EXTENSIONS OF DIFFEOMORPHISM ALGEBRAS FOR FLUID AND KINETIC THEORIES

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ABSTRACT

EXTENSIONS OF DIFFEOMORPHISM ALGEBRAS FOR FLUID AND KINETIC THEORIES

The main goal of this thesis is the extension of the Lie algebra of $Diff(Q)$ to the Lie algebra of $Diff\,(Q) \ltimes F\,(Q)$. For this purpose, we start with an arbitrary Lie algebra, and then extend it by a subalgebra of its derivation algebra and obtain the conditions for such an extension. Then, apply these procedure to $Diff(Q)$ and obtain the subalgebra of derivations that correspond to the algebra of functions $F(Q)$ on Q. Finally the Lie algebra cohomological aspects of this extension are studied.

ÖZET

AKIŞKAN VE KİNETİK TEORİ İLE İLGİLİ DİFEOMORFİZM CEBİRLERİNİN GENİŞLEMELERİ

Bu tezin temel amacı $Diff(Q)$ grubunun Lie cebirinin $Diff\,(Q) \ltimes F\,(Q)$ grubunun Lie cebirine genişletilmesinin incelenmesidir. Bu doğrultuda, verili herhangi bir Lie cebiriyle başlanarak, türev cebirinin bir altcebiriyle genişletilmesinin koşulları incelenmiştir. Ardından bu prosedür $Diff(Q)$ grubunun Lie cebirine uygulanmış ve türev cebirinin Q üzerindeki fonksiyon cebirine karşılık gelen altcebiriyle genişlemesi elde edilmiştir. Son olarak, bu genişlemenin kohomolojik özellikleri incelenmiştir.

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1. INTRODUCTION

In [Marsden and Weinstein 1982], the Poissson-Vlasov equations of collisionless plasma dynamics was shown to be Lie-Poisson Hamiltonian equations on the dual of Lie algebra of Hamiltonian vector fields identified with the space of densities on the particle configuration space. This formulation started with the configuration space $G = Diff_{can}(T^*Q)$, the group of canonical diffeomorphisms of the particle phase space $(T^*Q, \Omega_{T^*Q} = dq^i \Lambda dp_i)$ which is canonically symplectic.

In [Gümral 2010], a description of the Poisson equation in terms of a momentum map associated with gauge symmetries of Hamiltonian particle motion was given. It was pointed out that such a description requires the configuration space to be a product of the space $F(Q)$ of functions on particle configuration space and $Diff_{can}(T^*Q)$. By its symplectic action on T^*Q the group $F(Q)$ can be seen as a subgroup of $Diff_{can}(T^*Q)$, and this caused the algebraic structure of the product $F(Q) \times Diff_{can}(T^*Q)$ remain mysterious. Moreover, this space is expected to be more important in Lagrangian and in Euler-Poincare formulations of Poisson-Vlasov dynamics.

In [Esen and Gümral, 2012], Vlasov equations were obtained as vertical representative of complete cotangent lift of generators of particle motion (elements of the algebra of Hamiltonian vector fields). This geometric construction, when generalized to the algebra of symmetric tensors (with Schouten concominant), results in dynamics of Vlasov moments with Kuperschmidt-Manin bracket [Gibbons,1981], [Gibbons,Holm, Tronci, 2008]. It turns out that the algebra of symmetric tensors has three subalgebras $F(Q)$, $\mathfrak{X}(Q)$ and $\mathfrak{X}(Q) \times F(Q)$. The moment dynamics for the last one is the dynamics of compressible fluids whose configuration space is the semi-direct product $Diff(Q) \ltimes F(Q)$. A more direct description of this fluid-plasma relation was given as follows [Marsden, Weinstein, Ratiu, Schmid, Spencer, 1983]: the cotangent lifted action of $\mathfrak{X}(Q) \times F(Q)$ on T^*Q is Hamiltonian and hence is a Lie algebra homomorphism into the algebra $\mathfrak{X}_{ham}(T^*Q)$ of Hamiltonian vector fields. The dual of this homomorphism is a Poisson map from the Lie-Poisson structure of plasma dynamics to that of compressible fluid known as plasma-to-fluid map.

The problem of translations on particle phase space is related to the homomorphism between

algebra of Hamiltonian vector fields with Jacobi-Lie bracket and the algebra of functions with Poisson bracket. In [Van Hove, 1951] it was noticed that the correspondance $X_h \leftrightarrow h$ between Hamiltonian vector fields and Hamiltonian functions is defined only up to an additive constant. As a result, no canonical transformation exists for the function $1 = \{q, p\}$. Yet, the corresponding Hamiltonian vector fields commute $0 = [X_q, X_p]$. That means, phase space translations are commutative in $\mathfrak{X}_{ham}\left(T^*Q\right)$ and are non-commutative in $F\left(T^*Q\right)$.
In the preliminary section, basic definitions related with the Lie algebra extension are provided. In section 3, we present the Abelian, semi-direct/ direct extensions. In section 4, we have focused on the algebra of derivations, and the extention of a Lie algebra by a subalgebra of its derivations. In section 5, basic definitions related to Lie algebra cohomology are given and applied to the extension by a subalgebra of derivation algebra. In section 6, we want to understand the relationship between the extension of a subalgebra of a derivation algebra and the extension by the algebra of functions. To understand that connection, Hamiltonian vector fields are used as a tool. So, we start with the definition of Hamiltonian vector fields and focus on the algebra of Hamiltonian vector fields. Finally in the last chapter, we have studied on the extension of the Lie algebra of $Diff(Q)$ by functions.

2. PRELIMINARIES

In this chapter, some definitions which are necessary to understand the Lie algebra extensions are provided.

Definition 2.0.1. *[3] Let V be a set. The commutator of x and y is a binary operation $V \times V \to V$ denoted by*

$$(x, y) \longmapsto [x, y]$$

Definition 2.0.2. *[4] Let $\mathfrak{g}$ be a vector space endowed with $[.,.]$, then $\mathfrak{g}$ is called a **Lie algebra** over F if it satisfies the anti-symmetry, bilinearity and the Jacobi-identity.*

i. Bilinearity

$$\begin{aligned} [\alpha x + \beta y, z] &= \alpha [x, z] + \beta [y, z] \\ [z, \alpha x + \beta y] &= \alpha [z, x] + \beta [z, y] \end{aligned}$$

ii. Antisymmetry

$$\begin{aligned} [x, y] &= -[y, x] \\ [x, x] &= 0 \text{ for all } x \in \mathfrak{g} \end{aligned}$$

iii. Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for all } x, y, z \in \mathfrak{g}$$

Definition 2.0.3. *[3] Let $\mathfrak{s}$ be a subspace of $\mathfrak{g}$. If $\mathfrak{s}$ is closed under a Lie bracket, then $\mathfrak{s}$ is called a **Lie subalgebra** of $\mathfrak{g}$.*

Definition 2.0.4. *[4] If the Lie bracket vanishes, then the Lie algebra is called **abelian** .*

Definition 2.0.5. *[3] Let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$, then $\mathfrak{s}$ is called an **ideal of** $\mathfrak{g}$ if $[w, z] \in \mathfrak{s}$ for all $w \in \mathfrak{g}$ and $z \in \mathfrak{s}$.*

If $\mathfrak{s}$ is an ideal of $\mathfrak{g}$ such that $[w, z] \in \mathfrak{s}$ for all $w \in \mathfrak{g}$ and $z \in \mathfrak{s}$, then $-[z, w] \in \mathfrak{s}$ because

$[w, z] = -[z, w]$. Therefore, there is no meaning to distunguish the notion of the left and right ideals in Lie algebra.

Definition 2.0.6. *[10] If there is a linear mapping between two Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{a}$ *which preserves the Lie algebra structure, then the map* $f : \mathfrak{g} \to \mathfrak{a}$ *is called a Lie algebra homomorphism.*

Definition 2.0.7. *[10] Let* $f_i : \mathfrak{g}_i \to \mathfrak{g}_{i+1}$ *be a collection of Lie algebra homomorphisms,*

$$....., \mathfrak{g}_i \to \mathfrak{g}_{i+1} \to \mathfrak{g}_{i+2} \to .....$$

By the homomorphism property of f_i*, we have*

$$f_i\left([\alpha, \beta]_{\mathfrak{g}_i}\right) = [f_i(\alpha), f_i(\beta)]_{\mathfrak{g}_{i+1}}$$

where $\alpha, \beta \in \mathfrak{g}_i$. If $range\ f_i = ker\ f_{i+1}$, then the sequence f_i is called an exact sequence of Lie algebra homomorphisms.

3. LIE ALGEBRA EXTENSIONS

Definition 3.0.1. *[10] Let* $\mathfrak{g}, \mathfrak{h}$ *and* $\mathfrak{e}$ *be Lie algebras. If there are morphisms of Lie algebras*

$$i : \mathfrak{h} \to \mathfrak{e} \ \textit{and} \ \pi : \mathfrak{e} \to \mathfrak{g}$$

such that

$$Ker\,(\pi) = Image\,(i)$$

where the homomorphism

$$i : \mathfrak{h} \to \mathfrak{e}$$

is an injection and the homomorphism

$$\pi : \mathfrak{e} \to \mathfrak{g}$$

is a projection(*surjection*)*, then the Lie algebras will fit into an exact sequence as*

$$0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{g} \to 0 \tag{3.1}$$

Definition 3.0.2. *[10] The Lie algebra* $\mathfrak{e}$ *is called an extension of the Lie algebra* $\mathfrak{g}$ *by the Lie algebra* $\mathfrak{h}$*, if the Lie algebras* $\mathfrak{g}, \mathfrak{h}$ *and* $\mathfrak{e}$ *fit into an exact sequence as in* **??**.

Definition 3.0.3. *[12] A Lie algebra homomorphism splits if there is a Lie algebra homomorphism* $s : \mathfrak{g} \to \mathfrak{e}$ *such that* $\pi \circ s = Id_{\mathfrak{g}}$ (*the identity mapping in* $\mathfrak{g}$)

The main question is the preservation of the Lie algebra structure of the linear map $s : \mathfrak{g} \to \mathfrak{e}$. In general, the linear map $s : \mathfrak{g} \to \mathfrak{e}$ is not necessarily a homomorphism. The Lie algebra extension problem is to determination how much s differs from a homomorphism.

Proposition 3.0.4. *[12] Let* $\beta \in \mathfrak{e}$*,* $\eta \in \mathfrak{h}$*, then*

$$\pi\,[\beta, i\eta]_{\mathfrak{e}} = [\pi\beta, \pi i\eta]_{\mathfrak{g}} = 0$$

Proof. The first equality follows from the homomorphism property of π. In the second

equality, exactness of the sequence was used. □

Corollary 3.0.5.

$$\pi\left[\beta, i\eta\right]_{\mathfrak{e}} = 0 \tag{3.2}$$

for $\beta \in \mathfrak{e}$, $\eta \in \mathfrak{h}$ *means that* $\left[\beta, i\eta\right]_{\mathfrak{e}} \in Ker\left(\pi\right)$. *Since*

$$Ker\left(\pi\right) = Image\left(i\right)$$

$$\left[\beta, i\eta\right]_{\mathfrak{e}} \in Image\left(i\right)$$

So that $Image\left(i\right)$ *is an ideal in* $\mathfrak{e}$.

Definition 3.0.6. *[10]Let* $\mathfrak{g}$ *be a Lie algebra. The vector space V over the field K is* ***left*** $\mathfrak{g}$***-module*** *if there is an operator* $\rho : \mathfrak{g}\times V \rightarrow V$ *with the properties*

$$\begin{aligned} \textbf{i}.\rho_{\alpha}(v + v') &= \rho_{\alpha}v + \rho_{\alpha}v' \\ \textbf{ii}.\rho_{(\alpha+\alpha')v} &= \rho_{\alpha}v + \rho_{\alpha'}v \\ \textbf{iii}.\rho_{[\alpha,\alpha']}v &= [\rho_{\alpha}, \rho_{\alpha'}]v \end{aligned}$$

for $\alpha, \alpha' \in \mathfrak{g}$ *and* $v, v' \in V$. *The operator* ρ *is a left action of* $\mathfrak{g}$ *on V. For the right action, the bracket in* (iii) *incorporates a* $(-)$ *sign. Use* $\mathfrak{h}$ *as the vector space of the definition of left* $\mathfrak{g}$*-module. Define the left action of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *by* $\rho : \mathfrak{g}\times\mathfrak{h} \rightarrow \mathfrak{h}$ *as*

$$\rho_{\alpha}\eta := i^{-1}\left[\mathfrak{s}\alpha, i\eta\right]_{\mathfrak{e}} \tag{3.3}$$

for $\alpha \in \mathfrak{g}$ *and* $\eta \in \mathfrak{e}$.

Theorem 3.0.7. *[10] If the action* ρ *is a homomorphism, then* $\mathfrak{h}$ *is a left* $\mathfrak{g}$*-module. In this case, either* $\mathfrak{h}$ *is abelian or* s *is a homomorphism.*

Proof. (i)

$$\begin{aligned}\rho_\alpha(\eta+\eta')= i^{-1}\left[s\alpha, i\left(\eta+\eta^{'}\right)\right]_{\mathfrak{e}} &= i^{-1}\left[s\alpha, i\eta+i\eta'\right]_{\mathfrak{e}} \\ &= i^{-1}\left(\left[s\alpha, i\eta\right]_{\mathfrak{e}}+\left[s\alpha, i\eta^{'}\right]_{\mathfrak{e}}\right) \\ &= i^{-1}\left[s\alpha, i\eta\right]_{\mathfrak{e}}+i^{-1}\left[s\alpha, i\eta^{'}\right]_{\mathfrak{e}} \\ &= \rho_\alpha\eta+\rho_\alpha\eta^{'} \qquad \text{(by definition of } \rho)\end{aligned}$$

To prove (i) I use the fact that, i is a linear map. The inverse of a linear operator is linear if it exists.

(ii)

$$\begin{aligned}&\rho_{(\alpha+\alpha')}\eta{=}i^{-1}[s(\alpha+\alpha^{'}), i\eta]_{\mathfrak{e}}{=}i^{-1}[s\alpha{+}s\alpha^{'}, i\eta]_{\mathfrak{e}} \\ =\ & i^{-1}([s\alpha, i\eta]_{\mathfrak{e}}+[s\alpha^{'}, i\eta]_{\mathfrak{e}}){=}i^{-1}[s\alpha, i\eta]_{\mathfrak{e}}+i^{-1}[s\alpha^{'}, i\eta]_{\mathfrak{e}} \\ =\ & \rho_\alpha\eta+\rho_{\alpha'}\eta\end{aligned}$$

To prove (ii), the linearity of the map i^{-1} and the definition of ρ was used. (iii)

$$\begin{aligned}&[\rho_\alpha, \rho_\beta]\eta{=}(\rho_\alpha\rho_\beta-\rho_\beta\rho_\alpha)\eta{=}\rho_\alpha\rho_\beta\eta-\rho_\beta\rho_\alpha\eta \\ =\ & \rho_\alpha i^{-1}[s\beta, i\eta]_{\mathfrak{e}}-\rho_\beta i^{-1}[s\alpha, i\eta]_{\mathfrak{e}} \\ =\ & i^{-1}[s\alpha, ii^{-1}[s\beta, i\eta]_{\mathfrak{e}}]_{\mathfrak{e}}-i^{-1}[s\beta, ii^{-1}[s\alpha, i\eta]_{\mathfrak{e}}]_{\mathfrak{e}} \\ =\ & i^{-1}[s\alpha, [s\beta, i\eta]_{\mathfrak{e}}]_{\mathfrak{e}}-i^{-1}[s\beta, [s\alpha, i\eta]_{\mathfrak{e}}]_{\mathfrak{e}} \\ =\ & -i^{-1}[s\beta, [i\eta{,}s\alpha]_{\mathfrak{e}}]_{\mathfrak{e}}-i^{-1}[i\eta, [s\alpha{,}s\beta]_{\mathfrak{e}}]_{\mathfrak{e}}-i^{-1}[s\beta, [s\alpha, i\eta]_{\mathfrak{e}}]_{\mathfrak{e}} \\ =\ & i^{-1}\left[s\beta, [s\alpha, i\eta]_{\mathfrak{e}}\right]_{\mathfrak{e}}-i^{-1}[i\eta, [s\alpha{,}s\beta]_{\mathfrak{e}}]_{\mathfrak{e}}-i^{-1}[s\beta, [s\alpha, i\eta]_{\mathfrak{e}}]_{\mathfrak{e}} \\ =\ & -i^{-1}[i\eta, [s\alpha, s\beta]_{\mathfrak{e}}]_{\mathfrak{e}}{=}i^{-1}[[s\alpha{,}s\beta]_{\mathfrak{e}}, i\eta]_{\mathfrak{e}} \\ =\ & i^{-1}[s[\alpha, \beta]_{\mathfrak{g}}, i\eta]_{\mathfrak{e}}+i^{-1}[([s\alpha{,}s\beta]_{\mathfrak{e}}-s[\alpha, \beta]_{\mathfrak{g}}), i\eta]_{\mathfrak{e}} \\ =\ & \rho_{[\alpha,\beta]}\eta+i^{-1}[([s\alpha{,}s\beta]_{\mathfrak{e}}{-}s[\alpha, \beta]_{\mathfrak{g}}), i\eta]_{\mathfrak{e}}\end{aligned}$$

To prove (iii), I use the Jacobi identity and also by adding and subtracting $\mathfrak{s}[\alpha, \beta]$ and by the definition of ρ. □

We get the result

$$[\rho_\alpha, \rho_\beta]\eta{=}\rho_{[\alpha,\beta]}\eta + i^{-1}[([s\alpha,s\beta]_{\mathfrak{e}}{-}s[\alpha,\beta]_{\mathfrak{g}}), i\eta]_{\mathfrak{e}} \tag{3.4}$$

Thus, ρ is a homomorphism if

$$i^{-1}[([\mathfrak{s}\alpha,\mathfrak{s}\beta]_{\mathfrak{e}}{-}\mathfrak{s}[\alpha,\beta]_{\mathfrak{g}}), i\eta]_{\mathfrak{e}}$$

vanishes. Let us define [12] $w : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ by

$$w(\alpha,\beta) := i^{-1}([\mathfrak{s}\alpha,\mathfrak{s}\beta]_{\mathfrak{e}} - \mathfrak{s}[\alpha,\beta]_{\mathfrak{g}}) \tag{3.5}$$

Thus

$$[\rho_\alpha, \rho_\beta]\eta = \rho_{[\alpha,\beta]}\eta + [w(\alpha,\beta), \eta]_{\mathfrak{h}}$$

If

$$[w(\alpha,\beta), \eta]_{\mathfrak{h}} = 0$$

then ρ is a homomorphism. This is possible if

$$(\mathbf{i})\mathfrak{h} \text{ is Abelian,}$$

$$(\mathbf{ii})s \text{ is a homomorphism.}$$

If either of these conditions is satisfied, $\mathfrak{h}$ with the action ρ is a left $\mathfrak{g}$ module. In the next subsections we will consider these two cases separately.

3.1. Extension by an Abelian Lie Algebra

Assume that the homomorphism condition (i), $\mathfrak{h}$ is Abelian, is met. Therefore $\mathfrak{h}$ is a left $\mathfrak{g}$ -module, and we can define $\mathfrak{h}$-valued cochains on $\mathfrak{g}$. In particular, w defined by

$$w(\alpha,\beta) := i^{-1}([\mathfrak{s}\alpha,\mathfrak{s}\beta]_{\mathfrak{e}} - \mathfrak{s}[\alpha,\beta]_{\mathfrak{g}})$$

is a 2-cochain (anti-symmetric,bilinear map). The ”failure” of s to be a homomorphism is measured by this map.

Proposition 3.1.1. *[12] w is a 2-cocycle.*

Proof. [12] To prove it, the definition of the coboundary operator between cochains,

$$s_n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V), \tag{3.6}$$

defined by

$$\begin{aligned}(s_n w_n)(\alpha_1, \alpha_2, ..., \alpha_{n+1}) = \quad & \sum_{i=1}^{n+1} (-)^{i+1} \rho_{\alpha_i} w_n(\alpha_1, ..., \alpha_i^{\sim}, ..., \alpha_{n+1}) \\ & + \sum_{j,k=1}^{n+1} (-)^{j+k} ([\alpha_j, \alpha_k], \alpha_1, ..., \alpha_j^{\sim}, ..., \alpha_k^{\sim}, ..., \alpha_{n+1})\end{aligned}$$

where the caret $(\sim)$ means an argument is omitted, was used. From the above equation

$$\begin{aligned}(sw)(\alpha, \beta, \gamma) \quad = \quad & \rho_\alpha w(\beta, \gamma) + \rho_\beta w(\gamma, \alpha) + \rho_\gamma w(\alpha, \beta) \\ & - w\left([\alpha, \beta]_{\mathfrak{g}}, \gamma\right) - w\left([\beta, \gamma]_{\mathfrak{g}}, \alpha\right) - w\left([\gamma, \alpha]_{\mathfrak{g}}, \beta\right)\end{aligned}$$

By using the definition [12]

$$\rho_\alpha \eta := i^{-1}[s\alpha, i\eta]_{\mathfrak{e}} \tag{3.7}$$

and the definition of

$$w(\alpha, \beta) := i^{-1}([s\alpha, s\beta]_{\mathfrak{e}} - s[\alpha, \beta]_{\mathfrak{g}}) \tag{3.8}$$

we get the result as:

$$\begin{aligned}
(sw)(\alpha,\beta,\gamma) &= i^{-1}[s\alpha, i(i^{-1}([s\beta, s\gamma]_{\mathfrak{e}} - s[\beta,\gamma]_{\mathfrak{g}}))]_{\mathfrak{e}} \\
&\quad + i^{-1}[s\beta, i(i^{-1}([s\gamma, s\alpha]_{\mathfrak{e}} - s[\gamma,\alpha]_{\mathfrak{g}}))]_{\mathfrak{e}} \\
&\quad + i^{-1}[s\gamma, i(i^{-1}([s\alpha, s\beta]_{\mathfrak{e}} - s[\alpha,\beta]_{\mathfrak{g}}))]_{\mathfrak{e}} \\
&\quad - w\left([\alpha,\beta]_{\mathfrak{g}}, \gamma\right) - w\left([\beta,\gamma]_{\mathfrak{g}}, \alpha\right) - w\left([\gamma,\alpha]_{\mathfrak{g}}, \beta\right) \\
&= i^{-1}[s\alpha, [s\beta, s\gamma]_{\mathfrak{e}} - s[\beta,\gamma]_{\mathfrak{g}}]_{\mathfrak{e}} + i^{-1}[s\beta, [s\gamma, s\alpha]_{\mathfrak{e}} - s[\gamma,\alpha]_{\mathfrak{g}}]_{\mathfrak{e}} \\
&\quad + i^{-1}[s\gamma, [s\alpha, s\beta]_{\mathfrak{e}} - [s[\alpha,\beta]_{\mathfrak{g}}]_{e} \\
&\quad - w([\alpha,\beta]_{\mathfrak{g}}, \gamma) - w([\beta,\gamma]_{\mathfrak{g}}, \alpha) - w([\gamma,\alpha]_{\mathfrak{g}}, \beta) \\
&\quad + i^{-1}[s\gamma, [s\alpha, s\beta]_{\mathfrak{e}} - [s\,[\alpha,\beta]_{\mathfrak{g}}]_{\mathfrak{e}} - \\
&\quad i^{-1}\left([s[\alpha,\beta]_{\mathfrak{g}}, s\gamma]_{\mathfrak{e}} - s[[\alpha,\beta]_{\mathfrak{g}}, \gamma]_{\mathfrak{g}}\right) - \\
&\quad i^{-1}\left([s[\beta,\gamma]_{\mathfrak{g}}, s\alpha]_{\mathfrak{e}} - s[[\beta,\gamma]_{\mathfrak{g}}, \alpha]_{\mathfrak{g}}\right) \\
&\quad - i^{-1}\left([s[\gamma,\alpha]_{\mathfrak{g}}, s\beta]_{\mathfrak{e}} - s[[\gamma,\alpha]_{\mathfrak{g}}, \beta]_{\mathfrak{g}}\right) \\
&= i^{-1}[s\alpha, [s\beta, s\gamma]_{\mathfrak{e}}]_{\mathfrak{e}} + i^{-1}[s\beta, [s\gamma, s\alpha]_{\mathfrak{e}}]_{\mathfrak{e}} \\
&\quad + i^{-1}[s\gamma, [s\alpha, s\beta]_{\mathfrak{e}}]_{\mathfrak{e}} + i^{-1}\left(s[[\alpha,\beta]_{\mathfrak{g}}, \gamma]_{\mathfrak{g}}\right) \\
&\quad + i^{-1}\left(s[[\beta,\gamma]_{\mathfrak{g}}, \alpha]_{\mathfrak{g}}\right) + i^{-1}\left(s[[\gamma,\alpha]_{\mathfrak{g}}, \beta]_{\mathfrak{g}}\right) - i^{-1}[s\alpha, s[\beta,\gamma]_{\mathfrak{g}}]_{\mathfrak{e}} \\
&\quad + i^{-1}[s\alpha, s[\beta,\gamma]_{\mathfrak{g}}]_{\mathfrak{e}} - i^{-1}[s\beta, s[\gamma,\alpha]_{\mathfrak{g}}]_{\mathfrak{e}} \\
&\quad + i^{-1}[s\beta, s[\gamma,\alpha]_{\mathfrak{g}}]_{\mathfrak{e}} - i^{-1}[s\gamma, s[\alpha,\beta]_{\mathfrak{g}}]_{\mathfrak{e}} + i^{-1}[s\gamma, s[\alpha,\beta]_{\mathfrak{g}}]_{\mathfrak{e}} \\
&= 0
\end{aligned}$$

Because; the first three vanishes by the Jacobi identity in $\mathfrak{e}$, the other three vanishes by the Jacobi identity in $\mathfrak{g}$,and the others were cancelled in pairs. Hence, w is a 2-cocycle. □

Definition 3.1.2. *[12] If there exists a Lie algebra isomorphism σ such that $\sigma \circ i = i'$ and $\pi = \pi' \circ \sigma$ where*

$$i : \mathfrak{h} \to \mathfrak{e}\ ,\ i' : \mathfrak{h} \to \mathfrak{e}'\ ,\ \pi : \mathfrak{e} \to \mathfrak{g}\ ,\ \pi' : \mathfrak{e}' \to \mathfrak{g}\ ,\ \sigma : \mathfrak{e} \to \mathfrak{e}'$$

, then two extensions $\mathfrak{e}$ and $\mathfrak{e}'$ are equivalent. There will be an injection s associated with π and an injection s' associated with π'such that $\pi \circ s = 1_{\mathfrak{g}} = \pi' \circ s'$. The linear map $\nu = \sigma^{-1}s' - s$

must be from $\mathfrak{g}$ to $i\mathfrak{h}$. Consider ρ and ρ' defined using $\mathfrak{s}$, i and s', i' respectively by

$$\rho_\alpha \eta := i^{-1}[s\alpha, i\eta]_{\mathfrak{e}} \quad \rho'_\alpha \eta := i'^{-1}[s'\alpha, i'\eta]_{\mathfrak{e}'}$$

$$\begin{aligned}
(\rho_\alpha - \rho'_\alpha)\eta =& i^{-1}[\mathfrak{s}\alpha, i\eta]_{\mathfrak{e}} - i'^{-1}[\mathfrak{s}'\alpha, i'\eta]_{\mathfrak{e}} \\
=\ & i^{-1}[\mathfrak{s}\alpha, i\eta]_{\mathfrak{e}} - (\sigma \circ i)^{-1}[\sigma(\nu+\tau)\alpha, (\sigma \circ i)\eta]_{\mathfrak{e}'} \\
=\ & i^{-1}[\mathfrak{s}\alpha, i\eta]_{\mathfrak{e}} - i^{-1}\sigma^{-1}[\sigma(\nu+\tau)\alpha, \sigma i\mu]_{\mathfrak{e}'} \\
=\ & i^{-1}[\mathfrak{s}\alpha, i\eta]_{\mathfrak{e}} - i^{-1}[(\nu+\tau)\alpha, \sigma i\mu]_{\mathfrak{e}} \\
=\ & i^{-1}[\mathfrak{s}\alpha, i\eta]_{\mathfrak{e}} - i^{-1}[\nu\alpha, i\eta]_{\mathfrak{e}} - i^{-1}[\mathfrak{s}\alpha, i\eta]_{\mathfrak{e}} \\
=\ & -i^{-1}[\nu\alpha, i\eta]_{\mathfrak{e}} = 0
\end{aligned}$$

since $\mathfrak{h}$ is Abelian. Hence, τ and τ' define the same ρ.

Proposition 3.1.3. *[12] An element of the second cohomology group group $H^2_\rho(\mathfrak{g}; \mathfrak{h})$, is uniquely defined by the equivalent extensions.*

Proof. Now consider the 2-cocycles w and w' defined from s and s' by

$$w(\alpha, \beta) := i^{-1}([\mathfrak{s}\alpha,\mathfrak{s}\beta]_{\mathfrak{e}} - \mathfrak{s}[\alpha, \beta]_{\mathfrak{g}}) \tag{3.9}$$

$$\begin{aligned}
w'(\alpha,\beta) - w(\alpha,\beta) =& i'^{-1}([s'\alpha, s'\beta]_{\mathfrak{e}'} - s'[\alpha,\beta]_{\mathfrak{g}}) - i^{-1}([s\alpha,s\beta]_{\mathfrak{e}} - s[\alpha,\beta]_{\mathfrak{g}}) \\
=\ & i^{-1}\sigma^{-1}([\sigma(\nu+s)\alpha, \sigma(\nu+s)\beta]_{\mathfrak{e}'} - \sigma(\nu+s)[\alpha,\beta]_{\mathfrak{g}}) - i^{-1}([s\alpha,s\beta]_{\mathfrak{e}} - s[\alpha,\beta]_{\mathfrak{g}}) \\
=\ & i^{-1}([(\nu+s)\alpha, (\nu+s)\beta]_{\mathfrak{e}} - (\nu+s)[\alpha,\beta]_{\mathfrak{g}}) - i^{-1}[s\alpha,s\beta]_{\mathfrak{e}} + i^{-1}s[\alpha,\beta]_{\mathfrak{g}} \\
=\ & i^{-1}([(\nu+s)\alpha, (\nu+s)\beta]_{\mathfrak{e}}) - i^{-1}((\nu+s)[\alpha,\beta]_{\mathfrak{g}})i^{-1}[s\alpha,s\beta]_{\mathfrak{e}} + i^{-1}s[\alpha,\beta]_{\mathfrak{g}} \\
=\ & i^{-1}([(\nu+s)\alpha, (\nu+s)\beta]_{\mathfrak{e}} - \nu[\alpha,\beta]_{\mathfrak{g}} - [s\alpha,s\beta]_{\mathfrak{e}}) \\
=\ & i^{-1}([\nu\alpha, \nu\beta]_{\mathfrak{e}} + [\nu\alpha,\mathfrak{s}\beta]_{\mathfrak{e}} + [s\alpha, \nu\beta]_{\mathfrak{e}} + [\mathfrak{s}\alpha,s\beta]_{\mathfrak{e}} - \nu[\alpha,\beta]_{\mathfrak{g}} - [s\alpha,s\beta]_{\mathfrak{e}} \\
=\ & i^{-1}([\nu\alpha, \nu\beta]_{\mathfrak{e}} + [\nu\alpha,\mathfrak{s}\beta]_{\mathfrak{e}} + [s\alpha, \nu\beta]_{\mathfrak{e}} - \nu[\alpha,\beta]_{\mathfrak{g}}) \\
=\ & i^{-1}([s\alpha, \nu\beta]_{\mathfrak{e}} + [\nu\alpha,\mathfrak{s}\beta]_{\mathfrak{e}} - \nu[\alpha,\beta]_{\mathfrak{g}}) \\
=\ & \rho_\alpha(i^{-1}\nu\beta) - \rho_\beta(i^{-1}\nu\alpha) - i^{-1}\nu[\alpha,\beta]_{\mathfrak{g}}
\end{aligned}$$

where the definition

$$\rho_\alpha \eta := i^{-1}[s\alpha, i\eta]_{\mathfrak{e}} \text{ was used.}$$

By comparing with

$$(sw_1)(\alpha_1, \alpha_2) = \rho_{\alpha_1} w_1(\alpha_2) - \rho_{\alpha_2} w_1(\alpha_1) - w_1([\alpha_1, \alpha_2]) \tag{3.10}$$

We see that $w' - w = s(i^{-1}\nu)$. So, w and $w^{'}$ differ by a coboundary. Hence; they represent the same element in the second cohomology group ; $\mathfrak{h}$). □

3.1.1. Bracket for Abelian Extension

The main goal of this subsection is to write down explicitly the bracket in $\mathfrak{e}$. An element $\alpha \in \mathfrak{e}$ can be represented as a two-tuple:[12] $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \mathfrak{g}$ and $\alpha_2 \in \mathfrak{h}$ ($\mathfrak{e} = \mathfrak{g} \oplus \mathfrak{h}$) as a vector space. The injection i is $i \circ \alpha_2 = (0, \alpha_2)$, the projection π is $\pi \circ (\alpha_1, \alpha_2) = \alpha_1$, and since extension is independent of $\mathfrak{s}$ we take $\mathfrak{s} \circ (\alpha_1, 0) = (\alpha_1, 0)$. By linearity

$$[\alpha, \beta]_{\mathfrak{e}} = [(\alpha_1, 0), (\beta_1, 0)]_{\mathfrak{e}} + [(0, \alpha_2), (0, \beta_2)]_{\mathfrak{e}} + [(\alpha_1, 0), (0, \beta_2)]_{\mathfrak{e}} + (0, \alpha_2), (\beta_1, 0)]_{\mathfrak{e}}$$

We know that $[(0, \alpha_2), (0, \beta_2)]_{\mathfrak{e}} = 0$ since $\mathfrak{h}\alpha_1, \mathfrak{s}\beta_1]_{\mathfrak{e}}$ Therefore,

$$[\alpha, \beta]_{\mathfrak{e}} = [s\alpha_1, s\beta_1]_{\mathfrak{e}} + [(\alpha_1, 0), (0, \beta_2)]_{\mathfrak{e}} + (0, \alpha_2), (\beta_1, 0)]_{\mathfrak{e}} \tag{3.11}$$

To write this bracket more explicitly, we evaluate the brackets on the right-hand side of the above equation. By definition of the cocycle w,

$$w(\alpha, \beta) := i^{-1}([\mathfrak{s}\alpha, s\beta]_{\mathfrak{e}} - s[\alpha, \beta]_{\mathfrak{g}})$$

We have

$$[s\alpha_1, s\beta_1]_{\mathfrak{e}} = iw(\alpha_1, \beta_1) + s[\alpha_1, \beta_1]_{\mathfrak{g}} \tag{3.12}$$

Hence

$$[(\alpha_1, 0), (\beta_1, 0)]_{\mathfrak{e}} = [\mathfrak{s}\alpha_1, \mathfrak{s}\beta_1]_{\mathfrak{e}} = iw(\alpha_1, \beta_1) + \mathfrak{s}[\alpha_1, \beta_1]_{\mathfrak{g}} = ([\alpha_1, \beta_1]_{\mathfrak{g}}, w(\alpha_1, \beta_1)) \tag{3.13}$$

By the definition of ρ,

$$[(\alpha_1, 0), (0, \beta_2)]_{\mathfrak{e}} = [s\alpha_1, i\beta_2]_{\mathfrak{e}} = \rho_{\alpha_1}\beta_2$$

Similarly,

$$(0, \alpha_2), (\beta_1, 0)]_{\mathfrak{e}} = [i\alpha_2, s\beta_1]_{\mathfrak{e}} = -\rho_{\beta_1}\alpha_2 \tag{3.14}$$

So,

$$[\alpha, \beta]_{\mathfrak{e}} = ([\alpha_1, \beta_1]_{\mathfrak{g}}, \rho_{\alpha_1}\beta_2 - \rho_{\beta_1}\alpha_2 + w(\alpha_1, \beta_1)) \tag{3.15}$$

is the bracket for the bracket abelian extension [12].

3.2. Semi-Direct and Direct Extensions

Assume that w defined by [12]

$$w\,(\alpha, \beta) := i^{-1}([s\alpha, s\beta]_{\mathfrak{e}} - s[\alpha, \beta]_{\mathfrak{g}}) \tag{3.16}$$

is a coboundary. In that case, there exists an equivalent extension with $w = 0$, by $w^{'} - w = s(i^{-1}v)$. For that equivalent extension s is a homomorphism. Thus, the sequence $\mathfrak{e} \leftarrow \mathfrak{g} \leftarrow 0$ is an exact sequence of Lie algebra homomorphisms as well as the sequence given by

$$0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{e} \rightarrow \mathfrak{g} \rightarrow 0$$

where $i : \mathfrak{h} \rightarrow \mathfrak{e}$, $\pi : \mathfrak{e} \rightarrow \mathfrak{g}$, s: $\mathfrak{g} \rightarrow \mathfrak{e}$. The homomorphism i is an injection and π is a projection(surjection). We then say that the extension is a **semi-direct extension (or semi-direct sum of algebras).** Now consider the case where i^{-1} is a homomorphism and $keri^{-1} = ranges$. Then the sequence

$$0 \rightarrow \mathfrak{h} \leftrightarrow \mathfrak{e} \leftrightarrow \mathfrak{g} \rightarrow 0$$

where $i : \mathfrak{h} \rightarrow \mathfrak{e}$, $\phi : \mathfrak{e} \rightarrow \mathfrak{g}$, s: $\mathfrak{g} \rightarrow \mathfrak{e}$, $i^{-1} : \mathfrak{e} \rightarrow \mathfrak{h}$ is exact in both directions and both i and $\phi = s^{-1}$ are bijections. The action of $\mathfrak{g}$ on $\mathfrak{h}$ is

$$\rho_\alpha \eta = i^{-1}[s\alpha, i\eta]_{\mathfrak{e}} = [i^{-1}s\alpha, \eta]_h = 0 \tag{3.17}$$

since by exactness $i^{-1} \circ s = 0$ This is called a **direct sum**.

3.2.1. Bracket for Semi-Direct and Direct Extension

The main goal of this subsection is to derive the bracket in $\mathfrak{e}$ for a semi-direct sum. An element $\alpha \in \mathfrak{e}$ can be represented as as a two-tuple:[12] $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \mathfrak{g}$ and $\alpha_2 \in \mathfrak{h}$ ($\mathfrak{e} = \mathfrak{g} \oplus \mathfrak{h}$) as a vector space. The injection i is

$$i \circ \alpha_2 = (0, \alpha_2)$$

the projection π is

$$\pi \circ (\alpha_1, \alpha_2) = \alpha_1$$

and since extension is independent of $\mathfrak{s}$ we take $\mathfrak{s} \circ (\alpha_1, 0) = (\alpha_1, 0)$. Also, we know that s is a homomorphism. So that,

$$[(\alpha_1, 0), (\beta_1, 0)]_{\mathfrak{e}} = [s\alpha_1, s\beta_1]_{\mathfrak{e}} = s[\alpha_1, \beta_1]_{\mathfrak{g}} = ([\alpha_1, \beta_1]_{\mathfrak{g}}, 0)$$

and $\mathfrak{h}$ is not assumed Abelian

$$[(0, \alpha_2), (0, \beta_2)]_{\mathfrak{e}} = [i\alpha_2, i\beta_2]_{\mathfrak{e}} = i[\alpha_2, \beta_2]_{\mathfrak{h}} = (0, [\alpha_2, \beta_2]_{\mathfrak{h}}) \tag{3.18}$$

Also

$$[(\alpha_1, 0), (0, \beta_2)]_{\mathfrak{e}} = [s\alpha_1, i\beta_2]_{\mathfrak{e}} = \rho_{\alpha_1 \beta_2} \tag{3.19}$$

and

$$[(0, \alpha_2), (\beta_1, 0)]_{\mathfrak{e}} = [i\alpha_2, s\beta_1]_{\mathfrak{e}} = -[s\beta_1, i\alpha_2]_{\mathfrak{e}} = -\rho_{\beta_1} \alpha_2 \tag{3.20}$$

Thus,

$$[\alpha, \beta]_{\mathfrak{e}} = [(\alpha_1, 0), (\beta_1, 0)]_{\mathfrak{e}} + [(0, \alpha_2), (0, \beta_2)]_{\mathfrak{e}} + [(\alpha_1, 0), (0, \beta_2)]_{\mathfrak{e}} + (0, \alpha_2), (\beta_1, 0)]_{\mathfrak{e}} \tag{3.21}$$

can be written as

$$[\alpha, \beta]_{\mathfrak{e}} = ([\alpha_1, \beta_1]_{\mathfrak{g}}, 0) + (0, [\alpha_2, \beta_2]_{\mathfrak{h}}) + \rho_{\alpha_1 \beta_2} - \rho_{\beta_1 \alpha_2} \tag{3.22}$$

So,

$$[\alpha, \beta]_{\mathfrak{e}} = ([\alpha_1, \beta_1]_{\mathfrak{g}}, [\alpha_2, \beta_2]_{\mathfrak{h}} + \rho_{\alpha_1 \beta_2} - \rho_{\beta_1 \alpha_2}) \tag{3.23}$$

is the bracket for the semi-direct extension. By letting $\rho = 0$ in the bracket of semi-direct extension, we get the desired result for direct extension as:

$$[\alpha, \beta]_{\mathfrak{e}} = ([\alpha_1, \beta_1]_{\mathfrak{g}}, [\alpha_2, \beta_2]_{\mathfrak{h}}) \tag{3.24}$$

4. EXTENSIONS AND ALGEBRA OF DERIVATIONS

Definition 4.0.1. *[4] A smooth manifold Q with a group structure such that the multiplication $Q \times Q \to Q$ and the inversion $Q \to Q$ are smooth maps, is called a Lie group.*

Definition 4.0.2. *[4] An infinite dimensional manifold is a manifold modeled on infinite dimensional locally convex vector space just as a finite dimensional manifold is modeled on $\mathbb{R}^n$.*

Definition 4.0.3. *[4] A Frechet space is a complete locally convex Hausdorff metrizable vector space.*

Definition 4.0.4. *[4] A Frechet manifold is a Hausdorff space with a coordinate atlas taking values in a Frechet space such that all transition functions are smooth maps.*

Lemma 4.0.5. *[4] Let Q be a compact n dimensional manifold. Consider the set $Diff\,(Q)$ of diffeomorphisms of Q. It is an open subspace(the Frechet manifold of) all smooth maps from Q to Q. The composition and inversion are smooth maps, so that the set $Diff\,(Q)$ is a Frechet Lie group.*

In general, if Q is paracompact we may consider $Diff(Q)$ as a Lie group whose Lie algebra consists of vector fields with compact support. [14]

Lemma 4.0.6. *[4] $\mathfrak{X}(Q)$, the Lie algebra of smooth vector fields on Q is the Lie algebra of $Diff\,(Q)$.*

In this chapter, our aim is to extend the Lie algebra $\mathfrak{X}(Q)$ to the semidirect product Lie algebra $\mathfrak{X}(Q) \times F(Q)$ which is a Lie algebra of the Lie group $Diff\,(Q) \ltimes F(Q)$. If we are given a Lie algebra, there are various ways to extend it. But, we can consider the derivations on each Lie algebra. So, if a Lie algebra is given then the extension of a Lie algebra by the derivations on it is naturally done. Because of that, we will consider a Lie algebra extension by its derivation algebra on it.

Consider any exact sequence of homomorphisms of Lie algebras as in (3.1), where the ho-

momorphism

$$i : \mathfrak{h} \to \mathfrak{e}$$

is an injection and the homomorphism

$$\pi : \mathfrak{e} \to \mathfrak{g}$$

is a projection(surjection) . Consider a linear mapping

$$s : \mathfrak{g} \to \mathfrak{e}$$

with

$$\pi \circ s = Id_{\mathfrak{g}} \text{ (the identity mapping in } \mathfrak{g} \text{)}$$

Then s induces mappings [13]

$$\alpha : \mathfrak{g} \to der\left(\mathfrak{h}\right) \tag{4.1}$$

$$X \to \alpha_X$$

such that

$$\alpha_X\left(H\right) = \left[s\left(X\right), H\right]$$

where $X \in \mathfrak{g}$, $H \in \mathfrak{h}$. Let

$$\rho : \overset{2}{\Lambda}\mathfrak{g} \to \mathfrak{h} \tag{4.2}$$

be a Lie algebra valued bivector

$$(X, Y) \to \rho\left(X, Y\right)$$

such that

$$\rho\left(X, Y\right) = \left[s\left(X\right), s\left(Y\right)\right]_{\mathfrak{e}} - s\left(\left[X, Y\right]_{\mathfrak{g}}\right) \tag{4.3}$$

Proposition 4.0.7.

$$\rho\left(X, Y\right) \in \ Image\ \left(i\right)$$

Proof. Consider $\rho\left(X, Y\right)$ and apply the mapping π to $\rho\left(X, Y\right)$.

$$\pi\left(\rho\left(X, Y\right)\right) = \pi\left(\left[s\left(X\right), s\left(Y\right)\right]_{\mathfrak{e}} - s\left(\left[X, Y\right]_{\mathfrak{g}}\right)\right) \tag{4.4}$$

Since π is linear we have

$$\pi\left(\rho\left(X,Y\right)\right)=\pi\left(\left[s\left(X\right),s\left(Y\right)\right]_{\mathfrak{e}}\right)-\pi\left(s\left(\left[X,Y\right]_{\mathfrak{g}}\right)\right) \tag{4.5}$$

By the homomorphism property of πwe can write

$$\pi\left(\rho\left(X,Y\right)\right)=\left[\pi\circ s\left(X\right),\pi\circ s\left(Y\right)\right]_{\mathfrak{g}}-\pi\circ s\left(\left[X,Y\right]_{\mathfrak{g}}\right) \tag{4.6}$$

$$\pi\circ s=Id_{\mathfrak{g}}$$

$$\pi\left(\rho\left(X,Y\right)\right)=\left[X,Y\right]_{\mathfrak{g}}-\left[X,Y\right]_{\mathfrak{g}} \tag{4.7}$$

$$\pi\left(\rho\left(X,Y\right)\right)=0 \tag{4.8}$$

Thus,

$$\rho\left(X,Y\right)\in Ker\left(\pi\right)$$

Since

$$Ker\left(\pi\right)=Image(i)$$

the conclusion follows. □

How much s differs from a homomorphism is determined by the mapping

$$\rho\left(X,Y\right)=\left[s\left(X\right),s\left(Y\right)\right]_{\mathfrak{e}}-s\left(\left[X,Y\right]_{\mathfrak{g}}\right) \tag{4.9}$$

If the section is changed s to $s^{'}=s+b$ for linear $b:\mathfrak{g}\rightarrow\mathfrak{h}$. Then

$$\alpha_{X}^{'}\left(H\right)=\alpha_{X}\left(H\right)+ad_{b(X)}\left(H\right)$$

$$\rho^{'}\left(X,Y\right)=\rho\left(X,Y\right)+ad_{X}b\left(Y\right)-ad_{Y}b\left(X\right)+\left[b\left(X\right),b\left(Y\right)\right]-b\left(\left[X,Y\right]\right)$$

If there is a linear mapping $b:\mathfrak{g}\rightarrow\mathfrak{h}$ such that

$$\alpha_{X}^{'}\left(H\right)=\alpha_{X}\left(H\right)+ad_{b(X)}\left(H\right)$$

and

$$\rho^{'}(X,Y)=\rho(X,Y)+ad_{X}b(Y)-ad_{Y}b(X)+[b(X),b(Y)]-b([X,Y])$$

then, two data (α,ρ) and $\left(\alpha^{'},\rho^{'}\right)$ are equivalent A datum (α,ρ) corresponds to a split extension if and only if (α,ρ) is equivalent to a datum of the form $\left(\alpha^{'},0\right)$. This is the case if and only if there exists a mapping $b:\mathfrak{g}\rightarrow\mathfrak{h}$ such that

$$\rho(X,Y)=-ad_{X}b(Y)+ad_{Y}b(X)-[b(X),b(Y)]+b([X,Y])$$

Theorem 4.0.8. *[13] The Lie algebra structure on* $\mathfrak{e}=\mathfrak{h}\oplus\mathfrak{s}(\mathfrak{g})$ *can be completely described in terms of the maps* α *and* ρ *as*

$$[H_1+s(X_1),H_2+s(X_2)]=[H_1,H_2]-\alpha_{X_2}(H_1)+\alpha_{X_1}(H_2)+\rho(X_1,X_2)+s([X_1,X_2]) \tag{4.10}$$

Proof.

$$\begin{aligned}
[H_1+s(X_1),H_2+s(X_2)] &= [H_1,H_2+s(X_2)]+[s(X_1),H_2+s(X_2)] \\
&= [H_1,H_2]+[H_1,s(X_2)]+[s(X_1),H_2] \\
&\quad +[s(X_1),s(X_2)] \\
&= [H_1,H_2]-[s(X_2),H_1]+[s(X_1),H_2]+\rho(X_1,X_2) \\
&\quad +s([X_1,X_2]) \\
&= [H_1,H_2]-\alpha_{X_2}(H_1)+\alpha_{X_1}(H_2)+\rho(X_1,X_2) \\
&\quad +s([X_1,X_2])
\end{aligned}$$

□

Theorem 4.0.9. *The bracket*

$$[H_1+s(X_1),H_2+s(X_2)]=[H_1,H_2]-\alpha_{X_2}(H_1)+\alpha_{X_1}(H_2)+\rho(X_1,X_2)+s([X_1,X_2])$$

defines a Lie algebra structure on $\mathfrak{e}=\mathfrak{h}\oplus\mathfrak{s}(\mathfrak{g})$ *if the maps* α *and* ρ *satisfy the following*

properties:

$$\rho\left(X_1, X_2\right) = -\rho\left(X_2, X_1\right)$$

for all $X_1, X_2 \in \mathfrak{g}$, *and*

$$\left[\alpha_{X_1}, \alpha_{X_2}\right] - \alpha_{[X_1, X_2]} = ad_{\rho(X_1, X_2)}$$

for all $X_1, X_2 \in \mathfrak{g}$.

Proof.

$$\begin{aligned}
\left[H_1 + s\left(X_1\right), H_2 + s\left(X_2\right)\right] &= \left[H_1, H_2 + s\left(X_2\right)\right] + \left[s\left(X_1\right), H_2 + s\left(X_2\right)\right] \\
&= \left[H_1, H_2\right] + \left[H_1, s\left(X_2\right)\right] + \left[s\left(X_1\right), H_2\right] + \left[s\left(X_1\right), s\left(X_2\right)\right] \\
&= \left[H_1, H_2\right] - \left[s\left(X_2\right), H_1\right] + \left[s\left(X_1\right), H_2\right] + \rho\left(X_1, X_2\right) \\
&\quad + s\left(\left[X_1, X_2\right]\right) \\
&= \left[H_1, H_2\right] - \alpha_{X_2}\left(H_1\right) + \alpha_{X_1}\left(H_2\right) + \rho\left(X_1, X_2\right) \\
&\quad + s\left(\left[X_1, X_2\right]\right)
\end{aligned}$$

$$\begin{aligned}
\left[H_2 + s\left(X_2\right), H_1 + s\left(X_1\right)\right] &= \left[H_2, H_1 + s\left(X_1\right)\right] + \left[s\left(X_2\right), H_1 + s\left(X_1\right)\right] \\
&= \left[H_2, H_1\right] + \left[H_2, s\left(X_1\right)\right] + \left[s\left(X_2\right), H_1\right] + \left[s\left(X_2\right), s\left(X_1\right)\right] \\
&= \left[H_2, H_1\right] - \left[s\left(X_1\right), H_2\right] + \left[s\left(X_2\right), H_1\right] \\
&\quad + \rho\left(X_2, X_1\right) + s\left(\left[X_2, X_1\right]\right) \\
&= \left[H_2, H_1\right] - \alpha_{X_1}\left(H_2\right) + \alpha_{X_2}\left(H_1\right) + \rho\left(X_2, X_1\right) \\
&\quad + s\left(\left[X_2, X_1\right]\right)
\end{aligned}$$

Anti-symmetry implies that

$$\left[H_1 + s\left(X_1\right), H_2 + s\left(X_2\right)\right] = -\left[H_2 + s\left(X_2\right), H_1 + s\left(X_1\right)\right]$$

This is satisfied if and only if

$$\rho\left(X_1, X_2\right) = -\rho\left(X_2, X_1\right)$$

To prove the Jacobi- identity

$$[[H_1 + s(X_1), H_2 + s(X_2)], H_3 + s(X_3)] + [[H_2 + s(X_2), H_3 + s(X_3)], H_1 + s(X_1)]$$
$$+ [[H_3 + s(X_3), H_1 + s(X_1)], H_2 + s(X_2)] = 0$$

We proceed by computing

$$\begin{aligned}
[H_1 + s(X_1), H_2 + s(X_2)] &= [H_1, H_2 + s(X_2)] + [s(X_1), H_2 + s(X_2)] \\
&= [H_1, H_2] + [H_1, s(X_2)] + [s(X_1), H_2] + [s(X_1), s(X_2)] \\
&= [H_1, H_2] + [H_1, s(X_2)] - [H_2, s(X_1)] + [s(X_1), s(X_2)]
\end{aligned}$$

$$\begin{aligned}
&[[H_1 + s(X_1), H_2 + s(X_2)], H_3 + s(X_3)] = \\
&[[H_1, H_2], H_3 + s(X_3)] + [[H_1, s(X_2)], H_3 + s(X_3)] \\
&- [[H_2, s(X_1)], H_3 + s(X_3)] + [[s(X_1), s(X_2)], H_3 + s(X_3)]
\end{aligned}$$

$$\begin{aligned}
= \; & [[H_1, H_2], H_3] + [[H_1, H_2], s(X_3)] + [[H_1, s(X_2)], H_3] \\
& + [[H_1, s(X_2)], s(X_3)] - [[H_2, s(X_1)], H_3] - [[H_2, s(X_1)], s(X_3)] \\
& + [[s(X_1), s(X_2)], H_3] + [[s(X_1), s(X_2)], s(X_3)]
\end{aligned}$$

Interchanging

$$1 \to 2,\ 2 \to 3 \text{ and } 1 \to 3,\ 2 \to 1$$

$$\begin{aligned}
&[[H_2 + s(X_2), H_3 + s(X_3)], H_1 + s(X_1)] = \\
&[[H_2, H_3], H_1 + s(X_1)] + [[H_2, s(X_3)], H_1 + s(X_1)] \\
&- [[H_3, s(X_1)], H_1 + s(X_1)] + [[s(X_2), s(X_3)], H_1 + s(X_1)]
\end{aligned}$$

$$\begin{aligned}
= \; & [[H_2, H_3], H_1] + [[H_2, H_3], s(X_1)] + [[H_2, s(X_3)], H_1] \\
& + [[H_2, s(X_3)], s(X_1)] - [[H_3, s(X_2)], H_1] - [[H_3, s(X_2)], s(X_1)] \\
& + [[s(X_2), s(X_3)], H_1] + [[s(X_2), s(X_3)], s(X_1)]
\end{aligned}$$

and

$$\begin{aligned}
&[[H_3 + s(X_3), H_1 + s(X_1)], H_2 + s(X_2)] = \\
&[[H_3, H_1], H_2 + s(X_2)] + [[H_3, s(X_1)], H_2 + s(X_2)] \\
&- [[H_1, s(X_2)], H_2 + s(X_2)] + [[s(X_3), s(X_1)], H_2 + s(X_2)]
\end{aligned}$$

$$\begin{aligned}
= \; & [[H_3, H_1], H_2] + [[H_3, H_1], s(X_2)] + [[H_3, s(X_1)], H_2] \\
& + [[H_3, s(X_1)], s(X_2)] - [[H_1, s(X_3)], H_2] - [[H_1, s(X_3)], s(X_2)] \\
& + [[s(X_3), s(X_1)], H_2] + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}$$

Therefore

$$\begin{aligned}
&[[H_1 + s(X_1), H_2 + s(X_2)], H_3 + s(X_3)] \\
&+ [[H_2 + s(X_2), H_3 + s(X_3)], H_1 + s(X_1)] \\
&+ [[H_3 + s(X_3), H_1 + s(X_1)], H_2 + s(X_2)]
\end{aligned}$$

$$\begin{aligned}
= \; & [[H_1, H_2], H_3] + [[H_1, H_2], s(X_3)] + [[H_1, s(X_2)], H_3] \\
& + [[H_1, s(X_2)], s(X_3)] - [[H_2, s(X_1)], H_3] - [[H_2, s(X_1)], s(X_3)] \\
& + [[s(X_1), s(X_2)], H_3] + [[s(X_1), s(X_2)], s(X_3)] \\
& + [[H_2, H_3], H_1] + [[H_2, H_3], s(X_1)] + [[H_2, s(X_3)], H_1] \\
& + [[H_2, s(X_3)], s(X_1)] - [[H_3, s(X_2)], H_1] - [[H_3, s(X_2)], s(X_1)] \\
& + [[s(X_2), s(X_3)], H_1] + [[s(X_2), s(X_3)], s(X_1)] \\
& + [[H_3, H_1], H_2] + [[H_3, H_1], s(X_2)] + [[H_3, s(X_1)], H_2] \\
& + [[H_3, s(X_1)], s(X_2)] - [[H_1, s(X_3)], H_2] - [[H_1, s(X_3)], s(X_2)] \\
& + [[s(X_3), s(X_1)], H_2] + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}$$

$$
\begin{aligned}
= \; & [[H_1, H_2], s(X_3)] + [[H_1, s(X_2)], H_3] + [[H_1, s(X_2)], s(X_3)] - [[H_2, s(X_1)], H_3] \\
& - [[H_2, s(X_1)], s(X_3)] + [[s(X_1), s(X_2)], H_3] + [[s(X_1), s(X_2)], s(X_3)] \\
& + [[H_2, H_3], s(X_1)] + [[H_2, s(X_3)], H_1] + [[H_2, s(X_3)], s(X_1)] - [[H_3, s(X_2)], H_1] \\
& - [[H_3, s(X_2)], s(X_1)] + [[s(X_2), s(X_3)], H_1] + [[s(X_2), s(X_3)], s(X_1)] \\
& + [[H_3, H_1], s(X_2)] + [[H_3, s(X_1)], H_2] + [[H_3, s(X_1)], s(X_2)] - [[H_1, s(X_3)], H_2] \\
& - [[H_1, s(X_3)], s(X_2)] + [[s(X_3), s(X_1)], H_2] + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}
$$

$$
\begin{aligned}
= \; & [[H_1, H_2], s(X_3)] + [[H_1, s(X_2)], H_3] + [[H_1, s(X_2)], s(X_3)] - [[H_2, s(X_1)], H_3] \\
& - [[H_2, s(X_1)], s(X_3)] + [[s(X_1), s(X_2)], H_3] + [[s(X_1), s(X_2)], s(X_3)] \\
& + [[H_2, H_3], s(X_1)] + [[H_2, s(X_3)], H_1] + [[H_2, s(X_3)], s(X_1)] - [[H_3, s(X_2)], H_1] \\
& - [[H_3, s(X_2)], s(X_1)] + [[s(X_2), s(X_3)], H_1] + [[s(X_2), s(X_3)], s(X_1)] \\
& + [[H_3, H_1], s(X_2)] + [[H_3, s(X_1)], H_2] + [[H_3, s(X_1)], s(X_2)] - [[H_1, s(X_3)], H_2] \\
& - [[H_1, s(X_3)], s(X_2)] + [[s(X_3), s(X_1)], H_2] + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}
$$

$$
\begin{aligned}
= \; & [[H_2, s(X_3)], H_1] + [[H_1, H_2], s(X_3)] + [[H_1, s(X_2)], H_3] + [[H_3, H_1], s(X_2)] \\
& + [[H_3, s(X_1)], H_2] + [[H_2, H_3], s(X_1)] - [[H_2, s(X_1)], H_3] + [[H_1, s(X_2)], s(X_3)] \\
& - [[H_2, s(X_1)], s(X_3)] + [[s(X_1), s(X_2)], H_3] + [[s(X_1), s(X_2)], s(X_3)] \\
& + [[H_2, s(X_3)], s(X_1)] - [[H_3, s(X_2)], H_1] - [[H_3, s(X_2)], s(X_1)] \\
& + [[s(X_2), s(X_3)], H_1] + [[s(X_2), s(X_3)], s(X_1)] + [[H_3, s(X_1)], s(X_2)] \\
& - [[H_1, s(X_3)], H_2] - [[H_1, s(X_3)], s(X_2)] + [[s(X_3), s(X_1)], H_2] \\
& + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}
$$

$$
\begin{aligned}
[[H_2, s(X_3)], H_1] &= [H_2 \circ s(X_3) - s(X_3) \circ H_2, H_1] \\
&= [H_2 \circ s(X_3), H_1] - [s(X_3) \circ H_2, H_1]
\end{aligned}
$$

$$
\begin{aligned}
&= H_2 \circ s(X_3) \circ H_1 - H_1 \circ H_2 \circ s(X_3) - (s(X_3) \circ H_2 \circ H_1 - H_1 \circ s(X_3) \circ H_2) \\
&= H_2 \circ s(X_3) \circ H_1 - H_1 \circ H_2 \circ s(X_3) - s(X_3) \circ H_2 \circ H_1 + H_1 \circ s(X_3) \circ H_2
\end{aligned}
$$

$$
\begin{aligned}
[[H_2, H_1], s(X_3)] &= [H_2 \circ H_1 - H_1 \circ H_2, s(X_3)] \\
&= [H_2 \circ H_1, s(X_3)] - [H_1 \circ H_2, s(X_3)]
\end{aligned}
$$

$$
\begin{aligned}
&= H_2 \circ H_1 \circ s(X_3) - s(X_3) \circ H_2 \circ H_1 - (H_1 \circ H_2 \circ s(X_3) - s(X_3) \circ H_1 \circ H_2) \\
&= H_2 \circ H_1 \circ s(X_3) - s(X_3) \circ H_2 \circ H_1 - H_1 \circ H_2 \circ s(X_3) + s(X_3) \circ H_1 \circ H_2
\end{aligned}
$$

$$
\begin{aligned}
&[[H_2, s(X_3)], H_1] - [[H_2, H_1], s(X_3)] \\
= \; &H_2 \circ s(X_3) \circ H_1 - H_1 \circ H_2 \circ s(X_3) - s(X_3) \circ H_2 \circ H_1 + H_1 \circ s(X_3) \circ H_2 \\
&- (H_2 \circ H_1 \circ s(X_3) - s(X_3) \circ H_2 \circ H_1 - H_1 \circ H_2 \circ s(X_3) + s(X_3) \circ H_1 \circ H_2)
\end{aligned}
$$

$$
\begin{aligned}
&= H_2 \circ s(X_3) \circ H_1 - H_1 \circ H_2 \circ s(X_3) - s(X_3) \circ H_2 \circ H_1 + H_1 \circ s(X_3) \circ H_2 \\
&\quad -H_2 \circ H_1 \circ s(X_3) + s(X_3) \circ H_2 \circ H_1 + H_1 \circ H_2 \circ s(X_3) - s(X_3) \circ H_1 \circ H_2
\end{aligned}
$$

$$
\begin{aligned}
&= -H_2 \circ H_1 \circ s(X_3) + H_2 \circ s(X_3) \circ H_1 + H_1 \circ s(X_3) \circ H_2 - s(X_3) \circ H_1 \circ H_2 \\
&= -H_2 \circ (H_1 \circ s(X_3) - s(X_3) \circ H_1) + (H_1 \circ s(X_3) - s(X_3) \circ H_1) \circ H_2 \\
&= -H_2 \circ [H_1, s(X_3)] + [H_1, s(X_3)] \circ H_2 \\
&= [H_1, s(X_3)] \circ H_2 - H_2 \circ [H_1, s(X_3)] \\
&= [[H_1, s(X_3)], H_2]
\end{aligned}
$$

$$[[H_2, s(X_3)], H_1] - [[H_2, H_1], s(X_3)] = [[H_1, s(X_3)], H_2]$$

Therefore, we get

$$[[H_2, s(X_3)], H_1] + [[H_1, H_2], s(X_3)] = [[H_1, s(X_3)], H_2]$$

$$\begin{aligned}[[H_1, s(X_2)], H_3] &= [H_1 \circ s(X_2) - s(X_2) \circ H_1, H_3] \\ &= [H_1 \circ s(X_2), H_3] - [s(X_2) \circ H_1, H_3]\end{aligned}$$

$$\begin{aligned}&= H_1 \circ s(X_2) \circ H_3 - H_3 \circ H_1 \circ s(X_2) - (s(X_2) \circ H_1 \circ H_3 - H_3 \circ s(X_2) \circ H_1) \\ &= H_1 \circ s(X_2) \circ H_3 - H_3 \circ H_1 \circ s(X_2) - s(X_2) \circ H_1 \circ H_3 + H_3 \circ s(X_2) \circ H_1\end{aligned}$$

$$\begin{aligned}[[H_1, H_3], s(X_2)] &= [H_1 \circ H_3 - H_3 \circ H_1, s(X_2)] \\ &= [H_1 \circ H_3, s(X_2)] - [H_3 \circ H_1, s(X_2)]\end{aligned}$$

$$\begin{aligned}&= H_1 \circ H_3 \circ s(X_2) - s(X_2) \circ H_1 \circ H_3 - (H_3 \circ H_1 \circ s(X_2) - s(X_2) \circ H_3 \circ H_1) \\ &= H_1 \circ H_3 \circ s(X_2) - s(X_2) \circ H_1 \circ H_3 - H_3 \circ H_1 \circ s(X_2) + s(X_2) \circ H_3 \circ H_1\end{aligned}$$

$$\begin{aligned}&[[H_1, s(X_2)], H_3] - [[H_1, H_3], s(X_2)] \\ = \;& H_1 \circ s(X_2) \circ H_3 - H_3 \circ H_1 \circ s(X_2) - s(X_2) \circ H_1 \circ H_3 + H_3 \circ s(X_2) \circ H_1 \\ & - (H_1 \circ H_3 \circ s(X_2) - s(X_2) \circ H_1 \circ H_3 - H_3 \circ H_1 \circ s(X_2) + s(X_2) \circ H_3 \circ H_1)\end{aligned}$$

$$\begin{aligned}&= H_1 \circ s(X_2) \circ H_3 - H_3 \circ H_1 \circ s(X_2) - s(X_2) \circ H_1 \circ H_3 + H_3 \circ s(X_2) \circ H_1 \\ & -H_1 \circ H_3 \circ s(X_2) + s(X_2) \circ H_1 \circ H_3 + H_3 \circ H_1 \circ s(X_2) - s(X_2) \circ H_3 \circ H_1\end{aligned}$$

$$= H_1 \circ s(X_2) \circ H_3 + H_3 \circ s(X_2) \circ H_1 - H_1 \circ H_3 \circ s(X_2) - s(X_2) \circ H_3 \circ H_1$$

$$= -H_1 \circ H_3 \circ s(X_2) + H_1 \circ s(X_2) \circ H_3 + H_3 \circ s(X_2) \circ H_1 - s(X_2) \circ H_3 \circ H_1$$

$$= -H_1 \circ (H_3 \circ s(X_2) - s(X_2) \circ H_3) + (H_3 \circ s(X_2) - s(X_2) \circ H_3) \circ H_1$$

$$= -H_1 \circ [H_3, s(X_2)] + [H_3, s(X_2)] \circ H_1$$

$$= [H_3, s(X_2)] \circ H_1 - H_1 \circ [H_3, s(X_2)]$$

$$= [[H_3, s(X_2)], H_1]$$

$$[[H_1, s(X_2)], H_3] - [[H_1, H_3], s(X_2)] = [[H_3, s(X_2)], H_1]$$

Therefore, we get the result as

$$[[H_1, s(X_2)], H_3] + [[H_3, H_1], s(X_2)] = [[H_3, s(X_2)], H_1]$$

$$\begin{aligned} [[H_3, s(X_1)], H_2] &= [H_3 \circ s(X_1) - s(X_1) \circ H_3, H_2] \\ &= [H_3 \circ s(X_1), H_2] - [s(X_1) \circ H_3, H_2] \end{aligned}$$

$$\begin{aligned} &= H_3 \circ s(X_1) \circ H_2 - H_2 \circ H_3 \circ s(X_1) - (s(X_1) \circ H_3 \circ H_2 - H_2 \circ s(X_1) \circ H_3) \\ &= H_3 \circ s(X_1) \circ H_2 - H_2 \circ H_3 \circ s(X_1) - s(X_1) \circ H_3 \circ H_2 + H_2 \circ s(X_1) \circ H_3 \end{aligned}$$

$$\begin{aligned} [[H_2, H_3], s(X_1)] &= [H_2 \circ H_3 - H_3 \circ H_2, s(X_1)] \\ &= [H_2 \circ H_3, s(X_1)] - [H_3 \circ H_2, s(X_1)] \end{aligned}$$

$$\begin{aligned} &= H_2 \circ H_3 \circ s(X_1) - s(X_1) \circ H_2 \circ H_3 - (H_3 \circ H_2 \circ s(X_1) - s(X_1) \circ H_3 \circ H_2) \\ &= H_2 \circ H_3 \circ s(X_1) - s(X_1) \circ H_2 \circ H_3 - H_3 \circ H_2 \circ s(X_1) + s(X_1) \circ H_3 \circ H_2 \end{aligned}$$

$$\begin{aligned} &[[H_3, s(X_1)], H_2] + [[H_2, H_3], s(X_1)] \\ &= H_3 \circ s(X_1) \circ H_2 - H_2 \circ H_3 \circ s(X_1) - s(X_1) \circ H_3 \circ H_2 + H_2 \circ s(X_1) \circ H_3 \\ &\quad + H_2 \circ H_3 \circ s(X_1) - s(X_1) \circ H_2 \circ H_3 - H_3 \circ H_2 \circ s(X_1) + s(X_1) \circ H_3 \circ H_2 \end{aligned}$$

$$= H_3 \circ s(X_1) \circ H_2 - H_3 \circ H_2 \circ s(X_1) + H_2 \circ s(X_1) \circ H_3 - s(X_1) \circ H_2 \circ H_3$$

$$= -H_3 \circ H_2 \circ s(X_1) + H_3 \circ s(X_1) \circ H_2 + H_2 \circ s(X_1) \circ H_3 - s(X_1) \circ H_2 \circ H_3$$

$$= -H_3 \circ [H_2, s(X_1)] + [H_2, s(X_1)] \circ H_3$$

$$= [H_2, s(X_1)] \circ H_3 - H_3 \circ [H_2, s(X_1)]$$

$$= [[H_2, s(X_1)], H_3]$$

Therefore, we get the result as

$$[[H_3, s(X_1)], H_2] + [[H_2, H_3], s(X_1)] = [[H_2, s(X_1)], H_3]$$

$$\begin{aligned}
&[[H_1 + s(X_1), H_2 + s(X_2)], H_3 + s(X_3)] \\
&+ [[H_2 + s(X_2), H_3 + s(X_3)], H_1 + s(X_1)] \\
&+ [[H_3 + s(X_3), H_1 + s(X_1)], H_2 + s(X_2)]
\end{aligned}$$

$$\begin{aligned}
= \; & [[H_2, s(X_3)], H_1] + [[H_1, H_2], s(X_3)] + [[H_1, s(X_2)], H_3] + [[H_3, H_1], s(X_2)] \\
& + [[H_3, s(X_1)], H_2] + [[H_2, H_3], s(X_1)] - [[H_2, s(X_1)], H_3] + [[H_1, s(X_2)], s(X_3)] \\
& - [[H_2, s(X_1)], s(X_3)] + [[s(X_1), s(X_2)], H_3] + [[s(X_1), s(X_2)], s(X_3)] \\
& + [[H_2, s(X_3)], s(X_1)] - [[H_3, s(X_2)], H_1] - [[H_3, s(X_2)], s(X_1)] \\
& + [[s(X_2), s(X_3)], H_1] + [[s(X_2), s(X_3)], s(X_1)] + [[H_3, s(X_1)], s(X_2)] \\
& - [[H_1, s(X_3)], H_2] - [[H_1, s(X_3)], s(X_2)] + [[s(X_3), s(X_1)], H_2] \\
& + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}$$

$$\begin{aligned}
= \; & [[H_1, s(X_3)], H_2] + [[H_3, s(X_2)], H_1] + [[H_2, s(X_1)], H_3] \\
& - [[H_2, s(X_1)], H_3] + [[H_1, s(X_2)], s(X_3)] - [[H_2, s(X_1)], s(X_3)] \\
& + [[s(X_1), s(X_2)], H_3] + [[s(X_1), s(X_2)], s(X_3)] + [[H_2, s(X_3)], s(X_1)] \\
& - [[H_3, s(X_2)], H_1] - [[H_3, s(X_2)], s(X_1)] + [[s(X_2), s(X_3)], H_1] \\
& + [[s(X_2), s(X_3)], s(X_1)] + [[H_3, s(X_1)], s(X_2)] - [[H_1, s(X_3)], H_2] \\
& - [[H_1, s(X_3)], s(X_2)] + [[s(X_3), s(X_1)], H_2] + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}$$

$$\begin{aligned}
= \; & [[H_1, s(X_2)], s(X_3)] - [[H_2, s(X_1)], s(X_3)] + [[H_2, s(X_3)], s(X_1)] \\
& - [[H_3, s(X_2)], s(X_1)] + [[H_3, s(X_1)], s(X_2)] - [[H_1, s(X_3)], s(X_2)] \\
& + [[s(X_1), s(X_2)], H_3] + [[s(X_2), s(X_3)], H_1] + [[s(X_3), s(X_1)], H_2] \\
& + [[s(X_1), s(X_2)], s(X_3)] + [[s(X_2), s(X_3)], s(X_1)] + [[s(X_3), s(X_1)], s(X_2)]
\end{aligned}$$

$$
\begin{aligned}
&= [[H_3, s(X_1)], s(X_2)] - [[H_3, s(X_2)], s(X_1)] + [[s(X_1), s(X_2)], H_3] \\
&\quad + [[H_1, s(X_2)], s(X_3)] - [[H_1, s(X_3)], s(X_2)] + [[s(X_2), s(X_3)], H_1] \\
&\quad + [[H_2, s(X_3)], s(X_1)] - [[H_2, s(X_1)], s(X_3)] + [[s(X_3), s(X_1)], H_2]
\end{aligned}
$$

$$
\begin{aligned}
= \; & [[H_3, s(X_1)], s(X_2)] - [[H_3, s(X_2)], s(X_1)] + [\rho(X_1, X_2) + s([X_1, X_2]), H_3] \\
& + [[H_1, s(X_2)], s(X_3)] - [[H_1, s(X_3)], s(X_2)] + [\rho(X_2, X_3) + s([X_2, X_3]), H_1] \\
& + [[H_2, s(X_3)], s(X_1)] - [[H_2, s(X_1)], s(X_3)] + [\rho(X_3, X_1) + s([X_3, X_1]), H_2]
\end{aligned}
$$

$$
\begin{aligned}
= \; & [[H_3, s(X_1)], s(X_2)] - [[H_3, s(X_2)], s(X_1)] + [\rho(X_1, X_2), H_3] + [s([X_1, X_2]), H_3] \\
& + [[H_1, s(X_2)], s(X_3)] - [[H_1, s(X_3)], s(X_2)] + [\rho(X_2, X_3), H_1] + [s([X_2, X_3]), H_1] \\
& + [[H_2, s(X_3)], s(X_1)] - [[H_2, s(X_1)], s(X_3)] + [\rho(X_3, X_1), H_2] + [s([X_3, X_1]), H_2]
\end{aligned}
$$

$$
\begin{aligned}
&= [[H_1, s(X_2)], s(X_3)] - [[H_2, s(X_1)], s(X_3)] + [[H_2, s(X_3)], s(X_1)] \\
&\quad - [[H_3, s(X_2)], s(X_1)] + [[H_3, s(X_1)], s(X_2)] - [[H_1, s(X_3)], s(X_2)] \\
&\quad + [\rho(X_1, X_2), H_3] + [s([X_1, X_2]), H_3] + [\rho(X_2, X_3), H_1] \\
&\quad + [s([X_2, X_3]), H_1] + [\rho(X_3, X_1), H_2] + [s([X_3, X_1]), H_2]
\end{aligned}
$$

$$
\begin{aligned}
= \; & [[H_1, s(X_2)], s(X_3)] - [[H_1, s(X_3)], s(X_2)] + [s([X_2, X_3]), H_1] \\
& + [\rho(X_2, X_3), H_1] + [[H_2, s(X_3)], s(X_1)] - [[H_2, s(X_1)], s(X_3)] \\
& + [s([X_3, X_1]), H_2] + [\rho(X_3, X_1), H_2] + [[H_3, s(X_1)], s(X_2)] \\
& - [[H_3, s(X_2)], s(X_1)] + [s([X_1, X_2]), H_3] + [\rho(X_1, X_2), H_3]
\end{aligned}
$$

$$
\begin{aligned}
= \; & -[[s(X_2), H_1], s(X_3)] + [[s(X_3), H_1], s(X_2)] + [s([X_2, X_3]), H_1] \\
& + [\rho(X_2, X_3), H_1] - [[s(X_3), H_2], s(X_1)] + [[s(X_1), H_2], s(X_3)] \\
& + [s([X_3, X_1]), H_2] + [\rho(X_3, X_1), H_2] - [[s(X_1), H_3], s(X_2)] \\
& + [[s(X_2), H_3], s(X_1)] + [s([X_1, X_2]), H_3] + [\rho(X_1, X_2), H_3]
\end{aligned}
$$

$$
\begin{aligned}
= \; & [s(X_3), [s(X_2), H_1]] - [s(X_2), [s(X_3), H_1]] + [s([X_2, X_3]), H_1] \\
& + [\rho(X_2, X_3), H_1] + [s(X_1), [s(X_3), H_2]] - [s(X_3), [s(X_1), H_2]] \\
& + [s([X_3, X_1]), H_2] + [\rho(X_3, X_1), H_2] + [s(X_2), [s(X_1), H_3]] \\
& - [s(X_1), [s(X_2), H_3]] + [s([X_1, X_2]), H_3] + [\rho(X_1, X_2), H_3]
\end{aligned}
$$

$$
\begin{aligned}
= \; & \alpha_{X_3}\alpha_{X_2}H_1 - \alpha_{X_2}\alpha_{X_3}H_1 + \alpha_{[X_2,X_3]}H_1 + ad_{\rho(X_2,X_3)}H_1 \\
& + \alpha_{X_1}\alpha_{X_3}H_2 - \alpha_{X_3}\alpha_{X_1}H_2 + \alpha_{[X_3,X_1]}H_2 + ad_{\rho(X_3,X_1)}H_2 \\
& + \alpha_{X_2}\alpha_{X_1}H_3 - \alpha_{X_1}\alpha_{X_2}H_3 + \alpha_{[X_1,X_2]}H_3 + ad_{\rho(X_1,X_2)}H_3
\end{aligned}
$$

$$
\begin{aligned}
= \; & \alpha_{[X_2,X_3]}H_1 - \alpha_{X_2}\alpha_{X_3}H_1 + \alpha_{X_3}\alpha_{X_2}H_1 + ad_{\rho(X_2,X_3)}H_1 \\
& + \alpha_{[X_3,X_1]}H_2 - \alpha_{X_3}\alpha_{X_1}H_2 + \alpha_{X_1}\alpha_{X_3}H_2 + ad_{\rho(X_3,X_1)}H_2 \\
& + \alpha_{[X_1,X_2]}H_3 - \alpha_{X_1}\alpha_{X_2}H_3 + \alpha_{X_2}\alpha_{X_1}H_3 + ad_{\rho(X_1,X_2)}H_3
\end{aligned}
$$

$$
\begin{aligned}
= \; & \left(\alpha_{[X_2,X_3]} - \alpha_{X_2}\alpha_{X_3} + \alpha_{X_3}\alpha_{X_2}\right)H_1 + ad_{\rho(X_2,X_3)}H_1 \\
& + \left(\alpha_{[X_3,X_1]} - \alpha_{X_3}\alpha_{X_1} + \alpha_{X_1}\alpha_{X_3}\right)H_2 + ad_{\rho(X_3,X_1)}H_2 \\
& + \left(\alpha_{[X_1,X_2]} - \alpha_{X_1}\alpha_{X_2} + \alpha_{X_2}\alpha_{X_1}\right)H_3 + ad_{\rho(X_1,X_2)}H_3
\end{aligned}
$$

$$
\begin{aligned}
= \; & \left(\alpha_{[X_2,X_3]} - \alpha_{X_2}\alpha_{X_3} + \alpha_{X_3}\alpha_{X_2} + ad_{\rho(X_2,X_3)}\right)H_1 \\
& + \left(\alpha_{[X_3,X_1]} - \alpha_{X_3}\alpha_{X_1} + \alpha_{X_1}\alpha_{X_3} + ad_{\rho(X_3,X_1)}\right)H_2 \\
& + \left(\alpha_{[X_1,X_2]} - \alpha_{X_1}\alpha_{X_2} + \alpha_{X_2}\alpha_{X_1} + ad_{\rho(X_1,X_2)}\right)H_3
\end{aligned}
$$

So, the Jacobi identity is satisfied if and only if

$$\alpha_{[X_2,X_3]} - \alpha_{X_2}\alpha_{X_3} + \alpha_{X_3}\alpha_{X_2} + ad_{\rho(X_2,X_3)} = 0$$

$$\Rightarrow \quad ad_{\rho(X_2,X_3)} = \alpha_{X_2}\alpha_{X_3} - \alpha_{X_3}\alpha_{X_2} - \alpha_{[X_2,X_3]} = [\alpha_{X_2}, \alpha_{X_3}] - \alpha_{[X_2,X_3]}$$

for all $X_2, X_3 \in \mathfrak{g}$. $\square$

4.0.1. Special Case 1: s is a homomorphism

Assume that s is a homomorphism. In this case

$$\rho(X,Y) = [s(X), s(Y)]_{\mathfrak{e}} - s\left([X,Y]_{\mathfrak{g}}\right) = 0.$$

Also we have

$$ad_{\rho(X,Y)} = 0$$

and so

$$[\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = 0$$

That is;

$$[\alpha_X, \alpha_Y] = \alpha_{[X,Y]}$$

In that case, the Lie algebra structure on $\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{s}(\mathfrak{g})$ can be completely described in terms of the maps α and ρ as:

$$[H_1 + s(X_1), H_2 + s(X_2)] = [H_1, H_2] - \alpha_{X_2}(H_1) + \alpha_{X_1}(H_2) + s([X_1, X_2])$$

Theorem 4.0.10. *If*

$$\rho(X_1, X_2) = 0$$

then the bracket

$$[H_1 + s(X_1), H_2 + s(X_2)] = [H_1, H_2] - \alpha_{X_2}(H_1) + \alpha_{X_1}(H_2) + s([X_1, X_2]) \quad (4.11)$$

defines a Lie algebra structure on $\mathfrak{e} = \mathfrak{h} \oplus \mathfrak{s}(\mathfrak{g})$ *if the map* α *satisfies the following property:*

$$[\alpha_{X_1}, \alpha_{X_2}] = \alpha_{X_1}\alpha_{X_2} - \alpha_{X_2}\alpha_{X_1} \quad (4.12)$$

where $X_1, X_2 \in \mathfrak{g}$.

This theorem shows that, derivations form a Lie algebra.

4.0.2. Extension of a Lie Algebra by Its Derivation Algebra

Definition 4.0.11. *[1] A derivation is a linear map* $\delta : \mathfrak{g} \to \mathfrak{g}$ *of a Lie algebra* $\mathfrak{g}$ *if it also satisfies the Leibniz identity. That is,*

$$\delta([X,Y]) = [\delta(X), Y] + [X, \delta(Y)] \tag{4.13}$$

for all $X, Y \in \mathfrak{g}$.

Let us consider the Lie algebra $\mathfrak{g}Ⓢ Der_{\mathfrak{g}}$ where $Der_{\mathfrak{g}}$ denotes the algebra of derivations.

Consider $(\mathfrak{g}, Der_{\mathfrak{g}}, \lhd, \rhd)$ where $\mathfrak{g}$ and $Der_{\mathfrak{g}}$ are Lie algebras and

$$\rhd : Der_{\mathfrak{g}} \times \mathfrak{g} \to \mathfrak{g} \tag{4.14}$$

$$\lhd : Der_{\mathfrak{g}} \times \mathfrak{g} \to Der_{\mathfrak{g}} \tag{4.15}$$

such that $(\mathfrak{g}, \rhd)$ is a left $Der_{\mathfrak{g}}$ module and $(Der_{\mathfrak{g}}, \lhd)$ is a right $\mathfrak{g}$ module. Let us consider the linear maps

$$\lhd \ : \ Der\mathfrak{g} \times \mathfrak{g} \to Der\mathfrak{g}$$

$$(\xi, X) \to \xi \lhd X$$

$$\rhd \ : \ Der\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$(\xi, X) \to \xi \rhd X$$

Lemma 4.0.12. $\rhd$ *is a left action of* $Der\mathfrak{g}$ *on* $\mathfrak{g}$

$$\alpha \rhd (X + Y) = \alpha \rhd X + \alpha \rhd Y$$

$$(\alpha + \xi) \rhd X = \alpha \rhd X + \xi \rhd X$$

$$[\alpha, \xi] \rhd X = [\alpha \rhd X, \xi \rhd X]$$

Lemma 4.0.13. $\lhd$ *is a right action of* $\mathfrak{g}$ *on* $Der\mathfrak{g}$

$$(\alpha + \xi) \lhd X = \alpha \lhd X + \xi \lhd X$$

$$\alpha \lhd (X + Y) = \alpha \lhd X + \alpha \lhd Y$$

$$\alpha \lhd [X, Y] = [\alpha \lhd X, \alpha \lhd Y]$$

Definition 4.0.14. *The bracket on the system* $(\mathfrak{g}, Der_{\mathfrak{g}}, \lhd, \rhd)$ *is*

$$[(X, \alpha), (Y, \xi)] := ([X, Y] + \alpha \rhd Y - \xi \rhd X\ ,\ [\alpha, \xi] + \alpha \lhd Y - \xi \lhd X) \tag{4.16}$$

where $X, Y \in \mathfrak{g}$ *and* $\alpha, \xi \in Der_{\mathfrak{g}}$.

Theorem 4.0.15. *The system* $(\mathfrak{g}, Der_{\mathfrak{g}}, \lhd, \rhd)$ *with the bracket defined as in the above equation has a Lie algebra structure if and only if the following conditions hold:*

$i.$

$$[\alpha, \xi] \rhd X = \alpha \rhd (\xi \rhd X) - \xi \rhd (\alpha \rhd X)$$

for all $\alpha, \xi \in Der\mathfrak{g}$ *and* $X \in \mathfrak{g}$

$ii.$

$$\alpha \rhd [X, Y] = [X, \alpha \rhd Y] - [Y, \alpha \rhd X] + (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X$$

for all $\alpha \in Der\mathfrak{g}$ *and* $X, Y \in \mathfrak{g}$

$iii.$

$$[\alpha, \xi] \lhd X = [\alpha, \xi \lhd X] - [\xi, \alpha \lhd X] + \alpha \lhd (\xi \rhd X) - \xi \lhd (\alpha \rhd X)$$

for all $\alpha, \xi \in Der\mathfrak{g}$ *and* $X \in \mathfrak{g}$

$iv.$

$$\alpha \lhd [X, Y] = (\alpha \lhd X) \lhd Y - (\alpha \lhd Y) \lhd X$$

for all $\alpha \in Der\mathfrak{g}$ *and* $X, Y \in \mathfrak{g}$

The conditions (ii) and (iii) are important because they are related with the Lie algebra cohomology.

5. LIE ALGEBRA COHOMOLOGY

Definition 5.0.1. *[9] Let $\mathfrak{g}$ be a Lie K-algebra, $\mathfrak{g}$ is a $\mathfrak{g}$-module for the adjoint action, there is a resolution of $Hom_{\mathfrak{g}}(\mathfrak{g},\mathfrak{g})$ by*

$$\mathfrak{g} \to Hom(\mathfrak{g},\mathfrak{g}) \to Hom\left(\overset{2}{\Lambda}\mathfrak{g},\mathfrak{g}\right) \to Hom\left(\overset{3}{\Lambda}\mathfrak{g},\mathfrak{g}\right) \to .....$$

where the differential is given by

$$dX(Y) = [X,Y] \tag{5.1}$$

$$dh(X,Y) = [X,h(Y)] - h([X,Y]) + [h(X),Y] \tag{5.2}$$

$$\begin{aligned} dh(X,Y,Z) &= [X,h(Y,Z)] + [Y,h(Z,X)] + [Z,h(X,Y)] && (5.3)\\ &\quad -h([X,Y],Z) - h([Y,Z],X) - h([Z,X],Y) && (5.4)\end{aligned}$$

Lemma 5.0.2. *$dh(X,Y,Z) = 0$ if and only if*

$$[X,h(Y,Z)]+[Y,h(Z,X)]+[Z,h(X,Y)]-h([X,Y],Z)-h([Y,Z],X)-h([Z,X],Y) = 0$$

That is

$$[X,h(Y,Z)]-h([Y,Z],X) = -[Z,h(X,Y)]+h([X,Y],Z)+h([Z,X],Y)-[Y,h(Z,X)]$$

$$[X,h(Y,Z)]+h(X,[Y,Z]) = [h(X,Y),Z]+h([X,Y],Z)+h(Y,[X,Z])+[Y,h(X,Z)] \tag{5.5}$$

Definition 5.0.3. *Let $h : V \to V$ be an anti-symmetric linear map,then*

$$[X,h(Y,Z)]+h(X,[Y,Z]) = [h(X,Y),Z]+h([X,Y],Z)+h(Y,[X,Z])+[Y,h(X,Z)] \tag{5.6}$$

is called a Lie 2- cocycle in the Lie cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$.

Let us deal with being Lie algebra conditions (ii) and (iii). First of all, let us deal with Lie

algebra condition (ii) .

$$\alpha \rhd [X,Y] = [X, \alpha \rhd Y] - [Y, \alpha \rhd X] + (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X$$

That is

$$\alpha \rhd [X,Y] - [X, \alpha \rhd Y] + [Y, \alpha \rhd X] = (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X$$

$$\alpha \rhd [X,Y] - [X, \alpha \rhd Y] - [\alpha \rhd X, Y] = (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X \tag{5.7}$$

Let us define a map

$$N_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

as

$$(X,Y) \to N_\alpha (X,Y)$$

such that

$$N_\alpha (X,Y) = (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X \tag{5.8}$$

Theorem 5.0.4. *The map defined as*

$$N_\alpha (X,Y) = (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X$$

is a Lie 2- cocycle in the Lie cohomology of $\mathfrak{g}$ *with coefficients in* $\mathfrak{g}$.

Proof. The map defined as

$$N_\alpha (X,Y) = (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X$$

is a Lie 2- cocycle in the Lie cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$. since

$$\begin{aligned} dN_\alpha (X,Y,Z) &= [X, N_\alpha (Y,Z)] + [Y, N_\alpha (Z,X)] + [Z, N_\alpha (X,Y)] \\ &\quad - N_\alpha ([X,Y], Z) - N_\alpha ([Y,Z], X) - N_\alpha ([Z,X], Y) \\ &= 0 \end{aligned}$$

□

The map defined by

$$N_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$(X, Y) \to N_\alpha (X, Y)$$

such that

$$\begin{aligned} N_\alpha (X, Y) &= (\alpha \lhd X) \rhd Y - (\alpha \lhd Y) \rhd X \\ &= \alpha \rhd [X, Y] - [X, \alpha \rhd Y] - [\alpha \rhd X, Y] \end{aligned}$$

measures whether the action acts as a derivation on bracket.

Lemma 5.0.5. *If*

$$N_\alpha (X, Y) = 0$$

that is,

$$\alpha \rhd [X, Y] - [X, \alpha \rhd Y] - [\alpha \rhd X, Y] = 0$$

then in this case action acts as a derivation on bracket since

$$\alpha \rhd [X, Y] = [X, \alpha \rhd Y] + [\alpha \rhd X, Y] \tag{5.9}$$

Let us deal with the Lie algebra condition (iii) .

$$[\alpha, \xi] \lhd X = [\alpha, \xi \lhd X] - [\xi, \alpha \lhd X] + \alpha \lhd (\xi \rhd X) - \xi \lhd (\alpha \rhd X)$$

$$[\alpha, \xi] \lhd X - [\alpha, \xi \lhd X] + [\xi, \alpha \lhd X] = \alpha \lhd (\xi \rhd X) - \xi \lhd (\alpha \rhd X)$$

$$[\alpha, \xi] \lhd X - [\alpha, \xi \lhd X] - [\alpha \lhd X, \xi] = \alpha \lhd (\xi \rhd X) - \xi \lhd (\alpha \rhd X)$$

Let us define a map

$$N_X : Der\mathfrak{g} \times Der\mathfrak{g} \to Der\mathfrak{g}$$

as

$$(\alpha, \xi) \to N_X (\alpha, \xi)$$

such that

$$N_X (\alpha, \xi) = \alpha \lhd (\xi \rhd X) - \xi \lhd (\alpha \rhd X) \tag{5.10}$$

Theorem 5.0.6. *The map defined as*

$$N_X(\alpha, \xi) = \alpha \lhd (\xi \rhd X) - \xi \lhd (\alpha \rhd X) \tag{5.11}$$

is a Lie 2- cocycle in the Lie cohomology of $Der\mathfrak{g}$ with coefficients in $Der\mathfrak{g}$.

Proof. The map defined as

$$N_X(\xi, \tau) = \xi \lhd (\tau \rhd X) - \tau \lhd (\xi \rhd X)$$

is a Lie 2-cocycle in the Lie cohomology of $Der\mathfrak{g}$ with coefficients in $Der\mathfrak{g}$ since

$$\begin{aligned} dN_X(\alpha, \xi, \tau) &= [\alpha, N_X(\xi, \tau)] + [\xi, N_X(\tau, \alpha)] + [\tau, N_X(\alpha, \xi)] \\ &\quad -N_X([\alpha, \xi], \tau) - N_X([\xi, \tau], \alpha) - N_X([\tau, \alpha], \xi) \\ &= 0 \end{aligned}$$

□

Let us consider a map

$$\delta_\alpha : \mathfrak{g} \to \mathfrak{g}$$

$$X \to \delta_\alpha(X)$$

such that

$$\delta_\alpha(X) = \alpha \rhd X \tag{5.12}$$

Then

$$d\delta_\alpha(X, Y) = -N_\alpha(X, Y) \tag{5.13}$$

Therefore, this is a boundary map.

6. EXTENSION BY A SUBALGEBRA OF A DERIVATION ALGEBRA

In this chapter, we want to understand the relationship between the extension by a subalgebra of a derivation algebra and the extension by the algebra of functions. Hamiltonian vector fields will be a tool to understand that connection.

Definition 6.0.1. *If M which is equipped with a closed nondegenerate differential two form,then it is called a symplectic manifold.*

Suppose (M, w) is a symplectic manifold. The fiberwise-linear isomorphism

$$w : TM \to T^*M$$

between the tangent bundle TM and the cotangent bundle T^*M, with the inverse

$$\Omega : T^*M \to TM, \Omega = w^{-1}.$$

is set up by the symplectic form w since it is nondegenerate. Therefore, one forms on a symplectic manifold M may be identified with vector fields and every differentiable function $f : M \to \mathbb{R}$ determines a unique vector field X_f, called the Hamiltonian vector field with the Hamiltonian f. The Hamiltonian vector field X_f is defined as

$$X_f = w^{-1}df$$

6.0.1. Algebra of Hamiltonian Vector Fields

Definition 6.0.2. *[1] If there is a function $f : M \to \mathbb{R}$ such that*

$$i_X w = df$$

then a vector field X on M is called Hamiltonian.

Definition 6.0.3. *If $i_X w$ is closed, then a vector field X is called locally Hamiltonian vector*

field. This is equivalent to

$$\mathcal{L}_X w = 0$$

where $\mathcal{L}_X w$ denotes the Lie differentiation of w along the vector field X because

$$\mathcal{L}_X w = i_X dw + d i_X w$$

Definition 6.0.4. *[3] A smooth vector field X on M is a symplectic vector field if $\mathcal{L}_X w = 0$.*

Lemma 6.0.5. *[1] A vector field is X on M is symplectic if and only if $i_X w$ is closed.*

Proof. The proof of that lemma follows by Cartan's magic formula and the closedness of w.

$$\mathcal{L}_X w = i_X dw + d i_X w = d i_X w$$

Since

$$\mathcal{L}_X w = 0 \Leftrightarrow d i_X w = 0 \Leftrightarrow i_X w \text{ is closed.}$$

□

Remark 6.0.6. *Any Hamiltonian vector field is a symplectic vector field. If $H^1(M, \mathbb{R}) = 0$, then $i_X w$ on M is exact. So, in this case that any symplectic vector field is a Hamiltonian vector field.*

Lemma 6.0.7. *[3] The bracket of two locally Hamiltonian vector fields on a symplectic manifold (M, w) is locally Hamiltonian.*

Let us consider the cotangent bundle T^*M, with its intrinstic symplectic structure $w = dq^i \Lambda dp_i$. Let $\mathfrak{X}(T^*M)$ be the algebra of vector fields on T^*M and $\mathfrak{X}_{symp}(T^*M)$ be the algebra of symplectic elements and also $\mathfrak{X}_{ham}(T^*M)$ be the algebra of hamiltonian elements.

$$\mathfrak{X}_{symp}(T^*M) := \{X \in \mathfrak{X}(T^*M) : d i_X w = 0\}$$

$$\mathfrak{X}_{ham}(T^*M) : \{X \in \mathfrak{X}(T^*M) : i_X w = df\}$$

It can be easily shown that $\mathfrak{X}_{ham}(T^*M)$ is a Lie subalgebra of $\mathfrak{X}_{symp}(T^*M)$.

Let us consider the diagram

$$\begin{array}{ccc} \mathcal{L}_{X_f} & \rightarrow & \mathcal{L}_{X_{\mathcal{L}_{X_f}}} \\ \uparrow & & \uparrow \\ X_f & \rightarrow & X_{\mathcal{L}_{X_f}} \\ \uparrow & & \uparrow \\ f & \rightarrow & \mathcal{L}_{X_f} \end{array}$$

In this diagram, X_f denotes the Hamiltonian vector field and is defined as

$$X_f = w^{-1}df$$

and $X_{\mathcal{L}_{X_f}}$ is defined as

$$X_{\mathcal{L}_{X_f}} = w^{-1}\mathcal{L}_X w X_f$$

Lemma 6.0.8.

$$\mathcal{L}_{X_f}(g) = \{f, g\}$$

where $\{f, g\}$ denotes the Poisson bracket of f and g.

Proof. Let us consider the map

$$X_f \rightarrow \mathcal{L}_{X_f}$$

such that

$$\mathcal{L}_{X_f}(g) = i_{X_f} dg$$

Then

$$\begin{aligned} \mathcal{L}_{X_f}(g) &= i_{X_f} dg \\ &= i_{X_f}\left(i_{X_g} w\right) \\ &= w\left(X_f, X_g\right) \\ &= \{f, g\} \end{aligned}$$

□

Lemma 6.0.9.

$$\mathcal{L}_{X_{\mathcal{L}_X f}}(g) = \{\mathcal{L}_X f, g\}$$

Proof. Let us consider the map

$$\mathcal{L}_X f \to \mathcal{L}_{X_{\mathcal{L}_X f}}$$

such that

$$\mathcal{L}_{X_{\mathcal{L}_X f}}(g) = i_{X_{\mathcal{L}_X f}}(dg)$$

$$\begin{aligned}
\mathcal{L}_{X_{\mathcal{L}_X f}}(g) &= i_{X_{\mathcal{L}_X f}}(dg) \\
&= i_{X_{\mathcal{L}_X f}}\left(i_{X_g} w\right) \\
&= w\left(X_{\mathcal{L}_X f}, X_g\right) \\
&= w\left(w^{-1}\mathcal{L}_X w X_f, X_g\right) \\
&= w\left(w^{-1}\mathcal{L}_X w w^{-1} df, X_g\right) \\
&= w\left(w^{-1}\mathcal{L}_X df, X_g\right) \\
&= w\left(w^{-1} d\mathcal{L}_X f, X_g\right) \\
&= w\left(w^{-1} d\left(i_X df\right), X_g\right) \\
&= w\left(X_{i_X df}, X_g\right) \\
&= \{i_X df, g\} \\
&= \{\mathcal{L}_X f, g\}
\end{aligned}$$

□

Lemma 6.0.10. *The bracket of two Hamiltonian vector fields X_f and X_g are*

$$[X_f, X_g] = X_{\{f,g\}}$$

Lemma 6.0.11.

$$\left[\mathcal{L}_{X_f}, \mathcal{L}_{X_g}\right] = \mathcal{L}_{\left[X_f, X_g\right]}$$

Remark 6.0.12.

$$[X_f, X_g] = 0 \Rightarrow \mathcal{L}_{\left[X_f, X_g\right]} = \left[\mathcal{L}_{X_f}, \mathcal{L}_{X_g}\right] = 0$$

Lemma 6.0.13. *The Poisson bracket of $\mathcal{L}_X f$ and g is*

$$\{\mathcal{L}_X f, g\} = \mathcal{L}_{X_{\mathcal{L}_X f}}(g)$$

Proof.

$$\begin{aligned}
\{\mathcal{L}_X f, g\} &= w\left(X_{\mathcal{L}_X f}, X_g\right) \\
&= i_{X_{\mathcal{L}_X f}}\left(i_{X_g} w\right) \\
&= i_{X_{\mathcal{L}_X f}}(dg) \\
&= \mathcal{L}_{X_{\mathcal{L}_X f}}(g)
\end{aligned}$$

□

Lemma 6.0.14. *The Poisson bracket of f and $\mathcal{L}_X g$ is*

$$\{f, \mathcal{L}_X g\} = -\mathcal{L}_{X_{\mathcal{L}_X g}}(f)$$

Proof. The proof follows from the anti-symmetry property of the Poisson bracket. Since

$$\begin{aligned}
\{f, \mathcal{L}_X g\} &= -\{\mathcal{L}_X g, f\} \\
&= -w\left(X_{\mathcal{L}_X g}, X_f\right) \\
&= -i_{X_{\mathcal{L}_X g}} i_{X_f} w \\
&= -i_{X_{\mathcal{L}_X g}}(df) \\
&= -\mathcal{L}_{X_{\mathcal{L}_X g}}(f)
\end{aligned}$$

□

Lemma 6.0.15.

$$\mathcal{L}_X\left(\mathcal{L}_{X_f}(g)\right) = \mathcal{L}_{X_{\mathcal{L}_X f}}(g) - \mathcal{L}_{X_{\mathcal{L}_X g}}(f)$$

Proof. The proof follows from the fact that

$$\mathcal{L}_X\left(\mathcal{L}_{X_f}(g)\right) = \mathcal{L}_X(\{f, g\})$$

and

$$\mathcal{L}_X\left(\{f,g\}\right)=\{\mathcal{L}_Xf,g\}+\{f,\mathcal{L}_Xg\}$$

Therefore

$$\mathcal{L}_X\left(\mathcal{L}_{X_f}\left(g\right)\right)=\mathcal{L}_{X_{\mathcal{L}_Xf}}\left(g\right)-\mathcal{L}_{X_{\mathcal{L}_Xg}}\left(f\right)$$

□

Lemma 6.0.16.

$$\mathcal{L}_{X_f}\mathcal{L}_X\left(g\right)=\mathcal{L}_{X_{\mathcal{L}_Xf}}\left(g\right)-\mathcal{L}_{X_{\mathcal{L}_Xg}}\left(f\right)+\mathcal{L}_{\mathcal{L}_{X_f}X}$$

Proof.

$$[X,X_f]=\mathcal{L}_XX_f=-\mathcal{L}_{X_f}X$$

$$\left[\mathcal{L}_X,\mathcal{L}_{X_f}\right]=\mathcal{L}_{\left[X,X_f\right]}=\mathcal{L}_{-\mathcal{L}_{X_f}X}=-\mathcal{L}_{\mathcal{L}_{X_f}X}$$

$$\begin{aligned}\left[\mathcal{L}_X,\mathcal{L}_{X_f}\right]\left(g\right) &= \left(\mathcal{L}_X\mathcal{L}_{X_f}-\mathcal{L}_{X_f}\mathcal{L}_X\right)\left(g\right)\\ &= \mathcal{L}_X\mathcal{L}_{X_f}\left(g\right)-\mathcal{L}_{X_f}\mathcal{L}_X\left(g\right)\end{aligned}$$

Hence,

$$-\mathcal{L}_{\mathcal{L}_{X_f}X}=\mathcal{L}_X\mathcal{L}_{X_f}\left(g\right)-\mathcal{L}_{X_f}\mathcal{L}_X\left(g\right)$$

$$-\mathcal{L}_{\mathcal{L}_{X_f}X}=\mathcal{L}_{X_{\mathcal{L}_Xf}}\left(g\right)-\mathcal{L}_{X_{\mathcal{L}_Xg}}\left(f\right)-\mathcal{L}_{X_f}\mathcal{L}_X\left(g\right)$$

Therefore

$$\mathcal{L}_{X_f}\mathcal{L}_X\left(g\right)=\mathcal{L}_{X_{\mathcal{L}_Xf}}\left(g\right)-\mathcal{L}_{X_{\mathcal{L}_Xg}}\left(f\right)+\mathcal{L}_{\mathcal{L}_{X_f}X}$$

□

Let us introduce a bracket on $\mathfrak{X}\left(Q\right)\times H\left(Q\right)$ where $H\left(Q\right)$ is a subalgebra of the algebra of derivations as:

$$\left[\left(X,\mathcal{L}_{X_f}\right),\left(Y,\mathcal{L}_{X_g}\right)\right]=\left(\left[X,Y\right],\mathcal{L}_{X_{\mathcal{L}_Xg}}-\mathcal{L}_{X_{\mathcal{L}_Xf}}\right)$$

Theorem 6.0.17. *The bracket defined as*

$$\left[\left(X,\mathcal{L}_{x_f}\right),\left(Y,\mathcal{L}_{X_g}\right)\right]=\left(\left[X,Y\right],\mathcal{L}_{X_{\mathcal{L}_Xg}}-\mathcal{L}_{X_{\mathcal{L}_Xf}}\right)$$

on $\mathfrak{X}(Q) \times H(Q)$ *makes it a Lie algebra.*

Proof. The bracket defined as

$$\left[\left(X, \mathcal{L}_{x_f}\right), \left(Y, \mathcal{L}_{X_g}\right)\right] = \left(\left[X, Y\right], \mathcal{L}_{X_{\mathcal{L}_X g}} - \mathcal{L}_{X_{\mathcal{L}_X f}}\right)$$

satisfies the bilinearity and anti-symmetry. Also, Jacobi identity is satisfied because of the Jacobi identity on $\mathfrak{X}(Q) \times F(Q)$ where $F(Q)$ is the algebra of functions on a manifold Q. □

7. SEMI-DIRECT EXTENSIONS BY FUNCTIONS

Let us deal with the semi-direct product $Diff\left(Q\right) \ltimes F\left(Q\right)$.

$$\left(\Phi, f\right).\left(\Psi, g\right) = \left(\Phi \circ \Psi, \Phi_* g + f\right)$$

where a pair $\left(\Phi, f\right)$ consists of a diffeoorphism Φ and a smooth function f on the manifold Q, is the group multiplication is this semi-direct product. Also,

$$\Phi_* g = g \circ \Phi^{-1}$$

denotes the pushforward of the function g by the diffeomorphism Φ. The corresponding Lie algebra is the semi-direct product $\mathfrak{X}\left(Q\right) \ltimes F\left(Q\right)$, where $\mathfrak{X}\left(Q\right)$ is the Lie algebra of vector fields and $F\left(Q\right)$ is the algebra of functions. As a vector space, $\mathfrak{X}\left(Q\right) \oplus F\left(Q\right)$. The Lie bracket on the semi-direct product algebra $\mathfrak{X}\left(Q\right) \ltimes F\left(Q\right)$ is

$$\left[\left(X, f\right), \left(Y, g\right)\right] = \left(-\left[X, Y\right], \mathcal{L}_Y f - \mathcal{L}_X g\right)$$

where $X, Y \in \mathfrak{X}\left(Q\right)$, $f, g \in F\left(Q\right)$ and $\mathcal{L}_Y f$ denotes the Lie derivative of a function along the vector field Y.

Theorem 7.0.1. *$\mathfrak{X}\left(Q\right) \ltimes F\left(Q\right)$ with the bracket which is defined as above is a Lie algebra if and only if*

$$\mathcal{L}_X\left[\alpha, \xi\right] = \left[\alpha, \mathcal{L}_X \xi\right] - \left[\xi, \mathcal{L}_X \alpha\right]$$

$$\mathcal{L}_{[X,Y]}\alpha = \mathcal{L}_Y \mathcal{L}_X \alpha - \mathcal{L}_X \mathcal{L}_Y \alpha$$

holds.

For the semi-direct product algebra $\mathfrak{X}\left(Q\right) \ltimes F\left(Q\right)$

$$N_\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that

$$\left(X, Y\right) \to N_\alpha\left(X, Y\right)$$

is defined as

$$N_\alpha(X,Y) = \mathcal{L}_{[X,Y]}\alpha - \mathcal{L}_Y\mathcal{L}_X\alpha + \mathcal{L}_X\mathcal{L}_Y\alpha$$

This is trivially zero.

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