Ali KARAİSA

M.S. Thesis In Mathematics

July 2010

ON THE SPECTRUM OF SOME TRIANGLE OPERATORS OVER THE CLASSICAL SEQUENCE SPACE

by

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A thesis submitted to

the Graduate Institute of Sciences and Engineering

of

Fatih University

in partial fulfillment of the requirements for the degree of

Master of Science

in

The Graduate Institute of Arts and Science

July 2010
Istanbul, Turkey

APPROVAL PAGE

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Mustafa BAYRAM
Head of Department

This is to certify that I have read this thesis and that in my opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Assoc. Prof. Dr. Nurullah ARSLAN
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ABSTRACT

The present thesis consists of four chapters. In the second chapter, the definitions and basic properties of metric space, norm space and linear operator introduced by Spectral theory were discussed. In this study, some basic concepts related to the subject of spectrum were given by taking that subject into consideration. Also, the definition of sequence spaces was introduced and definitions and theorems related to matrix transformations were given in the first chapter.

In chapter 3, the fine spectrum of newly introduced Z^{α} operator on the sequence space c_0 was discussed. We examined the spectra and fine spectra of Zweier matrix which is a band matrix as an operator over the sequence spaces c_0 and c. additionally, we also determined the fine spectrum with Goldberg's classification of the operator Z^{α} on the sequence spaces c_0 and c.

In the fourth chapter, we essentially dealed with the fine spectrum of the generalized difference matrix $B(r_k, s_k)$ which is a band matrix over the sequence spaces c_0 and c.

Keywords: Spectral theory, Fine spectrum, Point Spectrum, Residual spectrum, Continuous spectrum, Sequence space on c_0 and c

BAZI ÜÇGEN OPERATÖRLERİN KLASİK DİZİ UZAYLARI ÜZERİNDEKİ SPEKTRUMU

Ali KARAİSA

Yüksek Lisans Tezi – Matematik

Temmuz 2010

Tez Danıışmanı: Prof. Dr. Feyzi BAŞAR

ÖZ

Bu tez 4 bölümden ibarettir. İkinci bölümde; metrik uzaylar, normlu uzaylar ve lineer dönüşümlerin tanımı verilerek özelliklerden bahsedilmiştir. Ayrıca bölümün ikinci kısmında spektrum konusu ele alınarak, konuyla ilgili bazı temel tanım ve kavramlar verilmiştir.

Tezin üçüncü bölümünde; c_0 ve c dizi uzayları üzerinde Z^{α} Zwier operatörü tanımlanarak bu operatörün ince spektrumu incelenmiştir. Ayrıca Z^{α} operatörünün c_0 ve c dizi uzayları üzerindeki spektrumu Goldberg sınıflandırmasına göre verilmiştir.

Tezin dördüncü bölümünde; aslî köşegeninde (r_k) ve ona paralel ikinci köşegeninde (s_k) dizilerinin terimlerini ihtiva eden $B(r_k, s_k)$ alt üçgen matris olmak üzere bu matrisin c_0 ve c dizi uzayları üzerindeki ince spektrumu incelenmiştir.

Anahtar Kelimeler: Spektrum, ince spektrum, artık spektrum, nokta spektrum, sürekli spektrum, c_0 ve c dizi uzayları.

DEDICATION

To my familiy

ACKNOWLEDGEMENT

I express sincere appreciation to Prof. Dr. Feyzi BAŞAR for his guidance and insight throughout the research and thanks go to the other faculty members.

I express my thanks and appreciation to my wife for her understanding, motivation and patience. Lastly, but in no sense the least, I am thankful to all colleagues and friends who made my stay at the university a memorable and valuable experience.

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$\mathbb{R}$: Reel sayıların cümlesi

$\mathbb{C}$: Kompleks sayılar cümlesi,

$B(X)$: X normlu uzayında tanımlı bütün sınırlı lineer operatörlerin cümlesi,

$R(T)$: T operatörünün görüntü cümlesi,

$D(T)$: T operatörünün tanım cümlesi,

$N(T)$: T operatörünün çekirdeği,

T^{-1} : T operatörünün tersi

c : Reel terimli yakınsak dizilerin uzayı

c_0 : Reel terimli sıfıra yakınsak dizilerin uzayı

$\overline{R(T)}$: $R(T)$ cümlesinin kapanışı

$\rho(T)$: T operatörünün resolventi

$\sigma(T)$: T operatörünün spektrumu

$\sigma_r(T)$: Artık spektrum

$\sigma_c(T)$: Sürekli spektrum

$r(T)$: T operatörünün spektral yarıçapı

T^* : T operatörünün adjointi

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ALI KARAISA

M S, Thesis-Mathematics

July 6, 2010

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ABSTRACT

The present thesis consists of four chapters. In the second chapter, the definitions and basic properties of metric space, norm space and linear operator introduced by Spectral theory were discussed. In this study, some basic concepts related to the subject of spectrum were given by taking that subject into consideration. Also, the definition of sequence spaces was introduced and definitions and theorems related to matrix transformations were given in the second chapter. In chapter 3, the fine spectrum of newly introduced operator on the sequence space c_0 was discussed. We examined the spectra and fine spectra of Zweier matrix which is a band matrix as an operator over the sequence spaces c_0 and c. Additionally, we also determined the fine spectrum with Goldberg's classification of the operator on the sequence spaces c_0 and c. In the fourth chapter, we essentially dealed with the fine spectrum of the generalized difference matrix which is a band matrix over the sequence spaces c_0 and c.

Keywords: Spectral theory, Fine spectrum, Point Spectrum, Residual spectrum, Continuous spectrum, Sequence space on c_0 and c

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ALI KARAISA

Yüksek Lisans Tezi-Matematik

July 6, 2010

Tez Yöneticisi: Prof. Dr. Feyzi Başar

ÖZ

Bu tez 4 bölümden ibarettir. Birinci bölümde tez haknda genel bilgi verildi. İkinci bölümde; metrik uzaylar, normlu uzaylar ve lineer dönüşümlerin tanımı verilerek özelliklerden bahsedilmiştir. Ayrca bölümün ikinci kısmında spektrum konusu ele alınarak, konuyla ilgili bazı temel tanım ve kavramlar verilmiştir. Tezin ikinci bölümünde; c_0 ve c dizi uzayları üzerinde Zweier operatörü tanımlanarak bu operatörün ince spektrumu incelenmiştir. Ayrıca Zweier operatörünün c_0 ve c dizi uzayları üzerindeki spektrumu Goldberg sınıflandırmasına göre verilmiştir. Tezin üçüncü bölümünde; asli köşegeninde (r_k) ve ona paralel ikinci köşegeninde (s_k) dizilerinin terimlerini ihtiva eden alt üçgen matris olmak üzere bu matrisin c_0 ve c dizi uzayları üzerindeki ince spektrumu incelenmiştir.

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INTRODUCTION

We summarize the knowledge in the existing literature concerning with the spectrum and the fine spectrum of the linear operators defined by some particular limitation matrices over some sequence spaces. ([Wenger R.B., 1975]) examined the fine spectrum of the integer power of the Cesàro operator in the space c. ([Reade J.B., 1985]) worked with the spectrum of the Cesàro operator in the sequence space c_0. ([Gonzalez M., 1985]) studied the fine spectrum of the Cesàro operator in the sequence space ℓ_p. ([Okutoyi J.T., 1994]) computed the spectrum of Cesàro operator on the sequence space bv. Recently, ([Yıldırım M., 1996]) worked the fine spectrum of Rhally operators acting on the space c and c_0. Lately, ([Coşkun C., 1997]) studied the spectrum and fine spectrum for $p-$Cesàro operator acting on the space c_0. ([Akhmedov A.M. and Başar F.,2004]) have recently determined, independently than that of ([Gonzalez M., 1985]) the fine spectrum of the Cesàro operator in the sequence spaces c_0, ℓ_p, ℓ_∞ by the different way, respectively, where $1 < p < \infty$. Quite recently,([Malafose B., 2002]) and ([Altay B. and Başar F., 2004])have respectively studied the spectrum and fine spectrum of the difference operator on the sequence spaces c, c_0, s_r; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by

$$\|x\|_{s_r} = \sup_{k\in\mathbb{N}} \frac{|x_k|}{r^k}, \quad (r > 0).$$

Also, ([Akhmedov A.M. and Başar F., 2004]) , ([Altay B. and Başar F., 2005]) have determined the fine spectrum with respect to Goldberg's classification of the difference operator Δ and generalized difference operator $B(r, s)$ over the sequence spaces ℓ_p, bv_p and c_0, c; receptively. Later Bilgiç ([H. and Furkan H., 2007]) worked on the spectrum of the operator $B(r, s, t)$, defined by a triple-band lower triangle matrix, over the sequence spaces ℓ_1 and bv.

CHAPTER 2

PRELIMINARIES, BACKGROUND AND NOTATION

Spectral theory is one of the main branches of modern functional analysis and its applications. Roughly speaking it is concerned with certain inverse operator, their general properties and their relation the original operator. Such inverse operator arise naturally in connection with the problem of solving equation (system of linear algebraic equation differential equation, integral equation) for instance, the investigation of boundary value problem by Sturm and Liouville, and Fredholm's famous theory of integral equations were important to development of the field.

In Chapter 2, we mention some properties and give an introduction to spectral theory of bounded linear operators.

2.1. Background

In this section, we give some required definitions related with the spectrum.

Definition 2.1. (Metric space) Let X be a non-empty set and d be a distance function from $X \times X$ to the set $\mathbb{R}^+$ of non-negative real numbers. Then the pair (X, d) is called a *metric space* and d is a metric for X, if the following metric axioms are satisfied for all elements $x, y, z \in X$:

(M.1) $d(x, y) = 0$ if and only if $x = y$.

(M.2) $d(x, y) = d(y, x)$, (the symmetry property).

(M.3) $d(x, z) \leq d(x, y) + d(y, z)$, (the triangle inequality).

Definition 2.2. (Normed space) Let X be a real or complex linear space and $\|\cdot\|$ be a function from X to the set $\mathbb{R}^+$ of non-negative real numbers. Then the pair $(X, \|\cdot\|)$ is called a *normed space* and $\|\cdot\|$ is a *norm* for X, if the following norm axioms are satisfied for all elements $x, y \in X$ and for all scalars α:

(N.1) $\|x\| = 0$ if and only if $x = \theta$.

(N.2) $\|\alpha x\| = |\alpha|\|x\|$, (the absolute homogenity property).

(N.3) $\|x + y\| \leq \|x\| + \|y\|$, (the triangle inequality).

Definition 2.3. (Banach space) A Banach space X is complete normed linear space. Completeness means that if $\|x_m - x_n\| \to 0$ as $m, n \to \infty$, where $x_n \in X$, then there exist $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 2.4. (Linear operator) In calculus we consider real line $\mathbb{R}$ and real-valued functions on $\mathbb{R}$ (or on a subset $\mathbb{R}$). Obviously, any such function is a mapping of its domain into $\mathbb{R}$. In functional analysis, we consider more general space, such metric space and normed space and mapping of these space in the case of vector space and in particular, normed space a mapping is called an operator.

A linear operator T is an operator such that

(i) the domain $D(T)$ of T is a vector space and range $R(T)$ lies in a vector space over the same field

(ii) for all $x, y \in D(T)$ and scalars α,

$$\begin{aligned} T(x + y) &= Tx + Ty \\ T(\alpha x) &= \alpha Tx \end{aligned}$$

Observe the notation; we write Tx instead of $T(x)$; this simplification is standard in functional analysis. Furthermore for the remainder of the thesis we shall use the fol- lowing notation:

$D(T)$ denotes the domain of T

$R(T)$ denotes the range of T

$N(T)$ denotes the null space of T

By definition the null space of T is set of all $x \in D(T)$ such that $Tx = 0$. We should also say something about the use of arrows in connection with operators. Let $D(T) \subset X$ and $R(T) \subset Y$, where X and Y are vector spaces, both real or complex. Then T is an operator from $D(T)$ onto $R(T)$, written

$$T : D(T) \to R(T)$$

or from $D(T)$ into Y

$$T : D(T) \to Y$$

If $D(T)$ is the whole space X, then and only then we write

$$T : X \to Y$$

Definition 2.5. (Bounded linear operator) Let X and Y be the normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded operator if there is a positive real number c such that

$$\|Tx\| = c\|x\|$$

for all $x \in D(T)$. Let X and Y be linear spaces. By $L(X, Y)$ and $B(X, Y)$, we denote the set of all linear operators and the set of all bounded linear operators from X into Y.

Theorem 2.1.1. *Let X, Y be the normed spaces and $T : X \to Y$ be linear operator. Then*

(i) T continuous at the origin implies Tuniformly continuous on X.

(ii) T is continuous on X if and only if it is bounded.

Proof. (i) We have $\|Tx\| < \epsilon$ whenever $\|x\| < \delta$. Hence, if $\|x-y\| < \delta$ then $\|T(x-y)\| = \| \ Tx - Ty\| < \epsilon$ by linearity of T.

(ii) First let T be bounded. $\|Tx\| = c\|x\|$ on X. Then

$$\| \ Tx - Ty\| = \|T(x-y)\| \leq c\|x-y\| < \epsilon$$

if $\|x-y\| < \frac{\epsilon}{c}$. Hence, T uniformly continuous on X. Then T is continuous at θ and so there exist $\delta = \delta(1)$ such that $\|Tx\| < 1$ whenever $\|x\| < \delta$. Take any $x \neq \theta$. Then

$$\left\|\frac{\delta x}{2\|x\|}\right\| = \frac{\delta}{2}$$

and so $\left\|T\left(\frac{\delta x}{2\|x\|}\right)\right\| < 1$, $\|Tx\| < \frac{2}{\delta}\|x\|$
If $x = \theta$ then $\|Tx\| = 0$ and so $\|Tx\| \leq 2\delta^{-1}\|x\|$ on X, i.e T is bounded. □

Definition 2.6. (Norm of a bounded operator) Let $T \in B(X,Y)$ then the norm of T is defined as

$$\|T\| = \sup_{x \neq \theta} \frac{\|Tx\|}{\|x\|} < \infty \tag{2.1}$$

The supremum on the right side of (2.1)is finite which follows from the fact that $\|Tx\| = c\|x\|$ when $T \in B(X,Y)$
Now we show that $\|Tx\|$ is indeed a norm on $B(X,Y)$ and also we prove that $B(X,Y)$ is a Banach space when Y is a Banach space, irrespective of whether X is a Banach space.

Definition 2.7. (Matrix Transformation) Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. We write

$$(AX)_n = \sum_{k=0}^{\infty} a_{nk}x_k; \quad (n \in \mathbb{N}, x \in D_{00}), \tag{2.2}$$

where $D_{00}(A)$ denotes the subspace w consisting of $x \in W$ for which the sum exist as a finite sum. More generally, if μ is a normed sequence space, we can write $D_\mu(A)$ for $x \in w$ for which the sum in (2.2) converges in the norm of μ. We will write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\}$$

for the space of those matrices send the whole of the sequence space λ into μ in this sense.

Lemma 1. *The matrix $A = (a_{nk})$ gives rise to a bounded operator $T \in B(c)$ from c to itself if and only if*

(1) the rows of A in ℓ_1 and their ℓ_1 norms are bounded,
(2) the columns of A are in c,
(3) the sequence of row sums of A is in c.

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Lemma 2. *The matrix $A = (a_{nk})$ gives rise to a bounded operator $T \in B(c_0)$ from c_0 to itself if and only if*

(1) the rows of A in ℓ_1 and their ℓ_1 norms are bounded,
(2) the columns of A are in c_0.

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Definition 2.8. (Spectral theory in finite dimensional normed space) Let X be finite dimensional normed space and $T : X \to Y$ a linear operator. Spectral theory of such operators is simpler than that of operators defined on infinite dimensional spaces. In fact we know that we can represent T by matrices which depends on the choice of bases for X and we shall see that spectral theory of T is essential matrix eigenvalue theory. So, let us begin matrices. For given real or complex n-square matrix $A = (\alpha_{jk})$ the concepts of eigenvalue and eigenvector are defined terms of the equation

$$Ax = \lambda x \tag{2.3}$$

as follows.

Definition 2.9. (Eigenvalues, eigenvectors, eigenspaces, spectrum, resolvent set of matrix) An eigenvalue of a square matrix $A = (\alpha_{jk})$ is a number λ such that (2.3) has a solution $x \neq \theta$. This x is called an eigenvector of A corresponding to that eigenvalue λ. The eigenvectors corresponding to that eigenvalue λ and the zero vector from a vector subspace of X which called eigenspace of A corresponding to that eigenvalue λ. The set $\sigma(A)$ of all eigenvalues of A is called the spectrum of A, its complement $\rho(A) = \mathbb{C} - \sigma(A)$ in the complex plane is called the resolvent set of A.

Theorem 2.1.2. *Let $T \in B(X, X)$, where X is a Banach space. If $\|T\| < 1$, then $(I - T)^{-1}$ exist as a bounded linear operator on the whole space X and*

$$(1 - T)^{-1} = \sum_{k=0}^{\infty} T^k = I + T + T^2 + \cdots$$

Theorem 2.1.3. *If $X \neq \{\theta\}$ is a complex Banach space and $T \in B(X, X)$, then $\sigma(T) \neq \emptyset$.*

Definition 2.10. (Spectral radius) The spectral radius $r_\sigma(T)$ of an operator $T \in B(X, X)$ on a complex Banach space X is the radius of the smallest closed disk centered at the origin of the complex λ-plane and containing $\sigma(T)$,i.e

$$r_\sigma(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$$

2.2. Purpose of Study

Let X and Y be the Banach spaces and $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, \ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$ then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a non trivial complex normed space and $T : \mathcal{D}(T) \to X$ a linear operator defined on subspace $\mathcal{D}(T) \subseteq X$. We do not assume that $\mathcal{D}(T)$ is dense in X, or that T has closed graph $\{(x, Tx) : x \in \mathcal{D}(T)\} \subseteq X \times X$. We mean by the expression "T is *invertible*" that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number α is perturbed operator

$$T_\alpha = T - \alpha I,$$

defined on the same domain $\mathcal{D}(T)$ as T. The *spectrum* $\sigma(T, X)$ consist of those $\alpha \in \mathbb{C}$ for which T_α is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T, X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\alpha \mapsto T_\alpha^{-1}$.

The name *resolvent* is appropriate, since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1}y$ provided T_α^{-1} exists. More important, the investigation of properties of T_α^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α's in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. A *regular value* α of T is a complex number such that T_α^{-1} exists and bounded and whose domain is dense in X. For our investigation of T, T_α and T_α^{-1}, we need some basic concepts in spectral theory which are given as follows ([Kreyszig E., 1978 see pp. 370-371]):

The *resolvent set* $\rho(T)$ of T is the set of all regular values α of T. Furthermore, the spectrum $\sigma(T)$ is partitioned into the following three disjoint sets

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_α^{-1} does not exist. An $\alpha \in \sigma_p(T)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_α^{-1} exists and is bounded and the domain of T_α^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but the domain of T_α^{-1} is not dense in X.

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite dimensional case.

From ([Goldberg S., 1985]) if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and T^{-1}:

(I) $R(T) = X$

(II) $R(T) \neq \overline{R(T)} = X$

(III) $\overline{R(T)} \neq X$

and

(1) T^{-1} exists and is continuous.

(2) T^{-1} exists but is discontinuous.

(3) T^{-1} does not exist.

Applying Goldberg's classification to T_α, we have three possibilities for T_α and T_α^{-1};

(I) T_α is surjective.

(II) $\quad R(T_\alpha) \neq \overline{R(T_\alpha)} = X$

(III) $\quad \overline{R(T_\alpha)} \neq X$

and

(1) $\quad T_\alpha$ is injective and T_α^{-1} is continuous.

(2) $\quad T_\alpha$ is injective and T_α^{-1} is discontinuous.

(3) $\quad T_\alpha$ is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labeled by: $I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2$ and III_3. If α is a complex number such that $T_\alpha \in I_1$ or $T_\alpha \in II_1$ then α is in the resolvent set $\rho(T, X)$ of T. The further classification gives rise to the fine spectrum of T. If an operator is in state II_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\alpha \in II_2\sigma(T, X)$.

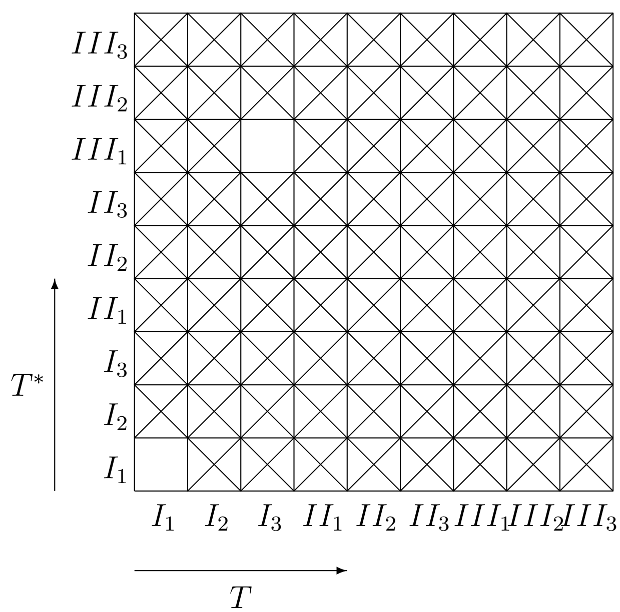

Fig.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X.

By the definitions given above, we can illustrate the subdivision of spectrum in the following table:

		1	2	3
		$R(\lambda;L)$ exists and is bounded	$R(\lambda;L)$ exists and is unbounded	$R(\lambda;L)$ does not exists
I	$R(\lambda I-L)=X$	$\lambda\in\rho(L)$	–	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$
II	$\overline{R(\lambda I-L)}=X$	$\lambda\in\rho(L)$	$\lambda\in\sigma_c(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$
III	$\overline{R(\lambda I-L)}\neq X$	$\lambda\in\sigma_r(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L)$	$\lambda\in\sigma_r(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L)$	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L)$

Table 1.2

From now on, we should note that the index p has different meanings in the notation of the spaces ℓ_p, ℓ_p^*, bv_p, bv_p^* and the point spectrums $\sigma_p(B(r,s,t),\ell_p)$, $\sigma_p(B(r,s,t)^*,\ell_p^*)$, $\sigma_p(B(r,s,t),bv_p)$, $\sigma_p(B(r,s,t)^*,bv_p^*)$ which occur in theorems given in Section 2 and Section 3.

By a *sequence space*, we understand a linear subspace of the space $w=\mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non–zero sequences, where $\mathbb{N}=\{0,1,2,\dots\}$. We write ℓ_∞, c, c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by ℓ_p, we denote the space of all p–absolutely summable sequences, where $1\leq p<\infty$.

The purpose of this study is to collect the knowledge on the spectrum of some triangle operators defined by certain summability matrices over the classical spaces ℓ_∞, c, c_0 and ℓ_p. Of course, to obtain the spectrum and fine spectrum of the generalized difference operator $B(r, s)$ over one of the classical spaces ℓ_∞, c, c_0 and ℓ_p may be studied by following ([A.M. Akhmedov and Baar F., 2004]) and ([Bilgi H. and Furkan H., 2007]).

CHAPTER 3

ON THE SPECTRUM AND FINE SPECTRUM OF THE ZWEIER MATRIX AS AN OPERATOR

3.1. Introduction

The purpose in this chapter is to determine the fine spectrum of the Zweier matrix which is the band matrix as an operator over the sequence spaces c_0 and c.

3.2. Zweier Matrices

Let us consider the zweier matrix Z^α represented by following band matrix

$$\mathbf{Z}^\alpha = \begin{pmatrix} \alpha & 0 & 0 & \cdots \\ 1-\alpha & \alpha & 0 & \cdots \\ 0 & 1-\alpha & \alpha & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\alpha \in \mathbb{R} \setminus \{0, 1\}$. We begin with determining that a matrix A induces a bounded operator from c to c

Corollary 3.2.1. *$Z^\alpha : c \to c$ is a bounded linear operator with the norm $\|Z^\alpha\|_{(c,c)} = |\alpha| + |1 - \alpha|$*

Corollary 3.2.2. *$Z^\alpha : c_0 \to c_0$ is a bounded linear operator with the norm $\|Z^\alpha\|_{(c,c)} = \|Z^\alpha\|_{(c_0,c_0)}$*

3.3. The Spectrum of the Operator Z^α On the Sequence Spaces c_0 and c

Theorem 3.3.1. $\sigma(Z^\alpha, c_0) = \lambda \in \mathbb{C} : |\lambda - \alpha| \leq |1-\alpha|$.

Proof. It is enough to prove that $(Z^\alpha - \lambda I)^{-1}$ exist and is in $(c_0 : c_0)$ for $\lambda \notin \sigma(Z^\alpha, c_0)$ and nextly show that the operator $Z^\alpha - \lambda I$ is not invertible for $\lambda \in \sigma(Z^\alpha, c_0)$. Let $\lambda \notin \sigma(Z^\alpha, c_0)$. Since $Z^\alpha - \lambda I$ is triangle, $(Z^\alpha - \lambda I)^{-1}$ exists. Therefore, solving the equation

$$\begin{aligned}
(Z^\alpha - \lambda I)x &= \begin{pmatrix} \alpha-\lambda & 0 & 0 & \cdots \\ 1-\alpha & \alpha-\lambda & 0 & \cdots \\ 0 & 1-\alpha & \alpha-\lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} x_0(\alpha-\lambda) \\ x_0(1-\alpha) + x_1(\alpha-\lambda) \\ x_1(1-\alpha) + x_2(\alpha-\lambda) \\ \vdots \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{pmatrix}
\end{aligned}$$

we derive x in terms of y that,

$$\begin{aligned}
x_0 &= \frac{y_0}{\alpha-\lambda}, \\
x_1 &= \frac{y_1}{\alpha-\lambda} - \frac{y_0}{(\alpha-\lambda)^2}, \\
x_2 &= \frac{y_0(1-\alpha)^2}{(\alpha-\lambda)^3} - \frac{y_1(1-\alpha)}{(\alpha-\lambda)^2} + \frac{y_0}{\alpha-\lambda} \\
&\vdots
\end{aligned}$$

which gives the matrix $(Z^\alpha - \lambda I)^{-1}$. The n^{th} row turns out to be,

$$\frac{(1-\alpha)^{n-k}}{(\alpha-\lambda)^{n-k+1}}$$

$$\begin{aligned}\|(Z^\alpha-\lambda I)^{-1}\|_{(c_0:c_0)} &= \sup_{n\in\mathbb{N}}\sum_{k=1}^{n}\left|\frac{1-\alpha}{\lambda-\alpha}\right|^{n-k}\left|\frac{1}{\lambda-\alpha}\right| \\ &= \left|\frac{1}{\lambda-\alpha}\right|\sum_{k=1}^{n}\left|\frac{1-\alpha}{\lambda-\alpha}\right|^{n}<\infty \qquad (3.1)\end{aligned}$$

that is $(Z^\alpha-\lambda I)^{-1}\in(c_0:c_0)$.

$\alpha\in\sigma(Z^\alpha,c_0)$ and $\lambda\neq\alpha$ since $Z^\alpha-\lambda I$ is triangle $(Z^\alpha-\lambda I)^{-1}$ exists and one can see by (3.1)That

$$\|(Z^\alpha-\lambda I)^{-1}\|_{(c_0:c_0)}=\infty$$

whenever $\alpha\in\sigma(Z^\alpha,c_0)$ and $(Z^\alpha-\lambda I)^{-1}$ is not in $B(c_0)$. If $\alpha=\lambda$ the operator $Z^\alpha-\lambda I$ is represented by the matrix,

$$\mathbf{Z}^\alpha=\begin{pmatrix}0&0&0&\cdots\\1-\alpha&0&0&\cdots\\0&0&1-\alpha&\cdots\\\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

since $(Z^\alpha-\lambda I)x=\theta$ implies $x=\theta$, $Z^\alpha-\lambda I:c_0\to c_0$ is injective but has not dense range. So, $Z^\alpha-\lambda I$ is not invertible. This completes the proof. □

Theorem 3.3.2. $\sigma_p(Z^\alpha,c_0)=\emptyset$.

Proof. Suppose $Z^\alpha x=\lambda x$ for $x\neq\theta=(0,0,0,\dots)$ in c_0. Then, by solving the system of linear equations

$$\begin{aligned} \alpha x_0 &= \lambda x_0 \\ (1-\alpha)x_0 + \alpha x_1 &= \lambda x_1 \\ (1-\alpha)x_1 + \alpha x_2 &= \lambda x_2 \\ &\vdots \\ (1-\alpha)x_k + \alpha x_{k+1} &= \lambda x_k \\ &\vdots \end{aligned}$$

we find that if x_0 is the first non-zero entry of sequence $x = (x_n)$ then $\lambda = \alpha$ and from the equation,

$$(1-\alpha)x_{n_0} + x_{n_0+1} = \lambda x_{n_0+1}$$

we set $(1-\alpha)x_{n_0} = 0$. Since $\alpha \neq 1$ we must have $\sigma_p(Z^\alpha, c_0) = \emptyset$, as desired.

$\square$

Theorem 3.3.3. $\sigma_p((Z^\alpha)^*, c_0^*) = \{\alpha \in \mathbb{C} : |\lambda - \alpha| \leq |1-\alpha|\}$

Proof. Suppose $(Z^\alpha)^* x = \lambda x$ for $x \neq \theta = (0,0,0,\ldots) \in c_o^* \cong \ell_1$.

$$\begin{pmatrix} \alpha & 0 & 0\ldots \\ 1-\alpha & 0 & 0\ldots \\ 0 & 1-\alpha & \alpha\ldots \\ \vdots & \vdots & \vdots\ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}$$

$$\begin{aligned}
\alpha x_0 + (1-\alpha)x_1 &= \lambda x_0 \\
\alpha x_1 + (1-\alpha)x_2 &= \lambda x_1 \\
\alpha x_2 + (1-\alpha)x_3 &= \lambda x_2 \\
&\vdots \\
\alpha x_k + (1-\alpha)x_{k+1} &= \lambda x_k \\
&\vdots \quad .
\end{aligned}$$

We obtain that $x_n = \frac{\lambda-\alpha}{1-\alpha}x_0$ which shows that $x = (x_k) \in c_o^*$ if and only if $|\lambda - \alpha| \leqslant |1-\alpha|$.

This completes the proof. $\square$

Lemma 3. *T has a dense range if and only if T^* is one to one.*

Lemma 4. *The adjoint operator T^* of T is onto if and only if T is a bounded operator.*

Theorem 3.3.4. $\sigma_r(Z^\alpha, c_0) = \{\alpha \in \mathbb{C} : |\lambda - \alpha| \leq |1-\alpha|\}$.

Proof. We show that $\overline{R(Z^\alpha - \lambda I)} \neq c_0$ for α satisfying $|\lambda - \alpha| \leq |1-\alpha|$ for $\lambda \neq \alpha$ the operator $Z^\alpha - \lambda I$ triangle and has an inverse for $\lambda = \alpha$ the operator $Z^\alpha - \alpha I$ is one to one hence has a inverse but $(Z^\alpha)^* - \lambda I$ not one to one by Theorem 3.3.3. Now, Lemma 4 yields the fact that the range of the operator $Z^\alpha - \lambda I$ is not dense in c_0.

This completes the proof. $\square$

Theorem 3.3.5. $\sigma_c(Z^\alpha, c_0) = \emptyset$.

Proof. Since $\sigma_c(Z^\alpha, c_0)$, $\sigma_p(Z^\alpha, c_0)$ and $\sigma_r(Z^\alpha, c_0)$ are disjoint union of the spectrum $\sigma(Z^\alpha, c_0)$, we must have $\sigma_c(Z^\alpha, c_0) = \emptyset$. $\square$

Theorem 3.3.6. $\sigma(Z^\alpha, c) = \lambda \in \mathbb{C} : |\lambda - \alpha| \leq |1-\alpha|$

Proof. This is obtained in the similar way that used in proof of Theorem 3.3.1 □

Theorem 3.3.7. $\sigma_p(Z^\alpha, c) = \emptyset$

Proof. The proof may be obtained by proceeding as in proving Theorem 3.3.2. □

If $T : c \to c$ is bounded matrix operator with A, then $T^* : c^* \to c^*$ acting on $\mathbb{C} \oplus \ell_1$ has a matrix representation of the form

$$\begin{pmatrix} \chi & 0 \\ b & A^t \end{pmatrix},$$

where χ denotes the characteristic of the matrix A and is the limit of the sequence of rows of A minus of the limit of the columns of A and b is the column vector whose kth entry is the limit of kth column of A for each $k \in \mathbb{N}$. $(Z^\alpha)^* \in B(\mathbb{C} \oplus \ell_1)$ is the following:

$$\begin{pmatrix} 1 & 1-\alpha & 0 & \cdots \\ 0 & \alpha & 1-\alpha & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Theorem 3.3.8. $\sigma((Z^\alpha)^*, c^*) = \lambda \in \mathbb{C} : |\lambda - \alpha| \leq |1 - \alpha|$.

Proof. Suppose $(Z^\alpha)^* x = \lambda x$ for $x \neq \theta \in c^* = \mathbb{C} \oplus \ell_1$,i.e,

$$\begin{pmatrix} 1 & 1-\alpha & 0 & \cdots \\ 0 & \alpha & 1-\alpha & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix}.$$

Then, by solving the system of linear equations,

$$\begin{aligned}
x_0 + (1-\alpha)x_1 &= \lambda x_0 \\
\alpha x_1 + (1-\alpha)x_2 &= \lambda x_1 \\
\alpha x_2 + (1-\alpha)x_3 &= \lambda x_2 \\
&\vdots \\
\alpha x_k + (1-\alpha)x_{k+1} &= \lambda x_k \\
&\vdots
\end{aligned}$$

we obtain that,

$$x_n = \left(\frac{\lambda-\alpha}{1-\alpha}\right)^n x_0$$

for all $n \in \mathbb{N}$. If $\frac{\lambda-\alpha}{1-\alpha} = 1$, $\alpha = 1$ then since $s \neq 1$ $x_1 = 0, x_2 = 0, x_3 = 0, \ldots, x_n = 0, \ldots$, that is $x = (x_0, 0, 0, \ldots)$ is an eigenvector corresponding to $\alpha = 1$ If $\frac{\lambda-\alpha}{1-\alpha} \neq 1$ then $x \in \mathbb{C} \oplus \ell_1$

$$\sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n-1} \left(\frac{\lambda-\alpha}{1-\alpha}\right)^k \left(\frac{\lambda-\alpha}{1-\alpha}\right) \right| < \infty$$

which leads us to consequence that,

$$\left|\frac{\lambda-\alpha}{1-\alpha}\right| \sup_{n\in\mathbb{N}} \left| \frac{1-\left(\frac{\lambda-\alpha}{1-\alpha}\right)^n}{1-\frac{\lambda-\alpha}{1-\alpha}} \right| < \infty$$

if and only if $|\lambda - \alpha| \leqslant |1-\alpha|$, as asserted. □

Theorem 3.3.9. *$\alpha \in III_1\sigma(Z^\alpha, c_0)$.*

Proof. By Theorem 3.3.3 and Lemma 3, $Z^\alpha - \alpha I \in III$. On the other hand, since $\sigma_p(Z^\alpha, c_0) = \emptyset$ by Theorem 3.3.2 $Z^\alpha - \alpha I$ has inverse then $1 \cup 2$ to show that $Z^\alpha - \alpha I \in 1$, it is enough to establish by Lemma 3 that$(Z^\alpha)^* - \alpha I$ is onto. Given

$y = (y_n) \in \ell_1$ such that $((Z^\alpha)^* - \alpha I)x = y$, therefore, calculation gives that

$$x_n = \frac{1}{1-s} y_{n-1}$$

for all $n \in \mathbb{N}$. This shows that $(Z^\alpha)^* - \alpha I$ is onto. □

Theorem 3.3.10. *If $\alpha \neq \lambda$, $\alpha \in III_2\sigma(Z^\alpha, c_0)$.*

Proof. Since $\alpha \neq \lambda$, the operator $Z^\alpha - \alpha I$ is triangle. Hence it has inverse by Theorem 3.3.2. The inverse of the operator $Z^\alpha - \alpha I$ is discontinuous. Therefore, $Z^\alpha - \alpha I \in 2$. By Theorem 3.3.3, $(Z^\alpha)^* - \alpha I$ is not one to one. By Lemma 3 $Z^\alpha - \alpha I$ does not have a dense range. Therefore, $Z^\alpha - \alpha I \in III$. □

CHAPTER 4

ON THE FINE SPECTRUM OF THE GENERALIZED DIFFERENCE OPERATOR $B(r_k, s_k)$

In this chapter, we essentially deal with the fine spectrum of the generalized difference matrix $B(r_k, s_k)$ which is a band matrix over the sequence spaces c_0 and c.

By $\mathcal{C}$ and $\mathcal{SD}$,throughout the text we denote the sets of all constant and strictly decreasing sequences of real numbers.

4.1. introduction

Let $r = (r_k)$ and $s = (s_k)$ be either constant or strictly decreasing sequence of positive real numbers satisfying the following conditions:

$$\begin{aligned} \lim_{k\to\infty} r_k &= L_1 > 0 \text{ and } \lim_{k\to\infty} s_k = L_2 > 0, \\ \sup_{k\in\mathbb{N}} r_k &\leq 2L_1 \text{ and } \sup_{k\in\mathbb{N}} s_k \leq 2L_2. \end{aligned}$$

Define the generalized difference matrix $B(r_k, s_k)$ via the sequences $r = (r_k)$ and $s = (s_k)$ by

$$B(r_k, s_k) = \begin{pmatrix} r_0 & 0 & 0 & \cdots \\ s_0 & r_1 & 0 & \cdots \\ 0 & s_1 & r_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Corollary 4.1.1. *$B(r_k, s_k) : c_0 \to c_0$ is a bounded linear operator and* $\|B(r_k, s_k)\| = 2(\sup_{k\in\mathbb{N}} r_k + \sup_{k\in\mathbb{N}} s_k)$

4.2. The Fine Spectrum of the Operator $B(r_k, s_k)$ on the Sequence Space c_0 and c

Theorem 4.2.1. $\sigma(B(r_k, s_k), c_0) = \left\{\alpha \in \mathbb{C} : \left|\frac{L_1-\alpha}{L_2}\right| \leq 1\right\}$.

Proof. We firstly prove that

$$\sigma(B(r_k, s_k), c_0) \subseteq \left\{\alpha \in \mathbb{C} : \left|\frac{L_1-\alpha}{L_2}\right| \leq 1\right\}, \tag{4.1}$$

which is equivalent to show that $\alpha \in \mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right| > 1$ implies $\alpha \notin \sigma(B(r_k, s_k), c_0)$. Secondly we prove that

$$\left\{\alpha \in \mathbb{C} : \left|\frac{L_1-\alpha}{L_2}\right| \leq 1\right\} \subseteq \sigma(B(r_k, s_k), c_0). \tag{4.2}$$

We get $B(r_k, s_k) - \alpha I = (a_{nk})$ is triangle and has an inverse. Thus,by solving the matrix equation

$$\begin{aligned}
B(r_k, s_k)x &= \begin{bmatrix} r_0-\alpha & 0 & 0 & \dots \\ s_0 & r_1-\alpha & 0 & \dots \\ 0 & s_1 & r_2-\alpha & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\
&= \begin{bmatrix} (r_0-\alpha)x_0 \\ s_0x_0 + (r_1-\alpha)x_1 \\ x_1s_1 + (r_2-\alpha)x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
x_0 &= \frac{y_0}{r_0-\alpha}, \\
x_1 &= \frac{y_1}{r_1-\alpha} + \frac{s_0y_0}{(r_1-\alpha)(r_0-\alpha)}, \\
x_2 &= \frac{y_2}{r_2-\alpha} + \frac{s_1y_1}{(r_2-\alpha)(r_1-\alpha)} + \frac{s_0s_1y_0}{(r_2-\alpha)(r_1-\alpha)(r_0-\alpha)}, \\
&\vdots \\
x_n &= \frac{y_n}{r_n-\alpha} + \frac{s_{n-1}y_{n-1}}{(r_n-\alpha)(r_{n-1}-\alpha)} + \frac{s_0s_1s_2\dots s_{n-1}y_0}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\dots(r_n-\alpha)}, \\
&\vdots
\end{aligned}
$$

Therefore we obtain $B = (b_{nk}) = (B(r_k, s_k) - \alpha I)^{-1}$ as follows:

$$
(b_{nk}) = \begin{bmatrix} \frac{1}{r_0-\alpha} & 0 & 0 & \dots \\ \frac{-s_0}{(r_1-\alpha)(r_0-\alpha)} & \frac{1}{r_1-\alpha} & 0 & \dots \\ \frac{s_0s_1}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)} & \frac{-s_1}{(r_2-\alpha)(r_1-\alpha)} & \frac{1}{r_2-\alpha} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}
$$

Hence $(B(r_k, s_k) - \alpha I)^{-1} \in (c_0, c_0)$ if and if only

1. The series $\sum_{k=0}^{\infty} |b_{nk}|$ is convergent for each $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} b_{nk} < \infty$;
2. $\lim_{n\to\infty} |b_{nk}| = 0$ for each $k \in \mathbb{N}$.

Now,we will show that the series $\sum_{k=0}^{\infty} b_{nk}$ is convergent for each $n \in \mathbb{N}$. Let

$$
S_n = \sum_{k=0}^{n} |b_{nk}|
$$

$$
S_n = \left|\frac{s_0s_1s_2\dots s_{n-1}}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\dots(r_n-\alpha)}\right| + \left|\frac{-s_{n-1}}{(r_n-\alpha)(r_{n-1}-\alpha)}\right| + \left|\frac{1}{r_n-\alpha}\right|.
$$

Clearly, $\sum_{k=0}^{\infty} b_{nk}$ is convergent for each $n \in \mathbb{N}$. Now, we show that $\sup_{n\in\mathbb{N}}$ is finite let $\theta = \lim_{n\to\infty} \left|\frac{s_{n-1}}{r_n-\alpha}\right|$ θ is continuous. Hence

$$
\theta = \left|\frac{L_1-\alpha}{L_2}\right|. \tag{4.3}
$$

Which shows $0 < \theta < 1$ and gives

$$\lim_{n\to\infty} \frac{1}{|r_n - \alpha|} = \lim_{n\to\infty} \left|\frac{s_{n-1}}{r_n - \alpha}\right| \left|\frac{1}{s_{n-1}}\right| = \frac{\theta}{L_2} \tag{4.4}$$

$$S_n = \left|\frac{s_{n-1}}{r_n - \alpha}\right| S_{n-1} + \left|\frac{1}{r_{n-1}}\right|. \tag{4.5}$$

Letting $n \to \infty$ in (4.5) with lies (4.3) and (4.4) one see that

$$\lim_{n\to\infty} S_n = \frac{\theta}{L_2}\Big(\frac{1}{1-\theta}\Big) < \infty \; so \; \sup_{n\in\mathbb{N}} S_n < \infty$$

since $\theta = \lim_{n\to\infty} \left|\frac{s_{n-1}}{r_n-\alpha}\right| < 1$. Therefore, $\left|\frac{s_{n-1}}{r_n-\alpha}\right| < 1$. Consequently;

$$\lim_{n\to\infty} |b_{n0}| = \left|\frac{s_0 s_1 s_2 \ldots s_{n-1}}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\ldots(r_n-\alpha)}\right| = 0.$$

Similarly we can show that $\lim_{n\to\infty} |b_{nk}| = 0$, for all $k = 1, 2, 3, \ldots$.
Thus;

$$(B(r_k, s_k) - \alpha I)^{-1} \in \; B(c_0) \; for \; \alpha \in \mathbb{C} \; with \; \left|\frac{L_1 - \alpha}{L_2}\right| > 1.$$

Now, we show that the domain at the operator $(B(r_k, s_k) - \alpha I)^{-1}$ is dense in c_0, this statement holds if and if only the range of operator $(B(r_k, s_k) - \alpha I)^{-1}$ is dense in c_0. Since we have $(B(r_k, s_k) - \alpha I)^{-1} \in (c_0, c_0)$. This show that,

$$\sigma(B(r_k, s_k), c_0) \subseteq \left\{\alpha \in \mathbb{C} : \left|\frac{L_1 - \alpha}{L_2}\right| \leq 1\right\}. \tag{4.6}$$

Now, we suppose that $\alpha \neq L_1$ and $\alpha \neq r_k$ for all k and let $\alpha \in \mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right| \leq 1$. Obvious, $B(r_k, s_k) - \alpha I$ is triangle and hence $(B(r_k, s_k) - \alpha I)^{-1}$ exists. So condition (R_2) is satisfied but (R_1) fails as we can see below.

$\alpha \in \mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right| < 1$

$$\lim_{n\to\infty}\left|\frac{s_{n-1}}{r_n-\alpha}\right| > 1.$$

Consequently;

$$\lim_{n\to\infty}|b_{n0}| = \left|\frac{s_0 s_1 s_2 \dots s_{n-1}}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\dots(r_n-\alpha)}\right| \neq 0.$$

Hence $(B(r_k, s_k) - \alpha I)^{-1} \notin \ B(c_0)$ for with $\left|\frac{L_1-\alpha}{L_2}\right| < 1$.
Now we consider $\left|\frac{L_1-\alpha}{L_2}\right| = 1$ so $|L_1-\alpha| = |L_2|$ which implies $|r_k-\alpha| = |s_k|$ for all k. Therefore; $\left|\frac{1}{s_k}\right| \leq \left|\frac{1}{r_k-\alpha}\right|$. We have,

$$S_n = \left|\frac{s_0 s_1 s_2 \dots s_{n-1}}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\dots(r_n-\alpha)}\right| + \left|\frac{-s_{n-1}}{(r_n-\alpha)(r_{n-1}-\alpha)}\right| + \left|\frac{1}{r_n-\alpha}\right| \geq \frac{n+1}{s_n}.$$

So we can write,

$$\sup_{n\in\mathbb{N}} S_n = \sup_{n\in\mathbb{N}}\left[\frac{n+1}{s_n}\right] = \infty.$$

So condition (R_2) fails. Hence $(B(r_k, s_k) - \alpha I)^{-1} \notin \ B(c_0)$ for $\alpha \in \mathbb{C}$ with

$\left|\frac{L_1-\alpha}{L_2}\right| = 1$ if $L_1 = \alpha$ as well as $r_k = \alpha$

$$\begin{aligned}
(B(r_k,s_k)-r_kI)x &= \begin{bmatrix} r_0-r_k & 0 & 0 & \dots \\ s_0 & r_1-r_k & 0 & \dots \\ 0 & s_1 & r_2-r_k & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix} \\
&= \begin{bmatrix} (r_0-r_k)x_0 \\ s_0x_0+(r_1-r_k)x_1 \\ s_1x_1+(r_2-r_k)x_2 \\ \vdots \\ s_{k-2}x_{k-2}+(r_{k-1}-r_k)x_{k-1} \\ s_kx_k+(r_{k+1}-r_k)x_{k+1} \\ \vdots \end{bmatrix}
\end{aligned}$$

if $(r_k) = L_1$ and $(s_k) = L_2$, constant $B(r_k,s_k)x = \theta$ so $x_0 = x_1 = x_2 \ldots = 0$ this one show $(B(r_k,s_k)-r_kI)$ one to one but $R(B(r_k,s_k)-r_kI)$ is not dense in c_0, since the condition (R_3) fails. Hence $L_1 \in \sigma(B(r_k,s_k),c_0)$ so if (r_k) and (s_k) are strictly decreasing sequence

$$B(r_k,s_k)x = \theta$$

$$x_0 = x_1 = x_2 \cdots = x_{k-1} = 0$$

$x_n = \left(\frac{s_n}{r_{n+1}-r_k}\right)x_n$ for all $n \geq k$. This show that $B(r_k,s_k)-r_kI$ not injective. Therefore condition (R_1) fails. So $r_k \in \sigma(B(r_k,s_k),c_0)$ for all $k \in \mathbb{N}$. When $|L_1-\alpha| = |L_2|$, which implies $|r_k-\alpha| = |s_k|$ for all k. Therefore, $\left|\frac{1}{s_k}\right| \leq \left|\frac{1}{r_k-\alpha}\right|$. We have,

$$S_n = \left|\frac{s_0s_1s_2\ldots s_{n-1}}{(r_0-\alpha)(r_1-\alpha)(r_2-\alpha)\ldots(r_n-\alpha)}\right| + \left|\frac{-s_{n-1}}{(r_n-\alpha)(r_{n-1}-\alpha)}\right| + \left|\frac{1}{r_n-\alpha}\right| \geq \frac{n+1}{s_n}.$$

So we can write,

$$\sup_{n\in\mathbb{N}} S_n = \sup_{n\in\mathbb{N}} \left[\frac{n+1}{s_n}\right] = \infty.$$

So,the condition (R_2) fails. Therefore,

$$(B(r_k, s_k) - \alpha I)^{-1} \notin \ B(c_0)$$

for $\alpha = L_1$ so $L_1 \in \sigma(B(r_k, s_k), c_0)$ thus in this case $r_k \in \sigma(B(r_k, s_k), c_0)$ for all $k \in \mathbb{N}$. This shows that

$$\left\{\alpha \in \mathbb{C} : \left|\frac{L_1 - \alpha}{L_2}\right| \leq 1\right\} \subseteq \sigma(B(r_k, s_k), c_0) \tag{4.7}$$

(4.7) and (4.6) from we get,

$$\sigma(B(r_k, s_k), c_0) = \left\{\alpha \in \mathbb{C} : \left|\frac{L_1 - \alpha}{L_2}\right| \leq 1\right\}$$

This complete proof. □

Theorem 4.2.2.

$$\sigma_p(B(r_k, s_k), c_0) = \begin{cases} \emptyset & , \quad (r_k), (s_k) \in \mathcal{C}, \\ \{(r_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}\} & , \quad (r_k), (s_k) \in \mathcal{SD}. \end{cases}$$

Proof. Proof of this theorem is dived two parts. Case1: Assume that (r_k) and (s_k) are the constant sequence. Consider $B((r_k, s_k), c_0)x = \alpha x$ for $x \neq \theta = (0, 0, 0, \ldots)$

in c_0 which gives

$$\begin{aligned} r_0x_0 &= \alpha x_0 \\ s_0x_0 + r_1x_1 &= \alpha x_1 \\ s_1x_1 + r_2x_2 &= \alpha x_2 \\ &\vdots \\ s_{k-1}x_{k-1} + r_kx_k &= \alpha x_k \\ &\vdots \end{aligned}$$

let x_m is the first non zero entry of the sequence $x = (x_n)$. So we get $s_mx_m + r_{m+1}x_{m+1} = \alpha x_{k+1}$ which implies $\alpha = x_m$ and from the equation

$$s_mx_m + r_{m+1}x_{m+1} = \alpha x_{m+1}$$

we get $x_m = 0$ which is contradiction our assumption. Therefore;

$$\sigma_p(B(r_k, s_k), c_0) = \emptyset$$

Case2: Assume (r_k) and (s_k) are strictly decreasing sequence. suppose $\alpha = r_0$ Consider
$B((r_k, s_k), c_0)x = \alpha x$ for $x \neq \theta = (0, 0, 0, \ldots)$ in c_0 which gives

$$x_k = \left(\frac{s_{k-1}}{r_k - r_0}\right)x_{k-1}$$

$$x_k = \left[\frac{s_0s_1s_2\ldots s_{n-1}}{(r_k - r_0)(r_{k-1} - r_0)(r_{k-2} - r_0)\ldots(r_1 - r_0)}\right]x_0 \; for\ all\ k \geq 1.$$

We take $x_0 \neq 0$ get non zero solution of the equation $B((r_k, s_k) - r_0I)x = \theta$ similarly if $r_k = \alpha$ for all $k \geq 1$. Then, $x_{k-1} = 0, x_{k-2} = 0, \ldots, x_0 = 0$ and

$$x_{n+1} = \left(\frac{s_n}{r_{n+1} - r_k}\right)x_n$$

$$x_{n+1} = \left[\frac{s_n s_{n-1} s_{n-2} \dots s_k}{(r_{n+1} - r_k)(r_n - r_k)(r_{n-1} - r_k) \dots (r_{k+1} - r_k)}\right] x_k \quad \text{for all} \quad n \geq k.$$

if we take $x_k \neq 0$, then, get non zero solution of the equation $B((r_k, s_k) - r_0 I)x = \theta$. Thus, $s_k = \alpha$. If we take $s_0 = \alpha$ and similarly if we take $s_k = \alpha$, we can find a sequence (s_k). So; $\sigma_p(B(r_k, s_k), c_0) = \{s_0, s_1, s_2 \cdots\}$

$$\sigma_p(B(r_k, s_k), c_0) = \{(r_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}\}$$

This complete the proof. □

Theorem 4.2.3. $\sigma_p(B(r_k, s_k)^*, c_0^*) = \left\{\alpha \in \mathbb{C} : \left|\frac{L_1 - \alpha}{L_2}\right| < 1\right\}$.

Proof. Suppose $(B(r_k, s_k))^* = \alpha f$ for $\theta \neq f \in c_0 \cong \ell_1$. Then since

$$(B(r_k, s_k))^* = \begin{bmatrix} r_0 & s_0 & 0 \dots \\ 0 & r_1 & s_1 \dots \\ 0 & 0 & r_2 \dots \\ \vdots & \vdots & \vdots \ddots \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} f_0 r_0 + s_0 f_1 \\ f_1 r_1 + s_1 f_2 \\ f_2 r_2 + s_2 f_3 \\ \vdots \end{bmatrix} = \alpha \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots, \end{bmatrix}$$

we have

$$\begin{aligned} r_0 f_0 + s_0 f_1 &= \alpha f_0 \\ r_1 f_1 + s_1 f_2 &= \alpha f_1 \\ r_2 f_2 + s_2 f_3 &= \alpha f_2 \\ &\vdots \\ r_{k-1} f_{k-1} + s_{k-1} f_k &= \alpha f_k \\ &\vdots \end{aligned}$$

This gives $f_k = \left(\frac{\alpha - r_k}{s_{k-1}}\right) f_{k-1}$ for all $k \geq 1$. Then ,$(f_k) \in \ell_1$ if and if only if $\left|\frac{\alpha - r_k}{s_{k-1}}\right| < 1$ which implies $\left|\frac{\alpha - L_1}{L_2}\right| < 1$. This complete the proof. □

Theorem 4.2.4.

$$\sigma_r(B(r_k,s_k),c_0)=\begin{cases}\left\{\alpha\in\mathbb{C}:\left|\frac{L_1-\alpha}{L_2}\right|<1\right\} & , \quad (r_k),(s_k)\in\mathcal{C},\\ \left\{\alpha\in\mathbb{C}:\left|\frac{L_1-\alpha}{L_2}\right|<1\right\}\backslash\{(r_k),(s_k)\} & , \quad (r_k),(s_k)\in\mathcal{SD}.\end{cases}$$

Proof. We prove the theorem by divided it into two parts.

Case1: $(r_k),(s_k)\in\mathcal{C}$ for $\alpha\in\mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right|<1$ the operator $B(r_k,s_k)-\alpha I$ is a triangle except $\alpha=L_1$ and consequently $B(r_k,s_k)-\alpha I$ has an inverse. By Theorem 4.2.2 the operator $B(r_k,s_k)-\alpha I$ is one to one for $\alpha=L_1$. So it has a inverse. By Theorem 4.2.3 the operator $(B(r_k,s_k)-\alpha I))^*=(B(r_k,s_k))^*-\alpha I$ is not one to one with $\alpha\in\mathbb{C}$ such that $\left|\frac{L_1-\alpha}{L_2}\right|<1$. Hence by Lemma 4 the range of the operator $B(r_k,s_k)-\alpha I$ is not dense in c_0. So,

$$\sigma_r(B(r_k,s_k),c_0)=\left\{\alpha\in\mathbb{C}:\left|\frac{L_1-\alpha}{L_2}\right|<1\right\}$$

Case2: Let (r_k) and (s_k) be the strictly decreasing sequences with $\lim_{k\to\infty}r_k=L_1$ and $\lim_{k\to\infty}s_k=L_2$ for $\alpha\in\mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right|<1$ the operator $B(r_k,s_k)-\alpha I$ is a triangle, except $\alpha=r_k$ and $\alpha=s_k$ for some $k\in\mathbb{N}$. So the operator $B(r_k,s_k)-\alpha I$ has an inverse. By Theorem 4.2.2 the operator $B(r_k,s_k)-\alpha I$ is not one to one for $\alpha=r_k$ and $\alpha=s_k$ for some $k\in\mathbb{N}$. Thus $(B(r_k,s_k)-\alpha I)^{-1}$ does not exist. But by Theorem 4.2.3
$(B(r_k,s_k)-\alpha I))^*=(B(r_k,s_k))^*-\alpha I$ is not one to one with $\alpha\in\mathbb{C}$ such that $\left|\frac{L_1-\alpha}{L_2}\right|<1$. Hence by Lemma 4 the range of the operator $B(r_k,s_k)-\alpha I$ is not dense in c_0. So,

$$\sigma_r(B(r_k,s_k),c_0)=\left\{\alpha\in\mathbb{C}:\left|\frac{L_1-\alpha}{L_2}\right|<1\right\}\backslash\{(r_k),(s_k)\}$$

This complete the proof. $\square$

Theorem 4.2.5.

$$\sigma_c(B(r_k, s_k), c_0) = \begin{cases} \left\{\alpha \in \mathbb{C} : \left|\frac{L_1-\alpha}{L_2}\right| = 1\right\} & , \quad (r_k), (s_k) \in \mathcal{C}, \\ \left\{\alpha \in \mathbb{C} : \left|\frac{L_1-\alpha}{L_2}\right| = 1\right\} \setminus \{r_0\} & , \quad (r_k), (s_k) \in \mathcal{SD}. \end{cases}$$

Proof. We prove that two case;

Case1: $(r_k), (s_k) \in \mathcal{C}$ for $\alpha \in \mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right| < 1$ the operator $B(r_k, s_k) - \alpha I$ is a triangle except $\alpha = L_1$ and consequently $B(r_k, s_k) - \alpha I$ has an inverse. By Theorem 4.2.2 the operator $B(r_k, s_k) - \alpha I$ is one to one for $\alpha = L_1$. So has an inverse. By Theorem 4.2.3 the operator $(B(r_k, s_k) - \alpha I))^* = (B(r_k, s_k))^* - \alpha I$ is not one to one with $\alpha \in \mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right| = 1$. Hence by Lemma 4 the range of the operator $B(r_k, s_k) - \alpha I$ is not dense in c_0. So,

$$\sigma_c(B(r_k, s_k), c_0) = \left\{\alpha \in \mathbb{C} : \left|\frac{L_1 - \alpha}{L_2}\right| = 1\right\}$$

Case2: Let (r_k) and (s_k) strictly decreasing sequence with $\lim_{k\to\infty} r_k = L_1$ and $\lim_{k\to\infty} s_k = L_2$ for $\alpha \in \mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right| = 1$. The operator $B(r_k, s_k) - \alpha I$ is a triangle except r_0. So, the operator $B(r_k, s_k) - \alpha I$ has an inverse. By Theorem 4.2.3 the operator $B(r_k, s_k) - \alpha I$ is not one to one for $\alpha = r_0$. Thus $(B(r_k, s_k) - \alpha I)^{-1}$ does not exist.

Nevertheless by Theorem 4.2.3. $(B(r_k, s_k) - \alpha I))^* = (B(r_k, s_k))^* - \alpha I$ is not one to one with $\alpha \in \mathbb{C}$ with $\left|\frac{L_1-\alpha}{L_2}\right| = 1$. Hence by Lemma 4 the range of the operator $B(r_k, s_k) - \alpha I$is not dense in c_0. So,

$$\sigma_c(B(r_k, s_k), c_0) = \left\{\alpha \in \mathbb{C} : \left|\frac{L_1 - \alpha}{L_2}\right| = 1\right\} \setminus \{r_0\}.$$

This completes the proof. ∎

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