REPUBLIC OF TURKEY

FIRAT UNIVERSITY

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INVERSE NODAL PROBLEM FOR p–LAPLACIAN DIFFERENTIAL OPERATORS

Master Thesis

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January-2017

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Date the dissertation was given to the institue	:	03.01.2017
The date of the thesis defence	:	23.01.2017

Supervisor: Associated Prof. Dr. Emrah YILMAZ (Firat U.)

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JANUARY-2017

ACKNOWLEDGMENTS

This master thesis is completed with the reinforcement of many people. I want to mean my indebtedness to all of them. Firstly, I am very grateful to my superviser, Associated Prof. Dr. Emrah YILMAZ, for his precious guiding, scholarly inputs and consistent exhortation I received throughout the research work. This achievement was feasible only because of the unconditional reinforcement provided by Sir Emrah YILMAZ person with a positive disposition, Sir has always made himself available to elucidate my doubts despite his intensive schedules and I consider it as an excellent chance to do my master thesis under his guiding and to learn from his research speciality. Thank you Sir, for all your contribution and support. I thank Research Assistant Dr. Tuba GULSEN, for her academic reinforcement, and I also express my indebtedness to her.

Beyond everything, I owe it all to almighty Allah for granting me the intelligence, health and energy to undertake this thesis and enabling me to its accomplishment.

MUDHAFAR HAMED HAMADAMEN

ELAZIG-2017

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SUMMARY

Inverse nodal problem for $p-$Laplacian differential operators

In this thesis, we have studied on $p-$Laplacian differential operators. The thesis contains five main chapters. In chapter 1, historical improvement of inverse spectral theory and $p-$Laplacian differential operators is given. Then, the fundamental definitions and theorems related to our thesis and some properties of generalized trigonometric functions are expressed in chapter 2. In chapter 3, $p-$Laplacian diffusion equation with boundary conditions which depends on spectral parameter polynomially is considered. By using generalized Prüfer substitution, asymptotic formulas of eigenparameters and reconstruction formulas for potential functions are obtained. In chapter 4, $p-$Laplacian Bessel equation where the boundary conditions have polynomially dependent spectral parameter is considered. Later, inverse nodal problem for this equation is solved by using generalized Prüfer substitution. Finally, some conclusions about thesis are expressed in chapter 5.

Key Words: Inverse Nodal problem, $p-$Laplacian Diffusion equation, $p-$Laplacian Bessel equation, Prüfer Substitution.

ÖZET

$p-$Laplacian diferensiyel operatörler için ters nodal problem

Bu tezde, $p-$Laplacian diferensiyel operatörleri çalışıldı. Tez beş temel kısımdan oluşmaktadır. Birinci bölümde, ters spektral teori ve $p-$Laplacian diferensiyel operatörlerinin tarihçesi verildi. Daha sonra, ikinci bölümde tez ile ilgili temel tanım ve teoremler ile genelleştirlmiş trigonometrik fonksiyonların bazı özellikleri ifade edildi. Üçüncü bölümde, sınır koşullarında polinom şeklindeki spektral parametreye sahip $p-$Laplacian difüzyon denklemi ele alındı. Genelleştirilmiş Prüfer dönüşümü kullanılarak, özparametreler için asimptotik formüller ve potansiyel fonksiyon için yapılandırma formülleri elde edildi. Dördüncü bölümde, sınır koşullarında polinom şeklindeki spektral parametreye sahip $p-$Laplacian bessel denklemi ele alındı. Daha sonra, bu denklem için ters nodal problem genelleştirilmiş Prüfer dönüşümü kullanılarak çözüldü. Son olarak beşinci bölümde tezle ilgili bazı sonuçlar ifade edildi.

Anahtar Kelimeler: Ters Nodal Problem, $p-$Laplacian Difüzyon denklemi, $p-$Laplacian Bessel Denklemi, Prüfer Dönüşümü.

SYMBOLS

$\mathbb{N}$	:	The set of Natural Numbers
$\mathbb{R}$	:	The set of Real Numbers
S_p	:	Generalized Sine Function
T_p	:	Generalized Tangent Function
π_p	:	Generalized Pi Number
$L^2(a,b)$	:	The space of all functions which are square integrable on (a,b)
$L^1(a,b)$	:	The space of all functions which are integrable on (a,b)
$W_2^1(0,1)$	:	Sobolev Space
q	:	Potential function
λ_n	:	n.th eigenvalue
x_j^n	:	j.th nodal point coressponding to n.th eigenvalue
l_j^n	:	j.th nodal length coressponding to n.th eigenvalue
O	:	Big oh
o	:	Small oh

1 INTRODUCTION

In mathematics, $p-$Laplacian, or the $p-$Laplace operator, is a quasilinear elliptic partial differential operator of second order. It is a nonlinear generalization of the Laplace operator, where p is allowed to range over $1 < p < \infty$. In the special case when $p = 2$, this operator reduces to the usual Laplacian. In general solutions of equations involving the $p-$Laplacian do not have second order derivatives in classical sense, thus solutions to these equations have to be understood as weak solutions.

In the classical theory of the Laplace equation several main parts of mathematics are joined in a fruitful way: calculus of variations, partial differential equations, potential theory, function theory (analytic functions), not to mention mathematical physics and calculus of probability. This is the strength of the classical theory. It is very remarkable that the $p-$Laplace equation occupies a similar position, when it comes to non-linear phenomena. Much of what is valid for the ordinary Laplace equation also holds for the $p-$harmonic equation, except that the principle of superposition is naturally lost. A non-linear potential theory has been created with all its requisites: $p-$superharmonic functions, Perron's method, barriers, Wiener's criterion and so on. In the complex plane a special structure related to quasiconformal mappings appears. Last but not least, the $p-$harmonic operator appears in physics: glacelogy, radiation of heat, plastic moulding etc.

In inverse spectral theory, application of $p-$Laplacian operators is a new and popular subject. There are many important studies about these type operators in literature. For instance, Delpino and Manasevich [2] considered the global bifurcation from the eigenvalues of the $p-$Laplacian in 1991. Fabry and Fayyad [3] studied the periodic solutions of the second order differential equations with a $p-$Laplacian and asymmetric nonlinearities in 1992. Reichel and Walter [4] found radial solutions of equations and inequalities involving the $p-$Laplacian in 1997. Walter [5] explained Sturm-Liouville theory for the radial Δ_p-operator in 1992. In 1999, Reichel and Walter [6] solved Sturm-Liouville type problems in $p-$Laplacian form under asymptotic non-resonanca conditions. Cuesta [7] studied eigenvalue problems for the $p-$Laplacian with indefinite weights in 2001. Binding and Drabek [8] considered Sturm-Liouville theory for $p-$Laplacian in 2003. Bonder and Pinasco [9] examined the asymptotic behaviour of the eigenvalues of the one dimensional weighted $p-$Laplacian operator in 2003. Brown and Reichel [10] studied the eigenvalues of the radially symmetric $p-$Laplacian in 2004. Bonder and Pinasco [11] considered eigenvalues of the $p-$Laplacian in fractal strings with indefinite weights in 2005. Binding and Volkmer [12] constructed Prüfer angle asymptotics for Atkinson's semi-definite Sturm-Liouville eigenvalue problem in 2005 and 2006. Binding, Boulton, Cepicka and Drabek [13] studied basic properties of eigenfunctions of the $p-$Laplacian in 2006. Pinasco [14] made a comparison of eigenvalues for the $p-$Laplacian with integral inequalities in 2006. Brown and Eastham [15] studied eigenvalues of the radial $p-$Laplacian with a potential on $(0, \infty)$ in 2007. Binding and Rynne [16] examined variational and non-variational eigenvalues of the $p-$Laplacian in 2008. Law, Lian and Wang [17] solved inverse nodal problem and the Ambarzumyan problem for the $p-$Laplacian in 2009. Wang [18] studied direct and inverse problems for one dimensional $p-$Laplacian operators in 2010. Wang, Cheng and Lian [19] solved inverse nodal problems for the $p-$Laplacian with eigenparameter dependent boundary conditions in 2011. Chen, Cheng and Law [20] studied Tikhanov regularization for the inverse nodal problem for $p-$Laplacian in 2012. Chen, Law, Lian and Wang [21] obtained some results about optimal upper bounds for the eigenvalue ratios of one dimensional $p-$Laplacian in 2013. Binding and Volkmer [22] gave a new approach about a Prüfer angle to semi definite Sturm-Liouville problem with coupling boundary conditions in 2013. Gulsen and Yilmaz [23] solved inverse nodal problem for $p-$Laplacian diffusion equation with polynomially dependent spectral parameter in 2016. Gulsen, Yilmaz and Koyunbakan [24] studied inverse nodal problem for $p-$Laplacian Dirac system in 2016. Law, Wang and Chuan Wang [25] examined Sturm-Liouville equation for Atkinson's semi definite $p-$Laplacian eigenvalue problems in 2016. There are many other studies in literature about $p-$Laplacian eigenvalue problems.

Now, let us say something about the difference of our work from other studies. As much as we know, there is not any study about $p-$ Laplacian Bessel equation with the boundary conditions which depends on spectral parameter polynomially. By cahnging conditions, we get new and original results about asymptotic expansions of nodal parameters and potential functions for $p-$Laplacian Bessel equation.

2 PRELIMINARIES

2.1 Some Basic Definitions and Theorems

In this section, we give some important concepts that we need througout this thesis.

Definition 2.1.1: (Big Oh Notation) [26] Let f and g be two functions defined on a subset of real numbers. Then, $f = O(g)$ as $x \to \infty$ if and only if there exists the numbers x_0 and M which satisfy

$$|f(x)| \leq M\,|g(x)|\,,$$

where $x > x_0$.

Definition 2.1.2: (Small Oh Notation) [26] Let f and g be two functions defined on a subset of real numbers. Then, $f(x) = o(g(x))$ if and only if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Definition 2.1.3 (Hilbert space) [27] A complete inner product space is called Hilbert sapce.

Definition 2.1.4 ($L^2\left(\Omega\right)$ Hilbert Space) [27]The space of all square integrable complex valued functions on Ω is called $L^2\left(\Omega\right)$ space. The inner product on $L^2\left(\Omega\right)$ is defined by

$$< f, g >= \int\limits_{\Omega} f(x)\overline{g(x)}dx,$$

where $f, g \in L^2\left(\Omega\right)$. The spectral theory of differential operators are studied on $L^2\left(\Omega\right)$.

Definition 2.1.6.(Sobolev Space) [28] Sobolov space was introduced by Russian mathematician Sergei Lvovich Sobolev in 1930 to examine elliptic differential equations. Suppose that Ω is a bounded domain on $\mathbb{R}^n$. Then

$$W^{1,2}\left(\Omega\right) = \left\{u : u \in L^2\left(\Omega\right) \text{ and } \frac{\partial u}{\partial x_i} \in L^2\left(\Omega\right)\right\},$$

is called sobolev space. We have an inclusion relation $W^{1,2}\left(\Omega\right) \subset L^2\left(\Omega\right)$. The inner product on the sobolev space $W^{1,2}\left(\Omega\right)$ is defined by

$$< f, g >= \int\limits_{\Omega} fg dx + \int\limits_{\Omega} \operatorname{grad} f \operatorname{grad} g dx,$$

where $f, g \in W^{1,2}\left(\Omega\right)$.

2.2 Generalized Trigonometric Functions

In this section, we will give some information about generalized trigonometric functions which are useful throghout this thesis. Trigonometric functions are one of the most important group of the elemantary functions. Using these functions, we can solve geometric problems, complex analytic problems and also problems which involve Fourier series. All the six trigonometric functions can be defined through the *sine* and *cosine* functions. In fact, one can develop all the properties of the functions using the differential properties

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad \sin(0) = 0, \cos(0) = 1,$$

or using the integral properties

$$x = \int_0^{\sin x} \left(1 - t^2\right)^{-\frac{1}{2}} dt, \quad |x| = \int_{\cos x}^{1} \left(1 - t^2\right)^{-\frac{1}{2}} dt.$$

There are some generalizations of these trigonometric functions [1]. They are called the generalized *sine* function $S_p(x)$ and generaized *cosine* function $C_p(x)$, with $p > 1$. They are essentially defined by their integral equations

$$x = \int_0^{S_p(x)} \left(1 - |t|^p\right)^{-\frac{1}{p}} dt, \quad |x| = \int_{C_p(x)}^{1} \left(1 - t^p\right)^{-\frac{1}{p}} dt. \tag{2.2.1}$$

When $p = 2$, the above formulas give the classical sine function and cosine function. However, the above definitions make sense when $p > 1$. Moreover, most of the trigonometric properties can be derived. In fact, the $S_p(x)$ function satisfies the identity

$$|S_p(x)|^p + \left|S_p'(x)\right|^p = 1, \tag{2.2.2}$$

and the differential equation

$$S_p''(x) = -\left|\frac{S_p(x)}{S_p'(x)}\right|^{p-2} S_p(x). \tag{2.2.3}$$

Note that for the general p, $S_p'(x)$ play a more important role than $C_p(x)$. But when $p = 2$, they are the same. It seems that Elbert [29], [30] first studied the $S_p(x)$ and $C_p(x)$ functions. He used the differential equation (2.2.3) and derived (2.2.1) and (2.2.2). In 1995, Lindqvist conducted a more detailed study about the similar $\sin_p(x)$ and $\cos_p(x)$ functions where $\sin_p(x)$ is used as

$$x = \int_0^{\sin_p(x)} \left(1 - \frac{t^p}{p-1}\right)^{-\frac{1}{p}} dt,$$

so that the related differential equation is

$$\left(\left|S'\right|^{p-2}S'\right)' + |S|^{p-2}S = 0.$$

Lindqvist [31], [32] also defined generalized tangent function $\tan_p(x)$ and discussed the relation between $\cos_p(x)$ and $\cos_p'(x)$. Elbert's definition of $S_p(x)$ and Lindqvist's definition of $\sin_p(x)$ are similar but not the same. We will adopt Elbert's definitions and most of his notations [33], [34].

Definition 2.2.1. (Generalized Pi number) [1] Generalized Pi number was defined by David Shelupsky as

$$\pi_p = 2\int_0^1 \frac{1}{\sqrt[p]{(1-s^p)^{p-1}}}ds, \tag{2.2.4}$$

which is shown to be the bounded area enclosed by the graph of $|x|^p + |y|^p = 1$. By taking $p = 2$, we get the classical Pi number as

$$\pi = 2\int_0^1 \frac{1}{\sqrt{(1-s^2)}}ds.$$

π is known as Euclidean number. This is the area of the unit circle enclosed by $x^2 + y^2 = 1$.

Definition 2.2.2 (Generalized Sine Function) [18]The solution of the following initial value problem

$$-\left[\left(S_p'(x)\right)^{(p-1)}\right]' = (p-1)S_p^{(p-1)}(x),$$

$$S_p(0) = 0, \quad S_p'(0) = 1,$$

is called generalized sine function where $f^{(p-1)} = |f|^{p-2}f$.

Theorem 2.2.1. [1] *For the fix number $p > 1$, the following statements are equivalent.*

a) *For any $x \in \left[-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right]$,*

$$x = \int_0^{S_p(x)} (1-|t|^p)^{-\frac{1}{p}}dt,$$

and for any $x \in \mathbb{R}$,

$$S_p(x + k\pi_p) = (-1)^k S_p(x).$$

b) $S_p(0) = 0, S_p'(0) = 1,$ *and*

$$|S_p(x)|^p + \left|S_p'(x)\right|^p = 1.$$

Theorem 2.2.2. [1] *For all $x \in \mathbb{R}$, it is known that*

$$S_p(-x) = -S_p(x).$$

Proof: We know that

$$x = \int_0^{S_p(x)} (1-|t|^p)^{-\frac{1}{p}}\, dt \tag{2.2.5}$$

for $x \in \left[-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right]$ by Theorem 2.2.1. By using the substitution $t = -r$ in 2.2.5., we get

$$x = \int_0^{S_p(x)} (1-|t|^p)^{-\frac{1}{p}}\, dt = x = -\int_0^{-S_p(x)} (1-|r|^p)^{-\frac{1}{p}}\, dr.$$

On the other hand, one can easily write the following equality from 2.2.5.

$$-x = \int_0^{S_p(-x)} (1-|t|^p)^{-\frac{1}{p}}\, dt \tag{2.2.6}$$

So, we finally conclude that

$$S_p(-x) = -S_p(x)$$

by considering 2.2.5 and 2.2.6 together. This completes the proof.

Theorem 2.2.3. [1] *For any $x \in \mathbb{R}, x \neq \left(k + \frac{\pi_p}{2}\right)$ for any $k \in \mathbb{Z}$,*

a)

$$S_p'' = -\left|\frac{S_p}{S_p'}\right|^{p-2} S_p$$

b)

$$\left(S_p S_p'^{(p-1)}\right)' = \left|S_p'\right|^p - (p-1)\left|S_p\right|^p = 1 - p\left|S_p\right|^p = 1 - p + p\left|S_p'\right|^p$$

c)

$$\left(\left|S_p'\right|^p - \left|S_p\right|^p\right)' = -2pS_p^{(p-1)}S_p'$$

Proof:

a) Let us consider the differential equation $-\left[\left(S_p'(x)\right)^{(p-1)}\right]' = (p-1)S_p^{(p-1)}(x)$. It is easy to get that

$$-(p-1)\left(S_p'(x)\right)^{p-2} S_p'' = (p-1)S_p^{p-1}(x)$$

and,

$$-\left(S_p'\right)^{p-2} S_p'' = S_p^{p-2}(x) S_p \Rightarrow S_p'' = -\left|\frac{S_p}{S_p'}\right|^{p-2} S_p.$$

It completes the proof.

b) By using the multiplication rule in derivative $\left(S_pS_p'^{(p-1)}\right)'$, we get

$$\left(S_pS_p'^{(p-1)}\right)' = S_p'S_p'^{(p-1)} + S_p\left(S_p'^{(p-1)}\right)'. \tag{2.2.7}$$

Then by using the relation

$$-\left[\left(S_p'(x)\right)^{(p-1)}\right]' = (p-1)S_p^{(p-1)}(x),$$

in 2.2.7., one can get

$$\begin{aligned}\left(S_pS_p'^{(p-1)}\right)' &= S_p'S_p'^{(p-1)} + S_p\left(S_p'^{(p-1)}\right)' \\ &= S_p'S_p'^{(p-1)} - S_p\left[(p-1)S_p^{(p-1)}(x)\right] \\ &= \left(S_p'\right)^p - (p-1)S_p^p.\end{aligned}$$

Finally, using the equality $|S_p(x)|^p + \left|S_p'(x)\right|^p = 1$ in the last relation, we get

$$\left(S_pS_p'^{(p-1)}\right)' = 1 - p\,|S_p(x)|^p = (1-p) + p\left|S_p'(x)\right|^p.$$

c) By considering $S_p'' = -\left|\frac{S_p}{S_p'}\right|^{p-2} S_p$ in a), we obtain

$$\begin{aligned}\left(\left|S_p'\right|^p - |S_p|^p\right)' &= -pS_p'^{(p-1)}\frac{S_p^{(p-1)}}{\left|S_p'^{(p-2)}\right|} - pS_p^{(p-1)}S_p' \\ &= -pS_p^{(p-1)}S_p' - pS_p^{(p-1)}S_p' \\ &= -2pS_p^{(p-1)}S_p'.\end{aligned}$$

Definition 2.2.3. (Generalized tangent function) [1] Using $S_p(x)$ and $S_p'(x)$, the generalized tangent function $T_p(x)$ can be defined as

$$T_p(x) = \frac{S_p(x)}{S_p'(x)}, \text{ for } x \neq \left(k + \frac{1}{2}\right)\pi_p,$$

where

$$T_p'(x) = 1 + |T_p(x)|^p, \text{ for } x \neq \left(k + \frac{1}{2}\right)\pi_p.$$

There are many other studies about generalized trigonometric functions in literature.

3 INVERSE NODAL PROBLEM FOR $p-$LAPLACIAN DIFFUSION EQUATION WITH POLYNOMIALLY DEPENDENT SPECTRAL PARAMETER

Let us consider following $p-$Laplacian diffusion eigenvalue problem [35]

$$-\left(y'^{(p-1)}\right)' = (p-1)\left(\lambda^2 - q_m(x) - 2\lambda r_m(x)\right) y^{(p-1)},\ 0 \le x \le 1, \tag{3.1}$$

with the boundary conditions

$$y(0) = 0, y'(0) = 1,$$
$$y'(1,\lambda) + f(\lambda) y(1,\lambda) = 0, \tag{3.2}$$

where $p > 1$ is a constant, [36]

$$f(\lambda) = a_1\lambda + a_2\lambda^2 + ... + a_m\lambda^m, a_i \in \mathbb{R}, a_m \ne 0, m \in \mathbb{Z}^+, \tag{3.3}$$

λ is a spectral parameter and $y^{(p-1)} = |y|^{(p-2)}\, y$. Throughout this study, we suppose that $q_m(x) \in L^2(0,1)$ and $r_m(x) \in W_2^1(0,1)$ are real-valued functions defined in the interval $0 \le x \le 1$ for all $m \in \mathbb{Z}^+$. Equation (3.1) becomes following well-known diffusion equation (or quadratic pencil)

$$-y'' + [q_m + 2\lambda r_m]\, y = \lambda^2 y, \tag{3.4}$$

for $p = 2$. Equation (3.4) is extremely important for both classical and quantum mechanics. For instance, such problems arise in solving the Klein-Gordon equations, which describe the motion of massless particles such as photons. Diffusion equations are also used for modelling vibrations of mechanical systems in viscous media (see [37]). In this type of problems, the spectral parameter λ is related to the energy of the system, and this motivates the terminology 'energy-dependent' used for the spectral problem of the form (3.4). Inverse problems for quadratic pencil have been studied by numerous authors (see [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54]).

Inverse spectral problem consists in recovering differential equation from its spectral parameters like eigenvalues, norming constants and nodal points (zeros of eigenfunctions). These type problems are divided into two parts as inverse eigenvalue problem and inverse nodal problem. They play important role and also have many applications in applied mathematics. Inverse nodal problem has been firstly studied by McLaughlin in 1988. She showed that the knowledge of a dense subset of nodal points is sufficient to determine the potential function of Sturm-Liouville problem up to a constant [55]. Also, some numerical results about this problem were given in [56]. Nowadays, many authors have given some interesting results about inverse nodal problems for different type operators (see [57], [58], [59], [60], [61]).

In this study, we concern ourselves with the inverse nodal problem for $p-$Laplacian diffusion equation with boundary condition polynomially dependent on spectral parameter. This problem was solved by [55]. Furthermore, we give asymptotics of eigenparameters and reconstructing formula for potential function.

The zero set $X_n = \left\{x_{j,m}^n\right\}_{j=1}^{n-1}$ of the eigenfunction $y_{n,m}(x,\lambda)$ corresponding to $\lambda_{n,m}$ is called the set of nodal points where $0 = x_{0,m}^{(n)} < x_{1,m}^{(n)} < ... < x_{n-1,m}^{(n)} < x_{n,m}^{(n)} = 1$ for all $m \in \mathbb{Z}^+$. And, $l_{j,m}^n = x_{j+1,m}^n - x_{j,m}^n$ is referred to the nodal length of $y_{n,m}$. The eigenfunction $y_{n,m}(x)$ has exactly $n-1$ nodal points on $(0,1)$.

This chapter is organized as follows: In 3.1, we give some asymptotic formulas for eigenvalues and nodal parameters for $p-$Laplacian diffusion eigenvalue problem (3.1)-(3.2) with polynomially dependent spectral parameter by using modified Prüfer substitution. In 3.2, we give a reconstruction formula of the potential functions for the problem (3.1)-(3.2).

3.1 Asymptotics of Some Eigenparameters for Diffusion equation

Here, we present some important results for the problem (3.1)-(3.2). To do this, we need to consider modified Prüfer's transformation which is one of the most powerful method for solution of inverse problem. Recalling that Prüfer's transformation for a nonzero solution y of (3.1) takes the form

$$\begin{aligned} y(x) &= R(x)S_p\left(\lambda^{2/p}\ \theta(x,\lambda)\right), \\ y'(x) &= \lambda^{2/p}R(x)S_p'\left(\lambda^{2/p}\ \theta(x,\lambda)\right), \end{aligned} \tag{3.1.1}$$

or

$$\frac{y'(x)}{y(x)} = \lambda^{2/p}\frac{S_p'\left(\lambda^{2/p}\ \theta(x,\lambda)\right)}{S_p\left(\lambda^{2/p}\ \theta(x,\lambda)\right)}, \tag{3.1.2}$$

where $R(x)$ is amplitude and $\theta(x)$ is Prüfer variable [62]. After some straightforward computations, one can get easily following relation [63]

$$\theta'(x,\lambda) = 1 - \frac{q_m(x)}{\lambda^2}S_p^p\left(\lambda^{2/p}\ \theta(x,\lambda)\right) - \frac{2}{\lambda}r_m(x)S_p^p\left(\lambda^{2/p}\ \theta(x,\lambda)\right). \tag{3.1.3}$$

Lemma 3.1. [19] *Define $\theta(x,\lambda_{n,m})$ as in (3.1.1) and $\phi_n(x) = S_p^p\left(\lambda_{n,m}^{2/p}\,\theta(x,\lambda_{n,m})\right) - \frac{1}{p}$. Then, for any $g \in L^1(0,1)$*

$$\int_0^1 \phi_n(x)g(x)dx = 0.$$

Theorem 3.1. [63] *The eigenvalues $\lambda_{n,m}$ of the $p-$ Laplacian diffusion eigenvalue problem given in (3.1)-(3.2) have the form*

$$\lambda_{n,1}^{2/p} = n\hat{\pi} - \frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p-2}{2}}} + \frac{1}{p\left(n\hat{\pi}\right)^{p-1}}\int_0^1 q_1(x)dx + \frac{2}{p\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}\int_0^1 r_1(x)dx + O\left(\frac{1}{n^{p-2}}\right), \tag{3.1.4}$$

$$\lambda_{n,2}^{2/p} = n\hat{\pi} - \frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p-2}{2}} + a_2\left(n\hat{\pi}\right)^{p-1}} + \frac{1}{p\left(n\hat{\pi}\right)^{p-1}}\int_0^1 q_2(x)dx + \frac{2}{p\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}\int_0^1 r_2(x)dx + O\left(\frac{1}{n^{p-1}}\right), \tag{3.1.5}$$

$$\lambda_{n,m}^{2/p} = n\hat{\pi} - \frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p-2}{2}} + \ldots + a_m\left(n\hat{\pi}\right)^{\frac{mp-2}{2}}} + \frac{1}{p\left(n\hat{\pi}\right)^{p-1}}\int_0^1 q_m(x)dx + \frac{2}{p\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}\int_0^1 r_m(x)dx + O\left(\frac{1}{n^{p-1}}\right), \tag{3.1.6}$$

for $m=1$, $m=2$ and $m \geq 3$, respectively as $n \to \infty$.

Proof: Let $\theta(0,\lambda) = 0$ for the problem (3.1)-(3.2). Integrating both sides of (3.1.3) with respect to x from 0 to 1, we get

$$\theta(1,\lambda) = 1 - \frac{1}{\lambda^2}\int_0^1 q_m(x)S_p^p\left(\lambda^{2/p}\,\theta(x,\lambda)\right)dx - \frac{2}{\lambda}\int_0^1 r_m(x)S_p^p\left(\lambda^{2/p}\,\theta(x,\lambda)\right)dx.$$

By lemma 3.1, one can obtain

$$\int_0^1 q_m(x)\left\{S_p^p\left(\lambda_n^{2/p}\,\theta(x,\lambda)\right) - \frac{1}{p}\right\}dx = o(1), \text{ as } n \to \infty.$$

Hence, we get

$$\lambda^{2/p}\theta(1,\lambda) = \lambda^{2/p} - \frac{1}{p\lambda^{2-\frac{2}{p}}}\int_0^1 q_m(x)dx - \frac{2}{p\lambda^{1-\frac{2}{p}}}\int_0^1 r_m(x)dx + O\left(\frac{1}{\lambda^{2-\frac{2}{p}}}\right). \tag{3.1.7}$$

Let $\lambda_{n,m}$ be an eigenvalue of the problem (3.1)-(3.2) for all m. Now, we will prove the theorem for $m=1$. By (3.1.2), we have

$$\lambda_{n,1}^{2/p}R(1)S_p'\left(\lambda_{n,1}^{2/p}\,\theta(1,\lambda_{n,1})\right) + a_1\lambda_{n,1}R(1)S_p\left(\lambda_{n,1}^{2/p}\,\theta(1,\lambda_{n,1})\right) = 0,$$

or

$$-\frac{\lambda_{n,1}^{\frac{2}{p}-1}}{a_1}=\frac{S_p\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right)}{S_p'\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right)}=T_p\left(\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})\right).$$

As n is sufficiently large, it follows

$$\lambda_{n,1}^{2/p}\ \theta(1,\lambda_{n,1})=T_p^{-1}\left(-\frac{\lambda_{n,1}^{\frac{2}{p}-1}}{a_1}\right)=n\hat{\pi}-\frac{\lambda_{n,1}^{\frac{2}{p}-1}}{a_1}+o\left(\lambda_{n,1}^{\frac{4}{p}-2}\right). \tag{3.1.8}$$

By considering (3.1.7) and (3.1.8) together, we get

$$\lambda_{n,1}^{2/p}=n\hat{\pi}-\frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}+\frac{1}{p\left(n\hat{\pi}\right)^{p-1}}\int_0^1 q_1(x)dx+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}\int_0^1 r_1(x)dx+O\left(\frac{1}{n^{p-2}}\right).$$

For the case $m=2$, by using the similar process as in $m=1$, we can easily obtain

$$\lambda_{n,2}^{2/p}R(1)S_p'\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)+\left(a_1\lambda_{n,2}+a_2\lambda_{n,2}^2\right)R(1)S_p\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)=0,$$

or

$$-\frac{\lambda_{n,2}^{\frac{2}{p}}}{a_1\lambda_{n,2}+a_2\lambda_{n,2}^2}=\frac{S_p\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)}{S_p'\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right)}=T_p\left(\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})\right),$$

and

$$\lambda_{n,2}^{2/p}\ \theta(1,\lambda_{n,2})=n\hat{\pi}-\frac{\lambda_{n,2}^{\frac{2}{p}}}{a_1\lambda_{n,2}+a_2\lambda_{n,2}^2}+o\left(\frac{\lambda_{n,2}^{\frac{4}{p}}}{\left(a_1\lambda_{n,2}+a_2\lambda_{n,2}^2\right)^2}\right). \tag{3.1.9}$$

Therefore, we have

$$\lambda_{n,2}^{2/p}=n\hat{\pi}-\frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p-2}{2}}+a_2\left(n\hat{\pi}\right)^{p-1}}+\frac{1}{p\left(n\hat{\pi}\right)^{p-1}}\int_0^1 q_2(x)dx+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}\int_0^1 r_2(x)dx+O\left(\frac{1}{n^{p-1}}\right),$$

by using (3.1.7) and (3.1.9). Finally, let us find the asymptotic expansion of $\lambda_{n,m}$ for $m\geq 3$. Similarly, by using (3.1.2), we have

$$\lambda_{n,m}^{2/p}R(1)S_p'\left(\lambda_{n,m}^{2/p}\ \theta(1,\lambda_{n,m})\right)+\left(a_1\lambda_{n,m}+...+a_m\lambda_{n,m}^m\right)R(1)S_p\left(\lambda_{n,m}^{2/p}\ \theta(1,\lambda_{n,m})\right)=0,$$

or

$$-\frac{\lambda_{n,m}^{\frac{2}{p}}}{a_1\lambda_{n,m}+...+a_m\lambda_{n,m}^m}=\frac{S_p\left(\lambda_{n,m}^{2/p}\ \theta(1,\lambda_{n,m})\right)}{S_p'\left(\lambda_{n,m}^{2/p}\ \theta(1,\lambda_{n,m})\right)}=T_p\left(\lambda_{n,m}^{2/p}\ \theta(1,\lambda_{n,m})\right). \tag{3.1.10}$$

By considering (3.1.7) and (3.1.10) together and using similar procedure, we deduce that

$$\lambda_{n,m}^{2/p}=n\hat{\pi}-\frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p-2}{2}}+...+a_m\left(n\hat{\pi}\right)^{\frac{mp-2}{2}}}+\frac{1}{p\left(n\hat{\pi}\right)^{p-1}}\int_0^1 q_m(x)dx+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p-2}{2}}}\int_0^1 r_m(x)dx+O\left(\frac{1}{n^{p-1}}\right).$$

Theorem 3.2. [63] A*symptotic estimates* of *the nodal points for the problem (3.1)-(3.2) satisfies*

$$x_{j,1}^n = \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}}} + \frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_1(t)dt + \frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_1(t)dt \quad (3.1.11)$$
$$+\frac{1}{(n\hat{\pi})^p}\int_0^{x_{j,1}^n} q_1(t)S_p^p dt + \frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_0^{x_{j,1}^n} r_1(t)S_p^p dt + O\left(\frac{j}{n^p}\right),$$

$$x_{j,2}^n = \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + a_2 n^p \hat{\pi}^p} + \frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_2(t)dt + \frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_2(t)dt \quad (3.1.12)$$
$$+\frac{1}{(n\hat{\pi})^p}\int_0^{x_{j,2}^n} q_2(t)S_p^p dt + \frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_0^{x_{j,2}^n} r_2(t)S_p^p dt + O\left(\frac{j}{n^{p+1}}\right),$$

and

$$x_{j,m}^n = \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + ... + a_m n^{\frac{mp}{2}+1} \hat{\pi}^{\frac{mp}{2}}} + \frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_m(t)dt + \frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_m(t)dt$$
$$+\frac{1}{(n\hat{\pi})^p}\int_0^{x_{j,m}^n} q_m(t)S_p^p dt + \frac{2}{(n\hat{\pi})^{\frac{p}{2}}}\int_0^{x_{j,m}^n} r_m(t)S_p^p dt + O\left(\frac{j}{n^{p+1}}\right), \quad (3.1.13)$$

for $m = 1$, $m = 2$ *and* $m \geq 3$, *respectively as* $n \to \infty$.

Proof: Integrating (3.1.3) from 0 to $x_{j,m}^n$ and letting $\theta(x_{j,m}^n, \lambda) = \dfrac{j\hat{\pi}}{\lambda_{n,m}^{2/p}}$, we have

$$x_{j,m}^n = \frac{j\hat{\pi}}{\lambda_{n,m}^{2/p}} + \frac{1}{\lambda_{n,m}^2}\int_0^{x_{j,m}^n} q_m(t)S_p^p dt + \frac{2}{\lambda_{n,m}}\int_0^{x_{j,m}^n} r_m(t)S_p^p dt. \quad (3.1.14)$$

Now, we will find the asymptotic estimate of nodal points for $m = 1$. From the formula (3.1.4), we deduce

$$\frac{1}{\lambda_{n,1}^{2/p}} = \frac{1}{n\hat{\pi}} - \frac{1}{a_1 (n\hat{\pi})^{\frac{p+2}{2}}} + \frac{1}{p(n\hat{\pi})^{p+1}}\int_0^1 q_1(t)dt + \frac{2}{p(n\hat{\pi})^{\frac{p}{2}+1}}\int_0^1 r_1(t)dt + O\left(\frac{1}{n^p}\right), \quad (3.1.15)$$

and therefore we obtain the formula (3.1.11) by using (3.1.14) and (3.1.15).

In (3.1.11), if we take $\dfrac{j}{n} \to 1$ as $n \to \infty$, we obtain

$$x_{j,1}^n = \frac{j}{n} - \frac{j}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}}} + \frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_1(t)dt + \frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_1(t)dt \quad (3.1.16)$$
$$+\frac{1}{p(n\hat{\pi})^p}\int_0^1 q_1(t)dt + \frac{2}{p(n\hat{\pi})^{\frac{p}{2}}}\int_0^1 r_1(t)dt + O\left(\frac{1}{n^{\frac{p}{2}+1}}\right).$$

By using (3.1.5), the asymptotic estimate of eigenvalues $1/\lambda_{n,2}^{2/p}$ for $m=2$ is considered as

$$\frac{1}{\lambda_{n,2}^{2/p}}=\frac{1}{n\hat{\pi}}-\frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p+2}{2}}+a_2\left(n\hat{\pi}\right)^{p+1}}+\frac{1}{p\left(n\hat{\pi}\right)^{p+1}}\int_0^1 q_2(t)dt+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p}{2}+1}}\int_0^1 r_2(t)dt+O\left(\frac{1}{n^{p+1}}\right), \tag{3.1.17}$$

and, we conclude the formula (3.1.12) by using (3.1.14) and (3.1.17).

In the formula (3.1.12), if we take $\dfrac{j}{n}\to 1$ as $n\to\infty$, we have

$$\begin{aligned} x_{j,2}^n &= \frac{j}{n}-\frac{j}{a_1 n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}+a_2 n^p\hat{\pi}^p}+\frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_2(t)dt+\frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_2(t)dt \qquad (3.1.18)\\ &\quad +\frac{1}{p\left(n\hat{\pi}\right)^p}\int_0^1 q_2(t)dt+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p}{2}}}\int_0^1 r_2(t)dt+O\left(\frac{1}{n^{\frac{p}{2}+1}}\right). \end{aligned}$$

For $m\geq 3$, from the formula (3.1.6), it can be easily obtain that

$$\frac{1}{\lambda_{n,m}^{2/p}}=\frac{1}{n\hat{\pi}}-\frac{1}{a_1\left(n\hat{\pi}\right)^{\frac{p+2}{2}}+...+a_m\left(n\hat{\pi}\right)^{\frac{mp+2}{2}}}+\frac{1}{p\left(n\hat{\pi}\right)^{p+1}}\int_0^1 q_m(t)dt+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p}{2}+1}}\int_0^1 r_m(t)dt+O\left(\frac{1}{n^{p+1}}\right), \tag{3.1.19}$$

and we get the formula (3.1.13) by using (3.1.14) and (3.1.19).

In (3.1.13), if we take $\dfrac{j}{n}\to 1$ as $n\to\infty$, we obtain

$$\begin{aligned} x_{j,m}^n &= \frac{j}{n}-\frac{j}{a_1 n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}+...+a_m n^{\frac{mp}{2}+1}\hat{\pi}^{\frac{mp}{2}}}+\frac{j}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_m(t)dt+\frac{2j}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_m(t)dt\\ &\quad +\frac{1}{p\left(n\hat{\pi}\right)^p}\int_0^1 q_m(t)dt+\frac{2}{p\left(n\hat{\pi}\right)^{\frac{p}{2}}}\int_0^1 r_m(t)dt+O\left(\frac{1}{n^{\frac{p}{2}+1}}\right). \qquad (3.1.20) \end{aligned}$$

Theorem 3.3. [63] *Asymptotic estimate of the nodal lengths for the problem (3.1)-(3.2) satisfies*

$$\begin{aligned} l_{j,1}^n &= \frac{1}{n}-\frac{1}{a_1 n^{\frac{p+2}{2}}\hat{\pi}^{\frac{p}{2}}}+\frac{1}{pn^{p+1}\hat{\pi}^p}\int_0^1 q_1(t)dt+\frac{2}{pn^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}}\int_0^1 r_1(t)dt \qquad (3.1.21)\\ &\quad +\frac{1}{\left(n\hat{\pi}\right)^p}\int_{x_{j,1}^n}^{x_{j+1,1}^n} q_1(t)S_p^p dt+\frac{2}{\left(n\hat{\pi}\right)^{\frac{p}{2}}}\int_{x_{j,1}^n}^{x_{j+1,1}^n} r_1(t)S_p^p dt+O\left(\frac{1}{n^p}\right), \end{aligned}$$

$$l_{j,2}^n = \frac{1}{n} - \frac{1}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + a_2 n^{p+1} \hat{\pi}^p} + \frac{1}{p n^{p+1} \hat{\pi}^p} \int_0^1 q_2(t) dt + \frac{2}{p n^{\frac{p}{2}+1} \hat{\pi}^{\frac{p}{2}}} \int_0^1 r_2(t) dt \tag{3.1.22}$$

$$+ \frac{1}{(n\hat{\pi})^p} \int_{x_{j,2}^n}^{x_{j+1,2}^n} q_2(t) S_p^p dt + \frac{2}{(n\hat{\pi})^{\frac{p}{2}}} \int_{x_{j,2}^n}^{x_{j+1,2}^n} r_2(t) S_p^p dt + O\left(\frac{1}{n^{p+1}}\right),$$

and

$$l_{j,m}^n = \frac{1}{n} - \frac{1}{a_1 n^{\frac{p+2}{2}} \hat{\pi}^{\frac{p}{2}} + \ldots + a_m n^{\frac{mp}{2}+1} \hat{\pi}^{\frac{mp}{2}}} + \frac{1}{p n^{p+1} \hat{\pi}^p} \int_0^1 q_m(t) dt + \frac{2}{p n^{\frac{p}{2}+1} \hat{\pi}^{\frac{p}{2}}} \int_0^1 r_m(t) dt$$

$$+ \frac{1}{(n\hat{\pi})^p} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) S_p^p dt + \frac{2}{(n\hat{\pi})^{\frac{p}{2}}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} r_m(t) S_p^p dt + O\left(\frac{1}{n^{p+1}}\right), \tag{3.1.23}$$

for $m = 1$, $m = 2$ *and* $m \geq 3$, *respectively as* $n \to \infty$.

Proof: For large $n \in \mathbb{N}$, integrating (3.1.3) in $[x_{j,m}^n, x_{j+1,m}^n]$ and using the definition of nodal lengths, we have

$$l_{j,m}^n = \frac{\hat{\pi}}{\lambda_{n,m}^{2/p}} + \frac{1}{\lambda_{n,m}^2} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) S_p^p dt + \frac{2}{\lambda_{n,m}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} r_m(t) S_p^p dt, \tag{3.1.24}$$

or

$$l_{j,m}^n = \frac{\hat{\pi}}{\lambda_{n,m}^{2/p}} + \frac{1}{p\lambda_{n,m}^2} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) dt + \frac{2}{p\lambda_{n,m}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} r_m(t) dt + O\left(\frac{1}{n^{\frac{p}{2}+1}}\right).$$

For $m = 1$, $m = 2$ and $m \geq 3$, we can obtain easily (3.1.21), (3.1.22) and (3.1.23) by using the formulas (3.1.15), (3.1.17), (3.1.19), respectively.

3.2 Reconstruction of the potential functions in Diffusion equation

In this section, we give an explicit formula for the potential functions of the diffusion equation (3.1) by using nodal lengths. The method used in the proof of the theorem is similar to classical problems; $p-$Laplacian Sturm-Liouville eigenvalue problem and $p-$Laplacian energy-dependent Sturm-Liouville eigenvalue problem (see [17], [18], [19], [35]).

Theorem 3.4. [63] *Let $q_m(x) \in L^2(0,1)$ and $r_m(x) \in W_2^1(0,1)$ are real-valued functions defined in the interval $0 \le x \le 1$ for all m. Then*

$$q_m(x) = \lim_{n\to\infty} \left(\frac{p\lambda_{n,m}^{\frac{2}{p}+2} l_{j,m}^n}{\hat{\pi}} - p\lambda_{n,m}^2 \right), \tag{3.2.1}$$

and

$$r_m(x) = \lim_{n\to\infty} \left(\frac{p\lambda_{n,m}^{\frac{2}{p}+1} l_{j,m}^n}{2\hat{\pi}} - p\frac{\lambda_{n,m}}{2} \right),$$

for $j = j_{n,m}(x) = \max\left\{ j : x_{j,m}^n < x \right\}$ and $m \in \mathbb{Z}^+$.

Proof: We need to consider Theorem 3.3 for proof. From (3.1.24), we have

$$\frac{p\lambda_{n,m}^{2/p+2}}{\hat{\pi}} l_{j,m}^n = p\lambda_{n,m}^2 + \frac{p\lambda_{n,m}^{2/p}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) S_p^p dt + \frac{2p\lambda_{n,m}^{2/p+1}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} r_m(t) S_p^p dt.$$

Then, we can use similar procedure as those in [17] for $j = j_{n,m}(x) = \max\left\{ j : x_{j,m}^n < x \right\}$ to show

$$\frac{\lambda_{n,m}^{2/p}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) dt \to q_m(x),$$

and

$$\frac{p\lambda_{n,m}^{2/p}}{\hat{\pi}} \int_{x_{j,m}^n}^{x_{j+1,m}^n} q_m(t) \left(S_p^p - \frac{1}{p} \right) dt \to 0,$$

pointwise almost everywhere. Hence, we get

$$q_m(x) = \lim_{n\to\infty} \left(\frac{p\lambda_{n,m}^{\frac{2}{p}+2} l_{j,m}^n}{\hat{\pi}} - p\lambda_{n,m}^2 \right).$$

By using similar way, we can easily get the asymptotic expansion of $r_m(x)$.

Theorem 3.5. [63] *Let $\left\{ l_{j,m}^{(n)} : j = 1, 2, ..., n-1 \right\}_{n=2}^{\infty}$ be a set of nodal lengths of the problem (3.1)-(3.2) where $q_m(x)$ and $r_m(x)$ are real-valued functions on $0 \le x \le 1$* for all m. *Let us define*

$$F_{n,1}(x) = p\,(n\hat{\pi})^p \left(n l_{j,1}^{(n)} - 1 \right) - \frac{p}{a_1} (n\hat{\pi})^{p/2} + \int_0^1 q_1(t) dt + 2\,(n\hat{\pi})^{p/2} \int_0^1 r_1(t) dt, \tag{3.2.2}$$

$$F_{n,2}(x) = p\,(n\hat{\pi})^p \left(n l_{j,2}^{(n)} - 1 \right) - \frac{p\,(n\hat{\pi})^{p/2}}{a_1 + a_2\,(n\hat{\pi})^{p/2}} + \int_0^1 q_2(t) dt + 2\,(n\hat{\pi})^{p/2} \int_0^1 r_2(t) dt, \tag{3.2.3}$$

$$F_{n,m}(x) = p\,(n\hat{\pi})^p \left(nl_{j,m}^{(n)} - 1\right) - \frac{p\,(n\hat{\pi})^{p/2}}{a_1 + ... + a_m\,(n\hat{\pi})^{\frac{mp-p}{2}}} + \int_0^1 q_m(t)dt + 2\,(n\hat{\pi})^{p/2} \int_0^1 r_m(t)dt. \quad (3.2.4)$$

and

$$G_{n,1}(x) = \frac{p\,(n\hat{\pi})^{\frac{p}{2}}}{2} \left(nl_{j,1}^{(n)} - 1\right) - \frac{p}{2a_1} + \frac{1}{2\,(n\hat{\pi})^{p/2}} \int_0^1 q_1(t)dt + \int_0^1 r_1(t)dt, \quad (3.2.5)$$

$$G_{n,2}(x) = \frac{p\,(n\hat{\pi})^{\frac{p}{2}}}{2} \left(nl_{j,2}^{(n)} - 1\right) - \frac{p}{2\left(a_1 + a_2\,(n\hat{\pi})^{\frac{p}{2}}\right)} + \frac{1}{2\,(n\hat{\pi})^{p/2}} \int_0^1 q_2(t)dt + \int_0^1 r_2(t)dt \quad (3.2.6)$$

$$G_{n,m}(x) = \frac{p\,(n\hat{\pi})^{\frac{p}{2}}}{2} \left(nl_{j,m}^{(n)} - 1\right) - \frac{p}{2\left(a_1 + ... + a_m\,(n\hat{\pi})^{\frac{mp-p}{2}}\right)} + \frac{1}{2\,(n\hat{\pi})^{\frac{p}{2}}} \int_0^1 q_m(t)dt + \int_0^1 r_m(t)dt \quad (3.2.7)$$

for $m = 1$, $m = 2$ *and* $m \geq 3$, *respectively. Then,* $\{F_{n,m}(x)\}$ *and* $\{G_{n,m}(x)\}$ *converge to* q_m *and* r_m *pointwise almost everywhere in* $L^1(0,1)$, *respectively, for all cases.*

Proof: We will prove this theorem only for $F_{n,1}$. Other cases can be shown similarly. By the asymptotic formulas of eigenvalues (3.1.4) and nodal lengths (3.1.21), we get

$$\begin{aligned} p\lambda_{n,1}^2 \left(\frac{\lambda_{n,1}^{2/p} l_{j,1}^n}{\hat{\pi}} - 1 \right) &= p\lambda_{n,1}^2 \left(nl_{j,1}^{(n)} - 1\right) - \frac{p}{a_1 \pi} (n\hat{\pi})^{p/2+1}\, l_{j,1}^{(n)} + nl_{j,1}^{(n)} \int_0^1 q_1(t)dt \\ &\quad + 2n\,(n\hat{\pi})^{p/2}\, l_{j,1}^{(n)} \int_0^1 r_1(t)dt + o(1). \end{aligned}$$

Considering $nl_{j,1}^{(n)} = 1 + o(1)$, as $n \to \infty$, we have

$$p\,(n\hat{\pi})^p \left(nl_{j,1}^{(n)} - 1\right) - \frac{p}{a_1} (n\hat{\pi})^{p/2} \to q_1(x) - \int_0^1 q_1(t)dt - 2\,(n\hat{\pi})^{p/2} \int_0^1 r_1(t)dt$$

pointwise almost everywhere in $L^1(0,1)$. By using similar way, it is not difficult to show that $\{G_{n,m}(x)\}$ converges to r_m pointwise almost everywhere in $L^1(0,1)$, respectively, for all $m \in \mathbb{Z}^+$.

4 INVERSE NODAL PROBLEM FOR $p-$LAPLACIAN BESSEL EQUATION WITH POLYNOMIALLY DEPENDENT SPECTRAL PARAMETER

Let us consider following $p-$Laplacian Bessel eigenvalue problem

$$-\left(u'^{(p-1)}\right)' = (p-1)\left(\lambda - w_0(x) - \frac{l(l+1)}{x^2}\right)u^{(p-1)}, x \in [1,a], \tag{4.1}$$

with the boundary conditions

$$u(1) = 0,\ u'(1) = 1,$$
$$u'(a) + f(\lambda)u(a) = 0, \tag{4.2}$$

where $p > 1$ is a constant, [64]

$$f(\lambda) = b_1\sqrt{\lambda} + b_2\left(\sqrt{\lambda}\right)^2 + ... + b_m\left(\sqrt{\lambda}\right)^m, b_i \in \mathbb{R}, i = \overline{1,m}, b_m \neq 0, m \in \mathbb{Z}^+, \tag{4.3}$$

λ is a spectral parameter and $u^{(p-1)} = |u|^{(p-2)}\,u$. During this study, we assume that $w_0(x) \in L^2[1,a]$ is a real-valued function defined on the interval $1 \leq x \leq a$ for all $m \in \mathbb{Z}^+$ and l is a positive integer or zero. $u(x,\lambda)$ denotes the eigenfunction of the problem (4.1)-(4.2). Equation (4.1) becomes following well-known Bessel equation

$$-u'' + w_m(x)u = \lambda u, \tag{4.4}$$

where $w_m(x) = w_0(x) + \frac{l(l+1)}{x^2}$ for $p = 2$ [65]. Equation (4.4) is extremely important in mathematical physics. Let us firstly give some information about physical meaning of Bessel equation and its historical improvement. In case of a wave function with spherical symmetry, the wave equation can be separated using polar coordinates and the equation for the radial component becomes equation (4.4). Physically, λ is proportional to the energy of the particle under consideration, $w_0(x)$ is proportional to the potential energy.

The first studies on the classical Bessel equations were given by Willson, Peirce [66] and Chessin [67]. Chessin obtained some important results about the general solution of the Bessel equation in 1894. Willson and Peirce gave a table of the first forty roots of Bessel equation in 1897. Then, Stashevskaya [68] studied inverse spectral problem for differential operators having singularity at zero in 1950's. For the case $l = 0$ in (4.4), there is a detailed inverse spectral theory in literature. In particular, Pöschel and Trubowitz [69] studied in detail the mapping sending the potential w_0 into the corresponding spectral data, and proved that this mapping is one to one and analytic in 1987. Then, Guillot and Ralston [70] extended the method of them to the case $l = 1$ in 1988. Serier [71] extended this analysis to the case of an arbitrary $l \in \mathbb{N}$. The spectra of the operator $L(l, w_0)$ for an arbitrary $l \in \mathbb{N}$ were also completely characterized in 1993 by Carlson [72]. Zhornitskaya and Serov [73] gave some uniqueness results for Bessel operator in 1994. To develop the ideas of Pöschel and Trubowitz, Carlson [74] proved several results on the unique reconstruction of $L(l, w_0)$ from the spectral data for all real $l \geq -1/2$ in 1997. Gasymov [75] announced the solution of the inverse spectral problem for the Bessel operator $L(l, w_0)$ with $l \in \mathbb{N}$ in his short paper in 1965. His approach was based on the double commutation method which allows the reduction of the inverse problem to the classical case $l = 0$. There are many other studies about Bessel type operators in classical spectral theory by several authors (see [76], [77], [78], [79], [80], [81], [82], [83], [84]).

Inverse spectral problem consists in recovering differential equation from its spectral parameters like eigenvalues, norming constants and nodal points (zeros of eigenfunctions). These type problems are divided into two parts as inverse eigenvalue problem and inverse nodal problem. They play important role and also have many applications in applied mathematics. Inverse nodal problem has been firstly studied by McLaughlin in 1988. She showed that the knowledge of a dense subset of nodal points is sufficient to determine the potential function of Sturm-Liouville problem up to a constant [55]. Also, some numerical results about this problem were given in [56]. Nowadays, many authors have given some interesting results about inverse nodal problems for different type operators (see [57], [58], [59], [60], [61]).

In this chapter, we concern ourselves with the inverse nodal problem for $p-$Laplacian Bessel equation with boundary condition polynomially dependent on spectral parameter. As far as we know, this problem has not been considered before. Furthermore, we give asymptotics of eigenparameters and reconstructing formula for potential function of $p-$ Laplacian Bessel eigenvalue problem.

The zero set $X_n = \left\{x_{j,m}^n\right\}_{j=1}^{n-1}$ of the eigenfunction $y_{n,m}(x, \lambda)$ corresponding to $\lambda_{n,m}$ is called the set of nodal points where $0 = x_{0,m}^{(n)} < x_{1,m}^{(n)} < ... < x_{n-1,m}^{(n)} < x_{n,m}^{(n)} = 1$ for all $m \in \mathbb{Z}^+$. And, $l_{j,m}^n = x_{j+1,m}^n - x_{j,m}^n$ is referred to the nodal length of $u_{n,m}$. The eigenfunction $u_{n,m}(x)$ has exactly $n - 1$ nodal points on $(1, a)$.

This chapter is organized as follows: In 4.1, we give some asymptotic formulas for eigenvalues and nodal parameters for $p-$Laplacian Bessel eigenvalue problem (4.1)-(4.2) with polynomially dependent spectral parameter by using modified Prüfer substitution. In 4.2, we give a reconstruction formula of the potential functions for the problem (4.1)-(4.2).

4.1 Asymptotics of Some Eigenparameters for p-Laplacian Bessel equation

Here, we present some important results for the problem (4.1)-(4.2). To do this, we need to consider modified Prüfer's transformation which is one of the most powerful method for solution of inverse problem. Recalling that Prüfer's transformation for a nonzero solution u of (4.1) takes the form

$$\begin{aligned} u(x) &= c(x)\, S_p\left(\lambda^{\frac{1}{p}}\theta(x,\lambda)\right), \\ u'(x) &= (l+1)\,\lambda^{\frac{1}{p}} c(x)\, S_p'\left(\lambda^{\frac{1}{p}}\theta(x,\lambda)\right), \end{aligned} \tag{4.1.1}$$

or

$$\frac{u'(x)}{u(x)} = (l+1)\,\lambda^{\frac{1}{p}} \frac{S_p'\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}{S_p\left(\lambda^{1/p}\ \theta(x,\lambda)\right)}, \tag{4.1.2}$$

where $c(x)$ is amplitude and $\theta(x,\lambda)$ is Prüfer's variable [62]. After some straightforward computations, one can get easily

$$\theta'(x,\lambda) = l+1+\left\{-(l+1)+(l+1)^{1-p}-\frac{(l+1)^{1-p}}{\lambda}w_m(x)\right\} S_p^p\left(\lambda^{1/p}\theta(x,\lambda)\right). \tag{4.1.3}$$

This equality plays an important role in this chapter.

Lemma 4.1. [19] *Define $\theta(x,\lambda)$ as in (4.1.1) and $\phi_n(x) = S_p^p\left(\lambda^{1/p}\ \theta(x,\lambda)\right) - \frac{1}{p}$. Then, for any $g \in L^1[1,a]$*

$$\int_1^a \phi_n(x) g(x) dx = 0.$$

Theorem 4.1. *The eigenvalues $\lambda_{n,m}$ of the $p-$ Laplacian Bessel eigenvalue problem given in (4.1)-(4.2) have the form*

$$\lambda_{n,1}^{1/p} = \frac{n\hat{\pi}}{\widetilde{l}\,(a-1)} - \frac{(l+1)\,\widetilde{l}^{\,p-3}(a-1)^{p-3}}{b_1\,(n\hat{\pi})^{p-2}} + \frac{(l+1)^{1-p}\,\widetilde{l}^{\,p-2}(a-1)^{p-2}}{p\,(n\hat{\pi})^{p-1}}\int_1^a w_1(r)\,dr + O\left(\frac{1}{n^{p-2}}\right), \tag{4.1.4}$$

$$\begin{aligned} \lambda_{n,m}^{1/p} &= \frac{n\hat{\pi}}{\widetilde{l}\,(a-1)} - \frac{(l+1)\,\widetilde{l}^{\,\frac{mp}{2}-2}(a-1)^{\frac{mp}{2}-2}}{(n\hat{\pi})^{\frac{p}{2}-1}\left(b_1\,\widetilde{l}^{\,\frac{mp-p}{2}}(a-1)^{\frac{mp-p}{2}} + \ldots + b_m\,(n\hat{\pi})^{\frac{mp-p}{2}}\right)} \\ &\quad + \frac{(l+1)^{1-p}\,\widetilde{l}^{\,p-2}(a-1)^{p-2}}{p\,(n\hat{\pi})^{p-1}}\int_1^a w_m(r)\,dr + O\left(\frac{1}{n^{2p-1}}\right) \end{aligned} \tag{4.1.5}$$

where $\widetilde{l} = \frac{(l+1)}{p}\left(p-1+\frac{1}{(l+1)^p}\right)$ for $m=1$ and $m \geq 2$, respectively as $n \to \infty$.

We shall devote most of our attention to $\theta(x,\lambda)$. This function is chosen so that $u(x,\lambda)=0$ if and only if $\theta(x,\lambda)=0(\operatorname{mod}\pi)$.

Proof: Let $\theta(1,\lambda)=0$ for the problem (4.1)-(4.2). Integrating both sides of (4.1.3) with respect to x from 1 to a, we get

$$\begin{aligned}\theta(a,\lambda) &= (l+1)(a-1)-(l+1)\int_1^a S_p^p\left(\lambda^{1/p}\,\theta(x,\lambda)\right)dx \\ &\quad + (l+1)^{1-p}\int_1^a S_p^p\left(\lambda^{1/p}\,\theta(x,\lambda)\right)dx-\frac{(l+1)^{1-p}}{\lambda}\int_1^a w_m(x)\,S_p^p\left(\lambda^{1/p}\,\theta(x,\lambda)\right)dx.\end{aligned}$$

By lemma 4.2, one can obtain

$$\int_1^a w_m(x)\left\{S_p^p\left(\lambda^{1/p}\,\theta(x,\lambda)\right)-\frac{1}{p}\right\}dx=o(1),\ \text{as } n\to\infty.$$

Hence, we get

$$\lambda^{\frac{1}{p}}\theta(a,\lambda)=\widetilde{l}\,(a-1)\,\lambda^{\frac{1}{p}}-\frac{(l+1)^{1-p}}{p\lambda^{1-\frac{1}{p}}}\int_1^a w_m(r)\,dr+O\left(\frac{1}{\lambda^{2-\frac{1}{p}}}\right). \tag{4.1.6}$$

Let $\lambda_{n,m}$ be an eigenvalue of the problem (4.1)-(4.2) for all m. Now, we will prove this lemma for $m=1$. By (4.2), we have

$$(l+1)\,\lambda_{n,1}^{1/p}c(a)S_p'\left(\lambda_{n,1}^{1/p}\,\theta(a,\lambda_{n,1})\right)+b_1\lambda_{n,1}c(a)S_p\left(\lambda_{n,1}^{1/p}\,\theta(a,\lambda_{n,1})\right)=0,$$

or

$$-\frac{(l+1)\,\lambda_{n,1}^{\frac{1}{p}-\frac{1}{2}}}{b_1}=\frac{S_p\left(\lambda_{n,1}^{1/p}\,\theta(a,\lambda_{n,1})\right)}{S_p'\left(\lambda_{n,1}^{1/p}\,\theta(a,\lambda_{n,1})\right)}=T_p\left(\lambda_{n,1}^{1/p}\,\theta(a,\lambda_{n,1})\right).$$

As n is sufficiently large, it follows

$$\lambda_{n,1}^{1/p}\,\theta(a,\lambda_{n,1})=T_p^{-1}\left(-\frac{(l+1)\,\lambda_{n,1}^{\frac{1}{p}-\frac{1}{2}}}{b_1}\right)=n\hat{\pi}-\frac{(l+1)\,\lambda_{n,1}^{\frac{1}{p}-\frac{1}{2}}}{b_1}+o\left(\lambda_{n,1}^{\frac{2}{p}-1}\right). \tag{4.1.7}$$

By considering (4.1.6) and (4.1.7) together, we get

$$\lambda_{n,1}^{1/p}=\frac{n\hat{\pi}}{\widetilde{l}\,(a-1)}-\frac{(l+1)\,\widetilde{l}^{\,p-3}(a-1)^{p-3}}{b_1\,(n\hat{\pi})^{p-2}}+\frac{(l+1)^{1-p}\,\widetilde{l}^{\,p-2}(a-1)^{p-2}}{p\,(n\hat{\pi})^{p-1}}\int_1^a w_1(r)\,dr+O\left(\frac{1}{n^{p-2}}\right). \tag{4.1.8}$$

For the case $m\geq 2$, by using the similar process as in $m=1$, we can easily obtain

$$(l+1)\,\lambda_{n,m}^{1/p}c(a)S_p'\left(\lambda_{n,m}^{1/p}\,\theta(a,\lambda_{n,m})\right)+\left(b_1\sqrt{\lambda_{n,m}}+...+b_m\left(\sqrt{\lambda_{n,m}}\right)^m\right)c(a)S_p\left(\lambda_{n,m}^{1/p}\,\theta(a,\lambda_{n,m})\right)=0,$$

or

$$-\frac{(l+1)\,\lambda_{n,m}^{\frac{1}{p}}}{b_1\sqrt{\lambda_{n,m}}+...+b_m\left(\sqrt{\lambda_{n,m}}\right)^m}=\frac{S_p\left(\lambda_{n,m}^{1/p}\;\theta(a,\lambda_{n,m})\right)}{S_p'\left(\lambda_{n,m}^{1/p}\;\theta(a,\lambda_{n,m})\right)}=T_p\left(\lambda_{n,m}^{1/p}\;\theta(a,\lambda_{n,m})\right). \quad (4.1.9)$$

By considering (4.1.6) and (4.1.9) together and using similar procedure, we conclude that

$$\begin{aligned}\lambda_{n,m}^{1/p} &= \frac{n\hat{\pi}}{\widetilde{l}\,(a-1)}-\frac{(l+1)\,\widetilde{l}^{\frac{mp}{2}-2}\,(a-1)^{\frac{mp}{2}-2}}{(n\hat{\pi})^{\frac{p}{2}-1}\left(b_1\,\widetilde{l}^{\frac{mp-p}{2}}\,(a-1)^{\frac{mp-p}{2}}+...+b_m\,(n\hat{\pi})^{\frac{mp-p}{2}}\right)}\\ &+\frac{(l+1)^{1-p}\,\widetilde{l}^{p-2}\,(a-1)^{p-2}}{p\,(n\hat{\pi})^{p-1}}\int_1^a w_m(r)\,dr+O\left(\frac{1}{n^{2p-1}}\right).\end{aligned}$$

This completes the proof.

Theorem 4.2. A*symptotic formulas* of *the nodal points for the problem (4.1)-(4.2) satisfies*

$$\begin{aligned}x_{i,1}^n &= 1+\frac{i\widetilde{l}\,(a-1)}{n\,(l+1)}+\frac{i\widetilde{l}^{p-1}\,(a-1)^{p-1}}{b_1 n^p\,\hat{\pi}^{p-1}}-\frac{i\widetilde{l}^{p}\,(a-1)^{p}}{p\,(l+1)^p\,n^{p+1}\hat{\pi}^p}\int_1^a w_1(r)\,dr \quad (4.1.10)\\ &+\int_1^{x_{i,1}^n} S_p^p dr-\frac{1}{(l+1)^p}\int_1^{x_{i,1}^n}\left(1-\frac{\widetilde{l}^{p}\,(a-1)^p\,w_1(r)}{(n\hat{\pi})^p}\right)S_p^p dr+O\left(\frac{i}{n^p}\right),\end{aligned}$$

and

$$\begin{aligned}x_{i,m}^n &= 1+\frac{i\widetilde{l}\,(a-1)}{n\,(l+1)}+\frac{i\widetilde{l}^{\frac{mp}{2}}\,(a-1)^{\frac{mp}{2}}}{n^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}\left(b_1\,\widetilde{l}^{\frac{mp-p}{2}}\,(a-1)^{\frac{mp-p}{2}}+...+b_m\,(n\hat{\pi})^{\frac{mp-p}{2}}\right)} \quad (4.1.11)\\ &-\frac{i\widetilde{l}^{p}\,(a-1)^p}{p\,(l+1)^p\,n^{p+1}\hat{\pi}^p}\int_1^a w_m(r)\,dr+\int_1^{x_{i,m}^{(n)}} S_p^p dr\\ &-\frac{1}{(l+1)^p}\int_1^{x_{i,m}^{(n)}}\left(1-\frac{\widetilde{l}^{p}\,(a-1)^p\,w_m(r)}{(n\hat{\pi})^p}\right)S_p^p dr \quad +O\left(\frac{i}{n^{2p+1}}\right)\end{aligned}$$

for $m=1$ and $m\geq 2$, respectively as $n\to\infty$.

Proof: Integrating (4.1.3) from 1 to $x_{i,m}^n$ and letting $\theta(x_{i,m}^n,\lambda)=\dfrac{i\hat{\pi}}{\lambda_{n,m}^{1/p}}$, we have

$$x_{i,m}^n=1+\frac{i\hat{\pi}}{(l+1)\,\lambda_{n,m}^{\frac{1}{p}}}+\int_1^{x_{i,m}^{(n)}} S_p^p dr-\frac{1}{(l+1)^p}\int_1^{x_{i,m}^n}\left(1-\frac{w_m(r)}{\lambda_{n,m}}\right)S_p^p dr. \quad (4.1.12)$$

Now, we will find the asymptotic formula of the nodal points for $m=1$. By the formula (4.1.4), we deduce

$$\frac{1}{\lambda_{n,1}^{1/p}}=\frac{\widetilde{l}\,(a-1)}{n\hat{\pi}}+\frac{(l+1)\,\widetilde{l}^{\,p-1}(a-1)^{p-1}}{b_1\,(n\hat{\pi})^p}-\frac{(l+1)^{1-p}\,\widetilde{l}^{\,p}(a-1)^p}{p\,(n\hat{\pi})^{p+1}}\int_1^a w_1(r)\,dr+O\left(\frac{1}{n^p}\right), \quad (4.1.13)$$

and therefore we obtain the formula (4.1.10) by using (4.1.12) and (4.1.13).

For $m\geq 2$, from the formula (4.1.5), it can be easily obtain that

$$\begin{aligned}\frac{1}{\lambda_{n,m}^{1/p}} &= \frac{\widetilde{l}\,(a-1)}{n\hat{\pi}}+\frac{(l+1)\,\widetilde{l}^{\,\frac{mp}{2}}(a-1)^{\frac{mp}{2}}}{(n\hat{\pi})^{\frac{p}{2}+1}\left(b_1\widetilde{l}^{\,\frac{mp-p}{2}}(a-1)^{\frac{mp-p}{2}}+\ldots+b_m\,(n\hat{\pi})^{\frac{mp-p}{2}}\right)} \\ &\quad -\frac{(l+1)^{1-p}\,\widetilde{l}^{\,p}(a-1)^p}{p\,(n\hat{\pi})^{p+1}}\int_1^a w_m(r)\,dr+O\left(\frac{1}{n^{2p+1}}\right),\end{aligned} \quad (4.1.14)$$

and we get the formula (4.1.11) by using (4.1.12) and (4.1.14).

Theorem 4.3. *Asymptotic formulas* of *the nodal lengths for the problem (4.1)-(4.2) satisfies*

$$\begin{aligned}l_{i,1}^n &= \frac{\widetilde{l}\,(a-1)}{n\,(l+1)}+\frac{\widetilde{l}^{\,p-1}(a-1)^{p-1}}{b_1 n^p\hat{\pi}^{p-1}}-\frac{\widetilde{l}^{\,p}(a-1)^p}{p\,(l+1)^p\,n^{p+1}\hat{\pi}^p}\int_1^a w_1(r)\,dr+\int_{x_{i,1}^{(n)}}^{x_{i+1,1}^{(n)}} S_p^p dr \\ &\quad -\frac{1}{(l+1)^p}\int_{x_{i,1}^{(n)}}^{x_{i+1,1}^{(n)}}\left(1-\frac{\widetilde{l}^{\,p}(a-1)^p\,w_1(r)}{(n\hat{\pi})^p}\right)S_p^p dr+O\left(\frac{1}{n^p}\right),\end{aligned} \quad (4.1.15)$$

and

$$\begin{aligned}l_{i,m}^n &= \frac{\widetilde{l}\,(a-1)}{n\,(l+1)}+\frac{\widetilde{l}^{\,\frac{mp}{2}}(a-1)^{\frac{mp}{2}}}{n^{\frac{p}{2}+1}\hat{\pi}^{\frac{p}{2}}\left(b_1\widetilde{l}^{\,\frac{mp-p}{2}}(a-1)^{\frac{mp-p}{2}}+\ldots+b_m\,(n\hat{\pi})^{\frac{mp-p}{2}}\right)} \\ &\quad -\frac{\widetilde{l}^{\,p}(a-1)^p}{p\,(l+1)\,n^{p+1}\hat{\pi}^p}\int_1^a w_m(r)\,dr+\int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} S_p^p dr \\ &\quad -\frac{1}{(l+1)^p}\int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}}\left(1-\frac{\widetilde{l}^{\,p}(a-1)^p\,w_m(r)}{(n\hat{\pi})^p}\right)S_p^p dr+O\left(\frac{1}{n^{2p+1}}\right),\end{aligned} \quad (4.1.16)$$

for $m=1$ and $m\geq 2$, *respectively as* $n\rightarrow\infty$.

Proof: For large $n \in \mathbb{N}$, integrating (4.1.3) in $[x_{i,m}^n, x_{i+1,m}^n]$ and using the definition of nodal lengths, we have

$$l_{i,m}^n = \frac{\hat{\pi}}{(l+1)\,\lambda_{n,m}^{1/p}} + \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} S_p^p dr - \frac{1}{(l+1)^p} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} \left(1 - \frac{w_m(r)}{\lambda_{n,m}^{1/p}}\right) S_p^p dr, \tag{4.1.17}$$

or

$$l_{i,m}^n = \frac{\hat{\pi}}{(l+1)\,\lambda_{n,m}^{1/p}} + \frac{1}{p} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} dr - \frac{1}{p\,(l+1)^p} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} dr + \frac{1}{p\,(l+1)^p\,\lambda_{n,m}^{1/p}} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} w_m(r)\,dr + O\left(\frac{1}{n^p}\right).$$

For $m = 1$ and $m \geq 2$, we can obtain easily (4.1.15) and (4.1.16) by using the formulas (4.1.13) and (4.1.14), respectively.

4.2 Reconstruction of the potential function in Bessel equation

In this chapter, we give an explicit formula for the potential functions of the Bessel equation (4.1) by using nodal lengths. The method used in the proof of the theorem is similar to classical problems; $p-$Laplacian Sturm-Liouville eigenvalue problem (see [17], [18], [19], [35]).

Theorem 4.4. *Let $w_m(x) \in L^2(1,a)$ are real-valued functions defined in the interval $1 \leq x \leq a$ for all m. Then*

$$w_m(x) = \lim_{n\to\infty} \left\{ p\,(l+1)^{p-1}\,\lambda_{n,m} \left(\frac{\lambda_{n,m}^{\frac{1}{p}}\,\widetilde{l}}{\hat{\pi}} l_{i,m}^{(n)} - 1 \right) \right\}, \tag{4.2.1}$$

for $i = i_{n,m}(x) = \max\left\{ i : x_{i,m}^n < x \right\}$ and $m \in \mathbb{Z}^+$.

Proof: We need to consider Theorem 4.3 to prove this theorem. From (4.1.16), we have

$$\begin{aligned} \frac{p\,(l+1)^p\,\lambda_{n,m}^{\frac{1}{p}+1}}{\hat{\pi}} l_{i,m}^{(n)} &= p\,(l+1)^{p-1}\,\lambda_{n,m} + \frac{p\,(l+1)^p\,\lambda_{n,m}^{\frac{1}{p}+1}}{\hat{\pi}} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} S_p^p dt \\ &\quad - \frac{p\lambda_{n,m}^{\frac{1}{p}+1}}{\hat{\pi}} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} \left(1 - \frac{w_m(t)}{\lambda_{n,m}}\right) S_p^p dt \ . \end{aligned}$$

Then, we can use similar procedure as those in [17] for $i = i_{n,m}(x) = \max\left\{i : x_{i,m}^n < x\right\}$ to show

$$\frac{\lambda_{n,m}^{1/p}}{\hat{\pi}} \int_{x_{i,m}^n}^{x_{i+1,m}^n} w_m(t)dt \to w_m(x),$$

and

$$\frac{p\,(l+1)^p\,\lambda_{n,m}^{\frac{1}{p}+1}}{\hat{\pi}} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} \left(S_p^p - \frac{1}{p}\right) dt \quad \to \quad 0,$$

$$\frac{p\lambda_{n,m}^{\frac{1}{p}+1}}{\hat{\pi}} \int_{x_{i,m}^{(n)}}^{x_{i+1,m}^{(n)}} \left(1 - \frac{w_m\,(t)}{\lambda_{n,m}}\right)\left(S_p^p - \frac{1}{p}\right) dt \quad \to \quad 0,$$

pointwise almost everywhere. Hence, we get

$$w_m(x) = \lim_{n\to\infty}\left\{p\,(l+1)^{p-1}\,\lambda_{n,m}\left(\frac{\lambda_{n,m}^{\frac{1}{p}}\,\widetilde{l}}{\hat{\pi}}\,l_{i,m}^{(n)} - 1\right)\right\}.$$

Theorem 4.5. *Let $\left\{l_{i,m}^{(n)} : j = 1, 2, ..., n-1\right\}_{n=2}^{\infty}$ be a set of nodal lengths of the problem (4.1)-(4.2) where $w_m(x)$ are real-valued functions on $1 \le x \le a$ for all m. Let us define*

$$F_{n,1}(x) = \frac{p\,(l+1)^{p-1}\,(n\hat{\pi})^p}{\widetilde{l}^{\,p}\,(a-1)^p}\left(\frac{n\,l_{i,1}^{(n)}}{a-1} - 1\right) - \frac{p\,(l+1)^p\,(n\hat{\pi})}{b_1\,\widetilde{l}^{\,2}\,(a-1)^2} + \frac{1}{\widetilde{l}\,(a-1)}\int_1^a w_1\,(t)\,dt\;, \tag{4.2.2}$$

$$\begin{aligned} F_{n,m}(x) \quad &= \quad \frac{p\,(l+1)^{p-1}\,(n\hat{\pi})^p}{\widetilde{l}^{\,p}\,(a-1)^p}\left(\frac{nl_{i,m}^{(n)}}{a-1} - 1\right) - \frac{p\,(l+1)^p\,\widetilde{l}^{\,\frac{mp-2p-p}{2}}\,(a-1)^{\frac{mp-2p-p}{2}}\,(n\hat{\pi})^{\frac{p}{2}}}{\left(b_1\,\widetilde{l}^{\,\frac{mp-p}{2}}\,(a-1)^{\frac{mp-p}{2}} + ... + b_m\,(n\hat{\pi})^{\frac{mp-p}{2}}\right)} \\ &\quad + \frac{1}{\widetilde{l}\,(a-1)}\int_1^a w_m\,(t)\,dt, \end{aligned} \tag{4.2.3}$$

for $m = 1$ and $m \ge 2$, respectively. Then, $\{F_{n,m}(x)\}$ converges to $w_m\,(x)$ pointwise almost everywhere in $L^1(1,a)$, respectively, for all cases.

Proof: We will prove this theorem only for $F_{n,1}$. Other case can be shown similarly. By the asymptotic formulas of eigenvalues (4.1.4) and nodal lengths (4.1.15), we get

$$\begin{aligned} \frac{p\,(l+1)^{p-1}\,\lambda_{n,1}}{\hat{\pi}}\left(\frac{\lambda_{n,1}^{1/p}\widetilde{l}}{\hat{\pi}}l_{i,1}^{(n)} - 1\right) \quad &= \quad p\,(l+1)^{p-1}\,\lambda_{n,1}\left(\frac{n\,l_{i,1}^{(n)}}{a-1} - 1\right) - \frac{p\,(l+1)^p\,l_{i,1}^{(n)}n^2\hat{\pi}}{b_1\,\widetilde{l}^{\,2}\,(a-1)^3} \\ &\quad + \frac{nl_{i,1}^{(n)}}{\widetilde{l}\,(a-1)^2}\int_1^a w_1\,(t)\,dt + o(1). \end{aligned}$$

Considering $nl_{i,1}^{(n)} = a - 1 + o(1)$, as $n \to \infty$, we have

$$\frac{p\,(l+1)^{p-1}\,(n\hat{\pi})^p}{\tilde{l}^p\,(a-1)^p}\left(\frac{n\;l_{i,1}^{(n)}}{a-1} - 1\right) - \frac{p\,(l+1)^p\,(n\hat{\pi})}{b_1\,\tilde{l}^2\,(a-1)^2} \to F_{n,1}(x) - \frac{1}{\tilde{l}\,(a-1)}\int_1^a w_1\,(t)\,dt$$

pointwise almost everywhere in $L^1(1,a)$.

5 Conclusion

In this thesis, we consider $p-$ Laplacian types differential operators. Recently, there are some studies about inverse problems for $p-$ Laplacian differential operators. For instance, some authors applied Ambarzumyan type theorems which is the first study about inverse eigenvalue problems to $p-$Laplacian type operators. Many important results about inverse problem of $p-$Laplacian Sturm-Liouville operator and $p-$Laplacian Diffusion operator and $p-$Laplacian Dirac operator are obtained by some authors. In these studies, they consider these equations with separeted boundary conditions. But, we consider these type $p-$Laplacian operators with quite different boundary conditions. Then, by using generalized Prüfer substitution, we solve inverse nodal problem for diffusion and Bessel equations. As we know, the inverse nodal problem for the classical Sturm–Liouville operator, Diffusion operator and Dirac operator is now quite well understood. To make clear the inverse nodal problem for $p-$Laplacian operators, we get some asymptotic formulas for eigenvalues, nodal parameters and potential function of the $p-$Laplacian diffusion eigenvalue problem and $p-$Laplacian Bessel eigenvalue problem with polynomially dependent spectral parameter, respectively. We see that the obtained results are generalizations of the classical problems for $p = 2$. This makes our study more important and original.

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