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Covering spaces

Przestrzenie nakrywające

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The purpose of this Master thesis is to analyze covering spaces, fundamental groups, and relation between them.

This thesis has 3 Chapters and 15 Sections. In the first chapter we consider what are homotopies and homotopic mappings.

In the second chapter, we study fundamental groups, liftings, induced homomorphisms, and homotopy types.

In the third chapter, we consider our main topic, i.e., covering spaces and their relations with fundamental groups.

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Introduction

In Mathematics, covering spaces theory is very important for topology, as well as for differential equations, Lie groups theory and Riemann spaces theory. Covering spaces theory is related to fundamental groups of topological spaces. Different problems of topology are converted to algebraic problems through algebraic topology. Therefore we have to learn basics both of fundamental groups and of covering spaces.

The purpose of this Master thesis is to analyze covering spaces, fundamental groups, and relation between them.

This thesis has 3 Chapters and 15 Sections. In the first chapter we will consider what are homotopies and homotopic mappings.

In the second chapter, we will consider fundamental groups, liftings, induced homomorphisms, and homotopy types.

In the third chapter, we will consider our main topic, i.e., covering spaces and their relations with fundamental groups.

CHAPTER 1

Homotopy

In this chapter we will learn about homotopies and their properties.

1.1. Notion of Homotopy

Definition 1.1.1. If for two continuous maps $f_0, f_1 : X \longrightarrow Y$ and there exist a continuous map $F : X \times I \longrightarrow Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for every $x \in X$, then these two continuous maps are called *homotopic*, and F is a *homotopy* between f_0 and f_1. We denote this with $f_0 \simeq f_1$.

Definition 1.1.2. If $k : A \longrightarrow B$ is a continuous mapping which image includes only a single point, then we say k is constant mapping.

Definition 1.1.3. A continuous map $f : A \longrightarrow B$ that is is homotopic to a constant map is called *null homotopic*.

Example 1. Suppose that $A = B = \mathbb{R}^n$ and $f_0(p) = p$ and $f_1(p) = 0$ for $p \in \mathbb{R}^n$. Assume $F : \mathbb{R}^n \times C \longrightarrow \mathbb{R}^n$ is defined by $F(p, t) = (1 - t)p$. Then F is a homotopy between f_0 and f_1, f_1 is constant, hence f_0 is null homotopic.

Example 2. Two constant maps or two null homotopic maps are not always homotopic. We can explain this with the following example.

Let X be connected and Y be not connected. Choose points y_0 and y_1 in distinct components of Y and suppose that $f_0(x) = y_0$ and $f_1(x) = y_1$ for all $x \in X$. The maps f_0 and f_1 are constant, $X \times I$ is connected but Y is not connected, hence f_0 and f_1 are not homotopic.

1.2. Properties of Homotopic Mappings

LEMMA 1.2.1. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous mappings then the composite transformation $h = gf : X \longrightarrow Z$ is continuous.*

PROOF. Inverse image of each open subset of Y under f is open in X and inverse image of each open subset Z under g is open in Y. Suppose that ϕ is an open set in Z. Then the continuity of h follows from the equality $h^{-1}(\phi) = f^{-1}\{g^{-1}(\phi)\}$. □

THEOREM 1.2.2. *If $p : A \times B \longrightarrow A$ and $q : A \times B \longrightarrow B$ are defined by the formulae $p(c_1, c_2) = c_1$ and $q(c_1, c_2) = c_2$ where $c_1 \in A$, $c_2 \in B$, then p and q are continuous.*

PROOF. Assume that ϕ is open in A. Then $p^{-1}(\phi)$ is the set of ordered pairs (c_1, c_2) where $c_1 \in \phi$ and $c_2 \in B$. Hence, $p^{-1}(\phi) = \phi \times B$, which is open in $A \times B$. Therefore p is continuous as well as q. □

Such p and q are called projections of the product $A \times B$ onto its factors.

1.2.1. Composition and Restriction.

PROPOSITION 1.2.3. *1) Assume that there are* $p_1, p_2 : X \longrightarrow Y$ *and* $t_1, t_2 : Y \longrightarrow Z$ *. If* $p_1 \simeq p_2$ *and* $t_1 \simeq t_2$*, then* $t_1 p_1 \simeq t_2 p_2$ *. 2) Assume that* $p, t : X \longrightarrow Y$ *and* $p \simeq t$*, then for any* $A \subset X$*,* $p|A \simeq t|A$*.*

1.2.2. Relative Homotopy.

DEFINITION 1.2.4. Suppose that $A \subset K$ and $f_0 : K \longrightarrow L$, $f_1 : K \longrightarrow L$ are continuous mapppings. Then f_0 and f_1 are called "homotopic relative" to A if there is a homotopy F between f_0 and f_1 such that $F(q,t)$ is independent of t for $q \in A$ i.e. $F(q,t) = f_0(q)$ for all $q \in A$ and for all $t \in C$.

It is obvious that $f_0(q) = f_1(q)$ for all $q \in A$. We call this homotopy relative, denoting with $f_0 \simeq f_1(relA)$ and, when F is to be referred, with $F : f_0 \simeq f_1(relA)$. So in this case $F(q,0) = f_0(q)$,$F(q,1) = f_1(q)$ for every $q \in K$ and $F(q_0,t) = f_0(q_0) = f_1(q_0)$ for each $q_0 \in A$ and for each $t \in C$.

THEOREM 1.2.5. *The relation of being homotopic relative to X is an equivalence relation (i.e., Reflexivity, Transitivity and Symmetry rules are valid.)*

1.2.3. Homotopy Types and Homotopy Equivalence.

DEFINITION 1.2.6. Given two spaces X and Y, we say they are homotopy equivalent, or of the same homotopy type, if there exist continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y. Such f and

g a called homotopy equivalences between X and Y. If X and Y are homotopy equivalent then they belong to the same homotopy type, i.e. homotopy type is the equivalence class with respect to homotopy equivalence.

CHAPTER 2

The Fundamental Group

2.1. Paths and Path Connected Spaces

DEFINITION 2.1.1. Assume that Q is a topological space and I denotes the unit interval $[0,1]$. A **Path** in Q is a continuous mapping $f : I \longrightarrow Q$. $f(0)$ is the initial point of path and $f(1)$ is the terminal point of path. We should know that path is the mapping f, not the image $f([0,1])$. If x is a fixed point of Q, we denote by ϵ_x the constant mapping $\epsilon_x : [0,1] \longrightarrow Q$ given by $\epsilon_x(t) = x$ for every $t \in D$. The path ϵ_x is called "null path".

DEFINITION 2.1.2. If for any two points $x, y \in A$ there is a path in A from x to y then the space A is called *path connected*. Path connected component of A is a maximal path connected subset of A.

2.2. Definition of Fundamental Group

2.2.1. Loops.

DEFINITION 2.2.1. A path $f : I \to X$ is said to be closed if $f(0) = f(1)$. If $f(0) = f(1) = x$ then we say that f is based at x, and x is called a basepoint of f.

Some books use the word "loop" for a closed path.

DEFINITION 2.2.2. A path α with $\alpha(t) = \alpha(0)$ for all $t \in [0, t]$ is called a constant loop or a trivial loop.

2.2.2. Homotopic Paths.

DEFINITION 2.2.3. Two paths α, β with common endpoints (i.e. $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$) are called *equivalent* or *homotopic* if there exists $H : I \times I \longrightarrow M$ (continuous function) such that $H(t, 0) = \alpha(t)$ $t \in [0, 1]$ $H(t, 1) = \beta(t)$ $t \in [0, 1]$ and $H(0, s) = \alpha(0) = \beta(0)$ $s \in [0, 1]$ $H(1, s) = \alpha(1) = \beta(1)$ $s \in [0, 1]$ H is giving us a family of paths connecting α and β . We denote this $\alpha \sim \beta$.

PROPOSITION 2.2.4. *Homotopy of paths is an equivalence relation, i.e.*

1) $\alpha \sim \alpha$

2) If $\alpha \sim \beta$ *then* $\beta \sim \alpha$ *and*

3) If $\alpha \sim \beta$ *and* $\beta \sim \gamma$ *then* $\alpha \sim \gamma$

The equivalence class of α is denoted $[\alpha]$.

2.2.3. Multiplication of Paths and Fundamental Group.

DEFINITION 2.2.5. If α, β paths in M with $\alpha(1) = \beta(0)$ then define $\alpha * \beta$:

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

LEMMA 2.2.6. *If* $\alpha \sim \alpha^{‘}$ *and* $\beta \sim \beta^{‘}$ *then* $\alpha * \beta \sim \alpha^{‘} * \beta^{‘}$.

Observe that the product $f * g$ is defined for any pair of closed paths based at some point $x \in X$. We denote the set of equivalence classes of closed paths based at $x \in X$ by $\pi_1(X,x)$. This set has a product defined by $[f]\cdot[g] = [f]*[g]$ for $[f],[g] \in \pi(X,x)$ which is well defined by the last lemma. The next result states that $\pi_1(X,x)$ is a group; we call it the *fundamental group* of X with base point x.

Recall that a set G with a binary operation $*$ is a group if:

1) There is a special element e with $e * x = x * e = x$ for any x [Identity]

2) For any x, There is a x^{-1} with $x * x^{-1} = x^{-1} * x = e$ [Inverse]

3) $x * (y * z) = (x * y) * z$ [Associativity]

Additionally, if $x * y = y * x$ for any x, y, we can say this group is commutative group.

THEOREM 2.2.7. *$\pi_1(X,x)$ is a group.*

LEMMA 2.2.8. *Suppose that f, g, h are three paths in X with $f(1) = g(0)$ and $g(1) = h(0)$. Then $(f*g)*h \sim f*(g*h)$.*

PROOF. Note first that,

$$(f*g)(h(t)) = \begin{cases} f(4t) & 0 \le t \le 1/4 \\ g(4t-1) & 1/4 \le t \le 1/2 \\ h(2t-1) & 1/2 \le t \le 1 \end{cases}$$

and

$$f*(g*h)(t) = \begin{cases} f(2t) & 0 \le t \le 1/4 \\ g(4t-2) & 1/2 \le t \le 3/4 \\ h(4t-3) & 3/4 \le t \le 1 \end{cases}$$

For example consider $(f*g) = h$; when $1/4 \le t \le 1/2$ we use g and compose it with a linear function that changes the interval $[1/4, 1/2]$ to

$[0,1]$, namely $t \longrightarrow 4t-1$. In fact any continuous function from $[1/4, 1/2]$ to $[0,1]$ which sends $1/4$ to 0 and $1/2$ to 1 will do but it is usually easiest to choose a linear function. For a given value of s we use f in the interval $[0,(s+1)/4]$,g in the interval $[(s+l)/4,(s+2)/4]$ and h in the interval $[(s+2)/4,1]$Using the method described above we are led to defining $F : I \times I \longrightarrow X$ by

$$F(t,s) = \begin{cases} f((4t)/(1+s)) & 0 \le t \le (s+1)/4 \\ g(4t-s-1) & (s+1)/4 \le t \le (s+2)/4 \\ h((4t-s-2)/(2-s)) & s+2/4 \le t \le 1 \end{cases}$$

The function F is continuous and $F(t,0) = ((f*g)*h)(t)$,$F(0,s) = f(0) = ((f*g)*h)(0)$,$F(t,1) = (f*(g*h))(t)$,$F(1,s) = h(1) = ((f*g)*h)(1)$ so that F provides the required homotopy. If $x \in X$ then we have defined $\epsilon_x : I \longrightarrow X$ as the constant path, i.e. $\epsilon_x(t) = x$. The equivalence path of the constant path behaves as a (left or right) identity,i.e. $[\epsilon_x][f] = [f] = [f][\epsilon_y]$ if f is a path that begins at x and ends at y.

Lemma 2.2.9. *If f is a path in X that begins at x and ends at y then $\epsilon_x * f \sim f$ and $f * \epsilon_y \sim f$.*

Proof. We shall only prove that $\epsilon_x * f \sim f$. The proof that $f * \epsilon_y \sim f$ is quite similar. Define $F : I \times I \longrightarrow X$by,

$$F(t,s) = \begin{cases} x & 0 \le t \le (1-s)/2 \\ f(2t+s-1)/(1+s) & (-s+1)/2 \le t \le 1 \end{cases}$$

Then $F(t,0) = \epsilon_x * f$,$F(t,1) = f(t)$ and F is a homotopy relative to $0,1$. Finally we would like inverses to paths (up to equivalence of paths). To this end recall that if f is a path then $\overline{f}$ is the path defined by $\overline{f} = f(1-t)$. Note that $f \sim g$ if and only iff $\overline{f} \sim \overline{g}$. The equivalence class of $\overline{f}$ acts as an inverse for the equivalence class off, i.e. $[f][\overline{f}] = [\epsilon_x]$,$[\overline{f}][f] = [\epsilon_y]$ for a path f beginning at x and ending at y. $\square$

Lemma 2.2.10. *Let f be a path in X that begins at x and ends at y.Then $f * \overline{f} \sim \epsilon_x$ and $\overline{f} * f \sim \epsilon_y$.*

Proof. We shall only prove that $f * \overline{f} \sim \epsilon_x$.The path $f * \overline{f}$ is given by ,

$$f * \overline{f}(t) = \begin{cases} f(2t) & 0 \leq t \leq 1/2 \\ f(2-2t) & 1/2 \leq t \leq 1 \end{cases}$$

It represents a path in which we travel along f for the first half of our time interval and then in the opposite direction along f for the second half. To make sure that we get from x to y and back to x we travel at speed 2 (i.e. twice 'normal' speed).If we now vary the speed proportionally to $(l-s)$ for $s \in I$ then for each s we get a path that starts at x, goes to $f(2(ls))$ and then returns to x. For $s = 0$ we get $f * \overline{f}$ and for $s = 1$ we get ϵ_x.Define therefore $F : I \times I \longrightarrow X$ by,

$$F(t,s) = \begin{cases} f(2t(1-s)) & 0 \leq t \leq 1/2 \\ f((2-2t)(1-s)) & 1/2 \leq t \leq 1 \end{cases}$$

F is obviously continuous and, $F(t,0) = (f * \overline{f})(t)$,$F(t,1) = f(0) = \epsilon_x(t) F(0,s) = f(0) = (f * \overline{f})(0)$,$F(l,s) = f(O) = (f * \overline{f})(1)$ so that $f * \overline{f} \sim \epsilon_x$ An alternative homotopy between $f * \overline{f}$ and ϵ_x is given by

$G : I \times I \longrightarrow X$ where,

$$G(t,s) = \begin{cases} f(2t) & 0 \le t \le (1-s)/2 \\ f(1-s) & (1-s)/2 \le t \le (1+s)/2 \\ f(2-2t) & (1+s)/2 \le t \le 1 \end{cases}$$

The idea here is that the time that we spend travelling along f is proportional to $(l-s)$. Thus we go along f for the first $(l-s)/2$ part of our time interval,then wait at the point $f(l-s)$ and then finally return along f for the last $(1-s)/2$ portion of our time. Thus when $s=0$ this is $f * \overline{f}$ but when $s=1$ we spend all our time waiting at x, i.e. ϵ_x. □

We have defined the product of two classes $[p] \in \pi_1(X, x_0)$ and $[s] \in \pi_1(X, x_0)$ by $[p][s] = [p*s]$. The definition of product is independent of the choice of representatives of $[f]$ and $[g]$ because, if $p \sim p_1$ and $s \sim s_1$ then $p*s \sim p_1 * s_1$ and so $[p_1][s_1] = [p_1 * s_1] = [p*s]$. Hence, the product $[p][s]$ is uniquely determined by $[p]$ and $[s]$. We show now that all the group axioms are satisfied.

(i) From definition it is obvious that $[p][s]$ is a homotopy class of paths at x_0.

(ii) Now,$([p][s])[h] = [p*s][h] = [(p*s)*h]$ and $[p]([s][h]) = [p][s*h] = [p*(s*h)]. (p*s)*h \sim p*(s*h)$. Thus, the operation is associative.

(iii) Let $[I]$ be the homotopy class of the null path at x_0, i.e. ϵ_{x_0} $[p][I] = [p] = [I][p]$. Thus, $[I]$ is a identity.

(iv) $[p][\overline{p}] = [p*\overline{p}] = [I]$ therefore that every element has an inverse so $\pi_1(X, x_0)$ is a group.

□

DEFINITION 2.2.11. Assume that for any $a_0 \in A$ if $\pi_1(A, a_0)$ is the identity group and path connected space, we can call this space *simply connected*. We usually state that $\pi_1(A, a_0)$ is the identity group by writing $\pi_1(A, a_0) = 0$

2.3. Induced Homomorphism

In this section we concerned with the effect a continuous map between topological spaces has upon fundamental groups. Suppose $\varphi : C \longrightarrow Y$ be a continuous map; (i) if v, t are paths in X then are paths in Y . (ii) if $v \sim t$ then $\varphi v \sim \varphi t$ (iii) if v is a closed path in C based at $c \in C$ then is a closed path in Y based at $p(c)$. Hence if $[v] \in \pi(C, c)$ then $[\varphi v]$ is a well-defined element of $\varphi(Y, (c))$. We denote $\varphi_* : \pi(C, c) \Rightarrow \pi(Y, \varphi(c))$ by $\varphi_*[v] = [\varphi v]$.

LEMMA 2.3.1. *φ_* is a homomorphism of groups.*

PROOF. $\varphi_*([v][t]) = \varphi_*[v * t] = [\varphi(v * t)] = [\varphi v * \varphi t] = [\varphi v][\varphi t] = \varphi_*[v]\varphi_*[t]$. □

DEFINITION 2.3.2. The homomorphism $\varphi_* : \pi_1(A, a_0) \longrightarrow \pi_1(B, p(a_0))$ determined with $\varphi_*[t] = [pt]$, where $\varphi : A \Rightarrow B$ is continuous, is called the *induced homomorphism*.

THEOREM 2.3.3. *If $p : (A, a_0) \longrightarrow (B, b_0)$ and $s : (B, b_0) \longrightarrow (C, c_0)$ are continuous, then $(s \circ p)_* = s_* \circ p_*$.*

If $i : (A, a_0) \longrightarrow (A, a_0)$ is the identity map then i_* is the identity isomorphism.

PROOF. The proof is a triviality. From Definition , $(s \circ p)_*([f]) = [(k \circ h) \circ f]$ $(s_* \circ p_*)([f]) = k_*(h_*([f])) = k_*([h \circ f]) = [k \circ (h \circ f)]$ Similarly, $i_*([f]) = [i \circ f] = [f]$ $\square$

COROLLARY 1. *Assume that* $p : (A, a_0) \longrightarrow (B, b_0)$ *is a homeomorphism of* A *with* B *then* p_* *is an isomorphism of* $\pi_1(A, a_0)$ *with* $\pi_1(B, b_0)$.

PROOF. Assume that $s : (B, b_0) \longrightarrow (A, a_0)$ be the inverse of p. Then, $s_* \circ p_* = (s \circ p)_* = i_*$ where i is the identity map of (A, a_0) and $p_* \circ s_* = (p \circ s)_* = j_*$ where j is the identity map of (B, b_0). Since i_* and j_* are the identity homomorphisms of the groups $\pi_1(A, a_0)$ and $\pi_1(B, b_0)$, s_* is the inverse of p_*. $\square$

Thus the fundamental group shows a way from topology to algebra. This has the following features; (i) For each topological space, we have a fundamental group. (ii) For each continuous map between topological spaces we have a induced homomorphism between groups. (iii) The composite of continuous maps causes the composite of the induced homomorphisms. (iv) The identity map causes the identity homomorphism. (v) A homeomorphism induces an isomorphism.

This gives a good example about what algebraic topology is. If the fundamental groups of two spaces are isomorphic it does not mean that the spaces are homeomorphic. On the other hand, if the fundamental groups are not isomorphic then the spaces cannot be homeomorphic.

2.4. Changing a Basepoint

THEOREM 2.4.1. *Assume* $c_0, c_1 \in C$. *If there is a path in* C *from* c_0 *to* c_1 *then the groups* $\pi_1(C, c_0)$ *and* $\pi_1(C, c_1)$ *are isomorphic.*

Proof. Let v be a path at c_0 and h is a path from c_0 to c_1. Then $t = (\overline{h} * v) * h$ is a path at c_1. Let θ be a transformation between the homotopy classes of paths at c_0 and the homotopy classes of paths at c_1 and showed, $\theta[v] = [t]$ $\theta[v]$ is uniquely specified by $[v]$. On the contrary $[v]$ is also uniquely specified by $\theta[v]$, because if $(\overline{h} * v_1) * h \sim (\overline{h} * v_2) * h$ then $v_1 \sim v_2$. Any path t at c_1 is homotopic to $\overline{h} * ((h * t) * \overline{h}) * h$ and thus each homotopy classes of paths at c_1 is of the form $\theta[v]$ for some $[v]$. It shows that θ is a one-one transformation of the members of $\pi_1(C, c_0)$ onto the members of $\pi_1(C, c_1)$. Assume that $[v_1]$ and $[v_2]$ are two elements of $\pi_1(C, c_0)$.Then, $\theta[v_1]\theta[v_2] = [(\overline{h} * v_1) * h][(\overline{h} * v_2) * h]$ $= [\overline{h} * v_1 * h * \overline{h} * v_2 * h] = [\overline{h} * v_1 * v_2 * h] = \theta[v_1 * v_2] = \theta([v_1][v_2])$. So θ is an isomorphism. □

2.5. Induced Homomorphisms for Homotopic Mappings

Theorem 2.5.1. *Assume that $\theta, \vartheta : K \longrightarrow L$ be continuous mappings between topological spaces and $F : \theta \simeq \vartheta$ is a homotopy . If $f : I \longrightarrow L$ is the path from $\theta(k_0)$ to $\vartheta(k_0)$ given by $f(t) = F(k_0, t)$ then the homomorphisms , $\theta_* : \pi(K, k_0) \longrightarrow \pi(L, \theta(k_0))$ and $\vartheta_* : \pi(K, k_0) \longrightarrow \pi(L, \vartheta(k_0))$ are linked via $\vartheta = u_f \theta_*$ where u_f is the isomorphism from $\pi(L, \theta(k_0))$ to $\pi(L, \vartheta(k_0))$ determined by the path f .*

Proof. We have to show that if $[g] \in \pi(K, k_0)$ then $[\vartheta g] = [\overline{f} * \theta g * f]$. In another words we should show that the paths $(\overline{f} * \theta g) * f$ and ϑg are equivalent. Observe that,

$$((\overline{f} * \vartheta g) * f))t) = \begin{cases} f(1-4t) & 0 \leq t \leq 1/4 \\ \vartheta g(4t-1) & 1/4 \leq t \leq 1/2 \\ f(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

which we will show as

$$((\overline{f} * \vartheta g) * f))t) = \begin{cases} F(x_0, 1-4t) & 0 \leq t \leq 1/4 \\ F(g(4t-1), 0) & 1/4 \leq t \leq 1/2 \\ F(x_0, 2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

The map P is clearly continuous with $P(t,0) = ((\overline{f} * \theta g * f)(t)$ $P(t,1) = ((\epsilon_X * \vartheta_g) * \epsilon_X)(t)$ $P(0,1) = F(x_0,1) = \vartheta(x_0)$ $P(1,s) = F(x_0,1) = \vartheta(x_0)$ Hence , $(\overline{f} * \theta g) * f \sim (\epsilon_K * \vartheta_g) * \epsilon_K \sim \vartheta_g$ which shows that $u_f \theta_* = \vartheta_*$ $\square$

2.6. Homotopy Equivalence and Fundamental Groups

THEOREM 2.6.1. *Assume that K and L be topological spaces such that there exist continuous mappings $f : K \longrightarrow L$ and $g : L \longrightarrow K$ satisfying $g(l_0) = k_0$ where $l_0 = f(k_0)$ and k_0 is some point of K. If gf is homotopic (rel k_0) to the identity I_K and fg is homotopic (rel l_0) to the identity I_L, then $f^* : \pi_1(K,k_0) \longrightarrow \pi_1(L,l_0)$ and $g^* : \pi_1(L,l_0) \longrightarrow \pi_1(K,k_0)$ are mutually inverse group isomorphisms.*

PROOF. We see that f, g, gf and fg induce the homomorphisms $f^* : \pi_1(K,k_0) \longrightarrow \pi_1(L,l_0)$ $g^* : \pi_1(L,l_0) \longrightarrow \pi_1(K,k_0)$ $(gf)^* : \pi_1(K,k_0) \longrightarrow \pi_1(K,k_0)$ and $(fg)^* : \pi_1(L,l_0) \longrightarrow \pi_1(L,l_0)$ respectively . Assume that α be a path at k_0 in K, then since $gf(k_0) = g(l_0) = k_0$, (gf), α is also a path at k_0 . But gf is homotopic to I_k and thus (gf) α is homotopic to α. This means that $(gf)^*$ is the identity transformation of $\pi_1(K,k_0)$.

Likewise $(fg)^*$ is the identity transformation of $\pi_1(Y, y_0)$ onto itself. But, $(gf)^* = g^*f^*$ and $(fg)^* = f^*g^*$. Hence g^*f^* and f^*g* are the identity transformations of $\pi_1(K, k_0)$ and $\pi_1(L, l_0)$ respectively. Conceive the path $(gf)\alpha : C \longrightarrow K$. This is the same like path $g(f\alpha) : C \longrightarrow K$. Thus, if $f^*[\alpha] = [\beta]$ then $g * [\beta] = g * f * [a]$ and because g^*f^* is the identity, we have $[\alpha] = g*[\beta]$. So, if $f^*[\alpha_1] = f^*[\alpha_2]$ then $[\beta_1] = [\beta_2]$ and this shows that $[\alpha_1] = [\alpha 2]$, i.e. f^* is a one-one transformation.Similarly we will learn that g^* is a one-one transformation. Further every element of $\pi_1(L, l_0)$ is of the form $f^*[\gamma]$ where $[\gamma]$ is an element of $\pi_1(K, k_0)$. Hence f^* is a one-one homomorphism, i.e. an isomorphism between $\pi_1(K, k_0)$ and $\pi_1(L, l_0)$. Similarly g^* is an isomorphism of $\pi_1(L, l_0)$ onto $\pi^1(K, k_0)$. □

This proof is instructive but in fact preservation of the points k_0 and l_0 with gf and fg respectively is not necessary for f_* and g_* being isomorphisms.

Theorem 2.6.2. *Suppose that A and B are of the same homotopy type and $\theta : A \longrightarrow B$ are a homotopy equivalence. Then, θ^* : $\pi_1(A, a) \longrightarrow \pi_1(B, \theta(a))$ is an isomorphism for any $a \in A$.*

Proof. Since ϕ is a homotopy equivalence, there exists a continuous mapping $\psi : B \to A$ such that $\phi\psi : B \to B$ is homotopic to I_b and $\psi\phi : A \to A$ is homotopic to I_a. From this, $\phi_f(\psi\phi)^* = I^*$ Since ϕ_f and I^* are isomorphisms, $(\psi\phi)^* = \phi^*\psi^*$ is an isomorphism. Therefore, ψ^* is an epimorphism and ϕ^* is a monomorphism. In the same way, $\phi^*\psi^*$ is an isomorphism and this shows that ϕ^* is an epimorphism and ψ^* is a monomorphism. □

2.7. Fundamental Group of a Circle is Infinite Cyclic

Denote S^1 the unit circle in the Euclidean plane $\mathbb{R}^2$:

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

(We can do this on $\mathbb{C}$ too). Let $f : I \longrightarrow S^1$ be the closed path that runs around the circle exactly once, defined by $f(t) = (\cos(2\pi t), \sin(2\pi t),$ $0 \leq t \leq 1$, and denote the equivalence class of f by the symbol α. Then each element of $\pi_1(S^1)$ is of the form α^n for a unique $n \in \mathbb{Z}$, by the following theorem.

Theorem 2.7.1. *(Poincaré) The fundamental group of the circle is isomorphic to the group* $(\mathbb{Z}, +)$.

CHAPTER 3

Covering Spaces

Covering Spaces are important in topology, differential geometry, Lie groups theory, Riemann surfaces theory etc. In this chapter we will learn about covering Spaces structures and properties. A lot of topological questions about covering spaces reduce to algebraic questions about the respective fundamental groups.

3.1. Notion of Covering and Some Examples

Let $\tilde{A}$ and A be topological spaces and $p : \tilde{A} \to A$ a continuous function. If for an open subset $U \subset A$ the preimage $p^{-1}(U)$ is the disjoint union of open subsets of $\tilde{A}$ each of which is mapped homeomorphically onto U by p, we say U is *evenly covered* by p. If each point $a \in A$ has an open neighbourhood evenly covered by p, the function $p : \tilde{A} \to A$ is said to be *a covering map*, $p : \tilde{A} \to A$, $\tilde{A}$ is called *a covering space* and A is *a base space.*

In other words, $p : \tilde{A} \to A$ is a covering if

- p is onto
- for all $a \in A$ there is an open neighbourhood U of a such that $p^{-1}(U) = \cup_{j\in J} U_j$

for some collection $U_j; j \in J$ of subsets of $\tilde{X}$ satisfying $U_i \cap U_j \neq \emptyset$ if $j \cap k$ and with $p \mid U_j : U_j \longrightarrow U$ being a homeomorphism for each $j \in J$.

Consider some examples of covering spaces.

Example 3. Verify that the mapping $p : \mathbb{R} \longrightarrow S^1$, $p(t) = e^{2\pi it}$ is a covering. 1) p is onto and continuous transformation. 2) For each point $z \in S^1$, the set $u = S^1 \setminus \{-z\}$ is open, and $p^{-1}(u) = p^{-1}(S^1 \setminus \{-z\}) = \mathbb{R} \setminus p^{-1}(-z)$. If $-z \in S^1$ is of the form $-z = e^{2\pi it_0}$, $t_0 \in [0,1]$, then $p^{-1}(-z)$ is the set of all $t \in \mathbb{R}$ such that $p(t) = -z = e^{2\pi it_0}$, i.e., $e^{2\pi it} = e^{2\pi it_0} \iff e^{2\pi(t-t_0)} = 1$, hence $t = t_0 + n, n \in \mathbb{Z}$. This implies $p^{-1} = \mathbb{R} \setminus \{t_0 + n; n \in \mathbb{Z}\} = \bigcup_{n\in\mathbb{Z}}(t_0+n, t_0+n+1)$, and each restriction $p \mid_{(t_0+n,t_0+n+1)} : (t_0+n, t_0+n+1) \to S^1 \setminus \{-z\}$ is a homeomorphism.

Example 4. Let W be a discrete space. We show that if $x : Q \times W \longrightarrow Q$ is the projection to the first component,then x is a covering map.

Let $b \in B$ and U_b a neighbourhood of b . Then $x^{-1}(U_b) = U_b \times W$. Since W is dicrete, $U_b \times W = \sqcup_{y\in W} U_b \times \{y\}$. Since $x|_{U_b\times\{y\}} U_b \times \{y\} \to U_b$ is a homeomorphism, p is a covering map .

Example 5. $P_n : S^1 \to S^1$, $n \in N$ and $p_n(z) = z^n$ is a covering space?

$p_n(z)$ is onto and continuous transformation. $z \in S^1$, $p^{-1}(u) = p^{-1}(S^1 \setminus \{-z\}) = S^1 \setminus p^{-1}(-z)$ $p^{-1}(-z) = \{x \in S^1; p(x) = x^n = -z\}$ For this we can take help from : $z = \mid z \mid (\cos(\alpha) + i\sin(\alpha))$ $\sqrt[n]{\mid z \mid} = W_0, ..., W_{n-1}$ $W_k = \sqrt[n]{\mid z \mid}(\cos(2\pi k + \alpha)/n) + i\sin(2\pi k + \alpha)/n)$, $k = 0, 1, ..., n-1$ $x^n = -z \neq 0$ $p_n^{-1}(-z) = \{W_0, ..., W_{n-1}\} = S^1 \setminus p^{-1}(-z) = p_n^{-1}(S^1) \setminus (-z)) = V_0 ... V_{n-1}$

EXAMPLE 6. $\tilde{X} = R^2$,$M = S^1 \times S^1$ $p : \mathbb{R}^2 \to M$,$p(x, y) = (e^{2\pi ix}, e^{2\pi iy})$ is a covering space?

p is onto and continuous transformation. $(a, b) \in S^1 \times S^1$, $u = S^1 \times S^1 \setminus (a, b)$ $p^{-1}(u) = p^{-1}$ $S^1 \times S^1 \setminus (a, b) = p^{-1}(S^1 \times S^1) \setminus p^{-1}(a, b) = R^2 \setminus p^{-1}(a, b)$ $p^{-1}(a, b) = \{(x, y); p(x, y) = (a, b) = (e^{2\pi ix_0}, e^{2\pi iy_0}), (e^{2\pi ix}, e^{2\pi iy})\} = (e^{2\pi ix_0}, e^{2\pi iy_0}) = e^{2\pi ix} = e^{2\pi ix_0}$ and $e^{2\pi iy} = e^{2\pi iy_0}$ $e^{2\pi i(x-x_0)} = 1$ and $e^{2\pi i(y-y_0)} = 1$

3.2. Properties of Coverings

THEOREM 3.2.1. *Any covering map $p : \tilde{A} \Rightarrow A$ is open.*

PROOF. Let $W \subset \tilde{A}$ is open . We will learn $p(W)$ is open in A. Let $x \in p(W)$ and There is an evenly covered open neighbourhood U of x. Admit that $\tilde{x} \in p^{-1}(x) \cap W$ and U_j in $\tilde{A}$ such that $\tilde{x} \in U_j$. We analized that $p(U_j \cap V)$ is an open subset of U . Now U is open in A and therefore $p(U_j \cap W)$ is open in A. Because, $x \in p(U_j \cap W) \subset p(W)$ and $p(W)$ is open in A. □

THEOREM 3.2.2. *If $p : \tilde{A} \longrightarrow A$ is a covering map then A has the quotient topology with respect to p.*

PROOF. Since p is a continuous open map, it come after that a subset C of A is open if and only if $p^{-1}(C)$ is open. □

PROPOSITION 3.2.3. *Let $x : A \longrightarrow B$ be a covering map. (a) If B is Hausdorff, then A is Hausdorff. (b) If B is compact and $x^{-1}(b)$ is finite for each $b \in B$, then A is compact.*

PROOF. **Solution a) :** Let a_1 and a_2 be distinct points in A . If $x(a_1) = x(a_2) = b$, then let U_b be a neighbourhood of b where $x^{-1}(U_b) =$

$\sqcup_c V_c$. Since each V_c is homeomorphic to U_b under x, a_1 and $a2$ must be involved in different V_c's, so a_1 and a_2 can be separated by open sets. If $x(a_1) =: b_1$ is explicit from $x(a_2) =: b_2$. Then there are neighbourhoods U_1 and U_2 of b_1 and b_2 such that $U_1 \cap U_2 = \emptyset$. Since x is a covering map, there are neighbourhoods V_1 and V_2 of b_1 and b_2 such that V_1 and V_2 are evenly covered by x. Let $O_i := V_i \cap U_i$ which is a neighborhood of b_i such that $O_1 \cap O_2 = \emptyset$ and O_i is evenly covered by x so that $x^{-1}(O_i) = \sqcap_{c_i} A_{c_i}$. If A_{c_i} is the open set including a_i, then since $O_1 \cap O_2 = \emptyset$, $A_{c_1} \cap A_{c_2} = \emptyset$ .So a_1 and a_2 can be separated by open sets.

Solution b) : First of all we demonstrate that for any open set $O \subset A$ including a fiber $x^{-1}(b)$, there is a small neighbourhood , W_b of b in B such that $p^{-1}(W_b) \subset O$. The proof of this statement is optional and you can use it to show the main statement. Let U_b be an equally covered neighbourhood of b, so that $\pi^{-1}(U_b) = V_{c_i} \sqcup ... V_{c_m}$ ($x^{-1}(b)$ is finite) . Let $W_{c_i} := x(O \cap V_{c_i})$. Let $W_b := \cap_{i=1}^{m} W_{c_i}$ then $x^{-1}(W_b) = \sqcup \tilde{W_{c_i}}$ where $\tilde{W_{c_i}} \subset O \cap V_{c_i} \subset O$. So $x^{-1}(W_b) \subset O$. Since x is a covering map, x is an open map.Assume that $\{U_c\}$ covers A.For each b, we have a finite subcovers $\{U_{b,1}, ..., U_{b,n_b}\} \subset \{U_c\}$ of $x^{-1}(b)$. Let W_b be a neighbourhood of b in B such that, $x^{-1}(W_b) \subset V_b := \cup_{i=1}^{n_b} U_{b_i}$. Since $\{W_b | b \in B\}$ covers B and B is compact, we have a finite subcover $\{W_{b_k}; k = 1; ...; m\}$. Since $\{x^{-1}(W_{b_k}), k = 1, ..., m\}$ covers A, $\{V_{b_k}, k = 1, ..., m\}$ covers A. Hence, $\{U_{b_k, i_{b_k}}, i_{b_k} = 1, ..., n_{b_k}, k = 1, ..., m\}$ is a required finite subcovering of $\{U_c\}$, that covers A. $\square$

DEFINITION 3.2.4. A mapping $f : X \longrightarrow Y$ then f is called a local homeomorphism if each point $x \in X$ has an open neighbourhood which is mapped homeomorphically by f onto an open subset of Y.

Note : Obviously a local homeomorphism is an open map.

THEOREM 3.2.5. *A covering map is a local homeomorphism.*

PROOF. Let $p : A \longrightarrow X$ be a covering map and $i \in A$. Let U be an open neighbourhood of $p(\tilde{i})$ which is evenly covered by p . From definition , $p^{-1}(B) = \cup_{j \in C} U_j$ with $B_j \cap B_k = \phi$ and each B_j is mapped homeomorphically onto B by p . Assume that $\tilde{B}$ is open set among $\{B_j\}$ which includes $\tilde{i}.\tilde{B}$ is an open neighbourhood of $\tilde{i}$ with the property that $p|\tilde{B}$ is a homeomorphism of $\tilde{B}$ onto B . It shows us that p is a local homeomorphism . After all a local homeomorphism does not need to be a covering map like example which we showed. □

EXAMPLE 7. Assume that $a_1 : (0,3) \longrightarrow A^1$ be the limitation of the map $a : R \longrightarrow A^1$ defined by $a(t) = e^{2\pi it}$ over open interval $(0,3)$. Since a is a covering map, it is a local homeomorphism and hence its limitation a_1 to the open set $(0,3)$ is a local homeomorphism. a_1 is a surjection too but since the complex number $1 \in A^1$ has no neighbourhood evenly covered by a_1, a_1 is not a covering map.

3.3. Liftings

DEFINITION 3.3.1. Let $p : \tilde{X} \longrightarrow X$ be a covering map. A continuous mapping $\tilde{f} : Y \to \tilde{X}$ is called a *lifting* of $f : Y \to X$ if $p\tilde{f} = f$.

3.3.1. Uniqueness.

Lemma 3.3.2. *Let $p : \tilde{A} \longrightarrow A$ is a covering map and $h, k : C \longrightarrow \tilde{T}$ are two lifts of $f : C \longrightarrow T$. Assume that C connected and there is $y_0 \in C$ such that $h(y_0) = k(y_0)$. Then $h = k$.*

Proof. Let C_1 is a set determined by $C_1 = \{y \in C; h(y) = k(y)\}$. C_1 is non-empty because $y_0 \in C_1$. We will show that C_1 is open and closed. If $y \in C$, There is an open neighbourhood V of $f(y)$ which is evenly covered by p and therefore , $p^{-1}(V) = \cup_{j \in J} V_j, V_j \cap V_i = \vartheta$ for $j \neq i$ and for each $j, p|V_j : V_j \longrightarrow V$ is a homeomorphism. If $y \in C_1$ then $h(y) = k(y) \in V + k$ for some k and we prove that $F = h^{-1}(V_k) \cap k^{-1}(V_k)$ is an open neighbourhood of y and is a subset of C_1. For this, Assume that $x \in F$ then $h(x)$ and $k(x) \in V_k$ and also $ph(x) = pk(x)$. Because $p|V_k$ is a homeomorphism, now we know that $h(x) = k(x)$. From here we can see each point of C_1 is an interior point of C_1 and C_1 is open. If $y \notin C_1$ then $h(y) \in V_k$ and $k(y) \in V_l$ for some k and l such that $k \neq l$. Finally, $h^{-1}(V_k) \cap k^{-1}(V_l)$ is an open neighbourhood of y. Hence C_1 is also closed. Since C is connected, $C = C_1$ and so $h = k$. □

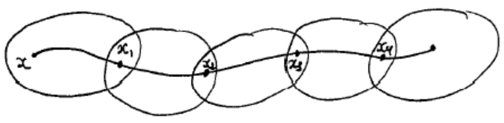
3.3.2. Path Lifting.

Proposition 3.3.3. *Let $f \in X$ and starting at a and any* lift *$\tilde{a}$ of a i.e. $p(\tilde{a}) = a$. There is only one* lift *of f to a path $\tilde{f} \in \tilde{X}$ starting a $\tilde{a}$,* *i.e., a unique path* $\overline{f} : I \Rightarrow \tilde{X}$ *such that* $\overline{f}(0) = \tilde{a}$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. We start with a path f in X and construct a *lift* as required.

Idea: p is only *locally a homeomorphism* on the B_αs display the covering space so we edge our way along, staying in one B_α at a time, We use 1) the definition of covering space, and 2) compactness of I



every cover has a finite subcover.

Step 1 : From definition of covering space we have a cover B_α of X such that for all α, $p^{-1}(B_\alpha)$ is a disjoint union of open sets in $\tilde{X}$,each of which maps homeomorphically to B_α under p. Now $f : I \longrightarrow X$ is continuous so $f^{-1}(B_\alpha)$ is an open cover of I . Hence for all of $t \in I$, there is some α with $t \in f^{-1}(U_\alpha)$ and there is an open ball $\circ A_r(t) \subseteq f^{-1}(B_\alpha)$.This gives a cover of I, so by compactness of I there is a finite subcover , $[a_0, b_0), (a_1, b_1), (a_2, b_2), ..., (a_m, b_m]$ where $a_0 = 0$, $b_m = 1$ and without loss of generality $a_0 < a_1 < b_0 < a_2 < b_1 < ... < b_m$ Put , $t_i = (a_i + b_{i-1})/2 \forall 1 \leq i \leq m$ So we have , $0 = t_0 < t_1 < ... < t_m < t_{m+1} = 1$ such that for each i, $f[t_i, t_{i+1}] \subset B_i$, say, where $B_i \in B_\alpha$.

Step 2 : We lift the path f on each of these subintervals one at a time. Conceive the first **patch** $[0, t_1]$ with $f[0, t_1]B_0$. We have

$f(0) = x$ and fix $\tilde{f}(0) = \tilde{x}$. Now $\tilde{x} \in p^{-1}(U_0)$ so has to lie in one homeomorphic copy of U_0 and called with $\tilde{U_0}$. Now We can define $\tilde{f}$ on $[0, t_1]$ by $[0, t_1] \overrightarrow{f} U_0 \overrightarrow{p^{-1}} \tilde{U_0} \subset \tilde{X}$ As in the beginning mentioned, We will go on to use our process for next each patch $[t_1, t_2], [t_2, t_3], ..., [t_m, 1]$. This constructs a lift $\tilde{f}$ of f .

Step 3 : In the end, we have to check that this lift is unique, by at heart the same argument. Suppose we have lifts $\tilde{f}, \tilde{f}_0 : I \to \tilde{X}$, so $p\tilde{f} = p\tilde{f}' = f$, and $\tilde{f}(0) = \tilde{f}'(0) = \tilde{x}$. As above, we choose a partition $0 = t_0 < t_1 < ... < t_{m+1} = 1$ such that for each i, $f[t_i, t_{i+1}] \subset$ some B_i . Now we *edge our way along* : $\tilde{f}$ is continuous and $[0, t_1]$ is connected, so $f[\tilde{0,} t_1]$ is connected in $\tilde{X}$ so $\tilde{f}[0, t_1]$ lies in one homeomorphic copy of U_0,We can say for it $\tilde{B_0}$. Also $\tilde{f}'[0, t_1]$ have to lie in the same copy as $\tilde{f}(0) = \tilde{f}'(0)$. But p gives a bijection $\tilde{B_0} \to B_0$ and we also know $p\tilde{f} = p\tilde{f}'$ so we should have $\tilde{f} = \tilde{f}'$ on $[0, t_1]$. We can go on to our process and *patch* all the way along . □

3.3.3. Homotopy Lifting.

PROPOSITION 3.3.4. *Let $\alpha : I \times A \longrightarrow X$ is a homotopy which stating at $\pi : A \longrightarrow X$ and any* lift $\tilde{\pi}$ *of* π *i.e.*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ A \arrow[ur, "\tilde{\phi}"] \arrow[r, "\phi"'] & X \end{tikzcd}$$

There is a unique lift of α to a homotopy $\tilde{\alpha}$ starting at $\tilde{\phi}$ i.e. a unique homotopy $I \times A \longrightarrow^{\tilde{a}} \tilde{X}$ such that $\tilde{a}(0,_) = \tilde{\Phi}$ and the following diagram *commutes:*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \times A \arrow[ur, "\tilde{a}"] \arrow[r, "a"'] & X \end{tikzcd}$$

3.3.4. Lifting Correspondence.

PROPOSITION 3.3.5. *Let's take a lift $\tilde{x}$ of x and $\phi_1(X, x) \longrightarrow p^{-1}(x)$ and $\tilde{\psi} \longrightarrow (1)$. and furthermore ,*

1) if $\tilde{X}$ is path-connected then this map is surjective,

2) if $\tilde{X}$ is simply connected then this map is also injective .

3.3.5. Sheets.

DEFINITION 3.3.6. If X is path-connected then the cardinality of $p^{-1}(x)$ is constant. This is called the number of *sheets* of the cover.

3.3.6. Important consequence.

DEFINITION 3.3.7. The group $\pi_1(\tilde{Y}, \tilde{y})$ is included as a subgroup of $\pi_1(Y, y)$ by the homomorphism $\pi_1(\tilde{Y}, \tilde{y}) \Rightarrow \pi_1(Y, y)$ under $\pi_1(p)$ function i.e. $\pi_1(p)$ is injective.

3.3.7. Morphisms of Covering Spaces.

DEFINITION 3.3.8. A morphism from a based covering $(\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x)$ by p_1 function to a based covering $(\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x)$ by p_2 function is a continuous function $f : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$ satisfying $p_1 = fp_2$.

Based covering spaces over $(X; x_0)$ form a category. The definition of morphism determines the notion of isomorphism: two based covering spaces $(\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x)$ by p_1 and$(\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x)$ by p_2 are isomorphic iff there are covering maps $f : \tilde{X}_1 \longrightarrow \tilde{X}_2$ and $g : \tilde{X}_2 \longrightarrow \tilde{X}_1$ so that $gf = id_{\tilde{X}_1}$ and $fg = id_{\tilde{X}_2}$.Notice that isomorphic covering spaces are homeomorphic topological spaces.

3.3.8. Deck transformations.

Definition 3.3.9. An isomorphism of covering spaces $\overline{X} \longrightarrow \overline{X}$ is called a *deck transformation* .

These form a group $G(\overline{X})$.

Definition 3.3.10. A covering space $\overline{X} \longrightarrow X$ is normal if for each $x \in X$ and each pair of lifts $\overline{x}, \overline{x}^{'} \in p^{-1}(x)$, there is a deck transformation taking $\overline{x}$ to $\overline{x}^{'}$.

Example 8. Conceive the universal covering of the Klein bottle and then explaining to all of $\mathbb{R}^2$ It is easy to see that the deck transformation group is $\mathbb{Z} \times \mathbb{Z}$ where $(m_1, n_1)(m_2, n_2) = (m_1 + (-1)^{n^{-1}} m_2, n_1 + n_2)$. Since $\mathbb{R}^2$ is simply connected,this group should be isomorphic to the fundamental group, which denoted to be $\{a, b | abab^{-1} = 1\}$ If We want show that these groups are isomorphic directly to confirm this.We can write: $\theta : \mathbb{Z} * \mathbb{Z} \Rightarrow \mathbb{Z} \times \mathbb{Z}$ is generated by $\theta(a) = (1, 0)$ and $\theta(b) = (0, 1)$.It follows that, $\theta(a^m) = (m, 0)$ $\theta(b^n = (0, n)$ $\theta(a^m, b^n) = (m, n)$ $\theta(abab^{-1}) = (0, 0)$ So we know that θ is a surjection and that the normal subgroup generated by $abab^{-1}$ is in the kernel. We can see that this is the entire kernel by considering the induced map, $\overline{\theta} : \{a, b | abab^{-1} = 1\} \longrightarrow$

$\mathbb{Z} \times \mathbb{Z}$ From the relation of $ab = ba^{-1}$ allows one to write any element of the group $\{a, b|abab^{-1} = 1\}$ as $a^m b^n$ and the map from these elements to $\mathbb{Z} \times \mathbb{Z}$ is injective, $\overline{\theta}$ is an isomorphism .

Proposition : If A is a covering space on space B; then: 1.1) The quotient map $p : B \longrightarrow B/A$ is a normal covering space. 1.2) A is the group of deck transformations if B is path connected. 1.3) A is isomorphic to $\pi_1(B/A)/p_*\pi_1(B)$ of B is path connected and locally path connected.

Proof. Given an open set U, the quotient will denote the disjoint sets $g(U)$. By the quotient topology, p limits a homeomorphism from $g(U)$ to its image $p(U)$ for each $g \in A$, and hence p is a covering space. Each element of A acts like deck transformation and openly one can get from any element in the fiber to any other (to get from $g(x_0)$ to $g'(x_0)$, simply use the element $g'g^{-1}$). Absolutely A is a subgroup of the deck transformation group. If B is path connected, then deck transformations are uniquely determined so G has to be the whole deck transformation group. □

3.3.9. Path Homotopy.

Definition 3.3.11. α, β :I⇒X paths of some end points . $\alpha \tilde{-}_p \beta$ if $F : I \times I \longrightarrow X$ such that;

<u>constant path</u> : $\epsilon p(s) = p$ for all s

<u>reverse path</u> : $\overline{a}(s)a(1 - s)$

concatenation:

$$(\alpha * \beta)(s) = \left\{ \begin{array}{ll} \alpha(2s) & s \in [0,1/2] \\ \beta(2s-1) & s \in [1/2,1] \end{array} \right\}$$

Proposition 3.3.12. *If* $\beta\tilde{\overline{a}}^{\sim}\beta'$, $\theta\overline{a}^{\sim}\theta'$ *and* $\beta(1) = \theta(0)$ *then* $\beta * \theta\overline{a}^{\sim}\beta' * \theta'$ *also* $\overline{\beta}\overline{a}^{\sim}\overline{\beta}'$

Proof. Fix $a \in X$. Let $\omega(X,a)$:="loops at a , i.e paths $\beta : I \longrightarrow$X such that $\beta(0) = \beta(1) = a$". Then, $\epsilon_a \in \omega(X,a)$, $\beta, \theta \in \omega(X,a) \longrightarrow \overline{\beta}, \beta * \theta \in \omega(X,a)$ □

Example 9. Let $f, g : A \longrightarrow B$ be a path from x_0 to x_1 and let $h : B \longrightarrow Y$ be a continuous map. 1) Show that if $F : f \longrightarrow_p g$ is a path homotopy, then $h \circ F : h \circ f \longrightarrow_p h \circ g$is a path homotopy . 2) Show that if $f(1) = g(0)$ then $h \circ (f * g) = (h \circ f) * (h \circ g)$.

Solution: $h \circ F : A \times [0,1] \longrightarrow Y$ is a continuous map . Since $h \circ F(c,0) = h \circ f(c)$ $h \circ F(c,1) = g(c)$, $h \circ F(0,t) = h(x_0)$ and $h \circ F(1,t) = h(x_1)$, $h \circ F$ is a path homotopy from $h \circ f$ to $h \circ g$.

$$(h \circ f) * (h \circ g) = \begin{cases} h \circ f(2c)\&c \in [0,1/2] \\ h \circ g(2c-1)\&c \in [1/2,1] \end{cases}$$

If $c \in [0,1/2]$, $h \circ (f * g)(c) = h(f * g)(c) = h(f(2c))$ and if $c \in [1/2,1]$, $h \circ (f * g)(c) = h(f * g)(c) = h(g(2c-1))$. Therefore $h \circ (f * g) = (h \circ f) * (h \circ g)$.

Example 10. Let $a, b : X \longrightarrow Y$ be continuous maps and assume $a(x_0) = y_0$ and $b(x_0) = y_0$. Show if there is a homotopy $A : X \times [0,1] \Rightarrow Y$ from a to b such that $A(x_0;t) = y_0$ for all $t \in [0,1]$, then $a_* = b_* : \pi_1(X,x_0) \Rightarrow \pi_1(Y,y_0)$.

Solution : Let $[f] \in \pi_1(X, x_0)$.We have to show that $[a \circ f] = [b \circ f] \in \pi_1(Y, y_0)$. Define $\tilde{A} : I \times [0,1] \Rightarrow Y$ by $\tilde{A}(s,t) = A(f(s),t)$.It is a composition of maps $A \circ (f, id_{[0;1]})$ so it is continuous. Now we can check, $\tilde{A}(s,0) = F(f(s),0) = a(f(s), \tilde{A}(s,1)) = F(f(s),1) = b(f(s))$ homotopy conditions, $\tilde{A}(0,t) = F(f(0),t) = F(x_0,t) = y_0, \tilde{A}(1,t) = A(f(1),t) = A(x_0,t) = y_0$ path-homotopy conditions. Thus $\tilde{A}$ is a path-homotopy from $a \circ f$ to $b \circ f$.

3.4. Classification of covering spaces

Definition 3.4.1. A space X is called *locally path-connected* if for any point $x \in X$ and any open set U containing x,there is a path-connected open set $V \subset U$ also containing x.

Definition 3.4.2. A space X is called *semi-locally simply-connected* if for any point $x \in X$ there is an open set U containing x in which every loop is nullhomotopic.

Remark. If A is path connected and *locally path connected*, then A is a *simply connected covering space* if and only if A is *semilocally simply connected.*

Proposition 3.4.3. *Assume that B is a path connected, locally path connected and semilocally simply connected then there exists bijection between the set of basepoint preserving isomorphism classes of path connected covering spaces. $p : (B, \overline{x_0}) \longrightarrow (B, x_0)$ and the set of subgroups of $\pi_1(B, x_0)$ attained by correlating the subgroup , $p_*(\pi_1(\overline{B}, \overline{x_0}))$ to the covering space $(\overline{B}, \overline{x_0})$. This gives a bijection between isomorphism classes* *of path connected covering spaces* $p : \overline{B} \longrightarrow B$ *and conjugacy classes of subgroups of* $\pi_1(\overline{B}, \overline{x_0})$.

3.4.1. Universal covering space.

Definition 3.4.4. If X is path connected, locally path connected, and semilocally simply connected, then X has a simply connected covering space called *the universal covering space* .

3.4.2. Connected covering spaces of S^1.

Definition 3.4.5. We knew the connected covering spaces of S^1 for each $x \in \mathbb{N}$ $S^1 \longrightarrow S^1$ and under s_x function $(1, \theta) \longrightarrow (1, x\theta)$ and $R \longrightarrow S^1$ under s function $\theta \longrightarrow (1, \theta)$ $\pi_1(S^1) = \mathbb{Z}$ so we need to take one connected covering space for each subgroup of $\mathbb{Z}$, i.e. for each $n\mathbb{Z}$ for $x \in N$ including 0 . Now, if we conceive the covering space, under s_x function $S^1 \longrightarrow S^1$ and $Im\pi_1 s_x$ is $n\mathbb{Z} \subset \mathbb{Z}$. In the end, under s, $R \longrightarrow S^1$ we get giving the subgroup : This gives all subgroups of $\mathbb{Z}$ therefore we should find all connected covering spaces of S^1 .

3.4.3. Connected covering spaces of $\mathbb{R}P^2$.

Definition 3.4.6. Assume that $\mathbb{R}P^2$ by S^2 by identifying antipodal points : $S^2 \longrightarrow^1 \mathbb{R}P^2$ $x \longrightarrow^1 [x] = \{x, -x\}$ We will see $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$ Z_2 has no non-trivial subgroups therefore this is the only connected covering space of $\mathbb{R}P^2$.

3.4.4. Connected covering spaces of S^2.

Definition 3.4.7. $\pi_1(S_2) = 0$, which has no non-trivial subgroups hence the unique connected covering space of S_2 is the unimportant one $S_2 \longrightarrow^1 S_2$.

3.4.5. Connected covering spaces of $\mathbb{R}P^2 \times \mathbb{R}P^2$.

Definition 3.4.8. Let π_1 conserves products therefore, $\pi(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \pi_1(\mathbb{R}P^2) \times \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ which has subgroups ,
1) 0 and $\mathbb{Z}_2 \times \mathbb{Z}_2$ and ,
2) subgroups of order 2 : $< (0,1) >, < (1,0) >, < (1,1) >$

And our covering space , $S^2 \times S^2 \longrightarrow \mathbb{R}P^2 \times \mathbb{R}P^2$ and $(\alpha,\beta) \longrightarrow ([\alpha],[\beta])$ and $\pi_1(S^2 \times S^2) \cong \pi_1(S^2) \times \pi_1(S^2) \cong 0$ Hence this corresponds to the trivial subgroup.

And also, $\mathbb{R}P^2 \times S^2 \longrightarrow \mathbb{R}P^2 \times \mathbb{R}P^2$ and $([\alpha],\beta) \longrightarrow ([\alpha],[\beta])$ and $S^2 \times \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2 \times \mathbb{R}P^2$ $(\alpha,[\beta]) \longrightarrow ([\alpha],[\beta])$ corresponding to $< (1,0) >$ and $< (0,1) >$. In the end,we look for the covering space corresponding to $< (1,1) >$. It can help at this point to think of $\mathbb{R}P^2 \times S^2$ as a quotient of $S^2 \times S^2$ by the relation , $(\alpha,\beta) \sim (-\alpha,\beta)$ and similarly $S^2 \times \mathbb{R}P^2$ is a quotient of $S^2 \times S^2$ by the relation , $(\alpha,\beta) \sim (-\alpha,\beta)$ and in a similar way $S^2 \times \mathbb{R}P^2$ is a quotient of $S^2 \times S^2$, $(\alpha,\beta) \sim (\alpha,-\beta)$ And Now we can see what the last covering space should be $X = (S^2 \times S^2)/\sim$ where $\sim$ is the equivalence relation defined by , $(\alpha,\beta) \sim (-\alpha,-\beta)$. This is not same with , $\mathbb{R}P^2 \times \mathbb{R}P^2 \cong S^2/\sim \times S^2/\sim \cong (S^2 \times S^2)/\sim$ with the equivalence relation , $(\alpha,\beta) \sim (\pm\alpha,\pm\beta)$ Our covering map is, $X \longrightarrow \mathbb{R}P^2 \times \mathbb{R}P^2$ and $[(\alpha,\beta)] \longrightarrow ([\alpha],[\beta])$ and this covering space corresponds to $< (1,1) >$.

3.5. Covering Space Constructions

3.5.1. Universal covering spaces.

Definition 3.5.1. Let start with (A, a) and we will make a new space with points of A together with the path . We can put a topology on , $\{[\gamma] | \gamma\}$ is a path in A starting at a . For instance, on S_1 if we turn the circle once we will not think about come "back where we started" because path is notvhomotopic to the constant path at a .

3.5.2. Non-universal covering spaces.

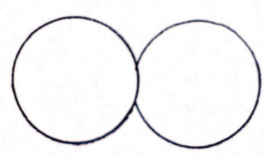
Definition 3.5.2. Let's take a covering space, $(\tilde{B}, \tilde{b}) \longrightarrow^{p} (B, b)$ this corresponds to the subgroup of $\pi_1(B, b)$ of "those loops that are still loops upstairs". Every Loop upstairs maps to a loop downstairs but some loops downstairs lift to paths upstairs that are not loops. We make non-universal covering spaces by quotienting the universal covering space just to make those loops we want still to be loops in $\tilde{B}$.

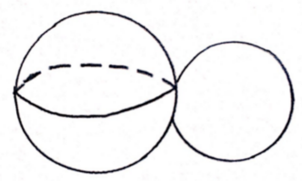
3.5.3. Wedge Sums.

Definition 3.5.3. Given based spaces (Q, q) and(C, c) their *wedge sum* is defined by steady Q and C at the basepoint $(Q, q) \vee (C, c) = (Q \sqcup C)/q \sim c$.

For example,

$S^1 \vee S^1$ is the familiar



$S^1 \vee S^2$ is 

Here we have an informal formula to make the universal covering space of $(Q, q) \vee (P, p)$.

Make the universal covers $\tilde{Q}$ of Q and $\tilde{C}$ of C .

Find all the preimages of the basepoint q in $\tilde{Q}$, and steady on a copy of $\tilde{C}$ at each one .

Find all the preimages of the basepoint c in each copy of $\tilde{C}$, and steady on a copy of $\tilde{Q}$ at each one .

Iterate ...

Negative Examples: Let Q be the space given by glueing two copies of $\mathbb{R}$ at the origin.

We have $S^1 \vee S^1$ but this is *not* a covering space. From neighbourhoods of the basepoint, It is easy to see This is not a covering space, This shows commonly to make a universal cover of $Q \vee C$ we can't just stick $\tilde{Q}$ to $\tilde{C}$, but we have to stick a copy of $\tilde{C}$ to every preimage of the basepoint in $\tilde{Q}$, and then repeat for $\tilde{C}$ and then iterate. This is **not** a covering space. $[0, 1) \longrightarrow S^1$ and $t \longrightarrow (cos2\pi t, sin2\pi t)$. This has the property of the cardinality of $p^{-1}(q)$ is constant, but this is **not** a covering space; From neighbourhoods of the point $(1, 0)$ It is easy to see.

3.6. Fundamental Group of the Covering Space

THEOREM 3.6.1. *Assume that $p : \tilde{A} \longrightarrow A$ be a covering map, where A is path connected . Let , $\tilde{a}_0, \tilde{a}_1 \in A$. Then there is a path f in A from $p(\tilde{a}_0)$ to $p(\tilde{a}_1)$ such that $\phi_f p * \pi_1(\tilde{A}, \tilde{a}_0) = p * \pi_1(\tilde{A}, \tilde{a}_1)$ (Here ϕ_f is the isomorphism determined and p^* is the induced homomorphism .)*

PROOF. Assume that g is a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$ and we know from fundamental groups ; Assume that X is a path connected space then for any $x, y \in X$, $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic. An isomorphism ϕ_g from , $\phi_1(\tilde{X}, \tilde{x}_0)$ to $\phi_1(\tilde{X}, \tilde{x}_1)$ Hence , $\phi_g \pi_1(\tilde{X}, \tilde{x}_0) = \pi_1(\tilde{X}, \tilde{x}_1)$ We use the induced homomorphism p^* and have , $p^* \phi_g \pi_1(\tilde{X}, \tilde{x}_0) = p^* \pi_1(\tilde{X}, \tilde{x}_1)$ We have that , $p^* \phi_g = \phi_{pg} p^*$ Thus if we write $f = pg$ then f is a path in X from $p(\tilde{x}_0)$ to $p(\tilde{x}_1)$. So We proved the theorem. □

Conclusions

In the end of the thesis we have learnt about what are covering spaces and fundamental groups and what is the relation between them. In the first chapter we analyzed what is notion of homotopy of mappings. It helped us to understand next chapters. In second chapter we considered fundamental groups of topological spaces, their properties and related topics, in particular, path structures, loops, induced homomorphisms and relations between them. This gave us a base to pass to covering spaces. In the third chapter, we analyzed covering space structure and we gave examples to understand it better. After we checked uniqueness of path liftings and homotopy liftings to understand covering spaces more deeply. Finally we have shown relation between fundamental groups and covering spaces. We believe this thesis shows base knowledge of fundamental groups and covering spaces.

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