REPUBLIC OF TURKEY
YUZUNCU YIL UNIVERSITY
INSTITUTE OF NATURAL AND APPLIED SCIENCES
DEPARTMENT OF MATHEMATICS

ON THE STABILITY, BOUNDEDNESS AND CONVERGENCE OF SOLUTIONS TO CERTAIN ORDINARY DIFFERENTIAL EQUATIONS OF FOURTH ORDER

MASTER THESIS

PRESENTED BY: Adnan Ahmad MAHMUD
SUPERVISED BY: PROF. DR. Cemil TUNÇ

VAN - 2016

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ACCEPTANCE and APPROVAL PAGE

This thesis entitled **"ON THE STABILITY, BOUNDEDNESS AND CONVERGENCE OF SOLUTIONS TO CERTAIN ORDINARY DIFFERENTIAL EQUATIONS OF FOURTH ORDER"** presented by **ADNAN AHMAD MAHMUD** under supervision of **Prof. Dr. Cemil TUNÇ** in Department of Mathematics has been accepted as a M. Sc. thesis according to Legislations of Graduate Higher Education on

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THESIS STATEMENT

All information presented in the thesis was obtained in the frame of ethical behavior and academic rules. In addition all kinds of information that does not belong to me have been cited appropriately in the thesis prepared by the thesis writing rules.

Signature

Adnan Ahmad MAHMUD

ÖZET

DÖRDÜNCÜ MERTEBEDEN DİFERANSİYEL DENKLEMLERDE KARARLILIK, SINIRLILIK VE YAKINSAKLIK

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Bu tez çalışmasının amacı, dördüncü mertebeden lineer olmayan belli formadaki diferansiyel denklemlerin çözümlerinin kararlılık, sınırlılık ve yakınsaklığı ile ilgili olarak literatürde yapılmış bulunan bazı çalışmaları araştırmacıların dikkatine sunmaktır. Bu tez çalışması altı bölümden oluşmaktadır. Birinci bölümde, tez konusu ile ilgili literatürde yapılan bazı çalışmalar özetle verildi, ikinci bölümde bu tezde kullanılacak materyal ve yöntem belirtildi. Üçüncü bölümde teze ait temel kavramlar, tanım ve teoremler sunuldu. Dördüncü bölümde dördüncü mertebeden lineer olmayan belli formadaki diferansiyel denklemlerin çözümlerinin karalılığı ve asimptotik davranışları ile ilgili olarak literatürde yer alan bazı çalışmalar ele alındı. Beşinci bölümde ise, dördüncü mertebeden lineer olmayan belli formadaki diferansiyel denklemlerin çözümlerinin sınırlılık davranışları ile ilgili olarak literatürde bulunan bazı çalışmalar sunuldu. Son olarak, altıncı bölümde ise, dördüncü mertebeden lineer olamayan belli formadaki diferansiyel denklemlerin çözümlerinin yakınsaklık davranışları ile ilgili olarak literatürde yer alan bazı çalışmalar verildi. Bu tez orijinal ve yeni sonuç içermemektedir. Bu tezdeki amacımız literatürde söz edilen konuları bir arada araştırmacıların dikkatine sunmaktır.

Anahtar kelimeler: Asimptotik davranış, Diferansiyel denklem, Dördüncü mertebe, Lyapunov fonksiyon, Kararlılık, Sınırlılık, Yakınsaklık.

ABSTRACT

ON THE STABILITY, BOUNDEDNESS AND CONVERGENCE OF SOLUTIONS TO CERTAIN ORDINARY DIFFERENTIAL EQUATIONS OF FOURTH ORDER

Adnan Ahmad MAHMUD
M.SC. mathematics science
SUPERVISOR: Prof. Dr. Cemil TUNÇ
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The aim of this thesis is to submit the scientific works done in the literature related to the stability, asymptotic behaviors, boundedness and convergences of solutions of the fourth order non-linear ordinary differential equations to the attention of the readers. This thesis consists of six chapters. In the first, we summarize the works done in the literature with respect to the subject of thesis, in the second chapter the materials and methods used in the thesis are given and in the third chapter the basic definitions and theorems which are related to the thesis are introduced. In the fourth chapter, we discuss the stability and asymptotic behaviors of solutions of certain nonlinear differential equations of fourth order, which exist in the literature. In the fifth chapter, we investigate the boundedness of solutions of certain nonlinear differential equations of fourth order in the literature. In the final chapter, chapter six, we study the convergence of solutions of certain nonlinear differential equations of fourth order, which are given in the literature. This thesis is a compilation and collection of works done on stability, asymptotic behaviors, boundedness and convergences of solutions of certain non-linear differential equations of third order in the literature. And it does not contain any original new result. Our aim is to unification and presents the mentioned topics in the literature to the researchers.

Keywords: Asymptotic behavior, Boundedness, Convergence, Differential equation, fourth order, Lyapunov function, Stability.

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LIST of SYMBOLS

Some symbols and abbreviations used in this study are presented below, along with explanations.

Symbols	**explanations**
$\boldsymbol{R}$	Set of real numbers,
$\boldsymbol{R^+}$	Non-negative real numbers,
$\boldsymbol{R^n}$	n-dimensional Euclidean space,
$\boldsymbol{C^1[R]}$	The space of continuous and differentiable functions on real numbers,
$\boldsymbol{CI}$	The family of continuous and increasing functions,
$\boldsymbol{CIP}$	The family of continuous, increasing and positive definite functions,
$\boldsymbol{\lvert x \rvert}$	Absolute value of x,
$\boldsymbol{\lVert x \rVert}$	Euclidian norm of x.

1. INTRODUCTION

1.1. Introduction

It is well known that the stability, boundedness and convergence of solutions of ordinary differential equations are very important problems in the theory and applications of the differential equations. So far, the most effective and important method to study the stability, boundedness and convergence of solutions of nonlinear differential equations is the Lyapunov's direct (or second) method. In the relevant literature, on fourth order nonlinear differential equations, many stability, boundedness and convergence results have been established by using this method. One can refer to the books of (Ahmad and Rao, 1999), (Yoshizawa, 1966), (Reissig et al., 1974) as a good survey, (Ross, 1984) and the papers of (Ezeilo, 1962), (Harrow, 1970), (Ezeilo and Tejumola, 1971), (Tunc and Ates, 2013),(Tunc, 1995;2001;2004;2005;2006;2007), (Omeike, 2007), (Tejumola, 1968;1972), (Ogundare and Okecha, 2007;2008), (Ogurtsov,1959), (Asmussen,1971), (Hara, 1974), (Skrapek and Lalli, 1977), (Yu Yuan-hong, 1990), (Wu and Xiong, 1998),(Adesina and Ogundare, 2012), (Afuwape, 1981; 1988; 2012), (E. Tunc, 2010), and theirs references for the works done on the mentioned topics.

1.2. Basic information and Preliminaries

Consider a system of differential equations

$$\frac{dx}{dt} = F(t,x), \tag{1.1}$$

where x is an n-vector. Suppose that $F(t,x)$is continuous in (t,x) on $I \times D$, where D is a connected open set in R^n, $I = [0,\infty), R = (-\infty,\infty)$ and R^n is the n-dimensional Euclidean space.

Definition 1.1. The solution $x(t) \equiv 0$ of (1.1) is stable, if for any $\varepsilon > 0$ and any $t_0 \in I$ there exists a $\delta(t_0,\varepsilon) > 0$ such that if $\|x_0\| < \delta(t_0,\varepsilon)$, we have $\|x(t;x_0,t_0)\| < \varepsilon$ for all $t \geq t_0$ (Yoshizawa, 1966).

Example 1.1. Consider the initial value problem

$$x'' + 5x' + 4x = 0$$
$$x(0) = -5, \quad x'(0) = -1.$$

It is clear the zero solution of this initial value problem is stable.

Definition 1.2. The zero solution of (1.1) is uniformly stable, if the δ in Definition 1.1 is independent of t_0 (Yoshizawa, 1966).

Example 1.2. Consider the initial value problem

$$x'' + x = 0$$

$$x(0) = c_1, \qquad x'(0) = c_2,$$

where c_1, c_2 are positive constants. It is clear that zero solution of the equation is uniformly stable.

Definition 1.3. The zero solution of (1.1) is asymptotically stable, if $x(t) \equiv 0$ is stable and if there exists a $\delta_0(t_0) > 0$ such that if $\|x_0\| < \delta_0(t_0)$, $\|x(t; x_0, t_0)\| \to 0$ as $t \to \infty$ (Yoshizawa, 1966).

In Example 1.1 when $t \to \infty$ each solution tends to zero. Hence the zero solution is asymptotically stable.

Definition 1.4. The zero solution of (1.1) is exponential- asymptotically stable, if there exists a $\lambda > 0$ and, given any $\varepsilon > 0$, there exist a $\delta(\varepsilon) > 0$ such that if $\|x_0\| < \delta(\varepsilon)$, $\|x(t; x_0, t_0)\| \leq \varepsilon \exp[-\lambda(t - t_0)]$ for all $t \geq t_0$ (Yoshizawa,1966).

Example 1.3. We take into consideration the initial value problem

$$x' + 2x = 0$$

$$x(0) = 2.$$

It can be easily shown that the zero solution of this initial value problem is exponential asymptotically stable.

Definition 1.5. The zero solution $x(t) \equiv 0$ of equation (1.1) is asymptotically stable in the large, if it is stable and if every solution of (1.1) tends to zero as $t \to \infty$ (Yoshizawa, 1966).

Definition 1.6. A solution $x(t; x_0, t_0)$ of (1.1) is bounded, if there exist a $\beta > 0$ such that $\|x(t; x_0, t_0)\| < \beta$ for all $t \geq t_0$, where β may depend on each solution (Yoshizawa, 1966).

In Example 1.1 each solution satisfies this definition so the solutions are bounded.

Definition 1.7. The solutions of (1.1) are uniformly bounded, if the β in Definition 1.6 is independent of t_0 (Yoshizawa, 1966).

Example 1.4. Let us consider the initial value problem

$$x'' + x = 0,$$
$$x(0) = 2, \qquad x'(0) = -2.$$

It follows that all solutions of this initial value problem are uniformly bounded.

Definition 1.8. The solutions of (1.1) are ultimately bounded for bound B, if there exist a $B > 0$ and a $T > 0$ such that for every solution $x(t; x_0, t_0)$ of (1.1), $\|x(t; x_0, t_0)\| < B$ for all $t \geq t_0 + T$, where B is independent of the particular solution while T may depend on each solution (Yoshizawa, 1966).

Example 1.5. Let us consider the initial value problem

$$x'' + x = 0,$$
$$x(0) = 1, \qquad x'(0) = -1.$$

It follows that all solutions of this initial value problem are ultimately bounded.

Definition 1.9. The solutions of (1.1) are equi-ultimately bounded, for bound B if there exists a $B > 0$ and if corresponding to any $\alpha > 0$ and $t_0 \in I$, there exist a $T(t_0, \alpha) > 0$ such that $x_0 \in \delta_a$ implies that $\|x(t; x_0, t_0)\| < B$ for all $t \geq t_0 + T(t_0, \alpha)$ (Yoshizawa, 1966).

Definition 1.10. The solutions of equation (1.1) are uniform-ultimately bounded for bound B, if the T in Definition 1.9 is independent of t_0 (Yoshizawa, 1966).

Definition 1.11. Consider the system

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y). \tag{1.2}$$

Assume that this system has an isolated critical point at the origin $(0,0)$ and that P and Q have continuous first partial derivatives for all (x, y).

Let $E(x, y)$ be positive definite for all (x, y) in domain D, containing the origin and such that the derivative $\dot{E}(x, y)$ of E with respect to system (1.2) is negative semi-definite for all $(x, y) \in D$. Then E is called a Lyapunov function for the system (1.2) in D, where D is a domain contains the origin $(0,0)$ (Ross, 1984).

Definition 1.12. A Lyapunov function V defined as $V: I \times R^n \to R$ is said to be complete if for $X \in R^n$ the following conditions hold:

(**i**) $V(t, X) \geq 0$,

(**ii**) $V(t,X)=0$ if and only if $X=0$, and

(**iii**) $\dot{V}(t,X)\leq -c|X|$, where c is any positive constant and $|X|$ is given by

$$|X|=\left(\sum(x_i^2)\right)^{\frac{1}{2}} \text{ such that } \quad |X|\to\infty \quad \text{as} \quad X\to\infty$$

(Ogundare, 2006).

Definition 1.13. A Lyapunov function V defined as $V: I\times R^n\to R$ is said to be incomplete if for $X\in R^n$, (i) and (ii) of Definition 1.12 are satisfied and in addition

(**iii**) $\dot{V}(t,X)\leq -c|X|_*$ where c is any positive constant and $|X|_*$ is given by

$$|X|_*=(x^2+y^2)^{\frac{1}{2}} \text{ or } |X|_*=(y^2+z^2)^{\frac{1}{2}} \text{ or } |X|_*=(x^2+z^2)^{\frac{1}{2}} \quad \text{or}$$

$$|X|_*=(x^2)^{\frac{1}{2}} \text{ or } |X|_*=(y^2)^{\frac{1}{2}} \text{ or } |X|_*=(z^2)^{\frac{1}{2}} \text{ such that } |X|_*\to\infty \text{ as } X\to\infty$$

(Ogundare, 2006).

The following are known as the Lyapunov's stability theorems.

Notice that CI and CIP are the set of continuous increasing and continuous increasing positive definite functions, respectively.

We reconsider the differential system (1.1) under assumption that $F(t,x)$ is continuous on $0\leq t<\infty$, $\|x\|<H$, $H>0$, and $F(t,0)\equiv 0$.

Theorem 1.1. Suppose that there exist a Lyapunov function $V(t,x)$ defined on

$0\leq t<\infty$, $\|x\|<H$, which satisfies the following conditions:

(**i**) $V(t,0)\equiv 0$,

(**ii**) $a(\|x\|)\equiv V(t,x)$, where $a(r)\in CIP$,

(**iii**) $\dot{V}(t,x)\leq 0$.

Then the zero solution of (1.1) is stable (Yoshizawa, 1966).

Theorem 1.2. If condition (ii) in Theorem 1.1 is replaced by $a(\|x\|)\leq V(t,x)\leq b(\|x\|)$, where $a(r)\in CIP$ and $b(r)\in CIP$, then the zero solution of (1.1) is uniformly stable (Yoshizawa, 1966).

Theorem 1.3. Under the same assumptions as in Theorem 1.2 if $\dot{V}(t,x) \leq -c(\|x\|)$, where $c(r)$ is continuous on $[0,H]$ and is positive definite, then zero solution of (1.1) is uniformly asymptotically stable (Yoshizawa, 1966).

We now give some theorems related to the boundedness of the solutions of (1.1).

Theorem 1.4. Suppose that there exists a Lyapunov function $V(t,x)$ defined on $I \times R^n$ which satisfies the following conditions:

(**i**) $a(\|x\|) \leq V(t,x)$, where $a(r) \in CI$, and $a(r) \to \infty$ as $r \to \infty$,

(**ii**) $\dot{V}(t,x) \leq 0$.

Then the solutions of (1.1) are equi-bounded (Yoshizawa, 1966).

Theorem 1.5. Suppose that there exists a Lyapunov function $V(t,x)$ defined on

$0 \leq t < \infty$, $\|X\| > R$, where R may be large, which satisfies the following conditions:

(**i**) $a(\|X\|) \leq V(t,X) \leq b(\|X\|)$, where $a(r), b(r) \in CI$, and $a(r) \to \infty$ as $r \to \infty$,

(**ii**) $\dot{V}(t,X) \leq 0$.

Then the solutions of (1.1) are uniform-bounded (Yoshizawa, 1966).

Now we consider the solutions of

$$L(D)x = x^{(n)} + a_1 x^{(n-1)} + \cdots + a_n x = 0, \tag{1.3}$$

where $a_1, a_2, \ldots, a_n$ are the real constants, defined on $-\infty < t < \infty$.

Theorem 1.6. (Hurwitz's theorem) A necessary and sufficient condition for the negativity of the real parts of all the roots of the polynomial

$$L(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n$$

with real coefficients is the positivity of all principal diagonal of the minors of the Hurwitz matrix

$$H_n = \begin{bmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

It should be noted that the principal diagonal of the Hurwitz matrix H_n exhibits the coefficients of the polynomial $L(\lambda)$ in the order of their numbers from a_1 to a_n. The

alternate columns in this matrix consist of coefficients with only odd indices or with only even indices (including the coefficient $a_0 = 1$) . Hence, the elements of Hurwitz matrix $H_n = (b_{ik})$ are given by $b_{ik} = a_{2i-k}$, the missing coefficients (that is, coefficients with indices greater than n or less than zero) being replaced by zero (Ahmad and Rao, 1999).

For example the Hurwitz conditions of $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$ reduce to $a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_4 > 0,\ a_1a_2 - a_3 > 0$ and $a_1a_2a_3 > a_3^2 + a_1^2a_4$.

Example 1.6. The characteristic polynomial

$$L(\gamma) = \gamma^4 + 3\gamma^3 + 3\gamma^2 + 2\gamma + K$$

is stable if $0 < K < \frac{14}{9}$. Therefore, all the solutions $x(t)$ of

$$x^{(4)} + 3x''' + 3x'' + 2x' + Kx = 0,$$

where K is a parameter, satisfy

$$\lim_{t\to\infty}|x(t)| = 0 \quad \text{provided} \quad 0 < K < \frac{14}{9}.$$

Theorem 1.7. If the roots of characteristic polynomial $L(\lambda)$ have negative real parts, then $a_1, \ldots, a_n$ are positive (Ahmad and Rao, 1999).

Definition 1.14. Let $x_1(t), x_2(t)$ be any two solutions of the fourth order differential equation with constants coefficients of the form:

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = p(t),$$

where the dots denotes the differentiation with respect to *t*. We say that $x_1(t), x_2(t)$ are converge to each other if

$$x_2(t) - x_1(t) \to 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \to 0,$$
$$\ddot{x}_2(t) - \ddot{x}_1(t) \to 0, \qquad \dddot{x}_2(t) - \dddot{x}_1(t) \to 0$$

as $t \to \infty$ (Afuwape, 2012).

Remark 1.8. If the relations just given are true of each arbitrary pairs of solutions, these lead that all solutions of this equation are converge.

Example 1.7. Consider the fourth order constant coefficients homogeneous linear differential equation of the form of

$$x^{(4)} + 10\dddot{x} + 35\ddot{x} + 50\dot{x} + 24x = 0.$$

It is clear to see that the linear independent solutions of homogenous differential equation are

$$x_1(t) = c_1 e^{-t}, \quad x_2(t) = c_2 e^{-2t}, \quad x_3(t) = c_3 e^{-3t}, \quad x_4(t) = c_4 e^{-4t}.$$

We can see that each of the above solutions satisfies the definition of convergence and hence all solutions of the considered differential equation are convergence.

2. LITERATURE REVIEW

Ogurtsov (1959) considered non-linear homogenous differential equation of the fourth order in the form

$$x^{(4)} + f(\ddot{x}) + c\ddot{x} + b\dot{x} + ax = 0$$

and investigated the stability property of the zero solution of this equation.

Some years after that, Asmussen (1971) studied the behavior of solutions of certain non- homogenous and non-linear differential equation of fourth order

$$x^{(4)} + f(\ddot{x})\ddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + h(x) = 0.$$

Asmussen (1971) found conditions on the functions f, g, h which will imply asymptotic stability.

In (1974), Hara investigated the asymptotic behaviors of the solutions of non-linear fourth order differential equation in the form

$$x^{(4)} + a(t)f(\ddot{x})\ddot{x} + b(t)\Phi(\dot{x}, \ddot{x}) + c(t)g(\dot{x}) + d(t)h(x) = 0,$$

where the functions appear are real valued functions.

Skrapek and Lalli (1977) investigated stability of solutions of the fourth order differential equation in the form

$$x^{(4)} + f(x, \dot{x}, \ddot{x}, \dddot{x})\dddot{x} = 0.$$

Yu Yuan-hong (1990) considered the certain non-linear fourth order differential equation of the form

$$x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = 0.$$

Yu Yuan-hong (1990) investigated the asymptotic stability in large of the trivial solution of this equation.

Wu and Xiong (1998) gave some remarks on the results of Chin (1989) and Ku (1964). Furthermore, the authors obtained some new criteria for asymptotic stability of the solutions.

Tunc (2006; 2007) investigated the stability of solutions of the following two fourth order non-linear differential equations described by

$$x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(x, \dot{x})\ddot{x} + g(\dot{x}) + h(x) = 0$$

and

$$x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(x, \dot{x}, \ddot{x}) + g(x, \dot{x}) + h(x) = 0,$$

respectively.

Adesina and Ogundare (2012) established some conditions on the globally asymptotic stability of trivial solution to a certain fourth order non-linear differential equation of the form

$$x^{(4)} + \Phi(\ddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = 0.$$

In (1962) Ezeilo considered the fourth order non-linear differential equation of the form

$$x^{(4)} + f(\ddot{x})\ddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t)$$

and he established some sufficient conditions for the boundedness of solutions, where α_2, α_4 are real positive constants.

In (1968) Tejumola considered the fourth order non-linear non homogeneous differential equation of the form

$$x^{(4)} + f(\ddot{x})\ddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}).$$

The author also gave sufficient conditions which guarantee the stability and boundedness of the solutions of the considered equation.

Harrow (1970) considered non homogeneous fourth order differential equation

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + h(x) = p(t).$$

Harrow (1970) established some new results on the stability and boundedness properties of solutions.

Reissig et al. (1974) collected the results obtained in the literature on the stability boundedness etc. of the solutions of certain nonlinear differential equations of fourth order in their book. This book can be considered as a survey for the results obtained in the literature on the subject by 1974.

Tunc (1995; 2001; 2004; 2005) studied the stability and boundedness of solutions for certain fourth order differential equations of the form

$$x^{(4)} + \varphi(x, \dot{x}, \ddot{x}, \dddot{x})\ddot{x} + f(\dot{x}, \ddot{x}) + g(x, \dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

$$x^{(4)} + \varphi(t, x, \dot{x}, \ddot{x}, \dddot{x})\ddot{x} + f(t, x, \dot{x}, \ddot{x}) + g(t, x, \dot{x}) + dx = p(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

$$x^{(4)} + \varphi(\ddot{x})\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x})$$

and

$$x^{(4)} + a(\ddot{x},\dddot{x})\dddot{x} + b(x,\dot{x})\ddot{x} + c(\dot{x}) + d(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}),$$

respectively. The author established some useful results about the stability and boundedness of solutions of these equations.

Omeike (2007) studied the boundedness of solutions of the fourth order non-linear differential equation of the form

$$x^{(4)} + a\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t).$$

Ogundare and Okecha (2008) studied the boundedness property of solution to certain non-linear fourth order differential equation of the form

$$x^{(4)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t).$$

In (2013), Tunc and Ates investigated boundedness properties of solutions of the fourth order non-linear differential equation of the form

$$x^{(4)} + a\dddot{x} + f(x,\dot{x},\ddot{x}) + g(x,\dot{x}) + h(x) = p(t).$$

Afuwape (1981) considered the fourth order differential equation

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x})$$

and discussed convergence of solutions of this equation, where a, b and c are positive constants, h and p are continuous, and p has the form $p = q(t) + r(t,x,\dot{x},\ddot{x},\dddot{x})$, with q, r are continuous.

Afuwape (1988) discussed the convergence of the solutions to the fourth order differential equations of the form

$$x^{(4)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}).$$

Ogundare and Okecha (2007) considered the fourth-order differential equation

$$x^{(4)} + a\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t)$$

and they gave sufficient conditions for the convergence of the solutions of this equation.

Tunc (2010) concerned with the fourth order non-linear differential equation of the form

$$x^{(4)} + f(\ddot{x}) + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

and established some results on the convergence of solutions.

Afuwape (2012) proved some results on the convergence of solutions of the fourth order non-linear differential equation of the form

$$x^{(4)} + \psi(\dddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}).$$

Also, Adesina and Ogundare (2012) obtained similar new results on the convergence of solutions for certain fourth order nonlinear differential equations. Throughout all the mentioned papers, the Lyapunov's second method used as a basic tool to verify the results therein.

3. MATERIALS and METHODS

In this study; while looking the introduction and literature review of this thesis, it can be seen that, till now, many important scientific results have been obtained in the relevant literature on qualitative behavior of solutions, stability, boundedness, convergence, etc., of certain nonlinear differential equations of fourth-order. Throughout that entire works discussed in this thesis, the Lyapunov's second (or direct) method has been used as a basic tool to show the mentioned results. However, if the solutions of any differential equation can be found explicitly, then one can show that stability, boundedness and convergence of solutions by means of the related definitions of these concepts as well as via the Lyapunov's second (or direct) method. On the other hand, when researchers use the Lyapunov's second (or direct) method to prove the proper stability, boundedness and convergence results of non-linear differential equations, they have to take into consideration the Routh-Hurwitz criteria during the establishment of the proper conditions. The Routh-Hurwitz criteria for nonlinear differential equations are expected as natural generalizations of that related linear differential equations with constant coefficients. The proofs of the results depend on the proper the Routh-Hurwitz criteria. To prove stability, boundedness and convergence of the solutions by one of these definitions is known as the Lyapunov first method in the literature. A.M. Lyapunov is a Russian Mathematician. This method was coined in the 1890. These days, this method is being studied in the scientific literature on a large scale; that is, this method is still used today actively. The material will be used in this thesis, because of the theoretical studies adopted in international articles, published books and publications considered to be similar in scientific studies. We can reach to the large scale of these sources with the help of scientific databases, benefiting from these resources in the library of Yuzuncu Yil University. While in doing the proofs of theorems, we take of consideration some basic theorems which are refer to the Lyapunov’s theorems.

4. STABILITY of SOLUTIONS

4.1. Stability in $x^{(4)} + f(\ddot{x}) + c\ddot{x} + b\dot{x} + ax = 0$

Ogurtsov (1959) considered non-linear differential equation of the fourth order in the form

$$x^{(4)} + f(\ddot{x}) + c\ddot{x} + b\dot{x} + ax = 0, \tag{4.1}$$

and investigated the stability of the zero solution of equation (4.1), where the function $f(\eta)$ is continuous and satisfies the Lipschitz condition for all real η, and it was also assumed that $f(0) = 0$.

Ogurtsov (1959) showed that a Lyapunov function can be constructed for equation (4.1),

Ogurtsov (1959) established the following results on the stability of zero solution of equation (4.1).

Theorem 4.1. If $a > 0, b > 0$, and the derivative of the function $f(\eta)$ is such that

$$\frac{df}{d\eta} > 0, \quad bc\frac{df}{d\eta} - b^2 - a\left(\frac{df}{d\eta}\right)^2 > 0 \tag{4.2}$$

hold clearly for all values of the argument, then the zero solution of equation (4.1) is asymptotically stable for arbitrary initial conditions (Ogurtsov, 1959).

Ogurtsov (1959) replaced equation (4.1) by the equivalent system of differential equations as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= a - f'(y)z, \\ \dot{u} &= -cz - by - ax \end{aligned} \tag{4.3}$$

and considered the continuously differentiable Lyapunov function V defined by:

$$2V = (b^2 + ac)x^2 + 2bcxy + (c^2 - 2a)y^2 + 4axz + 2byz + cz^2 + 2bxu$$
$$+2cyu + 2u^2 + 2b\int_0^y f'(y)ydy.$$

This function is positive definite and infinitely large. In fact, it is clear that

$$2V = (b^2 + ac)x^2 + 2bcxy + \left(c^2 - 2a + \frac{b^2}{c}\right)y^2 + 4axz + 2byz + cz^2$$
$$+2bxu + 2cyu + 2u^2 + 2b\int_0^y \left[f'(y) - \frac{b}{c}\right]ydy,$$

where

$$\int_0^y \left[f'(y) - \frac{b}{c}\right]ydy > 0 \quad \text{for} \quad y \neq 0,$$

this inequality holds clearly because by (4.2), $f'(y) > \frac{b}{c}$ and the quadratic form everywhere is positive definite.

The derivative of the function $V(x, y, z, u)$, with respect to time allowing for system (4.3) is

$$\frac{dV}{dt} = -abx^2 - 2af'(y)xz - cf'(y)z^2 + bz^2 .$$

It is easy to see that

$$\frac{dV}{dt} < 0 \text{ for } x \neq 0, \ z \neq 0 \quad \text{and} \quad \frac{dV}{dt} = 0 \quad \text{for } x = 0, \quad z = 0.$$

Obviously the yu −plane does not contain whole trajectories, except the origin of the coordinates. Thus, the zero solution of equation (4.1) is asymptotically stable for arbitrary initial perturbations.

4.2. Stability in $\boldsymbol{x^{(4)} + f(\ddot{x})\ddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + h(x) = 0}$

Asmussen (1971) investigated the behavior of solutions of certain fourth order non-linear differential equation of the form

$$x^{(4)} + f(\ddot{x})\ddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + h(x) = 0, \tag{4.4}$$

where α_2 is a positive constant and f, g and h are continuous real valued functions depending only on the arguments that displayed explicitly.

Asmussen (1971) worked to find sufficient conditions on f, g, h which imply asymptotic stability of all solutions of equation (4.4). The result deals with conditions under which all solutions of equation (4.4) tend to zero as $t \to \infty$.

Asmussen (1971) constructed his results about the stability of solutions as follows:

Theorem 4.2. Assume that g and h are differentiable on R and that

(i) there exists $\alpha_1 > 0$ such that $f(z) \geq \alpha_1$ for all z,

(ii) $h(0) = 0$ and there are positive constants α_4, A_4 such that $\frac{h(x)}{x} \geq \alpha_4,\ x \neq 0$

and for sufficiently small $\delta_1 > 0,\ A_4 - \delta_1 \leq h'(x) \leq A_4$ for all x,

(iii) $g(0) = 0$ and there exist positive constants α_3, Δ_0 such that

$$g'(y) \geq \alpha_3,$$

$$[\alpha_1\alpha_2 - g'(y)\,]\alpha_3 - \alpha_1 A_4 f(z) \geq \Delta_0 \quad \text{for all } y, z,$$

(iv) $g'(y) - \frac{g(y)}{y} \leq \delta_2$ for all $y \neq 0$, where δ_2 is a positive constant satisfying

$$\delta_2 < \frac{2A_4\Delta_0}{\alpha_1\alpha_3^2},$$

(v) $\frac{1}{z}\int_0^z f(s)ds - f(z) \leq \delta_3$ for all $z \neq 0$, where δ_3 is a positive constant satisfies

$$\delta_3 < \frac{2\Delta_0}{\alpha_1^2\alpha_3}\ .$$

Then zero solution of equation (4.4) is asymptotically stable.

Rather than deals with equation (4.4) directly, Asmussen (1971) considered the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -wf(z) - a_2 z - g(y) - h(x). \end{aligned} \tag{4.5}$$

For proving his result, Asmussen (1971) used the following Lyapunov function $V = V(x, y, z, w)$, where

$$2V = (a_2d_1 - d_2)z^2 + 2\int_0^z sf(s)ds + d_1w^2 + 2d_2y\int_0^z f(s)ds + 2d_2yw + 2zw$$

$$+2d_2\int_0^z f(s)ds + (a_2d_2 - A_4d_1)y^2 + 2\int_0^y g(s)ds + 2d_1zg(y) + 2yh(x)$$

$$+ 2d_1zh(x),$$

$$d_1 = \varepsilon + \frac{1}{\alpha_1}\ , \qquad d_2 = \varepsilon + \frac{A_4}{\alpha_3}$$

and $\varepsilon > 0$ is a constant.

Lemma 4.3. Under the hypotheses of Theorem 4.2, $V(0,0,0,0) = 0$ and there exist positive constants $E_1,\ E_2,\ E_3,\ E_4$ dependent only on $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ A_4,\ \Delta_0$ and ε such that

$$V \geq E_1x^2 + E_2y^2 + E_3z^2 + E_4w^2 \quad \text{for all } x, y, z, w \text{ (Asmussen, 1971).}$$

Lemma 4.4. Under the hypotheses of Theorem 4.2, $V(0,0,0,0) = 0$ and there exist positive constants $F_1,\ F_2,\ F_3,\ F_4$ dependent only on $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ A_4,\ \Delta_0$ and ε such that

$$\dot{V} \leq -(F_1y^2 + F_2z^2 + F_3w^2) + F_4(y^2 + z^2 + w^2)^{\frac{1}{2}} \left[\theta_1 + \theta_2(y^2 + z^2 + w^2)^{\frac{\alpha}{2}}\right]$$

for all x, y, z, w when Δ_1 is sufficiently small (Asmussen, 1971).

4.3. Stability in $\boldsymbol{x^{(4)} + a(t)f(\ddot{x})\ddot{x} + b(t)\Phi(\dot{x},\ddot{x}) + c(t)g(\dot{x}) + d(t)h(x) = 0}$

In (1974) Tadayuki Hara investigated the asymptotic behavior of the solutions of non-autonomous fourth order differential equation in the form

$$x^{(4)} + a(t)f(\ddot{x})\ddot{x} + b(t)\Phi(\dot{x},\ddot{x}) + c(t)g(\dot{x}) + d(t)h(x) = 0. \qquad (4.6)$$

Where the functions appear in the equation (4.6) are real valued functions. The dots indicated differentiation with respect to t, and all solutions considered are assumed to be real.

Hara (1974) gave conditions to ensure that all solutions of equation (4.6) tend to zero as $t \to \infty$.

Hara (1974) established the following results about the asymptotic stability of solutions of equation (4.6) as follows.

Theorem 4.5. Suppose that $a(t), b(t),\ c(t), d(t)$ are positive and continuously differentiable functions in $I = [0,\infty)$, each of $f(z), g(y), h(x)$ are continuously differentiable for all $x, y, z \in R$, $\Phi(y,z)$ and $\frac{\partial\Phi}{\partial y}(y,z)$ are continuous for all $(y,z) \in R^2$. And there exist positive constants such that the following assumptions hold:

(**i**) $A \geq a(t) \geq a_0 > 0,\ B \geq b(t) \geq b_0 > 0,\ C \geq c(t) \geq c_0 > 0,$

$$D \geq d(t) \geq d_0 > 0 \text{ for } t \in I,$$

(**ii**) $f(z) \geq f_0 > 0$ for all $z \in R$, and $f_1(z) = \frac{1}{z}\int_0^z f(s)ds$, $(z \neq 0), f_1(0) = f(0)$,

(**iii**) $g_1(y) \geq g_0 > 0$ for all $y \in R, g(0) = 0$, and

$$g_1(y) = \frac{g(y)}{y}, (y \neq 0), \quad g_1(0) = g'(0),$$

(**iv**) $xh(x) > 0 \quad (x \neq 0), \qquad H(x) \equiv \int_0^x h(s)ds \to \infty \quad \text{as } |x| \to \infty,$

$$h_0 - \frac{a_0 f_0 \delta_0}{2c_0 g_0 D} \leq h'(x) < h_0,$$

(**v**) $\emptyset_y(y,z) \leq 0, \quad \Phi(y,0) = 0 \quad \text{in } R^2, \qquad 0 \leq \frac{\Phi(y,z)}{z} - \emptyset_0 \leq \frac{\varepsilon_0 c_0^3 g_0^3}{BD^2 h_0^2} \quad (z \neq 0),$

where ε_0 is a sufficiently small positive constant,

(**vi**) $a_0 b_0 c_0 \Phi_0 f_0 g_0 - C^2 g_0 g'(y) - A^2 D f_0 h_0 f(z) \geq \delta_0 > 0$ for all $(y,z) \in R^2$,

(**vii**) $g'(y) - g_1(y) \leq \delta < \frac{2Dh_0\delta_0}{Ca_0 f_0 c_0^2 g_0^2}$,

(**viii**) $f_1(z) - f(z) \leq \frac{Cc_0 g_0 \delta}{Aa_0 f_0 D h_0}$,

(**ix**) $\int_0^\infty \{|a'(t)| + b'_+(t) + |c'(t)| + |d'(t)|\}dt < \infty,$

$$d'(t) \to 0 \quad \text{as} \quad t \to \infty, \quad \text{where} \quad b'_+(t) = max(b'(t), 0).$$

Then zero solution of equation (4.6) is asymptotically stable (Hara, 1974).

It is clear that the equation (4.6) can be written as equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -a(t)f(z)w - b(t)\Phi(y,z) - c(t)g(y) - d(t)h(x). \end{aligned} \qquad (4.7)$$

Hara (1974) used as a main tool the following continuously differentiable Lyapunov function $V = V(t,x,y,z,w)$ defined by

$$2V = 2\beta d(t)\int_0^x h(s)ds + 2c(t)\int_0^y g(\eta)d\eta + 2\alpha b(t)\int_0^z \Phi(y,s)ds + 2a(t)\int_0^z f(s)sds$$

$$+2\beta a(t)y\int_0^z f(s)ds+\{\beta\Phi_0 b(t)-\alpha h_0 d(t)\}y^2-\beta z^2+\alpha w^2+2d(t)h(x)y$$
$$+2\alpha d(t)h(x)z+2\alpha c(t)zg(y)+2\beta yw+2zw+k,$$

where $\alpha=\frac{1}{a_0 f_0}+\varepsilon$, $\beta=\frac{h_0 D}{c_0 g_0}+\varepsilon$ and ε, k are positive constants.

Moreover, Hara (1974) showed that there exist positive constants D_1, D_2 and D_6 such that

$$D_1\{H(x)+y^2+z^2+w^2\}\leq V\leq D_2\{H(x)+y^2+z^2+w^2+k\}.$$

And also

$$\dot{V}\leq -D_6(y^2+z^2+w^2),$$

which guarantees the stability of all the solutions $(x(t), y(t), z(t), w(t))$ of (4.7).

4.4. Stability in $x^{(4)}+f(x,\dot{x},\ddot{x},\dddot{x})\dddot{x}=0$

Skrapek and Lalli (1977) investigated asymptotic stability of trivial solution of the fourth order differential equation in the form

$$x^{(4)}+f(x,\dot{x},\ddot{x},\dddot{x})\dddot{x}=0, \tag{4.8}$$

in which the function f is continuous for all values of their respective arguments that displayed explicitly and the partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{\partial f}{\partial w},$$

exist and are continuous for all values of their arguments. Also, the existence and the uniqueness of the solutions of equation (4.8) are assumed.

Skrapek and Lalli (1977) used the following system which is equivalent to (4.8):

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=z,\\ \dot{z}&=w,\\ \dot{w}&=-f(x,y,z,w).\end{aligned} \tag{4.9}$$

Skrapek and Lalli (1977) obtained the results bellow on the stability of solutions.

Theorem 4.6. Let $\frac{\partial}{\partial w} f(0,0,z,w)|_{w=0} = f_1(z)$ and suppose the following conditions hold:

(i) $f(0,0,0,0) = 0, \quad f(x,y,0,w) - f(x,y,0,0) - f_1(0)w = 0$ for all x, y, w and there exist finite positive constants $\alpha_1,\ \alpha_2,\ \alpha_3$ and finite non negative constants m, n such that $f_1(z) \geq \alpha_1$ for all z,

(ii) $[f(x,y,0,0) - f(0,y,0,0)]sgnx > 0$ for $x \neq 0$ and

$[f(x,y,0,0) - f(0,y,0,0)]sgn\, x \to +\infty$ as $|x| \to +\infty$,

(iii) $\frac{\partial}{\partial y}\,[f(x,y,0,w) - f(x,y,0,0)] = 0$, and there exist finite positive constants A_4, Δ_0, Δ_1 and a finite constant A_5 such that

$$\frac{\partial}{\partial y} f(x,y,0,w) \leq A_4 \quad \text{and} \quad \frac{\partial}{\partial y}\,[f(x,y,0,w) - f(0,y,0,0)] \leq A_5 \quad \text{for all} \quad x, y,$$

$$\left[\alpha_1\alpha_2 - \frac{\partial}{\partial y} f(0,y,0,0)\right]\alpha_3 - \alpha_1 A_4 f_1(z) \geq \Delta_0 \qquad \text{for all} \quad y, z,$$

$$\frac{4\alpha_1\Delta_0}{\alpha_3} - \left[A_4 - \frac{\partial}{\partial x} f(0,y,0,0) + 4\alpha_1 A_5\right] \geq \Delta_1 \qquad \text{for all} \quad x, y,$$

(iv) $\frac{\partial}{\partial y} f(0,y,0,0) - \frac{f(0,y,0,0)}{y} \leq \delta_1 < \frac{2A_4\Delta_0}{\alpha_1\alpha_3^2}$ for $y \neq 0$,

(v) $\frac{1}{z}\int_0^z f_1(s)ds - f_1(z) \leq \delta_1 < \frac{2\Delta_0}{\alpha_1^2\alpha_3}$ for $z \neq 0$,

(vi) $\frac{\partial}{\partial y}\,[f(0,y,z,0) - f(0,y,0,0)] \leq 0$ for all y, z,

(vii) $[G(x,y,z,w)]^2 \leq \left(\frac{4\alpha_3}{A_4}\right)mn$, where

$$G(x,y,z,w) = \frac{f(x,y,0,w) - f(0,y,0,0) - f_1(z)w}{z} - \alpha_2$$

$$-\int_0^x \frac{1}{y}\frac{\partial}{\partial y}\,[f(s,y,0,0) - f(0,y,0,0)]ds - \frac{\alpha_3}{A_4}\frac{\partial}{\partial y}\,[f(x,y,0,0) - f(0,y,0,0)]$$

for $y \neq 0, z \neq 0$.

Then trivial solution of equation (4.9) is asymptotically stable.

For proving these results, Skrapek and Lalli (1977) used a scalar continuously differentiable Lyapunov function $V = V(x,y,z,w)$ which is defined by

$$2V = 2d_2 \int_0^x f(s,y,0,0)ds - 2d_2 f(0,y,0,0)x + (a_2 d_2 - A_4 d_1)y^2 + 2zw,$$

$$+2d_1 \int_0^x f(0,y,s,0)ds - 2d_1 f(0,y,0,0)z + 2\int_0^z f_1(s)sds - d_2 z^2 + d_1 w^2$$

$$+2f(x,y,0,0)y - 2f(0,y,0,0)y + 2d_1 f(x,y,0,0)z + 2d_2 y \int_0^z f_1(s)ds + 2d_2 yw,$$

where

$$d_1 = \varepsilon + \frac{1}{\alpha_1}\ , \qquad d_2 = \varepsilon + \frac{A_4}{\alpha_3}$$

and $\varepsilon > 0$ is a constant.

4.5. Stability in $\boldsymbol{x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = 0}$

Yu Yuan-hong (1990) considered the following non-linear fourth order differential equation of the form

$$x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = 0. \tag{4.10}$$

The purpose of (Yu Yuan-hong, 1990) is to investigate the asymptotic stability in large of trivial solution of equation (4.10).

Instead of (4.10) Yu Yuan-hong (1990) considered the equivalent system in the phase variables from

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -\varphi(z)w - f(z) - g(y) - h(x). \end{aligned} \tag{4.11}$$

In which the functions φ, f, g and h depend only on the arguments displayed and the dots denoted differentiation with respect to t .Where $\varphi(\ddot{x}), f(\ddot{x}), g'(\dot{x}), h'(x)$ are continuous for all values their respective arguments, and are such that the existence and the uniqueness of the solutions of (4.10) will be assumed.

Suppose

$$\varphi_1(z) = \begin{cases} \frac{1}{z}\int_0^z \varphi(s)ds, & z \neq 0, \\ \varphi(0), & z = 0, \end{cases}$$

$$f_1(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0, \\ f'(0), & z = 0 \end{cases}$$

and

$$g_1(y) = \begin{cases} \frac{g(y)}{y}, & y \neq 0, \\ g'(0), & y = 0. \end{cases}$$

Yu Yuan-hong (1990) established the following stability results.

Theorem 4.7. Suppose the following conditions hold:

(i) $f(0) = g(0) = h(0) = 0$,

(ii) there are positive constants a, b, c, d, δ and ε such that

$abc - cg'(y) - ad\varphi(z) \geq \delta > 0$ for all y and z,

(iii) $0 < d - \frac{a\delta}{4c} < h'(x) \leq d$ for all x, and $h(x)sgn\, x \to +\infty$, $|x| \to +\infty$,

(iv) $0 \leq g_1(y) - c < \frac{\delta}{8c}\sqrt{\frac{d}{2ac}}$ for all y,

(v) $0 \leq f_1(z) - b \leq \frac{\varepsilon_0 c^3}{d^2}$ for all z, where $0 < \varepsilon_0 < \varepsilon \leq \frac{\delta}{2acD}, D = ab + \frac{bc}{d}$,

(vi) $\varphi(z) \geq a$, $\varphi_1(z) - \varphi(z) < \frac{\delta}{2a^2c}$ for all z.

Then the trivial solution of system (4.11) is asymptotically stable in large (Yu Yuan-hong, 1990).

The main tool to prove the results of (Yu Yuan-hong, 1990) is the continuously differentiable Lyapunov function $V(x, y, z, w)$ defined by

$$2V(x, y, z, w) = 2\beta\int_0^z h(s)ds + (b\beta - d\alpha)y^2 + 2\int_0^y g(s)ds + 2\alpha\int_0^z f(s)ds$$

$$+2\int_0^z \varphi(s)ds - \beta z^2 + \alpha w^2 + 2h(x)y + 2\alpha h(x)z + 2\alpha g(y)z$$

$$+2\beta y\int_0^z \varphi(s)ds + 2\beta yw + 2zw,$$

where

$$\alpha = \varepsilon + \frac{1}{a}\ , \qquad \beta = \varepsilon + \frac{d}{c}\ .$$

Yu Yuan-hong (1990) improved and applied the following two lemmas to show that $V(x,y,z,w)$ is Lyapunov function to system (4.11).

Lemma 4.8. Suppose that all conditions of Theorem 4.7 hold. Then, there are positive constants $D_i \equiv D_i(a,b,c,d,\varepsilon,\delta),\ i = 1,2,3,4$, such that

$$V(x,y,z,w) \geq\ D_1\int_0^z h(s)ds + D_2y^2 + D_3z^2 + D_4w^2$$

for all x,y,z,w (Yu Yuan-hong, 1990).

Lemma 4.9. Suppose that all conditions of Theorem 4.7 hold. Then there are positive constants $D_i \equiv D_i(a,b,c,d,\varepsilon_0,\varepsilon,\delta),\ i = 5,6,7$, such that whenever (x,y,z,w) is only solution of system (4.11), then

$$\frac{d}{dt}V(x,y,z,w) \leq\ -(D_5y^2 + D_6z^2 + D_7w^2)$$

(Yu Yuan-hong, 1990).

4.6. Some Remarks on Stability Results to Certain Fourth Order Differential Equations

Wu and Xiong (1998) gave some remarks on the results of Chin (1989) and Ku (1964). Furthermore, some new criteria for local asymptotic stability are presented.

At the beginning Wu and Xiong (1998) discussed and compared the results of Chin (1989) and Cartwright (1956).

Wu and Xiong (1998) considered the fourth order differential equation in the form

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + f(x) = 0. \qquad (4.12)$$

It is clear that equation (4.12) can be rewritten as a system of differential equations

$$\dot{x} = y,$$
$$\dot{y} = z,$$
$$\dot{z} = w,$$
$$\dot{w} = -a_1 w - a_2 z - a_3 y - f(x). \tag{4.13}$$

Wu and Xiong (1998) showed that Chin (1989) worked by intrinsic method to constructing one new type of scalar continuously differentiable Lyapunov function $V = V(x, y, z)$ of equation (4.12) as follows

$$2V = (w + a_1 z + \Delta y)^2 + \frac{2}{a_1^2}\int_0^y A(x) y dy + \left(\frac{a_3}{a_1}\right)\left[z + a_1 y + \left(\frac{a_1}{a_3}\right) f(x)\right]^2$$
$$+ \frac{2}{a_1 a_3}\int_0^x f(x) A(x) dx \tag{4.14}$$

and the derivative of V along the solutions of (4.13) is given by

$$\dot{V} = -\frac{1}{a_1} A(x) y^2, \tag{4.15}$$

where

$$\Delta = a_2 - \frac{a_3}{a_1} \text{ and } A(x) = a_1 a_2 a_3 - a_3^2 - a_1^2 f'(x).$$

Also (Wu and Xiong, 1998) gave some notations on the results of Chin (1989) and Ku (1964), and they are consider the following fourth order differential equations which is the same as the equation that Chin (1989) studied

$$x^{(4)} + a_1 \dddot{x} + f(x, \dot{x})\ddot{x} + a_3 \dot{x} + a_4 x = 0, \tag{4.16}$$

and it can be written in phase variable from

$$\dot{x} = y,$$
$$\dot{y} = z,$$
$$\dot{z} = w,$$
$$\dot{w} = -a_1 w - f(x, y) z - a_3 y - a_4 x. \tag{4.17}$$

Wu and Xiong (1998) established the following results on the asymptotic stability of zero solution of (4.13).

Theorem 4.10. For the system (4.13), suppose that $a_j > 0,\ j = 1,2,3,\ f(x)$ is a twice continuously differentiable function with $f(0) = 0,\ xf(x) > 0,\ x \neq 0$, and satisfies one of the following conditions:

(**i**) $A(x) > 0$ and $\frac{1}{a_1}A(x) + \frac{1}{2}f''(x)y > 0$ for $|x| \leq l$ and $|y| \leq h\ \ (l, h > 0)$,

(**ii**) $A(x) \geq \delta > 0$ for $|x| \leq l, \quad (l > 0)$.

Then the zero solution of system (4.13) is asymptotically stable.

Proof. Clearly, the function V in (4.14) is positive definite, in addition, when condition (i) is true, V is negative semi-definite.

If condition (ii) holds, then there must exists a positive number h such that

$$\frac{a_1}{2}|f''(x)|h < \delta.$$

This means that for $|x| \leq l$ and $|y| \leq h$, we have

$$\dot{V} \leq -\left[\tfrac{1}{a_1}A(x) + \tfrac{1}{2}f''(x)h\right]y.$$

Hence, $\dot{V}$ is negative semi-definite. Furthermore, if condition (i) or condition (ii) is true, it follows from $\dot{V} \equiv 0$, that is, $y = 0$. Substitute this into (4.13) and then we obtain $x = z = w = 0$. Therefore

$$\{x(t), y(t), z(t), w(t) |\, \dot{V} = 0\}$$

does not contain any non-zero positive semi-orbit of system (4.13).

Wu and Xiong (1998) further study the asymptotic stability of solutions to system (4.17), and they show that equation (4.16) is rewritten as the same of Ku (1964), i.e.,

$$x^{(4)} + a\dddot{x} + f(x, \dot{x})\ddot{x} + c\dot{x} + dx = 0,$$

And this equation can be written in phase variable from as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -aw - f(x, y)z - cy - dx. \end{aligned} \tag{4.18}$$

They use the following Lyapunov function

$$2V = \frac{c}{a}\left(\frac{ad}{c}x + ay + z\right)^2 + \left(\frac{ad}{c}x + ay + z\right)^2 + \left[f(x,y) - \frac{c}{a} - \frac{ad}{c}\right]z^2 + \left[Y(x,y) - \frac{d}{c}\left(c + \frac{a^2 d}{c}\right)y^2\right] \quad (4.19)$$

Thus

$$\dot{V} = -\left[af(x,y) - c - \frac{a^2 d}{c} - \frac{1}{2}\left(\frac{\partial f}{\partial x}y + \frac{\partial f}{\partial y}z\right)\right]z^2 + \frac{ad}{c}y\int_0^y \frac{\partial f}{\partial x}ydy. \quad (4.20)$$

Let

$$Y(x,y) = \frac{2ad}{c}\int_0^y f(x,y)ydy, \qquad \varepsilon = \frac{ad}{c}\left(\frac{y}{z^2}\right)\int_0^y \frac{\partial f}{\partial y}ydy \quad \text{and} \quad \dot{f} = \frac{\partial f}{\partial x}y + \frac{\partial f}{\partial y}z.$$

Theorem 4.11. Suppose that a, c and d are all positive, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous for $(x,y) \in \Omega$ (a bounded domain in R^2) and

$$f(x,y) - \frac{c}{a} - \frac{ad}{c} > 0,$$

and one of the following conditions is satisfied:

(**i**) $af(x,y) - c - \frac{a^2 d}{c} - \frac{\dot{f}}{2} > \varepsilon,$

(ii) either

$$af(x,y) - c - \frac{a^2 d}{c} - \frac{\dot{f}}{2} > 0, \quad \text{and} \quad y\int_0^y \frac{\partial f}{\partial x}ydy \le 0$$

or

$$af(x,y) - c - \frac{a^2 d}{c} - \frac{\dot{f}}{2} \ge 0, \quad \text{and} \quad y\int_0^y \frac{\partial f}{\partial x}ydy < 0, \quad y \ne 0, \quad \text{where } z \in [-l, l],$$

(**iii**) $af(x,y) - c - \frac{a^2 d}{c} - \frac{1}{2}\frac{\partial f}{\partial x}y \ge \delta > 0$ and $\int_0^y \frac{\partial f}{\partial x}ydy \le 0.$

Then the zero solution of system (4.18) is asymptotically stable (Wu and Xiong, 1998).

Proof. Under condition (i) or (ii), it is obvious from (4.19) and (4.20) that the function V is positive definite and its derivative $\dot{V}$ is negative semi-definite. In addition, it is easy to show that when $\dot{V} = 0$, system (4.18) only admits the trivial solution

$\big(x(t), y(t). z(t), w(t)\big) \equiv (0,0,0,0)$, hence, we know that the conclusion of this theorem is true.

If the condition (iii) is true, then

$$\begin{aligned}\dot{V} &\leq -[af(x,y) - c - \frac{a^2 d}{c} - \frac{1}{2}\Big(\frac{\partial f}{\partial x}y + \frac{\partial f}{\partial y}z\Big)]z^2 \\ &\leq -\left[af(x,y) - c - \frac{a^2 d}{c} - \frac{1}{2}\frac{\partial f}{\partial x}y\right]z^2 - \frac{1}{2}\frac{\partial f}{\partial y}z^3 \\ &\leq -\delta z^2 - \tfrac{1}{2}\tfrac{\partial f}{\partial y}z^3. \qquad (4.21)\end{aligned}$$

Thus there must exist a number l such that

$$\frac{1}{2}\left|\frac{\partial f}{\partial x_2}\right| l < \delta$$

for any $(x, y) \in \Omega$. It therefore follows from (4.21) that $\dot{V} \leq 0$ for any $(x, y) \in \Omega$ and $z \in [-l, l]$ Furthermore, we then know that the zero solution of (4.18) is asymptotically stable.

Finally Wu and Xiong (1998) stated and proved the following theorem on the asymptotic stability of zero solution of system (4.18)

Theorem 4.12. Suppose that a, c and d are all positive, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous for $(x, y) \in \Omega$ (a bounded domain in R^2) and

$$f(x,y) - \frac{c}{a} - \frac{ad}{c} > \delta_1 > 0, \qquad y\int_0^y \frac{\partial f}{\partial x} y dy \leq 0,$$

as well as for any

$$(x, y, z, w) \in D, \quad af(x,y) - c - \frac{a^2 d}{c} - \frac{1}{2}\frac{\partial f}{\partial x}y \geq \delta_2 > 0,$$

and $\left|\frac{\partial f}{\partial y}\right| |z| \leq \delta_2$. Then for the solution $(x(t), y(t), z(t), w(t))$ of (4.18) starting at any initial point (x_0, y_0, z_0, w_0) in the domain D, we have

$$x^2(t) + y^2(t) + z^2(t) + w^2(t) \to 0 \text{ as } t \to +\infty.$$

In particular, the zero solution of system (4.18) is asymptotic stable (Wu and Xiong, 1998).

4.7. Stability in $x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = 0$

Tunc (2006) constructed some new Lyapunov functions to examine the stability of the solutions of fourth order non-linear differential equation described by

$$x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = 0. \tag{4.22}$$

It is clear from (4.22) that

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -\varphi(z)w - f(x,y)z - g(y) - h(x), \end{aligned} \tag{4.23}$$

in which the functions φ, f, g and h depend only on the arguments displayed and the dots denote differentiation with respect to t. The functions φ, f, g and h are assumed to be continuous on their respective domains. The derivatives

$$\frac{dg}{dy} \equiv g'(y) \qquad \text{and} \qquad \frac{dh}{dx} \equiv h'(x)$$

exist and are continuous.

Moreover, the existence and the uniqueness of the solutions of the equation (4.22) are assumed. That is, the functions φ, f, g and h so constructed such that the uniqueness theorem is valid. It is worth mentioning that the continuity of the functions φ, f, g, h guarantees at the least the existence of one solution of equation (4.22). Next, the existence and continuity of the derivatives

$$\frac{dg}{dy} \equiv g'(y) \qquad \text{and} \qquad \frac{dh}{dx} \equiv h'(x),$$

in a compact domain ensure that the functions g and h satisfy the locally Lipschitz condition in the closed domain. This guarantees the uniqueness of the solutions.

Now, for testing the stability of the equation (4.22), (Tunc, 2006) introduced the following notation

$$\varphi_1(z) = \begin{cases} \dfrac{1}{z}\displaystyle\int_0^z \varphi(\tau)d\tau, & z \neq 0 \\ \varphi(0), & z = 0 \end{cases}$$

and

$$g_1(y) = \begin{cases} \dfrac{g(y)}{y}, & y \neq 0 \\ g'(0), & y = 0. \end{cases}$$

The main stability result of (Tunc, 2006) is the following theorem.

Theorem 4.13. In addition to the basic assumptions on the functions φ, f, g and h, suppose that there are positive constants $a, b, c, d, \delta, \varepsilon$ and η such that the following conditions are satisfied:

(i) $h(0) = g(0) = 0$,

(ii) $abc - cg'(y) - ad\varphi(z) \geq \delta > 0$ for all y and z,

(**iii**) $0 < d - h'(x) < \dfrac{\sqrt{\delta\varepsilon a}}{4}$ for all x and $h(x)sgn\, x \to +\infty\ as\ |x| \to \infty$,

(**iv**) $0 \leq g_1(y) - c < \dfrac{\delta}{8c}\sqrt{\dfrac{d}{2ac}}$ and $g'(y) \geq c$ for all y,

(v) $0 \leq f(x,y) - b \leq \eta$ for all x and y where

$$\eta \leq min\left[\frac{c}{8d}\sqrt{\frac{\varepsilon\delta}{a}}, \frac{a}{8}\sqrt{\frac{\varepsilon\delta}{c}}\right], \qquad \varepsilon \leq \frac{\delta}{2acD}, \qquad D = ab + \frac{bc}{d},$$

(vi) $\varphi(z) \geq a, \varphi_1(z) - \varphi(z) < \frac{\delta}{2a^2c}$ for all z.

Then the zero solution of system (4.23) is asymptotically stable (Tunc, 2006).

The main tool for proving the above result is continuously differentiable scalar Lyapunov function $V = V(x, y, z, w)$ defined by:

$$2V = 2\beta\int_0^x h(\zeta)d\zeta + \beta by^2 - \alpha dy^2 + 2\int_0^y g(\rho)d\rho + \alpha bz^2 + 2\int_0^z \varphi(\tau)\tau d\tau$$
$$-\beta z^2 + \alpha w^2 + 2h(x)y + 2\alpha h(x)z + 2\alpha g(y)z$$
$$2\beta y\int_0^z \varphi(\tau)d\tau + 2\beta yw + 2zw,$$

where

$$\alpha = \varepsilon + \frac{1}{a}, \qquad \beta = \varepsilon + \frac{d}{c}.$$

Also (Tunc, 2006) showed that the following lemmas are used for proving that the function $V(x,y,z,w)$ is a Lyapunov function of system (4.23).

Lemma 4.14. Suppose that all the conditions of Theorem 4.13 hold. Then there are positive constants $D_i \equiv D_i(a,b,c,d,\varepsilon,\delta), (i = 1,2,3,4)$ such that for all x,y,z,w.

$$V \geq D_1 \int_0^x h(\zeta)d\zeta + D_2 y^2 + D_3 z^2 + D_4 w^2$$

(Tunc, 2006).

Lemma 4.15. Assume that all conditions of Theorem 4.13 hold. Then, there exist positive constants $D_i \equiv D_i(a,b,c,\varepsilon,\delta)$ $(i = 5,6,7)$ such that if $(x(t),y(t),z(t),w(t))$ is a solution of system (4.23), then

$$\frac{d}{dt}V(x,y,z,w) \leq -(D_5 y^2 + D_6 z^2 + D_7 w^2)$$

(Tunc, 2006).

4.8. Stability in $\boldsymbol{x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(x,\dot{x},\ddot{x}) + g(x,\dot{x}) + h(x) = 0}$

In (2007), Tunc studied the stability of the zero solution of certain fourth order differential equation in the form:

$$x^{(4)} + \varphi(\ddot{x})\ddot{x} + f(x,\dot{x},\ddot{x}) + g(x,\dot{x}) + h(x) = 0, \tag{4.24}$$

where the functions φ, f, g and h are continuous functions which depend only on the arguments displayed explicitly. The dots indicate differentiation with respect to the independent variable t and all solutions considered are assumed to be real. The derivatives,

$$\frac{\partial}{\partial x}g(x,\dot{x}) \equiv g_x(x,\dot{x}), \qquad \frac{\partial}{\partial \dot{x}}g(x,\dot{x}) \equiv g_{\dot{x}}(x,\dot{x}), \qquad \text{and} \qquad \frac{dh}{dx} \equiv h'(x)$$

exist and are continuous. Moreover, the existence and the uniqueness of the solutions of the equation (4.24) will be assumed.

Equation (4.24) can be written in the phase variables form as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -\varphi(z)w - f(x,y,z) - g(x,y) - h(x). \end{aligned} \tag{4.25}$$

Tunc (2007) assumed the following to investigate the stability of zero solution of equation (4.24) by setting

$$\varphi_1(z) = \begin{cases} \frac{1}{z}\int_0^1 \varphi(\zeta)d\zeta, & z \neq 0 \\ \varphi(0), & z = 0 \end{cases}$$

and

$$g_1(x,y) = \begin{cases} \frac{g(x,y)}{y}, & y \neq 0 \\ \frac{\partial}{\partial y} g(x,0), & y = 0. \end{cases}$$

The stability result that established by (Tunc, 2007) is the following theorem.

Theorem 4.16. Suppose the following conditions are satisfied:

(i) $h(0) = g(x,0) = 0,$

(ii) there are positive constants a, b, c, d, δ and ε such that

$abc - cg_y(x,y) - ad\varphi(z) \geq \delta > 0$ for all x, y and z,

$$\text{in which } \varepsilon \leq \frac{\delta}{2acD}, D = ab + \frac{bc}{d}$$

(**iii**) $0 < d - h'(x) \leq \frac{\sqrt{\delta\varepsilon a}}{4}$ for all x, and $h(x)sgnx \to +\infty$, as $|x| \to \infty$,

(**iv**)$0 \leq g_1(x,y) - c < \frac{\delta}{8c}\sqrt{\frac{d}{2ac}}$ for all $x\ and\ y \neq 0$, and $-\frac{\sqrt{\delta\varepsilon a}}{4} \leq yg_x(x,y) \leq 0$

for all x and y,

(**v**) $0 \leq \frac{f(x,y,z)}{z} - b \leq \min\left[\frac{c}{8d}\sqrt{\frac{\varepsilon\delta}{a}}, \quad \frac{a}{8}\sqrt{\frac{\varepsilon\delta}{a}}\right]$ for all x, y and z $\neq 0$, and

$yf_x(x,y,z) \leq 0$ for all x, y and z,

(**vi**) $\varphi(z) \geq a, \quad \varphi_1(z) - \varphi(z) < \frac{\delta}{2a^2c}$ for all z.

Then zero solution of system (4.25) is asymptotically stable (Tunc, 2007).

Throughout the main result (Tunc, 2007) used as a basic tool the scalar continuously differential Lyapunov function $V = V(x,y,z,w)$ defined by:

$$2V = 2\beta\int_0^x h(\zeta)d\zeta + \beta by^2 - \alpha dy^2 + 2\int_0^y g(x,\eta)d\eta + \alpha bz^2 +$$

$$+2\int_0^z \varphi(\zeta)\zeta d\zeta - \beta z^2 + \alpha w^2 + 2h(x)y + 2\alpha h(x)z$$

$$+2\alpha g(x,y)z + 2\beta y\int_0^z \varphi(\zeta)d\zeta \ + 2\beta yw + 2zw,$$

where

$$\alpha = \varepsilon + \frac{1}{a}, \qquad \beta = \varepsilon + \frac{d}{c}.$$

Tunc (2007) showed that the following lemmas are needed in the proof of Theorem 4.16

Lemma 4.17. Assume that all conditions of Theorem 4.16 hold. Then there are positive constants $D_i \equiv D_i(a,b,c,d,\varepsilon,\delta), (i = 1,2,3,4)$ such that

$$V \geq D_1\int_0^x h(\xi)d\xi + D_2y^2 + D_3z^2 + D_4w^2$$

for all x, y, z and w (Tunc, 2007).

Lemma 4.18. Assume that all conditions of Theorem 4.16 hold. Then there exist positive constants $D_i \equiv D_i(a,b,c,\varepsilon,\delta)$ $(i = 5,6,7)$ such that whenever $(x(t), y(t), z(t), w(t))$ is any solution of system (4.25), then

$$\frac{d}{dt}V(x,y,z,w) \leq -(D_5y^2 + D_6z^2 + D_7w^2)$$

(Tunc, 2007).

4.9. Stability in $x^{(4)} + \Phi(\ddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = 0$

Adesina and Ogundare in (2012) established some new conditions on the globally asymptotic stability of trivial solution to a certain fourth order non-linear differential equation of the form

$$x^{(4)} + \Phi(\ddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = 0, \tag{4.26}$$

where the functions Φ, f, g and h are continuous and depend at most only on the arguments displayed explicitly. Here and elsewhere, all the solutions considered and all the functions which appear supposed real. The dots indicate differentiation with respect to t.

Adesina and Ogundare (2012) state the considered equation with the associated system

$$\dot{x} = y,$$
$$\dot{y} = z,$$
$$\dot{z} = w,$$
$$\dot{w} = -\Phi(w) - f(z) - g(y) - h(x). \qquad (4.27)$$

Adesina and Ogundare (2012) gave the following result about the globally asymptotically stability of zero solution of equation (4.26).

Theorem 4.19. Let the functions Φ, f, g and h be continuous and the following conditions hold for some positive constants $a_0, b_0, c, d, \Delta_0, \Delta_1, K_0^*$ and $0 < K_1^* < 1$:

(**i**) $a \leq \dfrac{\Phi(w) - \Phi(0)}{w} = \Phi_0 \leq a_0 = \dfrac{\varepsilon + \delta(1-\varepsilon)D}{D}$, $(w \neq 0,\ \emptyset(0) = 0)$,

(**ii**) $b \leq \dfrac{f(z) - f(0)}{z} = F_0 \leq b_0 = \dfrac{\varepsilon + \gamma\delta(1-\varepsilon)D}{\delta(1-\varepsilon)D}$, $(z \neq 0\ ,\ f(0) = 0)$,

(**iii**) $\Delta_1 \leq \dfrac{g(z) - g(0)}{y} = G_0 \leq c = \dfrac{\varepsilon + \beta\gamma(1-\varepsilon)}{\gamma} \in I_1$, $\quad (y \neq 0 \text{ and } g(0) = 0)$,

$$\text{where } I_1 \equiv [\Delta_1, K_1^* ab],$$

(**iv**) $\Delta_0 \leq \dfrac{h(z) - h(0)}{x} = H_0 \leq d = \dfrac{\beta\varepsilon(1-\varepsilon)(\beta\gamma - 1)}{\gamma^2} \in I_0, (x \neq 0,\ h(0) = 0)$,

$$\text{where } I_0 \equiv \left[\Delta_0, K_0^* \left[\frac{(ab-c)c}{a^2}\right]\right],$$

(**v**) $\emptyset(0) = f(0) = g(0) = h(0) = 0$,

where $\varepsilon, \beta, \delta, \gamma$ and D are all positive with $(1-\varepsilon) > 0$ and $\beta\gamma > 1$. Furthermore, I_0 and I_1 are subintervals of the Routh- Hurwitz interval.

Then the trivial solution of equation (4.26) is globally asymptotically stable.

In the proof, Adesina and Ogundare (2012) used the following continuously differentiable scalar function $V = V(x, y, z, w)$ defined by

$$2V = [\beta(1-\varepsilon)x + \gamma y + \delta z + w]^2 + [(1-\varepsilon)D - 1](\delta z + w)^2$$
$$+\beta D[\varepsilon + (1-\varepsilon)D - 1]y^2 + \gamma(D-1)z^2 + \varepsilon D w^2$$

$$+\beta^2\varepsilon(1-\varepsilon)x^2 + 2\gamma\delta[(1-\varepsilon)^2 D - 1]yz,$$

where $0 < \varepsilon < 1 - \varepsilon < 1$, β, γ, δ are positive real numbers such that

$$\frac{\gamma\delta}{\beta} > 1 - \varepsilon, \quad D = 1 + \frac{\beta(1-\varepsilon)[\gamma\delta - \beta(1-\varepsilon)]}{\gamma\delta - \beta\varepsilon} \qquad \text{with} \qquad D > \frac{1}{(1-\varepsilon)^2} .$$

Also, (Adesina and Ogundare, 2012) showed that the following two lemmas are needed in the proofs of Theorem 4.19.

Lemma 4.20. Subject to the assumptions of Theorem 4.19, there exist positive constants $K_i = K_i(\varepsilon, \beta, \gamma, \delta, D)$, $i = 1,2$ such that

$$K_1(x^2 + y^2 + z^2 + w^2) \leq V(x, y, z, w) \leq K_2(x^2 + y^2 + z^2 + w^2)$$

(Adesina and Ogundare, 2012).

Lemma 4.21. If all assumptions of Theorem 4.19 are satisfied, then there are positive constants $K_3(\varepsilon, \beta, \gamma, \delta, D)$ such that for any solution (x, y, z, w) of system (4.27)

$$\frac{d}{dt}V(x, y, z, w) \leq -K_3(x^2 + y^2 + z^2 + w^2)$$

(Adesina and Ogundare, 2012).

5. BOUNDEDNESS of SOLUTIONS

5.1. Boundedness in $x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t)$

In (1962), Ezeilo concerned with fourth order non-linear differential equation of the form

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t) \tag{5.1}$$

to established some sufficient conditions for the boundedness of solutions. Where α_2, α_4 are constants and f, g, p depend only on the arguments displayed, the dots as usual denoting differentiation with respect to t. The functions f, g, p are such that $g'(y)$ exists and $f(z), g'(y), p(t)$ are continuous for all y, z and t.

Essentially, (Ezeilo, 1962) worked to established some sufficient conditions for the boundedness of solutions of equation (5.1) in the case $p \neq 0$,

Theorem 5.1. Suppose that

(**i**) α_2 and α_4 are both positive,

(**ii**) $g(0) = 0$ and there are finite constants $\alpha_1 > 0$, $\alpha_3 > 0$ such that

$$\frac{g(y)}{y} \geq \alpha_3 \ (y \neq 0) \quad \text{and} \quad f(z) \geq \alpha_1 \quad \text{for all } z,$$

(iii) there is a finite constant $\Delta_0 > 0$ such that

$$\{\alpha_1\alpha_2 - g'(y)\}\alpha_3 - \alpha_1\alpha_4 f(z) \geq \Delta_0 \quad \text{for all } y \text{ and } z,$$

(**iv**) $g'(y) - \dfrac{g(y)}{y} \leq \delta_1$ for all $y \neq 0$, where the constant δ_1 is such that

$$\delta_1 < \frac{2\alpha_4\Delta_0}{\alpha_1\alpha_3^2},$$

(**v**) $\left\{\dfrac{1}{z}\displaystyle\int_0^z f(\zeta)d\zeta\right\} - f(z) \leq \delta_2$ for all $z \neq 0$, where the constant δ_2 is such that

$$\delta_2 < \frac{2\Delta_0}{\alpha_1^2\alpha_3},$$

(**vi**) $\displaystyle\int_0^t |p(\tau)|d\tau \leq A < \infty$ for all $t \geq 0$.

Then given any finite x_0, y_0, z_0, w_0 there is a finite constant $D = D(x_0, y_0, z_0, w_0)$ such that the unique solution $x(t)$ of (5.1) which is determined by the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = y_0, \quad \ddot{x}(0) = z_0, \quad \dddot{x}(0) = w_0,$$

satisfies

$$|x(t)| \leq D, \quad |\dot{x}(t)| \leq D, \quad |\ddot{x}(t)| \leq D, \quad |\dddot{x}(t)| \leq D$$

for all $t \geq 0$ (Ezeilo, 1962).

Ezeilo (1962) considered the equivalent system of equation (5.1) as follows

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= u, \\ \dot{w} &= -wf(z) - \alpha_2 z - g(y) - \alpha_2 x + p(t). \end{aligned} \tag{5.2}$$

The proof of Theorem 5.1 (Ezeilo, 1962) depends on the continuously differentiable Lyapunov function $V = V(x, y, z, w)$ defined by

$$2V = \alpha_4 d_2 x^2 + (\alpha_2 d_2 - \alpha_4 d_1) y^2 + 2\int_0^y g(\eta)d\eta + (\alpha_2 d_1 - d_2) z^2 + 2\int_0^z \zeta f(\zeta)d\zeta$$

$$+ d_1 w^2 + 2\alpha_4 xy + 2\alpha_4 d_1 xz + 2d_2 y\int_0^z f(\zeta)d\zeta + 2d_1 z g(y) + 2d_2 yw + 2zw,$$

where

$$d_1 = \varepsilon + \frac{1}{\alpha_1}, \qquad d_2 = \varepsilon + \frac{\alpha_4}{\alpha_3},$$

and $\varepsilon > 0$ is a constant.

In the following two lemmas, it is shown that V is a Lyapunov function for equation (5.1).

Lemma 5.2. Let assumptions (i) - (v) of Theorem 5.1 hold. Then $V(0,0,0,0) = 0$ and further there are positive constants D_i $(i = 1,2,3,4)$ depending only on $\varepsilon, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \delta_1, \delta_2$ and Δ_0 such that

$$V \geq D_1 x^2 + D_2 y^2 + D_3 z^2 + D_4 w^2$$

for all x, y, z, w provided that $0 < \varepsilon \leq \varepsilon_1$, where ε_1 depends only on $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \delta_1, \delta_2$ and Δ_0 (Ezeilo, 1962).

Lemma 5.3. Under the hypotheses (i)-(v) of Theorem 5.1, there exist constants $D_6 > 0$, $D_7 > 0, D_8 > 0$ depending only on $\varepsilon, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ and Δ_0 such that, if (x, y, z, w) is any solution of the equation (5.1), then

$$\dot{V} \equiv \frac{d}{dt}V(x, y, z, w) \leq -(D_6 y^2 + D_7 z^2 + D_8 w^2)$$

provided that $0 < \varepsilon \leq \varepsilon_2$, where ε_2 is a constant depending only on $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and Δ_0 (Ezeilo, 1962).

5.2. Boundedness in $x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$

Tejumola (1968) considered the fourth order non-linear non homogeneous differential equation of the form

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{5.3}$$

where α_2, α_4 are constants and f, g, p depend only on the arguments displayed, the dots as usual denoting differentiation with respect to t. The functions f, g, p are such that $g'(y)$ exists and are continuous for all y.

Essentially, (Tejumola, 1968) worked to established some sufficient conditions for the boundedness of solutions of equation (5.3) in the case $p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \neq 0$.

The main result of (Tejumola, 1968) about boundedness of solutions of (5.3) is the following.

Theorem 5.4. Given equation (5.3). Suppose that

(**i**) $f(z) \geq \alpha_1$ for all z,

(**ii**) $g(0) = 0$ and there are finite constants $\alpha_3 > 0$ such that $g'(y) \geq \alpha_3$ for all y,

(iii) there is a finite constant $\delta_0 > 0$ such that

$$\{\alpha_1\alpha_2 - g'(y)\}\alpha_3 - \alpha_1\alpha_4 f(z) \geq \delta_0 \quad \text{for all } y \text{ and } z,$$

(**iv**) $g'(y) - \dfrac{g(y)}{y} \leq \delta_1$ for all $y \neq 0$, where the constant δ_1 is such that

$$\delta_1 < \frac{2\alpha_4\delta_0}{\alpha_1\alpha_3^2},$$

(**v**) $\left\{\frac{1}{z}\int_0^z f(\zeta)d\zeta\right\} - f(z) \leq \delta_2$ for all $z \neq 0$, where the constant δ_2 is such that

$$\delta_2 < \frac{2\delta_0}{\alpha_1^2\alpha_3},$$

(**vi**) there is a finite constant $\Delta > 0$ such that

$$|p(t,x,y,z,w)| \leq \Delta \text{ for all values of } t, x, y, z \text{ and } w.$$

Then there exist a finite constant $D_0 > 0$ whose magnitude depends only on the functions f, g and p as well as on $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \delta_0, \delta_1, \delta_2,$ and Δ such that every solution $x(t)$ of equation (5.3) satisfies

$$|x(t)| \leq D_0, \qquad |\dot{x}(t)| \leq D_0, \quad |\ddot{x}(t)| \leq D_0, \qquad |\dddot{x}(t)| \leq D_0$$

for all $t \geq t_0$ $(0 \leq t_0 < \infty)$ (Tejumola, 1968).

The proof of Theorem 5.4 (Tejumola, 1968) depends on the properties of the continuously differentiable Lyapunov function $V = V(x,y,z,w)$ defined by

$$V = V_1 + V_2,$$

$$2V_1 = \alpha_4 d_2 x^2 + (\alpha_2 d_2 - \alpha_4 d_1)y^2 + 2\int_0^y g(\eta)d\eta + (\alpha_2 d_1 - d_2)z^2 + 2\int_0^z \zeta f(\zeta)d\zeta$$

$$+ d_1 w^2 + 2\alpha_4 xy + 2\alpha_4 d_1 xz + 2d_2 y\int_0^z f(\zeta)d\zeta + 2d_1 z g(y) + 2d_2 yw + 2zw,$$

$$V_2 = \begin{cases} xsgn\, w, & if \quad |w| \geq |x| \\ wsgn\, x, & if \quad |w| \leq |x|, \end{cases}$$

where

$$d_1 = \varepsilon + \frac{1}{\alpha_1}, \qquad d_2 = \varepsilon + \frac{\alpha_4}{\alpha_3},$$

and $\varepsilon > 0$ is a constant.

After that Ezeilo and Tejumola (1971) studied some special cases of the equation (5.1) and they established the following results.

The first case of (Ezeilo and Tejumola, 1971) is the equation

$$x^{(4)} + \alpha_1\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t,x,\dot{x},\ddot{x},\dddot{x}), \tag{5.4}$$

in which $\alpha_1, \alpha_2, \alpha_4$ are positive constants the functions g, p continuous for all values of the respective arguments.

Theorem 5.5. Moreover the basic assumptions on $\alpha_1, \alpha_2, \alpha_4$ and the functions g, p suppose that:

(**i**) there is a positive constant η_0 such that $\frac{g(y)}{y} > 0 \ \ (|y| \geq \eta_0)$,

(ii) there is a constant $d_1 > 0$ such that

$$\alpha_1\alpha_2\frac{g(y)}{y} - \left[\frac{g(y)}{y}\right]^2 - \alpha_1^2\alpha_4 \geq d_1 \quad (|y| \geq \eta_0),$$

(**iii**) there is a finite constant A_0 such that $|p(t,x,y,z,u)| \leq A_0$ for all t, x, y, z and u.

Then there exists a finite constant D whose magnitude depends only on $\alpha_1, \alpha_2, \alpha_4, \eta_0, d_1, A_0$ and g such that every solution $x(t)$ of equation (5.4) ultimately satisfies

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq D$$

(Ezeilo and Tejumola, 1971).

The next case that (Ezeilo and Tejumola, 1971) studied the equation:

$$x^{(4)} + f(\ddot{x})\dddot{x} + \alpha_2\ddot{x} + \alpha_3\dot{x} + \alpha_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x}) \tag{5.5}$$

with $\alpha_2, \alpha_3, \alpha_4$ are positive constants, the functions f, p are continuous for all values of the respective arguments. And established the following result.

Theorem 5.6. In addition to the basic assumptions on $\alpha_2, \alpha_3, \alpha_4$ and the functions f, p, we suppose that:

(**i**) There is a positive constant ξ_0 such that $f(z) > 0 \ \ (|z| \geq \xi_0)$,

(ii) there is a constant $d_2 > 0$ such that

$$\alpha_2\alpha_3 f(z) - \alpha_3^2 - \alpha_4 f^2(z) \geq d_2 \quad (|z| \geq \xi_0),$$

(**iii**) there is a finite constant A_0 such that $|p(t,x,y,z,u)| \leq A_0$ for all t, x, y, z and u.

Then there exists a finite constant D whose magnitude depends only on $\alpha_2, \alpha_3, \alpha_4, \xi_0, d_2, A_0$ and g such that every solution $x(t)$ of equation (5.5) ultimately satisfies

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq D$$

(Ezeilo and Tejumola, 1971).

Also Ezeilo and Tejumola (1971) studied another form of fourth order differential equation that is in the form

$$x^{(4)} + f(\dot{x},\ddot{x})\dddot{x} + \alpha_2\ddot{x} + \alpha_3\dot{x} + \alpha_4 x = p(t,x,\dot{x},\ddot{x},\dddot{x}), \tag{5.6}$$

where $\alpha_2, \alpha_3, \alpha_4$ are positive constants and the function f is such that $\frac{\partial f}{\partial y}(y,z)$ exist, $f(y,z)$, $\frac{\partial f}{\partial y}(y,z), p(t,x,y,z,u)$ are continuous for all x, y, z, u and t.

Theorem 5.7. Moreover the basic assumptions on $\alpha_2, \alpha_3, \alpha_4$ and the functions f, p suppose that:

(**i**) there is a positive constant ξ_0 such that $f(y,z) > 0$ $(|z| \geq \xi_0)$,

(ii) there is a finite constant A such that $\max|f(y,z)| \leq A$ for all y and $|z| \leq \xi_0$,

(**iii**) $z\dfrac{\partial f}{\partial y}(y,z) \leq 0$ for all y, z,

(iv) there is a constant $d_2 > 0$ such that

$$\alpha_2\alpha_3 f(y,z) - \alpha_3^2 - \alpha_4 f^2(y,z) \geq d_2 \quad (|z| \geq \xi_0),$$

(**v**) there is a finite constant A_0 such that $|p(t,x,y,z,u)| \leq A_0$ for all t, x, y, z and u.

Then there exists a finite constant $D > 0$ whose magnitude depends only on $\alpha_2, \alpha_3, \alpha_4, \xi_0, d_2, A_0$ and f such that every solution $x(t)$ of equation (5.6) satisfies

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq D$$

(Ezeilo and Tejumola, 1971).

5.3. Boundedness in $x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + h(x) = p(t)$

Harrow (1970) considered non homogeneous fourth order differential equation of the form:

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + h(x) = p(t), \tag{5.7}$$

where a, b, c are given positive constants such that $ab > c$, $h(x)$ is differentiable and $p(t)$ is continuous in x and t, respectively, where the dots denote the differentiation with respect to t. Equation (5.7) can be written in the form of system as

$$\dot{x} = y,$$
$$\dot{y} = z,$$
$$\dot{z} = u,$$
$$\dot{w} = -aw - bz - cy - h(x) + p(t). \qquad (5.8)$$

Harrow (1970) established the following result on the boundedness of the solutions.

Theorem 5.8. Suppose that $h(0) = 0$ and that

(i) d is a positive constant such that $h'(x) \leq d$, $s = abc - c^2 - a^2 d > 0$, and δ_2 is a constant such that for all x

$$d - 2asc^{-1} < \delta_2 < h'(x) \leq d,$$

(ii) δ_1 is a positive constant such that for $x \neq 0$

$$d - \frac{ds}{s^*}\frac{c}{ab} < \delta_1 < \frac{h(x)}{x} \quad \text{where} \quad s^* = s + 2ad(ab - c)b^{-1}.$$

Then corresponding to any constant λ in the interval $0 \leq \lambda \leq \frac{1}{2}$ always there is a constant $\mu = \mu(a, b, c, d, \delta_1, \delta_2, \lambda) > 0$ such that every solution $x(t)$ of (5.7) determined by the initial conditions

$$x(t_0) = x_0, \quad \dot{x}(t_0) = y_0, \quad \ddot{x}(t_0) = z_0, \quad \dddot{x}(t_0) = w_0,$$

and defined for all $t \geq t_0$ satisfies

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq \left\{ e^{-\mu t}\left[D_1 + D_2 \int_{t_0}^{t} |p(\tau)|^{2(1-\lambda)} e^{\mu t} d\tau \right] \right\} \frac{1}{1-\lambda}$$

for all $t \geq t_0$, where $D_1 = D_1(a, b, c, d, \delta_1, \delta_2, t_0, x_0, y_0, z_0, w_0) > 0$ and $D_2 = D_2(a, b, c, d, \delta_1, \delta_2) > 0$ are constants (Harrow, 1970).

The proof of the above theorem depends on the continuously differential scalar Lyapunov function $V = V(x, y, z, w)$ defined by

$$V = M + N,$$

where

$$2M(x, y, z, w) = m_1(w + m_1 z + m_1 dc^{-1} y)^2 + c[z + m_1 y + m_1 c^{-1} h(x)]^2$$

$$+ m_1 d\sigma c^{-2}(y + cm_1 \propto \sigma^{-1} z)^2 + m_2 z^2,$$

where $\propto$ is a constant such that as is possible in view of condition $bc > ad$,

$$0 <\propto< s(bc - ad)^{-1}$$

and

$$m_1 = a-\propto, \quad m_2 = sc^{-1} + (ad - bc) \propto c^{-1}+\propto m_1^2\, m_3\sigma^{-1}+\propto dm_1c^{-1},$$
$$\sigma = m_1bc - m_1^2d - c^2, \quad m_3 = \sigma - d \propto m_1.$$

$$2N(x,y,z,w) = a[w + az + (ab - n_1)a^{-1}y + \beta x]^2 + n_1[z + ay + an_1^{-1}h(x]^2$$
$$+va^{-1}(y + an_1\beta v^{-1}x)^2 + n_2x^2 + 2k_1\int_0^x h(\tau)d\tau - dk_1x^2$$
$$+m_1^2c^{-1}[d^2x^2 - \{h(x)\}^2] + a^2n_1^{-1}[d^2x^2 - \{h(x)\}^2],$$

where β is a constant such that as possible in view of the condition $ab > c$,

$$0 < \beta < s(ab - c)^{-1}$$

and

$$n_1 = c - \beta, \qquad n_3 = v - \beta n_1,$$
$$v = abn_1 - n_1^2 - a^2d, \qquad k_1 = ab - n_1 + dm_1^2c^{-1},$$
$$n_2 = dsn_1^{-1} + a\beta n_1n_3v^{-1} - d\beta(ab - c)n_1^{-1} + d\beta,$$

Harrow (1970) proved that for certain positive constants D_3 and D_4,

$$D_3(x^2 + y^2 + z^2 + w^2) \le V(x,y,z,w) \le D_4(x^2 + y^2 + z^2 + w^2)$$

and that for certain positive constants D_5 and D_6,

$$\frac{dV}{dt} \le -D_5(x^2 + y^2 + z^2 + w^2) + D_6(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}}|p(t)|$$

(Harrow, 1970).

5.4. Boundedness in

$$\boldsymbol{x^{(4)} + \varphi(x,\dot{x},\ddot{x},\dddot{x})\dddot{x} + f(\dot{x},\ddot{x}) + g(x,\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x})}$$

In 1995, Tunc studied the boundedness of solutions for certain fourth order differential equation of the form

$$x^{(4)} + \varphi(x,\dot{x},\ddot{x},\dddot{x})\dddot{x} + f(\dot{x},\ddot{x}) + g(x,\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}), \qquad (5.9)$$

in which the functions φ, f, g, h and p depend at most on the arguments shown explicitly and the dots denote differentiation with respect to t. Further, it will be assumed that the functions φ, f, g, h and p are continuous for all values of their respective arguments and that derivatives

$$\frac{\partial}{\partial x}\varphi(x,y,z,u), \qquad \frac{\partial}{\partial y}\varphi(x,y,z,u), \qquad \frac{\partial}{\partial u}\varphi(x,y,z,u), \qquad \frac{\partial}{\partial y}f(y,z),$$

$$\frac{\partial}{\partial x}g(x,y), \qquad \frac{\partial}{\partial x}g(x,y)$$

and $h'(x)$ exist and are continuous for all x, y, z and u. All functions and solutions are supposed to be real. Moreover the existence and the uniqueness of the solutions of (5.9) will be assumed. It will be convenient in what follows to use the equivalent system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= u, \\ \dot{u} &= -\varphi(x,y,z,u)u - f(y,z) - g(x,y) - h(x) + p(t,x,y,z,u), \end{aligned} \tag{5.10}$$

which is obtained from (5.9) by setting $y = \dot{x},\ z = \ddot{x},\ u = \dddot{x}$.

Tunc (1995) established the following results on the boundedness of solutions of equation (5.9) in the case that $p(t,x,y,z,u) \neq 0$.

Theorem 5.9. Suppose the following conditions are satisfied:

(**i**) $f(y,0) = g(x,0) = h(0) = 0$,

(**ii**) there are positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and Δ_0 such that

$$\alpha_1\alpha_2\alpha_3 - \alpha_3\frac{g(x,y)}{y} - \alpha_1\alpha_4\varphi(x,y,z,0) \geq \Delta_0 \quad \text{for all } x, z \text{ and } y \neq 0,$$

(**iii**) $\varphi(x,y,z,u) \geq \alpha_1 > 0$ for all x, y, z and u, $\qquad \frac{f(y,z)}{z} \geq \alpha_2$ for all $y, z \neq 0$,

$$\frac{g(x,y)}{y} \geq \alpha_3 \text{ for all } x, y \neq 0, \qquad \frac{h(x)}{x} \geq \beta > 0 \text{ for all } x \neq 0,$$

(**iv**) $\left(\alpha_4 - \frac{\alpha_1\Delta_0}{4\alpha_3}\right) < h'(x) \leq \alpha_4$ for all x,

(**v**) $\left(\frac{\partial}{\partial y}g(x,y) - \frac{g(x,y)}{y}\right) \leq \delta_1$ for all $x, y \neq 0$, where $\delta_1 > 0$ and satisfying $\delta_1 < \frac{2\alpha_4\Delta_0}{\alpha_1\alpha_3^2}$,

(**vi**) $\frac{1}{z}\int_0^z \varphi(x,y,s,0)ds - \varphi(x,y,z,0) \le \delta_2$ for all x, y and $z \neq 0$,

where $\delta_2 > 0$ and satisfying $\delta_2 < \frac{2\Delta_0}{\alpha_1^2\alpha_3}$,

(**vii**) $\frac{\partial}{\partial y}f(y,z) \le 0, \quad y\frac{\partial}{\partial x}\varphi(x,y,z,0) \le 0, \quad z\frac{\partial}{\partial x}\varphi(x,y,z,0) \le 0,$

$y\frac{\partial}{\partial y}\varphi(x,y,z,0) \le 0$ and $z\frac{\partial}{\partial y}\varphi(x,y,z,0) \le 0$ for all x, y and z,

(**viii**) $\frac{f(y,z)}{z} - \alpha_2 \le \frac{\alpha_3^3\varepsilon_0}{\alpha_4^2}$ for all $y, z \neq 0$,

where ε_0 is a positive constant such that

$$\varepsilon_0 < \varepsilon \le min\left[\frac{1}{\alpha_1}, \frac{\alpha_4}{\alpha_3}, \frac{\Delta_0}{4\alpha_1\alpha_3 D_0}, \frac{\alpha_3}{4\alpha_4 D_0}\left(\frac{2\alpha_4\Delta_0}{\alpha_1\alpha_3^2} - \delta_1\right), \frac{\alpha_1}{4D_0}\left(\frac{2\Delta_0}{\alpha_1^2\alpha_3} - \delta_2\right)\right]$$

with $D_0 = \alpha_1\alpha_2 + \frac{\alpha_2\alpha_3}{\alpha_4}$,

(**ix**) $\left[\frac{\partial}{\partial x}g(x,y)\right]^2 \le \frac{\alpha_1\Delta_0(\varepsilon - \varepsilon_0)}{16}$ for all x and y,

and $\frac{1}{y}\int_0^y \frac{\partial}{\partial x}g(x,s)\,ds \le \frac{\sigma_3(\varepsilon - \varepsilon_0)}{4}$ for all $x, y \neq 0$,

(**x**) $z\frac{\partial}{\partial u}\varphi(x,y,z,u) + d_2 y\frac{\partial}{\partial u}\varphi(x,y,z,u) \ge 0$

for all x, y, z and u, where $d_2 = \frac{\alpha_4}{\alpha_3} + \varepsilon$,

and furthermore, the function $p(t,x,y,z,u)$ satisfies

$|p(t,x,y,z,u)| \le (1 + |y| + |z| + |u|)q(t)$, for all t, x, y, z and u, where $q(t)$ is a non negative and continuous function of t, and satisfies

$$\int_0^t q(s)ds \le A < \infty,$$

for all $t \ge 0$, with a positive constant A. Then for any given finite x_0, y_0, z_0, u_0, there exist a constant $D = D(x_0, y_0, z_0, u_0)$, such that the unique solution $x(t)$ of (5.9) is determined by the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = y_0, \quad \ddot{x}(0) = z_0, \quad \dddot{x}(0) = u_0,$$

satisfies

$$|x(t)| \leq D, \qquad |\dot{x}(t)| \leq D, \quad |\ddot{x}(t)| \leq D, \qquad |\dddot{x}(t)| \leq D,$$

for all $t \geq 0$ (Tunc, 1995).

Tunc (1995) established the following Lyapunov's function $V = V(x, y, z, u)$ as a main tool in the proof of Theorem 5.9:

$$2V = 2d_2 \int_0^x h(s)ds + [d_2\alpha_2 - d_1\alpha_4]y^2 + 2\int_0^y g(x,s)ds + 2\int_0^z [d_1 f(y,s) - d_2 s]ds$$

$$+2\int_0^z s\varphi(x,y,s,0)ds + 2d_2 y\int_0^z \varphi(x,y,s,0)ds + d_1u^2 + 2yh(x)$$

$$+2d_1zh(x) + 2d_1zg(x,y) + 2d_2yu + 2zu,$$

where

$$d_1 = \frac{1}{\alpha_1} + \varepsilon, \qquad d_2 = \frac{\alpha_4}{\alpha_3} + \varepsilon.$$

In the following two lemmas showed that the function $V(x, y, z, u)$ is a Lyapunov function of system (5.10).

Lemma 5.10. Suppose that the conditions of Theorem 5.9 hold. Then there is a positive constant D_1 such that

$$V \geq D_1(x^2 + y^2 + z^2 + u^2)$$

for all x, y, z and u (Tunc, 1995).

Lemma 5.11. Suppose that the conditions of Theorem 5.9 hold. Then there is a positive constant D_2 such that whenever (x, y, z, u) is any solution of (5.10), then

$$\dot{V} \equiv \frac{d}{dt}V(x,y,z,u) \leq -D_2(y^2 + z^2 + u^2)$$

(Tunc, 1995).

5.5. Boundedness in

$$\boldsymbol{x^{(4)} + \varphi(t,x,\dot{x},\ddot{x},\dddot{x})\ddot{x} + f(t,x,\dot{x},\ddot{x}) + g(t,x,\dot{x}) + dx = p(t,x,\dot{x},\ddot{x},\dddot{x})}$$

Tunc (2001) Considered a fourth order non-linear non homogeneous fourth order differential equation, namely,

$$x^{(4)} + \varphi(t,x,\dot{x},\ddot{x},\dddot{x})\dddot{x} + f(t,x,\dot{x},\ddot{x}) + g(t,x,\dot{x}) + dx = p(t,x,\dot{x},\ddot{x},\dddot{x}) \qquad (5.11)$$

or its equivalent system:

$$\dot{x} = y,$$
$$\dot{y} = z,$$
$$\dot{z} = u,$$
$$\dot{u} = -\varphi(t,x,y,z,u)u - f(t,x,y,z) - g(t,x,y) - dx + p(t,x,y,z,u). \qquad (5.12)$$

In which d is a constant, the functions φ, f, g and p depend only on the argument displayed and the dots denote differentiation with respect to t.The functions φ, f, g and p are continuous for all values of their respective arguments. The derivatives

$$\frac{\partial\varphi(t,x,y,z,u)}{\partial t} \equiv \varphi_t(t,x,y,z,u), \qquad \frac{\partial\varphi(t,x,y,z,u)}{\partial x} \equiv \varphi_x(t,x,y,z,u),$$
$$\frac{\partial\varphi(t,x,y,z,u)}{\partial y} \equiv \varphi_y(t,x,y,z,u), \qquad \frac{\partial\varphi(t,x,y,z,u)}{\partial u} \equiv \varphi_u(t,x,y,z,u),$$

and

$$\frac{\partial g(t,x,y)}{\partial t} \equiv g_t(t,x,y), \qquad \frac{\partial g(t,x,y)}{\partial x} \equiv g_x(t,x,y),$$

exist and are continuous.

Tunc (2001) investigated boundedness and uniform boundedness of all solutions of system (5.12). And established the following result.

Theorem 5.12. In addition to the fundamental assumptions on the functions φ, f, g and p suppose that there exist positive constants a, b, c, d, A, D and δ such that the following conditions are satisfied:

(i) $bc\varphi(t,x,y,z,u) - d\varphi^2(t,x,y,z,u) - c^2 \geq \delta$ for all t, x, y, z and u,

(ii) $\varphi_t(t,x,y,z,0) + y\varphi_x(t,x,y,z,0) + z\varphi_y(t,x,y,z,0) - z\varphi_u(t,x,y,z,0) \leq 0$ for all t, x, y and z,

(iii) $g(t,x,0) = 0, 0 \leq \frac{g(t,x,y)}{y} - c \leq \frac{\delta\sqrt{D}}{\alpha\sqrt{A}}$, and $\frac{1}{y}\int_0^y g_x(t,x,\eta)d\eta \leq -\left(\frac{3b^2\delta D}{4a^2A}\right)$

for all $t, , x$ and $y \neq 0$ and $yg_t(t,x,\dot{y}) \leq 0$ for all t, x and y,

(**iv**) $0 \leq \dfrac{f(t,x,y,z)}{z} - b \leq min\left\{\dfrac{\delta}{2b^2}, \dfrac{b^2c^2\delta D}{8a^2d^2A}\right\}$ for all t, x, y and $z \neq 0$,

(v) the function $p(t,x,y,z,u)$satisfies $|p(t,x,y,z,u)| \leq p(t)(x^2+y^2+z^2+u^2)^{\frac{1}{2}}$, where $p(t)$ is a non-negative continuous function and

$$\int_0^{\infty} P(t)dt < +\infty.$$

Then all solutions of system (5.12) are bounded. (Tunc, 2001).

Theorem 5.13. Suppose the following conditions are satisfied:

(i) Conditions (i) – (v) of Theorem 5.12 hold,

(**ii**) $\dfrac{rP(t)V}{\varepsilon} - \left(\dfrac{\delta}{c}\right)u^2 \leq 0$, where ε and r are positive constants.

Then all solutions of system (5.12) are uniformly bounded (Tunc, 2001).

The results of (Tunc, 2001) depends on a continuous scalar differentiable Lyapunov function $V(t,x,y,z,u)$.

This function and its time derivative satisfy some fundamental inequalities. The function $V = V(t,x,y,z,u)$ defined by:

$$2V = (2b)[bu+cz+2dy]^2 + b^2[bz+cy+2dx]^2 + (b^2-4d)[bz+cy]^2$$
$$+2d[(b^2-4d)b+2c^2]y^2 + 4bc\int_0^z [b\varphi(t,x,y,\xi,0)-c]\xi d\xi$$
$$+4b^2c\int_0^y [g(t,x,\eta)-c\eta]\,d\eta.$$

5.6. Boundedness in $\boldsymbol{x^{(4)} + \varphi(\ddot{x})\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x})}$

Tunc (2004) considered the certain fourth order non-linear non homogeneous differential equation of the form

$$x^{(4)} + \varphi(\ddot{x})\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x},\dddot{x}). \qquad (5.13)$$

The purpose of (Tunc, 2004) is to investigate the boundedness of all solutions of (5.13).

Tunc (2004) concerned with the differential equations of the form (5.13) or its equivalent system in the phase variables from

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= w,\\ \dot{w} &= -\varphi(z)w - f(x,y)z - g(y) - h(x) + p(t,x,y,z,w),\end{aligned} \tag{5.14}$$

in which the functions φ, f, g, h and p depend only on the arguments displayed and the dots denoted differentiation with respect to *t*. The functions φ, f, g, h and p are continuous for all values their respective arguments. The derivatives

$$\frac{\partial f(x,y)}{\partial x} \equiv f_x(x,y), \qquad \frac{dg}{dy} \equiv g'(y), \quad \text{and} \quad \frac{dh}{dx} \equiv h'(x),$$

exist and are continuous. Moreover the existence and the uniqueness of the solutions of (5.13) will be assumed.

Now for testing the boundedness of the equation (5.13), let

$$\varphi_1(z) = \begin{cases} \frac{1}{2}\int_0^z \varphi(z)dz, & z \neq 0,\\ \varphi(0), & z = 0 \end{cases}$$

and

$$g_1(y) = \begin{cases} \frac{g(y)}{y}, & y \neq 0,\\ g'(0), & y = 0. \end{cases}$$

In the case that $p \neq 0$ (Tunc, 2004) established the following results about the boundedness of solutions of equation (5.13).

Theorem 5.14. Suppose the following conditions are satisfied:

(i) $g(0) = 0$,

(ii) there are positive constants a, b, c, d, δ and ε such that

$abc - cg'(y) - ad\varphi(z) \geq \delta > 0$, for all y and z, in which $\varepsilon \leq \frac{\delta}{2acD}$, $D = ab + \frac{bc}{d}$,

(iii) $0 < d - \frac{a\delta}{4c} < h'(x) \leq d$ for all x,

(**iv**)$0 \le g_1(y) - c < \frac{\delta}{8c}\sqrt{\frac{d}{2ac}}$ for all y,

(**v**) $0 \le f(x,y) - b \le \frac{a}{8}\sqrt{\frac{\varepsilon\delta}{c}}$ for all x and y,

(**vi**) $\varphi(z) \ge a, \varphi_1(z) - \varphi(z) < \frac{\delta}{2a^2c}$ for all z,

(vii) $yf_x(x,y) \le 0 \ \ for\ all\ x\ and\ y$,

(viii) $|p(t,x,y,z,w)| \le (A + |y| + |z| + |w|)q(t)$,

where $q(t)$ is a non-negative and continuous function of t, and satisfies

$$\int_0^t q(s)\,ds \le B < \infty, \qquad \text{for all } t \ge 0,$$

A and B are positive constants. Then for any given finite constants x_0, y_0, z_0, w_0, there exist a constant $K = K(x_0, y_0, z_0, w_0)$, such that any solution $(x(t), y(t), z(t), w(t))$ of system (5.14) determined by

$$x(0) = x_0, \quad y(0) = y_0\ , \quad z(0) = z_0, \quad w(0) = w_0$$

satisfies for all $t \ge 0$,

$$|x(t)| \le K, \quad |y(t)| \le K, \quad |z(t)| \le K, \quad |w(t)| \le K \text{ (Tunc, 2004).}$$

The proof of Theorem 5.14 depends on some fundamental properties of a continuously differentiable Lyapunov function $V = V(x,y,z,w)$ defined by

$$2V = 2\beta\int_0^x h(x)dx + 2\beta\int_0^y f(x,y)ydy - \alpha dy^2 + 2\int_0^y g(y)dy$$

$$+\alpha bz^2 + 2\int_0^z \varphi(z)zdz - \beta z^2 + \alpha w^2 + 2h(x)y + 2\alpha h(x)z$$

$$+2\alpha g(y)z + 2\beta y\int_0^z \varphi(z)dz\ + 2\beta yw + 2zw, \tag{5.15}$$

where

$$\alpha = \varepsilon + \frac{1}{a}, \qquad \beta = \varepsilon + \frac{d}{c}. \tag{5.16}$$

ε, is a positive constants.

The first property of the function V is stated in the following.

Lemma 5.15. Subject to the assumption of Theorem 5.14, there are positive constants $D_i \equiv D_i(a,b,c,d,\varepsilon,\delta), i = 1,2,3,4$ such that

$$V \geq D_1x^2 + D_2y^2 + D_3z^2 + D_4w^2 \quad \text{for all } x,y,z,w.$$

Proof. We observe that $2V$ in (5.15) can be rearranged as

$$2V = \frac{1}{c}[h(x) + cy + \alpha cz]^2 + \frac{1}{\varphi_1(z)}[w + \varphi_1(z)z + \beta\varphi_1(z)y]^2$$

$$+\left[\alpha - \frac{1}{\varphi_1(z)}\right]w^2 + [\alpha b - \beta - \alpha^2 c]z^2 + 2\int_0^y [\beta f(x,y) - \alpha d]ydy$$

$$-\beta^2\varphi_1(z)y^2 + 2\int_0^y g(y)dy - cy^2 + 2\alpha[g_1(y) - c]yz$$

$$+2\beta\int_0^x h(x)dx - \left(\frac{1}{c}\right)h^2(x) + [2\int_0^z \varphi(z)zdz - \varphi_1(z)z^2].$$

In the light of the hypothesis of the theorem, and the use of (5.16) and the main value theorem (both for the derivative and integral) it can easily obtained that

$$2V \geq \varepsilon\left(d - \frac{a\delta}{4c}\right)x^2 + \left(\frac{\delta d}{2ac^2}\right)y^2 + \left(\frac{\delta}{4a^2c}\right)z^2 + \varepsilon w^2 + 2\alpha[g_1(y) - c]yz. \quad (5.17)$$

Now, consider the terms

$$W_5 \equiv \left(\tfrac{\delta d}{4ac^2}\right)y^2 + 2\alpha[g_1(y) - c]yz + \left(\tfrac{\delta}{8a^2c}\right)z^2,$$

which are contained in (5.17). In view of the inequality

$$\alpha^2[g_1(y) - c]^2 < \frac{4}{a^2}[g_1(y) - c]^2 < \frac{d\delta^2}{32a^3c^3} \quad \text{for all } y.,$$

it is evident that $W_5 \geq 0$. Hence

$$2V \geq \varepsilon\left(d - \frac{a\delta}{4c}\right)x^2 + \left(\frac{\delta d}{4ac^2}\right)y^2 + \left(\frac{\delta}{8ac^2}\right)z^2 + \varepsilon w^2$$

which is proves the lemma.

The next property of the function V is connected with its total time derivative and is contained in the following.

Lemma 5.16. Assume that all conditions of Theorem 5.14 hold. Then there exist positive constants $D_i \equiv D_i(a,b,c,\varepsilon,\delta)$ $(i = 5,6,7)$ such that whenever $(x(t), y(t), z(t), w(t))$ is any solution of system (5.14) then

$$\dot{V} \equiv \frac{d}{dt}V(x,y,z,w) \leq -(D_5 y^2 + D_6 z^2 + D_7 w^2).$$

Proof. From (5.15) and (5.14) it follows that

$$\dot{V} = -[d - h'(x)][y + \frac{\alpha z}{2}]^2 - [f(x,y) - \alpha g'(y) - \beta\varphi_1(z)]z^2 + \frac{\alpha^2}{4}[d - h'(x)]z^2$$

$$-[\alpha\varphi(z) - 1]w^2 - \left[\beta\frac{g(y)}{y} - d\right]y^2 - \alpha[f(x,y) - b]zw + \beta y\int_0^y f_x(x,y)ydy.$$

The conditions (ii)- (vi) and (5.16) show that

$$\dot{V} \leq -\left(\frac{\delta}{2ac}\right)z^2 - \frac{\alpha^2}{4}[d - h'(x)]z^2 - \varepsilon a w^2 - \varepsilon c y^2 - \alpha[f(x,y) - b]zw. \quad (5.18)$$

Consider the expression

$$W_6 \equiv \frac{\alpha^2}{4}[d - h'(x)]z^2 - \left(\frac{\varepsilon a}{4}\right)w^2 - \alpha[f(x,y) - b]zw - \left(\frac{\delta}{16ac}\right)z^2,$$

which is contained (5.18). Because of the inequalities

$$\frac{\alpha^2}{4}[d - h'(x)] < \frac{1}{a^2}[d - h'(x)] < \frac{\delta}{4ac} \text{ and } \frac{\alpha^2}{4}[f(x,y) - b]^2 \leq \frac{\varepsilon\delta}{64c}$$

We have from (5.18) that

$$W_6 \leq \left(\frac{\delta}{4ac}\right)z^2 - \left[\frac{\sqrt{\varepsilon a}}{2}w \pm \frac{1}{4}\sqrt{\frac{\delta z}{ac}}\right]^2 \leq \left(\frac{\delta}{4ac}\right)z^2.$$

A combination of the estimate for W_6 with (5.18) yields that

$$\dot{V} \leq -(\varepsilon c)y^2 - \left(\frac{3\delta}{16ac}\right)z^2 - \left(\frac{3\varepsilon a}{4}\right)w^2,$$

which is proves Lemma 5.16.

5.7. Boundedness in $x^{(4)} + a(\ddot{x}, \dddot{x})\dddot{x} + b(x, \dot{x})\ddot{x} + c(\dot{x}) + d(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$

In (2005) Tunc investigated the boundedness of all solutions of the certain fourth order differential equation in the form

$$x^{(4)} + a(\ddot{x}, \dddot{x})\dddot{x} + b(x, \dot{x})\ddot{x} + c(\dot{x}) + d(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{5.19}$$

with $p \neq 0$. It can be written in the phase variables form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= u, \\ \dot{u} &= -a(z, u)u - b(x, y)z - c(y) - d(x) + p(t, x, y, z, u), \end{aligned} \tag{5.20}$$

in which the functions a, b, c, d and p depend only on the arguments displayed and the dots denote differentiation with respect to t.The functions a, b, c, d and p are continuous for all values of their respective arguments. The derivatives

$$\frac{\partial}{\partial u}a(z, u) \equiv a_u(z, u), \quad \frac{\partial}{\partial x}b(x, y) \equiv b_x(x, y), \quad \frac{dc}{dy} \equiv c'(y) \text{ and } \frac{dd}{dx} \equiv d'(x).$$

Exist and are continuous. Moreover, the existence and the uniqueness of the solutions of (5.19) will be assumed.

In what follows (Tunc, 2005) used the following notations:

$$a_1(z, 0) = \begin{cases} \dfrac{1}{z}\displaystyle\int_0^z a(\tau, 0)d\tau, & z \neq 0, \\ a(0,0), & z = 0, \end{cases}$$

and

$$c_1(y) = \begin{cases} \dfrac{c(y)}{y}, & y \neq 0, \\ c'(0), & y = 0. \end{cases}$$

In the case $p \neq 0$, (Tunc, 2005) established the following results:

Theorem 5.17. Further to the basic assumptions on the functions a, b, c and d, assume that the following conditions are satisfied ($\alpha, \beta, \mu, \gamma, \delta, \eta, \varepsilon$ and ε_1 some positive constants)

(i) $0 \leq a(z, u) - \alpha \leq \varepsilon_1$ for all z and u,

(ii) $c_1(y) \geq \beta$ for all $y \neq 0, c(0) = 0$,

(**iii**) $0 \le b(x,y) - \mu \le \sqrt{\frac{\delta\varepsilon_1}{4\beta}}$ and $y\int_0^y b_x(x,y)ydy \le -(\frac{\beta^2}{\alpha\gamma})y^2$ for all x and y,

(**iv**) $d(x)x > 0\ for\ all\ x \ne 0, 0 \le \gamma - d'(x) \le \frac{\sqrt{\delta}}{2}$ for all x, and $d(0) = 0$,

(v) $\alpha\beta\mu - \beta c'(y) - \alpha\gamma a(z,u) \ge \delta$ for all y, z and u,

(**vi**)$c'(y) - c_1(y) \le \eta < \frac{2\delta\gamma}{\alpha\beta^2}$ for all $y \ne 0$, and $a_1(z,u) - a(z,u) \le \varepsilon < \frac{2\delta}{\alpha^2\beta}$

for all $z \ne 0$ and u,

(vii) $\gamma y a_u(z\,,u) + \beta z a_u(z\,,u) \ge 0$ for all y, z and u,

(vii) $|p(t,x,y,z,w)| \le (A + |y| + |z| + |w|)q(t)$,

where $q(t)$ is a non-negative and continuous function of t, and satisfies

$$\int_0^t q(s)\,ds \le B < \infty,$$

for all $t \ge 0$, A and B are some positive constants. Then for any given finite constants x_0, y_0, z_0 and u_0, there exist a constant $K = K(x_0, y_0, z_0, u_0)$, such that any solution $(x(t), y(t), z(t), u(t))$ of system (5.20) determined by

$$x(0) = x_0, \quad y(0) = y_0\ , \quad z(0) = z_0, \quad w(0) = w_0,$$

Satisfies for all $t \ge 0$,

$$|x(t)| \le K, \qquad |y(t)| \le K, \qquad |z(t)| \le K, \qquad |w(t)| \le K.$$

If p is a bounded function.Then the constant K above can be fixed independent of x_0, y_0, z_0 and u_0, (Tunc, 2005).

Theorem 5.18. Assume that conditions (i)-(vii) of Theorem 5.17 hold, and that $p(t,x,y,z,u)$ satisfies

$$|p(t,x,y,z,w)| \le A < \infty,$$

for all values of $t, x, y\,, z$ and u, where A is a positive constant. Then there exist a constant K_1 whose magnitude depends $\alpha, \beta, \mu, \gamma, \delta, \eta, \varepsilon$ and ε_1, as well as on the functions a, b, c and d such that every solution $(x(t), y(t), z(t), u(t))$ of system (5.20) ultimately satisfies

$$|x(t)| \le K_1, \qquad |y(t)| \le K_1, \qquad |z(t)| \le K_1, \qquad |w(t)| \le K_1$$

(Tunc, 2005).

The proofs of Theorem 5.17 depends on some certain fundamental properties of a continuously differentiable Lyapunov function $V = V(x, y, z, u)$ defined by:

$$V = \alpha\gamma \int_0^x d(x)dx + \alpha\gamma \int_0^y b(x,y)ydy - \left(\frac{\beta\gamma}{2}\right)y^2 + \alpha\beta \int_0^y c(y)dy$$
$$+ \left(\frac{\beta\mu}{2}\right)z^2 + \alpha\beta \int_0^z a(z,u)zdz - \left(\frac{\alpha\gamma}{2}\right)z^2 + \left(\frac{\beta}{2}\right)u^2 + \alpha\beta d(x)y$$
$$+\beta d(x)z + \beta c(y)z + \alpha\gamma y \int_0^z a(z,0)dz + \alpha\gamma yu + \alpha\beta zu.$$

The first property of the scalar function V is stated in the following.

Lemma 5.19. Assume that the conditions of Theorem 5.17 hold. Then
(i) $V(x, y, z, w) = 0$ at $x^2 + y^2 + z^2 + u^2 = 0$,
(ii) $V(x, y, z, w) > 0$ if $x^2 + y^2 + z^2 + u^2 > 0$, $\dot{V} \leq 0$ for all $t \geq 0$,
(iii) any of the positive semi trajectories of the system (5.20) is bounded,
(iv) the set $M = \{(x, y, z, u): \dot{V} = 0, (x, y, z, u) \in R^4\}$, except $(x, y, z, u) = 0$, does not contain the entire positive semi trajectory of the solution of system (5.20)(Tunc, 2005).

5.8. Boundedness in $x^{(4)} + a\ddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t)$

In (2007) Omeike studied and investigated boundedness of solutions of the fourth order non-linear non homogeneous differential equation of the form:

$$x^{(4)} + a\ddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t), \tag{5.21}$$

where $a > 0$, f, g, h and p, and their first derivatives are continuous functions depending on the arguments shown explicitly and dots means the derivative of the variable with respect to t .

Omeike (2007) gave the principal result of his study as follows.

Theorem 5.20. If there exist positive constants H, H', H_0, P, P', P_0, such that for $|x| > H_0$ and $t \geq 0$ the following conditions are satisfied:

(**i**) $|h(x)| \leq H, \quad |h'(x)| \leq H',$

(**ii**) $0 < \frac{f(\ddot{x})}{\ddot{x}} \leq b < \infty, \ f(0) = 0,$

(**iii**) $0 < \frac{g(\dot{x})}{\dot{x}} \leq c < \infty, \ g(0) = 0,$

(**iv**) $|p(t)| \leq P, \quad |p'(t)| \leq P', \quad \left|\int_0^t p(\tau)d\tau\right| \leq P_0, \quad \limsup_{t\to\infty}|p(t)| > 0,$

(**v**) $min[d(\bar{x}_k, \bar{x}_{k+1}), d(\bar{x}_k, \bar{x}_{k-1})] > \frac{(H+P)}{c_1}\left(\frac{4}{a} + \frac{2a}{b} + \frac{b}{c}\right) + \frac{P_0}{c_1},$

where $\bar{x}_k$ are roots of $h(x)$ with $h'(\bar{x}_k) > 0$ and $\bar{x}_{k-1}, \bar{x}_{k+1}$ denote the couple of adjacent roots of $\bar{x}_k$ $(k = 0, \mp 2, \mp 4, \dots)$, then all solutions $x(t)$ of (5.21) are bounded (Omeike, 2007).

Example 5.1. Consider the equation

$$x^{(4)}(t) + \frac{385}{16}\ddot{x}(t) + \frac{259\ddot{x}(t)}{2\left(1 + \left(\ddot{x}(t)\right)^2\right)} + \left(7\dot{x}(t) + \frac{\dot{x}(t)}{1 + \left(\ddot{x}(t)\right)^2}\right) + \frac{1}{10}\sin x(t) = \frac{1}{10}\cos\ t. \tag{5.22}$$

By comparing equation (5.22) with equation (5.21) it is clear that

$$a = \frac{385}{16}, \qquad f\left(\ddot{x}(t)\right) = \frac{259\ddot{x}(t)}{2\left(1 + \left(\ddot{x}(t)\right)^2\right)}, \qquad g\left(\dot{x}(t)\right) = 7\dot{x}(t) + \frac{\dot{x}(t)}{1 + \left(\ddot{x}(t)\right)^2},$$

$$h\left(x(t)\right) = \frac{1}{10}\sin x(t) \qquad \text{and} \qquad p(t) = \frac{1}{10}\cos\ t,$$

a simple calculation gives

$$H = 0.1, \ H' = 0.1, \ P = 0.1, \ P' = 0.1, \ P_0 = 0.1, \ b = \frac{259}{2}, \ c = 8 \text{ and } c_1 = 7.$$

It is obvious that conditions (i)- (iv) of Theorem 5.20 are satisfied. For condition (v), since $h\left(x(t)\right) = \frac{1}{10}\sin x(t)$ the roots of $h(x(t))$ are

$$\bar{x}_{k-1} = (k-1)\pi, \quad \bar{x}_k = k\pi, \qquad \bar{x}_{k+1} = (k+1)\pi, \quad (k = 0, \pm 2, \pm 4, \dots),$$

where $\bar{x}_{k-1}$ and $\bar{x}_{k+1}$ are the couple of adjacent roots of $\bar{x}_k = k\pi$. Thus,

$$min\,[d(\bar{x}_k, \bar{x}_{k+1}), d(\bar{x}_k, \bar{x}_{k-1})] = \pi$$

and

$$\frac{(H+P)}{c_1}\left(\frac{4}{a} + \frac{2a}{b} + \frac{b}{c}\right) + \frac{P_0}{c_1} < 1.$$

Since $\pi > 1$, then all conditions of Theorem 5.20 are satisfied, thus all solutions $x(t)$ of equation (5.22) are bounded.

5.9. Boundedness in $\boldsymbol{x^{(4)} + a\ddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t)}$

In (2008), (Ogundare and Okecha) together studied the boundedness property of solutions to certain non-linear non homogeneous fourth order differential equation of the form

$$x^{(4)} + a\ddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{5.23}$$

where a and b, are positive constants the functions g, h and p are continuous in the respective argument displayed explicitly, also dthe ots denoted the derivative of the variable with respect to t.

In continuation, the authors (Ogundare and Okecha, 2008) considered equation (5.23) with an equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -aw - bz - g(y) - h(x) + p(t), \end{aligned} \tag{5.24}$$

In the case where $p \neq 0$, (Ogundare and Okecha, 2008) established the following main results on boundedness of solutions of equation (5.23).

Theorem 5.21. In addition to the basic assumptions on g, h and p, suppose the following conditions are satisfied:

(**i**) $H_0 = \dfrac{h(x) - h(0)}{x} \leq d \in I_0$ with $I_0 = [\delta, \Delta], \quad x \neq 0, \quad d, \delta, \Delta > 0,$ and I_0

is the Routh Hurwitz interval.

(**ii**) $G_0 = \dfrac{g(y) - g(0)}{y} \leq c \in I_0\,, \qquad y \neq 0, \qquad c > 0,$

(**iii**) $h(0) = g(0) = 0,$

(**iv**) $|p(t)| \leq M$ (constant) for all $t \geq 0$ then there exists a constant $\mu, (0 < \mu < \infty)$ depending only on a, b, c, d and δ such that every solution of (5.23) satisfies

$$x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leq e^{-\frac{1}{2}\mu t}\left\{A_1 + A_2 \int_{t_0}^{t} |p(\tau)| e^{\frac{1}{2}\mu\tau} d\tau\right\}^2,$$

for all $t \geq t_0$, where the constant $A_1 > 0$, depends on a, b, c, d , δ as well as on $t_0, x(t_0), \dot{x}(t_0), \ddot{x}(t_0), \dddot{x}(t_0)$, and the constant $A_2 > 0$ depends on a, b, c, d and δ. (Ogundare and Okecha, 2008).

Ogundare and Okecha (2008) considered the case when $p(t)$ in (5.23) is replaced with $p(t, x, \dot{x}, \ddot{x}, \dddot{x})$, and its results with this case is the following thorem.

Theorem 5.22. Following the assumptions of Theorem 5.21 and condition (iv) replaced with $|p(t, x, \dot{x}, \ddot{x}, \dddot{x})| = (|x| + |y| + |z| + |w|)r(t)$, where $r(t)$ is a non negative and continuous function of t, and satisfies

$$\int_0^t r(s)ds \leq M < \infty,$$

where M, is a positive constant. Then there exists a constant K_0 which depends on M, K_1, K_2 and t_0 such that every solution $x(t)$ of equation (5.23) satisfies

$$|x(t)| \leq K_0, \qquad |\dot{x}(t)| \leq K_0, \quad |\ddot{x}(t)| \leq K_0, \qquad |\dddot{x}(t)| \leq K_0$$

for sufficiently large t.(Ogundare and Okecha, 2008).

Ogundare and Okecha (2008) used as a tool to prove them main results besides equation (5.23), which is a Lyapunov function $V = V(x, y, z, w)$ defined by

$$2V = Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy + 2Fxz + 2Gxw + 2Hyz + 2Iyw + 2jzw,$$

where

$$A = \frac{a\delta}{\Delta}\{(b + d)(c^2 + d^2)[d(1 - ad) - c] + d^3[a(b^2 + d^2) + L]\},$$

$$B = \frac{\delta}{\Delta}\{dL(adb + c) + a(b^2 + d^2)[b(d - c) + cd]$$

$$+[d(1 - ad) - c][ad(b^2 + c^2) - cd^2(b + 1) + a^2bc]\},$$

$$C = \frac{\delta}{\Delta}\{a(b^2+d^2)[d(1-ad+a^2c+d)-c]$$
$$+d[c(a^2+b^2)-ab][d(1-ad)-c]+dL(a^2c+d)\},$$
$$D = \frac{cd\delta}{\Delta}\{L+ab^2+(d-c)+ab[(1-ad)-c]\},$$
$$E = \frac{ac\delta}{\Delta}\{d^2L+(b^2+d^2)(d-c)\},$$
$$F = \frac{cd\delta}{bd\Delta}\{d^2L+ad^2(b^2+d^2)+[b(a^2+d^2)+d^2][ab^2d^2[d(1-ad)-c]]\},$$
$$G = \frac{abc[d(1-ad-c)]\delta}{\Delta},$$
$$H = \frac{abcd\delta}{\Delta}\{a(b^2+d^2)+L\},$$
$$I = \frac{a\delta}{\Delta}\{d^2L+bd[d(1-ad)-c]+(b^2+d^2)(d-c)\},$$
$$J = \frac{acd\delta}{\Delta}\{ab^2+d-c+L\},$$
$$\Delta = abcd[d(1-ad)-c],$$
$$L = b[ad+c[c(b+1)-c]],$$

with a, b, c, d positive and $[d(1-ad)-c] > 0$.

Lemma 5.23. Subject to the assumptions (i) – (iii) of Theorem 5.21, there exist positive constants $K_i = K_i(a,b,c,d,\delta)$, $i = 1,2$ such that

$$K_1(x^2+y^2+z^2+w^2) \le V(x,y,z,w) \le K_2(x^2+y^2+z^2+w^2)$$

(Ogundare and Okecha , 2008).

Lemma 5.24. Suppose all assumptions of Theorem 5.21, are satisfied. Then there are positive constants $K_j = K_j(a,b,c,d,\delta)$, $j = 3,4$ such that for any solution (x,y,z,w) of (5.24),

$$\dot{V} \le -K_3(x^2+y^2+z^2+w^2) + K_4(|x|+|y|+|z|+|w|)|p(t)|$$

(Ogundare and Okecha, 2008).

5.10. Boundedness in $x^{(4)} + a\ddot{x} + f(x,\dot{x},\ddot{x}) + g(x,\dot{x}) + h(x) = p(t)$

In (2013) Tunc and Ates, deals with the boundedness of solutions to a non-linear non-homogeneous differential equation of fourth order in the form:

$$x^{(4)} + a\ddot{x} + f(x, \dot{x}, \ddot{x}) + g(x, \dot{x}) + h(x) = p(t), \tag{5.25}$$

where $t \in [0, \infty), a > 0$, f, g, h and p, and their first derivatives are continuous functions depending on the arguments shown explicitly and dots means the derivative of the variable with respect to t .

Tunc and Ates (2013) gave the mains principal result of his study as follows

Theorem 5.25. If there exist positive constants H, H', H_0, P, P', P_0, such that for $|x| > H_0$ and $t \geq 0$ the following conditions are satisfied:

(i) $|h(x)| \leq H, \quad |h'(x)| \leq H',$

(ii) $0 < \dfrac{f(x, \dot{x}, \ddot{x})}{\ddot{x}} \leq b < \infty, \quad (\ddot{x} \neq 0), \quad f(x, \dot{x}, 0) = 0,$

(iii) $0 < \dfrac{g(x, \dot{x})}{\dot{x}} \leq c < \infty, (\dot{x} \neq 0), \quad g(x, 0) = 0,$

(iv) $|p(t)| \leq P, \quad |p'(t)| \leq P', \quad \left| \int_0^t p(\tau) d\tau \right| \leq P_0, \quad \lim_{t \to \infty} sup\, |p(t)| > 0,$

(v) $\min[d(\bar{x}_k, \bar{x}_{k+1}), d(\bar{x}_k, \bar{x}_{k-1})] > \dfrac{(H + P)}{c}\left(\dfrac{4}{a} + \dfrac{2a}{b} + \dfrac{b}{c}\right) + \dfrac{P_0}{c},$

where $\bar{x}_k$ are roots of $h(x)$ with $h'(\bar{x}_k) > 0$ and $\bar{x}_{k-1}, \bar{x}_{k+1}$ denote the couple of adjacent roots of $\bar{x}_k$ $(k = 0, \mp 2, \mp 4, \dots)$, then all solutions $x(t)$ of (5.25) are bounded (Tunc and Ates, 2013).

Example 5.2. Consider the differential equation

$$x^{(4)}(t) + 2\ddot{x}(t) + \frac{5\ddot{x}(t)}{4\left(1 + x^2(t) + \left(\dot{x}(t)\right)^2\right)} + \frac{\dot{x}(t)}{4 + x^2(t)} + \frac{1}{100}\sin x(t) = \frac{1}{100}\cos\, t. \tag{5.26}$$

By comparing equation (5.26) with equation (5.25) it is clear that

$$a = 2, \qquad b = \frac{5}{4}, \qquad c = \frac{1}{4},$$

$$h\big(x(t)\big) = \frac{1}{100}\sin x(t) \qquad \text{and} \qquad p(t) = \frac{1}{100}\cos\, t,$$

a simple calculation gives

$$H = 0.01,\ \ H' = 0.01,\ \ P = 0.01,\ \ P' = 0.01,\ \ P_0 = 0.01,$$

It is obvious that conditions (i)- (iv) of Theorem 5.25 are satisfied. For condition (v), since

$$h\big(x(t)\big) = \frac{1}{100}\sin x(t),$$

then the roots of $h(x(t))$ are

$$\bar{x}_{k-1} = (k-1)\pi, \quad \bar{x}_k = k\pi, \qquad \bar{x}_{k+1} = (k+1)\pi, \quad (k = 0, \pm 2, \pm 4, \dots),$$

where $\bar{x}_{k-1}$ and $\bar{x}_{k+1}$ are the couple of adjacent roots of $\bar{x}_k = k\pi$. Thus,

$$\min[d(\bar{x}_k, \bar{x}_{k+1}), d(\bar{x}_k, \bar{x}_{k-1})] \ = \pi,$$

and

$$\frac{(H+P)}{c}\left(\frac{4}{a} + \frac{2a}{b} + \frac{b}{c}\right) + \frac{P_0}{c}$$

$$= 4\left[(0.01 + 0.01)\left(2 + \frac{16}{5} + 5\right) + 0.01\right] = 0.856 < 1.$$

Since $\pi > 1$, then all conditions of Theorem 5.25 are satisfied, thus all solutions $x(t)$ of equation (5.26) are bounded.

6. CONVERGENCE of SOLUTIONS

To the best of our knowledge from the literature, first, in (1981) Afuwape considered the fourth order differential equation of the form

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

where a, b and c are positive constants, h and p are continuous, and p has the form $p = q(t) + r(t, x, \dot{x}, \ddot{x}, \dddot{x})$, with q, r are continuous. Two solutions $x_1(t), x_2(t)$ of the above differential equation are said to converge if

$$\lim_{t\to\infty}\left[x_2^{(i)}(t) - x_1^{(i)}(t)\right] = 0\,, \quad i = 0,1,2,3$$

In the special case that $h(x) = dx$, where d is a positive constant, and $p \equiv 0$, the Routh-Hurwitz conditions $ab - c > 0$ and $(ab - c) - a^2 d > 0$ imply that any two solutions are converge.

Here we do not give the details of the work of (Afuwape, 1981).

6.1. Convergence in $\boldsymbol{x^{(4)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})}$

After the work of (Afuwape, 1981), in (1988), Afuwape considered fourth order differential equations of the form:

$$x^{(4)} + a\dddot{x} + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \qquad (6.1)$$

in which a, b positive constants, functions g and h are continuous in their respective arguments. The function $p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ is assumed to have the form $p = q(t) + r(t, x, \dot{x}, \ddot{x}, \dddot{x})$ with the functions q and r depending explicitly on the arguments displayed, and continuous in their respective arguments. Further, it is assumed that $r(t, 0,0,0,0) = 0$ for all $t \geq 0$.

The convergence of solutions for equations of the form (6.1) was shown in (Afuwape ,1981) when $g(\dot{x}) = c\dot{x}$, with $c > 0$, and with the assumption that $h(x)$ is not necessarily differentiable, but with an incriminatory ratio $\eta^{-1}\{h(\xi + \eta) - h(\xi)\}, (\eta \neq 0)$, lying in a closed sub interval I_0 of the Routh-Hurwitz interval

$$\left(0, \ \frac{(ab - c)}{a^2}\right),$$

where

$$I_0 \equiv \left[\Delta_0, \frac{k(ab-c)c}{a^2}\right], \tag{6.2}$$

where $\Delta_0 > 0$ and $k < 1$.

The main purpose of the work of (Afuwape, 1988) was to give fourth order analogues of (Tejumola, 1972) as well as extending earlier results in (Afuwape, 1981) to equations of the form (6.1) with the additional condition that for $y_1 \neq y_2$

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq c_0, \tag{6.3}$$

for some constants $c_0 > 0$ and $c > 0$, satisfying

$$abc > c_0^2. \tag{6.4}$$

Moreover, while proving the convergence results for equation (6.1), (Afuwape, 1988) gave a general estimate for the constant $K < I$, from which a particular case is derived.

The main results of (Afuwape, 1988) are the following.

Theorem 6.1 Suppose that $g(0) = h(0)$ and that

(i) there are constants $c_0 > 0$, $c > 0$ such that $g(y)$ satisfies the inequalities (6.3) and (6.4),

(ii) there are constants $\Delta_0 > 0$, $k < 1$ such that for any $\xi, \eta, (\eta \neq 0)$ the incriminatory ratio for h satisfies

$$\eta^{-1}\{h(\xi + \eta) - h(\xi)\} \text{ lies in } I_0, \tag{6.5}$$

with I_0 as defined in (6.2),

(iii) there is a continuous function $\emptyset(t)$ such that

$$|r(t, x_2, y_2, z_2, w_2) - r(t, x_1, y_1, z_1, w_1)|$$
$$\leq \emptyset(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|\}, \tag{6.6}$$

holds for arbitrary $t, x_1, y_1, z_1, w_1, x_2, y_2, z_2$ and w_2. Then there exist a constant D_1 such that if

$$\int_0^t \emptyset^{\alpha}(\tau)d\tau \leq D_1 t, \tag{6.7}$$

for some α, in the range $1 \leq \alpha \leq 2$, then all solutions of (6.1) converge.

A very important step in the proof of Theorem 6.1 will be to give estimates for any two solutions of equation (6.l). This in itself, being of independent interest, is given as:

Theorem 6.2. Let $x_1(t),\ x_2(t)$ be any two solutions of equation (6.1). Suppose that all the conditions of Theorem 6.1 are satisfied, then for each fixed α, in the range $1 \leq \alpha \leq 2$, there exists constants D_2, D_3 and D_4 such that for $t_2 \geq t_1$,

$$S(t_2) \leq\ D_2 S(t_1) exp\left\{-D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \emptyset^{\alpha}(\tau) d\tau\right\}, \tag{6.8}$$

where

$$S(t) = [x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2 + [\dddot{x}_2(t) - \dddot{x}_1(t)]^2. \tag{6.9}$$

Let

$$Q(t) = \int_0^t q(\tau) d\tau.$$

For convenience, by setting $\dot{x} = y,\ \dot{y} = z,\ \dot{z} = w + Q(t)$, we replace the equation (6.1) by the equivalent system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w + Q(t), \\ \dot{w} &= -aw - bz - g(y) - h(x) + r\big(t, x, y, z, w + Q(t)\big) - aQ(t). \end{aligned} \tag{6.10}$$

Let $\big(x_i(t), y_i(t), z_i(t), w_i(t)\big)$, $(i = 1,2)$ be two solutions of system (6.10), such that

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq\ c_0, \tag{6.11}$$

and

$$\Delta_0 \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq\ \frac{k(ab - c)c}{a^2}, \tag{6.12}$$

where c, c_0, Δ_0, k are as defined in (6.2), (6.3) and (6.4).

The main tool in the proofs of convergence theorems is the following Lyapunov function:

$$W = W(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1,\ w_2 - w_1),$$

defined by

$$2W = c^2\varepsilon(1 - \varepsilon)(x_2 - x_1)^2 + ac(1 - \varepsilon)(D - 1)(\ y_2 - y_1)^2$$

$$+2c[\varepsilon+(D-1)](y_2-y_1)\,(z_2-z_1)+\varepsilon D(w_2-w_1)^2$$
$$+b(D-1)(z_2-z_1)^2+[(1-\varepsilon)D-1][a(z_2-z_1)+(w_2-w_1)]^2$$
$$+[c(1-\varepsilon)(x_2-x_1)+b(y_2-y_1)+(z_2-z_1)+(w_2-w_1)]^2, \qquad (6.13)$$

where

$$D-1=\frac{(\delta+c\varepsilon)}{(ab-c-\delta)}, \text{with } ab-c>\delta>0;\ 0<\varepsilon<1 \text{ and } ab\varepsilon(2-\varepsilon)=\delta.$$

This is an adaptation of the function V used in (Afuwape, 1981). Since $0<\varepsilon<1$, following the argument used in (Afuwape, 1981) we can easily verify the following for W.

Lemma 6.3. Suppose that

(i) $W(0,0,0,0)=0$,

(ii) there exist finite constants $D_6>0, D_7>0$ such that

$$D_6\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2\}\le W$$
$$\le D_6\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2\}. \qquad (6.14)$$

If the function $W(t)$ is defined by

$W\big(x_2(t)-x_1(t),y_2(t)-y_1(t),z_2(t)-z_1(t),w_2(t)-w_1(t)\big)$, and using the fact that the solutions $\big(x_i,y_i,z_i,w_i+Q(t)\big)$, $(i=1,2)$, satisfy (6.10), then $S(t)$ as defined in (6.9) becomes

$$S(t)=[x_2(t)-x_1(t)]^2+[y_2(t)-y_1(t)]^2+[z_2(t)-z_1(t)]^2$$
$$+[w_2(t)-w_1(t)]^2, \qquad (6.15)$$

The author prove the following result on the derivative of $W(t)$ with respect to t.

Lemma 6.4. Let the hypotheses (i) and (ii) of Theorem 6.1 hold. Then, there exists positive finite constants D_8 and D_9 such that

$$\frac{dW}{dt}\le -2D_8S+D_9S^{\frac{1}{2}}|\theta|, \qquad (6.16)$$

where $\theta=r(t,x_2,y_2,z_2,w_2+Q)-r(t,x_1,y_1,z_1,w_1+Q)$.

Proof . On using (6.10), a direct computation of $\frac{dW}{dt}$ gives after simplification

$$\frac{dW}{dt}=-W_1+W_2, \qquad (6.17)$$

where

$$W_1=\{c(1-\varepsilon)H(x_2,x_1)(x_2-x_1)^2+bc\varepsilon(y_2-y_1)^2+ab\varepsilon(1-\varepsilon)(z_2-z_1)^2$$
$$+a\varepsilon D(w_2-w_1)^2\}+\{G(y_2,y_1)-c\}\{c(1-\varepsilon)(x_2-x_1)$$

$$+b(y_2 - y_1) + a(1-\varepsilon)D(z_2 - z_1) + D(w_2 - w_1)(y_2 - y_1)$$
$$+H(x_2, x_1)\{\, b(y_2 - y_1) + a(1-\varepsilon)D(z_2 - z_1) + D(w_2 - w_1)\}(x_2 - x_1)$$

and

$$W_2 = \theta(t)\{c(1-\varepsilon)(x_2 - x_1) + b(y_2 - y_1) + a(1-\varepsilon)D(z_2 - z_1) + D(w_2 - w_1)\}$$

with

$$G(y_2, y_1) = \frac{g(y_2) - g(y_1)}{y_2 - y_1}, \qquad (y_2 \neq y_1), \tag{6.18}$$

$$H(x_2, x_1) = \frac{h(x_2) - h(x_1)}{x_2 - x_1}, \qquad (x_2 \neq x_1). \tag{6.19}$$

Let $\lambda = \{G(y_2, y_1) - c\} \geq 0$ for $y_2 \neq y_1$. Define

$$\sum_{i=1}^{5} \alpha_i = 1\,, \qquad \sum_{i=1}^{5} \beta_i = 1\,, \qquad \sum_{j=1}^{3} \gamma_j = 1\,, \qquad \sum_{j=1}^{3} \delta_j = 1\,,$$

where $\alpha_{i,}\beta_i, \gamma_j$ and δ_j are positive constants.

By choosing

$$2D_8 = \min\{\, c(1-\varepsilon)\Delta_0;\ bc\varepsilon; ab\varepsilon(1-\varepsilon)D;\ \ a\varepsilon D\},$$

clearly we have

$$W_1 \geq\ 2D_8 S, \tag{6.20}$$

also if we choose $D_9 = 2\max\{\, c(1-\varepsilon);\ b;\ D;\ \ a(1-\varepsilon)D; D\}$, then we can obtain:

$$W_2 \leq D_9 S^{\frac{1}{2}}|\theta|. \tag{6.21}$$

By following the way as shown in (Afwape, 1988) one can arrive at the result of Lemma 6.4. Therefore we omit the details.

Proof of Theorem 6.2. Let α be any constant in the range $1 \leq\ \alpha \leq 2$. Set $2\mu = 2 - \alpha$, so that $0 \leq 2\mu \leq 1$. In view of (Ezeilo, 1973) and on using inequality (6.16), it is clear that

$$\frac{dW}{dt} + D_8 S \leq D_9 S^{\mu} W^*,$$

where

$$W^* = \left(|\theta| - D_8 D_9^{-1} S^{\frac{1}{2}}\right) S^{\frac{1}{2}-\mu}. \tag{6.22}$$

We consider the following two cases:

(**i**) $|\theta| \leq \frac{D_8 S^{\frac{1}{2}}}{D_9}$,

(**ii**) $|\theta| > \frac{D_8 S^{\frac{1}{2}}}{D_9}$,

separately. We find that in either cases, there exists some constant $D_{11} > 0$ such that $W^* \leq D_{11}|\theta|^{2(1-\mu)}$. Thus using (6.6), inequality (6.22) becomes

$$\frac{dW}{dt} + D_8 S \leq D_{12} S^{\mu} \emptyset^{2(1-\mu)} S^{(1-\mu)}, \tag{6.23}$$

where $D_{12} \geq 2D_9 D_{11}$. This immediately gives

$$\frac{dW}{dt} + \left(D_{13} - D_{14}\emptyset^{\alpha}(t)\right)W \leq 0, \tag{6.24}$$

after using Lemma 6.3 on W, with D_{13} and D_{14} as some positive constants. Take integration of (6.24) from t_1 to t_2, $(t_2 \geq t_1)$, we get

$$W(t_1) \leq W(t_2) exp\left\{-D_{13}(t_2 - t_1) + D_{14}\int_{t_1}^{t_2} \emptyset^{\alpha}(\tau)d\tau\right\}. \tag{6.25}$$

Again, using Lemma 6.3, we obtain (6.8), with $D_2 = \frac{D_7}{D_6}$, $D_3 = D_{13}$ and $D_4 = D_{14}$.

This completes the proof of Theorem 6.2.

Proof of Theorem 6.1. This follows from the estimate (6.8) and the condition (6.7) on $\emptyset(t)$. Choose $D_1 = \frac{D_3}{D_4}$ in (6.7). Then, as $t = (t_2 - t_1) \to \infty$, $S(t) \to 0$, which proves that as $t \to \infty$,

$$x_2(t) - x_1(t) \to 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \to 0,$$
$$\ddot{x}_2(t) - \ddot{x}_1(t) \to 0, \qquad \dddot{x}_2(t) - \dddot{x}_1(t) \to 0.$$

This completes the proof of Theorem 6.1.

6.2. Convergence in $x^{(4)} + a\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t)$

Ogundare and Okecha (2007) considered the fourth-order non homogeneous differential equation of the form

$$x^{(4)} + a\dddot{x} + f(x,\dot{x})\ddot{x} + g(\dot{x}) + h(x) = p(t), \tag{6.26}$$

where $a > 0$, the functions f, g, h, p are continuous in the respective arguments displayed explicitly, $\dot{x} = \frac{dx}{dt}$. The conditions on $f, g, h,$ and p are such that the existence of solutions of (6.26) corresponding to any p reassigned initial solutions is guaranteed.

Ogundare and Okecha(2007) gave the following result on the convergence of the solutions:

Theorem 6.5. Suppose that $x_1(t), x_2(t)$ are two solutions of (6.26), suppose further that for arbitrary $\xi, \eta, (\eta \neq 0)$,

(**i**) $\dfrac{\{h(\xi+\eta) - h(\xi)\}}{\eta} \in I_0, \ \eta \neq 0,$

(**ii**) $\dfrac{\{g(\xi+\eta) - g(\xi)\}}{\eta} \neq 0,$

(iii) $h(0) = g(0) = 0,$

(**iv**) $|f(x,y)| \leq b$,

(**v**) $|p(t)| \leq \Lambda, \qquad (\Lambda \text{ constant}).$

Then there exists a positive constant K_5 such that

$$S(t_2) \leq S(t_1)\, e^{-K_5(t_2 - t_1)} \qquad \text{for } t_2 \geq t_1,$$

where

$$S(t) = [x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2 + [\dddot{x}_2(t) - \dddot{x}_1(t)]^2. \tag{6.27}$$

Furthermore, all solutions of (6.26) converge.

On setting $\dot{x} = y, \ \dot{y} = z, \ \dot{z} = w$, we replace the equation (6.26) by the equivalent system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -aw - f(x,y)z - g(y) - h(x) + p(t). \end{aligned} \tag{6.28}$$

Ogundare and Okecha (2007) defined a scalar continuously differential Lyapunov function as

$$2V(x,y,z,w) = Ax^2 + By^2 + Cz^2 + Dw^2 + 2Exy + 2Fxz + 2Ixw + 2Jyz + 2Myw + 2Nzw. \tag{6.29}$$

The proof rests mainly on the properties of the function
$W(t) \equiv V\big(x_2(t) - x_1(t), y_2(t) - y_1(t), z_2(t) - z_1(t), w_2(t) - w_1(t)\big)$, with $V\big(x(t), y(t), z(t), w(t)\big)$ written as $V(x, y, z, w)$, where

$$A = \frac{a\delta}{\Delta}\{(b+d)(c^2+d^2)[d(1-ad)-c] + d^3[a(b^2+d^2)+L]\},$$

$$B = \frac{\delta}{\Delta}\{dL(abd+c) + a(b^2+d^2)[b(d-c)+cd]$$
$$+[d(1-ad)-c][ad(b^2+c^2) - cd^2(b+1) + a^2bc]\},$$

$$C = \frac{\delta}{\Delta}\{a(b^2+d^2)[d(1-ad+a^2c+d)-c]$$
$$+d[c(a^2+b^2)-ab][d(1-ad)-c] + dL(a^2c+d)\},$$

$$D = \frac{cd\delta}{\Delta}\{L + ab^2 + (d-c) + ab[(1-ad)-c]\},$$

$$E = \frac{ac\delta}{\Delta}\{d^2L + (b^2+d^2)(d-c)\},$$

$$F = \frac{cd\delta}{bd\Delta}\{d^2L + ad^2(b^2+d^2) + [b(a^2+d^2)+d^2][ab^2d^2[d(1-ad)-c]]\},$$

$$I = \frac{abc[d(1-ad-c)]\delta}{\Delta},$$

$$J = \frac{abcds}{\Delta}\{a(b^2+d^2)+L\},$$

$$M = \frac{a\delta}{\Delta}\{d^2L + bd[d(1-ad)-c] + (b^2+d^2)(d-c)\},$$

$$N = \frac{acd\delta}{\Delta}\{ab^2 + d - c + L\},$$

$$\Delta = abcd[d(1-ad)-c],$$

$$L = b\big[ad + c[(b+1)-c]\big],$$

with a, b, c, d are positive constants and $[d(1-ad)-c] > 0$ were obtained after solving the equations that arose when constructing the Lyapunov function.
Thus W is equivalent to $V(x, y, z, w)$ with x, y, z, w replaced with $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$, $w_2 - w_1$, respectively.

Define W as follows

$$
\begin{aligned}
&2W(x_2-x_1, y_2-y_1, z_2-z_1,\ w_2-w_1)\\
&=\ A(x_2-x_1)^2\ +B(y_2-y_1)^2\ +C(z_2-z_1)^2\ +D(w_2-w_1)^2\\
&\qquad +2E(x_2-x_1)(y_2-y_1)\ +2F(x_2-x_1)(z_2-z_1)\\
&\qquad +2I(x_2-x_1)(w_2-w_1)+2J\ (y_2-y_1)(z_2-z_1)\\
&\qquad +2M(y_2-y_1)(w_2-w_1)+2N(z_2-z_1)(w_2-w_1). \qquad (6.30)
\end{aligned}
$$

Lemma 6.6. Suppose W is defined as in (6.30) and $W(0,0,0,0)=0$. Then there exist constants K_1 and K_2 such that the inequalities

$$
\begin{aligned}
&K_1((x_2-x_1)^2\ +(y_2-y_1)^2\ +(z_2-z_1)^2\ +(w_2-w_1)^2)\\
&\qquad \le W \le K_2((x_2-x_1)^2\ +(y_2-y_1)^2\ +(z_2-z_1)^2\ +(w_2-w_1)^2) \qquad (6.31)
\end{aligned}
$$

hold.

Proof. Clearly, $W(0,0,0,0)\equiv 0$.

And by rearranging (6.30), we have

$$
\begin{aligned}
&2W(x_2-x_1, y_2-y_1, z_2-z_1, w_2-w_1)\\
&\quad=\left(\frac{\delta}{\Delta}\right)\{a[d(1-ad)]\{b[c(x_2-x_1)+d(y_2-y_1)+(w_2-w_1)]^2\\
&+d^2[(y_2-y_1)+b^3d^2(x_2-x_1)]^2+b^2d[(y_2-y_1)+a^2bd(x_2-x_1)]^2\\
&+acd\left[(z_2-z_1)+\frac{b^2d^3}{a}(x_2-x_1)\right]^2\}+dL\{[(z_2-z_1)+ac(x_2-x_1)]^2\\
&+ac^2\left[(z_2-z_1)+\frac{1}{a}(w_2-w_1)\right]^2+c\left[(y_2-y_1)+\frac{ad}{c}(w_2-w_1)\right]^2\\
&+ad^2\left[(x_2-x_1)+\frac{c}{d}(y_2-y_1)\right]^2+abd\left[(y_2-y_1)+\frac{c}{d}(z_2-z_1)\right]^2\}\\
&+ad(b^2+d^2)\{ad^2\left[(x_2-x_1)+\frac{c(d-c)}{ad^3}(y_2-y_1)\right]^2\\
&+a^2c\left[(z_2-z_1)+\frac{d}{a}(x_2-x_1)\right]^2+\frac{c}{a(b^2+d^2)}[(w_2-w_1)+a(z_2-z_1)]^2\\
&+b(d-c)\left[(y_2-y_1)+\frac{(w_2-w_1)}{b}\right]^2+c[(y_2-y_1)+ab(z_2-z_1)]^2\}\\
&+\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2)\\
&-b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\}(x_2-x_1)^2
\end{aligned}
$$

$$+\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]$$
$$-ac^2dL-\frac{c^2(d-c)^2}{d^3}\}(y_2-y_1)^2$$
$$+\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2)$$
$$-abc^2L-a^2cd[ab^2+(d-c)]\}(z_2-z_1)^2$$
$$+\{L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\}(w_2-w_1)^2\},$$

from which obtained

$$2W(x_2-x_1,y_2-y_1,z_2-z_1,w_2-w_1)$$
$$\geq\left(\frac{\delta}{\Delta}\right)\{\{[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2)$$
$$-b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L\}(x_2-x_1)^2$$
$$+\{[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]$$
$$-ac^2dL-\frac{c^2(d-c)^2}{d^3}\}(y_2-y_1)^2$$
$$+\{ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2)$$
$$-abc^2L-a^2cd[ab^2+(d-c)]\}(z_2-z_1)^2$$
$$+\{L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\}(w_2-w_1)^2\}$$
$$\geq K_1((x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2),$$

where

$$K_1=\frac{\delta}{\Delta}min\left\{\left|[d(1-ad)-c](ad(c^2+d^2)+abd^2)-\frac{cd^3}{a}(b^2+d^2)\right.\right.$$
$$-b^4cd^3-a^5b^4d^3-ab^6d^4-a^2c^2d^2L|,$$
$$\left|[d(1-ad)-c][ad(b^2+c^2)-cd^2(b+1)+a^2bc-abd^2]-ac^2dL\right.$$
$$\left.-\frac{c^2(d-c)^2}{d^3}\right|,$$
$$|ad^2(b^2+d^2)+d(b^2c-ab)[d(1-ad)-c]-a^3b^2cd(b^2+d^2)$$
$$-abc^2L-a^2cd[ab^2+(d-c)]|,$$
$$\left.\left|L-ab[d(1-ad)-c]-\frac{a}{b}(b^2+d^2)(d-c)-\frac{a^2d^3}{c}-cdL\right|\right\}.$$

Therefore,

$$2W(x_2 - x_1, y_2 - y_1, z_2 - z_1, w_2 - w_1)$$
$$\geq K_1((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2). \quad (6.32)$$

By using Schwartz inequality $|xy| \leq \frac{1}{2}\|x^2 + y^2\|$ on (6.29), we have

$$2W(x_2 - x_1, y_2 - y_1, z_2 - z_1, w_2 - w_1)$$
$$\leq \left(\frac{\delta}{\Delta}\right)\{[A + E + F + I](x_2 - x_1)^2 + [B + E + J + M](y_2 - y_1)^2$$
$$+[C + F + J + N](z_2 - z_1)^2 + [D + I + M + N](w_2 - w_1)^2\} \quad (6.33)$$

$$\leq K_2((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2),$$

where

$$K_2 = \left(\frac{\delta}{\Delta}\right)\max\{[A + E + F + I], [B + E + J + M], [C + F + J + N], [D + I + M + N]\}$$
$$> 0.$$

From inequalities (6.32) and (6.33), we have

$$K_1((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2)$$
$$\leq W \leq K_2((x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 + (w_2 - w_1)^2).$$

This proves that (Ogundare and Okecha, 2007) Lyapunov function is positive definite which proves Lemma 6.6.

Lemma 6.7. Suppose that $(x_1(t), y_1(t), z_1(t), w_1(t))$ and $(x_2(t), y_2(t), z_2(t), w_2(t))$ are any two distinct solutions of system (6.28) such that

$$H(x_1, x_2) = \frac{h(x_1(t)) - h(x_2(t))}{x_1(t) - x_2(t)} \in I_0, \quad G(y_1, y_2) = \frac{g(y_1(t)) - g(y_2(t))}{y_1(t) - y_2(t)} \neq 0,$$

for all $t > 0$ $(0 < t < \infty)$, where I_0 carries its usual meaning as $I_0 = [\delta, \Delta]$, then the function

$$W = V(x_1 - x_2, y_1 - y_2, z_1 - z_2, w_1 - w_2),$$

satisfies

$$\dot{W} \leq -K_3 W \quad (6.34)$$

for some constant $K_3 > 0$ (Ogundare and Okecha, 2007).

Proof of Theorem 6.5. Indeed from inequality (6.34), it is clear that

$$\frac{dW}{dt} \leq -K_5 W. \tag{6.35}$$

Integrating this inequality from t_1 to t_2, we have

$$\ln\left(\frac{W(t_2)}{W(t_1)}\right) \leq -K_5(t_2 - t_1),$$

$$\frac{W(t_2)}{W(t_1)} \leq \exp[-K_5(t_2 - t_1)].$$

Therefore

$$W(t_2) \leq W(t_1)\exp[-K_5(t_2 - t_1)]. \tag{6.36}$$

From the inequality (6.31), it follows that

$$W_1 \leq K_3 S,$$

where S is defined in Theorem 6.5. From Lemma 6.6 we have that

$$W(t_1) \leq K_2[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2] = K_2 S(t_1),$$
$$W(t_2) \leq K_2[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + (w_1 - w_2)^2] = K_2 S(t_2).$$

Using this in inequality (6.36), we have

$$S(t_2) \leq S(t_1)\exp[-K_5(t_2 - t_1)] \tag{6.37}$$

for $t_2 \geq t_1$.

As $t \to \infty$, we have from inequality (6.36) that

$$\dot{W} \leq 0.$$

Also from inequality (6.37),

$$S(t_2) \to 0 \text{ as } t_2 \to \infty.$$

This implies that

$$x_2(t) - x_1(t) \to 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \to 0,$$
$$\ddot{x}_2(t) - \ddot{x}_1(t) \to 0, \qquad \dddot{x}_2(t) - \dddot{x}_1(t) \to 0.$$

Hence the proof of Theorem 6.5 is complete.

6.3. Convergence in $x^{(4)} + f(\ddot{x}) + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$

Tunc (2010) considered fourth order non-linear non homogeneous differential equation of the form of:

$$x^{(4)} + f(\ddot{x}) + b\ddot{x} + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}). \tag{6.38}$$

Where b is a positive constant, the functions $f, g, h,$ and p are real valued and continuous, and the dots denote differentiation with respect to t. Moreover, $f(0) = g(0) = h(0) = 0$. By using Lyapunov's second method, (Tunc, 2010) established the convergence of solutions in the case where the functions $f, g,$ and h are not necessarily differentiable.

The main convergence results of (Tunc, 2010) are the following:

Theorem 6.8. In addition to the fundamental conditions imposed on $f, g, h,$ and p, assume that

(i) there are positive constants a, a_0 such that

$$a \leq \frac{f(w_2) - f(w_1)}{w_2 - w_1} \leq a_0, \qquad w_1 \neq w_2, \tag{6.39}$$

(ii) there are positive constants c, c_0 such that

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq c_0 \qquad y_1 \neq y_2, \tag{6.40}$$

and

$$abc > c_0^2,$$

(iii) there are constants $\Delta_0 > 0, k < 1$ such that for any ξ, η $(\eta \neq 0)$ the incriminatory ratio for h satisfies the relation

$$\frac{\{h(\xi + \eta) - h(\xi)\}}{\eta} \in I_0, \tag{6.41}$$

where I_0 is defined as

$$I_0 \equiv \left[\Delta_0, \frac{k(ab - c)c}{a^2}\right],$$

(iv) there is a continuous function $\emptyset(t)$ such that

$$|p(t, x_2, y_2, z_2, w_2) - p(t, x_1, y_1, z_1, w_1)| \leq \emptyset(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|\},$$

holds for arbitrary $t, x_1, y_1, z_1, w_1, x_2, y_2, z_2$ and w_2.

If there exist a constant D_1 such that

$$\int_0^t \emptyset^v(\tau)d\tau \leq D_1 t, \tag{6.42}$$

for some v, in the range $1 \leq v \leq 2$, then all solutions of (6.38) converge.

Theorem 6.9. Assume that the conditions of Theorem 6.8 are satisfied. Let $x_1(t)$, $x_2(t)$ be any two solutions of (6.38). Then, for every fixed v, in the range $1 \leq v \leq 2$, there exists constants D_2, D_3 and D_4 such that for $t_2 \geq t_1$,

$$S(t_2) \leq D_2 S(t_1) exp\left\{-D_3(t_2 - t_1) + D_4 \int_{t_1}^{t_2} \emptyset^v(\tau)d\tau\right\}, \tag{6.43}$$

where

$$S(t) = [x_2(t) - x_1(t)]^2 + [\dot{x}_2(t) - \dot{x}_1(t)]^2 + [\ddot{x}_2(t) - \ddot{x}_1(t)]^2 + [\dddot{x}_2(t) - \dddot{x}_1(t)]^2. \tag{6.44}$$

It is convenient here to consider equation (6.38) as the following equivalent system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= w, \\ \dot{w} &= -f(w) - bz - g(y) - h(x) + p(t, x, y, z, w). \end{aligned} \tag{6.45}$$

Let $\left(x_i(t), y_i(t), z_i(t), w_i(t)\right)$, $(i = 1,2)$ be two solutions of the system (6.45) such that the inequalities (6.39) and (6.40) hold and

$$\Delta_0 \leq \frac{h(x_2) - h(x_1)}{x_2 - x_1} \leq \frac{k(ab - c)c}{a^2},$$

where c, c_0, Δ_0, k are as defined in the previous.

The basic tool in the proof of the convergence of the solutions will be the Lyapunov function

$$\begin{aligned} 2V = {} & c^2\varepsilon(1 - \varepsilon)x^2 + ac[(D - 1) + \varepsilon]y^2 + 2c[\varepsilon + (D - 1)]yz \\ & + \varepsilon D w^2 + b(D - 1)z^2 + 2\varepsilon a D z w + \varepsilon a^2 D z^2 \\ & + [(1 - \varepsilon)D - 1][az + w]^2 + [c(1 - \varepsilon)x + by + az + w]^2, \end{aligned} \tag{6.46}$$

where $0 < \varepsilon < 1$, $ab - c > \delta > 0$, $ab\varepsilon = \delta$, and

$$D - 1 = \frac{(\delta + c\varepsilon)}{(ab - c - \delta)}.$$

Indeed, one can rearrange the terms in (6.46) to obtain

$$2V = 2V_1 + 2V_2 + 2V_3,$$

where

$$2V_1 = ac[(D-1) + \varepsilon]y^2 + 2c[\varepsilon + (D-1)]yz + b(D-1)z^2,$$
$$2V_2 = \varepsilon a^2 Dz^2 + 2\varepsilon aDzw + \varepsilon Dw^2,$$

$$2V_3 = c^2\varepsilon(1-\varepsilon)x^2 + [(1-\varepsilon)D - 1][az + w]^2 + [c(1-\varepsilon)x + by + az + w]^2.$$

Note that V_3 is obviously positive definite. Also V_1 and V_2 regarded as quadratic forms in $y, z,$ and $w,$ are positive and nonnegative, respectively. Recall that a real 2×2 matrix

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

is positive definite if and only if its symmetric and the elements $a_1,\ a_4$ and $a_1a_4 - a_2a_3$ are non negative. Thus, one can rearrange the terms in V_1 as

$$(y \quad z)\begin{pmatrix} ac[(D-1)+\varepsilon] & c[\varepsilon + (D-1)] \\ c[\varepsilon + (D-1)] & b(D-1) \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}.$$

Hence, it follows that the conditions for positive definiteness are satisfied.

Similarly, V_2 is nonnegative. Hence, V is positive definite. We can therefore find a constant $D_6 > 0$ such that

$$D_6(x^2 + y^2 + z^2 + w^2) \leq V. \tag{6.47}$$

Furthermore, by using the Schwartz inequality

$$|y||z| \leq \frac{1}{2}(y^2 + z^2),$$

one can easily obtain

$$V \leq D_7(x^2 + y^2 + z^2 + w^2), \tag{6.48}$$

where D_7 is a positive constant.

Using the inequalities (6.47) and (6.48), we get

$$D_6(x^2+y^2+z^2+w^2) \le V \le D_7(x^2+y^2+z^2+w^2).$$

That is mean that we shown V is a positive and the following results can easily be verified for $W \equiv V$.

Lemma 6.10. Let a function $W(t) = W(x_2-x_1, y_2-y_1, z_2-z_1, w_2-w_1)$ be defined as follows:

$$\begin{aligned} 2W = {} & c^2\varepsilon(1-\varepsilon)(x_2-x_1)^2 + ac[(D-1)+\varepsilon](y_2-y_1)^2 \\ & +2c[\varepsilon+(D-1)](y_2-y_1)(z_2-z_1) + \varepsilon D(w_2-w_1)^2 \\ & +b(D-1)(z_2-z_1)^2 + 2\varepsilon aD(z_2-z_1)(w_2-w_1) + \varepsilon a^2 D(z_2-z_1)^2 \\ & +[(1-\varepsilon)D-1][a(z_2-z_1)+(w_2-w_1)]^2 \\ & +[c(1-\varepsilon)(x_2-x_1)+b(y_2-y_1)+a(z_2-z_1)+(w_2-w_1)]^2, \end{aligned}$$

where $0 < \varepsilon < 1$ and $W(0,0,0,0) = 0$. Then there exist finite constants $D_6 > 0$, $D_7 > 0$ such that

$$\begin{aligned} & D_6\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2\} \le W \\ & \le D_7\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2+(w_2-w_1)^2\}. \end{aligned} \tag{6.49}$$

If we denote the function $W(t)$ by

$W\big(x_2(t)-x_1(t), y_2(t)-y_1(t), z_2(t)-z_1(t), w_2(t)-w_1(t)\big)$, and using the fact that the solutions $(x_i, y_i, z_i, w_i), (i = 1,2)$, satisfy system (6.45), then $S(t)$ as defined in (6.44) takes the form

$$S(t) = [x_2(t)-x_1(t)]^2 + [y_2(t)-y_1(t)]^2 + [z_2(t)-z_1(t)]^2 + [w_2(t)-w_1(t)]^2,$$

(Tunc, 2010).

Lemma 6.11. Let conditions (i) - (iii) of Theorem 6.8 are satisfied. Then, there exists positive finite constants D_8 and D_9 such that

$$\frac{dW}{dt} \le -2D_8S + D_9S^{\frac{1}{2}}|\theta|, \tag{6.50}$$

where $\theta = p(t, x_2, y_2, z_2, w_2) - p(t, x_1, y_1, z_1, w_1)$ (Tunc, 2010).

Proof of Theorem 6.9. For this purpose, we assume that v is an arbitrary constant in the range $1 \le v \le 2$. Set $\sigma = 1 - \frac{v}{2}$, so that $0 \le \sigma \le \frac{1}{2}$. We rewrite (6.50) in the form

$$\frac{dW}{dt} + 2D_8S \le D_9S^{\sigma}W^*, \tag{6.51}$$

where

$$W^* = S^{\frac{1}{2}-\sigma}\left(|\theta| - D_{10}S^{\frac{1}{2}}\right). \tag{6.52}$$

and

$$D_{10} = \frac{D_8}{D_9}.$$

If $|\theta| < D_{10}S^{\frac{1}{2}}$, then $W^* < 0$.

On the other hand, if $|\theta| > D_{10}S^{\frac{1}{2}}$, then the definition of W^* in (6.52) gives at least

$$W^* = S^{\frac{1}{2}-\sigma}|\theta|$$

and also

$$S^{\frac{1}{2}} \leq \frac{|\theta|}{D_{10}}.$$

Thus,

$$S^{\frac{(1-2\sigma)}{2}} \leq \left[\frac{|\theta|}{D_{10}}\right]^{(1-2\sigma)}.$$

This, together with the relation for W^*, yields

$$W^* \leq D_{11}|\theta|^{2(1-\sigma)},$$

where $D_{11} = D_{10}^{(\sigma-1)}$. Using the estimate for W^* in inequality (6.51), we obtain

$$\frac{dW}{dt} + 2D_8S \leq D_9D_{11}S^{\sigma}|\theta|^{2(1-\sigma)} \leq D_{12}S^{\sigma}\emptyset^{2(1-\sigma)}S^{(1-\sigma)},$$

where $D_{12} > 2D_9D_{11}$, which follows from the relation

$$|p(t,x_2,y_2,z_2,w_2) - p(t,x_1,y_1,z_1,w_1)|$$
$$\leq \emptyset(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|\}.$$

In the view of the fact that $v = 2(1-\sigma)$, we get

$$\frac{dW}{dt} \leq -D_8S + D_{12}\emptyset^{v}S,$$

and using inequalities (6.49), we obtain

$$\frac{dW}{dt} + \left(D_{13} - D_{14}\emptyset^{v}(t)\right)W \leq 0, \tag{6.53}$$

for some constants D_{13} , D_{14}. Integrating (6.53) from t_1 to t_2, where $t_2 \geq t_1$, we get

$$W(t_1) \leq W(t_2)exp\left\{-D_{13}(t_2 - t_1) + D_{14}\int_{t_1}^{t_2} \emptyset^{v}(\tau)d\tau\right\}.$$

Using Lemma 6.10, we obtain (6.43) with $D_2 = D_7 D_6^{-1}$, $D_3 = D_{13}$, and $D_4 = D_{14}$. Theorem 6.9 is proved.

Proof of Theorem 6.8. The proof follows from the estimate (6.43) and the condition (6.42) on $\emptyset(t)$. Choose $D_1 = \frac{D_3}{D_4}$ in (6.42). Then, as $t = (t_2 - t_1) \to \infty$, $S(t) \to 0$, which proves that as $t \to \infty$, we have

$$x_2(t) - x_1(t) \to 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \to 0,$$
$$\ddot{x}_2(t) - \ddot{x}_1(t) \to 0, \qquad \dddot{x}_2(t) - \dddot{x}_1(t) \to 0.$$

The theorem is proved.

6.4. Convergence in $x^{(4)} + \psi(\ddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$

Afuwape in (2012), by using Lyapunov's second method, prove results on the convergence of solutions of the fourth order non-linear differential equation of the form

$$x^{(4)} + \psi(\ddot{x}) + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}), \tag{6.54}$$

in which $\psi(\ddot{x}), f(\ddot{x}), g(\dot{x})$ and $h(x)$ are continuous in their respective arguments. The function $p(t, x, \dot{x}, \ddot{x}, \dddot{x})$ is assumed to have a form $q(t) + r(t, x, \dot{x}, \ddot{x}, \dddot{x})$ with the function $q(t)$ and $r(t, x, \dot{x}, \ddot{x}, \dddot{x})$, depending explicitly on the argument displayed and continuous in their respective arguments, where dot denoted differentiation. Further, we shall assume that $r(t, 0,0,0,0) = 0$ for all t. Also (Afuwape, 2012) assumed that $\psi(0) = f(0) = g(0) = h(0) = 0$.

Theorem 6.12. Suppose that in equation (6.54), $\psi(0) = f(0) = g(0) = h(0) = 0$, and that

(i) there are positive constants $a, a_0, b, b_0, c, c_0, \Delta_0, \xi, \eta$ $(\eta \neq 0)$ such that ψ, f, g, h satisfy

$$\frac{\{h(\xi + \eta) - h(\xi)\}}{\eta} \in I_0$$

where I_0 is defined as

$$I_0 \equiv \left[\Delta_0, \quad \frac{k(ab-c)c}{a^2}\right], \qquad \Delta_0 > 0, \ \ k < 1,$$

$$a \leq \frac{\psi(\omega_2)-\psi(\omega_1)}{\omega_2-\omega_1} \leq a_0, \qquad \omega_1 \neq \omega_2,$$

$$b \leq \frac{f(z_2)-f(z_1)}{z_2-z_1} \leq b_0, \qquad z_1 \neq z_2,$$

$$c \leq \frac{g(y_2)-g(y_1)}{y_2-y_1} \leq c_0, \qquad y_1 \neq y_2,$$

with $abc > c_0^2$,

(ii) there is a continuous function $\emptyset(t)$ such that

$$|r(t,x_2,y_2,z_2,\omega_2+Q(t)) - r(t,x_1,y_1,z_1,\omega_1+Q(t))| \leq \emptyset(t)\{|x_2-x_1|+|y_2-y_1|+|z_2-z_1|+|\omega_2-\omega_1|\},$$

holds for arbitrary $t, x_1, y_1, z_1, \omega_1, x_2, y_2, z_2$ and ω_2.

If there exist a constant D_1 such that

$$\int_0^t \emptyset^{\alpha}(\tau)d\tau \leq D_1 t, \tag{6.55}$$

for some α, in the range $1 \leq \alpha \leq 2$, then all solutions of (6.54) are converge (Afuwape, 2012).

Theorem 6.13. Let $x_1(t)$, $x_2(t)$ be any two solutions of equation (6.54). Suppose that all the conditions of Theorem 6.12 are satisfied, then for each α, with $1 \leq \alpha \leq 2$, there exist constants D_2, D_3, and D_4 such that for $t_2 \geq t_1$,

$$S(t_2) \leq D_2 S(t_1) exp\left\{-D_3(t_2-t_1) + D_4 \int_{t_1}^{t_2} \emptyset^{\alpha}(\tau)d\tau\right\}, \tag{6.56}$$

where

$$S(t) = [x_2(t)-x_1(t)]^2 + [\dot{x}_2(t)-\dot{x}_1(t)]^2 + [\ddot{x}_2(t)-\ddot{x}_1(t)]^2 + [\dddot{x}_2(t)-\dddot{x}_1(t)]^2$$

(Afuwape, 2012).

The proof of the main results will follow the setting used in (Afuwape, 1988).

Let

$$Q(t) = \int_0^t q(s)ds.$$

An equivalent system of equation (6.54) is

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= w + Q(t),\\
\dot{w} &= -\psi(w + Q(t)) - f(z) - g(y) - h(x) + p\big(t, x, y, z, w + Q(t)\big). \qquad (6.57)
\end{aligned}$$

Let $\big(x_i(t), y_i(t), z_i(t), \omega_i(t)\big)$, $(i = 1,2)$, be two solutions of system (6.57), satisfy the condition (i) of Theorem 6.12.

The main tool in the proof of Afuwape (2012) is the following Lyapunov function $V = V(x, y, z, \omega)$ defined by

$$\begin{aligned}
2V = {} & c^2\varepsilon(1-\varepsilon)x^2 + ac\varepsilon(1-\varepsilon)Dy^2 + b(D-1)z^2 + \varepsilon D\omega^2 + 2c[\varepsilon + (D-1)]yz\\
& + [(1-\varepsilon)D - 1][az + \omega]^2 + [c(1-\varepsilon)x + by + az + \omega]^2, \qquad (6.58)
\end{aligned}$$

where $0 < \varepsilon < 1$, $ab - c > \delta > 0$, $ab\varepsilon(2-\varepsilon) = \delta$,

$$D = \frac{(ab - c + c\varepsilon)}{(ab - c - \delta)}, \qquad \text{and} \qquad D - 1 = \frac{(\delta + c\varepsilon)}{(ab - c - \delta)}.$$

Lemma 6.14. The function V defined in (6.58) is positive definite. Moreover, there exist constants D_6, D_7 such that

$$D_6(x^2 + y^2 + z^2 + \omega^2) \le V \le D_7(x^2 + y^2 + z^2 + \omega^2) \qquad (6.59)$$

hold.

Proof. Clearly by choosing

$$2D_6 = min\{c^2\varepsilon(1-\varepsilon), ac\varepsilon(1-\varepsilon)D, b(D-1), \varepsilon D\}$$

we have the left part of inequality (6.59), $D_6(x^2 + y^2 + z^2 + \omega^2) \le V$.

On the other hand, if we rewrite $2V$ as

$$2V = c^2\varepsilon(1-\varepsilon)x^2 + [ac\varepsilon(1-\varepsilon)D + b^2]y^2 + [b(D-1) + a^2(1-\varepsilon)D]z^2$$
$$+D\omega^2 + 2bc(1-\varepsilon)xy + 2[c(\varepsilon + D - 1) + ab]yz$$
$$+ 2a(1-\varepsilon)Dz\omega + 2c(1-\varepsilon)x\omega + 2by\omega + 2ac(1-\varepsilon)xz$$

and realize that by Schwartz inequality $2|u||v| \leq u^2 + v^2$, we have the right part of the inequality (6.59) for some $D_7 > 0$, with $D_7 \equiv D_7(a, b, c, \varepsilon, D)$, such that

$$V \leq D_7(x^2 + y^2 + z^2 + \omega^2).$$

Then, this inequality completes the proof of Lemma 6.14 .

Also, Afuwape (2012) showed another properties of function V, for any two distinct solutions $x_1(t), x_2(t)$ of equation (6.54) bysetting

$$W = W(t) = V(x_2 - x_1, y_2 - y_1, z_2 - z_1, \omega_2 - \omega_1).$$

It is clear that $W(t)$ satisfies the result of Lemma 6.14, and after some simplifications that the derivative of W, with respect to t, is

$$\frac{dW}{dt} \leq -2D_{10}S + D_{11}S^{\frac{1}{2}}|\theta|,$$

where D_{10}, D_{11} are positive constants and

$$|\theta| = |r(t, x_2, y_2, z_2, w_2 + Q) - r(t, x_1, y_1, z_1, w_1 + Q)|$$

(Afuwape, 2012).

Proof of Theorem 6.12. The proof follows from estimate (6.56) and condition (6.55) on $\emptyset(t)$. Then, as $t \equiv (t_2 - t_1) \to +\infty$, $S(t) \to 0$, which proves that as $t \to \infty$, we have

$$x_2(t) - x_1(t) \to 0, \qquad \dot{x}_2(t) - \dot{x}_1(t) \to 0,$$
$$\ddot{x}_2(t) - \ddot{x}_1(t) \to 0, \qquad \dddot{x}_2(t) - \dddot{x}_1(t) \to 0.$$

This completes the proof of Theorem 6.12.

Moreover, and in the addition of the work of (Afuwape, 2012), also in the same year Adesina and Ogundare together investigated the convergence of solutions of the equation (6.54).

(Adesina and Ogundare, 2012) established the following results.

Theorem 6.15. In addition to the basic assumptions, suppose that $\psi(0) = f(0) = g(0) = h(0) = 0$, the function $p(t,x,\dot{x},\ddot{x},\dddot{x}) = q(t) + r(t,x,\dot{x},\ddot{x},\dddot{x})$ with $r(t,0,0,0,0) = 0$ for all t, and that

(i) for some positive constants a, b, c, d, $(ab-c)c - a^2 d > 0$ and $ab - c > 0$,

(ii) for some positive constants $a, b, c, d, D, \Delta_0, \Delta_1, K_0$ and K_1, the intervals

$$I_0 \equiv \left[\Delta_0 \ , \ \frac{k(ab-c)c}{a^2}\right], \qquad I_1 \equiv \left[\Delta_1 \ , \ \frac{k(a^2 d + c^2)}{ac}\right],$$

are in the Ruth-Hurwitz interval,

(iii) there are positive constants $a, a_0, b, b_0, c,$ and c_0, such that

$$a \leq \frac{\psi(\omega_2) - \psi(\omega_1)}{\omega_2 - \omega_1} \leq a_0, \qquad \omega_1 \neq \omega_2,$$

$$b \leq \frac{f(z_2) - f(z_1)}{z_2 - z_1} \leq b_0, \qquad z_1 \neq z_2,$$

$$c \leq \frac{g(y_2) - g(y_1)}{y_2 - y_1} \leq c_0, \qquad y_1 \neq y_2,$$

(iv) for any ξ, η $(\eta \neq 0)$ the increment ratios for h and g satisfy

$$\frac{h(\xi+\eta) - h(\xi)}{\eta} \in I_0, \qquad \frac{g(\xi+\eta) - g(\xi)}{\eta} \in I_1,$$

(v) there is a continuous function $\emptyset(t)$ such that

$$|r(t,x_2,y_2,z_2,\omega_2) - r(t,x_1,y_1,z_1,\omega_1)| \leq \emptyset(t)\{|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |\omega_2 - \omega_1|\},$$

holds for arbitrary $t, x_1, y_1, z_1, \omega_1, x_2, y_2, z_2$ and ω_2.

If there exist a constant D_1 such that

$$\int_0^t \emptyset^{\alpha}(\tau) d\tau \leq D_1$$

for some α, in the range $1 \leq \alpha \leq 2$, then all solutions of (6.54) are converge (Adesina and Ogundare, 2012).

Also, (Adesina and Ogundare, 2012) showed and improved that Theorem 6.13 and Lemma 6.14 of (Afuwape, 2012) holds and results are true.

(Adesina and Ogundare, 2012) considered equation (6.54) as the equivalent system, for this supposed that

$$Q(t) = \int_0^t q(s)ds.$$

An equivalent system of equation (6.54) is

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= w,\\
\dot{w} &= -\emptyset(w) - f(z) - g(y) - h(x) + r(t,x,y,z,w) + Q(t). \qquad (6.60)
\end{aligned}$$

Adesina and Ogundare (2012) used a main tool in the proofs of convergence results as the following Lyapunov's function $V = V(x, y, z, \omega)$ defined by

$$2V = [\beta(1-\varepsilon)x + \gamma y + \delta z + \omega]^2 + [(1-\varepsilon)D - 1][\delta z + \omega]^2$$

$$\beta D[\varepsilon + (1-\varepsilon)D - 1]y^2 + \gamma(D-1)z^2 + \varepsilon D\omega^2$$

$$\beta^2\varepsilon(1-\varepsilon)x^2 + 2\gamma\delta[(1-\varepsilon)^2 D - 1]yz,$$

where

$0 < \varepsilon < 1-\varepsilon < 1$, $\frac{\gamma\delta}{\beta} > 1-\varepsilon$, β, γ, δ are positive real numbers and

$$D = 1 + \frac{\beta(1-\varepsilon)[\gamma\delta - \beta(1-\varepsilon)]}{\gamma\delta - \beta\varepsilon}, \qquad \text{with} \qquad D > \frac{1}{(1-\varepsilon)^2}.$$

In addition to the others results (Adesina and Ogundare, 2012) established the following results.

Lemma 6.16. The Lyapunov function $2V$ defined above is positive definite and moreover, there exist constants D_6 and D_7 such that

$$D_6(x^2 + y^2 + z^2 + w^2) \leq V \leq D_7(x^2 + y^2 + z^2 + w^2)$$

(Adesina and Ogundare, 2012).

Lemma 6.17. Let the hypotheses of Theorem 6.15 hold, and let the function $W(t) = W(x_2 - x_1, y_2 - y_1, z_2 - z_1, w_2 - w_1)$ be defined by

$$2W = [\beta(1-\varepsilon)(x_2 - x_1) + \gamma(y_2 - y_1) + \delta(z_2 - z_1) + (w_2 - w_1)]^2$$

$$+[(1-\varepsilon)D - 1][\delta(z_2 - z_1) + (w_2 - w_1)]^2 + \beta D[\varepsilon + (1-\varepsilon)D - 1](y_2 - y_1)^2$$

$$+\gamma(D-1)(z_2 - z_1)^2 + \varepsilon D(w_2 - w_1)^2 + \beta^2\varepsilon(1-\varepsilon)(x_2 - x_1)^2$$

$$+2\gamma\delta[(1-\varepsilon)^2 D - 1](y_2 - y_1)(z_2 - z_1).$$

Then there exist positive constants D_8, D_9 such that

$$\frac{dW}{dt} \leq -2D_8 S + D_9 S^{\frac{1}{2}}|\theta|,$$

where

$$|\theta| = |r(t, x_2, y_2, z_2, w_2) - r(t, x_1, y_1, z_1, w_1)|$$

(Adesina and Ogundare, 2012).

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APPENDIX INDEX

1. ÖZET

DÖRDÜNCÜ MERTEBEDEN DİFERANSİYEL DENKLEMLERDE KARARLILIK, SINIRLILIK VE YAKINSAKLIK

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Ağustos 2016, 95 sayfa

Bu tez çalışmasının amacı, dördüncü mertebeden lineer olmayan belli formadaki diferansiyel denklemlerin çözümlerinin kararlılık, sınırlılık ve yakınsaklığı ile ilgili olarak literatürde yapılmış bulunan bazı çalışmaları araştırmacıların dikkatine sunmaktır. Bu tez çalışması altı bölümden oluşmaktadır. Birinci bölümde, tez konusu ile ilgili literatürde yapılan bazı çalışmalar özetle verildi, ikinci bölümde bu tezde kullanılacak materyal ve yöntem belirtildi. Üçüncü bölümde teze ait temel kavramlar, tanım ve teoremler sunuldu. Dördüncü bölümde dördüncü mertebeden lineer olmayan belli formadaki diferansiyel denklemlerin çözümlerinin karalılığı ve asimptotik davranışları ile ilgili olarak literatürde yer alan bazı çalışmalar ele alındı. Beşinci bölümde ise, dördüncü mertebeden lineer olmayan belli formadaki diferansiyel denklemlerin çözümlerinin sınırlılık davranışları ile ilgili olarak literatürde bulunan bazı çalışmalar sunuldu. Son olarak, altıncı bölümde ise, dördüncü mertebeden lineer olamayan belli formadaki diferansiyel denklemlerin çözümlerinin yakınsaklık davranışları ile ilgili olarak literatürde yer alan bazı çalışmalar verildi. Bu tez orijinal ve yeni sonuç içermemektedir. Bu tezdeki amacımız literatürde söz edilen konuları bir arada araştırmacıların dikkatine sunmaktır.

Anahtar kelimeler: Asimptotik davranış, Diferansiyel denklem, Dördüncü mertebe, Lyapunov fonksiyon, Kararlılık, Sınırlılık, Yakınsaklık.

2. GİRİŞ

Diferansiyel denklemlerin teori ve uygulamasında adi diferansiyel denklemlerin çözümlerinin kararlılık, sınırlılık ve yakınsaklığının çok önemli problemler olduğu bilinir. Şimdiye kadar, lineer olmayan diferansiyel denklemlerin çözümlerinin kararlılık, sınırlılık ve yakınsaklığı çalışmasında en etkili ve önemli metot Lyapunov'un direk (veya ikinci) metodudur. İlgili literatürde, bu metodu kullanarak, dördüncü mertebeden lineer olmayan diferansiyel denklemlerde birçok kararlılık, sınırlılık ve yakınsaklık sonuçları kurulmuştur. Bahsedilen konu hakkında yapılan çalışmalar için, iyi bir araştırma olarak (Ahmad ve Rao, 1999), (Yoshizawa, 1966), (Reissig ve ark., 1974) kitaplarına, (Ross, 1984) ve (Ezeilo, 1962), (Harrow, 1970), (Ezeilo ve Tejumola, 1971), (Tunc ve Ates, 2013), (Tunc, 1995; 2001; 2004; 2005; 2006; 2007), (Omeike, 2007), (Tejumola, 1968; 1972), (Ogundare and Okecha, 2007; 2008), (Ogurtsov, 1959), (Asmussen, 1971), (Hara, 1974), (Skrapek ve Lalli, 1977), (Yu Yuan-hong, 1990), (Wu ve Xiong, 1998), (Adesina ve Ogundare, 2012), (Afuwape, 1981; 1988; 2012), (E. Tunc, 2010), makalelerine ve onların referanslarına bakılabilir.

3. MATERYAL ve YÖNTEM

Bu tez çalışmasında; giriş ve literatür bildirişlerine bakıldığında, ilgili literatürde dördüncü mertebeden lineer olmayan diferansiyel denklemlerin çözümlerinin, niteliksel davranışları, kararlılık, sınırlılık, yakınsaklık vb. için çok sayıda kayda değer bilimsel çalışmanın yapıldığı görülmektedir. Bu tezde ele alınan çalışmaların tamamında yöntem olarak Lyapunov'un ikinci (ya da doğrudan) metodu (yöntemi) kullanılmaktadır. Ancak ele alınan diferansiyel denklem çözülebiliyorsa ya da çözümleri tam olarak elde edilebiliyorsa, yöntem olarak ilgili kararlılık, sınırlılık, yakınsaklık tanımlarının yanı sıra Lyapunov'un ikinci (ya da doğrudan) metodu (yöntemi) da araştırmacıya aynı sonucu ispatlamaya imkan verir. Ancak, bu yöntem kullanıldığında, lineer olmayan diferansiyel denklemlerin kararlılık, sınırlılık ve yakınsaklıkları için kullanılan teoremlerin şartlarını oluşturma esnasında, Routh – Hurwitz şartları esas alınmaktadır. Lineer olmayan denklemler için şartların oluşturulmasında, sabit katsayılı denklemler için dikkate alınan şartların lineer olmayan genellemeleri belli kısıtlamalar altında yapılmaktadır. Buna bağlı olarak da ispat verilmektedir. Tanım kullanma, literatürde Lyapunov'un birinci metodu (yöntemi) olarak da bilinmektedir. A.M. Lyapunov bir Rus matematikçisidir. Bu yöntemi 1890'lı yıllarda ortaya atmıştır. Bu gün bu yöntem yardımıyla bilimsel literatürde çok sayıda kayda değer çalışma yapılmaktadır. Yani, yöntem bu gün aktif olarak kullanılmaktadır. Bu tezde kullanılacak materyel ise, yapılan çalışmanın teorik olması nedeni ile uluslarası makaleler, yayınlanmış kitaplar ve benzer bilimsel çalışmalar olarak düşünülebilir. Bu kaynakların büyük bir kısmına Yüzüncü Yıl Üniversitesi'ndeki bilimsel veri tabanları yardımıyla ulaşabilmekteyiz. İspatların yapılmasında, Lyapunov'a atfedilen teoremler esas alınmaktadır.

CURRICULUM VITAE

Preparer of this Thesis ADNAN AHMAD MAHMUD, born in 1-1-1979 at DUKAN-SULAYMANYAH- IRAQ, started primary and secondary school in his born place and completed, after that in 1998 started the study in Department of Mathematics Collage of science education in University of SALLAHADIN-ERBIL at 2002 finished it, from 2002 to 2014 worked as a teacher in secondary school in my town, in 2014 received an acceptation from Yuzuncu Yil University and in September 2014 started study master degree.