THE MONOMIAL BURNSIDE FUNCTOR

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Given a finite group G, we can realize the permutation modules by the linearization map defined from the Burnside ring $B(G)$ to the character ring of G, denoted $A_{\mathbb{K}}(G)$. But not all $\mathbb{K}G$-modules are permutation modules. To realize all the $\mathbb{K}G$-modules we need to replace $B(G)$ by the monomial Burnside ring $B_C(G)$. We can get information about monomial Burnside ring of G by considering subgroups or quotient groups of G. For this the setting of biset functors is suitable. We can consider the monomial Burnside ring as a biset functor and study the elemental maps: transfer, retriction, inflation, deflation and isogation. Among these maps, deflation is somewhat difficult and requires more consideration. In particular, we examine deflation for p-groups and study the simple composition factors of the monomial Burnside functor for 2-groups with the fibre group $\{\pm 1\}$.

Keywords: monomial Burnside functor, monomial Burnside ring.

ÖZET

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Sonlu grup G için permütasyon modüllerini $B(G)$ ile gösterilen Burnside halkasından, $A_{\mathbb{K}}(G)$ ile gösterilen karakter halkasına, tanımlanmış olan doğrusallaştırma fonksiyonu ile elde edebiliriz. Fakat bütün $\mathbb{K}G$-modülleri permütasyon modülü değildir. Bütün $\mathbb{K}G$-modüllerini elde etmek için $B(G)$'yi $B_C(G)$ ile ifade edilen tekil Burnside halkasıyla değiştirmek gerekir. G'nin alt gruplarına ve bölüm gruplarına bakarak $B_C(G)$ hakkında bilgi toplayabiliriz. Bunun için ikili küme izlecinin düzenlemesi uygundur. Tekil Burnside halklarını ikili küme izleci olarak düşünebiliriz. Özel olarak p-gruplar için deflasyon fonksiyonu ve 2-grupları için tekil Burnside izlecinin basit izleçleri incelendi.

Anahtar sözcükler: tekil Burnside izleci, tekil Burnside halkası.

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Chapter 1

Introduction

Biset functors introduced by Bouc [4] are equipped with the five elemental maps: transfer, restriction, inflation, deflation and a transfer of structure called isogation. Classical examples include ordinary representation rings and the Burnside ring $B(G)$, which is the Grothendieck ring for the category of G-sets.

We shall be considering Dress's monomial Burnside ring $B_C(G)$ where points of G-sets are replaced by fibres that are copies of a cyclic group C. The motive for this extension is that, embedding C as roots of unity in a characteristic zero field $\mathbb{K}$, Brauer's Induction Theorem says that the linearization morphism $B_C \to A_{\mathbb{K}}$ is an epimorphism when $\mathbb{K}$ is algebraically closed. The power of the theory of biset functors is that composition structures can be examined because the simple functors over $\mathbb{K}$ can be classified. They have the form $S_{H,V}$ where H is a group up to isomorphism and V is a $\mathbb{K}\mathrm{Out}(H)$-module. In [5] it is proved that, for p-groups the simple composition factors of $\mathbb{K}B$ are $S_{C_p \times C_p, \mathbb{K}}$ and $S_{1,\mathbb{K}}$. Our concern is with extension to the monomial Burnside functor $\mathbb{K}B_C$.

In Section 2.1 we recall some properties of monomial Burnside rings from Barker [2]. In Section 2.2 we define biset functors and the five elemental maps as discussed in Barker [1]. In chapter 3, we define a biset action on the monomial Burnside ring and examine the action of the five elemental maps on the transitive C-fibred G-sets and the primitive idempotents of $B_C(G)$, which is also studied by

R. Boltje and O. Coskun in [6]. Among the elemental maps, deflation is somewhat difficult. In the case of p-groups the calculation of the deflation map and the simple composition factors of B_C where C is cyclic of prime order, reduces to the case of elementary abelian groups. For p-groups, using p-binomial coefficients and the explicit evaluation of the Möbius function on the subgroup poset of p-groups, we give an explicit formula for the deflation map. In the case $p = 2$ the simple composition factors of $B_{\mathbb{R}}$, i.e B_C where $C = \{\pm 1\}$, are calculated.

Throughout the paper R denotes a commutative unital ring, $\mathbb{K}$ a field of characteristic zero, $\mathbb{K}^{\omega}$ the unit torsions of $\mathbb{K}$, G a finite group, C a cyclic group and the square bracket denotes the isomorphism class of the relevant object.

Chapter 2

Monomial Burnside rings and biset functors

2.1 Monomial Burnside rings

Given a finite group G and a cyclic group C, a C-fibred G-set is a C-free $C \times G$-set with finitely many orbits. For C-fibred G-sets CX and CY, let $[CX]$ and $[CY]$ denote the isomorphism classes of these C-fibred G-sets . Tensor product of two C-fibred G-sets, $CX \otimes CY$, is defined to be the C-orbits of $CX \times CY$ under C action given by

$$c(\alpha \times \beta) = c\alpha \times c^{-1}\beta$$

where $\alpha \times \beta \in CX \times CY$ and $c \in C$. Given $\xi \otimes \eta \in CX \otimes CY$, $c \in C$ acts such that $c\xi \otimes \eta = \xi \otimes c\eta$. We can also consider the disjoint union of two C-fibred G-sets, $CX \amalg CY$, which is regarded in an evident way as a C-fibred G-set . The monomial Burnside ring for G with the fibre group C, denoted by $B_C(G)$, is defined to be the ring, generated by the isomorphism classes of finite C-fibred G-sets, with multiplication and addition given by

$$[CX][CY] = [CX \otimes CY] \text{ and } [CX] + [CY] = [CX \amalg CY].$$

The identity element is the single fibre with trivial action and the zero element is the empty set.

To understand the additive structure of the monomial Burnside ring we need to consider transitive C-fibred G-sets. Since C-action and G-action on CX commute, CX is transitive as a C-fibred G-set if and only if $C\backslash CX$ is transitive as a G-set. Hence C-orbits of CX corresponds to transitive G-sets G/U for some subgroup U of G. Suppose U stabilizes the fibre Cx in CX, i.e. $ux = cx$ for some $c \in C$ and define $\mu\colon U \to C$ such that $\mu(u) = c$ then μ is a group homomorphism. Hence action of U on fibres is given by the linear character μ. Transitive C-fibred G-sets are denoted by $C_\mu G/U$ and the pair (U,μ) is called a C-subcharacter of G. C-subcharacters of G admit a G action by conjugation $g(U,\mu) = ({}^gU, {}^g\mu)$.

For subgroups $V \leq G \geq W$ and the linear characters $\nu\colon V \to C$ and $\omega\colon W \to C$, let $\nu.\omega$ denote the C-linear character of $V \cap W$ defined by $u \to \nu(u)\omega(u)$. Given C-subcharacters (V,ν) and (W,ω), we have [2]

$$[C_\nu G/V][C_\omega G/W] = \sum_{VgW \subseteq G} [C_{\nu.{}^g\omega} G/V \cap^g W]$$

where the summation runs over the representatives of double cosets of V and W in G.

Let $\mathrm{ch}(C,G) = \{(U,\mu) : U \leq G, \mu \in \widehat{U}\}$ denote C-subcharacters of G where $\widehat{U}$ =Hom(U,C). There is an evident bijective correspondence between the G-orbits of C-subcharacters of G and isomorphism classes of transitive C-fibred G-sets. Hence as an abelian group

$$B_C(G) = \bigoplus_{(U,\mu)\in_G \mathrm{ch}(C,G)} \mathbb{Z}[C_\mu G/U].$$

This equation also holds if we replace $\mathbb{Z}$ by the field $\mathbb{K}$. We denote $\mathbb{K} \otimes_{\mathbb{Z}} B_C(G)$ by $\mathbb{K}B_C(G)$. Then as $\mathbb{K}$-vector spaces

$$\mathbb{K}B_C(G) = \bigoplus_{(U,\mu)\in_G \mathrm{ch}(C,G)} \mathbb{K}[C_\mu G/U].$$

Let $O(G)$ denote the intersection of the kernels of the linear characters $G \to C$. The quotient $G/O(G)$ is the largest abelian quotient whose exponent divides the order of C. Then we have

$$\mathrm{Hom}(G, C) \cong \mathrm{Hom}(G/O(G), \mathbb{C}) \cong G/O(G).$$

In the next section we will describe the primitive idempotents of $B_C(G)$, for this let us introduce the set $\mathrm{el}(C, G) = \{(H, h)\colon H \leq G, hO(H) \in H/O(H)\}$ called *C-subelements* of G. The group G acts by conjugation on the C-subelements of G, $g(H, h) = ({}^gH, {}^gh)$. For a given subgroup $H \leq G$,

$$|\mathrm{Hom}(H, C)| = |H/O(H)|$$

then $|\mathrm{ch}(C, G)| = |\mathrm{el}(C, G)|$. Furthermore by [2, Lemma 3.3] we have $|G\backslash \mathrm{el}(C, G)| = |G\backslash \mathrm{ch}(C, G)|$. We will see that as the isomorphism classes of transitive C-fibred G-sets in $\mathbb{K}B_C(G)$ are parametrized by the G-conjugacy classes of $\mathrm{ch}(C, G)$, the primitive idempotents of $\mathbb{K}B_C(G)$ are parametrized by the G-conjugacy classes of $\mathrm{el}(C, G)$.

2.1.1 The primitive idempotents of the monomial Burnside ring

The idempotents of the Burnside algebra $\mathbb{Q}B(G)$ were calculated independently by Glück [7] and Yoshida [8]. We can express the connection between the basis $\{[G/U]\colon U \leq_G G\}$ and the primitive idempotents of $\mathbb{Q}B(G)$ using the Möbius inversion principle. In the case of the monomial Burnside ring we need to generalize the Möbius inversion. Let us recall some material from Barker [2]. For a proposition S, we define the *Kronecker value* of S to be

$$\lfloor S \rfloor = \begin{cases} 1 & \text{if } S \text{ holds} \\ 0 & \text{if } S \text{ fails} \end{cases}.$$

The *incidence function* of a partially ordered finite set P is the function $\zeta\colon P \times P \to \mathbb{Z}$ defined by the matrix I, whose columns and rows are indexed by P and entries are $I_{H,K} = \lfloor H \leq K \rfloor$ for $H, K \in P$. Note that I is invertable since it is upper-triangular with ones on the diagonal. Then the *Möbius function* of P is defined to be the function $\mu\colon P \times P \to \mathbb{Z}$ given by the inverse matrix I^{-1} .

Let A be an abelian group and θ, ϕ be functions $P \to A$. We define the *totient equation* and the *inversion equation* to be, respectively,

$$\theta(y) = \sum_{x \in P} \phi(x)\zeta(x, y)$$

$$\phi(x) = \sum_{y \in P} \theta(y)\mu(y, x).$$

If we write down these equations in the matrix form, we obtain the Möbius inversion principle, which states that the totient equation holds for all $y \in P$ if and only if the inversion equation holds for all $x \in P$. This is called the Möbius inversion principle.

We can generalize the incidence function and the Möbius function to the G-orbits of a G-poset P by defining the G-invariant functions

$$\zeta_G(x, y) = \sum_{x' =_G x} \zeta(x', y) \text{ and } \mu_G(x, y) = \sum_{x' =_G x} \mu(x', y)$$

where x' runs over elements in P which are G equivalent to x. It is clear that these functions are mutual inverses

$$\sum_{y \in_G P} \zeta_G(x, y)\mu_G(y, z) = \lfloor x =_G y \rfloor = \sum_{y \in_G P} \mu_G(x, y)\zeta_G(y, z).$$

We embed C into the torsion units $\mathbb{K}^\omega$ of $\mathbb{K}$. Now we can define a *monomial incidence function*

$$\zeta \colon \mathrm{el}(C, G) \times \mathrm{ch}(C, G) \to \mathbb{K} \text{ where } \zeta(H, h; V, \nu) = \nu(h)\zeta(H, V)$$

and a *monomial Möbius function*

$$\mu \colon \mathrm{ch}(C, G) \times \mathrm{el}(C, G) \to \mathbb{K} \text{ where } \mu(V, \nu; H, h) = \nu^{-1}(V \cap hO(H))\mu(V, H)/|V|$$

where ν^{-1} is the inverse of ν in $\widehat{V}$ and $\nu^{-1}(V \cap hO(H)) = \sum_{x \in V \cap hO(H)} \nu^{-1}(x)$. Similarly as before, we define G-invariant functions

$$\zeta_G(H, h; V, \nu) = \sum_{(H', h') \in [H, h]_G} \zeta(H', h'; V, \nu)$$

$$\mu_G(V, \nu; H, h) = \sum_{(V', \nu') \in [V, \nu]_G} \mu(V', \nu'; H, h)$$

where square bracket denotes the set of G-classes under the conjugation action of G. Next result is from Barker [2].

Theorem 2.1.1. (Monomial Möbius inversion). *Given G-invariant functions $\theta\colon \mathrm{ch}(C,G) \to A$ and $\phi\colon \mathrm{el}(C,G) \to A$, then the totient equation*

$$\theta(V,\nu) = \sum_{(H,h)\in_G \mathrm{el}(C,G)} \phi(H,h)\zeta_G(H,h;V,\nu)$$

holds for all $(V,\nu) \in \mathrm{ch}(C,G)$ if and only if the inversion equation

$$\phi(H,h) = \sum_{(V,\nu)\in_G \mathrm{ch}(C,G)} \theta(V,\nu)\mu_G(V,\nu;H,h)$$

holds for all $(H,h) \in \mathrm{el}(C,G)$

Let us describe the species of $\mathbb{K}B_C(G)$ (the algebra maps from $\mathbb{K}B_C(G)$ to the base field $\mathbb{K}$). The species of $\mathbb{K}B_C(G)$ are parametrized by the C-subelements of G. So for a C-subelement (H,h) and a C-fibred G-set CX we define $S^G_{H,h}[CX] = \sum_{Cx} \chi_x(h)$ where $\chi_x\colon H \to C$ is a linear character of H defined by $\chi_x(h) = c$ such that $hx = cx$ and Cx runs over the fibres of CX stabilized by H. We extend this map $\mathbb{K}$-linearly to $\mathbb{K}B_C(G)$.

To see $S^G_{H,h}$ is an algebra map observe that for C-fibred G-sets $[CX]$ and $[CY]$

$$(\sum_{Cx} \chi_x(h))(\sum_{Cy} \chi_y(h)) = \sum_{Cx\times Cy} \chi_x(h)\chi_y(h) = \sum_{Cx\times Cy} \chi_{xy}(h).$$

hence $S^G_{H,h}([CX])S^G_{H,h}([CY]) = S^G_{H,h}([CX \otimes CY])$. Note that if $C = 1$ then the species defined above reduces to the species of the Burnside algebra $\mathbb{K}B(G)$. Next result is due to Dress [9, Theorem 1′(c)], see also [2].

Lemma 2.1.2 (Dress)**.** *Recall that $\mathbb{K}$ is sufficiently large. Given C-subelements (H,h) and (I,i) of G, then $S^G_{H,h} = S^G_{I,i}$ if and only if $(H,h) =_G (I,i)$. Every species of $\mathbb{K}B_C(G)$ is of the form $S^G_{H,h}$, and the species span the dual space of $\mathbb{K}B_C(G)$.*

Let $e^G_{H,h} \in \mathbb{K}B_C(G)$ be such that $S^G_{I,i}(e^G_{H,h}) = \lfloor (I,i) =_G (H,h) \rfloor$. By Lemma 2.1.2, $\{e^G_{H,h}\colon (H,h) \in_G \mathrm{el}(C,G)\}$ is the set of primitive idempotents of $\mathbb{K}B_C(G)$ and as algebras over $\mathbb{K}$ we have

$$\mathbb{K}B_C(G) = \bigoplus_{(H,h)\in_G \mathrm{el}(C,G)} \mathbb{K}e^G_{H,h}.$$

This shows that $\mathbb{K}B_C(G)$ is semisimple with each summand $\mathbb{K}e^G_{H,h} \cong \mathbb{K}$. In [2], the transformation matrices between the basis $\{[C_\nu G/V] \colon (V,\nu) \in_G \mathrm{ch}(C,G)\}$ and the primitive idempotents $\{e^G_{H,h} \colon (H,h) \in_G \mathrm{el}(C,G)\}$ are calculated.

Theorem 2.1.3. (Idempotent formula). *Recall that $\mathbb{K}$ is sufficiently large. There is a bijective correspondence $e^G_{H,h} \leftrightarrow [H,h]_G$ between the primitive idempotents $e^G_{H,h}$ of $\mathbb{K}B_C(G)$ and the G-conjugacy classes $[H,h]_G$ of C-subelements (H,h) of G. We have*

$$|N_G(H,h)| e^G_{H,h} = \sum_{(V,\nu) \in_G \mathrm{ch}(C,G)} |V| \mu_G(V,\nu;H,h)[C_\nu G/V].$$

Now we can calculate the transformation matrices between the two basis. We express the relation by the equations

$$[C_\nu G/V] = \sum_{(H,h) \in_G \mathrm{el}(C,G)} m_G(H,h;V,\nu) e^G_{H,h} \text{ and}$$

$$e^G_{H,h} = \sum_{(V,\nu) \in_G \mathrm{ch}(C,G)} m^{-1}_G(V,\nu;H,h)[C_\nu G/V].$$

Using Theorem 2.1.3 and Theorem 2.1.1, the transformation matrices are

$$m_G(H,h;V,\nu) = \frac{|N_G(H,h)|}{|V|} \zeta_G(H,h;V,\nu) \text{ and}$$

$$m^{-1}_G(V,\nu;H,h) = \frac{|V|}{|N_G(H,h)|} \mu_G(V,\nu;H,h).$$

2.2 Biset functors

Biset functors were introduced by Bouc [4]. We recall some material from [1]. An *I-J-biset* is defined to be a finite left $I \times J$-set. We denote the action by

$$(i,j)x = ixj^{-1}.$$

Given an I-J-biset X and J-K-biset Y, consider the J-orbits of the $I \times J \times K$-set $X \times Y$, denoted by $X \times_J Y$. For $x \times_J y \in X \times_J Y$ we write

$$(i,k)(x \times_J y) = ix \times_J yk^{-1}.$$

We obtain a product between the Burnside ring of $I \times J$-sets and the Burnside ring of $J \times K$-sets,

$$B(I \times J) \times B(J \times K) \to B(I \times K).$$

This product is described more explicitely, by Bouc [4, 3.2] in terms of the transitive bisets. But first we need to describe the subgroups of a direct product of two groups.

For a subgroup $S \leq I \times J$, let $\uparrow S \leq I$ denote the image of the projection i_1 from S to I and $S \uparrow \leq J$ denote the image of the projection i_2 from S to J. We define normal subgroups $\downarrow S \trianglelefteq \uparrow S$ and $S \downarrow \trianglelefteq S \uparrow$, respectively to be the kernels of the projections i_2 and i_1. The two epimorphisms $\uparrow S / \downarrow S \leftarrow S \to S \uparrow / S \downarrow$ both have kernel $\downarrow S \times S \downarrow$, then

$$\frac{\uparrow S}{\downarrow S} \cong \frac{S}{\downarrow S \times S \downarrow} \cong \frac{S \uparrow}{S \downarrow}$$

Hence we obtain a composite isomorphism $\phi_S \colon \uparrow S / \downarrow S \leftarrow S \uparrow / S \downarrow$. With some further work it is easy to obtain *Goursat's Theorem*, which states that there is a bijective correspondence between the subgroups $S \leq I \times J$ and the quintubles $(I_1, I_2, \phi, J_2, J_1)$ such that $I_2 \trianglelefteq I_1 \leq I$ and $J_2 \trianglelefteq J_1 \leq J$ and ϕ is an isomorphism $I_1/I_2 \leftarrow J_1/J_2$, the correspondence is such that $S \leftrightarrow (\uparrow S, \downarrow S, \phi_S, S\downarrow, S\uparrow)$.

Given subgroups $S \leq I \times J$ and $T \leq J \times K$, the join of S and T is defined to be $S * T = \{(i,k) \in I \times K : \exists j \in J \text{ such that } (i,j) \in S \text{ and } (j,k) \in T\}$. $S * T$ is a subgroup of $I \times K$. Next result, due to Bouc [4, 3.2] , describes the product of two transitive bisets.

Theorem 2.2.1. (Generalized Mackey Product Theorem, Bouc). *Given finite groups I,J,K and subgroups $S \leq I \times J$ and $T \leq J \leq K$, then*

$$\left[\frac{I \times J}{S}\right] \left[\frac{J \times K}{T}\right] = \sum_{S\uparrow .j. \uparrow T \subseteq J} \left[\frac{I \times K}{S *^{(j,1)} T}\right] \tag{2.1}$$

where the notation indicates that j runs over representatives of the double cosets of $S\uparrow$ and $\uparrow T$ in J. The isomorphism class of the $I \times K$-set $(I \times K)/(S *^{(j,1)} T)$ depends only on the double coset $S \uparrow .j. \uparrow T$.

Let χ be a non-empty set of finite groups closed under subquotients up to isomorphism and R be a commutative unital ring. The alchemic algebra $R\Gamma^{\chi}$ for χ over R is defined as an R algebra,

$$R\Gamma^{\chi} = \bigoplus_{I,J\in\chi} R\Gamma(I,J)$$

where $R\Gamma(I,J) = R \otimes_Z B(I \times J)$. The multiplication operation comes from the product described in Theorem 2.2.1. A *biset functor* for χ over R is defined to be a locally unital $R\Gamma^{\chi}$-module, with morphisms defined to be the module homomorphisms. Let L be a biset functor for χ over R, since L is a locally unital $R\Gamma^{\chi}$-module we have

$$L = \bigoplus_{I\in\chi} L(I)$$

as R-modules. An I-J-biset X acts as an R-module homomorphism, $X\colon L(I) \to L(J)$. For details see [1].

In Barker [1] the five elemental maps isogation, transfer, restriction, inflation and deflation are discussed. Let $H \leq G$, transfer and restriction maps are defined by

$$\mathrm{tra}_{G,H} = \left[\frac{G \times H}{\Delta(H)}\right] \text{ and } \mathrm{res}_{H,G} = \left[\frac{H \times G}{\Delta(H)}\right]$$

where $\Delta(H) = \{(h,h) : h \in H\}$. For $N \trianglelefteq G$, inflation and deflation maps are defined by

$$\mathrm{inf}_{G,G/N} = \left[\frac{G \times G/N}{\Delta(G,G/N)}\right] \text{ and } \mathrm{def}_{G/N,G} = \left[\frac{G/N \times G}{\Delta(G/N,G)}\right].$$

Here $\Delta(G,G/N) = \{(g,gN) : g \in G\}$ and $\Delta(G/N,G) = \{(gN,g) : g \in G\}$. For an isomorphism $\theta\colon G \leftarrow G'$ where $G, G' \in \chi$, let $\Delta(G,\theta,G') = \{(\theta(x),x)\colon x \in G'\}$ then isogation map is defined by

$$\mathrm{iso}^{\theta}_{G,G'} = \left[\frac{G \times G'}{\Delta(G,\theta,G')}\right].$$

We call these maps the elemental maps since they generate the alchemic algebra $R\Gamma^{\chi}$. In Bouc [4, 3.3] it is proved that any transitive biset can be written as a product of a transfer, an inflation, an isogation, a deflation and a restriction maps in that order, through some suitable groups. Hence it is enough to consider the action of the five elemental maps instead of a general biset. For example to show that a morphism Φ between two biset functors is a biset morphism, it is enough to show that Φ commutes with the five elemental maps.

The simple biset functors were classified by Bouc [4, Section 4].

Theorem 2.2.2 (Bouc)**.** *Consider the pairs (H, V) such that H is a group in χ and V is a simple $R\mathrm{Out}(H)$-module. Two such pairs (H, V) and (H', V') are deemed to be equivalent provided $H \cong H'$ and the isomorphism $H \to H'$ transports V to V'. There is a bijective correspondence $(H, V) \leftrightarrow S_{H,V}$ between the equivalence classes of pairs (H, V) and the isomorphism classes of simple biset functors $S_{H,V}$. The correspondence is characterized by the condition that, with respect to subquotient relation, H is minimal among the groups J in χ satisfying $S_{H,V}(J) \neq 0$, and furthermore, $S_{H,V}(H) \cong V$ as $R\mathrm{Out}(H)$-modules.*

In the next chapter we will define a biset action on the monomial Burnside ring. Given an I-J-biset X, we will define an R-module homomorphism $RB_C(J) \to RB_C(I)$ and define the monomial Burnside functor.

Chapter 3

The Monomial Burnside functor

In this chapter we will define a biset action on the monomial Burnside ring and examine the action of the five elemental maps on the coordinate modules. Before we define the monomial Burnside functor, we shall introduce some notation. For a finite group G, CG denotes the direct product $C \times G$. A G-set T can be considered as a C-fibred G-set in an evident way, denoted by CT. Hence we have a ring monomorphism

$$\gamma_G^C \colon RB(G) \to RB_C(G) \text{ such that } [T] \to [CT].$$

For a C-fibred G-set Z, we denote the set of C-orbits of $z \in Z$ by Cz. We denote the stabilizer of Cz in G by $\mathrm{Stab}_G(Cz)$. For a $C \times G$-set F, let $(F)_{C\text{-free}}$ denote the largest subset of F on which C acts freely.

In general, given an I-J-biset X and a C-fibred J-set CY, let $CX \times_{CJ} CY$ denote the set of CJ-orbits of the $CI \times CJ$-set $CX \times CY$ under the action

$$(c, j)(\xi \times \eta) = c^{-1} j\xi \times cj\eta.$$

Let $X \otimes_{CJ} Y$ denote the set $(CX \times_{CJ} CY)_{C\text{-free}}$. Hence we have the following action

$$\theta \colon RB(I \times J) \times RB_C(J) \to RB_C(I) \text{ where } \theta(X, CY) = X \otimes_{CJ} Y.$$

We define the *monomial Burnside functor* RB_C for χ over R to be the biset functor

$$RB_C = \bigoplus_{J \in \chi} RB_C(J).$$

Given $I, J \in \chi$, an I-J-biset X and a C-fibred J-set CY, then the element $[X] \in RB(I \times J)$ sends $[CY]$ to the element $[X \otimes_{CJ} Y] \in RB_C(J)$. We need to check that this definition is actually a biset action.

3.1 Well-definedness of the biset action

We shall check the associativity of the biset action. First, let us examine the set $(CX \times_{CJ} CY)_{C\text{-free}}$. The action of J on CY gives rise to the linear character $\chi_\eta \colon Stab_J(C\eta) \to C$ defined by

$$\chi_\eta(j) = c \text{ such that } j\eta = c\eta.$$

More generally for $\xi \times_{CJ} \eta \in CX \times_{CJ} CY$, we can define a linear character $\chi_{\xi\times\eta} \colon Stab_I(C(\xi \times_{CJ} \eta)) \to C$ by

$$\chi_{\xi\times\eta}(i) = c \text{ such that } i\xi \times \eta = j\xi \times j\eta = \xi \times c\eta.$$

for some $j \in J$. This is well-defined, since if there exists j_1, j_2 such that $i\xi \times \eta = j_1\xi \times j_1\eta = j_2\xi \times j_2\eta$ then we have $\xi \times_{CJ} \eta = j_1 j_2^{-1}\xi \times_{CJ} j_1 j_2^{-1}\eta = \xi \times_{CJ} j_1 j_2^{-1}\eta$. Hence $j_1 j_2^{-1} \in \ker(\chi_\eta)$.

Proposition 3.1.1. *Let $(CX \times_{CJ} CY)_{C\text{-free}}$ denote the largest subset of $CX \times_{CJ} CY$ on which C acts freely. Then explicitely*

$$(CX \times_{CJ} CY)_{C\text{-free}} = \{\xi \times_{CJ} \eta \in CX \times_{CJ} CY \colon \mathrm{Stab_J(C\xi)} \cap \mathrm{Stab_J(C\eta)} \subseteq \ker(\chi_\eta)\}$$

Proof. Consider $c \in C$ action on $\xi \times_{CJ} \eta$,

$$c(\xi \times_{CJ} \eta) = c\xi \times_{CJ} \eta = \xi \times_{CJ} c\eta$$

Note that $c(\xi \times_{CJ} \eta) = \xi \times_{CJ} \eta$ if and only if there exists $j \in J$ such that

$$\xi \times_{CJ} c\eta = j\xi \times_{CJ} j\eta.$$

This implies that $j \in Stab_J(C\xi) \cap Stab_J(C\eta)$ and $\chi_\eta(j) = c$. Hence for C-free action $Stab_J(C\xi) \cap Stab_J(C\eta) \subseteq \ker(\chi_\eta)$. $\square$

Now we can check that given a I-J-biset X, J-K-biset Y and C-fibred K-set CZ the following holds

$$(XY)CZ = X(Y(CZ)).$$

Proposition 3.1.2. *Given a I-J-biset X, J-K-biset Y and C-fibred K-set CZ then*

$$(C(X \times_J Y) \times_{CK} CZ)_{C\text{-}free} = (CX \times_{CJ} (CY \times_{CK} CZ)_{C\text{-}free})_{C\text{-}free}$$

as C-fibred I-sets.

Proof. First observe that as $C \times C$-sets $C(X \times_J Y) = CX \times_{CJ} CY$. Let $\alpha \times \xi \times \eta \in CX \times_{CJ} CY \times_{CK} CZ$. We can write

$$(C(X \times_J Y) \times_{CK} CZ)_{C\text{-free}} = \{\alpha \times \xi \times \eta \colon Stab_K(C(\alpha, \xi)) \cap Stab_K(C\eta) \subseteq \ker(\chi_\eta)\}$$

and

$$(CX \times_{CJ} (CY \times_{CK} CZ)_{C\text{-free}})_{C\text{-free}} = \{\alpha \times \xi \times \eta \colon Stab_K(C\xi) \cap Stab_K(C\eta) \subseteq \ker(\chi_\eta)$$
$$\text{and } Stab_J(C\alpha) \cap Stab_J(C(\xi \times \eta)) \subseteq \ker(\chi_{\xi\times\eta})\}.$$

First assume that $\alpha \times \xi \times \eta \in (CX \times_{CJ} (CY \times_{CK} CZ)_{C\text{-free}})_{C\text{-free}}$ and $k \in Stab_K(C(\alpha \times \xi)) \cap Stab_K(C\eta)$. Then

$$\alpha \times_{CJ} k\xi \times_{CK} k\eta = \alpha \times_{CJ} j\xi \times_{CK} c\eta$$

such that $j \in Stab_J(C\alpha)$, $k\xi = j\xi$ and $k\eta = c\eta$. This implies that $j \in Stab_J(C\alpha) \cap Stab_J(C(\xi \times \eta))$ and $c = 1$ hence $k \in \ker(\chi_\eta)$. Other direction follows by a similar argument. Conversely, take $\alpha \times \xi \times \eta \in (C(X \times_J Y) \times_{CK} CZ)_{C\text{-free}}$ and $j \in Stab_J(C\alpha) \cap Stab_J(C(\xi \times \eta))$. Consider the action of j

$$j\alpha \times_{CJ} j\xi \times_{CK} \eta = \alpha \times_{CJ} k\xi \times_{CK} k\eta = \alpha \times_{CJ} k\xi \times_{CK} c\eta$$

such that $k \in Stab_K(C(\alpha \times \xi)) \cap Stab_K(C\eta)$ then $c = 1$. Hence $\alpha \times \xi \times \eta \in (CX \times_{CJ} (CY \times_{CK} CZ)_{C\text{-free}})_{C\text{-free}}$.

$\square$

3.2 Further properties

Obviously, we can apply the material in Section 2.2 to CI-CJ-bisets, i.e finite left $CI \times CJ$-sets. Recall that we have an action

$$\Theta\colon RB(CI \times CJ) \times RB(CJ) \to RB(CI)$$

where a CI-CJ-biset X sends a CI-set Y to the set of CJ-orbits of the $CI \times CJ$-set $X \times Y$.

Let $\overline{C}$ denote the quotient group $C \times C/\Delta(C)$ where $\Delta(C)$ is the diagonal subgroup $\{(c,c)\colon c \in C\}$. Given an I-J-biset X, consider the $CI \times CJ$-set $\overline{C}X = \{\bar{c}x\colon \bar{c} \in \overline{C}$ and $x \in X\}$ with left action. We have an injective R-module homomorphism

$$\Gamma^{C}_{I\times J}\colon RB(I \times J) \to RB(CI \times CJ)$$

such that $[X] \to [\overline{C}X]$.

Lemma 3.2.1. *Consider the cyclic group C as a $C \times C$-set with the action $(a,b)c = acb^{-1}$ and $\overline{C}$ as a $C \times C$-set with the left action. Then $\rho\colon \overline{C} \to C$ defined by $\rho(\overline{(c,t)}) = ct^{-1}$ where $\overline{(c,t)} \in \overline{C}$, is a $C \times C$-isomorphism.*

Proof. It is clear that ρ is a group isomorphism. Let $(a,b) \in C \times C$, then

$$(a,b)\rho(\overline{(c,t)}) = (a,b)ct^{-1} = act^{-1}b^{-1} = \rho((a,b)\overline{(c,t)}).$$

Hence ρ is $C \times C$-invariant. □

A C-fibred J-set CY can be considered as a $C \times J$-set, hence we have an R-module monomorphism

$$\iota^{C}_{J}\colon RB_C(J) \to RB(CJ).$$

In the next proposition, for simplicity of notation we treat this morphism as an inclusion and we omit it from the notation.

Proposition 3.2.2. *Given finite groups I,J, a cyclic group C and commutative unital ring R , consider the composition of R-module homomorphisms*

$$RB(I\times J)\times RB_C(J) \xrightarrow{\Gamma^C_{I\times J}} RB(CI\times CJ)\times RB(CJ) \xrightarrow{\Theta} RB(CJ) \xrightarrow{\pi} RB_C(J)$$

where $\pi(CZ)=(CZ)_{C\text{-}free}$. Then $\pi\circ\Theta\circ\Gamma^C_{I\times J}=\theta$.

Proof. Follows from Lemma 3.2.1. □

For a transitive C-fibred G-set $C_\mu G/U$, we will calculate the image $\iota^C_G([C_\mu G/U])$. First, we note a general remark.

Remark 3.2.3. *Let I and J be finite groups. In general for a transitive $I\times J$-set $I\times J/S$, let $\mathrm{Stab}_I(S)$ and $\mathrm{Stab}_J(S)$ denote the stabilizers of S in I and in J, respectively. By Goursat's Theorem S is defined by the quintuble $(\uparrow S,\downarrow S,\phi_S,S\downarrow,S\uparrow)$. Then $\downarrow S=\mathrm{Stab}_I(S)$ and $S\downarrow=\mathrm{Stab}_J(S)$. Indeed, $\downarrow S\subseteq \mathrm{Stab}_I(S)$ is clear and for $x\in\mathrm{Stab}_I(S)$ multiplication by $(x,1)$ is a bijection $S\to S$. Hence $(x^{-1},1)\in S$ also $(x,1)\in S$ which implies $x\in\downarrow S$.*

Proposition 3.2.4. *For a transitive C-fibred G-set $C_\mu G/U$, we have $\iota^C_G([C_\mu G/U]) = [C\times G/U_\mu]$ where U_μ is defined by the quintuble $(\mu^{-1}(U),1,\mu^{-1},\ker(\mu^{-1}),U)$ where μ^{-1} is the inverse of μ in $\widehat{G}$. .*

Proof. The kernel of the projection $\pi\colon U_\mu\to G$ is trivial by Remark 3.2.3. As a set we can express U_μ as the disjoint union

$$\coprod_{g\ker(\mu^{-1})\subseteq U}\mu^{-1}(g)\times g\ker(\mu^{-1}).$$

We write the action of $u\in U$,

$$\begin{aligned} uU_\mu &= \coprod_{g\ker(\mu^{-1})\subseteq U}\mu^{-1}(g)\times ug\ker(\mu^{-1})\\ &= \coprod_{x\ker(\mu^{-1})\subseteq U}\mu^{-1}(u^{-1}x)\times x\ker(\mu^{-1}) = \mu(u)U_\mu. \end{aligned}$$

Hence $u\in U$ acts on the fibres as left multiplication by $\mu(u)$. □

3.3 Biset action on the transitive C-fibred G-sets

Now we consider the action of the transitive biset $I \times J/S$ on the transitive C-fibred J-set $C_\mu J/U$. We will simply modify Theorem 2.2.1. We generalize the definition of join of two groups which was introduced in Section 2.2. Given $S \leq I \times J$ and $T \leq CJ$, let $\Delta(C, S)$ denote the subgroup $\{(c, i, c, j) \in CI \times CJ \colon (i, j) \in S\}$ of $CI \times CJ$ and the join $\Delta(C, S) * T$ denote the subgroup

$$\Delta(C, S) * T = \{(c, i) \in C \times I \colon \exists (c, t) \in T \text{ such that } (i, t) \in S\}.$$

Theorem 3.3.1. *Given a transitive I-J-set $I \times J/S$ and C-fibred J-set $C_\mu J/U$, the action of $I \times J/S$ on $C_\mu J/U$ is given by*

$$\left[\frac{I \times J}{S}\right][C_\mu J/U] = \sum_{S\uparrow jU \subseteq J \colon S\downarrow \cap {}^jU \subseteq \ker({}^j\mu)} \left[\frac{CI}{\Delta(C, S) *{}^jU_{{}^j\mu}}\right]. \tag{3.1}$$

Proof. By Propositions 3.2.2 and 3.2.4 and Theorem 2.2.1 , we have

$$\begin{aligned}
\theta(\left[\frac{I \times J}{S}\right], [C_\mu G/U]) &= \pi \circ \Theta \circ \Gamma^C_{I\times J}(\left[\frac{I \times J}{S}\right], [C_\mu G/U]) \\
&= \pi \circ \Theta(\left[\frac{CI \times CJ}{\Delta(C, S)}\right], \left[\frac{C \times G}{U_\mu}\right]) \\
&= \pi(\sum_{S\uparrow jU \subseteq J} \left[\frac{CI}{\Delta(C, S) *{}^jU_{{}^j\mu}}\right])
\end{aligned}$$

Finally by Proposition 3.1.1 the result follows. □

We could also use Remark 3.2.3 to find the largest C-free subset. C acts freely on $CI/(\Delta(C, S)*U_\mu)$ if and only if the kernel of the projection $\pi \colon \Delta(C, S)*U_\mu \to I$ is trivial. It is easy to calculate the kernel

$$\begin{aligned}
\ker(\pi) &= \{(c, 1) \in \Delta(C, S) * U_\mu \colon \exists u \in U, (1, u) \in S, c = \mu^{-1}(u)\} \\
&= \{(\mu^{-1}(u), 1) \in \Delta(C, S) * U_\mu \colon u \in S \downarrow \cap U\} \\
&\cong \mu^{-1}(S \downarrow \cap U) \cong \mu(S \downarrow \cap U).
\end{aligned}$$

To have C-free action $S\downarrow \cap U \subseteq \ker(\mu)$. Hence the subset of $CI/(\Delta(C,S)*U_\mu)$ with C-free action is either empty or itself.

Recall that in Section 2.2, the five elemental maps transfer, restriction, inflation, deflation and isogation are introduced. Suppose $U,H \leq G \trianglerighteq N$, $W \leq H$ and $\theta\colon G' \leftarrow G$ is a group isomorphism. Define the linear characters $\mu\colon U \to C$, $\omega\colon W \to C$ and $\lambda\colon U/N \to C$.

We will calculate the action of the elemental maps using equation (3.1). Consider the join of the groups in equation (3.1). It is clear that $\Delta(H)\downarrow = 1$, $\Delta(G,G/N)\downarrow = N/N$ and $\Delta(G/N,G)\downarrow = N$. In equation(3.1), only restriction has more than one summands. Consider the transfer map, then $\Delta(C,\Delta(H))*W_\omega = \{(c,x)\colon x \in H\cap W \text{ and } c = \omega(x)\} = W_\mu$ since $W \leq H$. Then

$$\mathrm{tra}_{G,H}[C_\omega H/W] = [C_\omega G/W].$$

Similar calculations for restriction and inflation gives $\Delta(C,\Delta(H)) *^g U_{^g\mu} = \{(c,x)\colon x \in H\cap^g U \text{ and } c =^g \mu(x)\} = (H\cap^g U)_{\mathrm{res}^g\mu}$ and $\Delta(C,\Delta(G,G/N))*\overline{U}_\lambda = \{(c,u)\colon uN \in U/N \text{ and } c = \lambda(uN)\} = U_{\mathrm{inf}\lambda}$. Here $\mathrm{res}^g\mu$ and $\mathrm{inf}\lambda$ denotes restriction to $H\cap^g U$ and inflation to U, respectively. Explicitely, $\mathrm{res}^g\mu(x) =^g \mu(x)$ and $\mathrm{inf}\lambda(u) = \lambda(uN)$ for $x \in H\cap^g U$ and $u \in U$. Hence we get

$$\mathrm{res}_{H,G}[C_\mu G/U] = \sum_{HgU\subseteq G}[C_{\mathrm{res}^g\mu}H/(H\cap^g U)]$$

and

$$\mathrm{inf}_{G,G/N}[C_\lambda \overline{G}/\overline{U}] = [C_{\mathrm{inf}\lambda}G/U]$$

where bar means quotient under N. Also, for isogation map it is straight forward to show that

$$\mathrm{iso}^\theta_{G',G}[C_\mu G/U] = [C_{\mathrm{iso}\mu}G'/U'].$$

where for $g' \in G'$, $\mathrm{iso}\mu(g') = \mu(\theta^{-1}(g'))$ and U' is the image of U under θ. Deflation is a bit tricky. Since $\Delta(G/N,G)\downarrow = N$,

$$\mathrm{def}_{G/N,G}[C_\mu G/U] = \begin{cases} [C_{\mathrm{def}_\mu}\overline{G}/\overline{UN}] & if N\cap U \subseteq \ker(\mu) \\ 0 & otherwise \end{cases}.$$

here $\mathrm{def}\mu$ is defined such that $\mathrm{def}\mu(uN) = \mu(u)$ where $u \in U$.

3.4 Biset action on the primitive idempotents

The primitive idempotents of $\mathbb{K}B_C(G)$ are described in Section 2.2.1. In this section we will calculate the action of transfer, restiction, inflation, deflation and isogation on the primitive idempotents $e^G_{H,h}$. Transfer and restriction are calculated in Barker [2].

Proposition 3.4.1. *Given $F \leq G$ and a C-subelement (H,h) of G, then*

$$\mathrm{res_{F,G}(e^G_{H,h}) = \sum_{(J,j)} e^F_{J,j}},$$

where (J,j) runs over representatives of the F-classes of C-subelements of F such that (J,j) is G-conjugate to (H,h).

Proposition 3.4.2. *Given $F \leq G$ and a C-subelement (J,j) of F, then*

$$\mathrm{tra_{G,F}(e^F_{J,j}) = |N_G(J,j)\colon N_F(J,j)| e^G_{J,j}}.$$

It is also straight forward to calculate isogation and inflation. Given a group isomorphism $\phi : G \to G'$

$$\mathrm{iso}_{G',G} e^G_{F,g} = e^{G'}_{F',g'} \text{ where } F' = \phi(F) \text{ and } g' = \phi(g).$$

Let $N \trianglelefteq G$ and $(\overline{F}, \overline{f}) \in \mathrm{el}(C, G/N)$ then

$$\mathrm{inf}_{G,\overline{G}} e^{\overline{G}}_{\overline{F},\overline{f}} = \sum_{(J,j)\in_G \mathrm{el}(C,G)\colon (\overline{JN},\overline{j}) =_{\overline{G}} (\overline{F},\overline{f})} e^G_{J,j}.$$

Deflation requires more work. Recall that we can identify $\mathbb{K}B(G)$ as a subring of $\mathbb{K}B_C(G)$ via the injection γ^C_G.

Lemma 3.4.3. *The ring monomorphism $\gamma^C_G \colon \mathbb{K}B(G) \to \mathbb{K}B_C(G)$ gives a morphism of biset functors $\gamma^C \colon \mathbb{K}B \to \mathbb{K}B_C$. And*

$$\gamma^C_G(e^G_H) = \sum_{j\in_{N_G(H)} H/O(H)} e^G_{H,j}$$

where e^G_H is a primitive idempotent of $\mathbb{K}B(G)$ and the summation runs over the $N_G(H)$-classes of the C-elements of H.

Proof. Transfer, restriction, inflation and isogation clearly commute with γ^C. Since G action on the image is trivial γ^C commutes with deflation as well. We have

$$S^G_{J,j}(\gamma^C_G(e^G_H)) = \lfloor J =_G H \rfloor.$$

□

Lemma 3.4.4. *Let $N \trianglelefteq G$ and $\nu\colon G \to C$ be a linear character of G such that $N \leq \ker(\nu)$. Then for all $[CX] \in \mathbb{K}B_C(G)$*

$$\mathrm{def}_{\overline{G},G}([C_\nu G/G][CX]) = \mathrm{def}_{\overline{G},G}[C_\nu G/G]\mathrm{def}_{\overline{G},G}[CX]$$

where $\overline{G} = G/N$.

Proof. It is clear that $N \leq \ker(\omega)$ if and only if $N \leq \ker(\nu.\omega)$. Hence we have

$$\mathrm{def}_{\overline{G},G}([C_\nu G/G][C_\omega G/W]) = \mathrm{def}_{\overline{G},G}[C_\nu G/G]\mathrm{def}_{\overline{G},G}[C_\omega G/W].$$

□

Observe that $[C_\nu G/G]e^G_{G,g} = \nu(g)e^G_{G,g}$. For the Burnside algebra $\mathbb{K}B(G)$ deflation on the primitive idempotents are calculated in Bouc [4, Lemma 16] and reviewed in Barker [3]. Let us recall the deflation numbers for G/N and G, we define

$$\beta(\overline{G}, G) = \frac{1}{|G|} \sum_{S \leq G\colon SN=G} |S|\mu(S, G)$$

and more generally for I/N and I where $N \trianglelefteq I \leq G$, we define

$$\beta_G(\overline{I}, I) = \frac{|N_{\overline{G}}(\overline{I})\colon \overline{I}|}{|N_G(I)\colon I|}\beta(\overline{I}, I).$$

Let e^G_I be a primitive idempotent of $\mathbb{K}B(G)$ then

$$\mathrm{def}_{\overline{G},G}e^G_I = \beta_G(\overline{I}, I)e^{\overline{G}}_{\overline{I}}.$$

We define the monomial deflation numbers $\beta^C(\overline{G}, G, g)$ for the monomial Burnside algebra, $\mathbb{K}B_C(G)$ by

$$\beta^C(\overline{G}, G, g) = \frac{1}{|G|} \sum_{V \leq G: VN=G} |V|\mu(V, G)\frac{|V \cap gO(G)|}{|(V \cap N)O(V)|}. \qquad (3.2)$$

The monomial deflation numbers depend on the fibre group C. Observe that using $\sum_{n\in N}|V\cap ngO(G)| = |O(G)(V\cap N)|$ we recover the deflation numbers for the Burnside ring

$$\frac{1}{|N\cap O(G)|}\sum_{n\in N}\beta^C(\overline{G},G,ng) = \beta(\overline{G},G).$$

We can write equation (3.2) in a simpler form,

$$\beta^C(\overline{G},G,g) = \frac{1}{|NO(G)|}\sum_{V\leq G:VN=G}|V\cap gO(G)|\mu(V,G).$$

Lemma 3.4.5. *Let $N\trianglelefteq G$ then,*

$$\mathrm{def}_{\overline{G},G}e^G_{G,g} = \beta^C(\overline{G},G,g)e^{\overline{G}}_{\overline{G},\overline{g}}.$$

Proof. Since γ^C is a morphism of biset functors we have

$$\mathrm{def}_{\overline{G},G}(\sum_{j\in_G G/O(G)}e^G_{G,j}) = \beta(\overline{G},G)\sum_{\overline{j}\in_{\overline{G}}\overline{G}/O(\overline{G})}e^{\overline{G}}_{\overline{G},\overline{j}}. \tag{3.3}$$

For a proper subgroup H of G, we have $S^{\overline{G}}_{\overline{H},\overline{h}}\mathrm{def}_{\overline{G},G}e^G_{G,g} = 0$. This follows from the commutation $\mathrm{res}_{\overline{H},\overline{G}}\mathrm{def}_{\overline{G},G} = \mathrm{iso}_{HN/N,H/H\cap N}\mathrm{def}_{H/H\cap N,H}\mathrm{res}_{H,G}$. Hence we can write

$$\mathrm{def}_{\overline{G},G}e^G_{G,g} = \sum_{\overline{j}\in\overline{G}/O(\overline{G})}\alpha_{\overline{j}}e^{\overline{G}}_{\overline{G},\overline{j}}.$$

Multiply both sides by $[C_\nu G/G]$ where ν is such that $N\leq\ker(\nu)$. Then by Lemma 3.4.4

$$\mathrm{def}_{\overline{G},G}([C_\nu G/G]e^G_{G,g}) = (\mathrm{def}_{\overline{G},G}[C_\nu G/G])(\mathrm{def}_{\overline{G},G}e^G_{G,g}).$$

Which results in the equation

$$\sum_{\overline{j}\in\overline{G}/O(\overline{G})}\alpha_{\overline{j}}(\nu(j)-\nu(g))e^{\overline{G}}_{\overline{G},\overline{j}} = 0$$

for all linear characters $\nu\colon G/N\to C$. Given two distinct elements $\overline{j}$ and $\overline{g}$ in $\overline{G}/O(\overline{G})$ there exists some linear character ν of G/N such that $\nu(\overline{j})\neq\nu(\overline{g})$.

Hence we get $\mathrm{def}_{\overline{G},G} e^G_{G,g} = \alpha_{\overline{g}} e^{\overline{G}}_{\overline{G},\overline{g}}$ for some $\alpha_{\overline{g}} \in \mathbb{K}$. Expressing $e^G_{G,g}$ in terms of the basis $\{[C_\nu G/V] \colon (V,\nu) \in_G \mathrm{ch}(C,G)\}$ we get

$$\begin{aligned} \alpha_{\overline{g}} = S^{\overline{G}}_{\overline{G},\overline{g}} \mathrm{def}_{\overline{G},G} e^G_{G,g} &= \sum_{(V,\nu) \in_G \mathrm{ch}(C,G)} m_G^{-1}(V,\nu;G,g)\nu(g) \\ &= \frac{1}{|G|} \sum_{V \leq G : VN = G} |V| \mu(V,G) \frac{|V \cap gO(G)|}{|(V \cap N)O(V)|}. \end{aligned}$$

□

Finally using the commutation

$$\mathrm{def}_{G/N,G} \mathrm{tra}_{G,H} = \mathrm{tra}_{G/N,H/N} \mathrm{iso}_{HN/N,H/(H \cap N)} \mathrm{def}_{H/(H \cap N),H}$$

we get the following result.

Proposition 3.4.6. *Given $N \trianglelefteq G \geq H$ and a C-subelement (H,h) of G, then*

$$\mathrm{def}_{\overline{G},G} e^G_{H,h} = \beta^C_G(\overline{H},H,h) e^{\overline{G}}_{\overline{HN},\overline{h}}$$

where $\beta^C_G(\overline{H},H,h) = \frac{|N \cap H|}{|N|} \frac{|N_{\overline{G}}(\overline{HN},\overline{h})|}{|N_G(H,h)|} \beta^C(H/(H \cap N),H,h)$.

3.5 Deflation numbers of p-groups

Recall that the *Frattini* subgroup $\Phi(G)$ of a finite group G is defined to be the intersection of all maximal subgroups of G. For a p-group P, the quotient $P/\Phi(P)$ is an elementary abelian p-group. Furthermore, if $N \trianglelefteq P$ such that P/N is an elementary abelian p-group then $N \geq \Phi(P)$. Hence $P/\Phi(P)$ is the maximal elementary abelian quotient of P.

For p-groups, the monomial deflation numbers can be calulated explicitely. Next theorem [10] gives the explicit formula of the Möbius fuction of the subgroup poset of a p-group. We denote the elementary abelian p-group of rank d by C_p^d.

Theorem 3.5.1. (Weisner's Theorem) *Suppose that P is a p-group. Then $\mu(1,P) = 0$ unless $P \cong C_p^d$ for some $d \geq 0$, in which case, $\mu(1,P) =$*

$(-1)^d p^{d(d-1)/2}$. *More generally,* $\mu(U,P) = 0$ *unless* $U \trianglelefteq P$ *and* $P/U \cong C_p^r$ *for some* $r \geq 0$*, in which case,* $\mu(U,P) = (-1)^r p^{r(r-1)/2}$.

There is a generalized version of the binomial coefficients called the *p-binomial coefficients* which are defined by

$$\begin{pmatrix} d \\ r \end{pmatrix}_p = \frac{(p^d-1)(p^d-p)...(p^d-p^{d-1})}{(p^r-1)(p^r-p)...(p^r-p^{r-1})} = \frac{(p^d-1)(p^{d-1}-1)...(p^{d-r+1}-1)}{(p^r-1)(p^{r-1}-1)...(p-1)}.$$

In the limit $p \to 1$ this reduces to the usual binomial coefficients. The number of subgroups of rank $r \leq d$ of C_p^d is equal to $\begin{pmatrix} d \\ r \end{pmatrix}_p$.

The C-subcharacters of a p-group depends on the Sylow p-subgroup of C, hence we can assume the fibre group to be a cyclic p-group. Let C_{p^n} denote the cyclic group of order p^n. We write $O_n(P)$ to denote the intersection of the kernels of C_{p^n}-characters of P and $\beta^n(\overline{P}, P, g)$ denotes the monomial deflation numbers for the fibre group C_{p^n} Observe that

$$P = O_0(P) \supseteq O_1(P) \supseteq ... \supseteq O_S(P)$$

where S is the exponent of P. In particular $O_1(P)$ is the Frattini subgroup $\Phi(P)$.

By Theorem 3.5.1, for p-groups the monomial deflation numbers simplifies

$$\begin{aligned} \beta^n(P/N, P, g) &= \frac{1}{|NO_n(P)|} \sum_{V \leq P : VN = P} |V \cap gO_n(P)| \mu(V, P) \\ &= \frac{1}{|NO_n(P)|} \sum_{\Phi(P) \leq V \leq P : VN = P} \lfloor g \in V \rfloor \mu(V, P) \\ &= \frac{|NO_1(P)|}{|NO_n(P)|} \beta^n(\overline{P}/\overline{N\Phi(P)}, \overline{P}, \overline{g}) \end{aligned}$$

where $\overline{g}$ is the image of g under the surjection $P \to P/\Phi(P)$ and bar means quotient under $\Phi(P)$. We reduced the problem to the elementary abelian p-groups. The deflation map is transitive, i.e. for $M \leq N$ and $N \trianglelefteq P \trianglerighteq M$ we have

$$\mathrm{def}_{P/N,P} = \mathrm{iso}_{P/N,(P/M)/(N/M)} \mathrm{def}_{(P/M)/(N/M),P/M} \mathrm{def}_{P/M,P}.$$

Hence we can restrict our attention to the case $P \cong C_p^d$ and $N \cong C_p$.

Lemma 3.5.2. *Let $P \cong C_p^d$ and $N \cong C_p$ then for all $n \geq 1$*

$$\beta^n(\overline{P}, P, g) = \begin{cases} 1/p & 1 \neq g \in N \\ (1-p^{d-1})/p & g = 1 \\ (1-p^{d-2})/p & otherwise \end{cases}.$$

Proof. Observe that for all $n \geq 1$, $O_n(P)$ is trivial. When $1 \neq g \in N$ the result is clear. Assume $g \notin N$, then $VN = P$ if and only if $V \cong C_p^{d-1}$ such that $N \nleq V$. We also require $g \in V$. Counting such subgroups using p-binomial coefficients and using Theorem 3.5.1 the formula of the monomial deflation numbers becomes

$$\beta^n(\overline{P}, P, g) = \frac{1}{p}(1 - \begin{pmatrix} d-1 \\ d-2 \end{pmatrix}_p + \begin{pmatrix} d-2 \\ d-3 \end{pmatrix}_p) = (1-p^{d-2})/p.$$

And by similar calculations for $g = 1$, we have

$$\beta^n(\overline{P}, P, g) = \frac{1}{p}(1 - \begin{pmatrix} d \\ d-1 \end{pmatrix}_p + \begin{pmatrix} d-1 \\ d-2 \end{pmatrix}_p) = (1-p^{d-1})/p$$

□

Let us introduce some notation. Given integers $0 \leq c < d$ we define the *monomial β-numbers* by

$$\beta_p(c, d) = \frac{1}{p^{d-c}} \prod_{i=c}^{d-1} (1-p^i) \text{ and } \beta_p(d, d) = 1.$$

By transitivity of the deflation map, we generalize Lemma 3.5.2.

Lemma 3.5.3. *Let $P \cong C_p^d$ and $N \cong C_p^{d-c}$ then for all $n \geq 1$*

$$\beta^n(\overline{P}, P, g) = \begin{cases} \beta_p(c, d) & if \quad g = 1 \\ \beta_p(c-1, d-1) & if \quad g \notin N \\ \frac{1}{p}\beta_p(c, d-1) & if \quad 1 \neq g \in N \end{cases}$$

Proof. The cases $g = 1$ and $g \notin N$ are clear. Assume that $1 \neq g \in N$, consider the chain of subgroups $< g > \leq C_p^2 \leq ... \leq C_p^{r-1} \leq C_p^r = N$. Using transitivity of deflation, the result follows immediately. □

More generally, for p-groups we have the following theorem.

Theorem 3.5.4. *Suppose that P is a p-group and $N \trianglelefteq P$. Let c and d denote the ranks of P/N and P respectively. Then*

$$|N\cap\Phi(P)|\frac{|N\cap O_n(P)|}{|O_n(P)|}\beta^n(P/N,P,g) = \begin{cases} \beta_p(c,d) & \text{if} \quad \overline{g}=\overline{1} \\ \beta_p(c-1,d-1) & \text{if} \quad \overline{g}\notin\overline{N\Phi(P)} \\ \frac{1}{p}\beta_p(c,d-1) & \text{if} \quad \overline{1}\neq\overline{g}\in\overline{N\Phi(P)} \end{cases}$$

where $\overline{g}$ is the image of g under the surjection $P \to P/\Phi(P)$.

Proof. Recall that for p-groups, we noted that

$$|NO_n(P)|\beta^n(P/N,P,g) = |\overline{N\Phi(P)}|\beta^n(\overline{P}/\overline{N\Phi(P)},\overline{P},\overline{g})$$

where $\overline{g}$ is the image of g under the surjection $P \to P/\Phi(P)$ and bar means quotient under $\Phi(P)$. Note that $\mathrm{rank}(\overline{P}/\overline{N\Phi(P)}) = \mathrm{rank}(P/N) = c$ and $\mathrm{rank}(P/\Phi(P)) = \mathrm{rank}(P) = d$. Hence we have

$$|\Phi(P)|\frac{|NO_n(P)|}{|N\Phi(P)|}\beta^n(P/N,P,g) = \beta^n(C_p^c, C_p^d, g)$$

which is calculated in Lemma 3.5.3. □

There is also a similar formula of Bouc-Thévenaz [5, eq 4.8,Lemma 8.1] for the Burnside algebra $\mathbb{Q}B(P)$.

3.6 Simple composition factors of the monomial Burnside functor for 2-groups

Let Ω_p denote the collection of p-groups. Let F be a subfunctor of $\mathbb{K}B_C$. We embed $\mathbb{K}B$ into $\mathbb{K}B_C$ via the injection γ^C.

Proposition 3.6.1. *Let F be a subfunctor of $\mathbb{K}B_C$, then for all $P \in \Omega_p$, $F(P)$ is a $\mathbb{K}B(P)$-module by left multiplication.*

Proof. Consider the action of the P-P-biset $P \times P/\Delta(W)$ on the C-fibred P-set $C_\mu P/U$

$$\begin{aligned}\left[\frac{P \times P}{\Delta(W)}\right][C_\mu P/U] &= \sum_{WgU \subseteq P} \left[\frac{CP}{\Delta(C, \Delta(W)) * {}^gU_{{}^g\mu}}\right] \\ &= \sum_{WgU \subseteq P} \left[\frac{CP}{(W \cap {}^gU)_{{}^g\mu}}\right] \\ &= \sum_{WgU \subseteq P} [C_{{}^g\mu} P/(W \cap {}^gU)] \\ &= [CP/W][C_\mu P/U]\end{aligned}$$

where $\Delta(W) \leq P \times P$ is the diagonal subgroup $\{(w, w) \in W\}$. $\square$

This suggests the definition of the following subalgebras, for a subgroup $H \leq P$ we define the subalgebra F_H^P of $F(P)$ by

$$F_H^P := \tau_H^P F(P) \text{ where } \tau_H^P = \sum_{h \in_{N_P(H)} H/O(H)} e_{H,h}^P.$$

Then we can write $F(P) = \oplus_{H \leq_P P} F_H^P$ as $\mathbb{K}$-algebras. In particular for elementary abelian p-groups, $Q \leq P$ we have

$$\mathrm{tra}_{P,Q} e_{Q,q}^Q = |P \colon Q| e_{Q,q}^P \text{ and } \mathrm{res}_{Q,P} e_{Q,q}^P = e_{Q,q}^Q.$$

which gives $\mathrm{tra}_{P,Q} F_Q^Q = F_Q^P$ and $\mathrm{res}_{Q,P} F_Q^P = F_Q^Q$. To simplify notation, let $\mathbb{K}B_n$ denote the monomial Burnside functor $\mathbb{K}B_C$ where C is the cyclic group of order p^n.

Lemma 3.6.2. *Assume that $S_{P,V}$ is a simple composition factor of $\mathbb{K}B_n$ where P is a p-group and V is a simple $\mathbb{K}Out(P)$-module, then P is an abelian p-group whose exponent divides p^n.*

Proof. Assume $O_n(P)$ is non-trivial. Let F and L be subfunctors of $\mathbb{K}B_n$ such that $F/L \cong S_{P,V}$. Observe that $F(P)/L(P) \cong S_{P,V}(P) \cong V \cong F_P^P/L_P^P$. Since $\mathrm{def}_{P/O_n(P),P} S_{P,V}(P) = 0$, we have $\mathrm{def}_{P/O_n(P),P} F_P^P \subseteq L_{\overline{P}}^{\overline{P}}$ where $\overline{P} = P/O_n(P)$. Consider the map $\tau_P^P \mathrm{inf}_{P,P/O_n(P)} \mathrm{def}_{P/O_n(P),P} \colon F_P^P \to F_P^P$, by Theorem 3.5.4 we have

$$\begin{aligned}\tau_P^P \mathrm{inf}_{P,P/O_n(P)}\mathrm{def}_{P/O_n(P),P} e_{P,x}^P &= \frac{1}{|O_n(P)|}\tau_P^P\left(\sum_{(J,j)\in_P \mathrm{el}(C_{p^n},P):\ (\overline{J},\overline{j})=(\overline{P},\overline{x})} e_{P,x}^P\right)\\ &= \frac{1}{|O_n(P)|}e_{P,x}^P.\end{aligned}$$

Observe that $\tau_P^P \mathrm{inf}_{P,P/O_n(P)}\mathrm{def}_{P/O_n(P),P}(F_P^P) \subseteq L_P^P$. Take $v \in F_P^P$ then

$$\tau_P^P \mathrm{inf}_{P,P/O_n(P)}\mathrm{def}_{P/O_n(P),P}(v) = \frac{1}{|O_n(P)|}v.$$

Hence $L_P^P = F_P^P$, but then $S_{P,V}(P) \cong F_P^P/L_P^P = 0$, which is a contradiction. $\square$

Let Γ_p^n denote the collection of finite abelian p-groups whose exponent divides p^n. As an immediate consequence of the lemma above, we have the following corollary.

Corollary 3.6.3. *The simple composition factors of $\mathbb{K}B_C^{\Omega_p}$ are as the same as the simple composition factors of $\mathbb{K}B_C^{\Gamma_p^n}$.*

We restrict our attention to 2-groups and fix the fibre group to $\{\pm 1\}$. Next theorem reveals the simple composition factors of $\mathbb{K}B_{\mathbb{R}}$ for 2-groups, i.e. $\mathbb{K}B_C$ where $C = \{\pm 1\}$.

Theorem 3.6.4. *For 2-groups, the simple composition factors of $\mathbb{K}B_{\mathbb{R}}$ are of the form $S_{C_2^2,1}$, $S_{C_2,1}$ and $S_{1,1}$ with multiplicities equal to 1.*

Proof. By Corollary 3.6.3 we consider elementary abelian 2-groups. The simple composition factors $S_{C_2^2,1}$ and $S_{1,1}$ of $\mathbb{K}B$ are contained in $\mathbb{K}B_{\mathbb{R}}$. Let F be a subfunctor of $\mathbb{K}B_{\mathbb{R}}$ which contains $\mathbb{K}B$ and the quotient $F/\mathbb{K}B$ is a simple biset functor. Say $F/\mathbb{K}B \cong S_{P,V}$ where $P \cong C_2^d$ and V is a simple $\mathbb{K}\mathrm{Out}(C_2^d)$-module. Let $N \cong C_2^{d-1}$. Take $0 \neq v \in (S_{P,V})_P^P$, recall that $(S_{P,V})_P^P$ is the subspace defined by $\tau_P^P S_{P,V}$. We can write $v = \sum_{x\in P}\alpha_x e_{P,x}^P$. By Lemma 3.5.3, we have

$$\mathrm{def}_{P/N,P}\nu = (\beta_2(1,d)\alpha_1 + \frac{1}{p}\beta_2(1,d-1)\sum_{x\in N-1}\alpha_x)e_{\overline{P},\overline{1}}^{\overline{P}}$$

since for $n \notin N$ the monomial deflation numbers $\beta_2(0, d-1)$ are zero. Then the coordinate module of $F/\mathbb{K}B$ at C_2 is non-zero, hence $P \cong C_2$.

Since the dimension of the quotient $\mathbb{K}B_{\mathbb{R}}(C_2)/\mathbb{K}B(C_2)$ is 1, $S_{C_2,1}$ appears only once as a composition factor. Observe that $F(C_2) = \mathbb{K}B_C(C_2)$, hence $F_{C_2}^{C_2}$ is spanned by $e_{C_2,1}^{C_2}$ and $e_{C_2,x}^{C_2}$. Let $P \cong C_2^d$ and N be a subgroup of P of rank 1. By an inductive argument on d and considering the inflation map $F_{\overline{P}}^{\overline{P}} \to F_P^P$, we will show that $F(P) = \mathbb{K}B_{\mathbb{R}}(P)$ and conclude that $F = \mathbb{K}B_{\mathbb{R}}$. Consider the action of the inflation map

$$\tau_P^P \mathrm{inf}_{P,\overline{P}} e_{\overline{P},\overline{p}}^{\overline{P}} = \tau_P^P \left(\sum_{J \leq P:\ JN = P} \sum_{n \in N} e_{J,pn}^P \right) = \sum_{n \in N} e_{P,pn}^P. \tag{3.4}$$

First assume $d = 2$. For $x, y \in P$, take $N =< xy >$ then by equation (3.4)

$$\tau_P^P \mathrm{inf}_{P,\overline{P}} e_{\overline{P},\overline{x}}^{\overline{P}} = \sum_{n \in N} e_{P,xn}^P = e_{P,x}^P + e_{P,y}^P.$$

Hence for all $x, y \in P$, we have $e_{P,x}^P + e_{P,y}^P \in F_P^P$. Since $\mathbb{K}B$ is a subfunctor of F, $\sum_{p \in P} e_{P,p}^P$ is also contained in F_P^P. Subtracting $\sum_{p \in P - \{1\}} (e_{P,1}^P + e_{P,p}^P)$ from $\sum_{p \in P} e_{P,p}^P$, we conclude that $e_{P,1}^P \in F_P^P$ hence for all $x \in P$, $e_{P,x}^P \in F_P^P$. Then $F_P^P =< e_{P,p}^P \colon p \in P >$. By induction on d we can show that $F_P^P =< e_{P,p}^P \colon p \in P >$ for all $d \geq 0$. For a subgroup Q of P, we have $\mathrm{tra}_{P,Q} F_Q^Q = F_Q^P$ and $\mathrm{res}_{Q,P} F_Q^P = F_Q^Q$. Then $F = \mathbb{K}B_{\mathbb{R}}$, hence we recovered all the simple composition factors of $\mathbb{K}B_{\mathbb{R}}$. $\square$

Bibliography

[1] L.Barker, Rhetorical biset functors, rational p-biset functors and their semisimplicity in characteristic zero, J.Algebra 319 (2008) 3810-3853

[2] L.Barker, Fibred permutation sets and the idempotents and units of monomial Burnside rings, J.Algebra 281 (2004) 535-566

[3] L.Barker, Tornehave morphisms II: the lifted Tornehave morphism and the dual of the Burnside functor

[4] S.Bouc, Foncteurs d'ensembles munis d'une double action, J. Algebra 183 (1996) 664-736

[5] S. Bouc, J. Thévenaz, The group of endo-permutation modules, Invent. Math. 139 (2000) 275-349 .

[6] R. Boltje, O. Coşkun, (unpublished notes, December 2008)

[7] D. Glück, Idempotent formula for the Burnside algebra with applications to the p- subgroup simplical complex, Illinois J. Math. 25, 63-67 (1981).

[8] T. Yoshida, Idempotents of Burnside rings and Dress induction theorem, J. Algebra 80 (1983) 90-105.

[9] A. Dress, The ring of monomial representations, I. Structure theory, J. Algebra 18 (1971) 137-157.

[10] L. Weisner, Some properties of prime-power groups, Trans. Amer. Math. Soc. 38 (1935) 485-492 .