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Absolutely Supplement and Absolutely Complement Modules

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ABSTRACT

We introduce and study absolutely supplement (respectively complement) modules. We call a module an absolutely supplement (respectively complement) if it is a supplement (respectively complement) in every module containing it. We show that a module is absolutely supplement (respectively complement) if and only if it is a supplement (respectively complement) in its injective envelope. The class of all absolutely supplement (respectively complement) modules is closed under extensions and under supplement submodules (respectively under factor modules by complement submodules). We also consider the dual notions of absolutely co-supplements (respectively co-complements).

ÖZ

Bu tezde mutlak tümleyen ve mutlak bütünleyen modülleri tanımladık ve inceledik. Bir modül kendisini içeren tüm modüllerin içerisinde tümleyen (sırasıyla bütünleyen) ise bu modüle mutlak tümleyen (sırasıyla bütünleyen) modül denir. Bir modülün mutlak tümleyen (sırasıyla bütünleyen) olması için gerek ve yeter koşul bu modülün kendi injektif bürümü içinde bir tümleyen (sırasıyla bütünleyen) olmasıdır. Bütün mutlak tümleyen (sırasıyla bütünleyen) modüller sınıfı genişlemeler ve tümleyen alt modüller (sırasıyla bütünleyen alt modüllere göre bölüm modulleri) altında kapalıdır. Ayrıca mutlak tümleyen (sırasıyla bütünleyen) modüllerin dualleri olan mutlak dual-tümleyen (sırasıyla dual-bütünleyen) modülleri ele aldık.

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Chapter 1

INTRODUCTION

The decomposition of modules into the direct sum of their submodules plays an important role in Module and Ring Theory. Supplements and complements are two generalizations of direct summands when only one of two conditions of the direct sum and some condition of minimality (respectively maximality) are satisfied. Namely, a submodule N of a module M is a supplement (respectively complement) of $K \leq M$ if $N + K = M$ $(N \cap K = 0)$ and N is minimal (respectively maximal) with respect to this property. The class of short exact sequences

$$0 \longrightarrow N \xrightarrow{f} M \longrightarrow L \longrightarrow 0$$

where $f(N)$ is a supplement (respectively complement) for some $K \leq M$, forms a proper class (purity in other terminology) in the sense of Buchsbaum and McLane, which we denote by $\mathcal{S}$ (respectively $\mathcal{C}$). One of the classical examples of proper classes is the Cohn class (Cohn purity) and one of the useful notions in the study of the Cohn class is the absolutely pure and flat modules. We introduce and study analogous notions for supplements and complements. Namely, N is an absolutely supplement (respectively absolutely complement) module if N is a supplement (respectively complement) of some submodule of each module M containing N. N is an absolutely co-supplement (respectively absolutely co-complement) module if in all cases when $M/L \cong N$ for some module M and its submodule L, L is a supplement (respectively complement) of some $K \leq M$.

We show that N is absolutely supplement if and only if N is a supplement in the injective envelope $E(N)$ of N. Every finite direct sum of absolutely supplement (respectively absolutely co-supplement) modules is absolutely supplement(respectively absolutely co-supplement). Every supplement submodule of an absolutely supplement module is absolutely supplement. If $N \leq M$ and N, M/N are absolutely supplement modules then M is an absolutely supplement module. We show also that M is absolutely co-supplement if and only if there is a projective module P and an epimorphism $f : P \longrightarrow M$ such that $\operatorname{Ker} f$

is a supplement in P. If M is an absolutely co-supplement module and N is a supplement submodule of M then M/N is also absolutely co-supplement. Conversely, if $N \leq M$ and N, M/N are absolutely co-supplement modules then M is an absolutely co-supplement module. It turns out that absolutely complement modules coincide with injective modules. Using this fact we prove that every complement submodule of an injective module is injective. If $N \leq M$ and N, M/N are absolutely complement modules then M is an absolutely complement module. A module M is absolutely co-complement if and only if there is an epimorphism $f : P \longrightarrow M$ with a projective module P and Ker f a complement submodule of P. If M is an absolutely co-complement module and N is a complement submodule of M then M/N is absolutely co-complement. Conversely, if $N \leq M$ and N, M/N are absolutely co-complement modules then M is an absolutely co-complement module.

Chapter 2

PRELIMINARIES

2.1 Isomorphism Theorems

Let R be a ring with identity 1 and M be an abelian group. Suppose there is a function $f : R \times M \longrightarrow M$ (we will denote $f(r, m)$ by rm) where $r \in R$ and $m \in M$. Then M is called a left *R-module* (or briefly a module) if the followings are satisfied.

1. $r(m + n) = rm + rn$ for every $r \in R$ and $m, n \in M$
2. $(r + s)m = rm + sm$ for every $r, s \in R$ and $m \in M$
3. $(rs)m = r(sm)$ for every $r, s \in R$ and $m \in M$
4. $1 \cdot m = m$ for every $m \in M$.

If the function $f : M \times R \longrightarrow M$ $(mr \in M)$ with similar conditions are given we will have right R-module. If M is a left R-module, right S-module and $(rm)s = r(ms)$ for every $r \in R$, $m \in M$, $s \in S$ then M is called an *R-S-module or bimodule.*

A subset N of an R-module M is called a submodule if N itself is a module with respect to the same operations. Notation: $N \leq M$.

Basic information about modules can be found in [3]. Throughout this study we will use the following definitions, theorems and propositions.

Definition 2.1.1. Let M be module and let N be a submodule of M. The set of cosets $M/N = \{x+N|x \in M\}$ is a module with respect to the addition and scalar multiplication defined by $(x + N) + (y + N) = (x + y) + N$, $r(x + N) = rx + N$. This module M/N is called a *factor module* of M by N.

Definition 2.1.2. If M and N are two modules then a function $f : M \longrightarrow N$ is a *homomorphism* in case for all $r, s \in R$ and $m, a \in M$ $f(rm+sa) = rf(m)+sf(a)$.

Definition 2.1.3. A homomorphism $f : M \longrightarrow N$ is called an *epimorphism* in case it is onto. It is called a *monomorphism* in case it is one to one.

Definition 2.1.4. *Kernel of* f: $\operatorname{Ker} f = \{m \in M | f(m) = 0\} \leq M$.
Image of f: $\operatorname{Im} f = \{f(m) | m \in M\} \leq N$.

So f is an epimorphism if and only if $\operatorname{Im} f = N$, and it can be easily verified that f is a monomorphism if and only if $\operatorname{Ker} f = 0$.

Definition 2.1.5. A homomorphism f is called an *isomorphism* if it is both an epimorphism and a monomorphism (i.e. it is a bijection).

Definition 2.1.6. If N is a submodule of M then the inclusion map $i : N \longrightarrow M$ is a monomorphism also called the *natural embedding* of N in M with image N. Let N be a submodule of M then the mapping $\sigma : M \longrightarrow M/N$ from M onto the factor module M/N defined by $\sigma(m) = m + N$, $m \in M$ is called *the natural (canonical) epimorphism* of M onto M/N. Clearly $\operatorname{Ker} \sigma = N$

Theorem 2.1.7. ***Fundamental Homomorphism Theorem***
Let M and N be left modules and $f : M \longrightarrow N$ be a homomorphism, then

$$M/\operatorname{Ker} f \cong \operatorname{Im} f$$

In particular if f is an epimorphism then $M/\operatorname{Ker} f \cong N$.

Proof. Define $\overline{f} : M/K \longrightarrow N$ (where $K = \operatorname{Ker} f$) by $\overline{f}(m + K) = f(m)$.
$m + K = n + K$ implies $m - n \in K$, so $f(m - n) = 0$, then $f(m) = f(n)$.
Thus $\overline{f}$ is well-defined.
Also $\overline{f}((m + K) + (n + K)) = \overline{f}((m + n) + K) = f(m + n) = f(m) + f(n) = \overline{f}(m + K) + \overline{f}(n + K)$.
$\overline{f}(r(m + K)) = \overline{f}(rm + K) = f(rm) = rf(m) = r\overline{f}(m + K)$.
So $\overline{f}$ is a homomorphism.
If $\overline{f}(m + K) = \overline{f}(n + K)$ then $f(m) = f(n) \Rightarrow f(m - n) = 0 \Rightarrow m - n \in K \Rightarrow m + K = n + K \Rightarrow \overline{f}$ is one-to-one.
At last for every $n \in N$ we have $n = f(m) = \overline{f}(m + K) \Rightarrow \overline{f}$ is onto.
So $\overline{f}$ is an isomorphism. □

Theorem 2.1.8. ***Second Isomorphism Theorem***
If N, K are submodules of M, then

$$(N + K)/K \cong N/(N \cap K)$$

Proof. Define $f : N \longrightarrow (N+K)/K$ by $f(n) = n + K$.
Since $(n+k)+K = n+K = f(n)$, f is an epimorphism.
$\operatorname{Ker} f = \{n \in N | n \in K\} = N \cap K$.
So by Fundamental Homomorphism Theorem $N/(N \cap K) \cong (N+K)/K$. □

Theorem 2.1.9. ***Third Isomorphism Theorem***
If $K \leq N \leq M$, then

$$(M/K)/(N/K) \cong M/N$$

Proof. Define $f : M/K \longrightarrow M/N$ by $f(m+K) = m+N$.
Suppose that $m_1 + K = m_2 + K$ then $m_1 - m_2 \in K \leq N \Rightarrow m_1 - m_2 \in N \Rightarrow m_1 + N = m_2 + N$.
Hence f is well-defined.
Also $f(r(m_1+K) + s(m_2+K)) = f((rm_1 + sm_2) + K) = (rm_1 + sm_2) + N = r(m_1+N) + s(m_2+N) = rf(m_1+K) + sf(m_2+K)$, i.e. f is a homomorphism.
Since for all $m + N \in M/N$ we have $f(m+K) = m+N$, f is an epimorphism.
$\operatorname{Ker} f = \{m + K | m \in N\} = N/K$.
So by Fundamental Homomorphism Theorem $(M/K)/(N/K) \cong M/N$. □

Lemma 2.1.10. ***Modular Law***
Let N, K, L be submodules of a module M and $K \leq N$, then

$$N \cap (K+L) = K + (N \cap L)$$

Proof. Any x from $N \cap (K+L)$ can be represented as $x = n = k + l$ for some $n \in N$, $k \in K$, and $l \in L$. Since $K \leq N$, $k \in N$. Therefore $l = n - k \in N \cap L$.
Thus $x = k + l \in K + (N \cap L)$. Converse is obvious. □

Definition 2.1.11. Let $\{N_i\}_{i \in I}$ be a family of submodules of a module M. M is the *internal direct sum* of submodules N_i if every element $m \in M$ can be uniquely represented as $m = \sum_{i \in I} n_i$; $n_i \in N_i$.

Proposition 2.1.12. *$M = \bigoplus_{i \in I} N_i$ if and only if $M = \sum_{i \in I} N_i$ and $N_i \cap (\sum_{i \neq j} N_j) = 0$ for every $i \in I$.*

Proof. ($\Rightarrow$) 1)For every $m \in M$ we have $m = \sum_{i \in I} n_i \in \sum_{i \in I} N_i$, $\Rightarrow M \subseteq \sum_{i \in I} N_i \Rightarrow M = \sum_{i \in I} N_i$.

2)Let $x = n_i = \sum_{i \neq j} n_j \in N_i \cap \sum_{j \neq i} N_j$ then by uniqueness of representation $x = n_i = 0$.

($\Leftarrow$) $\forall m \in M = \sum_{i \in I} N_i$, $m = \sum_{i \in I} n_i$. To prove uniqueness let $m = \sum_{i \in I} n_i = \sum_{i \in I} n_i^{'}$. For every $i \in I$ we have $n_i - n_i^{'} = \sum_{i \neq j}(n_j - n_j^{'}) \in N_i \cap (\sum_{i \neq j} N_j) = 0 \Rightarrow n_i = n_i^{'}$. $\square$

Definition 2.1.13. Let $\{N_i\}_I$ be a family of modules. The cartesian product $\prod_{i \in I} N_i$ is a module relative to the addition and scalar multiplication defined by $(n_i) + (m_i) = (n_i + m_i)$ and $r(n_i) = (rn_i)$, $n_i, m_i \in N_i$, $r \in R$. This module is called the *direct product* of the modules N_i. The subset $\bigoplus_{i \in I} N_i = \{(n_i) | n_i = 0$ for all but finite number of $i \in I\}$ is called the *external direct sum* of modules N_i.

Definition 2.1.14. If $M = N \oplus K$ then N, K are called *direct summands* of M.

2.2 Exact Sequences

Definition 2.2.1. A sequence

$$\mathbb{S}: \quad \cdots \longrightarrow M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \cdots$$

of modules $\{M_n\}_{n \in \mathbb{Z}}$ and homomorphisms $\{f_n\}_{n \in \mathbb{Z}}$ is called an *exact sequence* if for all $n \in \mathbb{Z}$ $\quad \operatorname{Im} f_{n-1} = \operatorname{Ker} f_n$.

Definition 2.2.2. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence.*

Theorem 2.2.3. *For a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the following conditions are equivalent.

1. *There is a homomorphism $h : B \longrightarrow A$ such that $h \circ f = 1_A$.*

2. $\operatorname{Im} f$ *is a direct summand of B i.e. $B = \operatorname{Im} f \oplus L$ for some $L \leq B$.*

3. *There is a homomorphism* $e : C \longrightarrow B$ *such that* $g \circ e = 1_C$.

Definition 2.2.4. If any of the conditions of this theorem is satisfied the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is said to be a *splitting short exact sequence.*

Remark 2.2.5. *If*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is splitting then $B \cong A \oplus C$.

Definition 2.2.6. Let A and B are modules. The set

$$\mathrm{Hom}(A, B) = \{f | f : A \longrightarrow B\}$$

of all module homomorphisms f of A into B is an abelian group under the addition defined for $f, g : A \longrightarrow B$ by $(f + g)(a) = f(a) + g(a)$. In case the ring R is commutative, $\mathrm{Hom}(A, B)$ can be regarded as a module when $tf : A \longrightarrow B$ is defined for $t \in R$ and $f : A \longrightarrow B$ by $(tf)(a) = t(f(a))$ for all $a \in A$. That tf is still a module homomorphism follows from the calculation
$(tf)(ra) = t(f(ra)) = tr(f(a)) = rt(f(a)) = r((tf)(a))$
which uses the commutativity of R.
Consider the effect of a fixed module homomorphism $\beta : B \longrightarrow B'$ on $\mathrm{Hom}(A, B)$. Each $f : A \longrightarrow B$ determines a composite $\beta \circ f : A \longrightarrow B'$ and $\beta \circ (f + g) = \beta \circ f + \beta \circ g$. Hence the correspondence $f \longrightarrow \beta \circ f$ is a homomorphism

$$\beta_* : \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, B') \tag{2.1}$$

of abelian groups. Explicitly, $\beta_*(f) = \beta \circ f$. If β is an identity, so is β_*, if β is a composite, so is β_*, in detail

$$(1_\beta)_* = 1_{\mathrm{Hom}(A,B)}, \ (\beta \circ \beta')_* = \beta_* \circ \beta'_* \tag{2.2}$$

the latter whenever the composite $\beta \circ \beta'$ is defined. We summarize (2.1) and (2.2) by the phrase: $\mathrm{Hom}(A, -)$ is a *covariant functor.*
For the first argument A a reverse in direction occurs. For a fixed module homomorphism $\alpha : A \longrightarrow A'$ each $f' : A' \longrightarrow B$ determines a composite

$f' \circ \alpha : A \longrightarrow B$ with $(f' + g') \circ \alpha = f' \circ \alpha + g' \circ \alpha$. Hence $f' \longrightarrow f' \circ \alpha$ is a homomorphism

$$\alpha^* : \mathrm{Hom}(A', B) \longrightarrow \mathrm{Hom}(A, B)$$

of abelian groups defined by $\alpha^*(f') = f' \circ \alpha$. Again $(1_A)^*$ is an identity map. If $\alpha : A \longrightarrow A'$ and $\alpha' : A' \longrightarrow A''$ the composite $\alpha' \circ \alpha$ is defined and $(\alpha' \circ \alpha)^* = \alpha^* \circ \alpha'^*$. Because of this reversal order $\mathrm{Hom}(-, B)$ is called a *contravariant functor.*

Theorem 2.2.7. *For any module M and any exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

the following sequence

$$0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C)$$

is exact.

Proof. Proof of this theorem can be found in [8]. □

Above theorem states that the functor $\mathrm{Hom}(M, -)$ for fixed M turns each exact sequence into a left exact sequence.

Theorem 2.2.8. *For any module M and any exact sequence*

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

the following sequence

$$0 \longrightarrow \mathrm{Hom}(C, M) \longrightarrow \mathrm{Hom}(B, M) \longrightarrow \mathrm{Hom}(A, M)$$

is exact.

Proof. For proof of this theorem see [8]. □

By the previous theorem, $\mathrm{Hom}(-, M)$ carries an exact sequence into a left exact sequence.

In this thesis without lose of generality, for a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

we will assume A as a submodule of B.

2.3 Projective and Injective Modules

Definition 2.3.1. A module I is called *injective* if for every monomorphism $f : A \longrightarrow B$ and homomorphism $g : A \longrightarrow I$ there exists a homomorphism $h : B \longrightarrow I$ such that $g = h \circ f$. We can show this by the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "h"] \\ & I & \end{tikzcd}$$

For $R = \mathbb{Z}$, i.e. for abelian groups D is injective if and only if D is a divisible group (László Fuchs, [5]), (i.e. for every $d \in D$ and non-zero integer n there is $d' \in D$ such that $d = nd'$).

Theorem 2.3.2. *If D is a divisible abelian group then* $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ *is an injective R-module.*

Proof. Proof of this theorem is given in [2]. □

Theorem 2.3.3. *For every module M there is a monomorphism $g : M \longrightarrow I$ with an injective module I.*

Proof. There is a divisible group D and monomorphism of abelian groups $f : M \longrightarrow D$. Then $e : M \longrightarrow \mathrm{Hom}(R, M)$ defined by $e(m)(r) = rm$ and $f_* : \mathrm{Hom}(R, M) \longrightarrow \mathrm{Hom}(R, D)$ are monomorphisms of modules. Now taking $I = \mathrm{Hom}(R, D)$ we will have monomorphism $f_* \circ e : M \longrightarrow I$ with injective module I. □

Theorem 2.3.4. *For a module I the following statements are equivalent.*

1. *I is injective.*

2. $\mathrm{Hom}(-, I)$ *is exact.*

3. *Every short exact sequence*

$$0 \longrightarrow I \xrightarrow{g} A \longrightarrow B \longrightarrow 0$$

is splitting.

4. *I is a direct summand of a module of the form* $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ *where D is a divisible group.*

Proof. 1. $\Rightarrow$ 2. Let

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

be a short exact sequence. Then the sequence

$$0 \longrightarrow \mathrm{Hom}(C, I) \longrightarrow \mathrm{Hom}(B, I) \xrightarrow{f^*} \mathrm{Hom}(A, I)$$

is exact. It remains only to show that f^* is an epimorphism. Let $\alpha \in \mathrm{Hom}(A, I)$. We have a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[dl, dashed, "\beta"] \\ & I & \end{tikzcd}$$

Since I is injective, there exists a homomorphism $\beta : B \longrightarrow I$ such that $\beta \circ f = \alpha$ i.e. $f^*(\beta) = \beta \circ f = \alpha$, $\beta \in \mathrm{Hom}(B, I)$. So f^* is an epimorphism. Thus

$$0 \longrightarrow \mathrm{Hom}(C, I) \longrightarrow \mathrm{Hom}(B, I) \xrightarrow{f^*} \mathrm{Hom}(A, I) \longrightarrow 0$$

is an exact sequence.

2. $\Rightarrow$ 3. Since $\mathrm{Hom}(-, I)$ is exact, the homomorphism $g^* : \mathrm{Hom}(A, I) \longrightarrow \mathrm{Hom}(I, I)$ is an epimorphism. Then $1_I \in \mathrm{Hom}(I, I)$ there is a homomorphism $h \in \mathrm{Hom}(A, I)$ such that $g^*(h) = 1_I \Rightarrow h \circ g = 1_I$. So the short exact sequence

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

is splitting.

3. $\Rightarrow$ 4. We know that there is a monomorphism $I \longrightarrow \mathrm{Hom}(R, D)$ for some divisible group D. Then we have a short exact sequence

$$0 \longrightarrow I \longrightarrow \mathrm{Hom}(R, D) \longrightarrow X \longrightarrow 0$$

By 3) this sequence is splitting $\mathrm{Hom}(R, D) \cong I \oplus X$.

4. $\Rightarrow$ 1. By the Theorem 2.3.2 $\mathrm{Hom}(R, D)$ is injective. So every direct summand of $\mathrm{Hom}(R, D)$ is injective. Thus I is injective. $\square$

Definition 2.3.5. Let M be a module. An injective module E together with a monomorphism $f : M \longrightarrow E$ is called an *injective envelope(hull)* of M if $\mathrm{Im}\, f \trianglelefteq E$ and denoted by $E(M)$.

Definition 2.3.6. A module P is called *projective* if for every epimorphism $f : A \longrightarrow B$ and homomorphism $g : P \longrightarrow B$ there exists a homomorphism $h : P \longrightarrow A$ such that $g = f \circ h$. We can show this by the following commutative diagram.

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

It is well-known that every vector space has a basis. Modules with bases are free modules.

Theorem 2.3.7. *Let F be a module and $X = \{x_\alpha\}_{\alpha \in A}$ be a subset of F. Then the followings are equivalent.*

1. *Every element $a \in F$ can be uniquely represented as $a = \sum_{\alpha \in A} r_\alpha x_\alpha$, where $r_\alpha \in R$ and $r_\alpha = 0$ for almost all $\alpha \in A$.*

2. *For each $\alpha \in A$ the function $f_\alpha : R \longrightarrow Rx_\alpha$ is defined by $f_\alpha(r) = rx_\alpha$ is an isomorphism and $F = \bigoplus_{\alpha \in A} Rx_\alpha$.*

Proof. 1. $\Rightarrow$ 2. f_α is an epimorphism. If $f_\alpha(r) = f_\alpha(s)$ i.e. $rx_\alpha = sx_\alpha$ by uniqueness of the representation $r = s$. So f_α is $1-1$. Thus f_α is an isomorphism. Every element can be uniquely represented as $a = \sum_{\alpha \in A} r_\alpha x_\alpha \in \sum_{\alpha \in A} Rx_\alpha$ therefore $F = \bigoplus_{\alpha \in A} Rx_\alpha$.
2. $\Rightarrow$ 1. $F = \bigoplus_{\alpha \in A} Rx_\alpha \Rightarrow \forall a \in F$, $a = \sum_{\alpha \in A} r_\alpha x_\alpha$. If $a = \sum_{\alpha \in A} r_\alpha x_\alpha = \sum_{\alpha \in A} r'_\alpha x_\alpha$ then $r_\alpha x_\alpha = r'_\alpha x_\alpha$. Since f_α is $1-1$ then $r_\alpha = r'_\alpha$. □

Definition 2.3.8. If F satisfies the condition of the Theorem 2.3.7 then it is said to be a *free module* and X is said to be a *basis* of F.

The following theorem shows that every function from a basis of a free module to any module can be uniquely extended to a homomorphism from the free module.

Theorem 2.3.9. *If F is a free R-module with a basis X then for every function $f : X \longrightarrow M$, where M is an R-module, there is a unique homomorphism $g : F \longrightarrow M$ such that $g|_X = f$ that is $g(x_\alpha) = f(x_\alpha)$ for all $x_\alpha \in X$.*

Proof. Define $g : F \longrightarrow M$ by $g(\sum_{\alpha\in A} r_\alpha x_\alpha) = \sum_{\alpha\in A} r_\alpha f(x_\alpha)$. g is a homomorphism and $g(x_\alpha) = g(1 \cdot x_\alpha) = 1 \cdot f(x_\alpha) = f(x_\alpha)$ i.e. $g|_X = f$. If $g' : F \longrightarrow M$ is a homomorphism such that $g'|_X = f$ then $\forall \sum_{\alpha\in A} r_\alpha x_\alpha \in F$ we have $g'(\sum_{\alpha\in A} r_\alpha x_\alpha) = \sum_{\alpha\in A} r_\alpha g'(x_\alpha) = \sum_{\alpha\in A} r_\alpha f(x_\alpha) = \sum_{\alpha\in A} r_\alpha g(x_\alpha) = g(\sum_{\alpha\in A} r_\alpha x_\alpha) \Rightarrow g = g'$ □

Corollary 2.3.10. *Every free module is projective.*

Proof. Let F be a free module with a basis $X = \{x_\alpha\}_{\alpha\in A}$ and $f : A \longrightarrow B$ be an epimorphism and $g : F \longrightarrow B$ be a homomorphism. Since f is an epimorphism for each element $g(x_\alpha) \in B$ there is an element $a_\alpha \in A$ such that $f(a_\alpha) = g(x_\alpha)$.

$$\begin{tikzcd} & F \arrow[dl, dashed, "u"'] \arrow[d, "g"] & \\ A \arrow[r, "f"] & B \arrow[r] & 0 \end{tikzcd}$$

Define a function $h : X \longrightarrow A$ by $h(x_\alpha) = a_\alpha$. By the Theorem 2.3.9 there is a homomorphism $u : F \longrightarrow A$ such that $u|_X = h$ i.e. $u(x_\alpha) = h(x_\alpha) = a_\alpha$. For every element $\sum_{\alpha\in A} r_\alpha x_\alpha \in F$, $(f \circ u)(\sum_{\alpha\in A} r_\alpha x_\alpha) = \sum_{\alpha\in A} r_\alpha f(u(x_\alpha)) = \sum_{\alpha\in A} r_\alpha f(a_\alpha) = \sum_{\alpha\in A} r_\alpha g(x_\alpha) = g(\sum_{\alpha\in A} r_\alpha x_\alpha) \Rightarrow f \circ u = g$. So F is projective. □

Lemma 2.3.11. *For every module M there is a free module F and an epimorphism $f : F \longrightarrow M$.*

Proof. Let $X = \{x_m\}_{m\in M}$ and $F = \{\sum_{m\in M} r_m x_m |$ where $r_m = 0$ for almost all $m \in M\}$. Define addition and multiplication by $\sum_{m\in M} r_m x_m + \sum_{m\in M} r'_m x_m = \sum_{m\in M} (r_m + r'_m) x_m$ and $s(\sum_{m\in M} r_m x_m) = \sum_{m\in M} (s r_m) x_m$. Then F is a free module with basis $X' = \{1 \cdot x_m\}$. Really $\forall \sum_{m\in M} r_m x_m \in F$ can be uniquely represented as $\sum_{m\in M} r_m x_m = \sum_{m\in M} r_m (1 \cdot x_m)$. Define a function $g : X \longrightarrow M$ by $g(x_m) = m$. By the Theorem 2.3.9 there is a homomorphism $f : F \longrightarrow M$ such that $f(x_m) = m$, $\forall m \in M$. So f is an epimorphism. □

Now we can formulate the following equivalent conditions for projective modules.

Theorem 2.3.12. *For a module P the following conditions are equivalent.*

1. *P is projective.*

2. $\mathrm{Hom}(P, -)$ *is exact.*

3. *Every exact sequence of the form*

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

is splitting.

4. *P is a direct summand of a free module, i.e. $F \cong P \oplus N$ for some free module F.*

Proof. 1. $\Rightarrow$ 2. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. Then the sequence

$$0 \longrightarrow \mathrm{Hom}(P, A) \xrightarrow{f_*} \mathrm{Hom}(P, B) \xrightarrow{g_*} \mathrm{Hom}(P, C)$$

is exact. It remains to show that g_* is an epimorphism. Let $\alpha \in \mathrm{Hom}(P, C)$. Since P is projective, there is a homomorphism $\beta : P \longrightarrow B$ such that $g \circ \beta = \alpha$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\beta"'] \arrow[d, "\alpha"] & \\ B \arrow[r, "g"] & A \arrow[r] & 0 \end{tikzcd}$$

i.e. $\exists \beta \in \mathrm{Hom}(P, B)$ such that $g_*(\beta) = g \circ \beta = \alpha$. So g_* is an epimorphism.
2. $\Rightarrow$ 3. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

be exact. By 2) the sequence

$$0 \longrightarrow \mathrm{Hom}(P, A) \longrightarrow \mathrm{Hom}(P, B) \xrightarrow{f_*} \mathrm{Hom}(P, P) \longrightarrow 0$$

is exact. So f_* is an epimorphism. For $1_P \in \mathrm{Hom}(P, P)$ there is $\alpha \in \mathrm{Hom}(P, B)$ such that $f_*(\alpha) = 1_P \Rightarrow f \circ \alpha = 1_P$. So the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

is splitting.
3. $\Rightarrow$ 4. By the Lemma 2.3.11 there is an epimorphism $f : F \longrightarrow P$ from a free module F. Then we have the following short exact sequence

$$0 \longrightarrow \mathrm{Ker}\, f \xrightarrow{i} F \xrightarrow{f} P \longrightarrow 0$$

By 3) this sequence is splitting. So $F \cong \operatorname{Ker} f \oplus P$.

4. $\Rightarrow$ 1. Follows from the Proposition 2.3.13. $\square$

Proposition 2.3.13. *A direct sum $P = \bigoplus_{\alpha \in A} P_\alpha$ of modules $\{P_\alpha\}_{\alpha \in A}$ is projective if and only if each P_α is projective.*

Proof. This proposition is proved by R. Alizade & A. Pancar in [2]. $\square$

2.4 Proper Classes

Sometimes it is convenient to study some homological properties with respect to some class of short exact sequences. Of course the class must satisfy some conditions. This leads to proper classes of short exact sequences.

Definition 2.4.1. Let $\mathcal{A}$ be a class of short exact sequences of modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

We assume that $\mathcal{A}$ is closed under isomorphisms: if $E \cong E'$, then $E \in \mathcal{A} \Leftrightarrow E' \in \mathcal{A}$. If $E \in \mathcal{A}$, we say that f is an $\mathcal{A}$-monomorphism ($f \in \mathcal{A}_m$) and g is an $\mathcal{A}$-epimorphism ($g \in \mathcal{A}_e$). $\mathcal{A}$ is said to be a *proper class* if it satisfies the following conditions:

$P1$. Every splitting short exact sequence is in $\mathcal{A}$;

$P2$. The composition $h \circ f$ of two $\mathcal{A}$-monomorphisms h, f is an $\mathcal{A}$-monomorphism if it is defined;

$P2'$. The composition $u \circ g$ of two $\mathcal{A}$-epimorphisms g, u is an $\mathcal{A}$-epimorphism if it is defined;

$P3$. If $h \circ f$ is an $\mathcal{A}$-monomorphism and h is a monomorphism then f is an $\mathcal{A}$-monomorphism;

$P3'$. If $u \circ g$ is an $\mathcal{A}$-epimorphism and g is an epimorphism then u is an $\mathcal{A}$-epimorphism.

Let $\mathcal{A}$ denote a proper class of R-modules.

Definition 2.4.2. An R-module P is said to be *$\mathcal{A}$-projective* if it is projective with respect to all short exact sequences in $\mathcal{A}$, that is, every diagram

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "h"'] \arrow[d, "u"] & \\ \mathbb{E}: \quad 0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0 \in \mathcal{A} \end{tikzcd}$$

can be embedded in a commutative diagram by choosing an R-module homomorphism $h : P \to B$ properly; equivalently,

$$\mathrm{Hom}(P, \mathbb{E}) : 0 \longrightarrow \mathrm{Hom}(P, A) \xrightarrow{f_*} \mathrm{Hom}(P, B) \xrightarrow{g_*} \mathrm{Hom}(P, C) \longrightarrow 0$$

is exact for every

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$. The class of all $\mathcal{A}$-projective modules is denoted by $\pi(\mathcal{A})$.

Definition 2.4.3. An R-module I is said to be *$\mathcal{A}$-injective* if it is injective with respect to all short exact sequences in $\mathcal{A}$, that is, every diagram

$$\begin{tikzcd} \mathbb{E}: \quad 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[dl, dashed, "h"] & C \arrow[r] & 0 \in \mathcal{A} \\ & I & & & \end{tikzcd}$$

can be embedded in a commutative diagram by choosing an R-module homomorphism $h : B \to I$ properly; equivalently,

$$\mathrm{Hom}(\mathbb{E}, I) : 0 \longrightarrow \mathrm{Hom}(C, I) \xrightarrow{f^*} \mathrm{Hom}(B, I) \xrightarrow{g^*} \mathrm{Hom}(A, I) \longrightarrow 0$$

is exact for every

$$\mathbb{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

in $\mathcal{A}$. The class of all $\mathcal{A}$-injective modules is denoted by $\iota(\mathcal{A})$.

For the study of proper classes it is useful to define some notions which have no nontrivial analogs for the class of all short exact sequences.

Definition 2.4.4. An R-module C is said to be *$\mathcal{A}$-coprojective* if every short exact sequence of R-modules with the term prior to the last 0 being C is in $\mathcal{A}$, that is, every short exact sequence of R-modules of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is in $\mathcal{A}$.

Definition 2.4.5. An R-module A is said to be *$\mathcal{A}$-coinjective* if every short exact sequence of R-modules with the term after the first 0 being A is in $\mathcal{A}$, that is, every short exact sequence of R-modules of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is in $\mathcal{A}$.

Definition 2.4.6. For a class $\mathcal{M}$ of modules M, $\pi^{-1}(\mathcal{M})$ denotes *the smallest proper class* $\mathcal{A}$ for which each $M \in \mathcal{M}$ is $\mathcal{A}$-projective, i.e.

$$\pi^{-1}(\mathcal{M}) = \{\mathbb{E} | \operatorname{Hom}(M, \mathbb{E}) \text{ is exact for all } M \in \mathcal{M}\}$$

Definition 2.4.7. For a class $\mathcal{M}$ of modules M, $\iota^{-1}(\mathcal{M})$ denotes *the smallest proper class* $\mathcal{A}$ for which each $M \in \mathcal{M}$ is $\mathcal{A}$-injective, i.e.

$$\iota^{-1}(\mathcal{M}) = \{\mathbb{E} | \operatorname{Hom}(\mathbb{E}, M) \text{ is exact for all } M \in \mathcal{M}\}$$

2.5 Pullback and Pushout Diagrams

We will use pullback and pushout diagrams for some following proofs. So we here will define pullback and pushout of a diagram and show their existence and uniqueness for R-modules. Extended information about these are given by R. Alizade in [1] and L.R.Vermani in [13].

Definition 2.5.1. An R-module G together with homomorphisms $\gamma : G \longrightarrow A$ and $\delta : G \longrightarrow B$ is said to be a *pullback* of the pair of homomorphisms $\alpha : A \longrightarrow C$ and $\beta : B \longrightarrow C$ if,

(i) the following diagram is commutative

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd} \tag{2.1}$$

(ii) if the following diagram is commutative

$$\begin{tikzcd} G' \arrow[r, "\gamma'"] \arrow[d, "\delta'"'] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"'] & C \end{tikzcd} \tag{2.2}$$

then there is a unique homomorphism $\theta : G' \longrightarrow G$ such that the diagrams

$$\begin{tikzcd} G' \arrow[r, "\theta"] \arrow[d, "\gamma'"'] & G \arrow[dl, dashed, "\gamma"] \\ A & \end{tikzcd} \qquad \begin{tikzcd} G' \arrow[r, "\theta"] \arrow[d, "\delta'"'] & G \arrow[dl, dashed, "\delta"] \\ B & \end{tikzcd}$$

are commutative i.e. $\gamma \circ \theta = \gamma'$ and $\delta \circ \theta = \delta'$. Also (2.1) is said to be a *pullback diagram.*

Theorem 2.5.2. ***Existence of Pullback***

Given the homomorphisms $\alpha : A \longrightarrow C$ and $\beta : B \longrightarrow C$ there is a pullback G.

Proof. Given α, β define G as the submodule of the direct sum $A \oplus B$ consisting of all (a, b) with $\alpha(a) = \beta(b)$ i.e. $G = \{(a, b) \in A \oplus B | \alpha(a) = \beta(b)\}$ and let $\gamma : G \longrightarrow A$ and $\delta : G \longrightarrow B$ be the corresponding projections :$\gamma(a, b) = a$ and $\delta(a, b) = b$. Then (2.1) will obviously be commutative. Assume (2.2) is a commutative diagram and define $\theta : G' \longrightarrow G$ as $\theta(g') = (\gamma'(g'), \delta'(g'))$ for $g' \in G'$. Since $(\alpha \circ \gamma')(g') = (\beta \circ \delta')(g')$ by commutativity of (2.2), we have $(\gamma'(g'), \delta'(g')) \in G$. Evidently, $(\gamma \circ \theta)(g') = \gamma'(g')$ and $(\delta \circ \theta)(g') = \delta'(g')$ for every $g' \in G'$, so θ is of the stated kind. $\operatorname{Ker} \gamma = \{(a, b) \in G | \gamma(a, b) = 0\} = \{(a, b) \in A \oplus B | a = 0, \beta(b) = \alpha(a) = 0\} = \{(0, b) | b \in \operatorname{Ker} \beta\} = 0 \oplus \operatorname{Ker} \beta$. So we have

$$\operatorname{Ker} \gamma = 0 \oplus \operatorname{Ker} \beta \tag{2.3}$$

Similarly, $\operatorname{Ker} \delta = \operatorname{Ker} \alpha \oplus 0$. Therefore, if $\theta' : G' \longrightarrow G$, also satisfies $\gamma \circ \theta' = \gamma'$ and $\delta \circ \theta' = \delta'$, then $\gamma(\theta - \theta') = 0 = \delta(\theta - \theta')$ and hence $\operatorname{Im}(\theta - \theta') \subset \operatorname{Ker} \gamma \cap \operatorname{Ker} \delta = 0$. This shows that $\theta - \theta' = 0$ and θ is unique. □

Theorem 2.5.3. ***Uniqueness***

This pullback G is unique up to isomorphism.

Proof. Let $\overline{G}$ be another pullback of the homomorphisms $\alpha : A \longrightarrow C$ and $\beta : B \longrightarrow C$ with homomorphisms $\overline{\gamma} : \overline{G} \longrightarrow A$ and $\overline{\delta} : \overline{G} \longrightarrow B$. So $\overline{G}$ has the same properties which are stated in the Definition 2.5.1. Then we have unique homomorphisms $\theta : \overline{G} \longrightarrow G$ and $\overline{\theta} : G \longrightarrow \overline{G}$ such that $\gamma \circ \theta = \overline{\gamma}$, $\delta \circ \theta = \overline{\delta}$, and $\overline{\gamma} \circ \overline{\theta} = \gamma$, $\overline{\delta} \circ \overline{\theta} = \delta$. Hence $\gamma \circ (\theta \circ \overline{\theta}) = \gamma$ and $\delta \circ (\theta \circ \overline{\theta}) = \delta$. On the other hand, $\gamma \circ 1_G = \gamma$ and $\delta \circ 1_G = \delta$. By uniqueness of the homomorphisms in (2.2) $\theta \circ \overline{\theta} = 1_G$. Similarly, $\overline{\theta} \circ \theta = 1_{\overline{G}}$. So θ is an isomorphism. □

Theorem 2.5.4. *If in the pullback diagram* (2.1), *β is a monomorphism, then so is γ. If β is an epimorphism, then so is γ; and δ induces an isomorphism* $\operatorname{Ker}\gamma \cong \operatorname{Ker}\beta$, *so we have the following commutative diagram with exact rows:*

$$\begin{tikzcd}
E: 0 \arrow[r] & \operatorname{Ker}\gamma \arrow[r] \arrow[d, "\delta'"'] & G \arrow[r, "\gamma"] \arrow[d, "\delta"'] & A \arrow[r] \arrow[d, "\alpha"] \arrow[dl, dashed, "\kappa"'] & 0 \\
E': 0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] & B \arrow[r, "\beta"'] & C \arrow[r] & 0
\end{tikzcd}$$

where $\delta' = \delta|_{\operatorname{Ker}\gamma}$. The short exact sequence E is splitting if and only if there exists a homomorphism $\kappa : A \longrightarrow B$ such that $\alpha = \beta \circ \kappa$.

Proof. Since G is unique up to isomorphism, and moreover $\operatorname{Im}\gamma$ and $\operatorname{Ker}\gamma$ is unique up to isomorphism, it suffices to prove the statement for G as constructed in the proof of Theorem 2.5.2. By equality (2.3) $\operatorname{Ker}\gamma = 0 \oplus \operatorname{Ker}\beta = 0$ if β is a monomorphism. If β is an epimorphism, then for every $a \in A$ there is $b \in B$ such that $\beta(b) = \alpha(a)$. Then $(a, b) \in G$ and $\gamma(a, b) = a$, so γ is an epimorphism by equality (2.3) $\operatorname{Ker}\gamma = 0 \oplus \operatorname{Ker}\beta \cong \operatorname{Ker}\beta$, hence the restriction of projection δ' to $\operatorname{Ker}\gamma$ is an isomorphism.

Now let E be splitting by Theorem 2.2.3 there is a homomorphism $\gamma' : A \longrightarrow G$ such that $\gamma \circ \gamma' = 1_A$. Since $\alpha \circ \gamma = \beta \circ \delta$, for the homomorphism $\kappa = \delta \circ \gamma' : A \longrightarrow B$ we have $\alpha \circ (\gamma \circ \gamma') = \beta \circ (\delta \circ \gamma')$. Therefore, $\alpha = \beta \circ \kappa$.

On the other hand if we have a homomorphism $\kappa : A \longrightarrow B$ with $\alpha = \beta \circ \kappa$, the function $\gamma' : A \longrightarrow G$ defined by $\gamma'(a) = (a, \kappa(a))$ is well-defined (since $\alpha(a) = (\beta \circ \kappa)(a)$ and $(a, \kappa(a)) \in G$) homomorphism. For every $a \in A, (\gamma \circ \gamma')(a) = \gamma(a, \kappa(a)) = a$, hence $\gamma \circ \gamma' = 1_A$ and by the Theorem 1.3.3. E is splitting. $\square$

Definition 2.5.5. An R-module G together with homomorphisms $\gamma : A \longrightarrow G$ and $\delta : B \longrightarrow G$ is said to be a *pushout* of the pair of homomorphisms $\alpha : C \longrightarrow A$ and $\beta : C \longrightarrow B$ if,

(i) the following diagram is commutative

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\gamma"] \\
B \arrow[r, "\delta"'] & G
\end{tikzcd} \tag{2.4}$$

(ii) if the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[d, "\gamma'"] \\ B \arrow[r, "\delta'"'] & G' \end{tikzcd} \tag{2.5}$$

then there is a unique homomorphism $\theta : G \longrightarrow G'$ such that the diagrams

$$\begin{tikzcd} A \arrow[d, "\gamma"] \arrow[dr, dashed, "\gamma'"] & \\ G \arrow[r, "\theta"] & G' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[d, "\delta"] \arrow[dr, dashed, "\delta'"] & \\ G \arrow[r, "\theta"] & G' \end{tikzcd}$$

are commutative i.e. $\theta \circ \gamma = \gamma'$ and $\theta \circ \delta = \delta'$. Also (2.4) is said to be a *pushout diagram.*

Theorem 2.5.6. ***Existence of Pushout***

Given the homomorphisms $\alpha : C \longrightarrow A$ and $\beta : C \longrightarrow B$ there is a pushout G.

Proof. Starting with α, β define G as the quotient module of $A \oplus B$ by the submodule H of the elements of the form $(\alpha(c), -\beta(c))$ for $c \in C$, and let $\gamma : A \longrightarrow G$ and $\delta : B \longrightarrow G$ be the maps induced by the injections $\gamma(a) = (a, 0) + H$, $\delta(b) = (0, b) + H$. Then $(\gamma \circ \alpha)(c) = (\delta \circ \beta)(c)$ for every $c \in C$ and (2.4) is commutative. If (2.5) is a commutative diagram define $\theta : G \longrightarrow G'$ as $\theta((a, b) + H) = \gamma'(a) + \delta'(b)$. If $(a, b) + H = (a', b') + H$, that is $(a, b) - (a', b') \in H$, then $a - a' = \alpha(c), b - b' = -\beta(c)$ for some $c \in C$. Since $\gamma' \circ \alpha = \delta' \circ \beta$, we have $\gamma'(a) + \delta'(b) = \gamma'(a') + (\gamma' \circ \alpha)(c) + \delta'(b') - (\delta' \circ \beta)(c) = \gamma'(a') + \delta'(b')$, so θ is well-defined. It is readily seen that $\theta \circ \gamma = \gamma'$, $\theta \circ \delta = \delta'$.
If $\theta' \circ \gamma = \gamma'$, $\theta' \circ \delta = \delta'$ for some $\theta' : G \longrightarrow G'$, then $(\theta - \theta')\gamma = 0 = (\theta - \theta')\delta$ and for every $(a, b) + H \in G$ we have $(\theta - \theta')((a, b) + H) = (\theta - \theta')(\gamma(a) + \delta(b)) = (\theta - \theta')\gamma(a) + (\theta - \theta')\delta(b) = 0$. Therefore $\theta - \theta' = 0$ and θ is unique. □

Theorem 2.5.7. ***Uniqueness***

This pushout G is unique up to isomorphism.

Proof. An argument analogous to Theorem 2.5.3 establishes the uniqueness of G. □

Theorem 2.5.8. *If in the pushout diagram* (2.4) α *is an epimorphism, then so is* δ*; if* α *is a monomorphism, then so is* δ *and* γ *induces an isomorphism* $\gamma' : A/Im\alpha \longrightarrow G/Im\delta$*. So we have the following commutative diagram with exact rows:*

$$\begin{tikzcd}
E' : 0 \arrow[r] & C \arrow[r, "\alpha"] \arrow[d, "\beta"'] & A \arrow[r, "\pi_1"] \arrow[d, "\gamma"] \arrow[dl, dashed, "\kappa"'] & A/Im\alpha \arrow[r] \arrow[d, "\gamma'"] & 0 \\
E : 0 \arrow[r] & B \arrow[r, "\delta"'] & G \arrow[r, "\pi_2"'] & G/Im\delta \arrow[r] & 0
\end{tikzcd}$$

where π_1, π_2 *are natural projections,* $\gamma'(a + Im\alpha) = \gamma(a) + Im\delta$*. The short exact sequence* E *is split if and only if there exists a homomorphism* $\kappa : A \longrightarrow B$ *such that* $\beta = \kappa \circ \alpha$*.*

Proof. If α is an epimorphism, for every $a \in A$ there is a $c \in C$ with $\alpha(c) = a$. Then for every element $(a, b) + H$ we have $(a, b) + H = (a - \alpha(c), b + \beta(c)) + H = (0, b + \beta(c)) + H = \delta(b + \beta(c))$. Hence δ is an epimorphism.
Let α be a monomorphism. Ker δ consists of all $b \in B$ for which $\delta(b) = (0, b) \in H$, i.e. there is a $c \in C$ with $\alpha(c) = 0$ and $-\beta(c) = b$. Since α is a monomorphism, $c = 0$ and $b = \beta(0) = 0$. So Ker $\delta = 0$ and δ is a monomorphism.
If E is splitting, there is $\delta' : G \longrightarrow B$ with $\delta' \circ \delta = 1_B$. Since $\delta \circ \beta = \gamma \circ \alpha$, for the homomorphism $\kappa = \delta' \circ \gamma$ we have $\delta' \circ (\delta \circ \beta) = \delta' \circ (\gamma \circ \alpha)$, or $\beta = \kappa \circ \alpha$.
If there is a homomorphism $\kappa : A \longrightarrow B$ with $\beta = \kappa \circ \alpha$, define the homomorphism $\delta'' : A \oplus B \longrightarrow B$ by $\delta''(a, b) = \kappa(a) + b$. For every $(\alpha(c), -\beta(c)) \in H$ we have $\delta''(\alpha(c), -\beta(c)) = (\kappa \circ \alpha)(c) - \beta(c) = 0$. Hence there exists a unique homomorphism $\delta' : G \longrightarrow B$ such that $\delta'((a, b) + H) = (\delta' \circ \pi)(a, b) = \delta''(a, b) = \kappa(a) + b$. Clearly, $(\delta' \circ \delta)(b) = \delta'((0, b) + H) = \kappa(0) + b = b$ for every $b \in B$. So $\delta' \circ \delta = 1_B$ and E is splitting. $\square$

2.6 Small Submodules

The most important notion in the study of supplements is the small submodule.

Definition 2.6.1. A submodule N of a module M is called *superfluous or small* if there is no proper submodule K of M such that $N + K = M$. Equivalently $N + K = M$ implies that $K = M$. It is denoted by $N \ll M$.

Some of properties of the following proposition can be found in [7]

Proposition 2.6.2. *Let M be a module*

1. *If $K \leq N \leq M$ and K is small in N then K is small in M.*
2. *Let N be a small submodule of a module M, then any submodule of N is also small in M.*
3. *If K is a small submodule of a module M and K is contained in a direct summand N of M then K is small in N.*
4. *$K \ll M$ and $N \ll M$ iff $K + N \ll M$.*
5. *If $K \leq N \leq M$, then $N \ll M$ iff $K \ll M$, $N/K \ll M/K$.*
6. *Finite sum of small submodules N_i of M is a small submodule of M.*
7. *Let $f : M \longrightarrow N$ be a homomorphism of modules M and N, let K be a submodule of M. If K is a small submodule of M, then $f(K)$ is a small submodule of N.*

Proof. 1. Let $K + L = M$ for a submodule L of M.
$N = N \cap M = N \cap (K+L) = K + (N \cap L)$. Since K is small in N, $N = N \cap L$ so $N \leq L$. $K \leq N$ and $N \leq L$ so $K \leq L$. Therefore $M = K + L = L$. Thus $K \ll M$.

2. Let K be a submodule of N and $K + L = M$ for a submodule L of M. Since $K \leq N$, $N + L = M$ and also since $N \ll M$, $L = M$. So $K \ll M$.

3. $K \leq N \leq M$,$K \ll M$ and $M = N \oplus L$ for a submodule L of M. Let $K + U = N$ for a submodule U of N. $M = N + L = K + U + L$ since $K \ll M$, $M = U + L$ and $U \cap L \leq N \cap L = 0$ implies $U \cap L = 0$ so $M = U \oplus L$. $N = N \cap M = N \cap (U \oplus L) = U \oplus (N \cap L) = U$. So $K \ll N$.

4. ($\Rightarrow$) Let $(K + N) + L = M$ for some $L \leq M$. Since $K + (N + L) = (K + N) + L = M$ and $K \ll M$, we have $N + L = M$. Since $N \ll M$, $L = M$.

   ($\Leftarrow$) $K \leq K + N \ll M$ by 2) $K \ll M$. Similarly $N \leq K + N \ll M$ by 2) $N \ll M$.

5. ($\Rightarrow$) Since $N \ll M$ by 2) $K \ll M$. Suppose that $N/K + X/K = M/K$ where X/K is a submodule of M/K, then $N + X = M$ by assumption $X = M$ i.e. $X/K = M/K$.

   ($\Leftarrow$) Let $N + X = M$ then $(N + X)/K = M/K$ i.e. $N/K + (X + K)/K = M/K$ or $N + X + K = M$. Since $N \ll M$ we have $X + K = M$. Now $K \ll M$ implies $X = M$.

6. Let $N = \sum_{i=1}^{n} N_i$ and let $N_1 + \cdots + N_n + X = M$ for some $X \leq M$. Since $N_i \ll M$, $N_1 + (N_2 + \cdots + N_n + X) = M$ then $N_2 + \cdots + N_n + X = M$ continuing in this way we obtain $N_n + X = M$ and since $N_n \ll M$, $X = M$.

7. Suppose that $f(K) + L = f(M)$ for some $L \leq f(M)$. Then $f^{-1}(f(K) + L) = f^{-1}(f(K)) + f^{-1}(L) = f^{-1}(f(M)) = M$ and therefore $M = K + \operatorname{Ker} f + f^{-1}(L) = K + f^{-1}(L)$. Since $K \ll M$, $f^{-1}(L) = M$, hence $f(f^{-1}(L)) = f(M)$ implies that $L \cap f(M) = f(M)$. So $L = f(M)$.

$\square$

2.7 Essential Submodules

The notion of essential submodule plays a key role in the study of complements. Some of the following propositions are given in [4] and [11].

Definition 2.7.1. A submodule N of an R-module M is said to be *essential (or big)* in M, if $N \cap K \neq 0$ for every non-zero submodule K of M. In other words $N \cap K = 0 \Rightarrow K = 0$. Notation: $N \trianglelefteq M$.

Proposition 2.7.2. *Let K and N be submodules of M. Then*

1. $N \trianglelefteq M \Leftrightarrow N \cap Rm \neq 0$ *for all* $0 \neq m \in M$.
2. *Given* $K \leq N$, $K \trianglelefteq M \Leftrightarrow K \trianglelefteq N$ *and* $N \trianglelefteq M$.
3. *If* $N \trianglelefteq M$ *then* $N \cap K \trianglelefteq K$.
4. $K \trianglelefteq M$ *and* $N \trianglelefteq M \Leftrightarrow N \cap K \trianglelefteq M$.
5. *Given* $K \subset N$ *if* $N/K \trianglelefteq M/K$ *the* $N \trianglelefteq M$.
6. *If* $M = \bigoplus_{i=1}^{n} M_i$ *and* $N_i \trianglelefteq M_i$ *for each* $i \in I$ *then* $\bigoplus_{i=1}^{n} N_i \trianglelefteq M$.

Proof. 1. ($\Rightarrow$) We know that $N \trianglelefteq M$. $0 \neq m \in M \Rightarrow 0 \neq Rm \leq M$ and since $N \trianglelefteq M$ then $N \cap Rm \neq 0$.

($\Leftarrow$) Let $N \cap K = 0$ for some $K \leq M$. Suppose $K \neq 0$ i.e. there is a non-zero element $x \in K$. By the condition $\exists r \in R$ such that $0 \neq rx \in N$. Since $x \in K$ $rx \in K$ so $0 \neq rx \in N \cap K = 0$. Contradiction. So $K = 0$.

2. ($\Rightarrow$) Let $T \leq N$ such that $K \cap T = 0$. Clearly $T \leq N \leq M \Rightarrow T \leq M$. Since $K \trianglelefteq M$ then $K \cap T = 0 \Rightarrow T = 0 \Rightarrow K \trianglelefteq N$. Let $S \leq M$ such that $N \cap S = 0$. $K \cap S \leq N \cap S = 0 \Rightarrow K \cap S = 0$. Since $K \trianglelefteq M$ then $S = 0 \Rightarrow N \trianglelefteq M$.

($\Leftarrow$) Let $K \cap L = 0$ for some $L \leq M \Rightarrow K \cap (L \cap N) = 0$. Since $K \trianglelefteq N$ and $L \cap N \leq N$, $L \cap N = 0$ and since $N \trianglelefteq M$, $L = 0$. So $K \trianglelefteq M$.

3. Let $(N \cap K) \cap T = 0$ for some $T \leq K \leq M$. $N \cap (K \cap T) = 0$. Since $N \trianglelefteq M$ and $K \cap T \leq M$, $K \cap T = 0$. Since $K \cap T = T, T = 0 \Rightarrow N \cap K \trianglelefteq K$.

4. ($\Rightarrow$)Let $(N \cap K) \cap T = 0$ for some $T \leq M \Rightarrow N \cap (K \cap T) = 0$. Since $N \trianglelefteq M$, $K \cap T = 0$ and also $K \trianglelefteq M, T = 0 \Rightarrow N \cap K \trianglelefteq M$.
($\Leftarrow$) $N \cap K \leq N \leq M$. Since $N \cap K \trianglelefteq M$ by 2), $N \trianglelefteq M$. Similarly $N \cap K \leq K \leq M$. Since $N \cap K \trianglelefteq M$ by 2), $K \trianglelefteq M$.

5. Let $N \cap T = 0$ for some $T \leq M \Rightarrow [(N \cap T) + K]/K = 0 \Rightarrow N/K \cap (T + K)/K = 0$ since $N/K \trianglelefteq M/K, (T + K)/K = 0 \Rightarrow T + K = K \Rightarrow T = 0$.

6. Suppose $(N_1 \oplus N_2) \cap K = 0$ and assume $K \neq 0$. $\exists 0 \neq x \in K$, $x \in M_1 \oplus M_2 \Rightarrow x = m_1 + m_2$ since $K \neq 0$ at least one of m_1, $m_2 \neq 0$. Say $m_1 \neq 0$. Since $N_1 \trianglelefteq M_1$ by 1) there is $r \in R$ such that $0 \neq rm_1 \in N_1$. $rx = rm_1 + rm_2$ if $rm_2 \in N_2$ then $rx \in (N_1 \oplus N_2) \cap K = 0 \Rightarrow rx = 0 \Rightarrow rm_1 = 0$. Contradiction. If $rm_2 \notin N_2$ since $N_2 \trianglelefteq M_2 \Rightarrow \exists s \in R$ such that $0 \neq sm_2 \in N_2$. $srx = srm_1 + srm_2 \in (N_1 \oplus N_2) \cap K = 0 \Rightarrow srm_2 = 0$. Contradiction. Thus $K = 0 \Rightarrow N_1 \oplus N_2 \trianglelefteq M_1 \oplus M_2$.

□

We can characterize essential submodules in the language of the elements and ideals.

Proposition 2.7.3. *A submodule N of an R-module M is essential in M if and only if for all $0 \neq m \in M$ and for the ideal $(N : m)$ of R, $(N : m)m \neq 0$.*

Proof. ($\Rightarrow$) Let $N \trianglelefteq M$. We know for all $0 \neq m \in M$, $Rm \cap N \neq 0 \Rightarrow \exists 0 \neq rm \in N \Rightarrow 0 \neq r \in (N : m) \Rightarrow (N : m) \neq 0 \Rightarrow (N : m)m \neq 0$.
($\Leftarrow$) For all $0 \neq m \in M$, $(N : m)m \neq 0 \Rightarrow \exists 0 \neq r \in (N : m) \leq R$ such that $0 \neq rm \in N \Rightarrow Rm \cap N \neq 0 \Rightarrow N \trianglelefteq M$. □

Definition 2.7.4. Let M be an R-module. A submodule K of M is said to be *closed* in M if K has no proper essential extension in M i.e. if L is a submodule of M such that $K \trianglelefteq L$ then $K = L$.

Definition 2.7.5. Let M be a module. A submodule N of module M is said to be a *complement* of a submodule L of M if $N \cap L = 0$ and N is maximal with respect to this property.

Example 2.7.6. If $M = N \oplus K$ then K is a complement of N. Really $N \cap K = 0$ and if for some $T \leq M$, $K \leq T$ such that $N \cap T = 0$ then by modular law $T = T \cap (N \oplus K) = K \oplus (N \cap T) \Rightarrow K = T$.

Complements always exist, moreover we have the following theorem.

Theorem 2.7.7. ***Existence of Complements***
Let L, N be submodules of a module M with $L \cap N = 0$. Then there exists a complement K of N such that $L \subset K$.

Proof. Let $\Gamma = \{T \leq M | L \leq T \text{ and } N \cap T = 0\}$. $\Gamma \neq \emptyset$ since $L \in \Gamma$. Let Ω be any chain in Γ, that is $\forall\, U, V \in \Omega$ either $U \leq V$ or $V \leq U$. Put $W = \bigcup\limits_{U \in \Omega} U$.
1) Let $x, y \in W$ and $x \in U \in \Omega$, $y \in V \in \Omega$ since Ω is a chain either $U \leq V$ or $V \leq U$. Let $U \leq V$ then $x, y \in V$ and $V \leq M \Rightarrow \forall r, s \in R$, $rx + sy \in V \leq M \Rightarrow W \leq M$.
2) For every $U \in \Omega$, $L \leq U \Rightarrow L \in \bigcup\limits_{U \in \Omega} U = W \Rightarrow L \leq W$. For ever $U \in \Omega$, $N \cap U = 0 \Rightarrow N \cap (\bigcup\limits_{U \in \Omega} U) = 0 \Rightarrow N \cap W = 0 \Rightarrow W \in \Gamma$.
3) For every $U \in \Omega$, $U \leq \bigcup\limits_{V \in \Omega} V = W \Rightarrow W$is an upper bound for Ω.
By Zorn's Lemma there is a maximal element K in Γ which is a complement of N containing L. □

Proposition 2.7.8. *If K is a complement of N in M then $N \oplus K \trianglelefteq M$.*

Proof. Let $0 \neq m \in M$. $(N \oplus K) \cap Rm \neq 0$ or $(N \oplus K) \cap Rm = 0$. If $(N \oplus K) \cap Rm = 0$, then for every $n = k + rm \in N \cap (K + Rm)$ we have $rm = n - k \in (N + K) \cap Rm = 0$. Therefore $rm = 0$ and $n = k \in N \cap K = 0$, so $n = 0$. Thus $N \cap (K + Rm) = 0$, but it contradicts the maximality of K. So $(N \oplus K) \cap Rm \neq 0$. □

Proposition 2.7.9. *If L is a complement of K in M then $E(M) = E(L) \oplus E(K)$.*

Proof. We know that $K \oplus L \trianglelefteq M \Rightarrow K \oplus L \trianglelefteq M \trianglelefteq E(M) \Rightarrow K \oplus L \trianglelefteq E(M)$. $K \oplus L \leq E(K \oplus L) \leq E(M) \Rightarrow E(K \oplus L) \trianglelefteq E(M)$. Since $E(K \oplus L)$ is injective, it is a direct summand, $E(M) = E(K \oplus L) \oplus T \Rightarrow E(K \oplus L) \cap T = 0 \Rightarrow T = 0 \Rightarrow E(M) = E(K \oplus L) = E(K) \oplus E(L)$. □

Proposition 2.7.10. *If L is a complement of K in M then $L = E(L) \cap M$, where $E(L)$ is injective envelope of L.*

Proof. Clearly $L \leq E(L) \cap M$. On the other hand $L \cap [(E(L) \cap M) \cap K] \leq L \cap K = 0$. But $L \trianglelefteq E(L)$, therefore $(E(L) \cap M) \cap K = 0$. By maximality of L we have $L = E(L) \cap M$. □

The following proposition shows that complements coincide with closed submodules.

Proposition 2.7.11. *Let K be a submodule of an R-module M. K is closed in $M \Leftrightarrow K$ is a complement of some submodule N of M.*

Proof. ($\Rightarrow$) Suppose that K is closed. Suppose $H \leq M$ such that $K \subset H$ and $H \cap N = 0$. Let L be a submodule of H such that $K \cap L = 0$ then $K \cap (L \oplus N) = 0 \Rightarrow N = L \oplus N$ i.e. $L = 0 \Rightarrow K \trianglelefteq H$ so $K = H$. Thus K is a complement of N in M.
($\Leftarrow$) Suppose that $K \trianglelefteq K' \leq M$. Since K is a complement of some submodule N in M, $(K' \cap N) \cap K = K' \cap (N \cap K) = 0$. Then $K' \cap N \leq K'$, $K \trianglelefteq K'$ implies $K' \cap N = 0$. By maximality of K with respect to property $K \cap N = 0$ we have $K = K'$ i.e. K is closed in M. □

Proposition 2.7.12. *Let L, K, N be submodules of an R-module M with $K \subset L$. Then*

1. *There exists a closed submodule H of M such that $N \trianglelefteq H$.*
2. *The submodule K is closed in M if and only if whenever $Q \trianglelefteq M$ such that $K \subset Q$ then $Q/K \trianglelefteq M/K$.*

Proof. 1. By Zorn's Lemma there exists $H \leq M$ maximal in the collection of submodules V of M with $H \trianglelefteq V$. Clearly H is closed.

2. ($\Rightarrow$) Let $Q \trianglelefteq M$, $K \subset Q$. Let P be a submodule of M such that $K \subset P$ and $(Q/K) \cap (P/K) = 0 \Rightarrow K = Q \cap P \trianglelefteq P$, (If $N \trianglelefteq M$ then $N \cap K \trianglelefteq K$). Since K is closed $K = P \Rightarrow P/K = 0 \Rightarrow Q/K \trianglelefteq M/K$.
($\Leftarrow$) $Q/K \trianglelefteq M/K$ for every essential submodule Q containing K. Suppose $K \trianglelefteq L$. Let T be a complement of K in $M \Rightarrow K \oplus T \trianglelefteq M \Rightarrow (K \oplus T)/K \trianglelefteq M/K$. But $L \cap T = 0 \Rightarrow ((K \oplus T)/K) \cap (L/K) = [(K \oplus T) \cap L]/K = [K \oplus (T \cap L)]/K = 0 \Rightarrow K = L \Rightarrow K$ is closed.

$\square$

Proposition 2.7.13. *If $A \leq B \leq C$ and A is a complement in C then A is also a complement in B.*

Proof. Let A be a complement of a submodule K of C. Then by the definition $A \cap K = 0$ and A is maximal with respect to this. Suppose that $A \leq A' \leq B$ and $A' \cap (K \cap B) = 0 \Rightarrow A' \cap K = 0 \Rightarrow A = A' \Rightarrow A$ is a complement of $K \cap B$ in B. $\square$

Lemma 2.7.14. *Let N, K, L be submodules of an R-module M such that $K \cap L = N$ and $K + L = M$. If N is a complement in K and $N \trianglelefteq L$, then $K \trianglelefteq M$ i.e.*

$$\begin{tikzcd} N \arrow[r, "\trianglelefteq"] \arrow[d, "compl"'] & L \arrow[d] \\ K \arrow[r, "\trianglelefteq ?"'] & M \end{tikzcd}$$

Proof. Suppose that $(K : m)m = 0$ for $0 \neq m \in M$ with $m = k + l$, where $k \in K$ and $l \in L$. We want to show that $N \trianglelefteq N + Rk = N_1$.
$(0 : m) = (K : m) = (N : l)$
Let $r \in (K : m)$. Since $(K : m)m = 0$, $rm = 0 \Rightarrow r \in (0 : m)$.
So $(K : m) \subseteq (0 : m)$. We know that $(0 : m) \subseteq (K : m)$. Thus $(0 : m) = (K : m)$.
$r \in (K : m) \Rightarrow r \in (0 : m) \Rightarrow rm = 0$, since $m = k + l$, $rk = -rl \in K \cap L = N \Rightarrow rl \in N \Rightarrow r \in (N : l) \Rightarrow (K : m) \subseteq (N : l)$.
$s \in (N : l) \Rightarrow sl \in N \Rightarrow sm = sk + sl \in K + N = K \Rightarrow s \in (K : m) \Rightarrow$
$(N : l) \subseteq (K : m)$. Thus $(N : l) = (K : m)$.
Let $x = n + rk \in N_1$, where $n \in N$, $k \in K$, $r \in R$, such that $rk \notin N \Rightarrow rm = rk + rl = \underbrace{rk + n}_{x} + \underbrace{rl + n}_{l'} = x + l'$.
$l' \notin N$: If $l' \in N$ then $rl \in N \Rightarrow r \in (N : l) = (0 : m) \Rightarrow rm = 0 \Rightarrow rk = -rl \in K \cap L = N \Rightarrow rk \in N$ contradiction.
$(0 : rm) \subseteq (N : x)$:
$s \in (0 : rm) \Rightarrow (sr)m = 0 \Rightarrow sr \in (0 : m) = (N : l) \Rightarrow srl \in N$, $sx = -sl' = sn - srl \in N \Rightarrow s \in (N : x) \Rightarrow (0 : rm) \subseteq (N : x)$.
$(0 : rm) = (N : l')$:
$s \in (0 : rm) \Rightarrow srm = 0 \Rightarrow sx + sl' = 0 \Rightarrow sx = -sl' = sn - srl \in N \Rightarrow$

$s \in (N : l')$.

$s \in (N : l') \Rightarrow sl' \in N \Rightarrow srl - sn \in N \Rightarrow sr \in (N : l) = (0 : m) \Rightarrow srm = 0 \Rightarrow$ $s \in (0 : rm)$.

$(N : l')l' \neq 0$ (since $N \trianglelefteq L$, $l' \notin N \Rightarrow \exists s \in (N : l') = (0 : rm)$ such that $sl' \neq 0 \Rightarrow sx \neq 0 \Rightarrow (N : x)x \neq 0 \Rightarrow N \trianglelefteq N_1 \leq K \Rightarrow N = N_1 \Rightarrow k \in N \Rightarrow$ $m \in L$.

$(K : m) = (N : l) = (N : k + l) = (N : m) \Rightarrow (N : m)m \neq 0$. Contradiction.

So $(K : m)m \neq 0$. □

The following theorem allows to use the properties of proper classes for the study of complements.

Theorem 2.7.15. (Generalov, [6]) *The class*

$$\mathcal{C} = \{0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \mid A \text{ is a complement of some } K \text{ in } B\}$$

is a proper class.

Proof. $P1$. If a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is splitting then $B = A \oplus C$, i.e. A is a complement in B. So every splitting exact sequence is in $\mathcal{C}$.

$P2$. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be $\mathcal{C}$-monomorphisms. We want to show that $g \circ f$ is a $\mathcal{C}$-monomorphism. In other words $A \leq B \leq C$, A is a complement in B and B is a complement in C. We want to show that A is a complement in C. We can draw the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "compl", "f"'] & B \arrow[r] \arrow[d, "g"', "compl"] & B/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "compl?", "g\circ f"'] \arrow[u, equal] & C \arrow[r] \arrow[d] & C/A \arrow[r] \arrow[d] & 0 \\
 & & C/B \arrow[r, equal] \arrow[d] & C/B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Suppose $A \trianglelefteq A_1 \leq C \Rightarrow A \trianglelefteq A_1 \cap B$ since A is a complement in B, $A_1 \cap B = A$.

Take $B_1 = B + A_1$.

$$\begin{tikzcd}
A \arrow[r, "\trianglelefteq"] \arrow[d, "compl"'] & A_1 \arrow[d] \\
B \arrow[r] & B + A_1
\end{tikzcd}$$

By Lemma 2.7.14 $B \trianglelefteq B + A_1$. B is a complement in $C \Rightarrow B = B + A_1 \Rightarrow A_1 \leq B$. Since $A \trianglelefteq A_1$ and A is a complement in B, $A = A_1$.

$P2'$. Suppose $K \leq A \leq B$. Let $\sigma : B \to B/K$ and $\overline{\sigma} : B/K \to B/A$ be $\mathcal{C}$-epimorphisms we want to show that $\overline{\sigma} \circ \sigma$ is a $\mathcal{C}$-epimorphism. That is as it can be seen on the following diagram we want to show that if K is complement in B and A/K is a complement in B/K then A is a complement in B.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "compl?"] & A/K \arrow[r] \arrow[d, "compl"] & 0 \\
0 \arrow[r] & K \arrow[r, "compl"] & B \arrow[r, "\sigma"] \arrow[d] & B/K \arrow[r] \arrow[d, "\overline{\sigma}"] & 0 \\
 & & B/A \arrow[r, equal] \arrow[d] & B/A \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Suppose $A \trianglelefteq C \leq B$. Since K is a complement in B, K is a complement in C. So K is closed in C. We know by the Proposition 2.7.12 If K is closed in C then whenever $A \trianglelefteq C$ then $A/K \trianglelefteq C/K$. Since A/K is a complement in B/K,$A/K = C/K$ so $A = C$. Hence A is a complement in B.

$P3$. Let $h \circ f$ be a $\mathcal{C}$-monomorphism and h be a monomorphism. We want to show that f is a $\mathcal{C}$-monomorphism.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "compl?", "f"'] \arrow[d, equal] & B \arrow[r] \arrow[d, "h"] & B/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "compl", "h\circ f"'] & C \arrow[r] \arrow[d] & C/A \arrow[r] \arrow[d] & 0 \\
 & & C/B \arrow[r, equal] \arrow[d] & C/B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Let $A \cap K = 0$ and A is a maximal with respect to this for some submodule K of C. We know this by the Proposition 2.7.13. Thus f is a $\mathcal{C}$-monomorphism.

$P3'$. Let $u \circ g$ be a $\mathcal{C}$-epimorphism and g be an epimorphism. We want to show that u is a $\mathcal{C}$-epimorphism.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d, "compl"] & K/A \arrow[r] \arrow[d, "compl?"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "g"] \arrow[d, "u\circ g"'] & B/A \arrow[r] \arrow[d, "u"] & 0 \\
 & & B/K \arrow[r, equal] \arrow[d] & B/K \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $u \circ g$ is a $\mathcal{C}$-epimorphism, K is a complement in B. We will show that K/A is a complement in B/A. Let $K \cap T = 0$ and K be a maximal with respect to this property for some submodule T of B. $K \cap (T + A) = A + (K \cap T) = A \Rightarrow K/A \cap (T + A)/A = 0$.

Suppose $K/A \leq K'/A \leq B/A$ and $K'/A \cap (T + A)/A = 0$.

$K \leq g^{-1}(K'/A)$. $g^{-1}(K'/A) \cap (T + A) = A + (g^{-1}(K'/A) \cap T) = A \Rightarrow g^{-1}(K'/A) \cap$

$T \leq A \leq K \Rightarrow g^{-1}(K'/A) \cap T \leq K \cap T = 0.$

Since K is maximal, $g^{-1}(K'/A) = K = g^{-1}(K/A) \Rightarrow K'/A = K/A$. Thus K/A is closed in B/A. Hence K/A is a complement in B/A. $\square$

Chapter 3

ABSOLUTELY SUPPLEMENT MODULES

3.1 Absolutely Supplement Modules

Let M be a module. A submodule N of module M is called a *supplement* of a submodule L of M if $N + L = M$ and N is minimal with respect to this property.

Proposition 3.1.1. (Wisbauer, [12]) *N is a supplement of L in M if and only if $N + L = M$ and $N \cap L \ll N$.*

Proof. ($\Rightarrow$) Let N be a supplement of L in M. Then we know that $M = N+L$ and N is minimal with respect to this property. For $K \leq N$ let $N = K + (N \cap L)$. By Modular Law $N = K + (N \cap L) = N \cap (K + L)$ that is $N \leq L + K$. $M = N + L = L + K$. By minimality of N we have $K = N$.
($\Leftarrow$) Let $M = L+K$ for some submodule K of N. $N = N \cap M = N \cap (K+L) = K + (N \cap L)$. Since $N \cap L \ll N$, $K = N$. So N is minimal with respect to $N + L = M$. □

Unlike complements, supplements need not exist always, but supplements give a proper class as well as complements.

Theorem 3.1.2. (Generalov, [6]) *The class*

$$\mathcal{S} = \{0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \,|\, A \text{ is a supplement in } B\}$$

is a proper class.

Proof. $P1$. If a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is splitting then A is a supplement in B. So every splitting exact sequence is in $\mathcal{S}$.

$P2$. Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be $\mathcal{S}$-monomorphisms. We must show that $\beta \circ \alpha$ is an $\mathcal{S}$-monomorphism. We can draw the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "suppl", "\alpha"'] \arrow[d, equal] & B \arrow[r] \arrow[d, "\beta"', "suppl"] & B/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "suppl?", "\beta o \alpha"'] & C \arrow[r] \arrow[d] & C/A \arrow[r] \arrow[d] & 0 \\
 & & C/B \arrow[r, equal] \arrow[d] & C/B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Let A be a supplement of a submodule K of B and B be a supplement of a submodule N of C.

$A + (K + N) = (A + K) + N = B + N = C$

Let $[A \cap (K + N)] + X = A$ for some submodule X of A. Then

$[A \cap (K + N)] + X + K = A + K = B$.

$[A \cap (K + N)] + X + K + N = B + N = C \Rightarrow X + K + N = C$. Since B is a supplement of N in C then $X + K = B \Rightarrow [A \cap (X + K)] = A \cap B = A \Rightarrow [X + (A \cap K)] = A$. Since $A \cap K \ll A$, $X = A$. So $A \cap (K + N) \ll A$. Thus A is a supplement of a submodule $(K + N)$ in C.

$P2'$. Let $\alpha : B \longrightarrow B/K$ and $\beta : B/K \longrightarrow B/A$ be $\mathcal{S}$-epimorphisms. We want to show that $\beta \circ \alpha$ is an $\mathcal{S}$-epimorphism. We can use the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "suppl?"] & A/K \arrow[r] \arrow[d, "suppl"] & 0 \\
0 \arrow[r] & K \arrow[r, "suppl"] & B \arrow[r, "\alpha"] \arrow[d, "\beta o \alpha"] & B/K \arrow[r] \arrow[d, "\beta"] & 0 \\
 & & B/A \arrow[r, equal] \arrow[d] & B/A \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Let K be a supplement of a submodule X of B and A/K be a supplement of a submodule Y/K of B/K where $K \leq Y \leq B$. We have an exact sequence

$$0 \longrightarrow (A \cap Y)/K \longrightarrow A/K \xrightarrow{\overline{\sigma}} B/Y \longrightarrow 0$$

Since $A \leq B$, $a + K \longmapsto a + Y$ and $\operatorname{Ker} \overline{\sigma} = (A \cap Y)/K$.
$b \in B = A+Y \Rightarrow b+Y = a+y+Y = a+Y = \overline{\sigma}(a+K)$. So $\overline{\sigma}$ is an epimorphism.
$K+X = B \Rightarrow Y = K+(X \cap Y) \Rightarrow A+Y = A+K+(X \cap Y) \Rightarrow A+(X \cap Y) = B$.
Now we will show that $A \cap (X \cap Y) \ll A$.
$\sigma : A/(K \cap X) \longrightarrow B/(X \cap Y)$
$B/(X \cap Y) = [A+(X \cap Y)]/(X \cap Y) \cong A/(A \cap X \cap Y)$. Since $(K \cap X) \leq (A \cap X \cap Y)$, we can define an epimorphism $A/(K \cap X) \longrightarrow A/(A \cap X \cap Y) \cong B/(X \cap Y)$. So σ is an isomorphism.
$\alpha : K/(K \cap X) \cong (K + X)/X = B/X$
$\beta : B/(X \cap Y) \cong Y/(X \cap Y) \oplus X/(X \cap Y)$ where since $X + K = B$ and $K \leq Y$, $X + Y = B$. Also by the second isomorphism theorem $Y/(X \cap Y) \cong B/X$ and $X/(X \cap Y) \cong B/Y$ so $\beta : B/(X \cap Y) \cong B/X \oplus B/Y$.
$\gamma : A/(K \cap X) = [K + (A \cap X)]/(K \cap X) \cong K/(K \cap X) \oplus (A \cap X)/(K \cap X)$.
Since $(A \cap X)/(K \cap X) = (A \cap X)/(A \cap X \cap K) \cong [K + (A \cap X)]/K = A/K$, $\gamma : A/(K \cap X) \cong K/(K \cap X) \oplus A/K$.
Now it can be seen that

$$\sigma = \beta^{-1} \circ (\alpha \oplus \overline{\sigma}) \circ \gamma$$

and so σ is a minimal epimorphism.

Thus $\operatorname{Ker}\sigma = (A \cap X \cap Y)/(K \cap X) \ll A/(K \cap X)$.

We know that

$(K \cap X) \leq (A \cap X \cap Y) \leq A$, $K \cap X \ll A$ and $(A \cap X \cap Y)/(K \cap X) \ll A/(K \cap X)$.

Hence $(A \cap X \cap Y) \ll A$.

$P3$. Let $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ be monomorphisms and $\beta \circ \alpha$ be an $\mathcal{S}$-monomorphism. We want to show that α is an $\mathcal{S}$-monomorphism. We can draw the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "suppl?", "\alpha"'] \arrow[d, equal] & B \arrow[r] \arrow[d, "\beta"] & B/A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "suppl", "\beta\circ\alpha"'] & C \arrow[r] \arrow[d] & C/A \arrow[r] \arrow[d] & 0 \\
 & & C/B \arrow[r, equal] \arrow[d] & C/B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Let A be a supplement of a submodule K of C. Then $A+K = C$ and $A \cap K \ll A$.

$B = B \cap C = B \cap (A+K) = A + (B \cap K)$.

$A \cap (B \cap K) \leq (A \cap K) \ll A \Rightarrow A \cap (B \cap K) \ll A$. Thus A is a supplement of a submodule $B \cap K$ of B.

$P3'$. Let $\alpha : C \longrightarrow C/A$ and $\beta : C/A \longrightarrow C/B$ be epimorphisms and $\beta \circ \alpha$ be an $\mathcal{S}$-epimorphism. We want to show that β is an $\mathcal{S}$-epimorphism. We can draw the following diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, "suppl"] & B/A \arrow[r] \arrow[d, "suppl?"] & 0 \\
0 \arrow[r] & A \arrow[r] & C \arrow[r, "\alpha"'] \arrow[d, "\beta o \alpha"'] & C/A \arrow[r] \arrow[d, "\beta"] & 0 \\
 & & C/B \arrow[r, equal] \arrow[d] & C/B \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Let B be a supplement of a submodule N of C.Then $B+N=C$ and $B\cap N \ll B$.
$C/A=\alpha(C)=\alpha(B+N)=\alpha(B)+\alpha(N)=B/A+\alpha(N)$.
$B/A\cap\alpha(N)=\alpha(B\cap N)$: $\alpha(B\cap N)\subseteq B/A\cap\alpha(N)$.
$\alpha(n)=\alpha(b)\in\alpha(B)\cap\alpha(N)\Rightarrow\alpha(b-n)=0\Rightarrow b-n=a\in A\Rightarrow n=b-a\in B\cap N\Rightarrow\alpha(n)\in\alpha(B\cap N)$. So $\alpha(B)\cap\alpha(N)=\alpha(B\cap N)\ll\alpha(B)$. □

Definition 3.1.3. A module M is called *absolutely supplement* if it is a supplement submodule of every module containing M.

Proposition 3.1.4. *For a module M the followings are equivalent*

1. *M is an absolutely supplement module.*
2. *M is a supplement in every injective module I containing M.*
3. *M is a supplement in its injective envelope $E(M)$.*

Proof. 1. $\Rightarrow$ 2. It is clear by the Definition 3.1.3.
2. $\Rightarrow$ 3. If M is a supplement in every injective module containing M and since $E(M)$ is an injective module containing M, M is a supplement submodule of $E(M)$.
3. $\Rightarrow$ 1. Let M be a submodule of a module N, $E(M)$ is an injective envelope of M; $f: M\longrightarrow N$ and $g: M\longrightarrow E(M)$ be inclusion maps. Then we have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[dl, "h"] \\ & E(M) & \end{tikzcd}$$

By (2) M is a supplement submodule of $E(M)$. Let M be a supplement of K in $E(M)$. Then $M+K = E(M)$ and $M \cap K \ll M$. $M = g(M) = h \circ f(M) \leq h(N)$, therefore by Modular Law $h(N) = M + (K \cap h(N))$. Let $n \in N$. Then $h(n) = m + h(n_1)$ where $m \in M$, $h(n_1) \in K \cap h(N)$ with $n_1 \in N$. Since the diagram is commutative, $m = g(m) = h(f(m)) \Rightarrow h(n) = m + h(n_1) = h(f(m)) + h(n_1) = h(f(m)+n_1) \Rightarrow h(n-f(m)-n_1) = 0 \Rightarrow y = n-f(m)-n_1 \in \operatorname{Ker} h \leq h^{-1}(K) \Rightarrow n = m + n_1 + y \in M + h^{-1}(K) \Rightarrow N = M + h^{-1}(K)$. Now to prove that M is a supplement of $h^{-1}(K)$ in N it remains to show that $M \cap h^{-1}(K) \ll M$. $M \cap h^{-1}(K) = f^{-1}(f(M) \cap h^{-1}(K)) = M \cap f^{-1}(h^{-1}(K)) = M \cap g^{-1}(K) = g^{-1}(M \cap K) = M \cap K \ll M$. □

Proposition 3.1.5. *Every finite direct sum of absolutely supplement modules is an absolutely supplement module.*

Proof. Let S_1 and S_2 be absolutely supplement modules. Since S_1 is absolutely supplement, S_1 is a supplement submodule of every injective module containing S_1. Similarly S_2 is a supplement submodule of every injective module containing S_2. Let S_1 be a supplement submodule of an injective module I_1 and S_2 be a supplement submodule of an injective module I_2. That is for suitable modules $K_1 \leq I_1$ and $K_2 \leq I_2$; $S_1 + K_1 = I_1$ and $S_1 \cap K_1 \ll S_1$, $S_2 + K_2 = I_2$ and $S_2 \cap K_2 \ll S_2$. We will show that $S_1 \oplus S_2$ is a supplement submodule of $I_1 \oplus I_2$ (Direct sum of finite number of injective modules is an injective module). $S_1 \oplus S_2 + K_1 + K_2 = I_1 \oplus I_2$. We must show that $(S_1 \oplus S_2) \cap (K_1 + K_2) \ll S_1 \oplus S_2$. Let $x \in (S_1+S_2) \cap (K_1+K_2) \Rightarrow x = s_1+s_2 = k_1+k_2$ where $s_1 \in S_1$, $s_2 \in S_2$, $k_1 \in K_1$, $k_2 \in K_2 \Rightarrow x = s_1-k_1 = k_2-s_2 \Rightarrow x \in I_1 \cap I_2 = 0 \Rightarrow x = 0 \Rightarrow s_1 = k_1$, $s_2 = k_2 \Rightarrow x \in (S_1 \cap K_1)+(S_2 \cap K_2) \Rightarrow (S_1+S_2) \cap (K_1+K_2) \leq (S_1 \cap K_1)+(S_2 \cap K_2) \ll S_1+S_2$. Since $S_1 \cap K_1 \ll S_1$ and $S_2 \cap K_2 \ll S_2 \Rightarrow (S_1 \oplus S_2) \cap (K_1+K_2) \ll S_1 \oplus S_2 \Rightarrow S_1 \oplus S_2$ is a supplement of $K_1 + K_2$ in $I_1 \oplus I_2$. by the Proposition 3.1.4 $S_1 \oplus S_2$ is an absolutely supplement module. □

Proposition 3.1.6. *If S_1 is a supplement in I_1 then S_1 is a supplement in $I_1 \oplus I_2$.*

Proof. Let S_1 be a supplement of K_1 in I_1. We will show that S_1 is a supplement of K_1+I_2 in $I_1 \oplus I_2$. $S_1+K_1+I_2 = I_1+I_2$. . Let $a \in S_1 \cap (K_1+I_2) \Rightarrow a = s = k+x$ where $s \in S_1$, $k \in K_1$, $x \in I_2 \Rightarrow x = s-k \Rightarrow x \in S_1+K_1 = I_1 \Rightarrow x \in I_1 \cap I_2 = 0 \Rightarrow x = 0 \Rightarrow a = s = k \Rightarrow a \in S_1 \cap K_1 \Rightarrow S_1 \cap (K_1+I_2) = S_1 \cap K_1 \ll S_1$ so S_1 is a supplement submodule of $I_1 \oplus I_2$. □

Proposition 3.1.7. *Every supplement submodule of an absolutely supplement module is absolutely supplement.*

Proof. Let M be an absolutely supplement module and N be a supplement submodule of M. Since M is a supplement in its injective envelope $E(M)$, N is a supplement in $E(M)$ by Theorem 3.1.2. We can assume that the injective envelope $E(N)$ of N is contained in $E(M)$. Then again by Theorem 3.1.2 N is a supplement in $E(N)$. Now by Theorem 3.1.4 N is an absolutely supplement module. □

Corollary 3.1.8. *Every direct summand of an absolutely supplement module is an absolutely supplement module.*

Proof. Let M be an absolutely supplement module and K be a direct summand of M. Let for a suitable submodule T of M, $M = K \oplus T$. Since M is absolutely supplement by Proposition 3.1.4, M is a supplement submodule of its injective envelope $E(M)$. Now it is sufficient to show that K is a supplement submodule of its injective envelope $E(K)$. □

Proposition 3.1.9. *For a submodule N of a module M if N and M/N are absolutely supplement then M is absolutely supplement.*

Proof. Since N is an absolutely supplement module, N is a supplement in every module containing it and since M/N is an absolutely supplement module, M/N is a supplement in every module containing it. So we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "suppl?"] & M/N \arrow[r] \arrow[d, "suppl"] & 0 \\
0 \arrow[r] & N \arrow[r, "suppl"] & E(M) \arrow[r, "\alpha"'] \arrow[d, "\beta\circ\alpha"'] & E(M)/N \arrow[r] \arrow[d, "\beta"] & 0 \\
 & & E(M)/M \arrow[r, phantom, "="] \arrow[d] & E(M)/M \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Where $E(M)$ is an injective envelope of M, α and β are two $\mathcal{S}$-epimorphisms so by Theorem 3.1.2 $\beta \circ \alpha$ is an $\mathcal{S}$-epimorphism i.e. M is a supplement in its injective envelope $E(M)$. Thus by Proposition 3.1.4 M is an absolutely supplement module. □

3.2 Absolutely Co-supplement Modules

Definition 3.2.1. If for all short exact sequences

$$0 \longrightarrow T \longrightarrow X \longrightarrow M \longrightarrow 0$$

T is a supplement submodule of X then M is called an *absolutely co-supplement module.*

Proposition 3.2.2. *For a module M the following conditions are equivalent*

1. *M is an absolutely co-supplement module i.e. for all short exact sequences*

$$0 \longrightarrow T \longrightarrow X \longrightarrow M \longrightarrow 0$$

T is a supplement submodule of X.

2. *There exists a short exact sequence*

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

with a projective module P such that N is a supplement submodule of P.

Proof. 1. $\Rightarrow$ 2. There is a short exact sequence

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

with a projective module P (see e.g. [2] or [9]). By (1) N is a supplement submodule of P.

2. $\Rightarrow$ 1. Let

$$0 \longrightarrow T \longrightarrow X \xrightarrow{s} M \longrightarrow 0$$

be a short exact sequence. By (2) there is a short exact sequence

$$0 \longrightarrow N \longrightarrow P \xrightarrow{g} M \longrightarrow 0$$

with a projective module P such that N is a supplement submodule of P. By Theorem 2.5.4 we have the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N \arrow[r, equal] \arrow[d] & N \arrow[d] & \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & Y \arrow[r, "f"] \arrow[d, "h"] & P \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & T \arrow[r] & X \arrow[r, "s"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since P is projective f is a splitting epimorphism. Then by Theorem 3.1.2 $s \circ h = g \circ f$ is an $\mathcal{S}$-epimorphism. Again by Theorem 3.1.2 s is an $\mathcal{S}$-epimorphism. It means that T is a supplement submodule of X. $\square$

Example 3.2.3. Every projective module is absolutely co-supplement. Really let P be a projective module. Then we know that every short exact sequence ending with P is splitting. Since the class $\mathcal{S} = \{0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \,|\, A$ is a supplement in $B\}$ is a proper class, every splitting short exact sequence is in $\mathcal{S}$. Thus P is an absolutely co-supplement module.

Proposition 3.2.4. *If M_1 and M_2 are absolutely co-supplement modules then $M_1 \oplus M_2$ is absolutely co-supplement.*

Proof. Since M_1 is an absolutely co-supplement module by the previous proposition there exists a short exact sequence

$$0 \longrightarrow N_1 \longrightarrow P_1 \longrightarrow M_1 \longrightarrow 0$$

with P_1 is projective and $P_1/N_1 \cong M_1$, such that N_1 is a supplement submodule of P_1 and similarly since M_2 is an absolutely co-supplement module there exists a short exact sequence

$$0 \longrightarrow N_2 \longrightarrow P_2 \longrightarrow M_2 \longrightarrow 0$$

with P_2 is projective and $P_2/N_2 \cong M_2$, such that N_2 is a supplement submodule of P_2. Now there exists a short exact sequence

$$0 \longrightarrow N_1 \oplus N_2 \longrightarrow P_1 \oplus P_2 \longrightarrow M_1 \oplus M_2 \longrightarrow 0$$

with $P_1 \oplus P_2$ is projective (direct sum of projective modules is projective) and $(P_1 \oplus P_2)/(N_1 \oplus N_2) \cong M_1 \oplus M_2$, such that $N_1 \oplus N_2$ is a supplement in $P_1 \oplus P_2$. Let N_1 be a supplement of a submodule K_1 of P_1 i.e. $N_1 + K_1 = P_1$ and $N_1 \cap K_1 \ll N_1$. $N_1 + K_1 + P_2 = P_1 + P_2$. Now we want to show that $N_1 \cap (K_1 + P_2) \ll N_1$. Let $a \in N_1 \cap (K_1 + P_2) \Rightarrow a = n = k + x$ for some $n \in N_1$, $k \in K_1$, $x \in P_2 \Rightarrow x = n - k \in N_1 \cap K_1 = P_1 \Rightarrow x \in P_1 \cap P_2 = 0 \Rightarrow a = n = k \Rightarrow a \in N_1 \cap K_1 \Rightarrow N_1 \cap (K_1 + P_2) = N_1 \cap K_1 \ll N_1$. Thus N_1 is a supplement of a submodule $K_1 + P_2$ of $P_1 \oplus P_2$. Similarly, N_2 is a supplement submodule of $P_1 \oplus P_2$. So it can be proved by similar way as in the last part of proof of Proposition 3.1.5 that $N_1 \oplus N_2$ is a supplement submodule of $P_1 \oplus P_2$. □

Corollary 3.2.5. *Every finite direct sum of absolutely co-supplement modules is absolutely co-supplement.*

An arbitrary factor module of an absolutely co-supplement module need not be absolutely co-supplement, but we have the following proposition.

Proposition 3.2.6. *If M is an absolutely co-supplement module and N is a supplement submodule of M then M/N is also absolutely co-supplement.*

Proof. If N is a supplement in M then

$$0 \longrightarrow N \xrightarrow{suppl} M \longrightarrow M/N \longrightarrow 0 \in \mathcal{S}$$

On the other hand since M is an absolutely co-supplement module, by the Proposition 3.2.2 there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with a projective module P such that N is a supplement in P. Then we have the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & T \arrow[r, "suppl?"] \arrow[d] & P \arrow[r, "g\circ f"] \arrow[d, "f"] & M/N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "suppl"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M/N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where $T = \mathrm{Ker}(g \circ f)$. Now f and g are $\mathcal{S}$-epimorphisms, therefore $g \circ f$ is also an $\mathcal{S}$-epimorphism since $\mathcal{S}$ is a proper class (see Theorem 3.1.2). It means that there is a short exact sequence

$$0 \longrightarrow T \longrightarrow P \longrightarrow M/N \longrightarrow 0 \in \mathcal{S}$$

Thus by Theorem 3.2.2 M/N is also an absolutely co-supplement module. □

The following proposition shows that absolutely co-supplement modules are closed under extensions.

Proposition 3.2.7. *For a module M, if a submodule N and a quotient module M/N of M are absolutely co-supplement, then M is also absolutely co-supplement.*

Proof. Let

$$0 \longrightarrow A \longrightarrow C \longrightarrow M \longrightarrow 0$$

be any short exact sequence. We have the following diagram with exact rows and columns.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \arrow[r, phantom, "="] \arrow[d, "f"', "suppl"] & A \arrow[d, "suppl?"] & & \\
0 \arrow[r] & B \arrow[r, "suppl", "g"'] \arrow[d] & C \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since N is absolutely co-supplement, A is a supplement in B and since M/N is absolutely co-supplement, B is a supplement in C. Then by Theorem 3.1.2 A is a supplement submodule of C. Thus M is absolutely co-supplement. $\square$

Chapter 4

ABSOLUTELY COMPLEMENT MODULES

4.1 Absolutely Complement Modules

Definition 4.1.1. A module M is called *absolutely complement* if it is a complement submodule of every module containing M.

Theorem 4.1.2. (Sharpe, [9]) *If I be an injective module then I is a direct summand of every extension of itself.*

Proof. Let M be an extension of I. Since I is an injective module, we have the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "1_I"'] & M \arrow[dl, dashed, "\theta"] \\ & I & \end{tikzcd}$$

such that $\theta \circ i = 1_I$. Let $m \in M \Rightarrow \theta(m) \in I$ so that $\theta(m) = \theta(\theta(m)) \Rightarrow m - \theta(m) \in \operatorname{Ker}\theta \Rightarrow m \in I + \operatorname{Ker}\theta \Rightarrow M = I + \operatorname{Ker}\theta$. But $I \cap \operatorname{Ker}\theta = 0$ so $M = I \oplus \operatorname{Ker}\theta$. □

Corollary 4.1.3. *Every injective module is absolutely complement.*

Proof. Really since an injective module is a direct summand of every extension of itself, it is a complement in every extension of itself. □

The following proposition shows that there are no non-trivial absolutely complement modules.

Proposition 4.1.4. *For a module M the followings are equivalent*

1. *M is an absolutely complement module.*

2. *M is a complement in every injective module I containing M.*

3. *M is a complement in its injective envelope $E(M)$.*

4. *M is an injective module.*

Proof. 1. $\Rightarrow$ 2. It is clear by the definition of absolutely complement modules.
2. $\Rightarrow$ 3. If M is a complement in every injective module containing M and since $E(M)$ is an injective module containing M, M is a complement submodule of $E(M)$.
3. $\Rightarrow$ 4. If M is a complement in its injective envelope $E(M)$ then M is closed in $M = E(M)$. So there is no proper essential extension of M in $E(M)$. Also we know that $M \trianglelefteq E(M)$. Thus $M = E(M)$ i.e. M is an injective module.
4. $\Rightarrow$ 1. Since every injective module is absolutely complement, M is an absolutely complement module. □

Corollary 4.1.5. *Every complement submodule of an injective module is injective.*

Proof. Let M be a complement submodule of an injective module I. Then the injective envelope $E(M)$ is a direct summand of I, therefore M is a complement submodule of $E(M)$. By Proposition 4.1.4 (3. $\Rightarrow$ 4.), M is an injective module. □

Proposition 4.1.6. *For a submodule N of a module M if N and M/N are absolutely complement then M is absolutely complement.*

Proof. Since N is an absolutely complement module, N is a complement in every module containing it and since M/N is an absolutely complement module, M/N is a complement in every module containing it. So we have the following commutative diagram with exact rows and columns.

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "compl?"] & M/N \arrow[r] \arrow[d, "compl"] & 0 \\
0 \arrow[r] & N \arrow[r, "compl"] & E(M) \arrow[r, "\alpha"'] \arrow[d, "\beta\circ\alpha"'] & E(M)/N \arrow[r] \arrow[d, "\beta"] & 0 \\
 & & E(M)/M \arrow[r, equal] \arrow[d] & E(M)/M \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

Where $E(M)$ is an injective envelope of M, α and β are two $\mathcal{C}$-epimorphisms so by Theorem 2.7.15 $\beta \circ \alpha$ is a $\mathcal{C}$-epimorphism i.e. M is a complement in its injective envelope $E(M)$. Thus by Proposition 4.1.4 M is an absolutely complement module. $\square$

4.2 Absolutely Co-complement Modules

Definition 4.2.1. If for all short exact sequences

$$0 \longrightarrow T \longrightarrow X \longrightarrow M \longrightarrow 0$$

T is a complement submodule of X then M is called an *absolutely co-complement module.*

Proposition 4.2.2. *For a module M the following conditions are equivalent*

1. *M is an absolutely co-complement module i.e. for all short exact sequences*

$$0 \longrightarrow T \longrightarrow X \longrightarrow M \longrightarrow 0$$

T is a complement submodule of X.

2. *There exists a short exact sequence*

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

with a projective module P such that N is a complement submodule of P.

Proof. 1. $\Rightarrow$ 2. is clear.

2. $\Rightarrow$ 1. Let

$$0 \longrightarrow T \longrightarrow X \longrightarrow M \longrightarrow 0$$

be any short exact sequence. The short exact sequences

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow T \longrightarrow X \longrightarrow M \longrightarrow 0$$

together with pullback diagrams (see 2.7) give the following commutative diagram with exact rows and columns.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & N \arrow[r, equal] \arrow[d] & N \arrow[d] & \\
0 \arrow[r] & T \arrow[r] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "e"] & P \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & T \arrow[r, "compl"] & X \arrow[r, "h"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since P is projective, the epimorphism g is splitting. By Theorem 2.7.15 $\mathcal{C}$ is a proper class, therefore $h \circ e = f \circ g$ is a $\mathcal{C}$-epimorphism. Again since $\mathcal{C}$ is a proper class h is a $\mathcal{C}$-epimorphism, i.e.

$$0 \longrightarrow T \longrightarrow X \longrightarrow M \longrightarrow 0 \in \mathcal{C}$$

So M is an absolutely co-complement module. $\square$

Example 4.2.3. Every projective module is absolutely co-complement. Really, if P be a projective module then every sequence ending with P is a splitting short exact sequence. Since the class $\mathcal{C} = \{0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 | A$ is a complement of some K in $B\}$ is a proper class, these short exact sequences are in $\mathcal{C}$ it means that projective modules are absolutely co-complement.

Corollary 4.2.4. *Every free module is an absolutely co-complement module.*

Absolutely co-complement modules are not closed under factor modules in general but we have the following proposition.

Proposition 4.2.5. *If M is an absolutely co-complement module and N is a complement submodule of M, then M/N is absolutely co-complement.*

Proof. If M is an absolutely co-complement module then by the Proposition 4.2.2 there exists a short exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{f} M \longrightarrow 0$$

with ap projective module P such that K is a complement in P. Also we have a short exact sequence

$$0 \longrightarrow N \longrightarrow M \xrightarrow{g} M/N \longrightarrow 0$$

such that N is a complement in M, that is f and g are $\mathcal{C}$-epimorphisms. We know that the composition $g \circ f$ of two $\mathcal{C}$-epimorphisms is a $\mathcal{C}$-epimorphism. Thus there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow P \xrightarrow{g \circ f} M/N \longrightarrow 0$$

with a projective module P such that T is a complement in P. Hence M/N is an absolutely co-complement module. □

The following proposition shows that absolutely co-complement modules are closed under extensions.

Proposition 4.2.6. *For a module M, if a submodule N of M and the quotient module M/N are absolutely co-complement, then M is also absolutely co-complement.*

Proof. Let

$$0 \longrightarrow A \longrightarrow C \longrightarrow M \longrightarrow 0$$

be any short exact sequence. We can use the following diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A \arrow[d, "compl"', "f"] \arrow[r, phantom, "="] & A \arrow[d, "compl?"] & & \\
0 \arrow[r] & B \arrow[r, "compl", "g"'] \arrow[d] & C \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/N \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

Since N is absolutely co-complement, A is a complement in B, M/N is absolutely co-complement, therefore B is a complement in C. Then by Theorem 2.7.15 A is a complement submodule of C. So M is absolutely co-complement. $\square$

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