SOME EXACT SOLUTIONS OF COLLIDING WAVES, BLACK HOLES AND THEIR THERMODYNAMICAL PROPERTIES IN DILATON GRAVITY

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By
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December 1998

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

SOME EXACT SOLUTIONS OF COLLIDING WAVES, BLACK HOLES AND THEIR THERMODYNAMICAL PROPERTIES IN DILATON GRAVITY

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Ph. D. in Mathematics
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December 1998

We consider all possible theories in spherically symmetric Riemannian geometry in D-dimensions. We find solutions to such theories, in particular black hole solutions of the low energy limit of the string theory in D-dimensions and study their uniqueness. We find for the first time in literature, exact solutions of colliding Dilaton-Einstein-Maxwell Plane Waves. We investigate thermodynamical properties of four and five dimensional black hole solutions of toroidally compactified string theory. We find the explicit expression of the first law of black hole thermodynamics. We calculate the temperature T, angular velocity Ω and the electromagnetic potentials Φ_i on the horizon using two different methods. Collision of plane waves in dilaton gravity theories and low energy limit of string theory is considered. The formulation of the problem and some exact solutions are presented

Keywords : Gravitation, colliding waves, black holes, thermodynamics, spherical symmetry.

ÖZET

DİLATON GRAVİTASYON TEORİSİNDE BAZI ÇARPIŞAN DALGA VE KARA DELİK ÇÖZÜMLERİ VE TERMODİNAMİK ÖZELLİKLERİ

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D boyutlu küresel simetrik Riemann geometrisinde mümkün olan bütün teorileri ele alarak çözümleri inceledik. Bir özel hal olarak sicim teorisinin düşük enerji limitinde kara delik çözümleri bulduk ve teklik durumunu inceledik. Literatürde ilk kez bir Einstein-Maxwell-Dilaton çarpışan düzlem dalga probleminin kesin çözümünü bulduk.

Toroidal indirgenmiş sicim teorisinde dört ve beş boyutlu kara deliklerin termodinamik özelliklerini inceledik. Kara delik termodinamiğinin ilk yasasının açık halini bulduk. Sıcaklık T, açısal hız Ω ve elektromagnetik potansiyel Φ_i 'nin ufuk üzerindeki değerlerini iki farklı metodla bulduk.

Dilaton gravitasyon teorisinde ve sicim teorisinin düşük enerji limitinde çarpışan düzlem dalga problemini inceledik. Problemin formülasyonuyla bazı çözümleri sunduk.

Anahtar Kelimeler : Gravitasyon, çarpışan dalgalar, kara delikler, termodinamik, küresel simetri.

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Chapter 1

Introduction

In this work, we are interested in finding solutions of field equations in low energy limit of string theory, extending some special solutions to higher dimensions, examining the uniqueness of static black holes, finding colliding gravitational plane wave solutions and analyzing thermodynamic properties of certain higher dimensional black holes.

In the classical relativity, the lagrangian contains only the Ricci scalar. On the other hand we learned from the low energy limit of string theory that the classical lagrangian contains all possible invariants constructed from the curvature tensor and the matter fields. Depending upon the order of the string tension parameter, this lagrangian is an infinite series expansion in these invariants, i.e.

$$L = \sqrt{-g}R + \sum_{n=1}^{\infty} \alpha^n L_n. \tag{1.1}$$

Here α is the inverse of the string tension and L_n are functions containing the invariants up to n^{th} order. There are several examples where the sum is terminated at some value n. For instance when $n = 2$ we have Gauss-Bonnet theory

$$L = \sqrt{-g}R + \alpha(R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2). \tag{1.2}$$

For different n we have the Lovelock theorem: In d-dimensions divergence-free second order symmetric tensors constructed from the metric and its first two derivatives are given by

$$A^i_j = \sum_{p=1}^{m-1} a_p \delta^{ih_1 \dots h_{2p}}_{jj_1 \dots j_{2p}} R^{j_1 j_2}_{h_1 h_2} R^{j_3 j_4}_{h_3 h_4} \cdots R^{j_{2p-1} j_{2p}}_{h_{2p-1} h_{2p}} + a\delta^i_j, \tag{1.3}$$

where $m = \frac{1}{2}n$ if n is even, $m = \frac{1}{2}(n+1)$ if n is odd.

Boulware and Deser [1] found two solutions for the Einstein plus Gauss-Bonnet lagrangian, one of them asymptotically flat, the other asymptotically anti-de Sitter.

Wiltshire [2] slightly generalized the previous results by including a Maxwell field. Wheeler [3] has considered the most general second-order gravity theory in arbitrary dimensions and analyzed asymptotically flat spherically symmetric static solutions, and cosmological solutions. Whitt [4] extended Wheeler's work to non-static space-times.

The Schwarzschild solution which describes the uncharged black holes in general relativity also describes (to a good approximation) uncharged black holes in string theory. But, this is not the case for charged black holes. The dilaton has a coupling to F^2, so the Reissner-Nordström solution is not even an approximate solution of string theory.

Charged black holes were first analyzed by Gibbons-Maeda [5] and GHS (Garfinkle, Horowitz, Strominger) [6] independently. These are classes of solutions of the Einstein field equations which are obtained through some complexification methods. These solutions describe the collision of plane waves in general relativity. The field equations arising from the low energy limit of the string theory admit also such kind of solutions preserving the singular structure of the collision region.

In chapters 2 and 3, we give a short review of Riemann geometry and some well-known solutions of Einstein's equations.

In chapter 4, we extend the Levi-Civita Bertotti-Robinson metrics to D-dimensions and prove that they solve all field equations arising from a variational principle.

Black hole solutions depend on dimension and number of physical parameters. In chapter 5 we find D-dimensional solutions with three independent parameters, mass M, electric charge Q, and the dilaton charge Σ. [7] We also study the uniqueness of static black holes in Einstein-Maxwell-Dilaton field theory.

Chapter 6 gives a summary of the black hole solutions found by Cvetic and Youm. [8, 9]. We will analyze the thermodynamical properties of these solutions in chapters 8 and 9.

Plane wave geometries are not only important in classical general relativity but also in string theory . It is now very well known that these geometries are the exact classical solutions of the string theory at all orders of string tension parameter [10]-[12]. It is also interesting that plane wave metrics in higher dimensions when dimensionaly reduced lead to exact extreme black hole solution in string theory [13].

In chapter 7 we will be interested in the head on collisions of these plane waves in the framework of Einstein-Maxwell-Dilaton theories with one $U(1)$ and two $U(1)$ abelian gauge fields [14]. Our formulation of the problem will also cover the low energy limit of the string theory for some fixed values of the dilaton coupling constants. Hence the solutions we present in this work are also exact solutions of the low energy limit of string theories.[15] We give the complete data for the colliding plane-shock waves. We formulate the collision of plane waves and give solutions for the colinear case. When the dilaton coupling constant vanishes one of our solutions reduces to the well known Bell-Szekeres solution [16] in Einstein Maxwell Theory.

For the collision problem in general relativity spacetime is divided into four regions with respect to the null coordinates u and v. The second and third (incoming) regions are the Cauchy data (characteristic initial data) for the field equations in the interaction region (IV. Region). For this purpose the specification of the data is quite important in the formulation of the collision problem [17]-[25]. We show that the future closing singularities appearing in classical solutions exist also in dilaton gravity and in the low energy limit of the string theory. This is due to focusing effect of the plane waves [26]. It is an open question whether this classical treatment of collision of plane waves can be extended to all orders in the string tension parameter [7], [27]. The Bell-Szekeres solution [16] in the interaction region is diffeomorphic to the Bertotti-Robinson spacetime [28] ,[25]. It is also known that string theory preserves the form of Bertotti-Robinson metric at all orders of the string parameter [29] , [30]. This does not necessarily lead to a conclusion that the Bell-Szkeres solution is an exact solution of the string theory. The reason is that the diffeomorphism is valid only in the interaction region $(u > 0, v > 0)$ and hence the field equations (at higher orders of the string parameter) are not satisfied on the hyperplanes $u = 0$ and $v = 0$.

The connection between black hole mechanics and thermodynamics is one of the most interesting developments in the past 30 years. The first law of black hole mechanics, as proved by Bardeen, Carter and Hawking [31], gives the variation of mass in terms of the variations in area and angular momentum. This relationship opened the way for the area of the black hole to be interpreted as its entropy and the surface gravity as its temperature.

With the discovery of the Hawking radiation, it was understood that the close parallel between the laws of thermodynamics and black hole mechanics was more than a coincidence and it had a physical basis. Black holes radiate with a black body spectrum at the temperature given by surface gravity.

Hawking radiation, while answering an important question, raised new ones like information loss, black hole evaporation and the microscopic origin of black hole entropy.

We are interested in finding the explicit form of the first law of black hole mechanics for two different black holes. In other words, given the metric, we want to find the coefficients T, Ω and Φ in

$$dM = TdS + \Omega dJ + \Phi_i \, dQ_i. \tag{1.4}$$

If we have a black hole solution, we can easily calculate the surface area of the outer horizon using the metric components. Entropy is given as $S = \frac{1}{4G_N} A$. Then, if the algebraic equations are tractable, we can isolate M, take derivatives, and obtain first law.

This procedure won't work if the solution is given in terms of parameters that can't be solved explicitly in terms of mass and charges. Then we have to use a roundabout way, using infinitesimal variations to make a change of variables, which, in general involve inverting a big matrix, and if all entries are nonzero, the results may be too complicated.

We expect the variation of mass with respect to area to be the temperature of the black hole, and the variation of mass with respect to angular momentum to be the angular velocity. These quantities can be computed from the metric. This will be an independent way to calculate the coefficients in the first law, and the results can be checked.

In chapters 8 and 9, we will follow the summarized procedure for two different types of black holes corresponding to four and five dimensional solutions of toroidally compactified string theory. We first write the area in terms of solution parameters, take the infinitesimal variation of the area, and replace the solution parameters by the physical ones using the jacobian matrix. Then we calculate Ω and κ using the metric, and compare the results with the first law.

We have also tried to obtain the Smarr's formula [32], but we do not know the mass as an explicit function, so Smarr's procedure of using the Euler theorem to obtain $M = 2TA + 2\Omega J + \Phi Q$ is not applicable in this case.

In Chapter 8, a four dimensional rotating black hole parametrized by ADM mass, four charges, and angular momentum, and in Chapter 9 a five dimensional black hole with two angular momenta and three charges will be analyzed. [33]

The appendices give the details of some straightforward calculations not included in the main text.

Chapter 2

Metric and Spherically Symmetric Riemannian Geometry in D-dimensions

The metric of a static and spherically symmetric D-dimensional spacetime is given by:

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 d\Omega^2_{D-2}, \tag{2.1}$$

where A,B,C depend only on r. $d\Omega^2_{D-2}$ is the metric on S_{D-2}. The metric can be rewritten as $g_{ij} = -A^2 t_i t_j + B^2 k_i k_j + C^2 h_{ij}$, where $t_i = \delta^t_i$, $k_i = \delta^r_i$, h_{ij} =metric on D-2 sphere for $i, j \geq 2$, $h_{0i} = h_{1i} = 0$.

Christoffel symbols are given by

$$\Gamma^i_{jm} = \frac{1}{2} g^{in}(g_{jn,m} + g_{nm,j} - g_{jm,n}) \tag{2.2}$$

$$\begin{aligned} \Gamma^i_{jm} \quad = \quad & -AA' t^i (t_j k_m + t_m k_j) + AA' k^i t_j t_m + BB' k^i k_j k_m - CC' k^i h_{jm} \\ & + CC' (h^i_j k_m + h^i_m k_j) + \Gamma^i_{(s)jm}, \end{aligned} \tag{2.3}$$

where $\Gamma_{(s)}$ is the christoffel symbol on $D-2$ sphere.

The Riemann tensor is given by

$$R^i_{jml} = \Gamma^i_{jl,m} - \Gamma^i_{jm,l} + \Gamma^i_{nm}\Gamma^n_{jl} - \Gamma^i_{nl}\Gamma^n_{jm} \tag{2.4}$$

$$
\begin{aligned}
R_{ijml} &= \left(AA'' - \frac{AA'B'}{B}\right) t_{[i}k_{j]}t_{[m}k_{l]} + \left(CC'' - \frac{CC'B'}{B}\right) k_{[i}h_{j][m}k_{l]} \\
&\quad + \frac{AA'CC'}{B^2} t_{[m}h_{l][j}t_{i]} - \frac{C^2C'^2}{B^2} h_{i[m}h_{l]j} + C^2R_{sijml}, \qquad (2.5)
\end{aligned}
$$

where $R_{sijml} = h_{i[m}h_{l]j}$.
Riemann tensor can be rewritten as

$$
R_{ijml} = g_{jl}S_{im} - g_{jm}S_{il} + g_{im}S_{jl} - g_{il}S_{mj} + \eta_2 H_{ijk\cdots n}H^{k\cdots n}_{ml} \qquad (2.6)
$$

where

$$
S_{ij} = \eta_0 M_{ij} + \eta_1 k_i k_j + \frac{1}{2}\eta_3 g_{ij}, \qquad (2.7)
$$

$$
M_{ij} = H_{im\cdots n}H_j^{m\cdots n} - \frac{1}{2(D-2)}H^2 g_{ij} \qquad (2.8)
$$

which turns out to be

$$
M_{ij} = \frac{(D-3)!}{C^{2(D-3)}}(h_{ij} - \frac{1}{2C^2}g_{ij}), \qquad (2.9)
$$

the tensor $H_{ij\ldots k}$ is the volume form of S_{D-2}, i.e.

$$
H_{ij\cdots k} = -\sqrt{h}\epsilon_{ij\cdots k}. \qquad (2.10)
$$

Here the scalars are given by

$$
\begin{aligned}
\eta_0 &= \frac{C^{2(D-2)}}{(D-3)!}\left(\frac{A''}{AB^2} - \frac{A'B'}{AB^3} - \frac{C''}{CB^2} + \frac{B'C'}{B^3C}\right), \\
\eta_1 &= \frac{A'C'}{AC} - \frac{C''}{C} + \frac{B'C'}{BC}, \qquad (2.11) \\
\eta_2 &= \frac{C^{2(D-3)}}{(D-4)!}\left(1 - \frac{C'^2}{B^2} + \frac{A'C'C}{AB^2} - \frac{A''C^2}{AB^2} + \frac{A'B'C^2}{AB^3} + \frac{CC''}{B^2} - \frac{B'C'C}{B^3}\right),
\end{aligned}
$$

$$\eta_3 = -\frac{A'C'}{AB^2C}. \tag{2.12}$$

The Ricci tensor, Ricci scalar and Einstein tensor can be computed from Riemann as follows:

$$\begin{aligned}
R_{ij} &= \left[\frac{(D-3)!}{2C^{2(D-2)}}(\eta_0(D-4)+\eta_2)+\frac{1}{B^2}\eta_1+\eta_3(D-1)\right] g_{ij} \\
&\quad +\eta_1(D-2)k_ik_j+[\eta_0(D-2)+\eta_2]M_{ij}, \qquad (2.13)
\end{aligned}$$

$$\begin{aligned}
R &= \frac{(D-3)!}{C^{2(D-2)}}[\eta_0(D-4)(D-1)+\eta_2(D-2)]+\frac{2\eta_1(D-1)}{B^2} \\
&\quad +\eta_3 D(D-1), \qquad (2.14)
\end{aligned}$$

$$\begin{aligned}
G_{ij} &= -\left[\frac{(D-3)!}{2C^{2(D-2)}}(\eta_0(D-4)(D-2)+\eta_2(D-3))+\frac{1}{B^2}\eta_1(D-2)\right. \\
&\quad \left.+\frac{1}{2}\eta_3(D-1)(D-2)\right] g_{ij}+\eta_1(D-2)k_ik_j \\
&\quad +[\eta_0(D-2)+\eta_2]M_{ij}. \qquad (2.15)
\end{aligned}$$

The covariant derivatives of $H_{ij...k}$ and k_i are given as

$$\nabla_l H_{ij...m} = -\rho[(D-2)H_{ij...m}k_l + k_iH_{lj...m} + k_jH_{il...m} + ... + k_mH_{ij...l}], \tag{2.16}$$

$$\nabla_i k_j = \rho_1 g_{ij} + \rho_2 M_{ij} + \rho_3 k_i k_j \tag{2.17}$$

where

$$\begin{aligned}
\rho &= \frac{C'}{C} & \rho_1 &= \frac{A'C+AC'}{2AB^2C} \\
\rho_2 &= \frac{C^{2(D-2)}}{(D-3)!}\frac{AC'-A'C}{AB^2C} & \rho_3 &= -\frac{(AB)'}{AB}. \qquad (2.18)
\end{aligned}$$

The covariant derivatives of $H_{ij...k}$ and k_i are expressed only in terms of themselves and the metric tensor. Riemann tensor is given in terms of $H_{ij...k}$, g_{ij} and k_i. Hence we have the following theorem:

Theorem 1 *Covariant derivatives of the Riemann tensor R_{ijkl}, the tensor H_{ij} and the vector k_i at any order are expressible only in terms of $H_{ij\cdots k}$, g_{ij} , k_i.*

Since contraction of k^i with $H_{ij\ldots k}$ vanishes, the only symmetric tensors constructable out of $H_{ij\cdots k}$, g_{ij} and k_i are M_{ij}, the metric tensor g_{ij} and $k_i k_j$. Then the following theorems hold:

Theorem 2 *Any second rank symmetric tensor constructed out of the Riemann tensor, anti-symmetric tensor $H_{ij\ldots k}$, dilaton field $\phi = \phi(r)$ and their covariant derivatives is a linear combination of M_{ij} , g_{ij} and $k_i k_j$.*

Let this symmetric tensor be E'_{ij}. Then we have

$$E'_{ij} = \sigma_1 M_{ij} + \sigma_2 g_{ij} + \sigma_3 k_i k_j \tag{2.19}$$

where σ_1, σ_2 and σ_3 are scalars which are functions of the metric functions, invariants constructed out of the curvature tensor R_{ijkl}, H_{ij} and the dilaton field.

Theorem 3 *Any vector constructed out of the Riemannian tensor R_{ijkl}, $H_{ij\ldots k}$ the dilaton field $\phi = \phi(r)$ and their covariant derivatives is proportional to k_i.*

Let this vector be E'_i. Hence

$$E'_i = \sigma k_i \tag{2.20}$$

where σ is a scalar like σ_1, σ_2, σ_3.

Theorem 2 has an application:

Theorem 4 *In a spherically symmetric, D-dimensional spacetime, the coefficients a_i in the identity*

$$a_1 R_{imnl} R_j^{mnl} + a_2 R_{im} R_j^m + a_3 R_{jmin} R^{mn} + a_4 R_{ij} + a_5 g_{ij} = 0 \tag{2.21}$$

can be found in terms of the η_i.

This can be easily seen if we write each of the symmetric tensors as a combination of M_{ij}, g_{ij}, k_ik_j. Let us denote these tensors as follows:

$$\begin{aligned} R_i^a R_{ja} &= \beta_1 g_{ij} + \beta_2 M_{ij} + \beta_3 k_i k_j \\ R_{jaib} R^{ab} &= \beta_4 g_{ij} + \beta_5 M_{ij} + \beta_6 k_i k_j \\ R_i^{abc} R_{jabc} &= \beta_7 g_{ij} + \beta_8 M_{ij} + \beta_9 k_i k_j \\ R_{ij} &= f_1 g_{ij} + f_2 M_{ij} + f_3 k_i k_j. \end{aligned} \tag{2.22}$$

Thus, we obtain

$$\begin{aligned} \beta_7 a_1 + \beta_1 a_2 + \beta_4 a_3 + f_1 a_4 + a_5 &= 0, \\ \beta_8 a_1 + \beta_2 a_2 + \beta_5 a_3 + f_2 a_4 &= 0, \\ \beta_9 a_1 + \beta_3 a_2 + \beta_6 a_3 + f_3 a_4 &= 0. \end{aligned} \tag{2.23}$$

Provided we know a_4 and a_5, we can compute a_1, a_2, a_3.

Let's denote

$$\begin{vmatrix} \beta_7 & \beta_1 & \beta_4 \\ \beta_8 & \beta_2 & \beta_5 \\ \beta_9 & \beta_3 & \beta_6 \end{vmatrix} = \frac{1}{\Psi}, \tag{2.24}$$

then,

$$a_1 = -\Psi[(\beta_2\beta_6 - \beta_3\beta_5)(f_1 a_4 + a_5) + (\beta_3\beta_4 - \beta_1\beta_6)(f_2 a_4) + (\beta_1\beta_5 - \beta_2\beta_4)(f_3 a_4)], \tag{2.25}$$

$$a_2 = -\Psi[(\beta_5\beta_9 - \beta_6\beta_8)(f_1 a_4 + a_5) + (\beta_6\beta_7 - \beta_4\beta_8)(f_2 a_4) + (\beta_4\beta_8 - \beta_5\beta_7)(f_3 a_4)], \tag{2.26}$$

$$a_3 = -\Psi[(\beta_3\beta_8-\beta_2\beta_9)(f_1a_4+a_5)+(\beta_1\beta_9-\beta_3\beta_7)(f_2a_4)+(\beta_2\beta_7-\beta_1\beta_8)(f_3a_4)]. \tag{2.27}$$

We will give β's explicitly in Appendix 1.

We may also write the Riemann tensor in the form

$$R_{ijml} = g_{jl}S_{im} - g_{jm}S_{il} + g_{im}S_{jl} - g_{il}S_{mj} + e_2F_{ij}F_{ml}, \tag{2.28}$$

where

$$F_{ij} = \frac{AB}{C^2}(t_ik_j - t_jk_i). \tag{2.29}$$

Other tensors are defined similar to the previous case, i.e.

$$S_{ij} = e_0\,\widetilde{M_{ij}} + e_1\,k_ik_j + \frac{1}{2}e_3\,g_{ij}, \tag{2.30}$$

$$\widetilde{M_{ij}} = F_{mj}F_i^m - \frac{1}{4}F^2\,g_{ij} = \frac{1}{C^2}(h_{ij} - \frac{1}{2\,C^2}\,g_{ij}). \tag{2.31}$$

The scalars are given as

$$\begin{aligned}
e_0 &= C^2 - \frac{C'^2C^2}{B^2} + \frac{C^3C'A'}{A\,B^2}, \\
e_1 &= \eta_1, \\
e_2 &= -C^2\left(1 - \frac{C'^2}{B^2} + \frac{A'C'C}{AB^2} - \frac{A''C^2}{AB^2} + \frac{A'B'C^2}{AB^3} + \frac{CC''}{B^2} - \frac{B'C'C}{B^3}\right), \\
e_3 &= \eta_3.
\end{aligned} \tag{2.32}$$

Notice that when $D = 4$, $\widetilde{M_{ij}} = M_{ij}$, and $e_2 = -\eta_2$.

The covariant derivatives of F_{ij} , k_i and t_i are given as

$$\nabla_lF_{ij} = -\frac{3C'}{C}F_{ij}k_l - \frac{AC'}{BC^3}(t_jg_{il} - t_ig_{jl}), \tag{2.33}$$

$$\nabla_ik_j = \rho_1g_{ij} + \tilde{\rho}_2\widetilde{M_{ij}} + \rho_3k_ik_j, \tag{2.34}$$

$$\nabla_i t_j = \frac{A'}{A}(t_i k_j + t_j k_i). \tag{2.35}$$

where $\tilde{\rho}_2 = \frac{(D-3)!}{C^{2(D-4)}}\,\rho_2$. The covariant derivatives of F_{ij} and k_i are expressed in terms of themselves, metric tensor and t_i. So we have a similar theorem for an extended set:

Theorem 5 *Covariant derivatives of the Riemann tensor R_{ijkl}, the anti-symmetric tensor F_{ij}, and the vectors k_i and t_i at any order are expressible only in terms of F_{ij}, k_i, t_i and g_{ij}.*

Chapter 3

Schwarzschild and Reissner-Nordström Solutions

These are the earliest spherically symmetric solutions, obtained in 1916, so they deserve a short rewiew here.

Let's take the line element as

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + r^2 d\Omega^2. \tag{3.1}$$

In other words, take $C(r) = r$. Our equation is

$$G_{ij} = 0. \tag{3.2}$$

We have already computed G_{ij} in terms of g_{ij}, $k_i k_j$ and M_{ij} (2.15), so equating the coefficients of these terms to zero, we obtain three equations:

$$\frac{(D-3)!}{2C^{2(D-2)}}[\eta_0(D-4)(D-2)+\eta_2(D-3)]+\frac{\eta_1}{B^2}(D-2)+\frac{\eta_3}{2}(D-1)(D-2)=0, \tag{3.3}$$

$$\eta_1(D-2) = 0, \tag{3.4}$$

$$\eta_0(D-2) + \eta_2 = 0, \tag{3.5}$$

(3.4) gives

$$\frac{A'}{A} + \frac{B'}{B} = 0 \tag{3.6}$$

so $AB = c_1$ but the boundary condition for the metric to be Lorentzian at infinity requires that $c_1 = 1$.

Thus

$$AB = 1. \tag{3.7}$$

Using (3.5), we can find η_2 in terms of η_0. Inserting η_2 and substituting $B = \frac{1}{A}$ in (3.3) gives

$$AA'' + A'^2 + \frac{(D-2)}{r} AA' = 0, \tag{3.8}$$

which is actually

$$u' + (D-2)\frac{u}{r} = 0 \tag{3.9}$$

where $u = A\,A'$.

Thus $A^2 = c_2 + c_3 r^{-D+3}$. Inserting this in (3.5), we find that $c_2 = 1$,

so

$$\begin{aligned} A^2 &= 1 - \frac{2m}{r^{D-3}}, \\ B^2 &= \left(1 - \frac{2m}{r^{D-3}}\right)^{-1}, \end{aligned} \tag{3.10}$$

where $2m$ is the integration constant c_3.

So we obtain the Schwarzschild solution. For Reissner-Nordström solution we have to take electrostatic charge Q into account.

Our equation is

$$G_{ij} = T_{ij} \tag{3.11}$$

where

$$T_{ij} = -2\left[F_{iN}F^{N}_{j} - \frac{1}{4}F_{MN}F^{MN}g_{ij}\right], \tag{3.12}$$

$$F_{tr} = Q\,\frac{AB}{r^{D-2}}. \tag{3.13}$$

T_{ij} turns out to be

$$T_{ij} = -2\,Q^2\frac{r^{2\,(D-4)}}{(D-3)!}\,M_{ij}. \tag{3.14}$$

Thus we obtain the same set of equations except for(3.5).

Proceeding in a similar manner, instead of (3.8) we obtain

$$AA'' + A'^2 + \frac{(D-2)}{r}AA' = -2\frac{(D-3)}{(D-2)}\frac{Q^2}{r^{2(D-2)}}, \tag{3.15}$$

which is actually

$$\frac{(r^{D-2}\,AA')'}{r^{D-2}} = -2\frac{(D-3)}{(D-2)}\frac{Q^2}{r^{2(D-2)}}, \tag{3.16}$$

so we find

$$A^2 = \frac{-2Q^2}{(D-2)(D-3)}\,r^{2(D-3)}. \tag{3.17}$$

But this is the non-homogeneous solution, we have to add the solution of (3.8) to this.

Thus

$$
\begin{aligned}
A^2 &= 1-\frac{2m}{r^{D-3}}-\frac{2\,Q^2}{(D-2)\,(D-3)\,r^{2(D-3)}}, \\
B^2 &= \left(1-\frac{2m}{r^{D-3}}-\frac{2\,Q^2}{(D-2)(D-3)\,r^{2(D-3)}}\right)^{-1}.
\end{aligned}
\tag{3.18}
$$

Here integration constants are found by the asymptotic behaviour of the solution:

$$
\begin{aligned}
\lim_{r\to\infty} r^{D-3}(A^2-1) &= -2M, \\
\lim_{r\to\infty} r^{D-2}F_{tr} &= Q.
\end{aligned}
\tag{3.19}
$$

Chapter 4

Levi-Civita Bertotti-Robinson Metric

4.1 Einstein's equation

An example is the Levi-Civita Bertotti-Robinson (LCBR) metric in D-dimensions

$$g_{ij} = \frac{q^2}{r^2}\left(-t_i t_j + c_0^2 k_i k_j + r^2 h_{ij}\right) \tag{4.1}$$

where h_{ij} is the metric on S_{D-2}, $t_i = \delta_i^t$, $k_i = \delta_i^r$. q and c_0 are constants.

This is the previous metric with the choice $A = \frac{q}{r}$, $B = \frac{qc_0}{r}$, $C = q$, then

$$F_{ij} = \frac{c_0}{r^2}(t_i k_j - t_j k_i), \tag{4.2}$$

$$\widetilde{M_{ij}} = F_{mj} F_i^m - \frac{1}{4} F^2 g_{ij}, \tag{4.3}$$

$e_1 = e_3 = 0 \rightarrow S_{ij} = e_0 \widetilde{M_{ij}}$, $e_0 = q^2$,

$$R_{ijkl} = q^2[g_{jl}\widetilde{M_{ik}} - g_{jk}\widetilde{M_{il}} + g_{ik}\widetilde{M_{jl}} - g_{il}\widetilde{M_{kj}}] + \frac{q^2(1-c_0^2)}{c_0^2} F_{ij} F_{ml}. \tag{4.4}$$

$$R_{ij} = q^2[(D-3) + \frac{1}{c_0^2}]\widetilde{M_{ij}} + \frac{1}{2q^2}[(D-3) - \frac{1}{c_0^2}]g_{ij}. \tag{4.5}$$

$$G_{ij} = q^2[(D-3) + \frac{1}{c_0^2}]\widetilde{M_{ij}} + \frac{1}{2q^2}[\frac{1}{c_0^2} - (D-3)^2]g_{ij}. \tag{4.6}$$

It is easily seen that

$$\nabla_l F_{ij} = 0 \quad \nabla_m R_{ijkl} = 0. \tag{4.7}$$

If we consider the Einstein's equations

$$G_{ij} = T_{ij} \tag{4.8}$$

where

$$T_{ij} = \widetilde{F_{mj}}\widetilde{F_i^m} - \frac{1}{4}\widetilde{F^2}\, g_{ij} \tag{4.9}$$

and $\widetilde{F_{ij}}(E.M) = eF_{ij}$, we obtain

$$\begin{aligned} q^2\left[(D-3) + \frac{1}{c_0^2}\right] &= e^2, \\ \frac{1}{2q^2}\left[\frac{1}{c_0^2} - (D-3)^2\right] &= \lambda \end{aligned} \tag{4.10}$$

where λ is the cosmological constant. To eliminate the cosmological constant, we let $\frac{1}{c_0} = |D-3|$. It is interesting that spacetime is conformally flat only for $D = 4$.

We can obtain the same equations using the expression (2.15). This time we have to choose

$$T_{ij} = \widetilde{H}_{im\cdots n}\widetilde{H}_j^{m\cdots n} - \frac{1}{2(D-2)}\widetilde{H^2} g_{ij} \tag{4.11}$$

and $\widetilde{H}_{ij\cdots k} = eH_{ij\cdots k}$. When the cosmological constant is set equal to zero, we obtain

$$c_0^2 = \frac{1}{(D-3)^2}. \tag{4.12}$$

Hence the higher dimensional ($D > 4$) Levi-Civita Bertotti-Robinson space-times without cosmological constants can not be conformally flat.

4.2 The Solutions of Lovelock Theory

In the following $\delta^{i_1 i_2 \cdots i_N}_{j_1 j_2 \cdots j_N}$ is the generalized Kronecker delta defined by

$$\delta^{i_1 i_2 \cdots i_N}_{j_1 j_2 \cdots j_N} = det \begin{vmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_N} \\ \vdots & & \vdots \\ \delta^{i_N}_{j_1} & \cdots & \delta^{i_N}_{j_N} \end{vmatrix} \tag{4.13}$$

According to a theorem by Lovelock, the only symmetric tensor $A^{ij} = A^{ij}(g_{rs}; g_{rs,t} : g_{rs,tu})$ for which $A^{ij}_{,j} = 0$ is

$$A^i_j = \sum_{p=1}^{m-1} a_p \delta^{ih_1 \ldots h_{2p}}_{jj_1 \cdots j_{2p}} R^{j_1 j_2}_{h_1 h_2} R^{j_3 j_4}_{h_3 h_4} \cdots R^{j_{2p-1} j_{2p}}_{h_{2p-1} h_{2p}} + a\delta^i_j \tag{4.14}$$

where a and a_p are arbitrary constants. We have found the equation for $n = 5, 6$

$$a_1 \delta^{i n_1 n_2}_{j m_1 m_2} R^{m_1 m_2}_{n_1 n_2} + a_2 \delta^{i n_1 n_2 n_3 n_4}_{j m_1 m_2 m_3 m_4} R^{m_1 m_2}_{n_1 n_2} R^{m_3 m_4}_{n_3 n_4} + a\delta^i_j = 0 \tag{4.15}$$

which is

$$G_{ij} + \alpha_0[(R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd})g_{ij} + 4(2R^a_i R_{ja} + 2R_{jaib}R^{ab} - RR_{ij} - R^{abc}_i R_{jabc})] = \lambda g_{ij} \tag{4.16}$$

where $\alpha_0 = -\frac{a_2}{a_1}$

We know that every symmetric tensor of rank two can be written as a linear combination of g_{ij}, M_{ij} and $k_i k_j$. To calculate these we need the following:

For the LCBR metric

$$ds^2 = \frac{q^2}{r^2}\left(-dt^2 + c_0^2\, dr^2 + r^2 d\Omega^2\right) \tag{4.17}$$

where q and c_0 are constants, we have

$$\begin{aligned} \eta_0 &= \frac{1}{\alpha c_0^2}, \\ \eta_1 &= 0, \\ \eta_2 &= \frac{(D-3)(c_0^2-1)}{\alpha c_0^2}, \\ \eta_3 &= 0. \end{aligned} \tag{4.18}$$

If we insert these values in t_1, t_2, t_3 we find

$$t_1 = \frac{1}{2q^2c_0^2} - \frac{(D-3)^2}{2q^2} + \frac{\alpha_0}{q^4}(D-3)(D-4)^2(D-5) - \frac{2\alpha_0}{q^4c_0^2}(D-3)(D-4), \tag{4.19}$$

$$t_2 = \frac{1}{\alpha}\left(D-3+\frac{1}{c_0^2}\right) - \frac{4\alpha_0(D-3)(D-4)}{\alpha q^2 c_0^2}\left[1 + c_0^2(D-5)\right], \tag{4.20}$$

$$t_3 = 0. \tag{4.21}$$

Field equations reduce to $t_1 = \lambda$, $t_2 = t_3 = 0$. This gives relations between constants of the theory and the constants of the metric.

For $D = 5$ we obtain

$$\begin{aligned} q^2 - 4q^2c_0^2 - 8\alpha_0 &= 2q^4c_0^2\,\lambda, \\ q^2 + 2q^2c_0^2 - 8\alpha_0 &= 0 \end{aligned} \tag{4.22}$$

which gives $c_0^2 = 0$ or $q^2 = -\frac{3}{\lambda}$

For $D = 6$ we obtain

$$
\begin{aligned}
q^2(1 - 9c_0^2) + 24\alpha_0(c_0^2 - 1) &= 2q^4 c_0^2\,\lambda, \\
3q^2 c_0^2 + q^2 - 24\alpha_0(1 + c_0^2) &= 0.
\end{aligned}
\tag{4.23}
$$

which gives $q^2 = 4\alpha_0$, $c_0^2 = -\frac{5}{3}$ for $\lambda = 0$ and

$$
\begin{aligned}
q^2 &= \frac{-3 + \sqrt{9 + 24\alpha_0\lambda}}{\lambda}, \\
c_0^2 &= \frac{q^2 - 24\alpha_0}{q^2(3 + \lambda)},
\end{aligned}
\tag{4.24}
$$

for $\lambda \neq 0$.

So there is no solution for 5 or 6 dimensional spacetimes without a cosmological constant.

4.3 Solution of the most general theory

The Lagrangian of the most general theory will be a scalar containing the Riemann tensor, metric tensor, and their derivatives, contractions and multiple products of all orders. But, according to theorem 2, all second rank symmetric tensors constructed out of these will be expressible in terms of *g_{ij}, M_{ij}, and $k_i k_j$.*

So, whatever the theory is, we'll obtain two equations for the LCBR metric, because the coefficient of the $k_i k_j$ term will automatically vanish. This will give us two algebraic equations for two unknown constants in the metric, namely q and c_0. These equations may or may not have a solution according to theory. For example, in the preceding case, in $D = 5, 6$ we have no solution.

Chapter 5

Solutions of the Low Energy Limit of the String Theory

In this chapter, we consider the Einstein Maxwell Dilaton field theory with arbitrary dilaton coupling constant α assuming spherical symmetry. For some fixed values of α this theory reduces to the low energy limit of the string theory and Kaluza Klein type of theories.

In 4-dimensions we have the Garfinkle-Horowitz-Strominger (GHS) solution [6]. Gibbons and Maeda (GM) [5] have given black hole solutions in arbitrary D-dimensions. In four dimensions GM solution describes the region $r > r_1$ of the GHS metric. In this work we find and classify the solutions of the dilatonic Einstein-Maxwell (and hence of the low energy limit of the string theory) for static and spherically symmetric space-times.

We also show that GHS-GM metric in four dimensions is the unique asymptotically flat static axially symmetric black hole solution of the Einstein Maxwell Dilaton filed theory.

In this chapter, we are going to obtain the Einstein-Maxwell-Dilaton Field equations and solve them, starting from the action:

$$L = \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{4}{(D-2)\kappa^2}(\nabla\phi)^2 - \frac{1}{4}\, e^{-\alpha_e\phi}F^2\right]. \qquad (5.1)$$

Dahl Park and Youngjai Kiem have studied the related problem of general static solutions of 2-dimensional Einstein-Dilaton-Maxwell-Scalar theories [44] , where they start with the action

$$I = \int d^2x\sqrt{-g}e^{-2\phi}[R+\gamma g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+\mu e^{2\lambda\phi}-\frac{1}{2}e^{-2\phi(\delta-1)}g^{\alpha\beta}\partial_\alpha f\partial_\beta f+\frac{1}{4}e^{\epsilon\phi}F^2], \tag{5.2}$$

where R denotes the 2-d scalar curvature and F, the curvature 2-form for an Abelian gauge field. ϕ and f represent a dilaton field and a massless scalar field, respectively. The parameters γ, μ, λ, ϵ and δ are assumed to be arbitrary real numbers.

The same authors also worked on the d-dimensional problem in [45] where they get the general static, spherically symmetric solutions of the d-dimensional Einstein-Maxwell-Dilaton theories by dimensionally reducing them to a class of 2-dimensional dilaton gravity theories. By studying the symmetries of the actions for the static equations of motion, we find field redefinitions that nearly reduce these theories to the d-dimensional Einstein-Maxwell-Scalar theories, and therefore enable us to get the exact solutions. We do not make any assumption about the asymptotic space-time structure. As a result, our 4-dimensional solutions contain the asymptotically flat Garfinkle-Horowitz-Strominger (GHS) solutions and the non-asymptotically flat Chan-Horne-Mann (CHM) solutions. Besides, we find some new solutions with a finite range of allowed radius of the transversal sphere. These results generalize to an arbitrary space-time dimension d ($d > 3$).

$$I = \int d^dx\sqrt{g^{(d)}}(R - \frac{1}{2}g^{(d)\alpha\beta}\partial_\alpha f\partial_\beta f + \frac{1}{4}e^{\chi f}F^2) \tag{5.3}$$

when the value of the constant χ is non-zero. Here F^2 is the square of the curvature 2-form of the $U(1)$ gauge field. The usual d-dimensional Einstein-Maxwell-Scalar theories correspond to (5.3) with $\chi = 0$.

5.1 Spherically Symmetric Solutions

The gravitational field equations obtained from the low energy limit of the string theory can be obtained from the following lagrangian

$$L = \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{4}{(D-2)\kappa^2}(\nabla\phi)^2 - \frac{1}{4}\, e^{-\alpha_e\phi}F^2\right]. \tag{5.4}$$

The field equations are

$$G_{ij} = \frac{8}{(D-2)}\left[\partial_i\phi\partial_j\phi - \frac{1}{2}(\nabla\phi)^2 g_{ij}\right] - \kappa^2 e^{-\alpha_e\phi}\left[F_i^m F_{jm} - \frac{1}{4}F^2 g_{ij}\right], \quad (5.5)$$

$$\nabla_i(e^{-\alpha_e\phi}F^{ij}) = 0, \quad (5.6)$$

$$\partial_i(\sqrt{-g}\, g^{ij}\, \partial_j\phi) + \frac{(D-2)\kappa^2\alpha_e\sqrt{-g}}{32} e^{-\alpha_e\phi}F^2 = 0, \quad (5.7)$$

where F_{ij} is the Maxwell and ϕ is the dilaton field. Here $i, j = 1, 2, ..., D \geq 4$.

In static spherically symmetric spacetimes, gravitational field equations first lead to

$$\frac{\eta_0(D-4)(D-2)!}{2C^{2(D-2)}}+\frac{\eta_2(D-3)(D-3)!}{2C^{2(D-2)}}+\frac{\eta_1(D-2)}{B^2}+\frac{\eta_3(D-1)(D-2)}{2}$$
$$-\frac{4\phi'^2}{(D-2)B^2}=0, \tag{5.8}$$

$$\eta_1(D-2)-\frac{8\phi'^2}{(D-2)}=0, \tag{5.9}$$

$$\eta_0(D-2)!+\eta_2(D-3)!-\frac{\kappa^2 Q^2 e^{\alpha_e\phi}}{A^2_{D-2}}=0. \tag{5.10}$$

Dilaton equation is:

$$\frac{8}{D-2}\left[\frac{A}{B}\,C^{D-2}\,\phi'\right]'-\frac{\alpha_e\,A\,B\,\kappa^2\,Q^2\,e^{\alpha_e\phi}}{2\,C^{D-2}\,A^2_{D-2}}=0. \tag{5.11}$$

¿From (5.8) and (5.9) we obtain

$$\left[\frac{\left(A\,C^d\right)'\,C}{B}\right]'=d^2\,A\,B\,C^{d-1} \tag{5.12}$$

where $d=D-3$. Using the freedom in choosing the r coordinate we can let

$$A\,B\,C^{d-1}=r^{d-1}, \tag{5.13}$$

by using (5.12) and (5.13) we obtain

$$A^2\,C^{2d}=r^{2d}-2b_1\,r^d+b_2 \tag{5.14}$$

where b_1 and b_2 are integration constants.

A combination of the dilaton (5.6) and gravitational field equations (5.5) gives

$$d\,T^2 - \frac{d-1}{r}T + T' + \frac{8}{(d+1)^2}\,\phi'^2 = 0, \tag{5.15}$$

where T is defined as

$$T = \frac{(r^d + c_1)\,r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} - \frac{16\phi'}{(d+1)^2\alpha_e}. \tag{5.16}$$

Defining now

$$\alpha_e\,\phi' = \frac{k_1\,d\,r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)}\,\psi(\rho), \tag{5.17}$$

the equation (5.15) becomes

$$\frac{r^{2d} - 2b_1 r^d + b_2}{d\,r^{d-1}}\,\frac{d\psi}{d\rho}\,\frac{d\rho}{dr} = (\psi + \mu)^2 + \nu^2. \tag{5.18}$$

The constants are given by

$$\begin{aligned} a &= \frac{(d+1)^2\,\alpha_e^2}{32\,d}, \\ \mu &= -(c_1 + b_1), \\ \nu^2 &= a\mu^2 - \Delta(a+1). \end{aligned} \tag{5.19}$$

Now, if we solve the auxiliary equation

$$\frac{r^{2d} - 2b_1 r^d + b_2}{d\,r^{d-1}}\frac{d\rho}{dr} = \rho \tag{5.20}$$

and insert ρ above, we obtain

$$\rho\frac{d\psi}{d\rho} = (\psi + \mu)^2 + \nu^2. \tag{5.21}$$

Note that ρ depends on the sign of $\Delta = b_1^2 - b_2$. ϕ can be found from ψ by

$$\phi = \frac{k_1}{\alpha_e} \int \frac{\psi(\rho)}{\rho} d\rho + \phi_0, \tag{5.22}$$

where $k_1 = \frac{2a}{a+1}$. The metric function C is connected to ϕ as

$$\frac{C'}{C} = \frac{(r^d + c_1) r^{d-1}}{(r^{2d} - 2b_1 r^d + b_2)} - \frac{16\phi'}{(d+1)^2 \alpha_e} \tag{5.23}$$

this gives us

$$\ln(\frac{C}{c_0})^d = \left(\int \frac{u + b_1 + c_1}{u^2 - \Delta} du \right) - \frac{\alpha_e}{2a} \phi \tag{5.24}$$

where $u = r^d - b_1$.

Metric functions A and B can be found from C through the equations (5.13) and (5.14).

We have three different cases according to the sign of Δ,

$$\begin{aligned}
&\text{Case 1} \quad \Delta > 0 \quad \ln \rho(r) = \frac{1}{r_1^d - r_2^d} \ln \left(\frac{r^d - r_1^d}{r^d - r_2^d} \right), \\
&\text{Case 2} \quad \Delta = 0 \quad \ln \rho(r) = -\frac{1}{r^d - r_3^d}, \\
&\text{Case 3} \quad \Delta < 0 \quad \ln \rho(r) = \frac{1}{\sqrt{-\Delta}} \left[\arctan \left(\frac{r^d - b_1}{\sqrt{-\Delta}} \right) - \frac{\pi}{2} \right]
\end{aligned} \tag{5.25}$$

where $r_1^d = b_1 + \sqrt{\Delta}$, $r_2^d = b_1 - \sqrt{\Delta}$, $r_3^d = b_1$.

The integration constants c_0 and ϕ_0 are determined through the asymptotic behaviour of the functions $C(r)$ and $\phi(r)$.

$$\lim_{r \to \infty} \phi = \phi_0 = 0, \tag{5.26}$$

$$\lim_{r \to \infty} \frac{C(r)}{r} = c_0 = 1. \tag{5.27}$$

To determine the remaining integration constants, we use the asymptotic behaviour of the metric, scalar field and the tensor field F_{ij} as follows:

$$
\begin{aligned}
\lim_{r\to\infty} r^d\,(A^2-1) &= -\frac{2\kappa^2\,M}{A_{d+1}\,(d+1)},\\
\lim_{r\to\infty} r^{d+1}\phi' &= -\frac{\kappa\sqrt{d+1}}{2A_{d+1}}\,\Sigma,\\
\lim_{r\to\infty} r^{d+1}\,F_{tr} &= \frac{Q}{A_{d+1}}.
\end{aligned}
\tag{5.28}
$$

CASE 1

$$\Delta > 0 \tag{5.29}$$

Which means there are two roots to the equation (5.14). According to the sign of ν^2 we have three distinct solutions

Type 1

$$
\begin{aligned}
\nu^2 &< 0 \qquad\qquad \lambda^2 = -\nu^2,\\
\psi &= \frac{\lambda-\mu+c_2(\lambda+\mu)\rho^{2\lambda}}{1-c_2\rho^{2\lambda}}.
\end{aligned}
\tag{5.30}
$$

The metric functions become

$$
\begin{aligned}
A^2 &= \frac{(r^d-r_1^d)(r^d-r_2^d)}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^d-r_1^d)(r^d-r_2^d)},\\
C^d &= (r^d-r_2^d)\left(\frac{1-c_2\,\rho^{2\lambda}}{1-c_2}\right)^{k_2}\rho^{k_3}.
\end{aligned}
\tag{5.31}
$$

Dilaton field is given as

$$e^{\alpha_e\,\phi} = \left[\frac{(1-c_2)\,\rho^{\lambda-\mu}}{1-c_2\,\rho^{2\lambda}}\right]^{k_1}. \tag{5.32}$$

The constants are given by

$$k_2 = \frac{1}{(a+1)}, \qquad k_3 = \frac{(r_1^d - r_2^d)}{2} - \frac{(\nu + a\mu)}{(a+1)}. \tag{5.33}$$

Undetermined integration constants are r_1, r_2, c_1 and c_2. From the boundary condition (5.28) we find that

$$\begin{aligned} 2M &= \left[\frac{1+c_2}{1-c_2}\lambda + a\,\mu\right] e_1, \\ \Sigma &= \left[\frac{1+c_2}{1-c_2}\lambda - \mu\right] e_2, \\ Q &= \lambda \frac{\sqrt{c_2}}{1-c_2} e_3. \end{aligned} \tag{5.34}$$

The constants e_i are given by

$$\begin{aligned} e_1 &= \frac{2\,A_{d+1}\,(d+1)}{(a+1)\kappa^2}, \\ e_2 &= \frac{A_{d+1}\,\alpha_e\,(d+1)^{\frac{3}{2}}}{8\,(a+1)\kappa}, \\ e_3 &= \frac{2\,A_{d+1}}{\kappa}\sqrt{\frac{(d+1)\,d}{(a+1)}}. \end{aligned} \tag{5.35}$$

Note that, we have four integration constants (c_1, c_2, r_1^d and r_2^d) but there exists only three equations to determine them. Also note that c_1 does not appear in the solution directly, so we have a freedom in c_1. ($c_2 \neq 1$) In order to complete the solution, we need to determine the integration constants in terms of the physical parameters M, Σ and Q. Let us define some auxiliary variables to solve the set of algebraic equations (5.34)

$$\begin{aligned} T_1 &= \frac{1}{a+1}\left(\frac{a\,\Sigma}{e_2} + \frac{2\,M}{e_1}\right), \qquad T_2 = \frac{1}{a+1}\left(\frac{2\,M}{e_1} - \frac{\Sigma}{e_2}\right), \\ T_3 &= \sqrt{1 - \frac{4\,Q^2}{e_3^2\,T_1^2}}. \end{aligned} \tag{5.36}$$

Then the integration constants are

$$
\begin{aligned}
\lambda &= \pm T_3 T_1, \\
\mu &= T_2, \\
c_2 &= -1 + \frac{2}{1 \pm T_3}.
\end{aligned}
\tag{5.37}
$$

and

$$
\Delta = \frac{a\mu^2 + \lambda^2}{a+1} \tag{5.38}
$$

The reality of T_3 imposes

$$
M + g\,\Sigma \geq s\,|Q| \tag{5.39}
$$

where g and s are given by

$$
g = \frac{\alpha_e\,(d+1)^{3/2}}{4d\,\kappa} \quad , \quad s = \frac{1}{\kappa}\sqrt{\frac{(d+1)(a+1)}{d}} \tag{5.40}
$$

Such an inequality has been found by Gibbons and Wells for D=4.

When $\Delta > 0$ we have two roots. In general these roots are the singular points of the space-time. If the integration constants satisfy some additional constraints one of these roots becomes regular. In this case we have a black hole solution carrying mass M, electric charge Q and scalar charge Σ. An invariant of the space-time is the scalar curvature is given by

$$
R = A_1 \frac{\rho^{z_1}\left[-\mu + \lambda + \frac{2c_2\lambda\rho^{2\lambda}}{1-c_2\rho^{2\lambda}}\right]^2}{[(r^d - r_1^d)(r^d - r_2^d)]^{1+\frac{1}{d}}(1 - c_2\rho^{2\lambda})^{\frac{2k_2}{d}}} + A_2 \frac{\rho^{z_1+2\lambda}\left(\frac{1-c_2}{1-c_2\rho^{2\lambda}}\right)^{\frac{2k_2}{d}+2}}{[(r^d - r_1^d)(r^d - r_2^d)]^{1+\frac{1}{d}}} \tag{5.41}
$$

where

$$
\begin{aligned}
z_1 &= \frac{2}{d(a+1)}(a\mu+\lambda), \\
A_1 &= \frac{8}{d+1}\frac{k_1^2 d^2}{\alpha_e^2}, \\
A_2 &= \frac{\kappa^2}{2}\frac{(D-4)}{(D-2)}.
\end{aligned}
\tag{5.42}
$$

As $r \to r_1$ we have a singularity unless we choose $\mu = \lambda$. By this choice, $\mu = \frac{r_1^d - r_2^d}{2}$ and hence $\rho^{z_1} \sim (r^d - r_1^d)^{\frac{2}{d}}$, $\rho^{2\lambda} \sim (r^d - r_1^d)^2$ around the horizon, so $R \to 0$.

If we insert these values for μ and λ in the solution, we obtain

$$
C^d = (r^d - r_2^d)\left(1 - \frac{c_2}{1-c_2}\frac{(r_2^d - r_1^d)}{r^d - r_2^d}\right)^{k_2}. \tag{5.43}
$$

At this point, the choice of $r_2 = 0$ gives Gibbons-Maeda solution, whereas the choice of $c_2 = \frac{r_2^d}{r_1^d}$ gives the GHS solution. It is easy to show that Gibbons-Maeda metric is the same as the GHS metric with $r > r_2$.

Type 2

$$
\begin{aligned}
\nu^2 &> 0 \\
\psi &= \nu\,\tan(c_2 + \nu\ln\rho) - \mu
\end{aligned}
\tag{5.44}
$$

$$
\int \frac{\psi(\rho)}{\rho}d\rho = -\ln\left[\cos(c_2 + \nu\ln\rho)\right] - \mu\ln\rho + c_3 \tag{5.45}
$$

After similar steps as the previous type, we arrive at the solution

$$
A^2 = \frac{(r^d - r_1^d)(r^d - r_2^d)}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^d - r_1^d)(r^d - r_2^d)}, \tag{5.46}
$$

$$
C^d = (r^d - r_2^d)\,\rho^{\frac{r_1^d - r_2^d}{2} - \frac{\mu k_1}{2}}\left[\frac{\cos(c_2 + \nu\ln\rho)}{\cos c_2}\right]^{k_2}. \tag{5.47}
$$

Scalar field is given as

$$e^{\alpha_e \phi} = \left[\frac{\cos c_2}{\rho^{\mu}\, \cos(c_2 + \nu \ln \rho)}\right]^{k_1}. \tag{5.48}$$

Physical parameters are found using (5.28)

$$\begin{aligned} 2M &= (a\mu + \nu \tan c_2)\, e_1, \\ \Sigma &= (-\mu + \nu \tan c_2)\, e_2, \\ Q &= \frac{\nu}{\cos c_2}\, e_3, \end{aligned} \tag{5.49}$$

$$\begin{aligned} \mu &= T_2, \\ \sin c_2 &= \frac{e_3 T_1}{Q}, \\ \nu \tan c_2 &= T_1. \end{aligned} \tag{5.50}$$

The condition $|\sin c_2| < 1$ imposes

$$M + g\,\Sigma < s\,|Q| \tag{5.51}$$

where g and s are defined in (5.40). We also have to check the sign of Δ.

$$(a+1)\Delta = \frac{1}{a+1}\left(\frac{a\,\Sigma^2}{e_2^2} + \frac{4\,M^2}{e_1^2}\right) - \frac{Q^2}{e_3^2}. \tag{5.52}$$

Here the sign of Δ puts a constraint on the physical variables.

Type 3

$$\begin{aligned} \nu^2 &= 0, \\ \psi &= -\mu - \frac{1}{\ln \rho + c_2}, \end{aligned} \tag{5.53}$$

$$\int \frac{\psi(\rho)}{\rho} d\rho = -\ln(\ln\rho + c_2) - \mu\ln\rho + c_3, \tag{5.54}$$

$$e^{\alpha_e \phi} = \left[\frac{c_2}{\rho^{\mu}(c_2 + \ln\rho)}\right]^{k_1}, \tag{5.55}$$

$$C^d = \left(\frac{c_2 + \ln\rho}{c_2}\right)^{k_2} [(r^d - r_1^d)(r^d - r_2^d)]^{\frac{1}{2}} \rho^{\frac{-\mu k_1}{2}}. \tag{5.56}$$

Physical parameters are found using (5.28)

$$\begin{aligned} 2M &= (a\mu - \frac{1}{c_2})\, e_1, \\ \Sigma &= (-\mu - \frac{1}{c_2})\, e_2, \\ Q &= \frac{e_3}{2c_2}. \end{aligned} \tag{5.57}$$

The solution is

$$\begin{aligned} \mu &= -T_2, \\ -\frac{1}{c_2} &= T_1, \end{aligned} \tag{5.58}$$

(5.59)

which gives

$$\frac{2Q}{e_3} = T_1. \tag{5.60}$$

This is the equality case of the inequality (5.39).

CASE 2

$$\Delta = 0 \tag{5.61}$$

Then there is one root to the equation (5.14). Denote it by r_1^d.

$$A\,C^d = (r^d - r_1^d) \tag{5.62}$$

$$A^2 = \frac{(r^d - r_1^d)^2}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^d - r_1^d)^2}, \tag{5.63}$$

$$C^d = (r^d - r_1^d)\,\rho^{-\frac{\mu k_1}{2}} \left[\frac{\cos(c_2 + \nu \ln \rho)}{\cos(c_2)}\right]^{k_2}. \tag{5.64}$$

Scalar field is given as

$$e^{\alpha_e \phi} = \left[\frac{\cos c_2}{\rho^\mu \cos(c_2 + \nu \ln \rho)}\right]^{k_1}. \tag{5.65}$$

CASE 3

$$\Delta < 0 \tag{5.66}$$

This case is similar to the previous one.

$$e^{\alpha_e \phi} = \left[\frac{\cos c_2}{\rho^\mu \cos(c_2 + \nu \ln \rho)}\right]^{k_1}, \tag{5.67}$$

$$A^2 = \frac{(r^{2d} - 2b_1 r^d + b_2)}{C^{2d}}, \qquad B^2 = \frac{r^{2d-2}\,C^2}{(r^{2d} - 2b_1 r^d + b_2)}, \tag{5.68}$$

$$C^d = (r^{2d} - 2b_1 r^d + b_2)^{\frac{1}{2}}\,\rho^{-\frac{\mu k_1}{2}} \left[\frac{\cos(c_2 + \nu \ln \rho)}{\cos(c_2)}\right]^{k_2}. \tag{5.69}$$

In cases 2 and 3, the relation of physical parameters to integration constants are exactly the same as case 1 type 2, the only difference being the sign of Δ, hence equation(5.52).

5.2 Static Solutions: Uniqueness of Black Hole Solutions

In this section we are interested in the black hole solutions of the Einstein Maxwell Dilaton theory in the static axially symmteric spacetimes in four dimensions. Using a different approach the uniqueness of static charged dilaton black black hole (GHS+GM black hole) has been recently shown in ref.[59]. In this proof the dilaton coupling contant was taken fixed ($a = 2$) which corresponds to the low energy limit of the string theory. Here we shall show that the static black holes of the Einstein Maxwell Dilaton theory are unique for arbitrary values of the dilaton coupling constant a. We are not going to solve the corresponding field equations but first formulate these field equations as a sigma model in two dimensions and use this formulation in the proof of the uniqueness of the solutions under the same boundary condtions. Uniqueness of the stationary black hole solutions of the Einstein theory is now a very well established concept [59]-[58]. This proof is based on the sigma model formulation of the stationary axially symmteric Einstein Maxwell field equations [56]-[57]. Here we shall follow the approach given by [56] and [58].

Rogatko, in his recent work [60] have used a sigma model formulation of field equations provided the proof of the uniqueness of black hole solutions in $N = 4, d = 4$supergravity subject to certain boundary conditions.

The line element of a static axially symmetric four dimensional spacetime is given by

$$ds^2 = e^{2\psi}\,[e^{2\gamma}\,(d\rho^2 + dz^2) + \rho^2\, d\phi^2\,] - e^{-2\psi}\,(dt - \omega d\phi)^2, \tag{5.70}$$

The field equations of the Einstein Maxwell Dilaton field theory with the above metric and $A_\mu = (A, 0, 0, 0)$ are

$$\nabla^2\,\psi + \frac{\kappa^2}{2}\,e^{2\,\psi - a\,\phi}\,(\nabla\,A)^2 = 0 \tag{5.71}$$

$$\nabla^2\,\phi - a\,\frac{\kappa^2}{8}\,e^{2\,\psi - a\,\phi}\,(\nabla\,A)^2 = 0 \tag{5.72}$$

$$\nabla^2 A + \nabla(2\psi - a\,\phi)\,\nabla A = 0 \tag{5.73}$$

$$\frac{1}{\rho}\,\gamma_{,z} = 2\,\psi_{,\rho}\,\psi_{,z} + 4\,\phi_\rho\,\phi_{,z} - \kappa^2\,e^{2\psi - a\,\phi}\,A_\rho\,A_{,z} \tag{5.74}$$

$$\frac{2}{\rho}\,\gamma_{,\rho} = 2\,\psi_{,\rho}^2 - 2\,\psi_{,z}^2 + 4\,\phi_{,\rho}^2 - 4\,\phi_{,z}^2 - \kappa^2\,e^{2\,\psi - a\,\phi}\,(A_{,\rho}^2 - \;A_{,z}^2) \tag{5.75}$$

let $E = \psi - \frac{a}{2}\phi$ and $B = \frac{\kappa'}{\sqrt{2}}\,A$ we then find

$$\nabla^2\,E + \;e^{2\,E}\,(\nabla\,B)^2 = 0 \tag{5.76}$$

$$\nabla^2\,B + 2\,\nabla\,E\,\nabla\,B = 0 \tag{5.77}$$

where

$$\kappa' = \kappa\sqrt{1 + \frac{a^2}{8}} \tag{5.78}$$

We wish to write the Eqs.(5.76 , 5.77) as a single complex equation by introducing a complex potential. In order to achieve this we introduce pseudopotential ω by use of (5.77)

$$\omega_\rho = \rho\,e^{-2E}\,B_z \quad , \quad \omega_z = -\rho\,e^{-2E}\,B_\rho \tag{5.79}$$

then the resulting equations can be written as the following single complex equation (the Ernst equation) for $\varepsilon = \rho\,e^E + i\omega$

$$Re(\varepsilon)\,\nabla^2\,\varepsilon = \nabla\,\varepsilon\,\nabla\,\varepsilon \tag{5.80}$$

Hence the above complex equation reperesents the Eqs.(5.76) and (5.77) if we let $\varepsilon = \rho\,e^E + i\,\omega$. The above Ernst equation (5.80) defines a sigma model on $SU(2)/U(1)$ with the equation of motion

$$\vec{\nabla}\left(P^{-1}\vec{\nabla}P\right)=0 \tag{5.81}$$

where

$$P=\frac{1}{\rho e^{E}}\begin{pmatrix}1 & -B\\ -B & \rho^{2}e^{2E}+B^{2}\end{pmatrix} \tag{5.82}$$

In the sequel we assume enough differentiability for the components of the matrix P in $V \cup \partial V$. Here V is a region in M with boundary ∂V. In our case V is the region $r>0$ (see section 2 type I solutions) and hence ∂V has two disconnected components.

We also assume that P is positive definite. Let P_1 and P_2 be two different solutions of (5.81). The difference of their equations satisfy

$$\vec{\nabla}\left(P_1^{-1}(\vec{\nabla}Q)\,P_2\right)=0. \tag{5.83}$$

where $Q=P_1\,P_2^{-1}$. Multiplying the both sides by $Q^{\dagger}$ (hermitian conjugation) and taking the trace we obtain

$$\vec{\nabla}\left[tr\left(Q^{\dagger}\,P_1^{-1}(\vec{\nabla}Q)\,P_2\right)\right]=tr\left[(\vec{\nabla}Q^{\dagger})\,P_1^{-1}\,(\vec{\nabla}Q)\,P_2\right] \tag{5.84}$$

The left hand side of the above equation can be simplified further and we obtain

$$\nabla^{2}q=tr\left[(\vec{\nabla}Q^{\dagger})\,P_1^{-1}(\vec{\nabla}Q)\,P_2\right] \tag{5.85}$$

n where $q=tr(Q)$. Using the hermiticity and positive definiteness properties of the matrices P_1 and P_2 we may let

$$P_i=A_i\,A_i^{\dagger},\ (i=1,2) \tag{5.86}$$

where A_1 and A_2 are nonsingular $n\times n$ matrices given by

$$A_i = \frac{1}{\sqrt{\rho} e^{\frac{E_i}{2}}} \begin{pmatrix} 1 & 0 \\ -B_i & \rho e^{E_i} \end{pmatrix} \tag{5.87}$$

With the aid of (5.86) Eq.(5.85) reduces to

$$\nabla^2 q = tr(\vec{J}^\dagger \vec{J}) \tag{5.88}$$

where

$$\vec{J} = A_1^{-1} (\vec{\nabla} Q) A_2 \tag{5.89}$$

Eq.(5.88) is a crucial step towards the proof of the uniqueness theorems. It is clear that the right hand side is positive definite at all point of V. Before going on let us give the scalar function q.

$$q = 2 + \frac{1}{\rho^2 e^{E_1+E_2}} \left[\rho^2 (e^{E_1} - e^{E_2})^2 + (B_1 - B_2)^2\right] \tag{5.90}$$

It is clear that $q = 2$ and its first derivatives vanish on the boundary ∂V of V.

Let M be an 2 dimensional manifold with local coordinates (ρ, z). Let V be a region in M with boundary ∂V. Let P be a hermitian positive definite 2×2 matrix with unit determinant and let P_1 and P_2 be two such matrices satisfying (5.81) in V with the same boundary conditions on ∂V then we have $P_1 = P_2$ at all points in region V . The proof is as follows. Integrating (5.88) in V we obtain

$$\int_{\partial V} \vec{\nabla} q \, d\vec{\sigma} = \int_V tr\,(\vec{J}^\dagger \vec{J})\, dV \tag{5.91}$$

and using the boundary condition $q = 2$ on ∂V we get

$$\int_V tr\,(\vec{J}^\dagger \vec{J})\, dV = 0 \tag{5.92}$$

Then the integrand in (5.92) vanishes at all points in V. This implies the vanishing of $\vec{J}$ which implies that $Q = Q_0 =$ a constant matrix in V. Since Q is the identity matrix I on ∂V then $Q = I$ in V. Hence $P_1 = P_2$ at all points in V. Another way to prove to obtain this result is to use (5.88) directly. The vanishing of the integrand in (5.92) implies that q is an harmonic function in V. Since $q = 2$ on the boundary ∂V of V then it must be equal to the same constant in V as well. This implies that $P_1 = P_2$ in V.

In four dimensions the Einstein Maxwell Dilaton field theory a static black hole should carry mass M , electric charge Q and dilaton charge Σ. Such a black hole solution exist which was found by Gibbons and Maeda for an arbitrary dilaton coupling parameter a. Here the above proof implies that all those solutions with the same black hole boundary conditions (asymptotically flat and regular horizons) as the GM solution are the same everywhere in spacetime.

Chapter 6

Cvetic-Youm Solution in 4 and 5 dimensions

In this chapter I will give the explicit expression of the metric for two different black hole solutions found by Cvetic and Youm. [8, 9] In chapter 8, I will analyze the thermodynamical properties of these black hole solutions.

6.1 The Metric and Physical Parameters in 4 Dimensions

Cvetic and Youm present an explicit form of the (generating) solution for the four-charge rotating black hole solution of four-dimensional $N = 4$ (or $N = 8$) superstring vacua. They choose to parameterize the generating solution in terms of the massless fields of the heterotic string compactified on a six-torus (or Neveu-Schwarz-Neveu-Schwarz (NS-NS) sector of the Type IIA string compactified on T^6). This solution has an equivalent parameterization (due to the string-string duality) in terms of the NS-NS fields of Type IIA compactified on $K3 \times T^2$ or T-dualized Type IIB string. Due to the T-duality (or U-duality) of the Type IIA string, the solutions parameterized in terms of the NS-NS charges have a map onto Ramond-Ramond (R-R) charges and thus an interpretation in terms of D-brane configurations.

A brief summary of toroidal compactification is as follows:

The compactification of the extra $(10 - D)$ spatial coordinates on a $(10 - D)$-torus can be achieved by choosing the following Abelian Kaluza-Klein

Ansatz for the ten-dimensional metric

$$\hat{G}_{MN} = \begin{pmatrix} e^{a\varphi} g_{\mu\nu} + G_{mn} A^{(1)m}_{\mu} A^{(1)n}_{\nu} & A^{(1)m}_{\mu} G_{mn} \\ A^{(1)n}_{\nu} G_{mn} & G_{mn} \end{pmatrix}, \tag{6.1}$$

where $A^{(1)m}_{\mu}$ ($\mu = 0, 1, ..., D-1$; $m = 1, ..., 10-D$) are D-dimensional Kaluza-Klein $U(1)$ gauge fields, $\varphi \equiv \hat{\Phi} - \frac{1}{2}\ln\det G_{mn}$ is the D-dimensional dilaton field, and $a \equiv \frac{2}{D-2}$. Then, the affective action is specified by the following massless bosonic fields: the (Einstein-frame) graviton $g_{\mu\nu}$, the dilaton e^{φ}, $(36 - 2D)$ $U(1)$ gauge fields $\mathcal{A}^{i}_{\mu} \equiv (A^{(1)m}_{\mu}, A^{(2)}_{\mu m}, A^{(3)I}_{\mu})$ defined as $A^{(2)}_{\mu m} \equiv \hat{B}_{\mu m} + \hat{B}_{mn} A^{(1)n}_{\mu} + \frac{1}{2}\hat{A}^{I}_{m} A^{(3)I}_{\mu}$, $A^{(3)I}_{\mu} \equiv \hat{A}^{I}_{\mu} - \hat{A}^{I}_{m} A^{(1)m}_{\mu}$, and the following symmetric $O(10-D, 26-D)$ matrix of the scalar fields (moduli):

$$M = \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}a^{T} \\ -C^{T}G^{-1} & G + C^{T}G^{-1}C + a^{T}a & C^{T}G^{-1}a^{T} + a^{T} \\ -aG^{-1} & aG^{-1}C + a & I + aG^{-1}a^{T} \end{pmatrix}, \tag{6.2}$$

where $G \equiv [\hat{G}_{mn}]$, $C \equiv [\frac{1}{2}\hat{A}^{(I)}_{m}\hat{A}^{(I)}_{n} + \hat{B}_{mn}]$ and $a \equiv [\hat{A}^{I}_{m}]$ are defined in terms of the internal parts of ten-dimensional fields. Then the effective D-dimensional effective action takes the form:

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1}{16\pi G_D}\sqrt{-g}[\mathcal{R}_g - \frac{1}{(D-2)} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{8} g^{\mu\nu}\mathrm{Tr}(\partial_\mu ML\partial_\nu ML) - \frac{1}{12} e^{-2a\varphi} g^{\mu\mu'} g^{\nu\nu'} g^{\rho\rho} \\ - \; & \frac{1}{4} e^{-a\varphi} g^{\mu\mu'} g^{\nu\nu'} \mathcal{F}^{i}_{\mu\nu}(LML)_{ij}\mathcal{F}^{j}_{\mu'\nu'}], \end{aligned}$$

where $g \equiv \det g_{\mu\nu}$, $\mathcal{R}_g$ is the Ricci scalar of $g_{\mu\nu}$, and $\mathcal{F}^{i}_{\mu\nu} = \partial_\mu \mathcal{A}^{i}_{\nu} - \partial_\nu \mathcal{A}^{i}_{\mu}$ are the $U(1)^{36-2D}$ gauge field strengths.

The D-dimensional effective action (6.3) is invariant under the $O(10 - D, 26 - D)$ transformations (T-duality) [53, 54]:

$$M \to \Omega M \Omega^{T}, \quad \mathcal{A}^{i}_{\mu} \to \Omega_{ij}\mathcal{A}^{j}_{\mu}, \quad g_{\mu\nu} \to g_{\mu\nu}, \quad \varphi \to \varphi, \quad B_{\mu\nu} \to B_{\mu\nu}, \tag{6.4}$$

where Ω is an $O(10 - D, 26 - D)$ invariant matrix, *i.e.*, with the following property:

$$\Omega^{T} L \Omega = L, \quad L = \begin{pmatrix} 0 & I_{10-D} & 0 \\ I_{10-D} & 0 & 0 \\ 0 & 0 & I_{26-D} \end{pmatrix}, \tag{6.5}$$

where I_n denotes the $n \times n$ identity matrix.

In particular, for $D = 4$ the field strength of the one-form field is self-dual and the corresponding equations of motion and Bianchi identities are invariant under the $SL(2,R)$ transformations (S-duality) [54]:

$$S \to \frac{aS+b}{cS+d}, \quad M \to M, \quad g_{\mu\nu} \to g_{\mu\nu}, \quad \mathcal{F}^i_{\mu\nu} \to (c\Psi+d)\mathcal{F}^i_{\mu\nu} + ce^{-2\varphi}(ML)_{ij}\tilde{\mathcal{F}}^j_{\mu\nu}, \tag{6.6}$$

where $\tilde{\mathcal{F}}^{i\,\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}^i_{\rho\sigma}$ and $a,b,c,d \in R$ satisfy $ad-bc=1$. Here, $S \equiv \Psi + ie^{-\varphi}$ is a complex scalar field

In order to obtain the explicit form of rotating charged solution, they employ the solution generating technique, by performing symmetry transformations on a known solution. In particular, they perform four $SO(1,1) \subset O(8,24)$ boosts [52] on the four-dimensional Kerr solution, specified by the ADM mass m and the rotational parameter l (angular momentum per unit mass)

Here $O(8,24)$ is a symmetry of the effective three-dimensional action for stationary solutions of toroidally compactified heterotic string [50]. The four boosts δ_{e1}, δ_{e2}, δ_{p1} and δ_{p2} induce two electric $Q_2^{(1),(2)}$ and two magnetic charges $P_1^{(1),(2)}$ of $U(1)$ gauge fields $A_{\mu 2}^{(1),(2)}$ and $A_{\mu 1}^{(1),(2)}$, respectively. The solution obtained in that manner is specified by the ADM mass, *four* $U(1)$ charges, and one rotational parameter

In addition, one can subsequently rescale the asymptotic values of the dilaton-axion field and the four toroidal moduli of two-torus, *i.e.*, the scalar fields that vary with spatial direction.

Thus, the starting point is the following four-dimensional Kerr solution:

$$\begin{aligned} ds^2 &= -\frac{r^2+l^2\cos^2\theta-2mr}{r^2+l^2\cos^2\theta}dt^2 + \frac{r^2+l^2\cos^2\theta}{r^2+l^2-2mr}dr^2 + (r^2+l^2\cos^2\theta)d\theta^2 \\ &+ \frac{\sin^2\theta}{r^2+l^2\cos^2\theta}[(r^2+l^2)(r^2+l^2\cos^2\theta)+2ml^2r\sin^2\theta]d\phi^2 \\ &-\frac{4mlr\sin^2\theta}{r^2+l^2\cos^2\theta}dtd\phi, \end{aligned} \tag{6.7}$$

where m is its ADM mass and l is the rotational parameter. The explicit sequence of the four boost transformations as well as technical details of relating the fields of the effective three-dimensional action and the four-dimensional fields are detailed in Ref. [52] (See also Ref. [51].).

The metric for four-dimensional rotating charged black hole solutions of toroidally compactified superstring theory, parametrized by the ADM mass, four charges and angular momentum, is given by [8] [1]:

[1]The four-dimensional Newton's constant is taken to be $G_N^{D=4} = \frac{1}{8}$ and the convention

$$
\begin{aligned}
ds_E^2 &= \Delta^{\frac{1}{2}}[-\frac{r^2-2mr+l^2\cos^2\theta}{\Delta}dt^2+\frac{dr^2}{r^2-2mr+l^2}+d\theta^2+\frac{\sin^2\theta}{\Delta}\{(r+2m\sinh^2\delta_1) \\
&\times (r+2m\sinh^2\delta_2)(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_4)+l^2(1+\cos^2\theta)r^2+W \\
&+ 2ml^2r\sin^2\theta\}d\phi^2-\frac{4ml}{\Delta}\{(\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4 \\
&- \sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4)r+2m\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4\}\sin^2\theta dt d\phi], \qquad (6.8)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta &\equiv (r+2m\sinh^2\delta_1)(r+2m\sinh^2\delta_2)(r+2m\sinh^2\delta_3)(r+2m\sinh^2\delta_4) \\
&+ (2l^2r^2+W)\cos^2\theta, \\
W &\equiv 2ml^2(\sinh^2\delta_1+\sinh^2\delta_2+\sinh^2\delta_3+\sinh^2\delta_4)r \\
&+ 4m^2l^2(2\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4 \\
&- 2\sinh^2\delta_1\sinh^2\delta_2\sinh^2\delta_3\sinh^2\delta_4-\sinh^2\delta_1\sinh^2\delta_2\sinh^2\delta_4 \\
&- \sinh^2\delta_1\sinh^2\delta_2\sinh^2\delta_3-\sinh^2\delta_2\sinh^2\delta_3\sinh^2\delta_4-\sinh^2\delta_1\sinh^2\delta_3\sinh^2\delta_4) \\
&+ l^4\cos^2\theta. \qquad (6.9)
\end{aligned}
$$

The outer and inner event horizons are at

$$r_\pm = m \pm \sqrt{m^2-l^2}, \qquad (6.10)$$

Here, m, the non-extremality parameter, is related to the mass of Kerr solution, l is related to the angular momentum of the Kerr solution and $\delta_{1,2,3,4}$ are boost parameters. Our aim is to write the variation of S in terms of the physical parameters ADM mass M, two electric charges Q_1, Q_2, two magnetic charges P_1, P_2, and the angular momentum J. The physical parameters can be expressed in terms of m, l and the boosts as follows:

$$
\begin{aligned}
M &= 4m(\cosh^2\delta_1+\cosh^2\delta_2+\cosh^2\delta_3+\cosh^2\delta_4)-8m, \\
Q_1 &= 4m\cosh\delta_1\sinh\delta_1, \\
Q_2 &= 4m\cosh\delta_2\sinh\delta_2, \qquad (6.11) \\
P_1 &= 4m\cosh\delta_3\sinh\delta_3, \\
P_2 &= 4m\cosh\delta_4\sinh\delta_4, \\
J &= 8\,l\,m\,(\cosh\delta_1\cosh\delta_2\cosh\delta_3\cosh\delta_4-\sinh\delta_1\sinh\delta_2\sinh\delta_3\sinh\delta_4).
\end{aligned}
$$

of, *e.g.*, Ref. [49], is followed for the definitions of the ADM mass, charges, dipole moments and angular momenta.

Note that we choose $G_N = \frac{\pi}{4}$.

6.2 Metric and Physical Parameters in 5 Dimensions

Another black hole solution found by Cvetic and Youm is in five dimensions. [9] The starting point for obtaining the generating five-dimensional charged rotating solution is the five-dimensional neutral solution with two rotational parameters $l_{1,2}$ [49]. We can write it in the following form:

$$\begin{aligned}
ds^2 &= -\frac{r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta - 2m}{r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta}dt^2 + \frac{r^2(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta)}{(r^2 + l_1^2)(r^2 + l_2^2) - 2mr^2}dr^2 \\
&+ (r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta)d\theta^2 + \frac{4ml_1l_2\sin^2\theta\cos^2\theta}{r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta}d\phi d\psi \\
&+ \frac{\sin^2\theta}{r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta}[(r^2 + l_1^2)(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta) + 2ml_1^2\sin^2\theta]d\phi^2 \\
&+ \frac{\cos^2\theta}{r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta}[(r^2 + l_2^2)(r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta) + 2ml_2^2\cos^2\theta]d\psi^2 \\
&- \frac{4ml_1\sin^2\theta}{r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta}dtd\phi - \frac{4ml_2\cos^2\theta}{r^2 + l_1^2\cos^2\theta + l_2^2\sin^2\theta}dtd\psi. \qquad (6.12)
\end{aligned}$$

The black hole solution given in this section is uniquely parameterized by the ADM mass (which can be traded for the non-extremality parameter $m > 0$), two angular momenta and 27 (conserved) electric charges. [55]. The solution is parametrized in terms of massless fields of the heterotic string compactified on a five-torus (T^5)

A generating technique is used by performing a subset of $O(8,24)$ transformations, *i.e.*, symmetry transformations of the effective three-dimensional action for stationary solutions, on the five-dimensional (neutral) rotating solution parameterized by the mass m and two rotating parameters l_1 and l_2. The explicit form of the generating solution is determined by three $SO(1,1) \subset O(8,24)$ boosts δ_{e1}, δ_{e2}, and δ_e, which specify respectively the two electric charges $Q_1^{(1)}$, $Q_1^{(2)}$ of the two $U(1)$ gauge fields, *i.e.*, the Kaluza-Klein $A_{\mu 1}^{(1)}$ and the two-form $A_{\mu 1}^{(2)}$ gauge fields associated with the first compactified direction, and the charge Q, *i.e.*, the electric charge of the vector field, whose field strength is dual to the field strength $H_{\mu\nu\rho}$ of the five-dimensional two form field $B_{\mu\nu}$.

The metric for five-dimensional rotating charged black holes of toroidally compactified string theory, specified by the ADM mass M, three charges Q_1, Q_2, Q_3 and two rotational parameters l_1, l_2 is given by [9]:

$$
\begin{aligned}
ds_E^2 &= g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta} d\theta^2 + 2g_{\phi\psi} d\phi d\psi + 2g_{\phi t} d\phi dt \\
&\quad +2g_{\psi t} d\psi dt + g_{\phi\phi} d\phi^2 + g_{\psi\psi} d\psi^2,
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
g_{tt} &= -\Delta^{\frac{-2}{3}} R(R-2m), \\
g_{rr} &= \frac{\Delta^{\frac{1}{3}}\, r^2}{(r^2+l_1^2)(r^2+l_2^2) - 2mr^2}, \\
g_{\theta\theta} &= \Delta^{\frac{1}{3}}, \\
g_{\phi\psi} &= \cos^2\theta \sin^2\theta \Delta^{\frac{-2}{3}} (L_1 k_3 + L_3 k_1), \\
g_{\phi t} &= -2m \sin^2\theta \Delta^{\frac{-2}{3}} (l_1 R c_1 c_2 c_3 + l_2 (2m - R) s_1 s_2 s_3), \\
g_{\psi t} &= -2m \cos^2\theta \Delta^{\frac{-2}{3}} (l_1 (2m - R) s_1 s_2 s_3 + l_2 R c_1 c_2 c_3), \\
g_{\phi\phi} &= \sin^2\theta \Delta^{\frac{-2}{3}} [\Delta + \sin^2\theta (L_1 k_1 + L_2 k_2 + L_3 k_3)], \\
g_{\psi\psi} &= \cos^2\theta \Delta^{\frac{-2}{3}} [\Delta + \cos^2\theta (L_1 k_1 - L_2 k_2 + L_3 k_3)],
\end{aligned}
\tag{6.14}
$$

where

$$
\begin{aligned}
L_1 &= l_1^2 + l_2^2, \\
L_2 &= l_1^2 - l_2^2, \\
L_3 &= 2 l_1 l_2, \\
k_1 &= mR - 2m^2 q - 4m^2 t, \\
k_2 &= R^2 + mR + 2mRp + 2m^2 q, \\
k_3 &= 4m^2\, c_1 c_2 c_3 s_1 s_2 s_3, \\
R &= r^2 + l_1^2 \cos^2\theta + l_2^2 \sin^2\theta, \\
p &= s_1^2 + s_2^2 + s_3^2, \\
q &= s_1^2 s_2^2 + s_1^2 s_3^2 + s_2^2 s_3^2, \\
t &= s_1^2 s_2^2 s_3^2,
\end{aligned}
\tag{6.15}
$$

and $s_i,\ c_i$ stand for $\sinh\delta_i,\ \cosh\delta_i,\ (i = 1, 2, 3)$ respectively.

Electromagnetic vector potentials are given as:

$$
\begin{aligned}
A^{(1)}_{t1} &= \frac{m\cosh\delta_1\sinh\delta_1}{r^2+2m\sinh^2\delta_1+l_1^2\cos^2\theta+l_2^2\sin^2\theta},\\
A^{(1)}_{\phi 1} &= m\sin^2\theta\frac{l_1\sinh\delta_1\sinh\delta_2\cosh\delta_3-l_2\cosh\delta_1\cosh\delta_2\sinh\delta_3}{r^2+2m\sinh^2\delta_1+l_1^2\cos^2\theta+l_2^2\sin^2\theta},\\
A^{(1)}_{\psi 1} &= m\cos^2\theta\frac{l_1\cosh\delta_1\sinh\delta_2\sinh\delta_3-l_2\sinh\delta_1\cosh\delta_2\cosh\delta_3}{r^2+2m\sinh^2\delta_1+l_1^2\cos^2\theta+l_2^2\sin^2\theta},\\
A^{(2)}_{t1} &= \frac{m\cosh\delta_2\sinh\delta_2}{r^2+2m\sinh^2\delta_2+l_1^2\cos^2\theta+l_2^2\sin^2\theta},\\
A^{(2)}_{\phi 1} &= m\sin^2\theta\frac{l_1\cosh\delta_1\sinh\delta_2\cosh\delta_3-l_2\sinh\delta_1\cosh\delta_2\sinh\delta_3}{r^2+2m\sinh^2\delta_2+l_1^2\cos^2\theta+l_2^2\sin^2\theta},\\
A^{(2)}_{\psi 1} &= m\cos^2\theta\frac{l_1\sinh\delta_1\cosh\delta_2\sinh\delta_3-l_2\cosh\delta_1\sinh\delta_2\cosh\delta_3}{r^2+2m\sinh^2\delta_2+l_1^2\cos^2\theta+l_2^2\sin^2\theta},\\
B_{t\phi} &= -2m\sin^2\theta(l_1\sinh\delta_1\sinh\delta_2\cosh\delta_3-l_2\cosh\delta_1\cosh\delta_2\sinh\delta_3)(r^2+l_1^2\cos^2\theta\\
&\quad +l_2^2\sin^2\theta+m\sinh^2\delta_1+m\sinh^2\delta_2)/[(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_1)\\
&\quad \times(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_2)],\\
B_{t\psi} &= -2m\cos^2\theta(l_2\sinh\delta_1\sinh\delta_2\cosh\delta_3-l_1\cosh\delta_1\cosh\delta_2\sinh\delta_3)(r^2+l_1^2\cos^2\theta\\
&\quad +l_2^2\sin^2\theta+m\sinh^2\delta_1+m\sinh^2\delta_2)/[(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_1)\\
&\quad \times(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_2)],\\
B_{\phi\psi} &= \frac{2m\cosh\delta_3\sinh\delta_3\cos^2\theta\sin^2\theta(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+m\sinh^2\delta_1+m\sinh^2\delta}{(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_1)(r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta+2m\sinh^2\delta_2}
\end{aligned}
\tag{6.16}
$$

$$\Delta = R^3+2mpR^2+4m^2qR+8m^3t \tag{6.17}$$

$$r_\pm^2 = m-\frac{1}{2}L_1\pm\frac{1}{2}\sqrt{L_2^2+4m(m-L_1)} \tag{6.18}$$

We choose $G_N=\frac{\pi}{4}$.

The physical quantities ADM mass M, three charges Q_1, Q_2, Q_3 and two angular momenta J_1, J_2 are given as

$$
\begin{aligned}
M &= 2m(\cosh^2\delta_1+\cosh^2\delta_2+\cosh^2\delta_3)-3m,\\
Q_1 &= 2m\cosh\delta_1\,\sinh\delta_1,\\
Q_2 &= 2m\cosh\delta_2\,\sinh\delta_2,
\end{aligned}
\tag{6.19}
$$

$$
\begin{aligned}
Q_3 &= 2m \cosh\delta_3 \sinh\delta_3, \\
J_1 &= 4\,m\,(l_1\, \cosh\delta_1 \cosh\delta_2 \cosh\delta_3 - l_2\, \sinh\delta_1 \sinh\delta_2 \sinh\delta_3), \\
J_2 &= 4\,m\,(l_2\, \cosh\delta_1 \cosh\delta_2 \cosh\delta_3 - l_1\, \sinh\delta_1 \sinh\delta_2 \sinh\delta_3),
\end{aligned}
$$

where m is the nonextremality parameter, $\delta_{1,2,3}$ are the boost parameters and $l_{1,2}$ are the angular momentum parameters.

Chapter 7

Dilaton-Maxwell Colliding Waves in Low Energy Limit of String Theory

We have found that plane wave geometries are exact solutions for the string theory to all orders of string tension parameter.

The study of the gravitational interaction coupled to the Maxwell and dilaton fields has been the subject of recent investigations related to the heterotic string theory. Dilaton fields coupled to Einstein-Maxwell fields appear as a result of a dimensional reduction of the Kaluza-Klein Lagrangian. The solutions of the low-energy effective field are different from those in classical relativity.

For the first time in literature, we have found a solution describing colliding plane waves in dilaton gravity. [15]. After our publication, Nora Bretón, Tonatiuh Matos and Alberto García have studied a related problem in [47, 48] They consider the action

$$S = \int d^4x\sqrt{-g}\{-R + 2(\nabla\Phi)^2 + e^{-2\alpha\Phi}F^2\}, \tag{7.1}$$

where $g = \det(g_{\mu\nu}), \mu, \nu = 0, 1, 2, 3$. R is the scalar curvature, $F_{\mu\nu}$ is the Maxwell field, and Φ is the dilaton field.

They present a class of solutions of the Einstein-Maxwell-Dilaton field equations within the metric structure endowed with two orthogonal space-like Killing vectors. Two explicitly given sub-classes of solutions bear an interpretation as colliding plane waves in the low-energy limit of the heterotic string theory.

An extension of our result was studied by Özay Gürtuğ and Mustafa Halilsoy in [61]. they also consider colliding non-orthogonally polarized Abelian gauge fields with a cross polarization.

An interesting connection between the gravitational wave solution we found and singular hypersurfaces were considered by C. Barrabès and G.F. Bressange [46]. They studied the junction conditions which have to be satisfied by the various fields at an arbitrary singular hypersurface separating two different spacetimes in scalar-tensor theories of gravity.

The lightlike case has interesting properties because it can at the same time describe a lightlike shell with surface energy density and surface stresses, and an impulsive gravitational wave which is accompanied by shock waves when discontinuous gauge fields are present.

In the next sections we shall give the form of the metrics in the incoming regions. These will constitute the data for field equations in the interaction region. In the second section we give the formulation of the problem for one $U(1)$ abelian gauge field with a solution generalizing the Bell-Szekeres solution in general relativity. In the third section we consider two abelian $U(1)$ gauge fields and give some interesting exact solutions of the collision of plane waves problem. In the Appendix 2 we reduce the Maxwell Dilaton field equations , in the collision of plane waves , to two dimensional Ernst equation.

7.1 Dilaton gravity with one $U(1)$ vector field

Einstein-Maxwell-Dilaton Gravity is derivable from a variational principle with the lagrangian density

$$L = \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{2}{\kappa^2}(\nabla\psi)^2 - \frac{1}{4}\, e^{-a\psi}F^2\right]. \tag{7.2}$$

where a is the dilaton coupling constant. The field equations are

$$G_{\mu\nu} = 4\,\left[\partial_\mu\psi\,\partial_\nu\psi - \frac{1}{2}(\nabla\psi)^2 g_{\mu\nu}\right] + \kappa^2 e^{-a\psi}\left[F_\mu^\alpha F_{\nu\alpha} - \frac{1}{4}F^2 g_{\mu\nu}\right], \tag{7.3}$$

$$\nabla_\mu(e^{-a\psi}F^{\mu\nu}) = 0, \tag{7.4}$$

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\psi) + \frac{\kappa^2\, a\sqrt{-g}}{16} e^{-a\psi}F^2 = 0, \tag{7.5}$$

A space time describing the collision of plane waves admits two space like Killing vector fields. In the general case these vectors are nonorthogonal but here in this work we consider them to be orthogonal. For such a case an appropriate form of the metric $g_{\mu\nu}$ and $U(1)$ gauge potential A_μ are given by

$$ds^2 = 2e^{-M}du\,dv + e^{-U-V}dy^2 + e^{-U+V}dz^2 \tag{7.6}$$

$$A_\mu = (0, 0, A, 0) \tag{7.7}$$

where $M = M(u,v)$, $U = U(u,v)$, $V = V(u,v)$, $A = A(u,v)$ and dilaton field $\psi = \psi(u,v)$. The field equations turn out to be

$$-2A_{,uv} = (V_u - a\,\psi_u)\,A_{,v} + (V_v - a\,\psi_v)\,A_{,u} \tag{7.8}$$

$$U_{uv} - U_u U_v = 0 \tag{7.9}$$

$$2M_{uv} = -2\,U_{uv} + U_u U_v + V_u V_v + 8\psi_u\psi_v \tag{7.10}$$

$$2V_{uv} - U_u V_v - U_v V_u - 2\kappa^2 e^{U+V-a\psi} A_{,u}A_{,v} = 0 \tag{7.11}$$

$$2\psi_{uv} - U_u\psi_v - U_v\psi_u + \frac{a\kappa^2}{4}e^{U+V-a\psi}A_{,u}A_{,v} = 0 \tag{7.12}$$

$$-2M_u U_u - 2\,U_{uu} + U_u^2 + V_u^2 + 8\psi_u^2 + 2\kappa^2 e^{U+V-a\psi}A_{,u}^2 = 0 \tag{7.13}$$

$$-2M_vU_v - 2\,U_{vv} + U_v^2 + V_v^2 + 8\psi_v^2 + 2\kappa^2 e^{U+V-a\psi} A_{,v}^2 = 0 \tag{7.14}$$

Note that (7.10) can be derived from the other equations. It is not independent. From (7.11) and (7.12), letting $E = V - a\psi$ we obtain

$$2E_{uv} - U_uE_v - U_vE_u - (2+\frac{a^2}{4})\kappa^2 e^{U+E} A_{,u}A_{,v} \tag{7.15}$$

Letting

$$B = \sqrt{2+\frac{a^2}{4}}\,\kappa\,A \tag{7.16}$$

(7.8) and (7.15) become

$$-2\,B_{,uv} = E_uB_{,v} + E_vB_{,u} \tag{7.17}$$

$$2E_{uv} - U_uE_v - U_vE_u - e^{U+E}\,B_{,u}\,B_{,v} = 0 \tag{7.18}$$

The above two equations are the real and imaginary parts of the following Ernst equation

$$Re(\varepsilon)\,\nabla^2\,\varepsilon = \nabla\varepsilon\,\nabla\varepsilon \tag{7.19}$$

where differential operators in (7.19) are defined with respect to the metric given by $ds^2 = 2\,du\,dv - e^{-2U}\,d\,\phi^2$ and

$$\varepsilon = e^{-\frac{1}{2}(E+U)} + i\frac{B}{\sqrt{2}} \tag{7.20}$$

The remaining part of the Einstein equations are given as follows

$$U_{uv} - U_uU_v = 0 \tag{7.21}$$

$$2\,X_{uv} - U_u X_v - U_v X_u = 0 \tag{7.22}$$

$$-\,2M_u U_u - 2\,U_{uu} + U_u^2 + \frac{1}{\alpha}\,(E_u^2 + 8\,X_u^2) + 2\kappa^2 e^{U+E} A_{,u}^2 = 0 \tag{7.23}$$

$$-\,2M_v U_v - 2\,U_{vv} + U_v^2 + \frac{1}{\alpha}\,(E_v^2 + 8\,X_v^2) + 2\kappa^2 e^{U+E} A_{,v}^2 = 0 \tag{7.24}$$

where

$$\psi = \frac{1}{\alpha}\,(X - \frac{a}{8}\,E)\;,\; V = \frac{1}{\alpha}\,(a\,X + \,E)\;,\; \alpha = 1 + \frac{a^2}{8} \tag{7.25}$$

Hence a solution of the Dilaton Gravity field equations depends upon a linear equation (7.22) and the Ernst equation (7.19). The integrability of the Ernst equation and its properties are now very well known [43] , but the characteristic initial value problem has not been solved yet.

The formulation of the collision of plane waves is as follows: The space time is divided into four disjoint regions by the null hyperplanes $u = 0$ and $v = 0$

The first region: $(u \leq 0, v \leq 0)$

$$ds^2 = 2du\,dv + dy^2 + dz^2 \tag{7.26}$$

This is the flat space time with $\psi = A = 0$.

The second region: $(u > 0, v \leq 0)$

$$ds^2 = 2e^{-M_2} du\,dv + e^{-U_2 - V_2} dy^2 + e^{-U_2 + V_2} dz^2 \tag{7.27}$$

where $M_2 = M_2(u)$, $U_2 = U_2(u)$, $V_2 = V_2(u)$, $\psi_2 = \psi_2(u)$ and $A_2 = A_2(u)$ constitute the data at $v \leq 0$. The only field equation is

$$-\,2M_{2,u} U_{2,u} - 2\,U_{2,uu} + U_{2,u}^2 + \frac{1}{\alpha}\,(E_{2,u}^2 + 8\,X_{2,u}^2) + 2\kappa^2\, e^{U_2 + E_2}\, A_{2,u}^2 = 0 \tag{7.28}$$

The third region: ($u \leq 0, v > 0$)

$$ds^2 = 2e^{-M_3} du\, dv + e^{-U_3 - V_3} dy^2 + e^{-U_3 + V_3} dz^2 \qquad (7.29)$$

where $M_3 = M_3(v)$, $U_3 = U_3(v)$, $V_3 = V_3(v)$, $\psi_3 = \psi_3(v)$ and $A_3 = A_3(v)$ constitute the data at $u \leq 0$. The only field equation is

$$- 2M_{3,v}U_{3,v} - 2\, U_{3,vv} + U_{3,v}^2 + \frac{1}{\alpha}\,(E_{3,v}^2 + 8\, X_{3,v}^2) + 2\kappa^2\, e^{U_3 + E_3}\, A_{3,v}^2 = 0 \quad (7.30)$$

The second and third regions are called the incoming regions and the corresponding space times are the plane wave geometries. Hence the functions $M_2 = M_2(u)$, $U_2 = U_2(u)$, $V_2 = V(u)$, $\psi_2 = \psi_2(u)$, $A_2 = A_2(u)$ and $M_3 = M_3(v)$, $U_3 = U_3(v)$, $V_3 = V_3(v)$, $\psi_3 = \psi_3(v)$, $A_3 = A_3(v)$ should be considered as the data on the hyperplanes $v = 0$ and $u = 0$ respectively.

The fourth region: ($u > 0, v > 0$)

$$ds^2 = 2e^{-M} du\, dv + e^{-U-V} dy^2 + e^{-U+V} dz^2 \qquad (7.31)$$

where $M = M(u,v)$, $U = U(u,v)$, $V = V(u,v)$, $\psi = \psi(u,v)$ and $A = A(u,v)$ such that in the incoming regions ($u \leq 0, v \leq 0$) the metric (7.31) reduces to the corresponding metrics in the related regions. The field equations are

$$\begin{aligned} U_{uv} - U_u U_v &= 0 & (7.32)\\ Re(\varepsilon)\, \nabla^2 \varepsilon - \nabla\varepsilon\, \nabla\varepsilon &= 0 & (7.33)\\ 2\, X_{uv} - U_u X_v - U_v X_u &= 0 & (7.34)\\ -2M_u U_u - 2\, U_{uu} + U_u^2 + \frac{1}{\alpha}\,(E_u^2 + 8\, X_u^2) + 2\kappa^2 e^{U+E} A_{,u}^2 &= 0 & (7.35)\\ -2M_v U_v - 2\, U_{vv} + U_v^2 + \frac{1}{\alpha}\,(E_v^2 + 8\, X_v^2) + 2\kappa^2 e^{U+E} A_{,v}^2 &= 0 & (7.36) \end{aligned}$$

where $V(u,v)$ and $\psi(u,v)$ are given in terms of $X(u,v)$ and $E(u,v)$ in (7.25). The problem is to find the solutions of the above equations in such a way that the following conditions must be satisfied.

$$M(u,v\le 0)=M_2(u), U(u,v\le 0)=U_2(u), V(u,v\le 0)=V_2(u) \quad (7.37)$$
$$\psi(u,v\le 0)=\psi_2(u), A(u,v\le 0)=A_2(u) \quad (7.38)$$

$$M(u\le 0,v)=M_3(v), U(u\le 0,v)=U_3(v), V(u\le 0,v)=V_3(v) \quad (7.39)$$
$$\psi(u\le 0,v)=\psi_3(v), A(u\le 0,v)=A_3(v) \quad (7.40)$$

An exact solution of the above problem is.

$$\begin{aligned}
U &= -\ln\cos(P+Q)-\ln\cos(P-Q) \\
E &= \ln\cos(P+Q)-\ln\cos(P-Q) && (7.41)\\
A_2 &= \rho\sin(P-Q) && (7.42)\\
X &= \frac{k_1}{2}\ln\frac{\cos Q-\sin P}{\cos Q+\sin P}+\frac{k_2}{2}\ln\frac{\cos P-\sin Q}{\cos P+\sin Q} && (7.43)
\end{aligned}$$

Here $P=a_2\,u\,\theta(u)$, $Q=a_3\,v\,\theta(v)$, where θ is the Heaviside step function , a_2 and a_3 are arbitrary constants.

$$\rho^2=\frac{16}{(8+a^2)\kappa^2} \quad (7.44)$$

There are two distinct solutions.

1.$k_1=k_2=k$ **and** $k^2=\frac{a^2}{16}$

Second region: $v\le 0, u>0$ or $Q=0$

$$\begin{aligned}
e^{a\,\psi_2} &= \left[\frac{1-\sin\,P}{1+\sin\,P}\right]^{\frac{a\,k}{2\,\alpha}} && (7.45)\\
e^{-U_2-V_2} &= \cos^2\,P\left[\frac{1-\sin\,P}{1+\sin\,P}\right]^{-\frac{a\,k}{2\,\alpha}} && (7.46)\\
e^{-U_2+V_2} &= \cos^2\,P\left[\frac{1-\sin\,P}{1+\sin\,P}\right]^{\frac{a\,k}{2\,\alpha}} && (7.47)\\
e^{-M_2} &= (\cos\,P)^{\frac{a^2}{8\,\alpha}} && (7.48)\\
A_2 &= \rho\,\sin\,P && (7.49)
\end{aligned}$$

Third region: $u \leq 0, v > 0$ or $P = 0$.

$$e^{a\psi_3} = \left[\frac{1-\sin\,Q}{1+\sin\,Q}\right]^{\frac{ak}{2\alpha}} \tag{7.50}$$

$$e^{-U_3-V_3} = \cos^2\,Q\,\left[\frac{1-\sin\,Q}{1+\sin\,Q}\right]^{-\frac{ak}{2\alpha}} \tag{7.51}$$

$$e^{-U_3+V_3} = \cos^2\,Q\,\left[\frac{1-\sin\,Q}{1+\sin\,Q}\right]^{\frac{ak}{2\alpha}} \tag{7.52}$$

$$e^{-M_3} = (\cos\,Q)^{\frac{a^2}{8\alpha}} \tag{7.53}$$

$$A_3 = -\rho\,\sin\,Q \tag{7.54}$$

Fourth region: $u > 0, v > 0$.

$$e^{a\psi} = \left[\frac{(\cos\,Q-\sin\,P)\,(\cos\,P-\sin\,Q)}{(\cos\,Q+\sin\,P)\,(\cos\,P+\sin\,Q)}\right]^{\frac{ak}{2\alpha}}\;(\frac{\cos\,(P-Q)}{\cos(P+Q)})^{\frac{a^2}{8\alpha}}$$

$$e^{-U-V} = (\cos\,(P{+}Q))^{1-\frac{1}{\alpha}}\,(\cos\,(P{-}Q))^{1+\frac{1}{\alpha}}\;\left[\frac{(\cos\,Q-\sin\,P)\,(\cos\,P-\sin\,Q)}{(\cos\,Q+\sin\,P)\,(\cos\,P+\sin\,Q)}\right]^{-\frac{ak}{2\alpha}}$$

$$e^{-U+V} = (\cos\,(P{+}Q))^{1+\frac{1}{\alpha}}\,(\cos\,(P{-}Q))^{1-\frac{1}{\alpha}}\;\left[\frac{(\cos\,Q-\sin\,P)\,(\cos\,P-\sin\,Q)}{(\cos\,Q+\sin\,P)\,(\cos\,P+\sin\,Q)}\right]^{\frac{ak}{2\alpha}}$$

$$e^{-M} = (\cos\,(P+Q))^{\frac{3\,a^2}{16\,\alpha}}\;(\cos\,(P-Q))^{\frac{-a^2}{16\,\alpha}}$$

$$A_2 = \rho\,\sin\,(P-Q)$$

2.$k_2 = -k_1 = -k$ **and** $k^2 = \frac{a^2}{16}$

Second region: $v \leq 0, u > 0$ or $Q = 0$

$$e^{a\psi} = \left[\frac{1-\sin\,P}{1+\sin\,P}\right]^{\frac{ak}{2\alpha}} \tag{7.55}$$

$$e^{-U-V} = \cos^2\,P\,\left[\frac{1-\sin\,P}{1+\sin\,P}\right]^{-\frac{ak}{2\alpha}} \tag{7.56}$$

$$e^{-U+V} = \cos^2 P \left[\frac{1-\sin P}{1+\sin P}\right]^{\frac{a\,k}{2\,\alpha}} \tag{7.57}$$

$$e^{-M} = (\cos P)^{\frac{a^2}{8\,\alpha}} \tag{7.58}$$

$$A_2 = \rho \sin P \tag{7.59}$$

Third region: $u \leq 0, v > 0$ or $P = 0$.

$$e^{a\,\psi} = \left[\frac{1+\sin Q}{1-\sin Q}\right]^{\frac{a\,k}{2\,\alpha}} \tag{7.60}$$

$$e^{-U-V} = \cos^2 Q \left[\frac{1+\sin Q}{1-\sin Q}\right]^{-\frac{a\,k}{2\,\alpha}} \tag{7.61}$$

$$e^{-U+V} = \cos^2 Q \left[\frac{1+\sin Q}{1-\sin Q}\right]^{\frac{a\,k}{2\,\alpha}} \tag{7.62}$$

$$e^{-M} = (\cos Q)^{\frac{a^2}{8\,\alpha}} \tag{7.63}$$

$$A_2 = -\rho \sin Q \tag{7.64}$$

Fourth region: $u > 0, v > 0$.

$$e^{a\,\psi} = \left[\frac{(\cos Q - \sin P)\,(\cos P + \sin Q)}{(\cos Q + \sin P)\,(\cos P - \sin Q)}\right]^{\frac{a\,k}{2\,\alpha}} \left(\frac{\cos\,(P-Q)}{\cos(P+Q)}\right)^{\frac{a^2}{8\,\alpha}}$$

$$e^{-U-V} = (\cos\,(P+Q))^{1-\frac{1}{\alpha}}\,(\cos\,(P-Q))^{1+\frac{1}{\alpha}} \left[\frac{(\cos Q - \sin P)\,(\cos P + \sin Q)}{(\cos Q + \sin P)\,(\cos P - \sin Q)}\right]^{-\frac{a\,k}{2\,\alpha}}$$

$$e^{-U+V} = (\cos\,(P+Q))^{1+\frac{1}{\alpha}}\,(\cos\,(P-Q))^{1-\frac{1}{\alpha}} \left[\frac{(\cos Q - \sin P)\,(\cos P + \sin Q)}{(\cos Q + \sin P)\,(\cos P - \sin Q)}\right]^{\frac{a\,k}{2\,\alpha}}$$

$$e^{-M} = (\cos\,(P-Q))^{\frac{3\,a^2}{16\,\alpha}}\,(\cos\,(P+Q))^{\frac{-a^2}{16\,\alpha}}$$

$$A_2 = \rho \sin\,(P-Q)$$

The spacetime in the fourth region is singular on the hyperplanes $a_2\,u \pm a_3\,v = \frac{\pi}{2}$.

7.2 Dilaton gravity with two $U(1)$ vector fields

A dimensionally reduced superstring theory in four dimensions can be described in terms of $N = 4$ supergravity [14]. There are two versions $N = 4$ supergravity , SO(4) and SU(4) versions. We shall only consider bosonic part of the theory with $U(1) \otimes U(1)$ vectors in each version and one real dilaton field. In the following lagrangian although $(a, b) = (2, -2)$ for $SO(4)$ case and $(a, b) = (2, 2)$ for $SU(4)$ case we shall keep these constants (couplings of dilaton field to each gauge field)

$$L = \sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{2}{\kappa^2}(\nabla\phi)^2 - \frac{1}{4}\left(e^{-a\phi}F^2 + e^{-b\phi}H^2\right)\right]. \tag{7.65}$$

The field equations are

$$G_{\mu\nu} = 4\left[\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}(\nabla\phi)^2 g_{\mu\nu}\right] + \kappa^2 e^{-a\phi}\left[F_\mu^\alpha F_{\nu\alpha} - \frac{1}{4}F^2 g_{\mu\nu}\right] + \kappa^2 e^{-b\phi}\left[H_\mu^\alpha H_{\nu\alpha} - \frac{1}{4}H^2 g_{\mu\nu}\right],$$

$$\nabla_\mu(e^{-a\phi}F^{\mu\nu}) = 0, \tag{7.66}$$

$$\nabla_\mu(e^{-b\phi}H^{\mu\nu}) = 0, \tag{7.67}$$

$$\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\phi) + \frac{\kappa^2\sqrt{-g}}{16}(a\,e^{-a\phi}F^2 + b\,e^{-b\phi}H^2) = 0, \tag{7.68}$$

where $F^2 = F^{\alpha\beta}F_{\alpha\beta}$ and $H^2 = H^{\alpha\beta}H_{\alpha\beta}$. Both $F_{\mu\nu}$ and $H_{\mu\nu}$ are obtained by the vector potentials A_μ and B_μ respectively , i.e., they are given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad , \quad H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{7.69}$$

In this section instead of giving the complete formulation of the problem we give a special solution of collision problem. We consider the same space-time structure as considered in the previous section with the line element

(7.6). In the general case none of the waves superpose due to the nonlinearities in the field equations. On the other hand existence of two different abelian gauge fields allow one to consider the following type of collision problem (such a solution does not exist with one abelian gauge field). Consider one of the gauge fields is zero in one of the incoming regions and the other gauge field is zero in the other incoming region. More specifically one of the $U(1)$ potential (A_μ) vanishes in one of the incoming regions and the other $U(1)$ potential (B_μ) vanishes in the other region. In the interaction region we have both fields. This implies a superposition in the gauge fields. Such an assumption simplifies the field equations considerably [35].

The field equations are as follows

$$U_{uv} - U_u\, U_v = 0 \tag{7.70}$$

$$2V_{uv} - U_u V_v - U_v V_u = 0 \tag{7.71}$$

$$-\,2M_u U_u - 2\,U_{uu} + U_u^2 + (1+\frac{8}{a^2})\,V_u^2 + 4\,\kappa^2\,B_u^2\,e^U = 0 \tag{7.72}$$

$$-\,2M_v U_v - 2U_{vv} + U_v^2 + (1+\frac{8}{a^2})\,V_v^2 + 4\,\kappa^2\,A_v^2\,e^U = 0 \tag{7.73}$$

Depending upon the choices of the $U(1)$ potentials we find the dilaton field ψ accordingly. We have two distinct cases

Case 1: b=a We have two subcases (in each cases we assume that a is different then zero).
(1.a): $\psi = \frac{1}{a}V$, $A_\mu = (0,0,0,A(v))$, $B_\mu = (0,0,0,B(u))$. The field equations are given above (7.70-7.73)

(1.b): $\psi = \frac{1}{a}V$, $A_\mu = (0,0,A(u),0)$, $B_\mu = (0,0,B(v),0)$. The field equations are exactly the same as in case (1.a) if A and B are interchanged in the equations (7.72) and (7.73).

Case 2: b=-a We have again two subcases
(2.a): $\psi = \frac{1}{a}V$, $A_\mu = (0,0,0,A(v))$, $B_\mu = (0,0,B(u),0)$. The field equations are exactly the same as in case (1.a).

(2.b): $\psi = -\frac{1}{a}V$, $A_\mu = (0,0,A(u),0)$, $B_\mu = (0,0,0,B(v))$. The field equations are exactly the same as in case (1.b).

The solutions of the equations (7.70 - 7.73) are given as follows [19]

$$e^{-U} = f(u) + g(v) \tag{7.74}$$

$$V = \frac{1}{\sqrt{f+g}}(R+S) \tag{7.75}$$

$$R = \int_f^{\frac{1}{2}} P_{-\frac{1}{2}} \left(1 + \frac{2(\xi - f)(\frac{1}{2} - g)}{(\xi + \frac{1}{2})(f+g)}\right) \frac{d}{d\xi}\left[\sqrt{\frac{1}{2} + \xi}\, V_2(\xi)\right] d\xi \tag{7.76}$$

$$S = \int_g^{\frac{1}{2}} P_{-\frac{1}{2}} \left(1 + \frac{2(\eta - g)(\frac{1}{2} - f)}{(\eta + \frac{1}{2})(f+g)}\right) \frac{d}{d\eta}\left[\sqrt{\frac{1}{2} + \eta}\, V_3(\eta)\right] d\eta \tag{7.77}$$

where f and g are functions of u and v respectively , $P_{-\frac{1}{2}}$ is the Legendre function of order $-\frac{1}{2}$. These functions are determined from the data . In the incoming regions we have $f = \frac{1}{2}$ $(u \leq 0)$ and $g = \frac{1}{2}$ $(v \leq 0)$ where

$$f = e^{U_2} - \frac{1}{2}\ , \ g = e^{U_3} - \frac{1}{2} \tag{7.78}$$

The functions $V_2(u)$ and $V_3(v)$ are the data for the function $V(u,v)$. The solutions may be summarized as follows. Here we are giving the case (1.a) explicitly. The other cases can be given easily by correct identifications.

Second region: $v \leq 0, u > 0$ or $g = \frac{1}{2}$

$$ds^2 = 2e^{-M_2(u)}du\, dv + e^{-U_2(u) - V_2(u)}dy^2 + e^{-U_2(u)+V_2(u)}dz^2 \tag{7.79}$$

The dilaton field $\psi_2 = \frac{1}{a}V_2$, the gauge potentials are given as $A_\mu = 0$ and $B_\mu = (0,0,0,B(u))$. The only field equation is given by

$$-2M_{2,u}U_{2,u} - 2\,U_{2,uu} + U_{2,u}^2 + (1 + \frac{8}{a^2})\,V_{2,u}^2 + 4\,\kappa^2\,B_u^2\,e^{U_2} = 0 \tag{7.80}$$

Third region: $u \leq 0, v > 0$ or $f = \frac{1}{2}$

$$ds^2 = 2e^{-M_3(v)} du\, dv + e^{-U_3(v)-V_3(v)} dy^2 + e^{-U_3(v)+V_3(v)} dz^2 \tag{7.81}$$

The dilaton field $\psi_3 = \frac{1}{a} V_3$, the gauge potentials are given as $A_\mu = (0,0,0,A(v))$ and $B_\mu = 0$. The only field equation is given by

$$-2M_{3,v}U_{3,v} - 2U_{3,vv} + U_{3,v}^2 + (1 + \frac{8}{a^2})\, V_{3,v}^2 + 4\,\kappa^2\, A_v^2\, e^{U_3} = 0 \tag{7.82}$$

Fourth region: $u > 0, v > 0$

$$ds^2 = 2e^{-M(u,v)} du\, dv + e^{-U(u,v)-V(u,v)} dy^2 + e^{-U(u,v)+V(u,v)} dz^2 \tag{7.83}$$

where the exact solutions of $U(u,v)$ and $V(u,v)$ are given in (7.74) and (7.75). The dilaton field $\psi(u,v) = \frac{1}{a} V(u,v)$, the gauge potentials are given as $A_\mu = (0,0,0,A(v))$ and $B_\mu = (0,0,0,B(u))$. The field equations to be solved are (7.72) and (7.73). Given the data $(V_2(u), V_3(v))$ one finds the function $V(u,v)$ from the integral formula (7.75). Given the data $(V_2(u), V_3(v))$ and $(A(v), B(u))$ one integrates the function $M(u,v)$ from (7.72) and (7.73).

A simple exact solution to the above problem is given as follows

$$V = m_1 \tanh^{-1} \left(\frac{\frac{1}{2} - f}{\frac{1}{2} + g}\right)^{\frac{1}{2}} + m_2 \tanh^{-1} \left(\frac{\frac{1}{2} - g}{\frac{1}{2} + f}\right)^{\frac{1}{2}} \tag{7.84}$$

with

$$V_2 = m_1 \tanh^{-1} (\frac{1}{2} - f)^{\frac{1}{2}} \tag{7.85}$$

$$V_3 = m_2 \tanh^{-1} (\frac{1}{2} - g)^{\frac{1}{2}} \tag{7.86}$$

where m_1 and m_2 are arbitrary constants. In the general case the initial data is loaded on the functions f and g. The determination of these functions is important in the integration of the function M. We find this function by following two different approaches. This means that we have two different solutions for two different data.

First Solution : The functions f and g are given by

$$f(u)=\frac{1}{2}-s_1\,u^{n_1}\,\theta(u)\quad,\quad g(v)=\frac{1}{2}-s_2\,v^{n_1}\,\theta(v)$$

where n_1 and n_2 are positive integers (≥ 2). This is not the complete data but the function $M(u,v)$ can be found as

$$2\,M=(1-\frac{b}{4}\,(m_1+m_2)^2)\,\ln(f+g)+\frac{b}{4}\,\left[m_1^2\,\ln(\frac{1}{2}+g)+m_2^2\,\ln(\frac{1}{2}+f)\right]+$$
$$\frac{b}{2}\,m_1\,m_2\,\ln(\frac{1}{2}+2\,f\,g+\frac{1}{2}\,\sqrt{(1-4\,f^2)(1-4\,g^2)})-4\,\kappa^2\,\left(\int_{\frac{1}{2}}^{f}\,B_\xi^2\,d\xi+\int_{\frac{1}{2}}^{g}\,A_\eta^2\,d\eta\right)$$

In the incoming regions we have

$$2\,M_2=(1-\frac{b}{4}\,m_1^2)\,\ln(\frac{1}{2}+f)-4\,\kappa^2\,\int_{\frac{1}{2}}^{f}\,B_\xi^2\,d\xi \tag{7.87}$$

$$2\,M_3=(1-\frac{b}{4}\,m_2^2)\,\ln(\frac{1}{2}+g)-4\,\kappa^2\,\int_{\frac{1}{2}}^{g}\,A_\eta^2\,d\eta \tag{7.88}$$

where the last two integrals in above expression are due to the initial values of the gauge fields on the null hyperplanes which are left arbitrary and

$$b=1+\frac{8}{a^2}\quad,\quad b\,m_i^2=8\,(1-\frac{1}{n_i})$$

with $i=1,2$. As far as the singularity structure is considered our solution given above looks like the vacuum Einstein solutions given by Szekeres [19]. They all suffer from a future closing spacetime singularity at $f+g=0$.

Second Solution: The functions f and g are determined by the following equations

$$\frac{2\,f_{uu}}{f_u^2}=\frac{1}{\frac{1}{2}+f}-b\,(\frac{1}{2}+f)\,\left(\frac{dV_2}{df}\right)^2-4\,\kappa^2\,\left(\frac{dB}{df}\right)^2 \tag{7.89}$$

$$\frac{2\,g_{vv}}{g_v^2}=\frac{1}{\frac{1}{2}+g}-b\,(\frac{1}{2}+g)\,\left(\frac{dV_3}{dg}\right)^2-4\,\kappa^2\,\left(\frac{dA}{dg}\right)^2 \tag{7.90}$$

where V_2 and V_3 are given in (7.85)and (7.86). Then the function $M(u,v)$ is found as

$$\begin{aligned}2\,M \;&=\; \left[1-b\,\frac{(m_1+m_2)^2}{4}\right]\,\ln(f+g)+\left[-1+b\,\frac{m_1^2+m_2^2}{4}\right]\,\ln\left((\frac{1}{2}+f)(\frac{1}{2}+g)\right)\\ &+b\,\frac{m_1\,m_2}{2}\,\ln\left(\frac{1}{2}+2fg+\frac{1}{2}\sqrt{(1-4f^2)(1-4g^2)}\right)\end{aligned} \tag{7.91}$$

The function M in the incoming regions vanish ($M_2=M_3=0$). Hence given the functions $A(g)$ and $B(f)$, we determine the functions f and g through (7.89) and (7.90) in terms of u and v. This completes the determination of the metric in the fourth region. For different set of functions $(A(g),B(f))$ we have different solutions. For all of these solutions the surface $f+g=0$ is singular.

Chapter 8

Thermodynamics of Black Holes in Four Dimensions

In this chapter, we will analyze the thermodynamical properties of Cvetic-Youm solution given in Chapter 6.1.

8.1 First Law of Black Hole Thermodynamics

The entropy is given by $\frac{1}{4G_N}A$ where A is the area of the outer horizon. In this case, S has the form: [8]

$$\begin{aligned} S &= 16\pi[(m^2 + m\sqrt{m^2 - l^2})\,(\cosh\delta_1 \cosh\delta_2 \cosh\delta_3 \cosh\delta_4) \\ &\quad +(m^2 - m\sqrt{m^2 - l^2})\,(\sinh\delta_1 \sinh\delta_2 \sinh\delta_3 \sinh\delta_4)]. \end{aligned} \tag{8.1}$$

We can write the variation of entropy in terms of the solution parameters as follows:

$$dS = \frac{\partial S}{\partial\delta_1}\,d\delta_1 + \frac{\partial S}{\partial\delta_2}\,d\delta_2 + \frac{\partial S}{\partial\delta_3}\,d\delta_3 + \frac{\partial S}{\partial\delta_4}\,\delta_4 + \frac{\partial S}{\partial m}\,dm + \frac{\partial S}{\partial l}\,dl, \tag{8.2}$$

But we want to write the variation in terms of the physical parameters:

$$dS = \Gamma_1\,dQ_1 + \Gamma_2\,dQ_2 + \Gamma_3\,dP_1 + \Gamma_4\,dP_4 + \Gamma_5\,dM + \Gamma_6\,dJ, \tag{8.3}$$

where

$$\Gamma_1 = \left(\frac{\partial S}{\partial \delta_1}\frac{\partial \delta_1}{\partial Q_1} + \frac{\partial S}{\partial \delta_2}\frac{\partial \delta_2}{\partial Q_1} + \frac{\partial S}{\partial \delta_3}\frac{\partial \delta_3}{\partial Q_1} + \frac{\partial S}{\partial \delta_4}\frac{\partial \delta_4}{\partial Q_1} + \frac{\partial S}{\partial m}\frac{\partial m}{\partial Q_1} + \frac{\partial S}{\partial l}\frac{\partial l}{\partial Q_1}\right), \tag{8.4}$$

etc.

By rearranging (8.3), we can now write the variation of M to obtain the explicit form of the first law for this black hole:

$$dM = TdS + \Omega dJ + \Phi_1\, dQ_1 + \Phi_2\, dQ_2 + \Phi_3\, dP_1 + \Phi_4\, dP_2, \tag{8.5}$$

To find the Γ_i's, we need the derivatives of solution parameters (δ_i's etc.) with respect to the physical parameters (Q_i's etc.). We cannot invert equation (6.11), so we cannot find the solution parameters explicitly in terms of the physical parameters. But we can always find the infinitesimal variations by inverting the Jacobian matrix. The details of this lengthy but straightforward calculation is in Appendix 3.

Thus, we can find the coefficients in S (8.5) as:

$$\begin{aligned} T &= \frac{4\sqrt{m^2 - l^2}}{S}, \\ \Omega &= \frac{8\pi l}{S}, \\ \Phi_i &= \frac{S_i}{S}. \end{aligned} \tag{8.6}$$

In his paper [32], Smarr obtained the formula $dM = TdA + \Omega dJ + \Phi dQ$ and from this, using the fact that M is homogeneous of degree $\frac{1}{2}$ in (A, L, Q^2), he obtained

$$M = 2TA + 2\Omega J + \Phi Q \tag{8.7}$$

But we are not in a position to repeat this, because we can not express M in terms of area and charges explicitly, so we are unable to find the analog of Smarr's formula (8.7).

8.2 Thermodynamical Quantities Derived From the Metric

Now, we determine the thermodynamical quantities entering (8.5) using the metric. The temperature T is related to the surface gravity κ as

$$2\pi T = \kappa = -\frac{1}{2}\frac{dg_{tt}}{dr}|_{r=r_+,\theta=0}. \tag{8.8}$$

Using the metric

$$\frac{dg_{tt}}{dr}|_{r=r_+,\theta=0} = \frac{-2\,(r_+ - m)}{\sqrt{\Delta}}, \tag{8.9}$$

where

$$\sqrt{\Delta}|_{r=r_+,\theta=0} = 2m\,(m\,K + \sqrt{m^2 - l^2}\,L) = \frac{S}{8\pi}. \tag{8.10}$$

Thus,

$$\kappa = \frac{8\pi\sqrt{m^2 - l^2}}{S}, \tag{8.11}$$

which is in agreement with First law (8.5).

The angular velocity of the black hole at the outer horizon is:

$$\Omega \equiv \frac{-g_{tt}}{g_{\phi t}}|_{r=r_+,\theta=0}. \tag{8.12}$$

¿From the metric (6.8) we can write

$$\frac{g_{tt}}{g_{\phi t}} = \frac{r^2 - 2mr + l^2 - l^2\sin^2\theta}{2ml\sin^2\theta\,(r\,L + m\,K - m\,L)}. \tag{8.13}$$

At the horizon, $r = r_+ = m + \sqrt{m^2 - l^2}$, which means $r^2 + l^2 - 2mr = 0$, so

$$\frac{g_{tt}}{g_{\phi t}}|_{r=r_+,\theta=0} = \frac{-l}{2m(\sqrt{m^2-l^2}\,L + m\,K)} = \frac{-8\pi l}{S}, \tag{8.14}$$

$$\Omega_H = \frac{8\pi l}{S}, \tag{8.15}$$

which is also in agreement with the first law(8.5).

Chapter 9

Thermodynamics of Black Holes in Five Dimensions

In this chapter, we will analyze the thermodynamical properties of Cvetic-Youm solution given in Chapter 6.1.

9.1 First Law of Black Hole Thermodynamics

The entropy is given by:[8]

$$\begin{aligned} S &= 4\pi m[\sqrt{2m-(l_1-l_2)^2}\,(\cosh\delta_1\cosh\delta_2\cosh\delta_3+\sinh\delta_1\sinh\delta_2\sinh\delta_3) \\ &\quad +\sqrt{2m-(l_1+l_2)^2}(\cosh\delta_1\cosh\delta_2\cosh\delta_3-\sinh\delta_1\sinh\delta_2\sinh\delta_3)] \end{aligned} \tag{9.1}$$

$$dS=\tilde{\Gamma}_1\,dQ_1+\tilde{\Gamma}_2\,dQ_2+\tilde{\Gamma}_3\,dQ_3+\tilde{\Gamma}_4\,dM+\tilde{\Gamma}_5\,dJ_1+\tilde{\Gamma}_6\,dJ_2. \tag{9.2}$$

The first law is of the form:

$$dM=T\,dS+\Phi_1\,dQ_1+\Phi_2\,dQ_2+\Phi_3\,dQ_3+\Omega_1\,dJ_1+\Omega_2\,dJ_2, \tag{9.3}$$

Once again, we need to invert a Jacobian matrix and find the derivatives of solution parameters with respect to the physical parameters, because we can not invert the algebraic equations in (6.19) and find the solution parameters explicitly. The details are in Appendix 4. The result of this calculation is:

$$\begin{aligned} T &= \frac{\alpha\beta}{S}, \\ \Phi_i &= \frac{S_{,i}}{S}(i=1,2,3), \qquad (9.4) \\ \Omega_1 &= -\pi\frac{\beta(l_2-l_1)-\alpha(l_1+l_2)}{S}, \\ \Omega_2 &= -\pi\frac{\beta(l_1-l_2)-\alpha(l_1+l_2)}{S}. \end{aligned}$$

$$(9.5)$$

where

$$\begin{aligned} \alpha &= \sqrt{2m-(l_1-l_2)^2}, \qquad (9.6) \\ \beta &= \sqrt{2m-(l_1+l_2)^2}. \end{aligned}$$

9.2 Thermodynamic Quantities Derived from the Metric

Now, we make an independent check for the coefficients in first law. Using the metric, we can calculate Ω_1.

$$\begin{aligned} \Omega_1 &\equiv \frac{-g_{tt}}{g_{\phi t}}|_{r=r_+,\theta=\frac{\pi}{2}}, \qquad (9.7) \\ &= \frac{R(R-2m)}{-2m\sin^2\theta(l_1Rc_1c_2c_3+l_2(2m-R)s_1s_2s_3)}. \end{aligned}$$

where $R=r^2+l_1^2\cos^2\theta+l_2^2\sin^2\theta$. We are considering the point on the outer horizon with, $r=r_+,\ \ \theta=\frac{\pi}{2}$.

$$R=-\frac{[\beta(l_2-l_1)-\alpha(l_2+l_1)]\,l_2}{\alpha-\beta},\quad R-2m=\frac{[\beta(l_2-l_1)-\alpha(l_2+l_1)]l_1}{\alpha+\beta} \qquad (9.8)$$

So

$$\Omega_1 = -2\pi \frac{\beta(l_2 - l_1) - \alpha(l_2 + l_1)}{S} \tag{9.9}$$

This result is in agreement with first law(9.3) except for a numerical factor. We can repeat the calculation for Ω_2. It is also in agreement with first law. Now, let us make an independent check for κ, the surface gravity of the outer horizon. [36]

$$2\pi T = \kappa = -\frac{1}{2}\sqrt{-g^{rr}\, g^{tt}}\, \frac{d\Sigma}{dr}|_{r=r_+,\theta=\frac{\pi}{2}}, \tag{9.10}$$

where

$$\Sigma = g_{tt} - \frac{(g_{\phi t} + g_{\psi t})^2}{g_{\phi\phi} + g_{\psi\psi} + 2g_{\phi\psi}}. \tag{9.11}$$

To find κ, we need to calculate g^{rr}, g^{tt} and put Σ into some manageable form. The details of these calculations are in Appendix 4. The result is:

$$\kappa = \frac{\sqrt{l_2^2 + 4m(m - L_1)}}{\sqrt{\Delta + L_1 k_1 + L_2 k_2 + L_3 k_3}} \tag{9.12}$$

At the horizon, $S = 4\pi\sqrt{\Delta + L_1 k_1 + L_2 k_2 + L_3 k_3}$.

So,

$$\kappa = \frac{4\pi\alpha\beta}{S} \tag{9.13}$$

We can check the potentials Φ_i for the special case $l_1 = l_2 = 0$. In this case, $r_+^2 = 2m$, and $\Phi_1 = A_{t1}^{(1)}$, $\Phi_2 = A_{t1}^{(2)}$.

Chapter 10

Conclusion

In this study, we considered D-dimensional spherically symmetric space times and obtained the Riemann tensor in compact form. Using this expression, we established certain theorems concerning any spherically symmetric theory of gravitation. As a special case, we considered the Lovelock theory in 5 and 6 dimensions with the LCBR metric, which simplifies most of the expressions. Then we considered the low energy limit of the string theory and obtained black hole solutions carrying mass, electrostatic charge, and dilaton charge. We also generalized an inequality concerning physical parameters of the black hole to D-dimensions.

We have given exact solutions of the colliding plane waves in the Einstein Maxwell Dilaton Gravity theories. Although the exact solutions we obtained in this work differ from the solutions of the vacuum Einstein and Einstein Maxwell theories, the singularity structures of the solutions of these different theories look the same. We have studied the collision of plane waves in four dimensions. Higher dimensional plane waves when dimensionally reduced (with some duality transformations) lead to the extreme black hole solutions in four dimensions. In this respect it is perhaps more interesting to investigate the colliding gravitational plane waves in higher dimensions.

Interpretation of the surface area of the black hole as its entropy was one of the breakthroughs of black hole thermodynamics. This made possible the analogy between the first law of black hole mechanics ($dM = TdS + \Omega dJ + \Phi_i \, dQ_i$.) and the first law of thermodynamics.

We calculated the temperature, angular velocity and the potentials that enter the first law for two different black holes from toroidally compactified string theory. We did this using the expression for entropy as a starting point and taking derivatives using the chain rule to make a change of variables.

Then, as a check, we calculated temperature and angular velocity directly using the metric. The values found using two different methods are in agreement.

Note that in four dimensions, we used the formula $\kappa = -\frac{1}{2}\frac{dg_{tt}}{dr}$, but in five dimensions we have to use the more general formula

$$\kappa = -\frac{1}{2}\sqrt{-g^{rr}\,g^{tt}}\,\frac{d}{dr}\left(g_{tt} - \frac{(g_{\phi t} + g_{\psi t})^2}{g_{\phi\phi} + g_{\psi\psi} + 2g_{\phi\psi}}\right)|_{r=r_+,\theta=\frac{\pi}{2}}, \tag{10.1}$$

Chapter 11

Appendix 1 Explicit form of β's

In chapter 2, we expressed contractions of Riemann tensor with itself in terms of three tensors $(g_{ij}, M_{ij}, k_i k_j)$ multiplied by certain β coefficients. Here, we are giving these coefficients explicitly.

$$\begin{aligned} M_n^n &= \frac{\alpha(D-4)}{2C^2} \\ M_{in}M_j^n &= \frac{\alpha^2}{4C^4} g_{ij} \\ M_{in}k^n &= -\frac{\alpha}{2C^2} k_i \end{aligned} \tag{11.1}$$

where

$$M_{ij} = \alpha(h_{ij} - \frac{1}{2C^2} g_{ij}), \quad \alpha = \frac{(D-3)!}{C^{2(D-3)}} \tag{11.2}$$

Using these, we can compute several contractions of S_{ij}, which in turn will be used to compute the contractions of the Riemann tensor. Now we'll define new variables that will simplify the coefficients.

$$\xi_0 = \frac{\eta_0 \alpha}{C^2}, \ \xi_1 = \frac{\eta_1}{B^2}, \ \xi_2 = \frac{\eta_2 \alpha}{C^2}, \ \xi_3 = \eta_3 \tag{11.3}$$

$$S_{ij} = \frac{C^2}{\alpha} \xi_0 \, M_{ij} + B^2 \xi_1 k_i k_j + \frac{\xi_3}{2} g_{ij}$$

$$
\begin{aligned}
S_n^n &= \xi_0 \frac{(D-4)}{2} + \xi_1 + \frac{1}{2}\xi_3 D \\
S_{in} S_j^n &= \frac{1}{4}(\xi_0^2 + \xi_3^2) g_{ij} + \xi_0\, \xi_3 \frac{C^2}{\alpha} M_{ij} + (\xi_1^2 + \xi_1\xi_3 - \xi_0\xi_1) B^2 k_i k_j \\
S_{in} k^n &= (-\frac{1}{2}\xi_0 + \xi_1 + \frac{1}{2}\xi_3) k_i \\
S_{in} M_j^n &= \frac{\alpha}{C^2}(\frac{1}{4}\xi_0 g_{ij} + \frac{C^2}{2\alpha}\xi_3 M_{ij} - \frac{1}{2}\xi_1 B^2 k_i k_j) \\
S_i^n h_{jn} &= \frac{1}{2}(\xi_0 + \xi_3) h_{ij} \\
S^{mn} h_{jn} h_{mi} &= (\xi_0 + \xi_3) \frac{h_{ij}}{2C^2}
\end{aligned} \tag{11.4}
$$

Now we can compute the second rank tensors in terms of g_{ij}, M_{ij} and $k_i k_j$

$$
\begin{aligned}
R_i^a R_{ja} &= \beta_1 g_{ij} + \beta_2 M_{ij} + \beta_3 k_i k_j \\
R_{jaib} R^{ab} &= \beta_4 g_{ij} + \beta_5 M_{ij} + \beta_6 k_i k_j \\
R_i^{abc} R_{jabc} &= \beta_7 g_{ij} + \beta_8 M_{ij} + \beta_9 k_i k_j \\
R_{ij} &= f_1 g_{ij} + f_2 M_{ij} + f_3 k_i k_j
\end{aligned} \tag{11.5}
$$

where

$$
\begin{aligned}
\beta_1 = {} & \frac{[(D-3)^2+1]}{2}\xi_0^2 + \xi_1^2 + (D-3)\xi_0\xi_2 + \xi_1\xi_2 + \frac{1}{2}\xi_2^2 + (D-1)(D-4)\xi_0\xi_3 \\
& +(D-4)\xi_0\xi_1 + 2(D-1)\xi_1\xi_3 + (D-1)\xi_2\xi_3 + (D-1)^2\xi_3^2
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 = {} & \frac{C^2}{\alpha}[(D-2)(D-4)\xi_0^2 + 2(D-2)\xi_0\xi_1 + 2(D-3)\xi_0\xi_2 + 2\xi_1\xi_2 + \xi_2^2 \\
& +2(D-1)(D-2)\xi_0\xi_3 + 2(D-1)\xi_2\xi_3
\end{aligned}
$$

$$
\beta_3 = B^2[-2(D-2)\xi_0\xi_1 + D(D-2)\xi_1^2 + 2(D-1)(D-2)\xi_1\xi_3]
$$

$$
\begin{aligned}
\beta_4 = {} & \frac{[(D-3)^2+1]}{2}\xi_0^2 - 2\xi_0\xi_1 + (D-1)\xi_1^2 + (D-3)\xi_0\xi_2 + \frac{1}{2}\xi_1\xi_2 + \frac{1}{2}\xi_2^2 \\
& +(\frac{3}{2}D-2)(D-4)\xi_0\xi_3 + (3D-4)\xi_1\xi_3 + \frac{3}{2}(D-2)\xi_2\xi_3 + (D-1)^2\xi_3^2
\end{aligned}
$$

$$\beta_5 = \frac{C^2}{\alpha}[(D-2)(D-4)\xi_0^2 + 2(D-2)\xi_0\xi_1 + 2(D-3)\xi_0\xi_2 + \xi_1\xi_2 + \xi_2^2$$
$$+(D-2)^2\xi_0\xi_3 + (D-2)\xi_2\xi_3]$$

$$\beta_6 = (D-2)B^2[(D-2)\xi_0\xi_1 + \xi_1\xi_2 + (D-2)\xi_1\xi_3]$$

$$\beta_7 = (D-2)\xi_0^2 - 2\xi_0\xi_1 + 2\xi_1^2 + 2\xi_0\xi_2 + \frac{1}{(D-3)}\xi_2^2 + 2(D-4)\xi_0\xi_3 + 4\xi_1\xi_3$$
$$+2\xi_2\xi_3 + 2(D-1)\xi_3^2$$

$$\beta_8 = \frac{C^2}{\alpha}\left[2(D-4)\xi_0^2 + 4\xi_0\xi_1 + 4\xi_0\xi_2 + \frac{2\xi_2^2}{(D-3)} + 4(D-2)\xi_0\xi_3 + 4\xi_2\xi_3\right]$$

$$\beta_9 = 2(D-2)B^2\xi_1(\xi_1 + 2\xi_3) \tag{11.6}$$

$$\begin{aligned} f_1 &= \frac{1}{2}(\xi_0(D-4) + \xi_2) + \xi_1 + \xi_3(D-1) \\ f_2 &= \frac{C^2}{\alpha}(\xi_0(D-2) + \xi_2) \\ f_3 &= B^2(D-2)\xi_1 \end{aligned} \tag{11.7}$$

If we insert these in (4.16), we obtain

$$\left[f_1 - \tfrac{1}{2}R + \alpha_0(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} + 8\beta_1 + 8\beta_4 - 4\beta_7 - 4Rf_1)\right] g_{ij}$$
$$+ [f_2 + \alpha_0(8\beta_2 + 8\beta_5 - 4\beta_8 - 4Rf_2)]\, M_{ij}$$
$$+ [f_3 + \alpha_0(8\beta_3 + 8\beta_6 - 4\beta_9 - 4Rf_3)]\, k_ik_j = \lambda g_{ij} \tag{11.8}$$

We can rewrite it as

$$t_1 g_{ij} + t_2 M_{ij} + t_3 k_i k_j = 0 \tag{11.9}$$

So, the field equations will reduce to $t_1 = \lambda$, $t_2 = t_3 = 0$, where these coefficients are given as

$$
\begin{aligned}
t_1 &= -\frac{1}{2}(D-4)(D-2)\xi_0 - (D-2)\xi_1 - \frac{1}{2}(D-3)\xi_2 - \frac{1}{2}(D-1)(D-2)\xi_3 \\
&+\alpha_0(D-3)(D-4)[(D-3)(D-6)\xi_0^2 + 4(D-3)\xi_0\xi_1 + 2(D-6)\xi_0\xi_2 \\
&+4\xi_1\xi_2 + \frac{(D-4)(D-5)}{(D-3)^2}\xi_2^2 + 2(D-4)(D-2)\xi_0\xi_3 + 4(D-2)\xi_1\xi_3 \\
&+2(D-3)\xi_2\xi_3 + (D-1)(D-2)\xi_3^2] \qquad (11.10)
\end{aligned}
$$

$$
\begin{aligned}
t_2 &= \frac{C^2}{\alpha}[(D-2)\xi_0 + \xi_2] - 4\frac{C^2}{\alpha}\alpha_0(D-3)(D-4)\left[(D-4)\xi_0^2 + \frac{(2D-9)}{(D-3)}\xi_0\xi_2\right. \\
&\left.+2\xi_0\xi_1 + \frac{2}{(D-3)}\xi_1\xi_2 + \frac{(D-5)}{(D-3)^2}\xi_2^2 + (D-2)\xi_0\xi_3 + \xi_2\xi_3\right] \qquad (11.11)
\end{aligned}
$$

$$
\begin{aligned}
t_3 &= -4\alpha_0 B^2(D-2)(D-4)\xi_1[\xi_0(D-3) + \xi_2 + (D-3)\xi_3] \\
&+B^2(D-2)\xi_1 \qquad (11.12)
\end{aligned}
$$

Chapter 12

Appendix 2 Ernst Equation

In Maxwell theory , due to the linearity the solution in the interaction region is just the superposition of the plane wave solutions in the secod and third regions. In Einstein theory such a superposition is not possible and hence to find exact solutions (solution of the charcteristic initial value problem) is possible yet. In this appendix we consider the collision of the Maxwell-Dilaton Plane Waves. The lagrangian of the coresponding theory is

$$L = \sqrt{-g}\left[\frac{2}{\kappa^2}(\nabla\psi)^2 + \frac{1}{4}\, e^{-a\psi}F^2\right]. \tag{12.1}$$

where a is the dilaton coupling constant. Here we kept the the constant κ which may be set equal to unity. The field equations are

$$\nabla_\mu(e^{-a\psi}F^{\mu\nu}) = 0, \tag{12.2}$$

$$\partial_\mu(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu\psi) + \frac{\kappa^2\, a\sqrt{-g}}{16} e^{-a\psi}F^2 = 0, \tag{12.3}$$

$$ds^2 = 2du\, dv + dy^2 + dz^2 \tag{12.4}$$

With the choice $A_\mu = (0, 0, A, 0)$, where $A = A(u, v)$ and dilaton field $\psi = \psi(u, v)$, the field equations turn out to be

$$a\psi_u A_{,v} + a\psi_v A_{,u} - 2A_{uv} = 0 \tag{12.5}$$

and

$$\psi_{uv} + \frac{a\kappa^2}{8} e^{-a\psi} A_{,u} A_{,v} = 0. \tag{12.6}$$

These equations are the real and imaginary parts of the following Ernst equation

$$Re(\varepsilon)\, \nabla^2 \varepsilon = \nabla\varepsilon\, \nabla\varepsilon \tag{12.7}$$

where differential operators in (12.7) are defined with respect to the metric given by $ds^2 = 2\, du\, dv$ and

$$\varepsilon = e^{\frac{1}{2}a\psi} + i\frac{a\kappa}{4} A \tag{12.8}$$

This can be rewritten as

$$\nabla(g^{-1}\, \nabla g) = 0 \tag{12.9}$$

where

$$g = \frac{2}{\varepsilon + \bar{\varepsilon}} \begin{bmatrix} 1 & \frac{i}{2}(\varepsilon - \bar{\varepsilon}) \\ \frac{i}{2}(\varepsilon - \bar{\varepsilon}) & \varepsilon\bar{\varepsilon} \end{bmatrix} \tag{12.10}$$

Eq.(12.9) is the two dimensional sigma model equation on $SU(2)/U(1)$. Although the complete solution of (12.7) is not known yet its integrability has been shown long time ago [37]. The soliton solutions and many intersting properties are known.

Chapter 13

Appendix 3 Γ Coefficients

Note that we can write S in the form

$$S = 16\pi(m^2 K + m\sqrt{m^2 - l^2} L), \tag{13.1}$$

where

$$\begin{aligned} K &= \cosh\delta_1 \cosh\delta_2 \cosh\delta_3 \cosh\delta_4 + \sinh\delta_1 \sinh\delta_2 \sinh\delta_3 \sinh\delta_4, \\ L &= \cosh\delta_1 \cosh\delta_2 \cosh\delta_3 \cosh\delta_4 - \sinh\delta_1 \sinh\delta_2 \sinh\delta_3 \sinh\delta_4. \end{aligned} \tag{13.2}$$

To determine the coefficients Γ_i, we have to invert the following matrix:

$$\begin{pmatrix} dQ_1 \\ dQ_2 \\ dP_1 \\ dP_2 \\ dM \\ dJ \end{pmatrix} = \begin{pmatrix} 4mw_1 & 0 & 0 & 0 & 2z_1w_1 & 0 \\ 0 & 4mw_2 & 0 & 0 & 2z_2w_2 & 0 \\ 0 & 0 & 4mw_3 & 0 & 2z_3w_3 & 0 \\ 0 & 0 & 0 & 4mw_4 & 2z_4w_4 & 0 \\ 4mz_1w_1 & 4mz_2w_2 & 4mz_3w_3 & 4mz_4w_4 & \hat{M} & 0 \\ 8lmL_1 & 8lmL_2 & 8lmL_3 & 8lmL_4 & 8lL & 8mL \end{pmatrix} \begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ d\delta_4 \\ dm \\ dl \end{pmatrix} \tag{13.3}$$

where

$$w_i = \cosh 2\delta_i,$$

$$
\begin{aligned}
z_i &= \tanh 2\delta_i, \\
\hat{M} &= \frac{M}{m}, \\
L_i &= \frac{\partial L}{\partial \delta_i} \quad (i = 1, \ldots, 4).
\end{aligned} \tag{13.4}
$$

The result is

$$
\begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ d\delta_4 \\ dm \\ dl \end{pmatrix} = \frac{1}{B} \begin{pmatrix} \frac{B}{4mw_1} + z_1^2 & z_1 z_2 & z_1 z_3 & z_1 z_4 & -z_1 & 0 \\ z_2 z_1 & \frac{B}{4mw_2} + z_2^2 & z_2 z_3 & z_2 z_4 & -z_2 & 0 \\ z_3 z_1 & z_3 z_2 & \frac{B}{4mw_3} + z_3^2 & z_3 z_4 & -z_3 & 0 \\ z_4 z_1 & z_4 z_2 & z_4 z_3 & \frac{B}{4mw_4} + z_4^2 & -z_4 & 0 \\ -2mz_1 & -2mz_2 & -2mz_3 & -2mz_4 & 2m & 0 \\ \frac{u_1}{L} & \frac{u_2}{L} & \frac{u_3}{L} & \frac{u_4}{L} & \frac{lP_L}{L} - 2l & \frac{B}{8mL} \end{pmatrix} \begin{pmatrix} dQ_1 \\ dQ_2 \\ dP_1 \\ dP_2 \\ dM \\ dJ \end{pmatrix} \tag{13.5}
$$

where

$$
\begin{aligned}
u_i &= -z_i l P_L + 2 z_i l L - \frac{l L_i B}{4mw_i} \quad (i = 1, \ldots, 4), \\
P_L &= L_1 z_1 + L_2 z_2 + L_3 z_3 + L_4 z_4, \\
B &= 2M - 4m(w_1 z_1^2 + w_2 z_2^2 + w_3 z_3^2 + w_4 z_4^2).
\end{aligned} \tag{13.6}
$$

Now, using this matrix, we can calculate the Γ_i's defined in (8.4).

$$
\begin{aligned}
\Gamma_5 &= \frac{S}{4\sqrt{m^2 - l^2}}, \\
\Gamma_6 &= \frac{-2\pi l}{\sqrt{m^2 - l^2}}, \\
\Gamma_i &= \frac{-S_i}{4\sqrt{m^2 - l^2}} \quad (i = 1, \ldots, 4),
\end{aligned} \tag{13.7}
$$

where

$$
T = \frac{1}{\Gamma_5},
$$

$$\begin{aligned}\Omega &= -\frac{\Gamma_6}{\Gamma_5}, \\ \Phi_i &= -\frac{\Gamma_i}{\Gamma_5} \quad (i = 1 \ldots 4).\end{aligned} \tag{13.8}$$

where $S_i \equiv \frac{\partial S}{\partial \delta_i}$.

Chapter 14

Appendix 4 Finding the $\tilde{\Gamma}$'s

To find the derivatives of boosts with respect to physical variables ($\frac{\partial \delta_1}{\partial Q_2}$ etc.) we need to invert the following matrix:

$$\begin{pmatrix} dQ_1 \\ dQ_2 \\ dQ_3 \\ dM \\ dJ_1 \\ dJ_2 \end{pmatrix} = \begin{pmatrix} 2mw_1 & 0 & 0 & z_1 w_1 & 0 & 0 \\ 0 & 2mw_2 & 0 & z_2 w_2 & 0 & 0 \\ 0 & 0 & 2mw_3 & z_3 w_3 & 0 & 0 \\ 2mz_1 w_1 & 2mz_2 w_2 & 2mz_3 w_3 & w_0 + w_1 + w_2 & 0 & 0 \\ J_{1,1} & J_{1,2} & J_{1,3} & J_1/m & 4mC & -4mE \\ J_{2,1} & J_{2,2} & J_{2,3} & J_2/m & -4mE & 4mC \end{pmatrix} \begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ dm \\ dl_1 \\ dl_2 \end{pmatrix} \tag{14.1}$$

where

$$\begin{aligned} w_i &= \cosh 2\delta_i \ \ (i = 1, 2, 3), \\ z_i &= \tanh 2\delta_i \ \ (i = 1, 2, 3), \\ C &= \cosh \delta_1 \cosh \delta_2 \cosh \delta_3, \\ E &= \sinh \delta_1 \sinh \delta_2 \sinh \delta_3. \end{aligned} \tag{14.2}$$

The result is

$$
\begin{pmatrix} d\delta_1 \\ d\delta_2 \\ d\delta_3 \\ dm \\ dl_1 \\ dl_2 \end{pmatrix} = \frac{1}{2mU} \begin{pmatrix} \frac{U}{w_1}+z_1^2 & z_1z_2 & z_1z_3 & -z_1 & 0 & 0 \\ z_2z_1 & \frac{U}{w_2}+z_2^2 & z_2z_3 & -z_2 & 0 & 0 \\ z_3z_1 & z_3z_2 & \frac{U}{w_3}+z_3^2 & -z_3 & 0 & 0 \\ -2mz_1 & -2mz_2 & -2mz_3 & 2m & 0 & 0 \\ t_{51} & t_{52} & t_{53} & -c & CU/(2h) & EU/(2h) \\ t_{61} & t_{62} & t_{63} & -d & EU/(2h) & CU/(2h) \end{pmatrix} \begin{pmatrix} dQ_1 \\ dQ_2 \\ dQ_3 \\ dM \\ dJ_1 \\ dJ_2 \end{pmatrix} \tag{14.3}
$$

where

$$
\begin{aligned}
U &= (w_1 + w_2 + w_3 - w_1z_1^2 - w_2z_2^2 - w_3z_3^2), \\
P_E &= E_1z_1 + E_2z_2 + E_3z_3, \\
P_C &= C_1z_1 + C_2z_2 + C_3z_3, \\
h &= (C-E)(C+E), \\
a &= \frac{E\,l_1 + C\,l_2}{h}, \\
b &= \frac{C\,l_1 + E\,l_2}{h}, \\
c &= 2l_1 + a\,P_E - b\,P_C, \\
d &= 2l_2 + b\,P_E - a\,P_C, \\
t_{5i} &= \frac{U}{w_i}(E_i\,a - C_i\,b) + z_ic, \\
t_{6i} &= \frac{U}{w_i}(E_i\,b - C_i\,a) + z_id,
\end{aligned} \tag{14.4}
$$

Using these results, we can calculate $\tilde{\Gamma}_i{}'s$ as follows:

$$
\begin{aligned}
\tilde{\Gamma}_i &= \frac{-S_i}{\alpha\beta}, \\
\tilde{\Gamma}_4 &= \frac{S}{\alpha\beta}, && (14.5) \\
\tilde{\Gamma}_5 &= \pi\left(\frac{l_2-l_1}{\alpha} - \frac{l_1+l_2}{\beta}\right), && (14.6) \\
\tilde{\Gamma}_6 &= \pi\left(\frac{l_1-l_2}{\alpha} - \frac{l_1+l_2}{\beta}\right),
\end{aligned}
$$

Chapter 15

Appendix 5 Finding κ

We know that $g^{rr} = g_{rr}^{-1}$ and $g^{tt} = (g_{\phi\phi}g_{\psi\psi} - g_{\phi\psi}^2)/D$, where

$$\begin{aligned} D &= \frac{Det(g_{ij})}{g_{rr}g_{\theta\theta}} \qquad (15.1) \\ D &= g_{tt}(g_{\phi\phi}g_{\psi\psi} - g_{\phi t}^2) + 2g_{\phi\psi}g_{\phi t}g_{\psi t} - g_{\psi t}^2 g_{\phi\phi} - g_{\phi t}^2 g_{\psi\psi} \end{aligned}$$

After some algebra, we find that

$$\begin{aligned} D &= \cos^2\theta\sin^2\theta[(2m-R)R + L_2(\cos^2\theta - \sin^2\theta)(R-m) - L_1 m + L_2^2\cos^2\theta\sin^2\theta] \\ &= -\cos^2\theta\sin^2\theta\,\Delta^{\frac{1}{3}}\,r^2\,g_{rr}^{-1} \qquad (15.2) \end{aligned}$$

$$\begin{aligned} g_{\phi\phi}g_{\psi\psi} - g_{\phi\psi}^2 &= \cos^2\theta\sin^2\theta\Delta^{-1/3}[\Delta + k_1L_1 + (\sin^2\theta - \cos^2\theta)k_2L_2 \\ &\quad + k_3L_3 - \cos^2\theta\sin^2\theta L_2^2(2m + 2mp + R)] \qquad (15.3) \end{aligned}$$

We know that $g^{rr} = g_{rr}^{-1}$, so

$$-g^{tt}\,g^{rr} = \frac{\Delta + k_1L_1 + (\sin^2\theta - \cos^2\theta)k_2L_2 + k_3L_3 - \cos^2\theta\sin^2\theta L_2^2(2m+2mp+R)}{r^2\,\Delta^{\frac{2}{3}}} \qquad (15.4)$$

$$\Sigma = \Delta^{\frac{1}{3}} \frac{R(2m-R) + R(\cos^2\theta - \sin^2\theta)L_2 - 2m(\cos^2\theta l_1 + \sin^2\theta l_2)^2}{\Delta + (L_1 k_1 + L_3 k_3)(\sin^4\theta + \cos^4\theta) + L_2 k_2(\sin^2\theta - \cos^2\theta) + 2\sin^2\theta\cos^2\theta(L_1 k_3 + I} \tag{15.5}$$

$$\Sigma = \Delta^{\frac{1}{3}} \frac{R(2m-R) + R(\cos^2\theta - \sin^2\theta)L_2 - 2m(\cos^2\theta l_1 + \sin^2\theta l_2)^2}{\Delta + (L_1 k_1 + L_3 k_3)(\sin^4\theta + \cos^4\theta) + L_2 k_2(\sin^2\theta - \cos^2\theta) + 2\sin^2\theta\cos^2\theta(L_1 k_3 + I} \tag{15.6}$$

Note that at the horizon, $R = m - \frac{1}{2}L_2 + \frac{1}{2}\sqrt{L_2^2 + 4m(m - L_1)}$ and $\Sigma = 0$.

We find

$$\kappa = \frac{\sqrt{l_2^2 + 4m(m - L_1)}}{\sqrt{\Delta + L_1 k_1 + L_2 k_2 + L_3 k_3}} \tag{15.7}$$

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