \frac{4XR}{3}\gamma_{,X} - 4H\dot{\gamma} + \frac{4\kappa^2}{3}\rho_m,$$

the second Friedmann equation is obtained

$$-2\gamma\dot{H} = \gamma_{,X}XR + \ddot{\gamma} - H\dot{\gamma} + \kappa^2\left(\rho_m + p_m\right). \tag{3.7}$$

The sum of the first (3.5) and second (3.7) Friedmann equations gives us

$$-\gamma\dot{H} - 3\gamma H^2 = \ddot{\gamma} + 2H\dot{\gamma} + \kappa^2 p_m.$$

Here, taking into account that pressure of the matter (dust) is zero [18]:

$$p_m = 0, \tag{3.8}$$

and rearranging the equation, results in

$$\frac{d^2\gamma(t)}{dt^2} + 2H\frac{d\gamma(t)}{dt} + \left(3H^2 + 2\dot{H}\right)\gamma(t) = 0. \tag{3.9}$$

This equation can be solved by introducing a designated Hubble parameter.

Reorganizing the first Friedmann equation (3.5) gives us

$$\gamma_{,X}X = \frac{3\gamma H^2 + 3H\dot{\gamma} - \kappa^2\rho_m}{R}. \tag{3.10}$$

Action (3.1) can be altered with a specific form of $\gamma(X,\phi)$, which has to satisfy (3.10) [16]. As an example to this alteration, let us consider the standard form of a scalar field Lagrangian

$$\gamma(X,\phi) = X - V(\phi). \tag{3.11}$$

This turns the equation (3.10) into

$$X = \frac{3\gamma H^2 + 3H\dot{\gamma} - \kappa^2\rho_m}{R}. \tag{3.12}$$

Using the continuity equation (2.61) and $p_m = 0$ we get

$$\frac{d\rho_m}{\rho_m} = -3Hdt \quad \rightarrow \quad \rho_m = e^{-3\int Hdt + C}.$$

Redefining $\phi = t$ and setting the constant term to $e^c = \rho_0$, we get

$$\rho_m = \rho_0 e^{-3\int d\phi H}. \tag{3.13}$$

Substituting this result into the equations (3.11) and (3.12), we can rewrite the kinetic term and the scalar potential as

$$\begin{aligned} X(\phi) &= \frac{3\gamma(\phi)H^2 + 3H\dot{\gamma}(\phi) - \kappa^2\rho_0 e^{-3\int d\phi H}}{R}, \\ V(\phi) &= X(\phi) - \gamma(\phi). \end{aligned} \tag{3.14}$$

Now we will consider several cosmological solutions.

3.1.1 de-Sitter reconstruction

In 1917, de Sitter found a solution to the modified Einstein equations that included the cosmological constant Λ [26]. Initially, this solution portrayed a static model describing a space with zero matter density (vacuum). However, it was later understood that this model could also represent the currently observed expansion of the universe [27].

In summary, inflation and late-time cosmic acceleration can be described as de Sitter solutions with perturbations, playing a crucial role in cosmological models. The solution is described by an exponential expansion of the scale factor $a(t) = a_0 e^{H_0 t}$, which gives a constant Hubble parameter [20,28,29]

$$H = H_0 = const. \tag{3.15}$$

So the equation (3.9) turns into

$$\frac{d^2\gamma(t)}{dt^2} + 2H_0\frac{d\gamma(t)}{dt} + 3H_0^2\gamma(t) = 0. \tag{3.16}$$

To solve this differantial equation, let us take $\gamma(t) = e^{\alpha t}$ where α is constant

$$\gamma(t) = e^{\alpha t} \;\; \rightarrow \;\; \dot{\gamma}(t) = \alpha e^{\alpha t} \;\; \rightarrow \;\; \ddot{\gamma}(t) = \alpha^2 e^{\alpha t}. \tag{3.17}$$

With these transformations, (3.16) turns into

$$\alpha^2 e^{\alpha t} + 2H_0\alpha e^{\alpha t} + 3H_0^2 e^{\alpha t} = 0,$$
$$\alpha^2 + 2H_0 + 3H_0^2 = 0.$$

Using the quadratic formula, we get two complex conjugate roots

$$\alpha_1 = -H_0 + i\sqrt{2}H_0, \quad \alpha_2 = -H_0 - i\sqrt{2}H_0 \tag{3.18}$$

therefore, the general solution of the differential equation takes the form

$$\gamma(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}, \tag{3.19}$$

$$\gamma(t) = e^{-H_0 t}\underbrace{\left(C_1 e^{i\sqrt{2}H_0 t} + C_2 e^{-i\sqrt{2}H_0 t}\right)}_{\chi(t)}. \tag{3.20}$$

Now, defining

$$\chi(t) = \left(C_1 e^{i\sqrt{2}H_0 t} + C_2 e^{-i\sqrt{2}H_0 t}\right), \tag{3.21}$$

$$C_1 = \lambda_1 e^{i\beta_1}, \quad C_2 = \lambda_2 e^{i\beta_2}, \tag{3.22}$$

$$\chi(t) = \lambda_1 e^{i\left(\beta_1 + \sqrt{2}H_0 t\right)} + \lambda_2 e^{i\left(\beta_2 - \sqrt{2}H_0 t\right)},$$

and using Euler's formula $e^{i\theta} = cos\theta + isin\theta$, one can see that

$$\chi(t) = \lambda_1 cos\left(\beta_1 + \sqrt{2}H_0 t\right) + i\lambda_1 sin\left(\beta_1 + \sqrt{2}H_0 t\right)$$
$$+ \lambda_2 cos\left(\beta_2 - \sqrt{2}H_0 t\right) + i\lambda_2 sin\left(\beta_2 - \sqrt{2}H_0 t\right).$$

To get rid of the imaginary components, we choose

$$\lambda_1 sin\left(\beta_1 + \sqrt{2}H_0 t\right) + \lambda_2 sin\left(\beta_2 - \sqrt{2}H_0 t\right) = 0,$$
$$\lambda_1 sin\left(\beta_1 + \sqrt{2}H_0 t\right) = \lambda_2 sin\left(\sqrt{2}H_0 t - \beta_2\right).$$

Hence $\lambda_1 = \lambda_2$ and $\beta_1 = -\beta_2$, and the equation turns into

$$\chi(t) = \lambda_1 cos\left(\beta_1 + \sqrt{2}H_0 t\right) + \lambda_1 cos\left(-\beta_1 - \sqrt{2}H_0 t\right),$$
$$\chi(t) = 2\lambda_1 cos\left(\beta_1 + \sqrt{2}H_0 t\right).$$

Here, setting the constants as $C = 2\lambda_1$ and $\beta_1 = -\sqrt{2}H_0 t_0$, leads us to

$$\chi(t) = C\ cos\left[\sqrt{2}H_0\left(t - t_0\right)\right]. \tag{3.23}$$

Substituting (3.23) into (3.20), we obtain

$$\gamma(t) = e^{-H_0 t} C\ cos\left[\sqrt{2}H_0\left(t - t_0\right)\right]. \tag{3.24}$$

To find the kinetic term and the scalar potential (3.14), first we need to differentiate $\gamma(t)$ with respect to time. Then set $\phi = t$, therefore

$$\dot{\gamma}(t) = -H_0 e^{-H_0 t} C\ cos\left[\sqrt{2}H_0\left(t - t_0\right)\right] - e^{-H_0 t} C\sqrt{2}H_0\ sin\left[\sqrt{2}H_0\left(t - t_0\right)\right],$$
$$X(\phi) = \frac{3\gamma(\phi)H_0^2 + 3\dot{\gamma}(\phi)H_0 - \kappa^2\rho_0 e^{-3\int d\phi H_0}}{R},$$
$$X(\phi) = \frac{Ce^{-H_0\phi}\left[-3\sqrt{2}H_0^2 sin\left(\sqrt{2}H_0\left(\phi - \phi_0\right)\right)\right] - \kappa^2\rho_0 e^{-3H_0\phi}}{R}.$$

For Hubble parameter $H = H_0$, the Ricci scalar is $R = 12H_0^2$, gives us the following results

$$X(\phi) = \frac{-3\sqrt{2}H_0^2 Ce^{-H_0\phi} sin\left[\sqrt{2}H_0\left(\phi - \phi_0\right)\right]}{12H_0^2} - \frac{\kappa^2\rho_0 e^{-3H_0\phi}}{12H_0^2},$$
$$X(\phi) = -\frac{C}{2\sqrt{2}}\ e^{-H_0\phi}\ sin\left[\sqrt{2}H_0\left(\phi - \phi_0\right)\right] - \frac{\kappa^2\rho_0}{12H_0^2}\ e^{-3H_0\phi}. \tag{3.25}$$

$$\begin{aligned} V(\phi) &= X(\phi) - \gamma(\phi), \\ V(\phi) &= -C\ e^{-H_0\phi}\left[\frac{1}{2\sqrt{2}}\ sin\left(\sqrt{2}H_0\left(\phi - \phi_0\right)\right) + cos\left(\sqrt{2}H_0\left(\phi - \phi_0\right)\right)\right], \\ &\quad - \frac{\kappa^2\rho_0}{12H_0^2}\ e^{-3H_0\phi} \\ V(\phi) &= -\frac{C}{4}\ e^{-H_0\phi}\left[\sqrt{2}\ sin\left(\sqrt{2}H_0\left(\phi - \phi_0\right)\right) + 4cos\left(\sqrt{2}H_0\left(\phi - \phi_0\right)\right)\right] \\ &\quad - \frac{\kappa^2\rho_0}{12H_0^2}\ e^{-3H_0\phi}. \end{aligned} \tag{3.26}$$

Unlike the standard General Relativity, this action results in a constant Hubble parameter in the presence of matter (dust) and scalar field, but without a cosmological constant.

3.1.2 Power-law reconstruction

Power law solutions hold great importance in cosmic history, as radiation-dominated, matter-dominated, and dark energy eras can be described by power law expansions. Additionally, power-law solutions can be applied to certain special cases, including the possible phantom epoch [12,29].

Now let us consider a power-law solution where the scale factor has a power-law dependence on time

$$a(t) = a_0\, t^n. \tag{3.27}$$

We can find the Hubble parameter and its derivative with respect to time

$$\dot{a}(t) = a_0\, n\, t^{n-1}, \qquad \ddot{a}(t) = a_0\, n(n-1)\, t^{n-2}$$

$$H = \frac{\dot{a}}{a} \;\rightarrow\; H = \frac{n}{t}, \tag{3.28}$$

$$\dot{H} = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \;\rightarrow\; \dot{H} = -\frac{n}{t^2}. \tag{3.29}$$

When we plug these into the equation (3.9), we get

$$\frac{d^2\gamma(t)}{dt^2} + 2\,\frac{n}{t}\,\frac{d\gamma(t)}{dt} + \left(3\,\frac{n^2}{t^2} - 2\,\frac{n}{t^2}\right)\gamma(t) = 0,$$

$$t^2\,\frac{d^2\gamma(t)}{dt^2} + 2nt\,\frac{d\gamma(t)}{dt} + n(3n-2)\gamma(t) = 0, \tag{3.30}$$

which is in the form of Cauchy-Euler equation. To solve this, we assume [30]

$$\gamma(t) = t^m \;\rightarrow\; \dot{\gamma}(t) = m\, t^{m-1} \;\rightarrow\; \ddot{\gamma}(t) = m(m-1)\, t^{m-2}.$$

Therefore

$$t^2\, m(m-1)\, t^{m-2} + 2nt\, m\, t^{m-1} + n(3n-2)t^m = 0,$$

$$m^2 + (2n-1)m + n(3n-2) = 0.$$

Solving the quadratic equation, we get two distinct roots

$$m_1 = \frac{1-2n+\sqrt{1+4n-8n^2}}{2}, \quad m_2 = \frac{1-2n-\sqrt{1+4n-8n^2}}{2}.$$

If we define

$$p = \frac{1-2n}{2}, \quad q = \frac{\sqrt{1+4n-8n^2}}{2} \tag{3.31}$$

then $m_1 = p+q$ and $m_2 = p-q$. Therefore our solution to (3.30) is

$$\begin{aligned}
\gamma(t) &= C_1\, t^{m_1} + C_2\, t^{m_2}, \\
\gamma(t) &= C_1\, t^{p+q} + C_2\, t^{p-q}, \\
\dot{\gamma}(t) &= C_1(p+q)\, t^{p+q-1} + C_2(p-q)\, t^{p-q-1}.
\end{aligned} \tag{3.32}$$

Redefining the scalar field to coincide with the cosmic time $\phi = t$, makes the Hubble parameter $H = n\phi^{-1}$ and the Ricci scalar $R = 6n(2n-1)\phi^{-2}$. Hence, the equation (3.14) transforms into

$$\begin{aligned}
X(\phi) =& \frac{3n\phi^{-2}\left[C_1 n\phi^{p+q} + C_2 n\phi^{p-q} + C_1(p+q)\phi^{p+q} + C_2(p-q)\phi^{p-q}\right]}{6n(2n-1)\phi^{-2}} \\
& - \frac{\kappa^2\rho_0 e^{-3\int \frac{n}{\phi} d\phi}}{6n(2n-1)\phi^{-2}},
\end{aligned}$$

$$X(\phi) = \frac{C_1(n+p+q)\phi^{p+q} + C_2(n+p-q)\phi^{p-q}}{2(2n-1)} - \frac{\kappa^2\rho_0\phi^2 e^{-3n\ ln\phi}}{6n(2n-1)}.$$

If we substitute p using (3.31) and notice that $e^{-3n\ ln\phi} = \phi^{-3n}$, we obtain the kinetic term and the scalar potential (3.14) as follows

$$X(\phi) = \frac{C_1(1+2q)\phi^{p+q} + C_2(1-2q)\phi^{p-q}}{4(2n-1)} - \frac{\kappa^2\rho_0\phi^{2-3n}}{6n(2n-1)},$$

$$X(\phi) = \frac{1}{4(2n-1)}\left[C_1(1+2q)\phi^{p+q} + C_2(1-2q)\phi^{p-q} - \frac{1}{3n}\, 2\kappa^2\rho_0\phi^{2-3n}\right]. \tag{3.33}$$

$$V(\phi) = \frac{C_1(1+2q)\phi^{p+q} + C_2(1-2q)\phi^{p-q}}{4(2n-1)} - \frac{\kappa^2\rho_0\phi^{2-3n}}{6n(2n-1)} - C_1\, t^{p+q} + C_2\, t^{p-q},$$

$$V(\phi) = \frac{1}{4(2n-1)}\left[C_1(5-8n+2q)\phi^{p+q} + C_2(5-8n-2q)\phi^{p-q} - \frac{1}{3n}\, 2\kappa^2\rho_0\phi^{2-3n}\right]. \tag{3.34}$$

Thus, we have obtained the gravitational action that can regenerate power-law solutions.

3.1.3 ΛCDM reconstruction

At first, we are going to redefine the matter (dust) density ρ_m in terms of the scale factor $a(t)$. Using the continuity equation (2.61) and the fact that matter pressure $p_m = 0$ (3.8), we get

$$\dot{\rho_m} = -3H\rho_m \;\; \rightarrow \;\; \frac{1}{\rho_m}\frac{d\rho_m}{dt} = -\frac{3}{a}\frac{da}{dt},$$
$$ln\rho_m = lna^{-3} + C.$$

If we define the contribution from the integration constant as $\rho_{m,0}$, we find the following relation

$$\rho_m = \rho_{m,0}\, a^{-3}. \tag{3.35}$$

Using the definitions of density parameter (2.69) and critical density (2.70), the matter density for today can be expressed as follows

$$\rho_{m,0} = \rho_{c,0}\, \Omega_{m,0}, \tag{3.36}$$

where

$$\rho_{c,0} = \frac{3H_0^2}{\kappa^2}. \tag{3.37}$$

Therefore the equation (3.35) turns into

$$\rho_m = \rho_{c,0}\, \Omega_{m,0}\, a^{-3}. \tag{3.38}$$

The continuity equation for cosmological constant Λ is

$$\dot{\rho_\Lambda} + 3H(\rho_\Lambda + p_\Lambda) = 0. \tag{3.39}$$

The relation between the density and the pressure of the cosmological constant is [18]

$$\rho_\Lambda = -p_\Lambda. \tag{3.40}$$

Using this relation and the continuity equation (3.39), one can see that

$$\frac{d\rho_\Lambda}{dt} = 0 \;\; \rightarrow \;\; \rho_\Lambda = \rho_{\Lambda,0}, \tag{3.41}$$

therefore

$$\rho_\Lambda = \rho_{c,0}\, \Omega_{\Lambda,0}. \tag{3.42}$$

Substituting the equations (3.42) and (3.38) into the first Friedmann equation (2.56)

$$3H^2 = \kappa^2 \left(\rho_{c,0}\, \Omega_{m,0}\, a^{-3} + \rho_{c,0}\, \Omega_{\Lambda,0}\right),$$

and using (3.37), we obtain

$$H^2 = H_0^2 \left(\Omega_{m,0}\, a^{-3} + \Omega_{\Lambda,0}\right). \tag{3.43}$$

The continuity equation is valid at all times, so it should also apply at the present time t_0

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)}, \tag{3.44}$$

$$3H_0^2 = \kappa^2(\rho_{m,0} + \rho_{\Lambda,0}).$$

$$1 = \frac{\kappa^2}{3H_0^2}\,(\rho_{m,0} + \rho_{\Lambda,0}) \;\rightarrow\; 1 = \frac{\rho_{m,0} + \rho_{\Lambda,0}}{\rho_{c,0}},$$

$$\Omega_{m,0} + \Omega_{\Lambda,0} = 1. \tag{3.45}$$

The redshift z is defined as

$$z = a^{-1} + 1. \tag{3.46}$$

Using (3.46) and (3.45) and plugging it to (3.43); we obtain the Hubble parameter in terms of redshift, which is used to express the ΛCDM model

$$H(z) = H_0\sqrt{\Omega_{m,0}(1+z)^3 + 1 - \Omega_{m,0}}. \tag{3.47}$$

Now we are going to reformulate the equation (3.9) in terms of redshift. First, let us see how to convert time derivatives to redshift derivatives. Using the equation (3.46), we get following relations

$$\frac{dz}{dt} = -a^{-2}\,\dot{a} = -(1+z)\,H, \tag{3.48}$$

$$\frac{d}{dt} = \frac{dz}{dt}\,\frac{d}{dz} = -(1+z)\,H\,\frac{d}{dz}. \tag{3.49}$$

To express the derivative with respect to redshift, using $\frac{d}{dz} = {}'$

$$\frac{d\gamma}{dt} = -(1+z)H\gamma', \tag{3.50}$$

$$\frac{d^2\gamma}{dt^2} = \frac{dz}{dt}\left(-(1+z)H\gamma'\right)',$$

$$\frac{d^2\gamma}{dt^2} = (1+z)H^2\gamma' + (1+z)^2HH'\gamma' + (1+z)^2H^2\gamma''. \tag{3.51}$$

Finally, for Hubble parameter, using (3.47)

$$H' = \frac{3}{2}\frac{H_0^2}{H}\,\Omega_{m,0}(1+z)^2, \tag{3.52}$$

$$\dot{H} = \frac{3}{2}\frac{H_0^2}{H}\,\Omega_{m,0}(1+z)^2\,\frac{dz}{dt},$$

$$\dot{H} = -(1+z)HH'. \tag{3.53}$$

When we plug (3.50), (3.51) and (3.53) into the equation (3.9), we get

$$(1+z)^2H^2\gamma'' + \left[(1+z)^2HH' - (1+z)H^2\right]\gamma' + \left[3H^2 - 2(1+z)HH'\right]\gamma = 0. \tag{3.54}$$

Now let us solve this equation as $\gamma(X,\phi) = \gamma(z)$ for ΛCDM model using (3.47) and (3.52), as follows

$$H_0^2(1+z)^2\left[\Omega_{m,0}(1+z)^3 + 1 - \Omega_{m,0}\right]\gamma''$$
$$+ H_0^2(1+z)\left[\frac{3}{2}\Omega_{m,0}(1+z)^3 - \Omega_{m,0}(1+z)^3 - 1 + \Omega_{m,0}\right]\gamma'$$
$$+ 3H_0^2\left[\Omega_{m,0}(1+z)^3 + 1 - \Omega_{m,0} - \Omega_{m,0}(1+z)^3\right]\gamma = 0,$$

$$2(1+z)^2\left[\Omega_{m,0}(1+3z+3z^2+z^3) + 1 - \Omega_{m,0}\right]\gamma''$$
$$+ (1+z)\left[\Omega_{m,0}(1+3z+3z^2+z^3) - 2 + 2\Omega_{m,0}\right]\gamma'$$
$$+ 6\left[\Omega_{m,0}(3z+3z^2+z^3) + 1 - \Omega_{m,0}(1+3z+3z^2+z^3)\right]\gamma = 0,$$

$$2(1+z)^2\left[\Omega_{m,0}\,z\,(3+3z+z^2) + 1\right]\gamma(z)'' + (1+z)\left[\Omega_{m,0}(3+3z+3z^2+z^3) - 2\right]\gamma(z)'$$
$$+ 6\left[1 - \Omega_{m,0}\right]\gamma(z) = 0. \tag{3.55}$$

This equation can't be solved exactly. But can be solved approximately, using numerical methods [16].

3.2 Minimally Coupled Field Model

Now let us consider another gravitational action where the scalar field is minimally coupled to the metric field. For this, we choose $f(R,X,\phi) = \alpha(R) + \gamma(X,\phi)$, resulting in an action of the form

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(\alpha(R) + \gamma(X,\phi)\right) + L_m\right]. \tag{3.56}$$

The variation of the action with respect to metric is found as follows

$$\begin{aligned}\delta S = \int d^4x \Bigg[\frac{1}{2\kappa^2}\Bigg(&\alpha(R)\ \delta\sqrt{-g} + \sqrt{-g}\ \frac{\partial\alpha(R)}{\partial R}\ \delta R + \gamma(X,\phi)\ \delta\sqrt{-g} \\ &+\sqrt{-g}\ \frac{\partial\gamma(X,\phi)}{\partial X}\ \delta X\Bigg) + \delta\left(\sqrt{-g}\ L_m\right)\Bigg].\end{aligned} \tag{3.57}$$

Applying equations (2.8), (2.17) and (2.18), the equation (3.57) turns into

$$\begin{aligned}\delta S = \int d^4x \Bigg[\frac{1}{2\kappa^2}\Bigg(&-\frac{\sqrt{-g}}{2}\ g_{\mu\nu}\ \delta g^{\mu\nu}\alpha + \sqrt{-g}\ \alpha_{,R}\left(R_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\delta g^{\mu\nu} \\ &-\frac{\sqrt{-g}}{2}\ g_{\mu\nu}\ \delta g^{\mu\nu}\gamma - \frac{\sqrt{-g}}{2}\ \gamma_{,X}\ \delta g^{\mu\nu}\ \partial_\mu\phi\partial_\nu\phi\Bigg) + \delta\left(\sqrt{-g}\ L_m\right)\Bigg].\end{aligned}$$

Here, using integration by parts and remembering that $\delta S = 0$, we obtain

$$\begin{aligned}\delta S = \int d^4x\ \delta g^{\mu\nu}\frac{\sqrt{-g}}{2}\Bigg[\frac{1}{2\kappa^2}\Big(&-g_{\mu\nu}\alpha + 2\alpha_{,R}R_{\mu\nu} + 2g_{\mu\nu}\Box\alpha_{,R} - 2\nabla_\mu\nabla_\nu\alpha_{,R} - g_{\mu\nu}\gamma \\ &-\gamma_{,X}\partial_\mu\phi\partial_\nu\phi\Big) + \frac{2}{\sqrt{-g}}\ \frac{\delta\left(\sqrt{-g}\ L_m\right)}{\delta g^{\mu\nu}}\Bigg] = 0.\end{aligned}$$

Substituting the definition of matter energy-momentum (2.24) and Einstein tensors (2.25) gives us the field equation

$$\alpha_{,R}\ G_{\mu\nu} = \frac{g_{\mu\nu}}{2}\left(\alpha - \alpha_{,R}R\right) - g_{\mu\nu}\Box\alpha_{,R} + \nabla_\mu\nabla_\nu\alpha_{,R} + \frac{g_{\mu\nu}}{2}\gamma + \frac{1}{2}\gamma_{,X}\partial_\mu\phi\partial_\nu\phi + \kappa^2 T_{\mu\nu}. \tag{3.58}$$

Taking into account a flat FLRW metric in spherical coordinates, let us find Friedmann equations. In the first instance, we shall consider the 00 components. Using the equations from (2.34) through (2.44), and substituting them into the field equation (3.58) gives us

$$\alpha_{,R}\left(R_{00} - \frac{1}{2}g_{00}R\right) = -\frac{1}{2}\left(\alpha - \alpha_{,R}R\right) + \nabla_0\nabla^0\alpha_{,R} + \ddot{\alpha}_{,R} - \frac{\gamma}{2} + \gamma_{,X}X + \kappa^2\rho_m.$$

Here

$$\nabla_0\nabla^0\alpha_{,R} = \frac{1}{\sqrt{-g}}\partial_0\left(\sqrt{-g}g^{00}\partial_0\alpha_{,R}\right) = -3H\dot{\alpha}_{,R} - \ddot{\alpha}_{,R}.$$

Thus, we obtain the first Friedmann equation

$$3H^2\alpha_{,R} = \gamma_{,X}X + \frac{1}{2}\left(\alpha_{,R}R - \alpha - \gamma\right) - 3H\dot{\alpha}_{,R} + \kappa^2\rho_m. \tag{3.59}$$

Multiplying the field equation (3.58) with $g^{\mu\nu}$ gives us its trace

$$\begin{aligned} g^{\mu\nu}\alpha_{,R}\,G_{\mu\nu} = {} & \frac{g^{\mu\nu}g_{\mu\nu}}{2}\left(\alpha - \alpha_{,R}R\right) - g^{\mu\nu}g_{\mu\nu}\Box\alpha_{,R} + g^{\mu\nu}\nabla_\mu\nabla_\nu\alpha_{,R} + \frac{g^{\mu\nu}g_{\mu\nu}}{2}\gamma \\ & + \frac{g^{\mu\nu}}{2}\gamma_{,X}\partial_\mu\phi\partial_\nu\phi + \kappa^2 g^{\mu\nu}T_{\mu\nu}. \end{aligned}$$

Using the equations (2.9), (2.48) and (2.49)

$$\alpha_{,R}\,g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 2\left(\alpha - \alpha_{,R}R + \gamma\right) - 3\Box\alpha_{,R} - \frac{1}{2}\gamma_{,X}\partial^\nu\phi\partial_\nu\phi + \kappa^2 T,$$

so the trace of the field equation reads

$$\alpha_{,R}R = 2(\alpha + \gamma) - 3\Box\alpha_{,R} - \frac{1}{2}\gamma_{,X}\partial^\nu\phi\partial_\nu\phi + \kappa^2 T. \tag{3.60}$$

Rearranging this equation as follows

$$\frac{g_{\mu\nu}}{2}(\alpha + \gamma) = \frac{g_{\mu\nu}}{4}\left(\alpha_{,R}R + 3\Box\alpha_{,R} + \frac{1}{2}\gamma_{,X}\partial^\nu\phi\partial_\nu\phi - \kappa^2 T\right),$$

and plugging it into field equations (3.58), gives us

$$\begin{aligned} \alpha_{,R}\,G_{\mu\nu} = {} & \frac{g_{\mu\nu}}{4}\left(\alpha_{,R}R + 3\Box\alpha_{,R} + \frac{1}{2}\gamma_{,X}\partial^\nu\phi\partial_\nu\phi - \kappa^2 T\right) - \frac{g_{\mu\nu}}{2}\alpha_{,R}R - g_{\mu\nu}\Box\alpha_{,R} \\ & + \nabla_\mu\nabla_\nu\alpha_{,R} + \frac{1}{2}\gamma_{,X}\partial_\mu\phi\partial_\nu\phi + \kappa^2 T_{\mu\nu}. \end{aligned}$$

Now we are going to look at the 00 components of this equation, in the same manner as we did above

$$\begin{aligned} 3\alpha_{,R}\left(\dot{H} + H^2\right) &= \frac{1}{4}\left(\alpha_{,R}R - 3\ddot{\alpha}_{,R} - 9H\dot{\alpha}_{,R} + \gamma_{,X}X - \kappa^2 T\right) + 3H\dot{\alpha}_{,R} - \gamma_{,X}X - \kappa^2\rho_m, \\ -4\alpha_{,R}\left(\dot{H} + H^2\right) &= -2\dot{H}\alpha_{,R} - 4H^2\alpha_{,R} + \ddot{\alpha}_{,R} + 3H\dot{\alpha}_{,R} - \frac{1}{3}\gamma_{,X}X + \frac{1}{3}\kappa^2 T - 4H\dot{\alpha}_{,R} \\ & \quad + \frac{4}{3}\gamma_{,X}X + \frac{4}{3}\kappa^2\rho_m, \\ -2\alpha_{,R}\dot{H} &= \gamma_{,X}X + \ddot{\alpha}_{,R} - H\dot{\alpha}_{,R} + \frac{\kappa^2}{3}(3p_m - \rho_m) + \frac{4\kappa^2}{3}\rho_m. \end{aligned}$$

The pressure of the dust matter $p_m = 0$, therefore our second Friedmann equation

$$-2\alpha_{,R}\dot{H} = \gamma_{,X}X + \ddot{\alpha}_{,R} - H\dot{\alpha}_{,R} + \kappa^2\rho_m. \tag{3.61}$$

When we subtract the second Friedmann equation (3.61) from the first Friedmann equation (3.59), we get

$$\gamma = -\alpha + \left(R - 6H^2 - 4\dot{H}\right)\alpha_{,R} - 4H\dot{\alpha}_{,R} - 2\ddot{\alpha}_{,R}. \tag{3.62}$$

This equation can be expressed differently using the following transformations

$$\dot{\alpha}_{,R} = \dot{R}\ \alpha_{,RR}, \tag{3.63}$$

$$\ddot{\alpha}_{,R} = \ddot{R}\ \alpha_{,RR} + \dot{R}^2\ \alpha_{,RRR}. \tag{3.64}$$

Therefore,

$$\gamma(t) = -\alpha + \left(R - 6H^2 - 4\dot{H}\right)\alpha_{,R} - \left(4H\dot{R} + 2\ddot{R}\right)\alpha_{,RR} - 2\dot{R}^2\alpha_{,RRR}. \tag{3.65}$$

This equation can be solved by determining $\alpha(R)$ and again, introducing a designated Hubble parameter.

Readjusting the first Friedmann equation gives us

$$\gamma_{,X}X = \frac{\alpha}{2} + \left(3H^2 - \frac{R}{2}\right)\alpha_{,R} + 3H\dot{\alpha}_{,R} - \kappa^2\rho_m + \frac{\gamma}{2}. \tag{3.66}$$

$\gamma(X,\phi)$ is obtained by considering a specific form of the scalar field Lagrangian which satisfies the equation (3.66).

3.2.1 Starobinsky inflation

Inflation can be regenerated solely in terms of gravity, without the need for additional fields like scalar fields. An example of this is Starobinsky's inflationary model.

In 1980, coinciding with Guth's development of his model, Alexei Starobinsky proposed his inflationary theory [31]. In his papers, Starobinsky emphasized the role of quantum corrections to General Relativity in the early universe. These corrections are generally quadratic terms of the curvature (R^2), incorporated into the Einstein-Hilbert action. The inclusion of these terms may lead to an effective cosmological constant,

resulting in an inflationary phase that replaces the initial singularity of the universe [28].

Nowadays, the inflationary scenario associated to an action including an R^2 term is commonly known as the Starobinsky inflation [32,33].

Now, with the aim of reconstructing some specific solutions, let us consider

$$\alpha(R) = R + aR^2, \tag{3.67}$$

where a is a constant with proper dimensions. Using the equation (3.67), we can turn the equation (3.65) into

$$\gamma = a\left(R^2 - 8H\dot{R} - 4\ddot{R}\right) - \left(6H^2 + 4\dot{H}\right)(1 + 2aR). \tag{3.68}$$

3.2.2 de-Sitter reconstruction

For de-Sitter reconstruction, we will define the Hubble parameter as the Hubble constant $H = H_0$, like we did in (3.15). So the Ricci scalar will be $R = 12H_0^2$ and then the equation (3.68) becomes

$$\gamma = -6H_0^2. \tag{3.69}$$

Considering the standard form of a scalar field Lagrangian (3.11), second Friedmann equation (3.61) turns into

$$X = -2\alpha_{,R}\dot{H} - \ddot{\alpha}_{,R} + H\dot{\alpha}_{,R} - \kappa^2\rho_m. \tag{3.70}$$

Using the equations (3.13) and (3.67); also redefining the Hubble constant as $H = H_0$ and the scalar field as $\phi = t$, we get

$$X(\phi) = -\kappa^2\rho_0\, e^{-3H_0\phi}. \tag{3.71}$$

Then choosing

$$X_0 = -\kappa^2\rho_0, \tag{3.72}$$

we find the kinetic term and the scalar potential as

$$\begin{aligned} X(\phi) &= X_0\, e^{-3H_0\phi}, \\ V(\phi) &= X_0\, e^{-3H_0\phi} + 6H_0^2. \end{aligned} \tag{3.73}$$

3.2.3 Power-law reconstruction

For the equation (3.68), when we put the Ricci scalar and its derivatives

$$R = 6\dot{H} + 12H^2 \;\rightarrow\; \dot{R} = 6\ddot{H} + 24H\dot{H} \;\rightarrow\; \ddot{R} = 6\dddot{H} + 24\dot{H}^2 + 24H\ddot{H}, \tag{3.74}$$

we obtain

$$\gamma = -108a\dot{H}^2 - 216a\dot{H}H^2 - 144aH\ddot{H} - 6H^2 - 4\dot{H} - 24a\dddot{H}. \tag{3.75}$$

When the Hubble parameter is chosen as $H(t) = \frac{n}{t}$ just like in (3.28), the following transformations can be derived

$$\begin{aligned} H = \frac{n}{t} \;&\rightarrow\; \dot{H} = -\frac{n}{t^2} \;\rightarrow\; \ddot{H} = 2\,\frac{n}{t^3} \;\rightarrow\; \dddot{H} = -6\,\frac{n}{t^4}, \\ H^2 = \frac{n^2}{t^2}, \quad \dot{H}^2 &= \frac{n^2}{t^4}, \quad \dot{H}H^2 = -\frac{n^3}{t^4}, \quad H\ddot{H} = 2\,\frac{n^2}{t^4}. \end{aligned} \tag{3.76}$$

Therefore, the equation (3.75) becomes

$$\begin{aligned} \gamma &= 144\,a\,\frac{n}{t^4} - 396\,a\,\frac{n^2}{t^4} + 216\,a\,\frac{n^3}{t^4} + 4\,\frac{n}{t^2} - 6\,\frac{n^2}{t^2}, \\ \gamma(t) &= 2n\,\frac{18a\left(4 - 11n + 6n^2\right) + (2-3n)t^2}{t^4}. \end{aligned} \tag{3.77}$$

Since

$$\begin{aligned} \alpha = R + aR^2, \quad \dot{\alpha}_{,R} &= 2a\dot{R} \\ \alpha_{,R} = 1 + 2aR, \quad \ddot{\alpha}_{,R} &= 2a\ddot{R} \end{aligned} \tag{3.78}$$

the equation (3.70) can be written as

$$X = -2\dot{H}(1 + 2aR) + 2a(H\dot{R} - \ddot{R}) - \kappa^2\rho_m. \tag{3.79}$$

Applying the derivatives in (3.74) we find

$$X = -2\dot{H} - 72a\dot{H}^2 - 12a\dddot{H} - 36aH\ddot{H} - \kappa^2\rho_m. \tag{3.80}$$

Then using the transformations in (3.76) and the equation (3.13) while redefining $\phi = t$, the equation (3.80) turns into

$$X = 2\,\frac{n}{\phi^2} + 72a\,\frac{n}{\phi^4} - 144a\,\frac{n^2}{\phi^4} - \kappa^2\rho_0\,e^{-3\int d\phi\,\frac{n}{\phi}}. \tag{3.81}$$

Finally, using the definition in (3.72) and the fact that $e^{-3\int d\phi\,\frac{n}{\phi}} = \phi^{-3n}$, we obtain the kinetic term and the scalar potential as

$$\begin{aligned} X(\phi) &= X_0\phi^{-3n} + 2n\,\frac{18a(2-4n) + \phi^2}{\phi^4}, \\ V(\phi) &= X_0\phi^{-3n} + 2n\,\frac{18a(-2 + 7n - 6n^2) + (-1+3n)\phi^2}{\phi^4}. \end{aligned} \tag{3.82}$$

3.2.4 ΛCDM reconstruction

Since $H = \frac{\dot{a}}{a}$, using the equations (3.43) and (3.45), one can easily see that

$$\int \frac{da}{\sqrt{(1-\Omega_{\Lambda,0})a^{-1} + \Omega_{\Lambda,0}a^2}} = \int_{t_0}^{t} H_0 dt. \tag{3.83}$$

With u-substitution, we find

$$arcsinh\left(a^{3/2}\sqrt{\frac{\Omega_{\Lambda,0}}{1-\Omega_{\Lambda,0}}}\right) = \frac{3\sqrt{\Omega_{\Lambda,0}}}{2} H_0(t-t_0), \tag{3.84}$$

and rearranging this result gives us the scale factor as

$$a = \left(\frac{1-\Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}\right)^{1/3} sinh^{2/3}\left(\frac{3\sqrt{\Omega_{\Lambda,0}}}{2} H_0(t-t_0)\right). \tag{3.85}$$

Density of the cosmological constant is [18]

$$\rho_\Lambda = \rho_{\Lambda,0} = \frac{\Lambda}{\kappa^2}. \tag{3.86}$$

Using the equation (3.37) and the definition of density parameter (2.69) we get

$$\Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2}. \tag{3.87}$$

Therefore the equation (3.85) turns into

$$a = \left(\frac{1-\Omega_{\Lambda,0}}{\Omega_{\Lambda,0}}\right)^{1/3} sinh^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right), \tag{3.88}$$

and the Hubble parameter can be written as

$$H(t) = \sqrt{\frac{\Lambda}{3}}\, coth\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right). \tag{3.89}$$

Through the application of the following derivatives

$$\begin{aligned}
\dot{H} &= -\frac{\Lambda}{2} csch^2\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right), \\
\ddot{H} &= \frac{\sqrt{3}}{2}\Lambda^{3/2} coth\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right) csch^2\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right), \\
\dddot{H} &= -\frac{3\Lambda^2}{2} csch^4\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right)\left[\frac{1}{2} + cosh^2\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right)\right],
\end{aligned} \tag{3.90}$$

to the equation (3.75) and using the fact that $coth^2(x) - csch^2(x) = 1$, we obtain

$$\gamma = -2\Lambda - 9a\Lambda^2 csch^4\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right). \tag{3.91}$$

Now we are going to find the kinetic term and scalar potential. For this purpose, taking the Friedmann equation $3H^2 = \kappa(\rho_m + \rho_\Lambda)$ into account and using the equation (3.86), we can see that

$$\kappa^2\rho_m = 3H^2 - \Lambda.$$

Implementing this with previously found Hubble parameter (3.89) and its derivatives (3.90) to the equation (3.80), we obtain the kinetic term and the scalar potantial as

$$\begin{aligned} X(\phi) &= -9a\Lambda^2 csch^4\left(\frac{\sqrt{3\Lambda}}{2}(t-t_0)\right), \\ V(\phi) &= 2\Lambda. \end{aligned} \tag{3.92}$$

As it can be seen, the scalar potential has been reduced to the cosmological constant. Furthermore, the way the scalar field changes over time compensates for the additional terms in the gravitational sector.

4. CONCLUSIONS

Despite its remarkable success in accordance with observations, the standard model of cosmology falls short in explaining certain problems. Although some of its shortcomings can be resolved through inflation, the accelerated expansion of the universe at an unexpectedly high rate is still a conundrum.

In this review thesis, we examined the $f(R,\phi,X)$ gravity; a modified gravity model to provide an explanation for late-time cosmic acceleration. The article by Bahamonde, Böhmer, Lobo and Sáez-Gómez was taken as reference and the order presented in the article has been followed.

The theory includes various modified gravity and dark energy models. In particular, high-order derivative models in both gravitational and matter sectors. In this study, with the aim of regenerating the late-time cosmic acceleration, we introduced certain Hubble parameters and reconstructed the corresponding $f(R,\phi,X)$ action. With the application of the reconstruction techniques, the gravitational action can be derived effortlessly when some restrictions are imposed on $f(R,\phi,X)$.

Static solutions of modified gravity theories with various symmetries are also widely studied in the literature. One of our aims will be to examine spherically symmetric static star-like solutions of the models considered above.

REFERENCES

[1] **Guth, A.H.** (1981). Inflationary universe: A possible solution to the horizon and flatness problems, *Physical Review D*, *23*, 347–356.

[2] **Perivolaropoulos, L. and Skara, F.** (2022). Challenges for ΛCDM: An update, *New Astronomy Reviews*, *95*, 101659.

[3] **Tong, D.** (2019). *Lectures on Cosmology, Part II Mathematical Tripos*, `https://www.damtp.cam.ac.uk/user/tong/cosmo.html`.

[4] **Linde, A.D.** (1982). A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems, *Physics Letters B*, *108*(6), 389–393.

[5] **Linde, A.D.** (1983). Chaotic inflation, *Physics Letters B*, *129*(3), 177–181.

[6] **Riess, A.**, **Filippenko, A.**, **Challis, P.**, **Clocchiatti, A.**, **Diercks, A.**, **Garnavich, P.**, **... and Tonry, J.** (1998). Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, *The Astronomical Journal*, *116*(3), 1009–1038.

[7] **Perlmutter, S.**, **Alderin, G.**, **Goldhaber, G.**, **Knop, R.**, **Nugent, P.**, **... and Project, T.S.C.** (1999). Measurements of Ω and Λ from 42 High-Redshift Supernovae, *The Astrophysical Journal*, *517*(2), 565–586.

[8] **Sotiriou, T. and Faraoni, V.** (2010). $f(R)$ theories of gravity, *Reviews of Modern Physics*, *82*(1), 451–497.

[9] **De Felice, A. and Tsujikawa, S.** (2010). $f(R)$ Theories, *Living Reviews in Relativity*, *13*(1).

[10] **Harko, T. and Lobo, F.S.N.** (2020). Beyond Einstein's General Relativity: Hybrid metric-Palatini gravity and curvature-matter couplings, *International Journal of Modern Physics D*, *29*(13), 2030008.

[11] **Harko, T.**, **Lobo, F.**, **Nojiri, S. and Odintsov, S.** (2011). $f(R,T)$ gravity, *Physical Review D*, *84*(2), 024020.

[12] **Sharif, M. and Zubair, M.** (2012). Energy conditions constraints and stability of power law solutions in $f(R,T)$ gravity, *Journal of the Physical Society of Japan*, *82*(1), 014002.

[13] **Alves, M.**, **Moraes, P.**, **De Araujo, J. and Malheiro, M.** (2016). Gravitational waves in $f(R,T)$ and $f(R,T^{\phi})$ theories of gravity, *Physical Review D*, *94*(2), 024032.

[14] **Myrzakulov, R.**, **Sebastiani, L. and Vagnozzi, S.** (2015). Inflation in $f(R,\phi)$-theories and mimetic gravity scenario, *The European Physical Journal C*, *75*, 444.

[15] **Panda, S.**, **Rana, A. and Thakur, R.** (2023). Constant-roll inflation in modified $f(R,\phi)$ gravity model using Palatini formalism, *The European Physical Journal C*, *83*, 297.

[16] **Bahamonde, S.**, **Böhmer, C.**, **Lobo, F. and Sáez-Gómez, D.** (2015). Generalized $f(R,\phi,X)$ Gravity and the Late-Time Cosmic Acceleration, *Universe*, *1*(2), 186–198.

[17] **Carroll, S.** (1997). *Lecture Notes on General Relativity*, `https://doi.org/10.48550/arXiv.gr-qc/9712019`.

[18] **Amendola, L.** (2023). *Lecture notes: Cosmology*, `https://www.thphys.uni-heidelberg.de/~amendola/teaching/cosmology.pdf`.

[19] **Guarnizo, A.**, **Castañeda, L. and Tejeiro, J.M.** (2010). Boundary term in metric $f(R)$ gravity: Field equations in the metric formalism, *General Relativity and Gravitation*, *42*(11), 2713–2728.

[20] **Nojiri, S. and Odintsov, S.** (2008). Modified non-local-$F(R)$ gravity as the key for the inflation and dark energy, *Physics Letters B*, *659*(4), 821–826.

[21] **Harko, T.**, **Koivisto, T.S.**, **Lobo, F.S.N.**, **Olmo, G.J. and Rubiera-Garcia, D.** (2018). Coupling matter in modified Q gravity, *Physical Review D*, *98*(8), 084043.

[22] **Goheer, N.**, **Larena, J. and Dunsby, P.** (2009). Power-law cosmic expansion in $f(R)$ gravity models, *Physical Review D*, *80*(6), 061301.

[23] **Benetti, M.**, **Santos da Costa, S.**, **Capozziello, S.**, **Alcaniz, J.S. and De Laurentis, M.** (2018). Observational constraints on Gauss–Bonnet cosmology, *International Journal of Modern Physics D*, *27*(8), 1850084.

[24] **Brans, C. and Dicke, R.H.** (1961). Mach's Principle and a Relativistic Theory of Gravitation, *Physical Review D*, *124*(3), 925–935.

[25] **Tsujikawa, S.** (2010). Modified gravity models of dark energy. In Lectures on Cosmology: Accelerated Expansion of the Universe, Springer Berlin Heidelberg, p.99–145.

[26] **de Sitter, W.** (1917). On Einstein's Theory of Gravitation and its Astronomical Consequences. Third Paper., *Monthly Notices of the Royal Astronomical Society*, *78*(1), 3–28.

[27] **Cotsakis, S. and Yefremov, A.P.** (2022). 100 years of mathematical cosmology: Models, theories, and problems, Part A, *Phil. Trans. R. Soc. A*, *380*(20210191).

[28] **L'opez-Revelles, A.J.** (2013). *Theories of modified gravity and reconstruction schemes of cosmological models*, Universitat de Barcelona, (Doctoral dissertation). Retrieved from https://diposit.ub.edu/.

[29] **Sáez-Gómez, D.** (2011). *On Friedmann-Lemaître-Robertson-Walker cosmologies in non-standard gravity*, Universitat de Barcelona, (Doctoral dissertation). Retrieved from https://arxiv.org/pdf/1104.0813.pdf.

[30] **Kreyszig, E.** (2011). Advanced Engineering Mathematics, chapter 2, Wiley, 10 edition, p. 71.

[31] **Starobinsky, A.A.** (1980). A new type of isotropic cosmological models without singularity, *Physics Letters B*, *91*(1), 99–102.

[32] **Bamba, K. and Odintsov, S.** (2015). Inflationary Cosmology in Modified Gravity Theories, *Symmetry*, *7*(1), 220–240.

[33] **Cuzinatto, R.R.**, **Medeiros, L.G. and Pompeia, P.J.** (2019). Higher-order modified Starobinsky inflation, *Journal of Cosmology and Astroparticle Physics*, *2019*(02), 055.

CURRICULUM VITAE

Büşra TERZİ:

EDUCATION:

- **B.Sc.:** 2018, Uludağ University, Faculty of Arts and Science, Department of Physics