DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

APPLICATIONS OF LEGENDRE-BERNSTEIN BASIS TRANSFORMATIONS

by

Semra AVŞAR

October, 2016
İZMİR

APPLICATIONS OF LEGENDRE-BERNSTEIN BASIS TRANSFORMATIONS

A Thesis Submitted to the
Graduate School of Natural And Applied Sciences of Dokuz Eylül University
In Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

by
Semra AVŞAR

October, 2016
İZMİR

M.Sc THESIS EXAMINATION RESULT FORM

We have read the thesis entitled " **APPLICATIONS OF LEGENDRE-BERNSTEIN BASIS TRANSFORMATIONS** " completed by **SEMRA AVŞAR** under supervision of **PROF. DR. HALİL ORUÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

....

Prof. Dr. Halil ORUÇ

Supervisor

....

Prof. Dr. İbrahim ÇANAK

Jury Member

....

Doç. Dr. Başak KARPUZ

Jury Member

Prof. Dr. Emine İlknur CÖCEN
Director
Graduate School of Natural and Applied Sciences

ACKNOWLEDGEMENTS

Firstly, I would like to express my deepest gratitude to my supervisor, Prof. Dr. Halil ORUÇ, for his excellent guidance, caring, patience, and providing me with an excellent atmosphere for doing research.

Special thanks go to M.Sc. İsmail GÜZEL who as a good friend, was always willing to help and give his best suggestions. Also, there are a lot of his contributions towards improving the technical design of this thesis on LaTeX. I would also like to express my gratitude to all members of Dokuz Eylül University Department of Mathematics.

Finally, I would also like to thank my parents. They were always supporting me and encouraging me with their best wishes.

Semra AVŞAR

APPLICATIONS OF LEGENDRE-BERNSTEIN BASIS TRANSFORMATIONS

ABSTRACT

This thesis investigates relations between q-Bernstein and little q-Legendre polynomials. We first construct a determinant characterization of the little q-Legendre polynomials. Then we give Gram matrix for the q-Bernstein polynomials with respect to q-inner product. We also derive the transformation matrices that transform little q-Legendre and q-Bernstein polynomial basis of degree n, when the parameter q is nonzero. Furthermore, we study the relationships between degree elevation and degree reduction matrices from the view point of geometric modelling.

Keywords: q-Legendre basis, Bernstein basis, basis transformations, degree elevation, q-Bernstein basis.

LEGENDRE-BERNSTEIN BAZ DEĞİŞİMLERİNİN UYGULAMALARI

ÖZ

Bu tez q-Bernstein ve küçük q-Legendre polinomları arasındaki ilişkileri araştırır. İlk olarak, küçük q-Legendre polinomlarının bir determinant karakterizasyonu kurulmuştur. Sonra q-Bernstein polinomları için q-iç çarpımına göre Gram matris verilmiştir. Ayrıca, derecesi n olan küçük q-Legendre ve q-Bernstein polinom baz dönüşüm matrisleri, q parametresi sıfırdan farklıyken elde edilmiştir. Buna ek olarak, derece yükseltme ve derece azaltma matrisleri arasındaki ilişkiler geometrik modelleme açısından çalışılmıştır.

Anahtar kelimeler: q-Legendre baz, Bernstein baz, baz dönüşümleri, derece yükseltme, q-Bernstein baz.

CONTENTS

LIST OF FIGURES

Page

CHAPTER ONE
INTRODUCTION

One of the most elegant proofs of Weierstrass Approximation theorem was given by S. N. Bernstein in 1912. He introduced the Bernstein polynomials by

$$B_n(f;x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k}x^k(1-x)^{n-k}$$

for a function $f(x)$ on $[0,1]$ and proved that if $f \in C[0,1]$, then the sequence of $B_n(f;x)$ converges uniformly to $f(x)$. These polynomials possess many significant analytic and geometric properties. We now state some of them.

Firstly, it is easily verified that the operators B_nf are *linear monotone* operators on $C[0,1]$ for $n = 1,2,\ldots$. Namely, if $f(x) \geq g(x)$, then $B_n(f;x) \geq B_n(g;x)$. Therefore, they satisfy the conditions of Bohman-Korovkin theorem, Cheney (1982), which states that if the sequence of *linear monotone* operator $\mathcal{L}_nf$ on $C[a,b]$ converges uniformly to f for the functions $f(x) = 1, x, x^2$, then the sequence of $\mathcal{L}_nf$ converges uniformly to f for all function in $C[a,b]$. Thereby, we can say that B_nf converges uniformly to f for all f in $[0,1]$. Furthermore, derivatives of B_nf of any order converge uniformly to derivatives of f. Namely, if $f \in C^k[0,1]$ for any $k = 0,1,...$, then

$$\lim_{n\to\infty} B_n^{(k)}(f;x) = f^{(k)}(x)$$

uniformly on $[0,1]$, see Davis (1975). Another useful property is the asymptotic formula:

$$\lim_{n\to\infty} n(B_n(f;x) - f(x)) = \frac{x(1-x)}{2}f''(x),$$

due to Voronovskaya, demonstrate that for bounded f on $[0,1]$ and $f''(x)$ exists for some $x \in [0,1]$, the rate of convergence is very slow, like the sequence $1/n$.

Another remarkable property is *variation-diminishing* property which yields some shape-preserving properties. That is, the number of sign changes in $B_n(f;x)$ does not exceed the number of sign changes in $f(x)$. Moreover, *monotonic* and *convex* functions yield *monotonic* and *convex* Bernstein polynomials respectively.

Due to their many remarkable properties, such as variation-diminishing, shape-presevation, rate of convergence and convexity, an area of the widespread research for the Bernstein polynomials was occured. However, Bernstein basis polynomials found its true vocation in exploiting computers for geometric design applications. Using Bernstein basis for this purpose led to born a new theory. This important theory was builted on Bézier curve and surface.

We first give some well-known definitions and properties of the Bézier curves. Subsequently, we present a brief summary of the q-calculus. We also mention a generalization of Bernstein polynomials introduced by Phillips (1997) and q-version of the generalized Legendre polynomials given by Schmidt (1993).

Our purpose is to investigate relations between q-Bernstein and little q-Legendre polynomials in view of geometric modelling.

1.1 Bézier Curve

Bézier curves and surfaces techniques have been fundamentals of computer graphics and design. Bézier curves are devised separately by Paul de Casteljau at Citroën and Pierre Bézier at Renault around 1960's. A Bézier curve is represented explicitly as a linear combination of Bernstein basis polynomials by

$$P(t) = \sum_{i=0}^{n} p_i B_i^n(t), \qquad 0 \leq t \leq 1, \quad p_i \in \mathbb{R}^d,$$

where

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, \ldots, n.$$

The coefficients p_i's are called control points and the polygon which is formed by connecting the points consecutively $p_0, \ldots, p_n$ is called control polygon of the curve $P(t)$. The coefficients p_i give precious insight into geometric behavior of the curve

$P(t)$. This makes Bezier curve so popular for CAGD applications.

The de Casteljau algortihm is the most important recursive algorithm for the evaluation of Bézier cuves and surface. This algorithm is based on the repeated linear interpolation as follows:

de Casteljau's algorithm:

Given: Control points p_i for $i = 0, \ldots, n$ and $t \in [0, 1]$.

Compute: $p_i^{(r)}(t) = (1-t)p_i^{(r-1)}(t) + tp_{i+1}^{(r-1)}(t) \quad \begin{cases} r = 1, \ldots, n \\ i = 0, \ldots, n-r, \end{cases}$

where $\quad p_i^0(t) = p_i(t)$.

The consequence of the algorithm is $p_0^n(t) = P(t)$.

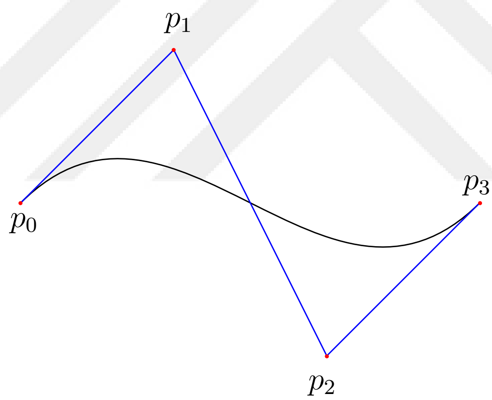

Figure 1.1 Cubic Bézier curve from four control points.

Figure 1.1 shows a cubic Bézier curve for the control points $p_0 = (0, 0)$, $p_1 = (1, 1)$, $p_2 = (2, -1)$ and $p_3 = (3, 0)$.

The important properties of Bézier curves are summarized below:

i) Affine invariance: Due to partition of unity property of Bernstein polynomials, that is $\sum_{i=0}^{n} B_i^n(t) = 1$, a Bézier curve does not change under affine transformations. Namely, if T is an affine transformation, then

$$T\left(\sum_{i=0}^{n} p_i B_i^n(t)\right) = \sum_{i=0}^{n} T(p_i) B_i^n(t).$$

ii) Convex hull property: For $t \in [0,1]$, Bézier curve $P(t)$ lies in the convex hull of the control points.

iii) Variation-diminishing Property: Bézier curve cannot intersect any line more times than its control polygon can.

iv) End-point interpolation: Bézier curve passes through the end points. In other words, $P(0) = p_0$ and $P(1) = p_n$.

Accordingly, Bézier curve mimics the shape of its control polygon. One can manipulate shape of curve or surface by changing the control polygon (or control net).

Degree elevation is a process to increase the flexibility of the polygon by adding a new control point. Thanks to this process, a Bezier curve of degree n can be represented as a curve of degree $(n+1)$. At the same time, shape of the curve does not change. New control points $p_i^{(1)}$ are computed recursively by

$$p_i^{(1)} = \frac{i}{n+1} p_{i-1} + \left(1 - \frac{i}{n+1}\right) p_i, \qquad i = 0, \ldots, n+1.$$

Degree elevation process satisfy end-point interpolation, $p_0^{(1)} = p_0$ and $p_{n+1}^{(1)} = p_n$.

After r times repeated degree elevation, the new control points $p_i^{(r)}$ is written recursively by

$$p_i^{(r)} = \frac{i}{n+r} p_{i-1}^{(r-1)} + \left(1 - \frac{i}{n+r}\right) p_i^{(r-1)}, \qquad i = 0, \ldots, n+r.$$

In Farin (2002), these news control points may be obtained from the initial control points by the formula

$$p_i^{(r)} = \sum_{j=0}^{n} \binom{n}{j} \frac{\binom{r}{i-j}}{\binom{n+r}{i}} p_j.$$

If we apply the degree elevation process infinitely many times, that is $r \to \infty$, the control polygons converge to the Bézier curve. However, the rate of convergence is very slow, see (Farin, 2002). Figure 1.2 presents an example of the limit behavior of the polygons.

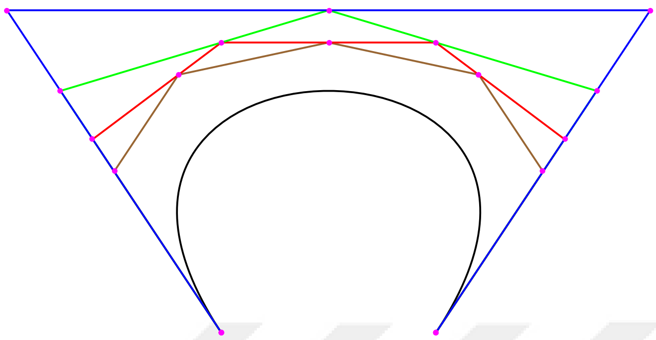

Figure 1.2 Repeated degree elevation: from cubic to sextic Bezier curve successively.

1.2 q-Calculus

We now give a brief background on q-Calculus following the book Gasper & Rahman (2004).

Let $q \neq 1$ be a fixed real number. For any real number a and any nonnegative integer n, the q-Pochhammer symbol is defined by

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n \in \mathbb{N}. \end{cases} \tag{1.1}$$

Then we define q-integer $[r]$ by

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1, \end{cases}$$

and q-factorial $[r]!$ by

$$[r]! = \begin{cases} [r][r-1]\dots[1], & r \geq 1, \\ 1, & r = 0. \end{cases}$$

The q-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$ is given by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}$$

for $n \geq r \geq 0$, and as zero otherwise. It satisfies the following Pascal type relations

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ r \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

Note that $\begin{bmatrix} n \\ r \end{bmatrix}$ is a polynomial of degree $r(n-r)$ in q. When $q \to 1$, the q-binomial coefficients turn into the usual binomial coefficients.

The following identity which will be needed in Section 2.2 is called Euler identity:

$$(1-x)(1-qx)\dots(1-q^{n-1}x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} (-1)^r q^{r(r-1)/2} x^r. \tag{1.2}$$

It is easily verified by induction on n.

We also require the q-Derivative $\mathcal{D}_q f(x)$ which is defined by

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{x(1-q)}, \qquad q \neq 1 \text{ and } x \neq 0.$$

Obviously, if $f(x)$ is differentiable, then

$$\lim_{q \to 1} \mathcal{D}_q f(x) = \frac{df(x)}{dx}.$$

For example, if we take $f(x) = x^n$, for $n \in \mathbb{N}$

$$\mathcal{D}_q x^n = \frac{x^n - (qx)^n}{x(1-q)} = \frac{1-q^n}{1-q} x^{n-1} = [n] x^{n-1}.$$

Notice that this operator is a linear operator. In other words, for any constants a and b,

$$\mathcal{D}_q(af(x) + bg(x)) = a\mathcal{D}_q f(x) + b\mathcal{D}_q g(x).$$

The formula for the q-Derivative $\mathcal{D}_q$ of the product of functions is

$$\mathcal{D}_q(f(x)g(x)) = f(qx)\mathcal{D}_q g(x) + \mathcal{D}_q f(x) g(x).$$

The product rule can be generalized to a q-analogue of Leibniz's rule:

$$\mathcal{D}_q^n(f(x)g(x)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \mathcal{D}_q f(xq^{n-k}) \mathcal{D}_q^{n-k} g(x) \quad \text{for} \quad n \in \mathbb{N}.$$

Jackson (1910) introduced the q-integral by the following series expansion:

$$\int_0^1 f(x) d_q x = (1-q) \sum_{j=0}^{\infty} q^j f(q^j),$$

provided that the series is convergent. A more general definition is that (see Gasper & Rahman, 2004)

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

where

$$\int_0^b f(x) d_q x = (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b).$$

If f is continuous function on $[0, a]$, then

$$\lim_{q \to 1} \int_0^a f(x) d_q x = \int_0^a f(x) dx.$$

1.3 q-Bernstein Polynomials

Due to development of q-Calculus, new generalizations of Bernstein polynomials have been discovered. Firstly, Lupaş introduced a generalization of the Bernstein polynomials which is a sequence of rational functions involving q-integers in 1987.

Phillips introduced another generalization of the Bernstein polynomials based on the q-integers, called q-Bernstein polynomials in (Phillips, 1997).

The q-Bernstein polynomials are defined as follows. For each positive integer n,

$$\mathcal{B}_n(f;q,x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x), \tag{1.3}$$

where empty product is 1, $f_r = f\left([r]/[n]\right)$ and q is a positive real number.

The basis functions which appear in (1.3) as

$$\mathcal{B}_r^n(x;q) = \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1-q^s x), \quad r = 0,1,\ldots,n, \tag{1.4}$$

form a basis for $\mathbb{P}_n$.

The basic properties of q-Bernstein polynomials are given below:

i) We recover the classical Bernstein polynomials, when $q=1$.

ii) They satisfy end-point interpolation

$$\mathcal{B}_n(f;q,0) = f(0) \quad \text{and} \quad \mathcal{B}_n(f;q,1) = f(1).$$

iii) They reproduce linear functions, in the sense that

$$\mathcal{B}_n(ax+b;q;x) = ax+b, \quad a,b \in \mathbb{R}.$$

iv) $\mathcal{B}_n(f;q,x)$ is a monotone linear operator, when $0 < q \leq 1$.

The q-Bernstein polynomials have been investigated by several authors because of many attractive properties. Some properties of the classical Bernstein polynomials may be extended to the q-Bernstein polynomials. For instance, q-Bernstein polynomials have end-point interpolation, shape preserving properties when $0 < q < 1$ and reproduce linear functions.

On the other hand, the convergence properties of the q-Bernstein polynomials are different from the classical Bernstein polynomials on $[0,1]$. It is proven in Oruç & Tuncer (2002) and Ostrovska (2003) that for a fixed q and $0 < q < 1$, $\mathcal{B}_n(f;q,x)$ converges uniformly to f if and only if f is linear on $[0,1]$. Moreover, for fixed real number $q \geq 1$, $\mathcal{B}_n f \to f$ as $n \to \infty$, for all polynomial f. It is also shown in Ostrovska (2003) that for $q > 1$, $\mathcal{B}_n f \to f$ uniformly if f is analytic.

The q-Bézier curves are proposed in Oruç (1998) as follows:

$$P(x) = \sum_{r=0}^{n} b_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \quad b_r \in \mathbb{R}^d. \tag{1.5}$$

Here, classical Bézier curve is recovered when we set $q = 1$. Recent researches focus blossom and subdivision procedures for q-Bézier curve in Goldman (2003).

1.4 q-Legendre Polynomials

The Heine's series or, in view of the base q, the basic hypergeometric series ${}_r\phi_s$ were introduced by Heine in 1846 as follows:

$${}_r\phi_s\left(a_1,\ldots,a_r;b_1,\ldots,b_s;q,z\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_r;q)_k}{(b_1;q)_k \ldots (b_s;q)_k (q;q)_k} [(-1)^k q^{\binom{k}{2}}]^{(1+s-r)} z^k,$$

where $q \neq 0$. It is assumed that the parameters $b_1, \ldots, b_s$ never take any values of q^{-k}.

The little q-Jacobi polynomials are a family of basic hypergeometric polynomials introduced by Hahn in 1949, see Andrews & Askey (1977). They are given in terms of basic hypergeometric functions by

$$\mathcal{P}_n(x;a,b;q) = {}_2\phi_1\left(q^{-n}, abq^{n+1}; aq; q, qx\right) \quad \text{for} \quad a,b \in \mathbb{C}. \tag{1.6}$$

The special case of little q-Jacobi polynomials obtained by $a = b = 1$ in (1.6) are known as little q-Legendre polynomials $\mathcal{P}_n(x;q)$. They are defined in terms of basic hypergeometric functions and Pochhammer symbols by

$$\begin{aligned}\mathcal{P}_n(x;q) &= {}_2\phi_1\left(q^{-n}, q^{n+1}; q; q, qx\right)\\ &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1};q)_k q^k x^k}{(q;q)_k (q;q)_k}, \quad 0<q<1.\end{aligned}$$

By using the q-binomial coefficients, this can be turn into the following

$$\mathcal{P}_n(x;q) = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k, \quad 0<q<1, \; x \in [0,1]. \tag{1.7}$$

Little q-Legendre polynomials reduce to Legendre polynomials on $[0,1]$ when $q \to 1$.

Some fundamental properties of little q-Legendre polynomials are the following:

i) The three-term recurrence relation for the little q-Legendre polynomial $\mathcal{P}_n(x;q)$ is

$$\begin{aligned} q^n(1-q^{n+1})(1+q^n)\mathcal{P}_{n+1} &= (1-q^{2n+1})(2q^n - (1+q^n)(1+q^{n+1})x)\mathcal{P}_n \\ &\quad - q^n(1-q^n)(1+q^{n+1})\mathcal{P}_{n-1} \end{aligned} \tag{1.8}$$

for $n = 0, 1, \ldots$, with initial conditions $\mathcal{P}_{-1}(x;q) = 0$ and $\mathcal{P}_0(x;q) = 1$.

ii) The generating function of the little q-Legendre polynomials is given by Koekoek et al. (2010)

$${}_0\phi_1\left(-, q; q, qxt\right) {}_2\phi_1\left(x^{-1}, 0; q; q, xt\right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}}{(q,q;q)_n} \mathcal{P}_n(x;q) t^n,$$

where $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \ldots (a_m; q)_n$ and the q-Pochammer symbol $(a; q)_n$ is defined by (1.1).

iii) q-Rodrigues formula:

$$\mathcal{P}_n(x; q) = \frac{(1-q)^n}{(q; q)_n} \mathcal{D}_q^n (x^n (x; q^{-1})_n), \qquad \text{for } n = 0, 1, \ldots.$$

iv) Orthogonality relation:

$$\int_0^1 \mathcal{P}_m(x; q) \mathcal{P}_n(x; q) d_q x = \frac{q^n}{[2n+1]} \delta_{mn}, \qquad \text{for } m, n = 0, 1, \ldots. \tag{1.9}$$

Thus the first few terms are given by

$\mathcal{P}_0(x; q) = 1,$

$\mathcal{P}_1(x; q) = 1 - (1+q)x,$

$\mathcal{P}_2(x; q) = 1 - \frac{(1+q)(1+q+q^2)}{q} x + \frac{(1+q^2)(1+q+q^2)}{q} x^2.$

Figure 1.3 demonstrates four cubic little q-Legendre polynomials on $[0, 1]$ for different values of q.

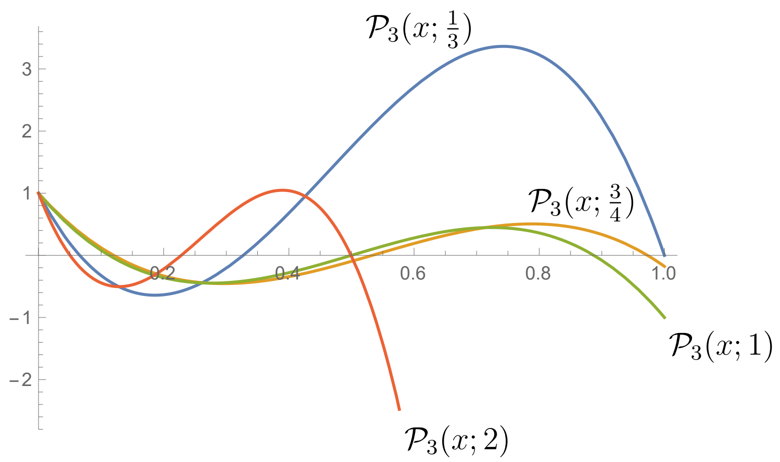

Figure 1.3 Cubic little q-Legendre polynomial $\mathcal{P}_3(x; q)$ on $[0, 1]$ for $q = \frac{1}{3}, \frac{3}{4}, 1, 2$.

CHAPTER TWO
Q-LEGENDRE POLYNOMIALS

Bernstein-Bezier representation of a polynomial has a great importance in CAGD because of its valuable geometrical behaviors. On the other hand, due to orthogonality of Legendre polynomials, they play important role in least square problems. Since Bernstein polynomials are not orthogonal, the basis transformation is significant. Farouki (2000) derived an explicit form of the basis transformation between Legendre and Bernstein basis. Our purpose is to relate the transformations between little q-Legendre and q-Bernstein basis.

Firstly, we construct a determinant characterization of little q-Legendre polynomials. Then, we derive the Gram matrix for the q-Bernstein polynomial basis. Furthermore, we discuss the conversion matrix of the the little q-Legendre polynomial basis into the the q-Bernstein polynomial basis on $[0,1]$ and vice versa, when the parameter $q \neq 0$.

2.1 Determinant Characterization of $\mathcal{P}_{n+1}(x;q)$

As a consequence of Favard's theorem, see Chihara (1978), we give the following determinant identity for the little q-Legendre polynomials on $[0,1]$.

Proposition 1. *For each $n = 1, 2, \ldots$ the little q-Legendre polynomial $\mathcal{P}_{n+1}(x;q)$ is the determinant of $(n+1)\times(n+1)$ tridiagonal matrix A_{n+1} with the entries*

$$a_{ij} = \begin{cases} 1 + \dfrac{[i]^2[2i+2]}{[i+1]^2[2i]} - \dfrac{[2i+1][2i+2]}{q^i[i+1]^2}x, & \textit{if } i = j > 0, \\ 1, & \textit{if } j = i+1, \\ \dfrac{[i]^2[2i+2]}{[i+1]^2[2i]}, & \textit{if } j = i-1, \\ 0, & \textit{otherwise}, \end{cases}$$

where $\det(A_0) = 1$ *and* $\det(A_1) = 1 - (1+q)x$ *for* $i, j = 0, 1, \ldots, n$ *and we define* $\det(A_{-1}) = 0$.

Proof. We prove it by induction on n. Inital step is obvious by the definition of $\mathcal{P}_0(x;q)$ and $\mathcal{P}_1(x;q)$. For the inductive step, we assume the truth of the statement for $(n-1)$ and n. Let us consider the matrix A_{n+1}. We expand $\det(A_{n+1})$ with respect to the last row and hence

$$\det(A_{n+1}) = (-1)^{2n-1} a_{n,n-1} \det(B) + (-1)^{2n} a_{n,n} \det(A_n), \tag{2.1}$$

where the $n \times n$ matrix B has entries

$$b_{ij} = \begin{cases} 1 + \dfrac{[i]^2[2i+2]}{[i+1]^2[2i]} - \dfrac{[2i+1][2i+2]}{q^i[i+1]^2}x, & \text{if } i = j \quad \text{for} \quad i = 1, \dots, n-2, \\ 1, & \text{if } j = i+1 \quad \text{for} \quad i = 0, \dots, n-2, \\ \dfrac{[i]^2[2i+2]}{[i+1]^2[2i]}, & \text{if } j = i-1, \\ 0, & \text{otherwise} \end{cases}$$

with $b_{0,0} = 1 - (1+q)x$ and $b_{n-1,n-1} = 1$.

To find the determinant of B, we expand $\det(B)$ with respect to last column. Therefore, we obtain

$$\det(B) = \det(A_{n-1}). \tag{2.2}$$

Combining (2.1) and (2.2) and using inductive hypotheses, we show that $\det(A_{n+1})$ and $\mathcal{P}_{n+1}$ satisfy the same recurrence relation and initial values given in (1.8). This completes the proof. □

2.2 Gram Matrix for q-Bernstein Polynomials

We can solve approximation problems in inner product spaces without using orthonormal bases.

Let M be a finite dimensional vector space and $\{v_1, v_2, \dots, v_n\}$ be a basis for a subspace V. Let $f \in M$ and $v \in V$. In order to compute a best approximation v to f, necessary and sufficient condition is that $\langle v - f, v_i \rangle = 0$ for $i = 1, 2, \dots, n$. On substituting $v = \sum_{j=1}^{n} a_j v_j$, we obtain

$$\sum_{j=1}^{n} a_j \langle v_j, v_i \rangle = \langle f, v_i \rangle.$$

They generate a system of n linear equations and n unknowns $a_1, a_2, \ldots, a_n$ such that

$$\begin{bmatrix} \langle v_1, v_1 \rangle & \langle v_2, v_1 \rangle & \cdots & \langle v_n, v_1 \rangle \\ \langle v_1, v_2 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_n, v_2 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle v_1, v_n \rangle & \langle v_2, v_n \rangle & \cdots & \langle v_n, v_n \rangle \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle v_1, f \rangle \\ \langle v_2, f \rangle \\ \vdots \\ \langle v_n, f \rangle \end{bmatrix}.$$

The coefficient matrix is called Gram matrix with the entries $G_{i,j} = \langle v_j, v_i \rangle$. The Gram matrix is determined by the chosen inner product and basis for V. Gram matrix is symmetric, that is $G_{i,j} = G_{j,i}$ and also positive semi-definite, meaning that $b^t G b \geq 0$ for every vector $b \in \mathbb{R}^n$. Note that if $\{v_1, \ldots, v_n\}$ is an orthonormal basis, $\langle v_i, v_j \rangle = \delta_{ij}$, then the unique solution becomes $a_i = \langle v_i, f \rangle$ for $i = 1, \ldots, n$. See for example Kincaid & Cheney (1996).

We compute the Gram matrix of q-Bernstein basis with respect to the following q-inner product on $[0, 1]$:

$$\langle f, g \rangle_{L_2} = \int_0^1 f(x) g(x) d_{1/q} x = (1 - q^{-1}) \sum_{k=0}^{\infty} q^{-k} f(q^{-k}) g(q^{-k}), \quad q > 1, \tag{2.3}$$

or

$$\langle f, g \rangle_{L_2} = \int_0^1 f(x) g(x) d_q x = (1 - q) \sum_{k=0}^{\infty} q^k f(q^k) g(q^k), \quad 0 < q < 1. \tag{2.4}$$

See Ait-Haddou & Goldman (2015).

Proposition 2. *The elements of the Gram matrix G_n of the q-Bernstein basis on $[0, 1]$ are given by*

$$G_n(i, j) = \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \sum_{m=0}^{n-i} \sum_{s=0}^{n-j} (-1)^{m+s} q^{\binom{m}{2}} q^{\binom{s}{2}} \frac{\begin{bmatrix} n-i \\ m \end{bmatrix} \begin{bmatrix} n-j \\ s \end{bmatrix}}{[i + j + m + s + 1]} \tag{2.5}$$

for $0 < q < 1$ *and* $i, j = 0, \ldots, n$.

Proof. By using definition (1.4) of q-Bernstein basis and Euler identity (1.2), we can write the $\mathcal{B}_i^n(x;q)$ and $\mathcal{B}_j^n(x;q)$ in the given q-inner product as follows

$$\begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \sum_{m=0}^{n-i} \sum_{s=0}^{n-j} (-1)^{m+s} q^{\binom{m}{2}} q^{\binom{s}{2}} \begin{bmatrix} n-i \\ m \end{bmatrix} \begin{bmatrix} n-j \\ s \end{bmatrix} \int_0^1 x^{(i+j+m+s)} d_q x.$$

Taking q-integral over $[0,1]$, the latter result is obtained as

$$\begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} (1-q) \sum_{k=0}^{\infty} \sum_{m=0}^{n-i} \sum_{s=0}^{n-j} (-1)^{m+s} q^{\binom{m}{2}} q^{\binom{s}{2}} q^{k(i+j+m+s+1)} \begin{bmatrix} n-i \\ m \end{bmatrix} \begin{bmatrix} n-j \\ s \end{bmatrix}.$$

In view of $0 < q < 1$, the above geometric series converges to (2.5). This completes the proof. □

2.3 Basis Conversion Matrix

We begin with the following proposition that demonstrates the relation between the little q-Legendre and q-Bernstein basis.

Proposition 3. *The little q-Legendre polynomials have the following representation in the q-Bernstein polynomial basis:*

$$\mathcal{P}_k(x;q) = \sum_{i=0}^{k} \sum_{m=0}^{i} (-1)^m q^{\binom{m+1}{2} - mk} \begin{bmatrix} k+m \\ m \end{bmatrix} \begin{bmatrix} i \\ m \end{bmatrix} \mathcal{B}_i^k(x;q). \tag{2.6}$$

Proof. We use the identity

$$x^m = \sum_{i=m}^{k} \frac{\begin{bmatrix} i \\ m \end{bmatrix}}{\begin{bmatrix} k \\ m \end{bmatrix}} \mathcal{B}_i^k(x;q)$$

obtained in Oruç & Phillips (2003) in equation (1.7). We can easily see that the term $\begin{bmatrix} k \\ m \end{bmatrix}$ is cancelled. Finally, rearranging the order of summations completes the proof. □

The q-Bernstein form presents intuitive characterization of little q-Legendre polynomials. Namely, using the control polygon associated with the q-Bézier coefficients of $\mathcal{P}_k(x;q)$, we obtain geometrical behavior of the graph of $\mathcal{P}_k(x;q)$ for $x \in [0,1]$. But q-Bernstein polynomials are not orthogonal. Thus, they are not convenient for the least square approximation. On the other hand, little q-Legendre polynomials constitute an orthogonal basis which enables to finding the polynomial of best approximation. Hence, basis transformations are important.

We are now in a position to state the basis conversion matrices. Firstly, we can write a polynomial $Q_n(x;q)$, $x \in [0,1]$ in the form of little q-Legendre and q-Bernstein polynomial basis when the parameter $q \neq 0$:

$$Q_n(x;q) = \sum_{j=0}^{n} c_j \mathcal{B}_j^n(x;q) = \sum_{k=0}^{n} l_k \mathcal{P}_k(x;q). \tag{2.7}$$

We will find the transformation matrix M that transforms the little q-Legendre ceofficients $\{l_k\}_{k=0}^n$ into the q-Bézier coefficients $\{c_j\}_{j=0}^n$

$$c_j = \sum_{k=0}^{n} M_{jk} l_k \tag{2.8}$$

and vice versa. Let $c = (c_0 \dots c_n)^t$ and $l = (l_0 \dots l_n)^t$. Then (2.8) can be written in matrix form as

$$c = Ml. \tag{2.9}$$

Also, we can express little q-Legendre polynomials as a linear combination of the q-Bernstein polynomial by

$$\mathcal{P}_k(x;q) = \sum_{j=0}^{n} \Lambda_{kj} \mathcal{B}_j^n(x;q) \tag{2.10}$$

where Λ is the $(n+1) \times (n+1)$ basis conversion matrix. We substitute (2.10) into (2.7) and rearrange the order of summation to obtain the entries of c_j as

$$c_j = \sum_{k=0}^{n} l_k \Lambda_{kj} \quad \text{for} \quad j = 0, \dots, n. \tag{2.11}$$

Comparing (2.8) with (2.11), we see that $M = \Lambda^t$.

The following proposition gives the entries of basis conversion matrix.

Proposition 4. *The basis conversion matrix Λ that transforms to q-Bernstein polynomial basis of degree n into little q-Legendre polynomials on $[0,1]$ has the entries*

$$\Lambda_{kj} = \frac{1}{\begin{bmatrix} n \\ j \end{bmatrix}} \sum_{i=0}^{k} \sum_{m=0}^{i} (-1)^m q^{\binom{m+1}{2} - km} q^{(j-i)(k-i)} \begin{bmatrix} k+m \\ m \end{bmatrix} \begin{bmatrix} i \\ m \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-k \\ j-i \end{bmatrix} \tag{2.12}$$

for $k, j = 0, 1, \ldots, n$.

Proof. We first require $(n-k)$ times degree elevation formula (see Oruç & Phillips, 2003). By applying $(n-k)$ times degree elevation on the q-Bernstein basis of degree k, we obtain

$$\mathcal{B}_i^k(x;q) = \sum_{j=i}^{n-k+i} q^{(j-i)(k-i)} \frac{\begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-k \\ j-i \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}} \mathcal{B}_j^n(x;q) \qquad \text{for} \quad i = 0, \ldots, k. \tag{2.13}$$

Substituting (2.13) into (2.6) and rearranging the order of summations, we obtain

$$\mathcal{P}_k(x;q) = \sum_{j=0}^{n} \sum_{i=0}^{k} \sum_{m=0}^{i} (-1)^m q^{\binom{m+1}{2} - mk} \begin{bmatrix} k+m \\ m \end{bmatrix} \begin{bmatrix} i \\ m \end{bmatrix} q^{(j-i)(k-i)} \frac{\begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-k \\ j-i \end{bmatrix}}{\begin{bmatrix} n \\ j \end{bmatrix}} \mathcal{B}_j^n(x;q).$$

Therefore

$$\mathcal{P}_k(x;q) = \sum_{j=0}^{n} \Lambda_{kj} \mathcal{B}_j^n(x;q) \qquad \text{for} \quad k = 0, \ldots, n,$$

where Λ_{kj} is given by (2.12) for $k, j = 0, 1, \ldots, n$. □

Note that we have $\Lambda^t = M$. For subsequent work, we construct the matrix M by orthonormal little q-Legendre polynomials. That is, the elements M_{jk} of the matrix M are given by

$$M_{jk} = \frac{\sqrt{[2j+1]}}{\begin{bmatrix} n \\ j \end{bmatrix} \sqrt{q^j}} \sum_{i=0}^{k} \sum_{m=0}^{i} (-1)^m q^{\binom{m+1}{2} - km} q^{(j-i)(k-i)} \begin{bmatrix} k+m \\ m \end{bmatrix} \begin{bmatrix} i \\ m \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} n-k \\ j-i \end{bmatrix}$$

for $k, j = 0, 1, \ldots, n$.

We are interested in the inverse transformation matrix giving the q-Bernstein basis polynomials in terms of the little q-Legendre polynomials. Hence, we state the following proposition to obtain the elements of the inverse basis conversion matrix.

Proposition 5. *The elements $(\Lambda^{-1})_{kj}$ of the matrix Λ^{-1} that transforms the little q-Legendre polynomial basis of degree n into the q-Bernstein polynomial basis are given by*

$$\sum_{i=0}^{j}\sum_{m=0}^{i}\sum_{l=0}^{j-i}\sum_{t=0}^{n-k}(-1)^{m+l+t}q^{\binom{m+1}{2}-mj}q^{\binom{l}{2}}q^{\binom{t}{2}}\frac{\begin{bmatrix}j+m\\m\end{bmatrix}\begin{bmatrix}i\\m\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}j\\i\end{bmatrix}\begin{bmatrix}j-i\\l\end{bmatrix}\begin{bmatrix}n-k\\t\end{bmatrix}}{[l+t+k+i+1]} \tag{2.14}$$

for $k, j = 0, 1, \ldots, n$.

Proof. Let us write the transformation of little q-Legendre polynomials on $[0, 1]$ into q-Bernstein basis of degree n as

$$\mathcal{B}_k^n(x;q) = \sum_{i=0}^{n}(\Lambda^{-1})_{ki}\mathcal{P}_i(x;q) \qquad \text{for} \quad k = 0, 1, \ldots, n. \tag{2.15}$$

Multiplying of the each side of (2.15) by $\mathcal{P}_j(x;q)$ and taking q-integral on $[0, 1]$, we can write

$$\int_0^1 \mathcal{B}_k^n(x;q)\mathcal{P}_j(x;q)d_qx = \sum_{i=0}^{n}(\Lambda^{-1})_{ki}\int_0^1 \mathcal{P}_i(x;q)\mathcal{P}_j(x;q)d_qx.$$

From the orthogonality relation of little q-Legendre polynomials (1.9), we obtain

$$(\Lambda^{-1})_{kj} = \frac{[2j+1]}{q^j}\int_0^1 \mathcal{B}_k^n(x;q)\mathcal{P}_j(x;q)d_qx \qquad \text{for} \quad k, j = 0, 1, \ldots, n.$$

Finally, setting (2.6) into the above expression and using (2.5), we obtain the elements of Λ^{-1} given by (2.14). □

Note that we have $(\Lambda^{-1})^t = M^{-1}$. Similarly, we construct the matrix M^{-1} by orthonormal little q-Legendre polynomials. Namely, the elements $(M^{-1})_{jk}$ of the matrix M^{-1} are given by

$$\sqrt{\frac{[2j+1]}{q^j}}\sum_{i=0}^{j}\sum_{m=0}^{i}\sum_{l=0}^{j-i}\sum_{t=0}^{n-k}(-1)^{m+l+t}q^{\binom{m+1}{2}-mj}q^{\binom{l}{2}}q^{\binom{t}{2}}\frac{\begin{bmatrix}j+m\\m\end{bmatrix}\begin{bmatrix}i\\m\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}j\\i\end{bmatrix}\begin{bmatrix}j-i\\l\end{bmatrix}\begin{bmatrix}n-k\\t\end{bmatrix}}{[l+t+k+i+1]}$$

for $k, j = 0, 1, \ldots, n$.

CHAPTER THREE
LEAST SQUARES AND Q-BERNSTEIN POLYNOMIALS

After the transformations between the Legendre and Bernstein bases on $[0,1]$ are derived by Farouki (2000), the relationship among the basis transformation matrices, degree elevation and degree reduction matrices are studied in Lee et al. (2002). The main purpose of this chapter is to study connection between the basis conversion matrices of little q-Legendre and q-Bernstein polynomials via degree elevation and reduction matrices of q-Bernstein polynomials. We obtain the q-Bernstein - little q-Legendre coefficient transformation matrix M_n, the little q-Legendre - q-Bernstein transformation coefficient matrix M_n^{-1} and degree elevation matrix.

3.1 Degree Elevation

Degree elevation algorithm can be applied to the q-Bernstein polynomials. This algorithm computes a new set of control points by selecting a convex combination of the old set of control points. The end points remain the same. The new vertices $b_i^{(1)}$ are obtained by the following recursive formula

$$b_i^{(1)} = \left(1 - \frac{[n+1-i]}{[n+1]}\right) b_{i-1} + \frac{[n+1-i]}{[n+1]} b_i, \qquad i = 0, \ldots, n+1. \tag{3.1}$$

See (Oruç & Phillips, 2003).

We now rewrite the degree elevation process, equation (3.1), in the matrix form

$$b^{(1)} = T_n b,$$

where $b = (b_0, b_1, \ldots, b_n)^t$, $b^{(1)} = (b_0^{(1)}, b_1^{(1)}, \ldots, b_{n+1}^{(1)})^t$ and the $(n+2)\times(n+1)$ matrix

T_n is given by

$$T_n = \frac{1}{[n+1]} \begin{bmatrix} [n+1] & 0 & 0 & \cdots & 0 & 0 & 0 \\ q^n[1] & [n] & 0 & \cdots & 0 & 0 & 0 \\ 0 & q^{n-1}[2] & [n-1] & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q^2[n-1] & [2] & 0 \\ 0 & 0 & 0 & \cdots & 0 & q[n] & [1] \\ 0 & 0 & 0 & \cdots & 0 & 0 & [n+1] \end{bmatrix}.$$

By repeating this process, we get a sequence of control points. After r degree elevations, we have $b^{(r)} = T_{n,r}b$, where the $(n+r+1) \times (n+1)$ matrix

$$T_{n,r} = T_{n+r-1}T_{n+r-2}\ldots T_{n+1}T_n$$

has elements

$$T_{n,r}(i,j) = q^{(i-j)(n-j)} \frac{\begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} r \\ i-j \end{bmatrix}}{\begin{bmatrix} n+r \\ i \end{bmatrix}} \tag{3.2}$$

for $i = 0, 1, \ldots, n+r$ and $j = 0, 1, \ldots, n$. See (Oruç & Phillips, 2003).

Similarly, the degree elevation of a polynomial of degree n with little q-Legendre polynomial basis with coefficients $l = (l_0, l_1, \ldots, l_n)^t$ gives a polynomial of degree $(n+1)$ with coefficients $l^{(1)} = (l_0, l_1, \ldots, l_n, 0)^t$.

After r degree elevations, we have a linear system

$$l^{(r)} = I_{n,r}l,$$

where the matrix $I_{n,r}$ of dimension $(n+r+1) \times (n+1)$ has elements

$$I_{n,r} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

After converting the q-Bezier coefficients into the little q-Legendre coefficients by using M_n^{-1}, the r times degree elevation by $I_{n,r}$, and finding the q-Bernstein coefficients by M_{n+r}, we can write $T_{n,r}$ as

$$T_{n,r} = M_{n+r} I_{n,r} M_n^{-1}.$$

3.2 Least Square Approximation

Best approximations with respect to square norm are called *least square approximations*.

Let $f(x)$ be a continuous function on $[a,b]$. Using square norm

$$||f(x)||_2 = \left(\int_a^b [f(x)]^2 dx \right)^{1/2},$$

we want to find $p^*(x)$ such that

$$||f(x) - p^*(x)||_2 \leq ||f(x) - p(x)||_2$$

for all $p(x) \in \mathbb{P}_n$ and $x \in [a,b]$. Let us write

$$p^*(x) = \sum_{i=0}^{n} a_i \phi_i(x),$$

where $\{\phi_0, \ldots, \phi_n\}$ is a basis for $\mathbb{P}_n$. We wish to find the coefficients a_i's such that $||f(x) - p^*(x)||_2$ is minimized. Consider

$$E(a_0, \ldots, a_n) = \left(\int_a^b [f(x) - p^*(x)]^2 dx \right)^{1/2},$$

as a function of the coefficients $a_0, \ldots, a_n$. It is necessary that

$$\frac{\partial E}{\partial a_i} = 0 \quad \text{for} \quad i = 0, \ldots, n.$$

This yields a system of linear equations to determine the coefficients a_i. Hence, solving this linear system, we find the a_i that minimizes $||f(x) - p^*(x)||_2$.

3.3 Degree Reduction

We start with two q-inner products. Let q be a real number. We denote the following q-inner products on $\mathbb{P}_n$:

$$\langle f, g\rangle_{L_2} = \int_0^1 f(x)g(x)d_{1/q}x = (1-q^{-1})\sum_{k=0}^{\infty} q^{-k} f(q^{-k})g(q^{-k}), \quad q > 1, \tag{3.3}$$

and the q-Euclidean inner product of the q-Bezier coefficients as

$$\langle B_q^n b, B_q^n c\rangle_{E_2} = \sum_{i=0}^{n} q^i b_i c_i, \tag{3.4}$$

where $B_q^n b(x) = \sum_{k=0}^n b_k B_k^n(x;q)$ for a vector $b = (b_0, \ldots, b_n)$. See (Ait-Haddou & Goldman, 2015).

Here, we need to give the following result from (Ait-Haddou & Goldman, 2015) for our subsequent work.

Corollary 1: Given a polynomial P of degree n, the approximation problem

$$\min_{Q \in P_m} ||P - Q||$$

has the same minimizer for the norm induced either by the inner product (3.3) or the inner product (3.4).

Our aim is to use the best approximation in the sense of the q-L_2 norm to obtain the degree reduction of q-Bézier curves.

For a given set of control points $\{a_i\}_{i=0}^n$, we can define the q-Bézier curve of degree n

$$a^n(t) = \sum_{i=0}^{n} a_i \mathcal{B}_i^n(t;q).$$

Then we find the approximative q-Bézier curve $b^m(t)$ of degree m

$$b^m(t) = \sum_{i=0}^{m} b_i \mathcal{B}_i^m(t;q),$$

where $m < n$, so that the q-L_2 norm between the curves b^m and a^n is minimized. We can use (3.3) as the q-L_2 norm of b^m and a^n due to Corollary 1:

$$||b^m - a^n||_{L_2}^2 = \int_0^1 |b^m(t) - a^n(t)|^2 d_{1/q}t.$$

Using the q-degree elevation matrix $T_{m,r}$, we can raise the degree of b^m from m to n by $b^{(r)} = T_{m,r}b$, where $r = n - m$.

So we have the curve b^m of degree m as a curve of degree n

$$b^m(t) = b^{(r)}(t) = \sum_{i=0}^{n} b_i^{(r)} \mathcal{B}_i^n(t;q).$$

Therefore, we obtain the q-L_2 norm between the q-Bézier curve b^m of degree m and the q-Bézier curve a^n of degree n as:

Proposition 6. *The q-L_2 norm between the two q-Bézier curve b^m and a^n is*

$$||b^m - a^n||_{L_2}^2 = ||b^{(r)} - a^n||_{L_2}^2 = A^t G_n A, \tag{3.5}$$

where $A = a - T_{m,r}b$, $b = (b_0, b_1, \ldots, b_m)^t$ and $a = (a_0, a_1, \ldots, a_n)^t$.

To obtain degree reduction of q-Bézier curve, we want to find the best approximation in the sense of q-L_2 norm. Therefore, substituting $A = a - T_{m,r}b$ in (3.5) gives

$$A^t G_n A = a^t G_n a - 2b^t T_{m,r}^t G_n a + b^t T_{m,r}^t G_n T_{m,r} b. \tag{3.6}$$

To find the vector $\hat{b}$ that minimizes the error function (3.6) with respect to vector b, we use the least square approximation method. Taking the first partial derivatives

$\partial(A^tGA)/\partial b$ and equating to zero, we obtain normal equations

$$T^t_{m,r}G_nT_{m,r}\hat{b} = T^t_{m,r}G_na.$$

We clearly see that $T_{m,r}$ is 1-banded matrix. From definition of $T_{m,r}$ and G_n, the matrix product $T^t_{m,r}G_nT_{m,r}$ reduces to G_m. Thus, $T^t_{m,r}G_nT_{m,r} = G_m$. This matrix is real symmetric positive definite matrix and $(T^t_{m,r}G_nT_{m,r})^{-1}$ exists. Hence, it possesses the unique solution for $\hat{b}$

$$\hat{b} = (T^t_{m,r}G_nT_{m,r})^{-1}T^t_{m,r}G_na. \tag{3.7}$$

The q-Bézier curve by using the points (3.7) is the best approximation with respect to q-L_2 norm. A similar solution is given in Lee et al. (2002) for the Legendre and Bernstein basis transformation.

The degree reduction of a polynomial of degree n with respect to q-L_2 norm with little q-Legendre polynomial basis and coefficients $l = (l_0, l_1, \ldots, l_n)^t$ is reduced to a polynomial of degree $(n-1)$ with coefficients $l^{(-1)} = (l_0, l_1, \ldots, l_{n-1})^t$. After r degree reductions, we have a linear system

$$l^{(-r)} = I_{n,-r}l,$$

where the matrix $I_{n,-r}$ of dimension $(n-r+1)\times(n+1)$ is

$$I_{n,-r} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

After converting the q-Bézier coefficients into the little q-Legendre coefficients by using M_n^{-1}, the r times degree reduction by $I_{n,-r}$, and finding the q-Bernstein coefficients by M_m, we get the following expression that gives r times degree reduction matrix $(T^t_{m,r}G_nT_{m,r})^{-1}T^t_{m,r}G_n$. Namely, the degree reduction matrix can be expressed as

$$(T_{m,r}^t G_n T_{m,r})^{-1} T_{m,r}^t G_n = M_m I_{n,-r} M_n^{-1}.$$

CHAPTER FOUR
CONCLUSIONS

Having transformations between the Legendre and Bernstein bases on $[0,1]$ investigated by Farouki (2000), these transformations are used for degree elevation, degree reduction and least square purposes in curves and surface design in Lee et al. (2002). Based on these works, we derive the transformations between the little q-Legendre and q-Bernstein polynomial bases on $[0,1]$ when the parameter $q \neq 0$. We obtain a determinant characterization of little q-Legendre polynomials. Furthermore, we give gram matrix G_n for q-Bernstein polynomials with respect to q-L_2 norm. Moreover, we obtain relationships between the coefficient transformation matrices, degree elevation and degree reduction matrices. For future work, investigation of the eigenvalues of Gram matrix G_n can be worth investigating.

REFERENCES

Ait-Haddou, R., & Goldman, R. (2015). Best polynomial degree reduction on q-lattices with applications to q-orthogonal polynomials. *Applied Mathematics and Computation*, *266*, 267-276.

Andrews, G. E., & Askey, R. (1977). Enumeration of partitions: The role of Eulerian series and q-orthogonal polynomials. In *Higher combinatorics* (3-26). Netherlands: Springer.

Cheney, E. (1982). *Introduction to approximation theory*. Rhode Island: American Mathematical Society.

Chihara, T. S. (1978). *An introduction to orthogonal polynomials*. New York: Gordon and Breach Science Publishers.

Davis, P. J. (1975). *Interpolation and approximation*. New York: Dover Publications.

Farin, G. (2002). *Curves and surfaces for computer-aided geometric design: A practical guide* (5th ed.). New York: Academic Press.

Farouki, R. T. (2000). Legendre-Bernstein basis transformations. *Journal of Computational and Applied Mathematics*, *119*(1), 145-160.

Gasper, G., & Rahman, M. (2004). *Basic hypergeometric series* (2nd ed.). Cambridge: Cambridge University Press.

Goldman, R. (2003). *Pyramid algorithms: A dynamic programming approach to curves and surfaces for geometric modeling*. San Francisco: Morgan Kaufmann.

Jackson, F. (1910). On q-definite integrals. *Quarterly Journal of Pure and Applied Mathematics*, *41*, 193-203.

Kincaid, D., & Cheney, W. (1996). *Numerical Analysis: The mathematics of scientific computing*. Pacific Grove, California: Brooks/Cole.

Koekoek, R., Lesky, P. A., & Swarttouw, R. F. (2010). *Hypergeometric orthogonal polynomials and their q-analogues*. Berlin: Springer.

Lee, Byung, G., Park, Y., & Yoo, J. (2002). Application of Legendre–Bernstein basis transformations to degree elevation and degree reduction. *Computer Aided Geometric Design*, *19*(9), 709-718.

Oruç, H. (1998). *Generalized Bernstein polynomials and total positivity*. PhD Thesis, University of St Andrews, School of Mathematical and Computational Sciences, Scotland, U.K.

Oruç, H., & Phillips, G. M. (2003). q-Bernstein polynomials and Bézier curves. *Journal of Computational and Applied Mathematics*, *151*, 1-12.

Oruç, H., & Tuncer, N. (2002). On the convergence and iterates of q-Bernstein polynomials. *Journal of Approximation Theory*, *117*(2), 301-313.

Ostrovska, S. (2003). q-Bernstein polynomials and their iterates. *Journal of Approximation Theory*, *123*(2), 232-255.

Phillips, G. M. (1997). Bernstein polynomials based on the q-integers. *Annals of Numerical Mathematics*, *4*, 511-518.

Schmidt, A. L. (1993). Generalized q-Legendre polynomials. *Journal of Computational and Applied Mathematics*, *49*(1-3), 243-249.