

**CLASSIFICATION OF DOUBLE-CIRCULANT
SELF-DUAL CODES OVER R_k**

by

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APPROVAL PAGE

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ABSTRACT

The main goal in this dissertation is to search conditions to construct binary self-dual codes of length $2^k.n$, as images of self-dual double-circulant codes over R_k of length n . In this work, the structure of the ring R_k and self-dual codes are studied. Projections and lifts in R_k are investigated. The lifts of double-circulant self-dual codes over R_k are mapped to binary self-dual codes via the Gray map. As a result, double-circulant self-dual codes over the ring R_k are classified.

Keywords: Self-dual codes, the ring R_k , lifts and projections, Gray map, double-circulant self-dual codes over R_k .

R_k HALKASI ÜZERİNDE SELF-DUAL ÇİFT-DEVİRLİ KODLARIN SINIFLANDIRILMASI

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ÖZ

Bu bilimsel çalışmadaki ana hedef, $2^k \cdot n$ uzunluğundaki ikili self-dual kodların, R_k halkası üzerinde n uzunluğundaki çift-devirli self-dual kodların görüntüsü olarak inşa edilmesi için gerekli olan şartların araştırılmasıdır. Bu çalışmada R_k halkasının ve self-dual kodların yapısı çalışıldı. R_k halkasındaki izdüşümler ve taşınmalar araştırıldı. R_k daki çift-devirli self-dual kodların taşınmaları Gray map üzerinden ikili self-dual kodlarla eşleştirildi. Sonuç olarak, R_k halkası üzerinde çift-devirli self-dual kodlar sınıflandırıldı.

Anahtar Kelimeler: Self-Dual kodlar, R_k halkası, taşınmalar ve izdüşümler, Gray fonksiyonu, R_k üzerinde çift devirli self-dual kodlar

To my family

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LIST OF SYMBOLS AND ABBREVIATION

SYMBOL/ABBREVIATION

M.S	Master Of Science
s.d.	self-dual
DC	Double-Circulant
PDC	Pure Double-Circulant
BDC	Bordered Double-Circulant

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CHAPTER 1

INTRODUCTION

1.1 HISTORY

Coding theory began with Claude Shannon's 1948 paper "A Mathematical Theory Of Communication" for the transmission problem of digital information securely. So that errors that occur during the transfer can be recognized and detected.

In the early periods, the theory of error correcting codes was all done over the binary field \mathbb{F}_2 which has two elements and called binary codes. Linear codes are convenient in constructing, encoding and decoding codes. Hence they are the most widespread studied types of codes. In the beginning of 90's codes over rings have been started to use.

After the publication of "The \mathbb{Z}_4 - linearity of Kerdock, Preparata, Goethals and related codes" by Hammons, Calderbank, Kumar, Sloane and Sole in 1994, codes over rings have been studied extensively. This paper introduced a way obtaining good nonlinear binary codes from linear codes over \mathbb{Z}_4 by using a weight preserving Gray Map which was defined from \mathbb{Z}_4^n to \mathbb{Z}_2^{2n} .

In 1999, Dougherty, Gaborit, Harada and Sole defined a Gray Map which was a linear isometry from R_1^n onto \mathbb{F}_2^{2n} ($R_1 = \mathbb{F}_2 + u\mathbb{F}_2$) in the paper "Type II codes over $\mathbb{F}_2 + u\mathbb{F}_2$ ". The Gray Map which preserves weight and self-orthogonality was defined as $\phi(x + uy) = (y, x + y)$ in this paper where $x, y \in \mathbb{F}_2^n$.

In 2010, the ring R_2 , which is not a chain ring but is a commutative Frobenius ring was studied and linear codes over this ring are defined by Yildiz and Karadeniz. Then they studied self-dual codes over R_k .

In this thesis, using the defined Gray map we will search the conditions to construct binary self-dual codes of length $2^k.n$, as images of self-dual double-circulant codes over R_k of length n .

1.2 BASIC DEFINITIONS

Firstly, we start with basic definitions of coding theory. We refer to [16] and [17] for more.

Let \mathbb{F}_q be the field on q elements. A linear code C of length n is a subspace of the vector space \mathbb{F}_q^n . If the dimension of the code is k then C is called $[n, k]$ -code over \mathbb{F}_q and the q^k elements are called *codewords* which are denoted by $\bar{c} \in C$. The code C is named a q -ary code. If $q = 2$, the code is called *binary* and if $q = 3$ then *ternary* code.

The *Hamming weight* $wt(x)$ of a codeword $x = (x_1, x_2, \dots, x_n)$ is the number of nonzero coordinates and the *Hamming distance* (minimum distance) is defined to be the number of coordinates that differ between two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ i.e,

$$d_H(x, y) = |\{i \mid x_i \neq y_i\}|$$

where $x, y \in \mathbb{F}_q^n$. Notice that

$$d_H(x, y) = w_H(x - y), \quad \forall x, y \in \mathbb{F}_q^n.$$

The *minimum weight* of a linear code is C defined as below:

$$w(C) = \min \{w(\bar{c}) \mid \bar{c} \in C, \bar{c} \neq 0\}$$

Example 1.1. Consider the binary linear code $C = \{0000, 1011, 0101, 1110\}$.

We can see that

$$w_H(1011) = 3, w_H(0101) = 2, w_H(1110) = 3$$

Hence $d(C) = 2$

Example 1.2. Let $A = \{0, 1, 2\}$ and $x = 21101$, $y = 20001$ and $z = 12201$ then

$$d(x, y) = 2, d(x, z) = 3, d(y, z) = 3$$

The minimum distance of a code is an important and useful characteristic which gives us information about the error-correcting capacity of the code; a code whose minimum distance d can detect $d - 1$ errors and correct up to $e = \left\lfloor \frac{d-1}{2} \right\rfloor$ errors.

Let C be a code of length n , size M and distance d , then C is said to be (n, M, d) -code. The numbers n , M and d are called the *parameters* of the code and the linear code C of length n which has dimension k , is denoted by $[n, k, d]$ -code. The dimension k determines the size of the code for linear codes. In addition to this, the minimum distance equals to the minimum weight for linear codes. Using this result we can easily find $d(C)$ without being compelled to do lots of comparisons between the codewords.

The inner product of two codewords \bar{x} and \bar{y} in \mathbb{F}_q^n is defined as

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

where the operations are done in \mathbb{F}_q .

A matrix whose rows form a basis of C is called *generator matrix* of this code. The *dual code* of C is defined to be

$$C^\perp = \left\{ x \in \mathbb{F}_q^n \mid \langle x, y \rangle = \sum_{i=1}^n x_i y_i = 0, \forall y \in C \right\}$$

Example 1.3. Let $S = \{v_1, v_2, v_3\} \subseteq \mathbb{F}_2^5$ and the vectors v_1, v_2, v_3 are as below:

$$v_1 = (1, 1, 0, 1, 0), \quad v_2 = (0, 1, 0, 1, 1) \quad \text{and} \quad v_3 = (1, 1, 1, 1, 0)$$

Now, let us find the dual code S^\perp of the code S .

Let $v = (x_1, x_2, x_3, x_4, x_5) \in S^\perp$, then

$$\langle v, v_1 \rangle = 0 \Rightarrow x_1 + x_2 + x_4 = 0$$

$$\langle v, v_2 \rangle = 0 \Rightarrow x_2 + x_4 + x_5 = 0$$

$$\langle v, v_3 \rangle = 0 \Rightarrow x_1 + x_2 + x_3 + x_4 = 0$$

The results are below:

$$x_3 = 0, \quad x_1 = a, \quad x_2 = b, \quad x_4 = a + b, \quad x_5 = a \quad \text{where} \quad a, b \in \mathbb{F}_2.$$

So $v = (a, b, 0, a + b, a)$ and $S^\perp = \{(a, b, 0, a + b, a) \mid a, b \in \mathbb{F}_2\}$.

1.3 OVERVIEW OF THE THESIS

In Chapter 2, basic information about self-dual codes are given. You can see a few examples at the end of the chapter.

In Chapter 3, the ring R_k is introduced and the properties of self-dual codes over the ring are investigated. Lee weight of the elements and the distance preserving natural

Gray map from the ring to the binary field are defined. The basics of the rings R_1 and R_2 are examined specifically. Projections and lifts of codes are defined.

In Chapter 4, definitions and kinds of circulant matrices are defined. Special structures of double-circulant and bordered double-circulant matrices which can be used to construct self-dual codes are given, respectively. A new method to construct self-dual double-circulant codes over R_k as lifts of binary self-dual codes is introduced.

In Chapter 5, we tabulated our results which are obtained by the construction method we developed.

CHAPTER 2

SELF-DUAL CODES

In this chapter, we give some of the elementary properties of linear codes especially self-dual codes which are interesting class of codes and related to many different branches of study such as designs, lattice theory, invariant theory and cryptography.

Definition 2.1. A, $k \times n$ generator matrix of a code C is shown as $G = [I_k | A]$ in standard form where I_k denotes the $k \times k$ identity matrix.

If $C \subseteq C^\perp$ then C is called *self-orthogonal* and C is said to be *self-dual* if $C = C^\perp$

Since the dimension of a code plus that of its dual add up to the length, a self-dual code must be a $[2k, k]$ linear code for some k . Therefore self-dual codes exist at even lengths whereas self-orthogonal codes may exist at any length.

Some properties of self-orthogonal and self-dual codes are given below:

- If a code C is self-orthogonal then $\dim(C) \leq \frac{n}{2}$
- If a code C is self-dual then $\dim(C) = \frac{n}{2}$
- The information rate for a self-dual code is always $\frac{1}{2}$.

- In a self-dual code each codeword is orthogonal to itself thus every codeword has an even weight. Hence, the minimum Hamming weight of a self-dual code is even.
- The generator matrix of a self-dual code is equivalent to a matrix in the form

$$[I_{n/2} | A] \text{ where } A \text{ is the } \frac{n}{2} \times \frac{n}{2} \text{ matrix such that } AA^T = I_{n/2}.$$

Proposition 2.1. If C is a linear $[n, k]$ code with generating matrix $G = [I_k, A]$ then C^\perp is a linear $[n, n-k]$ code with generating matrix $H = [-A^T, I_{n-k}]$. Moreover, G whose rows form a basis for C is a parity check matrix for C^\perp . In other words, $G.H^T = \bar{0}$.

Now, let us see an example:

Example 2.1. Let C be the binary code with generating matrix $G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$.

Using the proposition 2.1., we obtain the matrix $H = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ as the generating matrix of C^\perp .

The rows of G and the rows of H generate the same subspace. Therefore, $C = C^\perp$ which means that C is self-dual.

Definition 2.2. A self-dual binary code is called *singly even (Type I)* in which there is at least one codeword with weight not divisible by 4.

A self-dual binary code is called *doubly even (Type II)* in which the weight of each codeword is divisible by 4.

A doubly even self-dual code exists if and only if $n \equiv 0 \pmod{8}$.

An upper bound for the minimum distance d of an $[n, n/2]$ self-dual code is given by Rains in the following theorem:

Theorem 2.1. [25]

$$d \leq \begin{cases} 4 \left\lfloor \frac{n}{24} \right\rfloor + 6 & , n \equiv 22 \pmod{24} \\ 4 \left\lfloor \frac{n}{24} \right\rfloor + 4 & , \text{otherwise} \end{cases}$$

A self-dual code which meets the bound is called *extremal* and a self-dual code whose d is the maximum possible minimum distance among all self-dual codes of a given length is called *optimal*.

The *weight enumerator* of a self-dual code is the polynomial that shows the number of codewords of each weight and gives us information about minimum weight of C :

The weight enumerator $W = W(C)$ of a code C is the polynomial

$$W_C(x, y) = \sum_{i=0}^n A_i x^{n-i} y^i$$

where A_i is the number of codewords in C with weight i and x, y are indeterminates.

Note that $wt(\bar{u})$ denotes the Hamming weight and

$$W_C(x, y) = \sum_{\bar{u} \in C} x^{n-wt(\bar{u})} y^{wt(\bar{u})}$$

And by writing $x = 1$, we can write the weight enumerator in one indeterminate y ,

$$A(y) = W_C(1, y) = W_C(y) = \sum_{i=0}^n A_i y^i$$

The weight enumerator of the dual code C^\perp is

$$\sum_{k=0}^n A_i x^{n-k} y^k = \frac{1}{|C|} \sum_{i=0}^n A_i (x+y)^{n-i} (x-y)^i$$

Theorem 2.2. [18]

If C is an $[n, k]$ binary linear code with its dual C^\perp , then

$$W_{C^\perp}(x, y) = \frac{1}{|C|} W_C(x + y, x - y) \quad (2.1)$$

where $|C|$ is the size of the code C . Equivalently,

$$\sum_{k=0}^n A_k x^{n-k} y^k = \frac{1}{|C|} \sum_{i=0}^n A_i (x + y)^{n-i} (x - y)^i \quad (2.2)$$

Equations (2.1) - (2.2) are called the *MacWilliams Identities*.

We complete this part with examples of binary self-dual codes.

Example 2.4. The code $C = \{(0000), (1001), (0110), (1111)\}$ is generated by the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \text{ is a } [4, 2, 2] \text{ binary self-dual singly even (Type I) code.}$$

Example 2.5. This matrix below in standard form generates $[8, 4, 4]$ extended binary Hamming code.

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Extended Hamming code which is generated by G contains 16 codewords which are given below:

$$C = \begin{cases} 00000000 & 10001110 & 01000111 & 00101011 \\ 00011101 & 11001001 & 10100101 & 10010011 \\ 01101100 & 01011010 & 00110110 & 11110000 \\ 11010100 & 01110001 & 10111000 & 11111111 \end{cases}$$

CHAPTER 3

THE RING R_k AND SELF-DUAL CODES OVER R_k

In [2], it was shown that interesting binary codes could be found as images of linear codes over \mathbb{Z}_4 under a non-linear Gray map and another Gray map was defined in [3] from a ring of order 4 called $\mathbb{F}_2 + u\mathbb{F}_2$. The Gray map for this ring is linear so the binary images are linear codes that contain fixed point in their automorphism group that correspond to multiplication by the unique non-trivial unit $1+u$ in the ring. This ring and its corresponding Gray map was generalized to codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ with a linear Gray map by Yildiz and Karadeniz in 2010.

In this chapter we introduce the ring R_k and define self-dual codes over this ring. The ring R_k is defined in [4] as follows:

$$R_k = \mathbb{F}_2[u_1, u_2, \dots, u_k] / \langle u_i^2 = 0, u_i u_j = u_j u_i \rangle$$

If $k = 0$ the ring is $R_0 = \mathbb{F}_2$

If $k = 1$ the ring is $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$

If $k = 2$ the ring is $R_2 = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$

The ring R_k can also be defined inductively as

$$R_k = R_{k-1}[u_k] / \langle u_k^2 = 0, u_k u_j = u_j u_k \rangle = R_{k-1} + u_k R_{k-1}, \quad j = 1, 2, \dots, k-1$$

The order of the ring R_k is given by the following lemma:

Lemma 3.1. [4] The ring R_k is a commutative ring with $|R_k| = 2^{\binom{k}{2}}$.

This ring is a Frobenius ring which was proved in [1]. Codes over Frobenius rings satisfy both Mac Williams theorems which was shown in [5]. See [5] for results on codes over Frobenius rings.

Consider the ideals $\langle u_1 \rangle$ and $\langle u_2 \rangle$ neither ideal contains the other and hence are not in a chain giving that the ring is not a chain ring for $k > 1$. R_k is not principal for $k > 1$. since it is easy to see that the maximal ideal is not generated by a single element.

Lemma 3.2. An element of R_k is a *unit* if and only if it contains the constant term 1. Moreover, each unit is its own multiplicative inverse. And also

$$\forall a \in R_k, \quad a \cdot (u_1, u_2 \dots u_k) = \begin{cases} u_1, u_2 \dots u_k & , \text{ if } u \text{ is a unit} \\ 0 & , \text{ otherwise .} \end{cases}$$

For example, the units in R_1 are 1 and $1+u$ and $0, u$ are non-units in R_1 . And also the multiplicative inverse of 1 is 1 and the multiplicative inverse of $1+u$ is itself.

Lemma 3.3. [1] The ring R_k is a local ring with unique maximal ideal $m_k = I_{u_1, u_2, \dots, u_k}$, this ideal contains all non-units and has $|m_k| = \frac{|R_k|}{2}$

The inner product over R_k is defined as $[v, w]_k = \sum v_i w_i$ and the dual C^\perp of C is $C^\perp = \{v \in R_k^n \mid [v, w]_k = 0 \text{ for all } w \in C\}$. By the results in [5], we have that since this ring is a Frobenius ring any linear code C satisfies $|C| \cdot |C^\perp| = |R_k|^n$.

Before defining Gray map we introduce some of the well-known weight functions:

The hamming weight, which is mentioned before, can be formulized as

$$w_H(x) := \begin{cases} 0 & , \text{ if } x=0 \\ 1 & , \text{ otherwise .} \end{cases}$$

The Lee weight on \mathbb{Z}_4 , which we will denote by w_L is defined as

$$w_L(x) := \begin{cases} 0 & , \text{ if } x=0 \\ 2 & , \text{ if } x=2 \\ 1 & , \text{ otherwise .} \end{cases}$$

The Lee weight w_L of a codeword is the Hamming weight of the image of the codeword under a linear weight preserving Gray map ϕ_k from R_k^n to $\mathbb{F}_2^{2^k n}$ which maps self-dual codes over R_k of length n to binary self-dual codes of length $2^k n$. We can define the Gray map as

$$\phi_k(\bar{c}) = (\phi_{k-1}(\bar{c}_2), \phi_{k-1}(\bar{c}_1) + \phi_{k-1}(\bar{c}_2))$$

where $\bar{c} \in R_k^n$, $\bar{c} = \bar{c}_1 + u_k \bar{c}_2$ and $\bar{c}_1, \bar{c}_2 \in R_{k-1}^n$

Here ϕ_0 is the identity map on \mathbb{F}_2 . Note that an inductive argument shows that ϕ_k is one-to-one and that $w_L(u_A) = 2^{|A|}$ for each $A \subseteq \{1, 2, \dots, k\}$. See [1] for details. The Gray images of self-dual codes over R_k are binary self-dual codes by the following lemma from [10]:

Lemma 3.4. [10] If C is a self-dual code over R_k of length n , then $\phi_k(C)$ is a binary self-dual code of length $2^k n$. The Lee weight distribution of C and the Hamming weight distribution of $\phi_k(C)$ are the same. In particular, if C is Type I, then so is $\phi_k(C)$ and the same is true for Type II codes as well.

By using the lemma above, we can easily say the following:

If C is a self-dual code over R_k of length n ; for $k=1$, $\phi_1(C)$ is a binary self-dual code of length $2n$ and for $k=2$, $\phi_2(C)$ is a binary self-dual code of length $4n$.

The following theorem is about the dual of $\phi(C)$:

Theorem 3.1. [4] Let C be a linear code over R_k of length n . Then

$$\phi(C^\perp) \subseteq (\phi(C))^\perp$$

with $(\phi(C))^\perp$ denoting the ordinary dual of $(\phi(C))$ as a binary code.

Now, we introduce the basics of two rings R_1 and R_2 specifically.

3.1 THE RING R_1

The ring R_1 is a commutative chain ring $\mathbb{F}_2 + u\mathbb{F}_2 = \{0, 1, u, 1+u\}$ with $u^2 = 0$. This ring of characteristic 2 were used to construct quaternionic unimodular lattices in [6]. For $x, y, z, t \in \mathbb{F}_2$ we can define addition and multiplication as

$$(x+uy) + (z+ut) = (x+z) + u(y+t)$$

and

$$(x+uy)(z+ut) = xz + u(xt + yz)$$

The only units in R_1 are 1 and $1+u$. R_1 is a local ring with a maximal ideal given by $I_u = \{0, u\}$. A linear Gray map from R_1^n to \mathbb{F}_2^{2n} was defined in terms of vectors as

$$\phi(x+uy) = (y, x+y)$$

which turned out to be a linear distance preserving map where $\bar{x}, \bar{y} \in \mathbb{F}_2^n$.

Using this map we obtain the images of the elements of R_1 as follows:

$$\phi_1(0) = (00), \phi_1(1) = (01), \phi_1(u) = (11), \phi_1(1+u) = (10)$$

And the corresponding Lee weights are $w_L(0) = 0$, $w_L(1) = 1$, $w_L(u) = 2$, $w_L(1+u) = 1$. It is obviously seen that the Lee weight of a codeword is the same as the Hamming weight of the image of the codeword under the Gray map.

Example 3.4. Let $\bar{c} \in R_1^6 = \bar{x} + u\bar{y}$ where $\bar{x}, \bar{y} \in \mathbb{F}_2^6$.

$$\bar{c} = (1, u) = \left(\underbrace{1, 1, 1, 0, 0, 1}_{\bar{x}} \right) + u \left(\underbrace{0, 1, 0, 0, 1, 0}_{\bar{y}} \right)$$

Then the Gray image of \bar{c} is

$$\phi_1(\bar{c}) = \phi_1(\bar{x} + u\bar{y}) = (0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1)$$

Observe that $w_L(\bar{c}) = w_H(\phi_1(\bar{c})) = 6$

In [2] it was shown that any code over R_1 is permutation-equivalent to a code C with the generator matrix

$$G = \begin{pmatrix} I_{k_1} & A & B_1 + uB_2 \\ 0 & uI_{k_2} & uD \end{pmatrix}$$

where A , B_1 , B_2 and D are matrices over \mathbb{F}_2 .

It is still an open problem to generalize the generator matrix of a code C which is permutation-equivalent to any code over R_k for $k > 1$.

The authors in [3] associated two binary codes: the residue code $C_{(1)}$ and the torsion code $C_{(2)}$ as follows:

$$C_{(1)} = \{x \in \mathbb{F}_2^n \mid \exists y \in \mathbb{F}_2^n \mid x+uy \in C\}$$

and

$$C_{(2)} = \{x \in \mathbb{F}_2^n \mid ux \in C\}$$

A generator matrix of $C_{(1)}$ is

$$G_1 = \begin{pmatrix} I_{k_1} & A & B_1 \end{pmatrix}$$

and a generator matrix of $C_{(2)}$ is

$$G_2 = \begin{pmatrix} I_{k_1} & A & B_1 \\ 0 & I_{k_2} & D \end{pmatrix}$$

The order of C is

$$|C| = |C_1| \cdot |C_2| = 2^{k_1} 2^{k_1+k_2} = 2^{2k_1+k_2} = 4^{k_1} 2^{k_2}.$$

3.2 THE RING R_2

The ring $R_2 = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ which is an extension of R_1 was first introduced by Yildiz and Karadeniz in 2010 and later they studied self-dual codes over R_2 . The ring R_2 is a characteristic 2 ring with the restrictions $u^2 = v^2 = 0$ and $uv = vu$. So R_2 is defined as follows:

$$R_2 = \{a + bu + cv + duv \mid a, b, c, d \in \mathbb{F}_2, u^2 = v^2 = 0 \text{ and } uv = vu\}$$

By the definition above it can be seen that R_2 is a 4-dimensional algebra over \mathbb{F}_2 .

And also it can be defined in terms R_1 as

$$R_2 = R_1[v] / \langle u^2 = v^2 = 0, uv = vu \rangle.$$

The 16 elements of the ring R_2 can be listed as $\{0, 1, u, v, uv, 1+u, 1+v, 1+uv, u+v, u+uv, v+uv, 1+u+v, u+v+uv, 1+u+uv, 1+v+uv, 1+u+v+uv\}$.

Addition and multiplication in R_2 are given by:

For $x = a_1 + b_1u + c_1v + d_1uv$ and $y = a_2 + b_2u + c_2v + d_2uv \in R_2$,

$$x + y = (a_1 + a_2) + u(b_1 + b_2) + v(c_1 + c_2) + uv(d_1 + d_2)$$

and

$$x \cdot y = a_1a_2 + u(a_1b_2 + a_2b_1) + v(a_1c_2 + a_2c_1) + uv(b_1c_2 + b_2c_1 + a_1d_2 + a_2d_1)$$

The units of $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ can easily be found to be the following:

$$\{1, 1+u, 1+v, 1+u+v, 1+u+uv, 1+v+uv, 1+uv, 1+u+v+uv\}$$

The ideals of the ring R_2 can be described as

$$\begin{aligned} I_0 &= \{0\}, \\ I_{uv} &= uv(R_2) = \{0, uv\}, \\ I_u &= u(R_2) = \{0, u, uv, u+uv\}, \\ I_v &= v(R_2) = \{0, v, uv, v+uv\}, \\ I_{u+v} &= (u+v)(R_2) = \{0, u+v, uv, u+v+uv\}, \\ I_{u,v} &= \{0, u, v, uv, u+v, u+uv, v+uv, u+v+uv\}. \end{aligned}$$

It is obvious to see that

$$I_0 \subseteq I_{uv} \subseteq I_u, I_v, I_{u+v} \subseteq I_{u,v} \subseteq I_1 = R_2$$

We can observe that $I_u \not\subseteq I_v$ and $I_v \not\subseteq I_u$ thus R_2 is not a chain ring. $I_{u,v}$ is the only maximal ideal of R_2 so it is a local ring.

In [7], Yildiz and Karadeniz, defined the map $\phi_2 : R_2^n \rightarrow \mathbb{F}_2^{4n}$ given by

$$\phi_2(\bar{x} + u\bar{y} + v\bar{z} + uv\bar{t}) = (\bar{t}, \bar{z} + \bar{t}, \bar{y} + \bar{t}, \bar{x} + \bar{y} + \bar{z} + \bar{t})$$

as the Gray map from R_2^n to \mathbb{F}_2^{4n} by extending from the Gray map on R_1 from [3]. Now, let us see an example:

Example 3.5. Let $\bar{c} \in R_2^4 = (\bar{x} + u\bar{y} + v\bar{z} + uv\bar{t})$ where $\bar{x}, \bar{y}, \bar{z}, \bar{t} \in \mathbb{F}_2^4$.

$$\bar{c} = (1, u, v, uv) = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\bar{x}} + u \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\bar{y}} + v \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\bar{z}} + uv \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\bar{t}}$$

Then the Gray image of \bar{c} is

$$\phi_2(\bar{c}) = \phi_2(\bar{x} + u\bar{y} + v\bar{z} + uv\bar{t}) = (0, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 1)$$

It is clear that ϕ_2 is a linear map that takes a linear code over R_2 of length n to a binary linear code of length $4n$. By using the map ϕ_2 , the Lee weight w_L was defined in [4] as follows:

Definition 3.1. For any element $x + uy + vz + uv t \in \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$,

$$w_L(x + uy + vz + uv t) = w_H(x + y + z + t, z + t, y + t, t)$$

By using the definition 3.1, we can see that there is only one element (0) of R_2 whose weight is 0 and

there are four elements whose weights are 1 $(1, 1+u, 1+v, 1+u+v+uv)$,

six elements whose weights are 2 $(u, v, u+v, u+uv, v+uv, u+v+uv)$,

four elements whose weights are 3 $(1+uv, 1+u+uv, 1+v+uv, 1+u+v)$

and one element whose weight is 4 (uv) .

3.3 SELF-DUAL CODES OVER R_k

Codes over rings were first studied in [4]. Later when rings became more popular in coding theory, the scope of self-dual codes extended to rings, too. We can refer to [7], [11], [19] and [20] for some of these works. See [8] for the works about the theoretical background of self-dual codes over Frobenius rings.

In this section, we give important properties of self-dual codes over R_k .

Definition 3.2. [7] Let C be a linear code over R_k of length n . C is said to be self-orthogonal if $C \subseteq C^\perp$ and self-dual if $C = C^\perp$

The following 3 theorems are about the existence of self-dual codes over R_k :

The existence of a self-dual code implies by Lemma 3.2 in [8], that

Theorem 3.1. Self-dual codes over R_k exist for all lengths and for all $k \geq 1$.

We observe that the theorem above, implies that Type I codes over R_k of all lengths exist. For Type II codes we have the two following theorems:

Theorem 3.2. [11] Type II codes over R_k of all lengths exist for any $k \geq 3$.

Theorem 3.3. [11] Type II codes exist over R_2 for all even lengths.

And now we give a corollary about the upper bounds for the minimum Lee weights of Type I and Type II codes over R_k :

Corollary 3.1. Let $d_L(n, I)$ and $d_L(n, II)$ denote the minimum distance of a Type I and Type II code over R_k of length n , respectively. Then for $k \geq 2$, we have

$$d_L(n, I), d_L(n, II) \leq 4 \cdot \left\lfloor \frac{2^{k-2}n}{6} \right\rfloor + 4$$

Another observation appears as Proposition 3.5 in [3]:

Theorem 3.4. If C is a self-dual code over R_k then C contains the all u_1, u_2, \dots, u_k vector.

The following lemma [11] is about obtaining a self-dual code over R_{k-1} by using the self-dual code C over R_k :

Lemma 3.5. If C is a self-dual code over R_k of length n then $\phi_k(C)$ is a self-dual code over R_{k-1} of length $2n$.

Now, we are ready to see some examples.

Example 3.6. For $n=1$: We take $C_1 = \langle (u) \rangle$. C_1 is self-dual of size 4 with Lee weight enumerator $W_{C_1}(z) = 1 + 2z^2 + z^4$ and $\phi_2(C_1)$ is a binary Type I code with parameters $[4, 2, 2]$ which is the best possible binary Type I code.

For $n=2$: Take $C_2 = \langle (1, 1) \rangle$. C_2 is self-dual with Lee weight enumerator $W_{C_2}(z) = 1 + 4z^2 + 6z^4 + 4z^6 + z^8$ and therefore $\phi_2(C_2)$ is a binary Type I code with parameters $[8, 4, 2]$ which has the highest possible minimum distance for a binary Type I code of length 8. See the results in [7] for $n=3, 4, 5, 6, 7, 8$.

Next, we consider Type II codes over R_2 for $n=2$ and $n=4$.

Example 3.7. $n=2$: Take $D_2 = \langle (1, 1 + uv) \rangle$. D_2 is self-dual, with Lee weight enumerator $W_{D_2}(z) = 1 + 14z^4 + z^8$ and thus $\phi_2(D_2)$ is a binary Type II code of

parameters $[8,4,4]$ which is extremal and unique, as it is the extended Hamming code of order 3.

$n = 4$: Take $D_4 = \langle (1,1+uv,1+u,1+u+uv), (0,0,1+u,1+u+uv) \rangle$. D_4 is self-dual, with Lee weight enumerator $W_{D_4}(z) = 1 + 28z^4 + 198z^8 + 28z^{12} + z^{16}$ and thus $\phi_2(D_4)$ is a binary Type II code of parameters $[16,8,4]$ which is extremal by Corollary 3.1. See the results in [7] for $n = 6, 8$.

3.4 PROJECTIONS AND LIFTS IN R_K

$R_{k,i} = R_k / \langle u_i \rangle$ was defined in [10] as follows:

For $1 \leq i \leq k$,

$$R_{k,i} = \mathbb{F}_2[u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_k] / \langle u_1^2, \dots, u_{i-1}^2, u_{i+1}^2, \dots, u_k^2 \rangle$$

Notice that $R_{k,k} = R_{k-1}$ and the ring $R_{k,i}$ is isomorphic to R_{k-1} for any $i = 1, 2, \dots, k$. For example,

$$\begin{aligned} R_{2,2} &= R_1 = \mathbb{F}_2 + u_1\mathbb{F}_2, \\ R_{3,1} &= \mathbb{F}_2 + u_2\mathbb{F}_2 + u_3\mathbb{F}_2 + u_2u_3\mathbb{F}_2, \\ R_{3,2} &= \mathbb{F}_2 + u_1\mathbb{F}_2 + u_3\mathbb{F}_2 + u_1u_3\mathbb{F}_2, \\ R_{4,4} &= \mathbb{F}_2 + u_1\mathbb{F}_2 + u_2\mathbb{F}_2 + u_3\mathbb{F}_2 + u_1u_2\mathbb{F}_2 + u_1u_3\mathbb{F}_2 + u_2u_3\mathbb{F}_2 + u_1u_2u_3\mathbb{F}_2 \end{aligned}$$

Now, we define projections $\pi_{k,i} : R_k \rightarrow R_{k,i}$ which is the canonical projection where $\pi_{k,i}(a) \equiv a \pmod{u_i}$. For example,

$$\begin{aligned} \pi_{3,1}(1 + u_1 + u_2 + u_1u_2 + u_2u_3 + u_1u_2u_3) &= 1 + u_2 + u_2u_3 \\ \pi_{3,2}(1 + u_1 + u_2 + u_1u_2 + u_2u_3 + u_1u_2u_3) &= 1 + u_1 \\ \pi_{3,3}(1 + u_1 + u_2 + u_1u_2 + u_2u_3 + u_1u_2u_3) &= 1 + u_1 + u_2 + u_1u_2 \end{aligned}$$

The map $\pi_{k,i}$ can be extended to R_k^n . Hence, if we have a linear code C over R_k of length n , then $\pi_{k,i}(C)$ is a linear code over $R_{k,i}$ of length n .

Definition 3.3. [10] Let C be a linear code over R_k and D be a linear code over $R_{k,i}$ such that $\pi_{k,i}(C) = D$ for some i . Then we say D is a projection of C and C is a lift of D .

Theorem 3.5. [10] *If C is a self-dual code over R_k of length n , then $\pi_{k,i}(C)$ is self-orthogonal for any $i = 1, 2, \dots, k$.*

See [10] for the proof.

The projection of a self-dual code over R_k need not to be self-dual over $R_{k,i}$. For example, $C = \langle u_1 \rangle$ is a self-dual code over R_2 but $\pi_{2,1}(C)$ is the zero code which is not self-dual.

The following corollary is about the projection of a free code C :

Corollary 3.2. [10] *If C is a self-dual code generated over R_k by a matrix of the form $[I_{n/2} | A]$ then the projections of C are self-dual in $R_{k,i}$ for all $i = 1, 2, \dots, k$.*

The following theorem contains the conditions of having a common lift in R_k for given k elements, one in each $R_{k,i}$ for $i = 1, 2, \dots, k$.

Theorem 3.6. [10] *Let a_1, a_2, \dots, a_k be elements in $R_{k,1}, R_{k,2}, \dots, R_{k,k}$ respectively. Then there exists $a \in R_k$ such that $\pi_{k,i}(a) = a_i$ for $i = 1, 2, \dots, k$ if and only if for any $0 \leq j < k$ and for any $\{i_1, i_2, \dots, i_j\} \subset \{1, 2, \dots, k\}$, the term $u_{i_1}u_{i_2}\dots u_{i_j}$ appears in either none of the a_i 's. or in exactly $k - j$ of the a_i 's. Here $j = 0$ corresponds to the term 1.*

For example, if $a_1 = 1 + u_2 + u_3 \in R_{3,1}$, $a_2 = 1 + u_3 + u_1u_3 \in R_{3,2}$ and $a_3 = 1 + u_2 + u_1u_2 \in R_{3,3}$, then it is seen that all the conditions of the theorem are satisfied and we see that $a = 1 + u_2 + u_3 + u_1u_2 + u_1u_3$ and $\pi_{3,i}(a) = a_i$ for $i=1,2,3$

The existence of how many different lifts is given by the following theorem:

Theorem 3.7. [10] *Suppose that $a_i \in R_{k,i}$ for $i=1,2,\dots,k$ are given such that they have a common lift in R_k . Then there exist exactly two lifts of a_1, a_2, \dots, a_k to R_k , denoted by a and $a' = a + u_1u_2 \dots u_k$.*

Observe that the projections of a self-dual code are also self-dual but not all lifts are self-dual.

CHAPTER 4

DOUBLE-CIRCULANT SELF-DUAL CODES

4.1 CIRCULANT MATRICES AND CODES

Definition 4.1. The *circulant matrix* $V = \text{circ}\{v\}$ associated to the vector $v \in R^n$ is the $n \times n$ matrix whose rows are given by iterations of the shift operator $T(v) = (v_{n-1}, v_0, v_1, \dots, v_{n-2})$ acting on v ; its k^{th} row is $T^{k-1}v$, for $k = 1, 2, \dots, n$.

$$V = \begin{bmatrix} v_0 & v_1 & \cdots & v_{n-2} & v_{n-1} \\ v_{n-1} & v_0 & \cdots & v_{n-3} & v_{n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ v_2 & v_3 & \cdots & v_0 & v_1 \\ v_1 & v_2 & \cdots & v_{n-1} & v_0 \end{bmatrix} = \begin{bmatrix} T^0(v) \\ T^1(v) \\ \vdots \\ T^{n-1}(v) \end{bmatrix}$$

Definition 4.2. Let R be a commutative ring, k a natural number and $\alpha \in R$. An $(k \times k)$ matrix A is called α -*circulant*, if A has the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{k-2} & a_{k-1} \\ \alpha a_{k-1} & a_0 & a_1 & \cdots & a_{k-3} & a_{k-2} \\ \alpha a_{k-2} & \alpha a_{k-1} & a_0 & \cdots & a_{k-4} & a_{k-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha a_1 & \alpha a_2 & \alpha a_3 & \cdots & \alpha a_{k-1} & a_0 \end{pmatrix}$$

with $a_i \in R$ for $i \in \{0, \dots, k-1\}$. For $\alpha = 1$, A is called *circulant*, for $\alpha = -1$, A is called *nega-circulant* or *skew-circulant* and for $\alpha = 0$, A is called *semi-circulant*. An α -*circulant matrix* A is completely determined by its first row $v = (a_0, a_1, \dots, a_{k-1}) \in R^k$.

Now, we give the definition of *double-circulant codes* by the following:

Definition 4.3. Let D_p and D_b be codes with generator matrices of the form

$$[I \ R]$$

and

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ I & \vdots & R' & \\ 1 & & & \\ 1 & & & \end{bmatrix}$$

respectively, where I is the identity matrix and R and R' are circulant matrices. An $n \times n$ circulant matrix has the form

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}$$

so that each successive row is a cyclic shift of the previous one. The codes D_p and D_b are called *pure double-circulant* and *bordered double-circulant*, respectively. These two families of codes are collectively called *double-circulant (DC)* codes. The bordered DC construction is used only when $n \equiv 0 \pmod{4}$. You can find more about on double α -circulant and bordered α -circulant codes and more details about these codes in [20].

Proposition 4.1. [23] There exists no bordered double-circulant singly-even self-dual codes of length $n \equiv 0 \pmod{8}$.

By using the proposition above, we can easily say that for the case of singly even codes, it is sufficient to find extremal bordered DC codes only for lengths $\equiv 4 \pmod{8}$.

The following lemma is about equivalence of codes:

Lemma 4.1. [21] Let C and C' be self-dual $[2n, n]$ codes with generator matrices of the form $[I_n, A]$ and $[I_n, A^T]$, where A is an $n \times n$ $(1, 0)$ matrix and A^T is the transpose of A . Then C and C' are equivalent.

4.2 DOUBLE-CIRCULANT AND BORDERED DOUBLE-CIRCULANT CONSTRUCTIONS FOR SELF-DUAL CODES OVER R_k

In this section, we give three special double-circulant and bordered double-circulant constructions for self-dual codes over the ring R_k which are mainly taken from [22].

4.2.1 The Double-Circulant Construction

A *double-circulant matrix* over R_k , is a matrix of the form $G = [I_m \mid A]$, with A being a $m \times m$ circulant matrix over R_k . Then we consider codes over R_k obtained from a double-circulant generating matrix. To have G generate a self-dual code we will put some restrictions on the entries of A . We divide this into two cases:

Odd m :

Let $m = 2n + 1$. Considering that any circulant matrix is uniquely determined by its first row, we will just describe the first row of the matrix A :

Theorem 4.1. [11] Let C be a linear code over R_k of length $4n + 2$ generated by $G = [I_{2n+1} \mid A]$ where A is a circulant $(2n + 1) \times (2n + 1)$ matrix obtained by its first row

$$A_1 = \{a_1, a_2, \dots, a_n, a_n, a_{n-1}, \dots, a_1, x\}$$

where a_i 's are arbitrary non-units and x is a unit in R_k . Then C is a self-dual code.

Even m :

Let $m = 2n$. In this case, we have the following theorem:

Theorem 4.2. [11] Let C be a linear code over R_k of length $4n$ generated by $G = [I_{2n} | B]$ where B is a circulant $(2n) \times (2n)$ matrix obtained by its first row

$$B_1 = \{b_1, b_2, \dots, b_{n-1}, b_n, b_{n-1}, b_{n-2}, \dots, b_1, y\}$$

where either

- (i) b_i 's are arbitrary units and y is a non-unit in R_k or
- (ii) b_i 's are arbitrary non-units and y is a unit.

Then C is a self-dual code.

You can see details and the codes obtained by this construction in [11] and [22].

4.2.2 The Bordered Double-Circulant Construction

A bordered double-circulant matrix with a special structure over R_k which can be used to construct self-dual codes is given in the following theorem:

Theorem 4.3. [22] Let C be a linear code of length $4m$ over R_k , generated by a bordered double-circulant matrix of the form

$$G = \left[\begin{array}{c|cccc} & x & y & y & \cdots & y \\ \hline I_{2m} & z & & & & \\ & z & & & & D \\ & \vdots & & & & \\ & z & & & & \end{array} \right]$$

where x is an arbitrary non-unit in R_k ; y and z are arbitrary units in R_k and D is a circulant $(2m-1) \times (2m-1)$ matrix over R_k with the first row given by

$$D_1 = \{d_1, d_2, \dots, d_{m-1}, d_{m-1}, d_{m-2}, \dots, d_1, xyz\}$$

Then C is a self-dual code over R_k .

4.3 LIFTING BINARY SELF-DUAL DC CODES TO R_k

Now, we will use lifts of binary s.d. PDC and BDC codes to classify s.d. DC codes over R_k . Let $R_0 = \mathbb{F}_2$, $R_1 = \mathbb{F}_2 + u\mathbb{F}_2$, $R_2 = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$ and so on. With the help of a recursive algorithm, we are able to find all s.d. DC codes over R_k from s.d. DC codes over R_{k-1} .

4.3.1 Lifting Binary Self-Dual PDC Codes To R_1

Let $M_1 = [I, A + u_1B]$ be the generator matrix of a s.d. PDC code C_1 over R_1 . It is obvious that A and B are circulant matrices over \mathbb{F}_2 . By corollary 3.2, we know that the projection C_0 of C_1 onto R_0 is a s.d. code generated by $\pi(M) = M_0 = [I, A]$. In a series of papers, all s.d. PDC codes were classified. See [21],[22] and [23]. Now, the question is how to determine s.d. lifts of C_0 to R_1 i.e. determining binary circulant matrices B 's such that $[I, A + uB]$ generates a s.d. code. Note that projection of a s.d. code is again s.d. but the reverse is not true. Therefore we need to start with C_0 to classify C_1 's.

Given that $M_0 = [I, A]$ is the generator matrix of a s.d. PDC code over R_0 of length n and let M_1 be a lift of M_0 where

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ x_n & x_1 & \cdots & x_{n-1} \\ \vdots & \vdots & & \vdots \\ x_2 & x_3 & \cdots & x_1 \end{bmatrix}$$

There are 2^n possible PDC lifts of M_0 to R_1 . We can extract out the s.d. ones by the following procedure:

Since M_1 generates a s.d. code over R_1 , $M_1 M_1^T = 0$. We get

$$\begin{aligned} [I, A + u_1 B] \begin{bmatrix} I \\ A^T + u_1 B^T \end{bmatrix} &= 0, \\ I + AA^T + u_1 (AB^T + BA^T) &= 0 \end{aligned}$$

It is known that $I + AA^T = 0$, we end up with the condition $AB^T + BA^T = \bar{0}$ which is equivalent to solving a binary linear system of equations in n -unknowns. The solutions of the system which gives us s.d. lifts can be found easily. We use MAGMA package [26] to solve the linear system.

Example 4.1. Let C_0 be a binary s.d. PDC code generated by

$$M_0 = [I_3 \mid A] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \text{ and } C_1 \text{ is of the form}$$

$$M_1 = [I_3 \mid A + uB] = \left[\begin{array}{ccc|ccc} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] & \Bigg| & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] + u \begin{bmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{bmatrix} \end{array} \right]. \text{ By using}$$

the above condition, the circulant matrices B satisfying the equation $AB^T + BA^T = \bar{0}$ are sought. We have,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{bmatrix} + \begin{bmatrix} x_1 & x_3 & x_2 \\ x_2 & x_1 & x_3 \\ x_3 & x_2 & x_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{bmatrix} + \begin{bmatrix} x_1 & x_3 & x_2 \\ x_2 & x_1 & x_3 \\ x_3 & x_2 & x_1 \end{bmatrix} = \bar{0}$$

$$\begin{bmatrix} 0 & x_2 + x_3 & x_2 + x_3 \\ x_2 + x_3 & 0 & x_2 + x_3 \\ x_2 + x_3 & x_2 + x_3 & 0 \end{bmatrix} = \bar{0}.$$

So the solutions are of the form (x_1, x_2, x_2) where $x_1, x_2 \in \mathbb{F}_2$.

It is possible to narrow the solution space by a factor of 2 by eliminating ones. The idea is as follows:

Let $\bar{c} \in R^n$ and define $T(\bar{c}) = (a_n, a_1, a_2, \dots, a_{n-1})$ to be the circular shift of \bar{C} and

$$\text{Circ}(\bar{C}) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix}$$

It is known that $\left[I, \text{circ}(\bar{c}) \right]$ and $\left[I, \text{circ}\left(T^k(\bar{c})\right) \right]$, $k=0,1,\dots,n-1$ generates equivalent codes. Without loss of generality, we may assume $a_1=1$. Multiplying columns of $[I, A+uB]$ by $(1+u)$, we get equivalent codes so it is always possible to have 1 in the first position of the circulant matrix $A+uB$. That is to say, we may set x_1 to be 0. In the previous example, there are 4 solutions but they may be reduced down to 2 if we take $x_1=0$.

4.3.2 Lifting Self-Dual PDC- R_1 Codes To R_2

In this section, we will employ similar tools as was used in the previous section recursively to construct s.d. PDC codes over R_2 . Let $M_2 = [I_n, A+u_1B+u_2C+u_1u_2D]$ be the generator matrix of a s.d. PDC code C_2 over R_2 . The projections C_{u_1} and C_{u_2} of C_2 onto R_{u_1} and R_{u_2} respectively must be s.d. codes. Note that $\pi_{u_1}(M_2) = [I_n, A+u_2C]$ and $\pi_{u_2}(M_2) = [I_n, A+u_1B]$ are generator matrices of two s.d. codes over R_1 which are

lifts of C_0 . We already know how to find matrices B, C from the previous section. The only point remains to be settled down is how to determine the matrix D . Next, we show that D can be found easily by combining two s.d. R_1 -lifts of $[I, A]$ and by solving a linear system of equations.

Theorem 4.4. Let $M_2 = [I_n, A + u_1B + u_2C + u_1u_2D]$ generate a s.d. PDC code C_2 over R_2 . Then $AD^T + BC^T + CB^T + DA^T = 0$

Proof: $M_2M_2^T = 0$ implies that

$$[I, A + u_1B + u_2C + u_1u_2D] \begin{bmatrix} I \\ A^T + u_1B^T + u_2C^T + u_1u_2D^T \end{bmatrix} = 0.$$

By multiplying matrices we get,

$$(I + AA^T) + (AB^T + BA^T)u_1 + (AC^T + CA^T)u_2 + (AD^T + BC^T + CB^T + DA^T)u_1u_2 = \bar{0}$$

Since $I + AA^T = 0$, $AB^T + BA^T = 0$, $AC^T + CA^T = 0$, the result follows immediately.

Let $D = \text{Circ}(y_1, y_2, \dots, y_n)$. Since the matrices A, B and C are known D 's can be found by solving the linear system of equations satisfying $AD^T + DA^T = BC^T + CB^T$.

4.3.3 Lifting Binary Self-Dual BDC Codes To R_1 And R_2

It can be done exactly the same way as we classify s.d. PDC codes over R_1 and R_2 but this time we start with a binary s.d. BDC code and look for s.d. lifts of it by solving a linear system.

Example 4.2. Let C_0 be a binary s.d. BDC code generated by

$$M_0 = [I_4 \mid A] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array} \right] \text{ and the generating matrix of } C_1 \text{ is of}$$

the form

$$M_1 = [I_4 \mid A + uB] = \left[\begin{array}{c|c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & x_3 & x_4 \\ 0 & x_4 & x_2 & x_3 \\ 0 & x_3 & x_4 & x_2 \end{bmatrix} & \end{array} \right] + u \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & x_3 & x_4 \\ 0 & x_4 & x_2 & x_3 \\ 0 & x_3 & x_4 & x_2 \end{bmatrix}. \quad \text{Now,}$$

we need the condition for $AB^T + BA^T = \bar{0}$. We have,

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & x_4 & x_3 \\ 0 & x_3 & x_2 & x_4 \\ 0 & x_4 & x_3 & x_2 \end{bmatrix} + \begin{bmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & x_4 & x_3 \\ 0 & x_3 & x_2 & x_4 \\ 0 & x_4 & x_3 & x_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \bar{0}$$

$$\begin{bmatrix} 0 & x_2 + x_3 + x_4 & x_2 + x_3 + x_4 & x_2 + x_3 + x_4 \\ x_1 & x_3 + x_4 & x_2 + x_3 & x_2 + x_4 \\ x_1 & x_2 + x_4 & x_3 + x_4 & x_2 + x_3 \\ x_1 & x_2 + x_3 & x_2 + x_4 & x_3 + x_4 \end{bmatrix} + \begin{bmatrix} 0 & x_1 & x_1 & x_1 \\ x_2 + x_3 + x_4 & x_3 + x_4 & x_2 + x_3 & x_2 + x_4 \\ x_2 + x_3 + x_4 & x_2 + x_4 & x_3 + x_4 & x_2 + x_3 \\ x_2 + x_3 + x_4 & x_2 + x_3 & x_2 + x_4 & x_3 + x_4 \end{bmatrix} = \bar{0}$$

$$\begin{bmatrix} 0 & x_1 + x_2 + x_3 + x_4 & x_1 + x_2 + x_3 + x_4 & x_1 + x_2 + x_3 + x_4 \\ x_1 + x_2 + x_3 + x_4 & 0 & 0 & 0 \\ x_1 + x_2 + x_3 + x_4 & 0 & 0 & 0 \\ x_1 + x_2 + x_3 + x_4 & 0 & 0 & 0 \end{bmatrix} = \bar{0}$$

So the solutions are of the form (x_1, x_1, x_3, x_3) , where $x_1, x_3 \in \mathbb{F}_2$.

By continuing the same procedure recursively, it is possible to classify all DC self-dual codes over R_k . In order to find DC self-dual over R_k , we only need self-dual R_{k-1} lifts of C_0 and a binary linear system to be solved.

CHAPTER 5

RESULTS AND CONCLUSION

To classify double-circulant self-dual codes over R_k , we firstly determine the conditions for the lifts to be self-dual. Afterwards by using these conditions we searched the lifts whose Gray image have the best minimum distance that can be found. In this chapter, we list the results of our searches. In order to fit the results to the page we represent the elements of R_2 as follows:

$$0 \rightarrow 0$$

$$1 \rightarrow 1$$

$$u \rightarrow 2$$

$$1+u \rightarrow 3$$

$$v \rightarrow 4$$

$$1+v \rightarrow 5$$

$$u+v \rightarrow 6$$

$$1+u+v \rightarrow 7$$

$$uv \rightarrow 8$$

$$1+uv \rightarrow 9$$

$$u+uv \rightarrow A$$

$$1+u+uv \rightarrow B$$

$$v+uv \rightarrow C$$

$$1+v+uv \rightarrow D$$

$$u+v+uv \rightarrow E$$

$$I+u+v+uv \rightarrow F$$

These representations also contains the elements of R_1 . We also represent *units* with x_i 's and *non-units* with y_i 's .

5.1 PURE DOUBLE-CIRCULANT SELF-DUAL CODES OVER R_K

For $n = 12$;

Let $P_{12} = [I_6 | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [x_1, x_2, y_1, x_3, x_4, x_5]$. By using V , we obtain the following conditions for the lifts of P_{12} on R_k to be self dual:

$$x_1 = x_4$$

$$x_2 = x_3$$

The generators of the lifts of P_{12} to R_2 whose binary images $\phi_2(C_{12})$ have the parameters $[48,24,8]$ are given below:

Table 1 The list of pure double-circulant self-dual codes of length 12

#	C_{12}	Type	$ Aut \phi_2(C_{12}) $
1	118111	I	$2^8.3^3.5$
2	118113	I	$2^8.3^2$
3	916199	I	$2^{10}.3$
4	916197	I	$2^{10}.3^3.5$
5	116111	II	$2^8.3^3.5$
6	11A113	II	$2^8.3^2$
7	912191	II	$2^{10}.3^3.5$
8	912193	II	$2^{10}.3$
9	914193	II	$2^{20}.3^2$
10	918193	II	$2^{21}.3^2$
11	91A193	II	$2^{21}.3^3$
12	912199	II	$2^{22}.3^2$
13	910199	II	$2^{22}.3^2.7$
14	918199	II	$2^{10}.3^2$
15	918197	II	$2^{10}.3^3.5.7$

For $n=16$;

Let $P_{16} = [I_8 | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, x_1, x_2, x_3, y_2, x_4, y_3, x_5]$. By using V , we obtain the following conditions for the lifts of P_{16} on R_k to be self dual:

$$x_1 = x_3$$

$$x_4 = x_5$$

$$y_1 = y_2$$

By using these conditions above, we obtained 34 generators of the lifts of P_{16} to R_2 whose binary images $\phi_2(C_{16})$ have the parameters $[64,32,8]$. 2 of these generators are given below:

Table 2 The list of pure double-circulant self-dual codes of length 16

#	C_{16}	Type	$ Aut \phi_2(C_{16}) $
1	21112181	I	$2^{13}.3^2$
2	21112161	II	$2^{13}.3^2$

For $n = 18$;

Let $P_{18} = [I_9 | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, y_2, x_1, y_3, x_2, x_3, y_4, x_4, x_5]$. By using V , we obtain the following conditions for the lifts of P_{18} on R_k to be self dual:

$$x_1 = x_5 + y_2 + y_4$$

$$x_2 = x_4$$

$$x_3 = x_5 + y_3 + y_4$$

$$y_1 = y_2 + y_3 + y_4$$

By using the conditions above, we obtained the generators of the lifts of P_{18} to R_2 whose binary images $\phi_2(C_{18})$ are best-known codes with parameters $[72,36,12]$ which are given below:

Table 3 The list of pure double-circulant self-dual codes of length 18

#	C_{18}	Type	$ Aut \phi_2(C_{18}) $
1	C2B61F811	I	$2^3.3^2$
2	E2B41D811	II	$2^3.3^2$
3	025213611	II	$2^3.3^2$
4	E2781D411	II	$2^3.3^2$
5	C2581F611	II	$2^3.3^2$
6	403615211	II	$2^3.3^2$
7	C0B61DA11	II	$2^3.3^2$
8	E8D217411	II	$2^3.3^2$
9	C8F215611	II	$2^3.3^2$

For $n = 20$;

Let $P_{20-1} = [I_{10} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_1 = [y_1, x_1, y_2, y_3, x_2, x_3, x_4, y_4, y_5, x_5]$. By using V_1 , we obtain the following conditions for the lifts of P_{20} on R_k to be self dual:

$$x_1 = x_5$$

$$x_2 = x_4$$

$$y_2 = y_5$$

$$y_3 = y_4$$

The generators of the lifts of P_{20-1} to R_k whose binary images $\phi_1(C_{20-1})$ are extremal with parameters $[40,20,8]$ are given below:

Table 4.1 The list of pure double-circulant self-dual codes for length 20-1

#	C_{20-1}	Type	β	$ Aut \phi_1(C_{20-1}) $
1	0102111201	I	10	$2^{16}.3^3.5^2$
2	0120111021	I	10	$2^{14}.3.5$
3	2102111201	II		$2^{14}.3.5$
4	2120111021	II		$2^{16}.3^3.5^2$

For $n = 20$;

Let $P_{20-2} = [I_{10} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_2 = [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, y_1, x_9]$. By using V_2 , we obtain the following conditions for the lifts of P_{20-2} to R_k to be self dual:

$$x_1 = x_7$$

$$x_2 = x_6$$

$$x_3 = x_5$$

$$x_8 = x_9$$

The generators of the lifts of P_{20-2} to R_k whose binary images $\phi_1(C_{20-2})$ are extremal with parameters $[40,20,8]$ are given below:

Table 4.2 The list of pure double-circulant self-dual codes for length 20-2

#	C_{20-2}	Type	β	$ Aut \phi_1(C_{20-2}) $
1	1111111303	I	10	$2^{14}.3.5$
2	3111113101	I	10	$2^{16}.3^3.5^2$
3	1111111323	II		$2^{14}.3.5$
4	3111113121	II		$2^{16}.3^3.5^2$

For $n = 22$;

Let $P_{22} = [I_{11} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, x_1, x_2, x_3, y_4, x_4, y_5, y_6, x_5]$. By using V , we obtain the following conditions for the lifts of P_{22} to R_k to be self dual:

$$x_1 = x_5 + y_2 + y_3$$

$$x_2 = x_5 + y_3 + y_6$$

$$x_3 = x_5 + y_3 + y_4$$

$$x_4 = x_5 + y_2 + y_4 + y_5 + y_6$$

$$y_1 = y_2 + y_3 + y_4 + y_5 + y_6$$

The generators of the lifts of P_{22} to R_1 whose binary images $\phi_1(C_{22})$ are extremal with parameters $[44,22,8]$ are given below:

Table 5.1 The list of pure double-circulant self-dual codes of length 22

#	C_{22}	Type	β	$ Aut \phi_1(C_{22}) $
1	02213123201	I	0	$2^2 \cdot 11$
2	02213303001	I	0	$2^2 \cdot 11$
3	22213301201	I	22	$2^2 \cdot 11$
4	02031101201	I	44	$2^2 \cdot 11$
5	22211121221	II		$2^{16} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2$

Additionally, by using the same conditions above, we obtained the generators of the lifts of P_{22} to R_2 whose binary images $\phi_2(C_{22})$ are extremal Type II codes with parameters $[88,44,16]$ and best-known Type I codes with parameters $[88,44,14]$ which are given below in the following two tables consecutively:

Table 5.2 The list of pure double-circulant self-dual codes of length 22

#	C_{22}	Type	$ Aut \phi_1(C_{22}) $
1	42215B87A61	II	$2^3 \cdot 11$
2	0221B563E81	II	$2^3 \cdot 11$
3	E221556DE61	II	$2^3 \cdot 11$
4	422179A7A41	II	$2^3 \cdot 11$
5	2247F9C72A1	II	$2^3 \cdot 11$
6	02473BE5E61	II	$2^3 \cdot 11$
7	028B3D494A1	II	$2^3 \cdot 11$
8	028BF189461	II	$2^3 \cdot 11$
9	E26B5F67EC1	II	$2^3 \cdot 11$
10	628BD3AF241	II	$2^3 \cdot 11$
11	02655BC7A21	II	$2^3 \cdot 11$
12	02A95B0B6E1	II	$2^3 \cdot 11$
13	40239567AA1	II	$2^3 \cdot 11$
14	C001B10D6A1	II	$2^3 \cdot 11$
15	A001B76B6A1	II	$2^3 \cdot 11$
16	20895B2B4C1	II	$2^3 \cdot 11$
17	E089D907241	II	$2^3 \cdot 11$
18	A067DF8DEA1	II	$2^3 \cdot 11$
19	C82BB74FA81	II	$2^3 \cdot 11$
20	082B79A3441	II	$2^3 \cdot 11$
21	A625D9A9AE1	II	$2^3 \cdot 11$
22	C6077BAD661	II	$2^3 \cdot 11$
23	A607DBABAC1	II	$2^3 \cdot 11$
24	226DD96D229	II	$2^3 \cdot 11$

Table 5.3 The list of pure double-circulant self-dual codes of length 22

#	C_{22}	Type	$ Aut \phi_1(C_{22}) $
1	62215125261	I	$2^3 \cdot 11$
2	4221F747CC1	I	$2^3 \cdot 11$
3	C2215B8F261	I	$2^3 \cdot 11$
4	82215B8B661	I	$2^3 \cdot 11$
5	C221B56F281	I	$2^3 \cdot 11$
6	E221956D2A1	I	$2^3 \cdot 11$
7	E203976F281	I	$2^3 \cdot 11$
8	C2473D89461	I	$2^3 \cdot 11$
9	8247B36D6E1	I	$2^3 \cdot 11$
10	024759C5A01	I	$2^3 \cdot 11$
11	628BF90FA61	I	$2^3 \cdot 11$
12	E28B31876A1	I	$2^3 \cdot 11$
13	E265B7096C1	I	$2^3 \cdot 11$
14	0265FF87481	I	$2^3 \cdot 11$
15	42A93B0F481	I	$2^3 \cdot 11$
16	42A9BD6FA01	I	$2^3 \cdot 11$
17	E023D74D6E1	I	$2^3 \cdot 11$
18	C0899B25601	I	$2^3 \cdot 11$
19	8067D16F2A1	I	$2^3 \cdot 11$
20	882BB56BC81	I	$2^3 \cdot 11$
21	6809B7672A1	I	$2^3 \cdot 11$
22	2625D9A12E1	I	$2^3 \cdot 11$
23	268FDB2BA41	I	$2^3 \cdot 11$
24	26E9D96DE21	I	$2^3 \cdot 11$

For $n = 24$;

Let $P_{24} = [I_{12} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, x_1, x_2, x_3, y_2, y_3, x_4, y_4, y_5, y_6, y_7, x_5]$. By using V , we obtain the following conditions for the lifts of P_{24} to R_k to be self dual:

$$x_1 = x_5 + y_4 + y_6$$

$$x_2 = x_4$$

$$x_3 = x_5 + y_3 + y_6$$

$$y_1 = y_5$$

$$y_2 = y_7$$

The generators of the lifts of P_{24} to R_1 whose binary images $\phi_1(C_{24})$ have the parameters $[48,24,8]$ are given below:

Table 6 The list of pure double-circulant self-dual codes of length 24

#	C_{24}	Type	$ Aut \phi_1(C_{24}) $
1	231322122021	I	$2^4 \cdot 3$
2	231122102221	I	$2^7 \cdot 3^2$
3	031122100221	I	$2^{11} \cdot 3$
4	211120102021	I	$2^{20} \cdot 3^2$
5	011120100021	I	$2^7 \cdot 3 \cdot 5$
6	211122122221	II	$2^{21} \cdot 3^6 \cdot 5^2$
7	211102122201	II	$2^7 \cdot 3 \cdot 5$
8	011122120221	II	$2^8 \cdot 3 \cdot 5$
9	011102120201	II	$2^{20} \cdot 3^3$
10	211322102021	II	$2^4 \cdot 3$
11	211302102001	II	$2^4 \cdot 3$
12	011322100021	II	$2^4 \cdot 3$
13	011302100001	II	$2^4 \cdot 3$
14	231120122021	II	$2^8 \cdot 3^3$
15	231100122001	II	$2^{11} \cdot 3^2$
16	031120120021	II	$2^{12} \cdot 3$
17	031100120001	II	$2^7 \cdot 3^2$

For $n = 26$;

Let $P_{26} = [I_{13} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [x_1, x_2, y_1, x_3, y_2, y_3, x_4, x_5, x_6, y_4, x_7, x_8, x_9]$. By using V , we obtain the following conditions for the lifts of P_{26} to R_k to be self dual:

$$x_1 = x_4 + x_6 + x_9$$

$$x_2 = x_4 + x_6 + x_8 + y_3 + y_4$$

$$x_3 = x_6 + x_8 + x_9 + y_3 + y_4$$

$$x_5 = x_8 + y_2 + y_3$$

$$x_7 = x_8 + y_2 + y_4$$

$$y_1 = y_2 + y_3 + y_4$$

The generators of the lifts of P_{26} to R_1 whose binary images $\phi_1(C_{26})$ are extremal Type I codes with parameters $[52,26,10]$ are given below:

Table 7.1 The list of pure double-circulant self-dual codes of length 26

#	C_{26}	β	$ Aut \phi_1(C_{26}) $
1	1303201312111	0	$2^2 \cdot 13$
2	1121201310311	0	$2^2 \cdot 13$
3	1101021312311	0	$2^2 \cdot 13$
4	1323021310111	0	$2^2 \cdot 13$
5	3321201110133	0	$2^2 \cdot 13$
6	3301021112133	0	$2^2 \cdot 13$

Additionally, by using the same conditions above, we obtained the generators of the lifts of P_{26} to R_2 whose binary images $\phi_2(C_{26})$ are Type I codes with parameters $[104,52,16]$ which are given below:

Table 7.2 The list of pure double-circulant self-dual codes of length 26

#	C_{26}	$ Aut \phi_2(C_{26}) $
1	1FCF261518B11	$2^3 \cdot 13$
2	1525621516111	$2^3 \cdot 13$
3	15256A1D1E911	$2^3 \cdot 13$
4	3143441114113	$2^3 \cdot 13 \cdot 13$
5	3D8F46131AF13	$2^3 \cdot 13$
6	31434A1F1AF13	$2^3 \cdot 13$
7	33614E1B1C913	$2^3 \cdot 13$
8	3341681F1AD13	$2^3 \cdot 13$
9	3163661116113	$2^3 \cdot 13 \cdot 13$
10	31636C1B1CB13	$2^3 \cdot 13$
11	37C5A41F12913	$2^3 \cdot 13$
12	9DE5241718B19	$2^3 \cdot 13$
13	9B8324171ED19	$2^3 \cdot 13$
14	79AF24171CF17	$2^3 \cdot 13$
15	79AF2C1F14717	$2^3 \cdot 13$
16	75C38A131E719	$2^3 \cdot 13$
17	75E3A21716D17	$2^3 \cdot 13$
18	792FA41F1C717	$2^3 \cdot 13$
19	796FEA1512D17	$2^3 \cdot 13$
20	15254A1D1CB31	$2^3 \cdot 13$
21	1F8F4E1912531	$2^3 \cdot 13$
22	11614E191CB31	$2^3 \cdot 13$
23	13636E1B1EB31	$2^3 \cdot 13$
24	3341441314333	$2^3 \cdot 13$
25	3DAF44131AD33	$2^3 \cdot 13$
26	3DAF461118F33	$2^3 \cdot 13$
27	33414E191E933	$2^3 \cdot 13$
28	31A924171CF11	$2^3 \cdot 13$

Furthermore, by using the same conditions, we obtained 203 generators of the lifts of P_{26} to R_2 whose binary images $\phi_2(C_{26})$ have the parameters $[104,52,16]$. 1 of these generators is given below:

Table 7.3 The list of pure double-circulant self-dual codes of length 26

#	C_{26}	Type	$ Aut \phi_2(C_{26}) $
1	19A924171CF11	II	$2^3.13$

For $n = 30$;

Let $P_{30} = [I_{15} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [x_1, y_1, y_2, x_2, x_3, y_3, y_4, x_4, y_5, y_6, x_5, x_6, x_7, x_8, x_9]$. By using V , we obtain the following conditions for the lifts of P_{30} to R_k to be self dual:

$$x_1 = x_5 + x_8 + x_9 + y_5 + y_6$$

$$x_2 = x_5 + x_8 + x_9 + y_3 + y_6$$

$$x_3 = x_5 + y_3 + y_5$$

$$x_4 = x_9 + y_3 + y_5$$

$$x_6 = x_8 + y_4 + y_6$$

$$x_7 = x_8 + y_2 + y_6$$

$$y_1 = y_2 + y_3 + y_4 + y_5 + y_6$$

The generator of the lift of P_{30} to R_1 whose binary image $\phi_1(C_{30})$ is extremal with parameters $[60,30,12]$ is given below:

Table 8 The list of pure double-circulant self-dual code of length 30

#	C_{30}	Type	β	$ Aut \phi_1(C_{30}) $
1	120332030011111	I	10	$2^2.3.5$

For $n = 34$;

Let $P_{34} = [I_{17} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, y_6, y_7, x_4, x_5, y_8, x_6, x_7, x_8, x_9]$. By using V , we obtain the following conditions for the lifts of P_{34} to R_k to be self dual:

$$x_1 = x_9 + y_6 + y_7$$

$$x_2 = x_9 + y_2 + y_3 + y_7 + y_8$$

$$x_3 = x_7 + y_4 + y_5$$

$$x_4 = x_7 + y_5 + y_6 + y_7 + y_8$$

$$x_5 = x_9 + y_3 + y_4 + y_5 + y_6$$

$$x_6 = x_9 + y_2 + y_4 + y_5 + y_6$$

$$x_8 = x_9 + y_6 + y_8$$

$$y_1 = y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8$$

The generators of the lifts of P_{34} to R_1 whose binary images $\phi_1(C_{34})$ are extremal with parameters $[68,34,12]$ are given below:

Table 9 The list of pure double-circulant self-dual codes of length 34

#	C_{34}	Type	β	γ	$ Aut \phi_1(C_{34}) $
1	02222331203121111	I	170	0	$2^2 \cdot 17$
2	02202133001121131	I	204	0	$2^2 \cdot 17$
3	22200311203101131	I	204	0	$2^2 \cdot 17$
4	22022311203321111	I	136	0	$2^2 \cdot 17$
5	02020333021321131	I	136	0	$2^2 \cdot 17$
6	02002113223103131	I	102	0	$2^2 \cdot 17$
7	22000331021123131	I	102	0	$2^2 \cdot 17$
8	00200131223123111	I	102	0	$2^2 \cdot 17$
9	20022111221323111	I	68	0	$2^2 \cdot 17$
10	00022131223303131	I	136	0	$2^2 \cdot 17$
11	00022111003101111	I	136	0	$2^2 \cdot 17$
12	20020133221101131	I	204	0	$2^2 \cdot 17$
13	20002113003303111	I	170	0	$2^2 \cdot 17$
14	02202133201101113	I	170	0	$2^2 \cdot 17$
15	02022331001321113	I	136	0	$2^2 \cdot 17$
16	02000311223123133	I	102	0	$2^2 \cdot 17$
17	22000131201123133	I	34	0	$2^2 \cdot 17$
18	20222331223301113	I	204	0	$2^2 \cdot 17$
19	20222131203321133	I	272	0	$2^2 \cdot 17$
20	20202133021301133	I	136	0	$2^2 \cdot 17$
21	00022111203121133	I	238	0	$2^2 \cdot 17$
22	20020113201323133	I	170	0	$2^2 \cdot 17$
23	00002313001121113	I	102	0	$2^2 \cdot 17$

For $n = 38$;

Let $P_{38} = [I_{19} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, y_4, x_1, y_5, x_2, y_6, x_3, x_4, x_5, x_6, y_7, y_8, x_7, y_9, y_{10}, x_8, x_9]$. By using V , we obtain the following conditions for the lifts of P_{38} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_9 + y_2 + y_4 + y_5 + y_6 + y_7 + y_8 \\
x_2 &= x_9 + y_4 + y_6 + y_7 + y_9 \\
x_3 &= x_9 + y_3 + y_5 + y_9 + y_{10} \\
x_4 &= x_9 + y_4 + y_6 + y_8 + y_9 \\
x_5 &= x_9 + y_6 + y_{10} \\
x_6 &= x_9 + y_2 + y_4 + y_6 + y_9 \\
x_7 &= x_9 + y_2 + y_3 + y_4 + y_6 + y_7 + y_8 \\
x_8 &= x_9 + y_4 + y_{10} \\
y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10}
\end{aligned}$$

The generators of the lifts of P_{38} to R_1 whose binary images $\phi_1(C_{38})$ have the parameters $[76,38,12]$ are given below:

Table 10 The list of pure double-circulant self-dual codes of length 38

#	C_{38}	Type	$ Aut \phi_1(C_{38}) $
1	0222323211332030031	I	$2^2 \cdot 19$
2	2222123213110012211	I	$2^2 \cdot 19$
3	0222121231130010211	I	$2^2 \cdot 19$
4	0222321011112230031	I	$2^2 \cdot 19$
5	2222123011110210031	I	$2^2 \cdot 19$
6	2222321031130032031	I	$2^2 \cdot 19$
7	0222323013110030031	I	$2^2 \cdot 19$
8	0222303233110012211	I	$2^2 \cdot 19$
9	0222301231330010031	I	$2^2 \cdot 19$
10	0222301033330212211	I	$2^2 \cdot 19$
11	0222303031110210031	I	$2^2 \cdot 19$
12	0222101011130032031	I	$2^2 \cdot 19$
13	0220121233330212011	I	$2^2 \cdot 19$
14	2220123013132210011	I	$2^2 \cdot 3 \cdot 19$
15	2220321033112032011	I	$2^2 \cdot 19$
16	0220123033110012011	I	$2^2 \cdot 19$
17	2220121011130010011	I	$2^2 \cdot 19$
18	0220101211330032011	I	$2^2 \cdot 19$
19	2220103233310030011	I	$2^2 \cdot 19$
20	0220103011110232011	I	$2^2 \cdot 19$
21	0220301031130010011	I	$2^2 \cdot 19$
22	2202101031130012031	I	$2^2 \cdot 19$
23	0202103013110010031	I	$2^2 \cdot 19$
24	0200103213310010011	I	$2^2 \cdot 19$
25	0200303033110032011	I	$2^2 \cdot 19$

For $n = 40$;

Let $P_{40-1} = [I_{20} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, y_3, y_4, y_5, x_1, y_6, y_7, y_8, x_2, x_3, y_9, x_4, x_5, y_{10}, x_6, y_{11}, x_7, x_8, x_9]$. By using V_1 , we obtain the following conditions for the lifts of P_{40-1} to R_k to be self dual:

$$\begin{aligned} x_1 &= x_6 \\ x_2 &= x_9 + y_4 + y_7 \\ x_3 &= x_8 + y_1 + y_6 + y_8 + y_{10} \\ x_4 &= x_8 + y_6 + y_{11} \\ x_5 &= x_9 + y_7 + y_9 \\ x_7 &= x_9 + y_4 + y_9 \\ y_2 &= y_4 + y_7 + y_9 \\ y_3 &= y_6 + y_8 + y_{11} \\ y_5 &= y_6 + y_{10} + y_{11} \end{aligned}$$

By using these conditions above, we obtained 23 generators of the lifts of P_{40-1} to R_1 whose binary images have the parameters $[80,40,12]$. 2 of these generators are given below:

Table 11 The list of pure double-circulant self-dual codes of length 40-1

#	C_{40-1}	Type	$ Aut \phi_1(C_{40-1}) $
1	22020122211231210111	I	$2^4 \cdot 5$
2	20222120231213212111	II	$2^4 \cdot 5$

For $n = 40$;

Let $P_{40-2} = [I_{20} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, y_4, y_5, y_6, x_1, y_7, x_2, y_8, y_9, x_3, x_4, x_5, x_6, x_7, y_{10}, x_8, y_{11}, x_9]$. By using V_2 , we obtain the following conditions for the lifts of P_{40-2} to R_k to be self dual:

$$x_1 = x_6$$

$$x_2 = x_4$$

$$x_3 = x_9 + y_2 + y_8$$

$$x_5 = x_9 + y_2 + y_7$$

$$x_7 = x_9 + y_2 + y_6$$

$$x_8 = x_9 + y_2 + y_4$$

$$y_1 = y_9$$

$$y_3 = y_{11}$$

$$y_5 = y_{10}$$

By using these conditions above, we obtained 24 generators of the lifts of P_{40-2} to R_1 whose binary images have the parameters $[80,40,12]$. 2 of these generators are given below:

Table 12 The list of pure double-circulant self-dual codes of length 40-2

#	C_{40-2}	Type	$ Aut \phi_1(C_{40-2}) $
1	22222212102311112121	I	$2^4.5$
2	22022212122111112101	II	$2^5.5$

For $n = 40$;

Let $P_{40-3} = [I_{20} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_3 = [y_1, y_2, y_3, y_4, x_1, y_5, y_6, y_7, x_2, y_8, y_9, x_3, x_4, x_5, y_{10}, x_6, x_7, x_8, y_{11}, x_9]$. By using V_3 , we obtain the following conditions for the lifts of P_{40-3} to R_k to be self dual:

$$x_1 = x_7$$

$$x_2 = x_4$$

$$x_3 = x_9 + y_2 + y_8$$

$$x_5 = x_9 + y_2 + y_7$$

$$x_8 = x_9 + y_2 + y_4$$

$$y_1 = y_9$$

$$y_3 = y_{11}$$

$$y_6 = y_{10}$$

By using these conditions above, we obtained 23 generators of the lifts of P_{40-3} to R_1 whose binary images have the parameters $[80,40,12]$. 2 of these generators are given below:

Table 13 The list of pure double-circulant self-dual codes of length 40-3

#	C_{40-3}	Type	$ Aut \phi_1(C_{40-3}) $
1	22221222102311211121	I	$2^4 \cdot 5$
2	22021222122111211101	II	$2^5 \cdot 5$

For $n = 40$;

Let $P_{40-4} = [I_{20} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_4 = [y_1, y_2, y_3, y_4, x_1, y_5, y_6, y_7, x_2, y_8, x_3, x_4, x_5, x_6, y_9, y_{10}, x_7, x_8, y_{11}, x_9]$. By using V_4 , we obtain the following conditions for the lifts of P_{40-4} to R_k to be self dual:

$$x_1 = x_7 + y_6 + y_9$$

$$x_2 = x_7 + y_3 + y_9$$

$$x_3 = x_7 + y_1 + y_9$$

$$x_4 = x_9$$

$$x_5 = x_7 + y_9 + y_{11}$$

$$x_6 = x_8$$

$$y_2 = y_8$$

$$y_4 = y_7$$

$$y_5 = y_{10}$$

By using these conditions above, we obtained 25 generators of the lifts of P_{40-4} to R_1 whose binary images have the parameters $[80,40,12]$. 2 of these generators are given below:

Table 14 The list of pure double-circulant self-dual codes of length 40-4

#	C_{40-4}	Type	$ Aut \phi_1(C_{40-4}) $
1	22221222121131221101	I	$2^4 \cdot 5$
2	20221222101111221121	II	$2^5 \cdot 5$

For $n = 40$;

Let $P_{40-5} = [I_{20} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_5 = [y_1, y_2, y_3, x_1, y_4, y_5, y_6, x_2, y_7, y_8, x_3, x_4, x_5, y_9, y_{10}, x_6, x_7, y_{11}, x_8, x_9]$. By using V_5 , we obtain the following conditions for the lifts of P_{40-5} to R_k to be self dual:

$$x_1 = x_9 + y_5 + y_8$$

$$x_2 = x_9 + y_2 + y_8$$

$$x_3 = x_8$$

$$x_4 = x_9 + y_8 + y_{11}$$

$$x_5 = x_7$$

$$x_6 = x_9 + y_8 + y_9$$

$$y_1 = y_7$$

$$y_3 = y_6$$

$$y_4 = y_{10}$$

By using these conditions above, we obtained 30 generators of the lifts of P_{40-5} to R_1 whose binary images have the parameters $[80,40,12]$. 2 of these generators are given below:

Table 15 The list of pure double-circulant self-dual codes of length 40-5

#	C_{40-5}	Type	$ Aut \phi_1(C_{40-5}) $
1	22212221221312211011	I	$2^4 \cdot 5$
2	22230223201112031011	II	$2^4 \cdot 5$

For $n = 40$;

Let $P_{40-6} = [I_{20} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_6 = [y_1, y_2, y_3, x_1, y_4, y_5, x_2, y_6, y_7, y_8, x_3, x_4, y_9, y_{10}, x_5, x_6, y_{11}, x_7, x_8, x_9]$. By using V_6 , we obtain the following conditions for the lifts of P_{40-6} to R_k to be self dual:

$$x_1 = x_9 + y_2 + y_5$$

$$x_2 = x_8$$

$$x_3 = x_5$$

$$x_4 = x_9 + y_5 + y_{10}$$

$$x_6 = x_9 + y_5 + y_6$$

$$x_7 = x_9 + y_5 + y_6$$

$$y_1 = y_4$$

$$y_3 = y_9$$

$$y_7 = y_{11}$$

By using these conditions above, we obtained 23 generators of the lifts of P_{40-6} to R_1 whose binary images have the parameters $[80,40,12]$. 2 of these generators are given below:

Table 16 The list of pure double-circulant self-dual codes of length 40-6

#	C_{40-6}	Type	$ Aut \phi_1(C_{40-6}) $
1	222122212221320112111	I	$2^4.5$
2	222122212021122110111	II	$2^5.5$

For $n = 44$;

Let $P_{44-1} = [I_{22} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, x_1, x_2, y_8, y_9, x_3, y_{10}, x_4, y_{11}, y_{12}, x_5, y_{13}, x_6, x_7, x_8, x_9]$. By using V_1 , we obtain the following conditions for the lifts of P_{44-1} to R_k to be self dual:

$$x_1 = x_9 + y_8 + y_{12}$$

$$x_2 = x_8 + y_7 + y_9 + y_{10} + y_{11}$$

$$x_3 = x_9 + y_4 + y_6 + y_8 + y_{13}$$

$$x_4 = x_9 + y_{12} + y_{13}$$

$$x_5 = x_8 + y_5 + y_7 + y_{10} + y_{11}$$

$$x_6 = x_8 + y_5 + y_9$$

$$x_7 = x_9 + y_6 + y_{12}$$

$$y_1 = y_5 + y_9 + y_{10}$$

$$y_2 = y_4 + y_6 + y_8 + y_{12} + y_{13}$$

$$y_3 = y_5 + y_7 + y_9$$

By using the conditions above, we obtained the generators of the lifts of P_{44-1} to R_1 whose binary images $\phi_1(C_{44-1})$ are extremal Type II codes with parameters $[88, 44, 16]$ which are given below:

Table 17.1 The list of pure double-circulant self-dual codes of length 44-1

#	C_{44-1}	$ Aut \phi_1(C_{44-1}) $
1	2002222130030302103111	$2^3 \cdot 11$
2	2202222130030300123311	$2^3 \cdot 11$
3	2002222130010100103311	$2^3 \cdot 11$
4	0022220132032302103111	$2^3 \cdot 11$
5	2022200312010100303111	$2^3 \cdot 11$
6	2002022312210300323311	$2^3 \cdot 11$
7	0202022330212302103111	$2^3 \cdot 11$
8	0202022130232300123311	$2^3 \cdot 11$
9	0222020110232300323311	$2^3 \cdot 11$
10	2020222312212120101311	$2^3 \cdot 11$
11	0020222332210120301311	$2^3 \cdot 11$
12	0020222330230322301111	$2^3 \cdot 11$
13	0200220312230320121311	$2^3 \cdot 11$
14	0000220312210120101311	$2^3 \cdot 11$
15	2000220130212320321311	$2^3 \cdot 11$
16	0220202332230120301111	$2^3 \cdot 11$
17	0200200112210122121311	$2^3 \cdot 11$
18	0200200310210322101311	$2^3 \cdot 11$
19	2020022332012120301311	$2^3 \cdot 11$
20	0000020332010120301311	$2^3 \cdot 11$
21	2000020310032322101111	$2^3 \cdot 11$
22	2220002132012122321311	$2^3 \cdot 11$
23	0220002310010322101311	$2^3 \cdot 11$
24	2000000112032322101311	$2^3 \cdot 11$

And also by using the same conditions, we obtained the generators of the lifts of P_{44-1} to R_1 whose binary images $\phi_1(C_{44-1})$ are Type I codes with parameters $[88,44,14]$ which are given below:

Table 17.2 The list of pure double-circulant self-dual codes of length 44-1

#	C_{44-1}	$ Aut \phi_1(C_{44-1}) $
1	0222222110230300121311	$2^3 \cdot 11$
2	2002220312212300121311	$2^3 \cdot 11$
3	0002220130210100301311	$2^3 \cdot 11$
4	0222202312230300121111	$2^3 \cdot 11$
5	2222202330212102321311	$2^3 \cdot 11$
6	0202200330210102321311	$2^3 \cdot 11$
7	2022022312012300121311	$2^3 \cdot 11$
8	2022022110012100101311	$2^3 \cdot 11$
9	2002020330032102321111	$2^3 \cdot 11$
10	0222002132010302301311	$2^3 \cdot 11$
11	2222002310012102121311	$2^3 \cdot 11$
12	2022002310032302101311	$2^3 \cdot 11$
13	2202000332032300321111	$2^3 \cdot 11$
14	2202000330012102321311	$2^3 \cdot 11$
15	0202000310010102121311	$2^3 \cdot 11$
16	2000202112030322303311	$2^3 \cdot 11$
17	2020200132030322103311	$2^3 \cdot 11$
18	2020200332010320123111	$2^3 \cdot 11$
19	0220200310012322303311	$2^3 \cdot 11$
20	0200022310212122323111	$2^3 \cdot 11$
21	2200002130230320123111	$2^3 \cdot 11$
22	0220000132212122123311	$2^3 \cdot 11$
23	0020000132232322103311	$2^3 \cdot 11$
24	0220000130232320123111	$2^3 \cdot 11$

For $n = 44$;

Let $P_{44-2} = [I_{22} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, y_4, x_1, y_5, x_2, x_3, y_6, x_4, y_7, x_5, x_6, y_8, y_9, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_2 , we obtain the following conditions for the lifts of P_{44-2} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{12} + y_6 + y_7 \\
x_2 &= x_{12} + y_1 + y_7 \\
x_3 &= x_9 + x_{11} + x_{13} + y_4 + y_8 \\
x_4 &= x_{13} + y_5 + y_8 \\
x_5 &= x_9 + y_5 + y_8 \\
x_6 &= x_{12} + y_7 + y_9 \\
x_7 &= x_{11} + y_5 + y_8 \\
x_8 &= x_{12} + y_3 + y_7 \\
x_{10} &= x_{12} + y_1 + y_3 + y_6 + y_9 \\
y_2 &= y_4 + y_5 + y_8
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{44-2} to R_1 whose binary images $\phi_1(C_{44-2})$ are extremal Type II codes with parameters $[88,44,16]$ which are given below:

Table 18.1 The list of pure double-circulant self-dual codes of length 44-2

#	C_{44-2}	$ Aut \phi_1(C_{44-2}) $
1	2202121121213201311111	$2^3.11$
2	0200321123031003113111	$2^3.11$
3	0200121103033023113111	$2^3.11$
4	2222303123011001313113	$2^3.11$
5	2000121123213201311113	$2^3.11$
6	2000101323213001311113	$2^3.11$
7	0202121303013221113113	$2^3.11$
8	2000321101211223311311	$2^3.11$
9	0220301123033221313311	$2^3.11$
10	0020301321013023313311	$2^3.11$
11	2202123103011203113313	$2^3.11$
12	2202101323213003311313	$2^3.11$
13	0222321123013223313313	$2^3.11$
14	0020121303011203313313	$2^3.11$
15	0220103321233201111313	$2^3.11$

And also by using the same conditions, we obtained the generators of the lifts of P_{44-2} to R_1 whose binary images $\phi_1(C_{44-2})$ are Type I codes with parameters $[88,44,14]$ which are given below:

Table 18.2 The list of pure double-circulant self-dual codes of length 44-2

#	C_{44-2}	$ Aut \phi_1(C_{44-2}) $
1	2022101123233203113111	$2^3.11$
2	2022301103231223113111	$2^3.11$
3	0020101101013021311111	$2^3.11$
4	2020101323213001113113	$2^3.11$
5	0222121303013221311113	$2^3.11$
6	0222101103013021311113	$2^3.11$
7	0000323103211221313113	$2^3.11$
8	2202321301213203313311	$2^3.11$
9	2002321103233001313311	$2^3.11$
10	0222301121011003311311	$2^3.11$
11	2002123301033021111313	$2^3.11$
12	2202101323211023313313	$2^3.11$
13	2000301103213003313313	$2^3.11$
14	0020121303013223311313	$2^3.11$
15	0020103123211023113313	$2^3.11$

For $n = 44$;

Let $P_{44-3} = [I_{22} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_3 = [y_1, y_2, y_3, y_4, x_1, y_5, x_2, y_6, y_7, x_3, x_4, y_8, x_5, x_6, x_7, x_8, x_9, x_{10}, y_9, x_{11}, x_{12}, x_{13}]$. By using V_3 , we obtain the following conditions for the lifts of P_{44-3} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_7 + y_3 + y_9 \\
x_2 &= x_5 + x_7 + x_{12} + y_3 + y_7 \\
x_3 &= x_{13} + y_2 + y_5 \\
x_4 &= x_5 + y_3 + y_9 \\
x_6 &= x_{13} + y_2 + y_6 \\
x_8 &= x_{13} + y_2 + y_4 \\
x_9 &= x_{12} + y_3 + y_9 \\
x_{10} &= x_{13} + y_4 + y_5 + y_6 + y_8 \\
x_{11} &= x_{13} + y_2 + y_8 \\
y_1 &= y_3 + y_7 + y_9
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{44-3} to R_1 whose binary images $\phi_1(C_{44-3})$ are extremal Type II codes with parameters $[88,44,16]$ which are given below:

Table 19.1: The list of pure double-circulant self-dual codes of length 44-3

#	C_{44-3}	$ Aut \phi_1(C_{44-3}) $
1	0220103203121113112111	$2^3 \cdot 11$
2	0200323221301113312311	$2^3 \cdot 11$
3	2200321001321313312111	$2^3 \cdot 11$
4	2002321203301313332111	$2^3 \cdot 11$
5	2200121021121313310131	$2^3 \cdot 11$
6	0200301023301313112331	$2^3 \cdot 11$
7	2022321003321113130331	$2^3 \cdot 11$
8	0002123003121113330331	$2^3 \cdot 11$
9	0222323001101331312331	$2^3 \cdot 11$
10	2202323021101331310331	$2^3 \cdot 11$
11	0200321201101133310331	$2^3 \cdot 11$

And also by using the same conditions, we obtained the generators of the lifts of P_{44-3} to R_1 whose binary images $\phi_1(C_{44-3})$ are Type I codes with parameters $[88,44,14]$ which are given below:

Table 19.2 The list of pure double-circulant self-dual codes of length 44-3

#	C_{44-3}	$ Aut \phi_1(C_{44-3}) $
1	0220301023321313330111	$2^3.11$
2	0022301221301313310111	$2^3.11$
3	2022303001321113310311	$2^3.11$
4	2002123023101113110111	$2^3.11$
5	2220103223101113332331	$2^3.11$
6	0200301223301113132331	$2^3.11$
7	2022321203321313110331	$2^3.11$
8	0022101201101313312131	$2^3.11$
9	0222303203121131332131	$2^3.11$
10	0020303201121331312331	$2^3.11$
11	2000303221121331310331	$2^3.11$

For $n = 44$;

Let $P_{44-4} = [I_{22} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_4 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, x_1, x_2, x_3, y_{10}, y_{11}, y_{12}, x_4, x_5, x_6, y_{13}, x_7, x_8, x_9]$. By using V_4 , we obtain the following conditions for the lifts of P_{44-4} to R_k to be self dual:

$$x_1 = x_9 + y_2 + y_4 + y_8 + y_{11}$$

$$x_2 = x_8 + y_5 + y_9 + y_{12} + y_{13}$$

$$x_3 = x_9 + y_4 + y_6$$

$$x_4 = x_9 + y_6 + y_{11}$$

$$x_5 = x_8 + y_5 + y_{10} + y_{12} + y_{13}$$

$$x_6 = x_9 + y_6 + y_8$$

$$x_7 = x_9 + y_2 + y_6$$

$$y_1 = y_9 + y_{10} + y_{12}$$

$$y_3 = y_5 + y_9 + y_{10}$$

$$y_7 = y_9 + y_{10} + y_{13}$$

By using the conditions above, we obtained the generators of the lifts of P_{44-4} to R_1 whose binary images $\phi_1(C_{44-4})$ are extremal Type II codes with parameters $[88,44,16]$ which are given below:

Table 20.1 The list of pure double-circulant self-dual codes of length 44-4

#	C_{44-4}	$ Aut \phi_1(C_{44-4}) $
1	0202222021310023130111	$2^3.11$
2	0222202223332001332311	$2^3.11$
3	0202202223330021130311	$2^3.11$
4	2202000203330021330311	$2^3.11$
5	0200222201132023310111	$2^3.11$
6	0022222001110201132311	$2^3.11$
7	0002200223130223332111	$2^3.11$
8	2022002223330203130111	$2^3.11$
9	2020220001130023130311	$2^3.11$
10	0020220001330003330311	$2^3.11$
11	0000202023310223110111	$2^3.11$

And also by using the same conditions, we obtained the generators of the lifts of P_{44-4} to R_1 whose binary images $\phi_1(C_{44-4})$ are Type I codes with parameters $[88,44,14]$ which are given below:

Table 20.2 The list of pure double-circulant self-dual codes of length 44-4

#	C_{44-4}	$ Aut \phi_1(C_{44-4}) $
1	2222202201330223332311	$2^3.11$
2	2222020203312003112111	$2^3.11$
3	0202002221132203132311	$2^3.11$
4	0222002001332021110311	$2^3.11$
5	0200220003332023132111	$2^3.11$
6	2200200221310001132311	$2^3.11$
7	2200000221112021130311	$2^3.11$
8	2022000021130203312111	$2^3.11$
9	0020222023332201332311	$2^3.11$
10	2020020003332201132311	$2^3.11$
11	2000020003330221330311	$2^3.11$

For $n = 44$;

Let $P_{44-5} = [I_{22} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_5 = [y_1, x_1, y_2, x_2, y_3, x_3, x_4, x_5, y_4, x_6, x_7, x_8, x_9, x_{10}, y_5, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}]$. By using V_5 , we obtain the following conditions for the lifts of P_{44-5} to R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_{10} + x_{11} + x_{13} \\
 x_2 &= x_8 + x_{10} + x_{13} + x_{15} + x_{17} \\
 x_3 &= x_{11} + x_{13} + x_{15} \\
 x_4 &= x_{16} + y_3 + y_5 \\
 x_5 &= x_8 + x_{11} + x_{13} \\
 x_6 &= x_8 + x_{10} + x_{11} + x_{15} + x_{17} \\
 x_7 &= x_{16} + y_1 + y_2 + y_3 + y_4 \\
 x_9 &= x_{16} + y_2 + y_5 \\
 x_{12} &= x_{16} + y_4 + y_5 \\
 x_{14} &= x_{16} + y_1 + y_5
 \end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{44-5} to R_1 whose binary images $\phi_1(C_{44-5})$ are extremal Type II codes with parameters $[88,44,16]$ which are given below:

Table 21.1 The list of pure double-circulant self-dual codes of length 44-5

#	C_{44-5}	$ Aut \phi_1(C_{44-5}) $
1	0103211123113121113113	$2^3.11$
2	2123033103111121311311	$2^3.11$
3	0123231103111121313311	$2^3.11$
4	0103233103311101111311	$2^3.11$
5	2123211123111123131111	$2^3.11$

And also by using the same conditions, we obtained the generators of the lifts of P_{44-5} to R_1 whose binary images $\phi_1(C_{44-5})$ are Type I codes with parameters $[88,44,14]$ which are given below:

Table 21.2 The list of pure double-circulant self-dual codes of length 44-5

#	C_{44-5}	$ Aut \phi_1(C_{44-5}) $
1	2123011103113101113113	$2^3.11$
2	2123033123311121111311	$2^3.11$
3	2103031123111101313311	$2^3.11$
4	0222002001332021110311	$2^3.11$
5	0103011103111103131111	$2^3.11$

For $n = 46$;

Let $P_{46} = [I_{23} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, y_9, y_{10}, x_2, x_3, x_4, x_5, y_{11}, y_{12}, x_6, x_7, y_{13}, x_8, y_{14}, x_9]$. By using V , we obtain the following conditions for the lifts of P_{46} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_9 + y_4 + y_6 + y_8 + y_{11} + y_{13} + y_{14} \\
x_2 &= x_9 + y_2 + y_8 + y_9 + y_{12} \\
x_3 &= x_9 + y_2 + y_4 + y_6 + y_7 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14} \\
x_4 &= x_9 + y_4 + y_8 + y_9 + y_{10} + y_{12} + y_{13} \\
x_5 &= x_9 + y_2 + y_7 + y_8 + y_9 + y_{11} + y_{13} \\
x_6 &= x_9 + y_2 + y_6 + y_7 + y_8 + y_{11} + y_{12} + y_{13} + y_{14} \\
x_7 &= x_9 + y_6 + y_7 + y_8 + y_9 + y_{12} + y_{14} \\
x_8 &= x_9 + y_2 + y_8 + y_{10} + y_{11} + y_{12} + y_{14} \\
y_1 &= y_2 + y_7 + y_8 + y_{10} + y_{11} \\
y_3 &= y_4 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{14} \\
y_5 &= y_6 + y_7 + y_8 + y_{10} + y_{11} + y_{12} + y_{13}
\end{aligned}$$

By using these conditions above, we obtained 247 generators of the lifts of P_{46} to R_1 whose binary images have the parameters [92,46,14]. 1 of these generators is given below:

Table 22 The list of pure double-circulant self-dual codes of length 46

#	C_{46}	$ Aut \phi_1(C_{46}) $
1	22020222322313120112101	$2^2 \cdot 23$

For $n = 48$;

Let $P_{48} = [I_{24} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, x_1, y_8, x_2, y_9, y_{10}, x_3, y_{11}, y_{12}, x_4, y_{13}, y_{14}, y_{15}, x_5, x_6, x_7, x_8, x_9]$. By using V , we obtain the following conditions for the lifts of P_{48} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_9 + y_4 + y_6 + y_{11} + y_{14} \\
x_2 &= x_9 + y_6 + y_{10} \\
x_3 &= x_8 + y_{12} + y_{13} \\
x_4 &= x_9 + y_4 + y_{10} \\
x_5 &= x_9 + y_{10} + y_{11} \\
x_6 &= x_8 + y_7 + y_{13} \\
x_7 &= x_9 + y_6 + y_7 + y_8 + y_9 + y_{12} + y_{14} \\
y_1 &= y_7 + y_9 + y_{12} + y_{13} + y_{15} \\
y_2 &= y_4 + y_6 + y_{10} + y_{11} + y_{14} \\
y_3 &= y_7 + y_8 + y_9 + y_{12} + y_{13} \\
y_5 &= y_8 + y_9 + y_{15}
\end{aligned}$$

For $n = 50$;

Let $P_{50-1} = [I_{25} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, x_1, y_8, y_9, y_{10}, x_2, x_3, x_4, y_{11}, y_{12}, y_{13}, x_5, x_6, y_{14}, y_{15}, x_7, x_8, y_{16}, x_9]$. By using V_1 , we obtain the following conditions for the lifts of P_{50-1} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_9 + y_6 + y_8 + y_{11} + y_{12} + y_{13} + y_{15} \\
x_2 &= x_9 + y_6 + y_8 + y_9 + y_{10} + y_{13} + y_{16} \\
x_3 &= x_9 + y_5 + y_8 + y_9 + y_{10} + y_{12} + y_{13} + y_{14} + y_{16} \\
x_4 &= x_9 + y_7 + y_8 + y_{10} + y_{12} + y_{13} + y_{14} \\
x_5 &= x_9 + y_9 + y_{11} + y_{12} + y_{13} + y_{14} + y_{16} \\
x_6 &= x_9 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} \\
x_7 &= x_9 + y_5 + y_{10} + y_{12} + y_{13} + y_{15} + y_{16} \\
x_8 &= x_9 + y_6 + y_8 + y_9 + y_{11} + y_{13} + y_{14} + y_{15} + y_{16} \\
y_1 &= y_6 + y_7 + y_8 + y_{11} + y_{13} + y_{14} + y_{16} \\
y_2 &= y_6 + y_7 + y_8 + y_{10} + y_{12} + y_{15} + y_{16} \\
y_3 &= y_8 + y_{11} + y_{12} + y_{13} + y_{14} \\
y_4 &= y_5 + y_6 + y_7 + y_9 + y_{11} + y_{12} + y_{13} + y_{14} + y_{16}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{50-1} to R_1 whose binary images $\phi_1(C_{50-1})$ are best-known Type I codes with parameters $[100,50,16]$ which are given below:

Table 23 The list of pure double-circulant self-dual codes of length 50-1

#	C_{50-1}	$ Aut \phi_1(C_{50-1}) $
1	0220222322233122231201101	$2^2 \cdot 5^2$
2	2222222122233302013223121	$2^2 \cdot 5^2$
3	2020222322231300233203301	$2^2 \cdot 5^2$
4	0000222122031122233023321	$2^2 \cdot 5^2$
5	2200222322031120233221121	$2^2 \cdot 5^2$
6	2200222320213322231001101	$2^2 \cdot 5^2$
7	0002222120233102211203321	$2^2 \cdot 5^2$
8	0220222102213300033221321	$2^2 \cdot 5^2$
9	2222222302231300013201301	$2^2 \cdot 5^2$
10	0200222302233100031001101	$2^2 \cdot 5^2$
11	0202222302011122213223321	$2^2 \cdot 5^2$
12	2222222302011320013021101	$2^2 \cdot 5^2$
13	0002222102033302013203121	$2^2 \cdot 5^2$
14	2202222300233322213223301	$2^2 \cdot 5^2$
15	0200222100211322233203321	$2^2 \cdot 5^2$
16	2022222300213122213021321	$2^2 \cdot 5^2$
17	0002222300231100211221101	$2^2 \cdot 5^2$
18	0220222300033322033201301	$2^2 \cdot 5^2$
19	0222222100033100213003321	$2^2 \cdot 5^2$
20	2002220122213122231021321	$2^2 \cdot 5^2$
21	2000220122211120013023301	$2^2 \cdot 5^2$
22	0200220122213302011021101	$2^2 \cdot 5^2$
23	2200220322031100011221301	$2^2 \cdot 5^2$
24	2202220120213120233203301	$2^2 \cdot 5^2$
25	0222220320211102031201121	$2^2 \cdot 5^2$
26	2202220320211300231003301	$2^2 \cdot 5^2$
27	0222220302231322233021121	$2^2 \cdot 5^2$
28	2222220102031300031223321	$2^2 \cdot 5^2$

For $n = 50$;

Let $P_{50-2} = [I_{25} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, x_1, y_4, y_5, x_2, x_3, x_4, x_5, x_6, x_7, y_6, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, y_7, x_{15}, y_8, x_{16}, x_{17}]$. By using V_2 , we obtain the following conditions for the lifts of P_{50-2} to R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_{11} + x_{13} + x_{15} + y_2 + y_7 \\
 x_2 &= x_{11} + x_{13} + x_{14} + x_{15} + x_{16} + y_3 + y_7 \\
 x_3 &= x_{11} + x_{15} + x_{17} + y_2 + y_3 + y_6 + y_7 \\
 x_4 &= x_{11} + x_{14} + x_{15} + x_{16} + x_{17} + y_4 + y_6 \\
 x_5 &= x_{11} + x_{14} + x_{17} + y_2 + y_6 \\
 x_6 &= x_{13} + x_{15} + x_{16} + y_3 + y_8 \\
 x_7 &= x_{14} + y_2 + y_4 + y_5 + y_7 \\
 x_8 &= x_{11} + x_{13} + x_{14} + x_{15} + x_{17} + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \\
 x_9 &= x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + y_5 + y_8 \\
 x_{10} &= x_{15} + y_2 + y_3 + y_5 + y_6 \\
 x_{12} &= x_{13} + x_{14} + x_{16} + y_4 + y_5 + y_6 + y_7 \\
 y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8
 \end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{50-2} to R_1 whose binary images $\phi_1(C_{50-2})$ are best-known Type I codes with parameters $[100, 50, 16]$ which are given below:

Table 24 The list of pure double-circulant self-dual codes of length 50-2

#	C_{50-2}	$ Aut \phi_1(C_{50-2}) $
1	0223203133110331131101211	$2^2.5^2$
2	0203201333110313131101011	$2^2.5^2$
3	0021203331330113131101011	$2^2.5^2$
4	0021023311330131131101011	$2^2.5^2$
5	0021003311110133111101211	$2^2.5^2$
6	0003003133132113111121011	$2^2.5^2$
7	2221221333112331111121213	$2^2.5^2$
8	0223223311330113111101013	$2^2.5^2$
9	2203201111310313131101213	$2^2.5^2$
10	2201023331130311111121013	$2^2.5^2$
11	0023221131132333111121213	$2^2.5^2$
12	2021023133130131131101213	$2^2.5^2$
13	2003203331112133131121013	$2^2.5^2$
14	2003003311332113111121213	$2^2.5^2$
15	2203223131132133111101231	$2^2.5^2$
16	0023203111310113131121231	$2^2.5^2$
17	2001223111110333131101231	$2^2.5^2$
18	2003201133312333111121031	$2^2.5^2$
19	2221223131110333111121033	$2^2.5^2$
20	2221023333132331111121033	$2^2.5^2$
21	0203223313332133111101033	$2^2.5^2$
22	0201001311310113111121033	$2^2.5^2$
23	2023023313110131131121033	$2^2.5^2$
24	2001223333110113131101233	$2^2.5^2$
25	0001223333310333131101033	$2^2.5^2$
26	0223203113130133111123011	$2^2.5^2$
27	2203001111312333111123011	$2^2.5^2$
28	0001201111310333111123011	$2^2.5^2$

For $n = 52$;

Let $P_{52-1} = [I_{26} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, y_9, x_2, y_{10}, x_3, x_4, y_{11}, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, y_{12}, y_{13}, x_{12}, x_{13}]$. By using V_1 , we obtain the following conditions for the lifts of P_{52-1} to R_k to be self dual:

$$x_1 = x_{12} + y_3 + y_5 + y_7 + y_{12}$$

$$x_2 = x_{10} + y_5 + y_7$$

$$x_3 = x_{12} + y_3 + y_5 + y_7 + y_{11}$$

$$x_4 = x_{13} + y_9 + y_{10}$$

$$x_5 = x_{13} + y_6 + y_8 + y_9 + y_{13}$$

$$x_6 = x_{10} + y_3 + y_5$$

$$x_7 = x_{13} + y_4 + y_6 + y_9 + y_{13}$$

$$x_8 = x_{12} + y_{11} + y_{12}$$

$$x_9 = x_{13} + y_4 + y_6 + y_8 + y_9$$

$$x_{11} = x_{13} + y_6 + y_{10}$$

$$y_1 = y_3 + y_5 + y_7 + y_{11} + y_{12}$$

$$y_2 = y_6 + y_9 + y_{10}$$

By using the conditions above, we obtained the generators of the lifts of P_{52-1} to R_1 whose binary images $\phi_1(C_{52-1})$ are Type I codes with parameters $[104,52,16]$ which are given below:

Table 25.1 The list of pure double-circulant self-dual codes of length 52-1

#	C_{52-1}	$ Aut \phi_1(C_{52-1}) $
1	0222222321211231331110011	$2^3 \cdot 13$
2	2222222321231031311110011	$2^3 \cdot 13$
3	00222220121033031133132211	$2^3 \cdot 13$
4	20222220321033031113130211	$2^3 \cdot 13$
5	22222202323211031331112011	$2^3 \cdot 13$
6	00222200323033231113132211	$2^3 \cdot 13$
7	02222200103011011311130211	$2^3 \cdot 13$
8	22222020121013211311112211	$2^3 \cdot 13$
9	02222002103213031311130011	$2^3 \cdot 13$
10	00220222323033233311132011	$2^3 \cdot 13$
11	00220220103213013311110211	$2^3 \cdot 13$
12	20220202101213213113112011	$2^3 \cdot 13$
13	22220202301031013113130011	$2^3 \cdot 13$
14	22220200121211233113112211	$2^3 \cdot 13$
15	22220022103233233331130011	$2^3 \cdot 13$
16	20220020323211013331132211	$2^3 \cdot 13$
17	00220020303031233113112211	$2^3 \cdot 13$
18	22220002321033013113110011	$2^3 \cdot 13$
19	20220002101011233311112011	$2^3 \cdot 13$
20	02220000101233033133132211	$2^3 \cdot 13$
21	00202222301213231131110211	$2^3 \cdot 13$
22	20202222301233031111110211	$2^3 \cdot 13$
23	22202220121211211111112011	$2^3 \cdot 13$
24	00202220321013211131130011	$2^3 \cdot 13$
25	02202220101031031333132011	$2^3 \cdot 13$
26	02202200123211011111110011	$2^3 \cdot 13$
27	20202022121211231111132211	$2^3 \cdot 13$

By using the same conditions above, we obtained 187 generators of the lifts of P_{52-1} to R_1 whose binary images $\phi_1(C_{52-1})$ are Type II codes with parameters $[104,52,16]$. 1 of these generators is given below:

Table 25.2 The list of pure double-circulant self-dual codes of length 52-1

#	C_{52-1}	$ Aut \phi_1(C_{52-1}) $
1	02222222121231011131112211	$2^3 \cdot 13$

For $n = 52$;

Let $P_{52-2} = [I_{26} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, x_1, y_8, y_9, y_{10}, x_2, x_3, x_4, x_5, x_6, y_{11}, x_7, x_8, x_9, y_{12}, x_{10}, y_{13}, x_{11}, x_{12}, x_{13}]$. By using V_2 , we obtain the following conditions for the lifts of P_{52-2} to R_k to be self dual:

$$x_1 = x_9 + x_{10} + x_{13} + y_4 + y_6$$

$$x_2 = x_{13} + y_4 + y_9$$

$$x_3 = x_{12} + y_{11} + y_{13}$$

$$x_4 = x_9 + x_{10} + x_{11}$$

$$x_5 = x_{12} + y_7 + y_8 + y_{11} + y_{12}$$

$$x_6 = x_9 + x_{11} + x_{13} + y_4 + y_6$$

$$x_7 = x_{13} + y_6 + y_9$$

$$x_8 = x_{12} + y_7 + y_8 + y_{12} + y_{13}$$

$$y_1 = y_7 + y_8 + y_{10}$$

$$y_2 = y_4 + y_6 + y_9$$

$$y_3 = y_8 + y_{10} + y_{12}$$

$$y_5 = y_7 + y_8 + y_{11} + y_{12} + y_{13}$$

By using the conditions above, we obtained the generators of the lifts of P_{52-2} to R_1 whose binary images $\phi_1(C_{52-2})$ are Type I codes with parameters $[104,52,16]$ which are given below:

Table 26.1 The list of pure double-circulant self-dual codes of length 52-2

#	C_{52-2}	$ Aut \phi_1(C_{52-2}) $
1	20022221202331110331012111	$2^3 \cdot 13$
2	02222221022111310131010111	$2^3 \cdot 13$
3	22222221020131110131212111	$2^3 \cdot 13$
4	00222221002311112311012111	$2^3 \cdot 13$
5	00220221002331112331010111	$2^3 \cdot 13$
6	2222201220111112111012111	$2^3 \cdot 13$
7	2222201220111310131010111	$2^3 \cdot 13$
8	2022201200311112311012111	$2^3 \cdot 13$
9	20220201200331112331010111	$2^3 \cdot 13$
10	2202201022111112111212111	$2^3 \cdot 13$
11	00022201000331110331012111	$2^3 \cdot 13$
12	0002203220111330331210111	$2^3 \cdot 13$
13	2022203020131332311210111	$2^3 \cdot 13$
14	0222203002311132111012111	$2^3 \cdot 13$
15	02220023002331330111012111	$2^3 \cdot 13$
16	22220003200331132131010111	$2^3 \cdot 13$
17	20002223222331332111010111	$2^3 \cdot 13$
18	22002223202131332311010111	$2^3 \cdot 13$
19	02002223200111132311212111	$2^3 \cdot 13$
20	00200222302331330111012111	$2^3 \cdot 13$
21	00200203222311332131212111	$2^3 \cdot 13$
22	20000203022331132131210111	$2^3 \cdot 13$
23	02002021220311112311212111	$2^3 \cdot 13$
24	02200021022331112331010111	$2^3 \cdot 13$
25	02202021022311310331010111	$2^3 \cdot 13$
26	22002001022311112311212111	$2^3 \cdot 13$
27	20002001002111112111212111	$2^3 \cdot 13$
28	22022223222331130331012113	$2^3 \cdot 13$
29	00020223200131132131210113	$2^3 \cdot 13$
30	00022223200111330131210113	$2^3 \cdot 13$
31	00222223002111132111012113	$2^3 \cdot 13$
32	22022203022311330331210113	$2^3 \cdot 13$
33	00022203000131130131012113	$2^3 \cdot 13$

By using the same conditions above, we obtained 163 generators of the lifts of P_{52-2} to R_1 whose binary images $\phi_1(C_{52-2})$ are Type II codes with parameters $[104,52,16]$. 1 of these generators is given below:

Table 26.2 The list of pure double-circulant self-dual codes of length 52-2

#	C_{52-2}	$ Aut \phi_1(C_{52-2}) $
1	22220221222131112131210111	$2^3 \cdot 11$

For $n = 52$;

Let $P_{52-3} = [I_{26} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_3 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, x_1, y_8, y_9, x_2, y_{10}, x_3, y_{11}, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, y_{12}, y_{13}, x_{12}, x_{13}]$. By using V_3 , we obtain the following conditions for the lifts of P_{52-3} to R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_{13} + y_9 + y_{11} \\
 x_2 &= x_{10} + y_3 + y_5 + y_7 + y_{12} \\
 x_3 &= x_{10} + y_3 + y_5 + y_7 + y_8 \\
 x_4 &= x_{12} + y_3 + y_5 \\
 x_5 &= x_{13} + y_2 + y_6 + y_{11} + y_{13} \\
 x_6 &= x_{10} + y_8 + y_{12} \\
 x_7 &= x_{13} + y_6 + y_{10} + y_{11} + y_{13} \\
 x_8 &= x_{12} + y_3 + y_7 \\
 x_9 &= x_{13} + y_2 + y_{10} + y_{11} + y_{13} \\
 x_{11} &= x_{13} + y_9 + y_{13} \\
 y_1 &= y_3 + y_5 + y_7 + y_8 + y_{12} \\
 y_4 &= y_9 + y_{11} + y_{13}
 \end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{52-3} to R_1 whose binary images $\phi_1(C_{52-3})$ are Type I codes with parameters $[104,52,16]$ which are given below:

Table 27.1 The list of pure double-circulant self-dual codes of length 52-3

#	C_{52-3}	$ Aut \phi_1(C_{52-3}) $
1	22202223201012111313132211	$2^3 \cdot 13$
2	02222223001032133111112011	$2^3 \cdot 13$
3	02202201203030111333112011	$2^3 \cdot 13$
4	22222021221212131311112211	$2^3 \cdot 13$
5	22222021023232131311110211	$2^3 \cdot 13$
6	02222001223232131331112211	$2^3 \cdot 13$
7	22222003221230133331130011	$2^3 \cdot 13$
8	02200223223030331111112211	$2^3 \cdot 13$
9	22200221201030313313110011	$2^3 \cdot 13$
10	22220221023012313313112211	$2^3 \cdot 13$
11	22220201221012311333112211	$2^3 \cdot 13$
12	02200203203012313333130211	$2^3 \cdot 13$
13	02220023223230331311132011	$2^3 \cdot 13$
14	22220001221212331331112211	$2^3 \cdot 13$
15	22200001221212311133132011	$2^3 \cdot 13$
16	02022221203030331131132211	$2^3 \cdot 13$
17	02022221021012311333110211	$2^3 \cdot 13$
18	02002221021012331131130011	$2^3 \cdot 13$
19	22022223023010313333132011	$2^3 \cdot 13$
20	02022223001012331131110011	$2^3 \cdot 13$
21	02002201223012333111130011	$2^3 \cdot 13$
22	22022021023212333331112211	$2^3 \cdot 13$
23	02002003021230313113112211	$2^3 \cdot 13$
24	22000221221012131131132011	$2^3 \cdot 13$
25	02020221203010133131130211	$2^3 \cdot 13$
26	02000223021030133131112211	$2^3 \cdot 13$
27	22020223023030111333130011	$2^3 \cdot 13$

By using the same conditions above, we obtained 211 generators of the lifts of P_{52-3} to R_1 whose binary images $\phi_1(C_{52-3})$ are Type II codes with parameters $[104,52,16]$. 1 of these generators is given below:

Table 27.2 The list of pure double-circulant self-dual codes of length 52-3

#	C_{52-3}	$ Aut \phi_1(C_{52-3}) $
1	02202221223212133313130011	$2^3 \cdot 11$

For $n = 52$;

Let $P_{52-4} = [I_{26} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_4 = [y_1, y_2, y_3, y_4, y_5, y_6, x_1, x_2, x_3, y_7, x_4, y_8, x_5, x_6, x_7, y_9, y_{10}, x_8, x_9, y_{11}, y_{12}, y_{13}, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_4 , we obtain the following conditions for the lifts of P_{52-4} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{12} + y_3 + y_{10} \\
x_2 &= x_{13} + y_6 + y_7 + y_9 + y_{11} \\
x_3 &= x_{10} + y_5 + y_{12} \\
x_4 &= x_{12} + y_1 + y_{10} \\
x_5 &= x_{12} + y_1 + y_3 \\
x_6 &= x_{13} + y_8 + y_9 \\
x_7 &= x_{10} + y_1 + y_3 + y_5 + y_{10} \\
x_8 &= x_{13} + y_6 + y_9 + y_{11} + y_{13} \\
x_9 &= x_{10} + y_1 + y_3 + y_{10} + y_{12} \\
x_{11} &= x_{13} + y_8 + y_{11} \\
y_2 &= y_6 + y_7 + y_{13} \\
y_4 &= y_8 + y_9 + y_{11}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{52-4} to R_1 whose binary images $\phi_1(C_{52-4})$ are Type I codes with parameters $[104, 52, 16]$ which are given below:

Table 28.1 The list of pure double-circulant self-dual codes of length 52-4

#	C_{52-4}	$ Aut \phi_1(C_{52-4}) $
1	22222231123211320132221111	$2^3 \cdot 13$
2	20222213321013122130001111	$2^3 \cdot 13$
3	20202231123013320332201311	$2^3 \cdot 13$
4	22222231103013320130201111	$2^3 \cdot 13$
5	22222231103011300132201311	$2^3 \cdot 13$
6	20202033123013320332221311	$2^3 \cdot 13$
7	22202033123011300130201111	$2^3 \cdot 13$
8	22202033103213300132221111	$2^3 \cdot 13$
9	20202031103013320132201311	$2^3 \cdot 13$
10	22220211321211322112221111	$2^3 \cdot 13$
11	20220231123213100310001311	$2^3 \cdot 13$
12	20200211321011302310201111	$2^3 \cdot 13$
13	22220211301013322110201111	$2^3 \cdot 13$
14	20200233103013120112021311	$2^3 \cdot 13$
15	20220231103013120310021111	$2^3 \cdot 13$
16	20220013321211322312221111	$2^3 \cdot 13$
17	20200031123211120110021311	$2^3 \cdot 13$
18	20200011321213302112221111	$2^3 \cdot 13$
19	22220011301211322312221111	$2^3 \cdot 13$
20	20200013301213302312201111	$2^3 \cdot 13$
21	22220031103213100310021311	$2^3 \cdot 13$
22	22220033103011100112021311	$2^3 \cdot 13$
23	22022231121233302130221311	$2^3 \cdot 13$
24	20022231121233302330201311	$2^3 \cdot 13$
25	22022211323233100130021311	$2^3 \cdot 13$
26	22002213303033120332001311	$2^3 \cdot 13$
27	20022231101031302332221311	$2^3 \cdot 13$
28	22022013323231120132001111	$2^3 \cdot 13$
29	20022031121033322130221111	$2^3 \cdot 13$
30	22002013323031100130001111	$2^3 \cdot 13$
31	22002011303031100330021111	$2^3 \cdot 13$
32	20000231101233102312021111	$2^3 \cdot 13$

By using the same conditions above, we obtained 197 generators of the lifts of P_{52-4} to R_1 whose binary images $\phi_1(C_{52-4})$ are Type II codes with parameters $[104,52,16]$. 1 of these generators is given below:

Table 28.2 The list of pure double-circulant self-dual codes of length 52-4

#	C_{52-4}	$ Aut \phi_1(C_{52-4}) $
1	22202213121211122310221311	$2^3 \cdot 11$

For $n = 52$;

Let $P_{52-5} = [I_{26} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_5 = [y_1, y_2, y_3, y_4, y_5, x_1, y_6, y_7, y_8, x_2, y_9, x_3, y_{10}, y_{11}, y_{12}, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, y_{13}, x_{11}, x_{12}, x_{13}]$. By using V_5 , we obtain the following conditions for the lifts of P_{52-5} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_8 + y_7 + y_{11} \\
x_2 &= x_8 + x_{10} + x_{11} + y_4 + y_7 \\
x_3 &= x_8 + y_4 + y_{11} \\
x_4 &= x_{10} + x_{11} + x_{13} \\
x_5 &= x_{12} + y_8 + y_9 + y_{12} + y_{13} \\
x_6 &= x_8 + x_{10} + x_{13} + y_4 + y_7 \\
x_7 &= x_{12} + y_6 + y_8 + y_9 + y_{13} \\
x_9 &= x_{12} + y_6 + y_{12} \\
y_1 &= y_6 + y_8 + y_9 + y_{12} + y_{13} \\
y_2 &= y_4 + y_7 + y_{11} \\
y_3 &= y_8 + y_9 + y_{10} \\
y_5 &= y_9 + y_{10} + y_{13}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{52-5} to R_1 whose binary images $\phi_1(C_{52-5})$ are Type I codes with parameters $[104,52,16]$ which are given below:

Table 29.1 The list of pure double-circulant self-dual codes of length 52-5

#	C_{52-5}	$ Aut \phi_1(C_{52-5}) $
1	00200122232322213331110111	$2^3 \cdot 13$
2	00020322212300013111312111	$2^3 \cdot 13$
3	22202322230100011331312111	$2^3 \cdot 13$
4	00222322012300013111310111	$2^3 \cdot 13$
5	22220122010122211111112111	$2^3 \cdot 13$

By using the same conditions above, we obtained 27 generators of the lifts of P_{52-5} to R_1 whose binary images $\phi_1(C_{52-5})$ are Type II codes with parameters $[104,52,16]$. 1 of these generators is given below:

Table 29.2 The list of pure double-circulant self-dual codes of length 52-5

#	C_{52-5}	$ Aut \phi_1(C_{52-5}) $
1	02222122212122013111312111	$2^3 \cdot 11$

For $n = 52$;

Let $P_{52-6} = [I_{26} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_6 = [y_1, y_2, y_3, y_4, x_1, y_5, y_6, y_7, y_8, x_2, y_9, x_3, y_{10}, x_4, x_5, y_{11}, x_6, x_7, y_{12}, x_8, x_9, x_{10}, x_{11}, x_{12}, y_{13}, x_{13}]$. By using V_6 , we obtain the following conditions for the lifts of P_{52-6} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{11} + y_9 + y_{13} \\
x_2 &= x_{12} + y_5 + y_7 \\
x_3 &= x_{12} + y_5 + y_{11} \\
x_4 &= x_{13} + y_2 + y_5 + y_7 + y_{11} \\
x_5 &= x_{11} + y_6 + y_8 + y_{12} + y_{13} \\
x_6 &= x_{11} + y_8 + y_{10} + y_{12} + y_{13} \\
x_7 &= x_{12} + y_7 + y_{11} \\
x_8 &= x_{13} + y_2 + y_4 \\
x_9 &= x_{11} + y_9 + y_{12} \\
x_{10} &= x_{13} + y_4 + y_5 + y_7 + y_{11} \\
y_1 &= y_6 + y_8 + y_{10} \\
y_3 &= y_9 + y_{12} + y_{13}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{52-6} to R_1 whose binary images $\phi_1(C_{52-6})$ are Type I codes with parameters $[104,52,16]$ which are given below:

Table 30.1 The list of pure double-circulant self-dual codes of length 52-6

#	C_{52-6}	$ Aut \phi_1(C_{52-6}) $
1	02023222212303301321131101	$2^3 \cdot 13$
2	22221222210323303321331101	$2^3 \cdot 13$
3	22223222210323303301131121	$2^3 \cdot 13$
4	02023222010323303321331121	$2^3 \cdot 13$
5	22223220232123121301331101	$2^3 \cdot 13$
6	02021220232103321301331121	$2^3 \cdot 13$
7	22021220230123121301131101	$2^3 \cdot 13$
8	02221220230103321321331101	$2^3 \cdot 13$
9	22221220032103321321131121	$2^3 \cdot 13$
10	02223220030123121301131121	$2^3 \cdot 13$
11	22021202212303101301331121	$2^3 \cdot 13$
12	02221202210323103321331101	$2^3 \cdot 13$
13	22223202210303101301131121	$2^3 \cdot 13$

By using the same conditions above, we obtained 39 generators of the lifts of P_{52-6} to R_1 whose binary images $\phi_1(C_{52-6})$ are Type II codes with parameters $[104,52,16]$. 1 of these generators is given below:

Table 30.2 The list of pure double-circulant self-dual codes of length 52-6

#	C_{52-6}	$ Aut \phi_1(C_{52-6}) $
1	22223222212121121101311101	$2^3 \cdot 11$

For $n = 54$;

Let $P_{54-1} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, x_1, y_3, y_4, y_5, y_6, x_2, y_7, x_3, y_8, x_4, x_5, y_9, y_{10}, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}]$. By using V_1 , we obtain the following conditions for the lifts of P_{54-1} to R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_{14} + x_{15} + x_{17} + y_2 + y_3 + y_4 + y_5 + y_7 + y_{10} \\
 x_2 &= x_{16} + y_3 + y_4 + y_6 + y_8 \\
 x_3 &= x_{13} + y_2 + y_5 + y_7 + y_{10} \\
 x_4 &= x_{13} + x_{14} + x_{16} + y_3 + y_4 + y_7 + y_8 \\
 x_5 &= x_{13} + x_{15} + x_{17} + y_4 + y_5 + y_7 + y_8 + y_9 + y_{10} \\
 x_6 &= x_{14} + x_{15} + x_{16} + y_3 + y_4 + y_7 + y_8 + y_9 + y_{10} \\
 x_7 &= x_{13} + x_{15} + x_{17} + y_2 + y_3 + y_5 + y_6 + y_9 + y_{10} \\
 x_8 &= x_{14} + x_{15} + x_{16} + y_2 + y_3 + y_4 + y_5 + y_6 + y_9 \\
 x_9 &= x_{13} + x_{14} + x_{17} + y_4 + y_5 + y_8 + y_9 \\
 x_{10} &= x_{13} + x_{15} + x_{16} + y_7 + y_8 \\
 x_{11} &= x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + y_2 + y_8 \\
 x_{12} &= x_{13} + x_{14} + x_{17} + y_2 + y_4 + y_5 + y_6 + y_7 + y_9 \\
 y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10}
 \end{aligned}$$

By using the conditions above, we obtained 107 generators for the lifts of P_{54-1} to R_1 whose binary images $\phi_1(C_{54-1})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 31 The list of pure double-circulant self-dual codes of length 54-1

#	C_{54-1}	$ Aut \phi_1(C_{54-1}) $
1	2212222101231201311131311111	$2^2 \cdot 3^3$

For $n = 54$;

Let $P_{54-2} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, x_1, y_4, x_2, x_3, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, x_4, y_{12}, y_{13}, x_5, y_{14}, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_2 , we obtain the following conditions for the lifts of P_{54-2} to R_k to be self dual:

$$x_1 = x_{10} + y_3 + y_4 + y_5 + y_7 + y_8 + y_{12} + y_{13} + y_{14}$$

$$x_2 = x_{13} + y_5 + y_6 + y_8 + y_9 + y_{12} + y_{14}$$

$$x_3 = x_{13} + y_4 + y_5 + y_6 + y_8 + y_{10} + y_{13}$$

$$x_4 = x_{10} + y_3 + y_6 + y_8 + y_9 + y_{11} + y_{14}$$

$$x_5 = x_{10} + y_4 + y_8 + y_9 + y_{10} + y_{11} + y_{13}$$

$$x_6 = x_{13} + y_4 + y_5 + y_7 + y_{13}$$

$$x_7 = x_{13} + y_5 + y_9 + y_{11} + y_{12}$$

$$x_8 = x_{13} + y_3 + y_4 + y_8 + y_9 + y_{10} + y_{14}$$

$$x_9 = x_{10} + y_3 + y_7 + y_8 + y_{11} + y_{12} + y_{13}$$

$$x_{11} = x_{13} + y_3 + y_7 + y_9 + y_{10}$$

$$x_{12} = x_{13} + y_4 + y_8 + y_{10} + y_{11}$$

$$y_1 = y_3 + y_7 + y_9 + y_{10} + y_{11} + y_{13} + y_{14}$$

$$y_2 = y_4 + y_5 + y_6 + y_8 + y_{12}$$

By using the conditions above, we obtained 109 generators for the lifts of P_{54-2} to R_1 whose binary images $\phi_1(C_{54-2})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 32 The list of pure double-circulant self-dual codes of length 54-2

#	C_{54-2}	$ Aut \phi_1(C_{54-2}) $
1	222321322222203201233111131	$2^2.3^3$

For $n = 54$;

Let $P_{54-3} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_3 = [y_1, y_2, y_3, x_1, y_4, y_5, y_6, x_2, y_7, y_8, y_9, y_{10}, x_3, x_4, y_{11}, y_{12}, y_{13}, x_5, x_6, y_{14}, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_3 , we obtain the following conditions for the lifts of P_{54-3} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{13} + y_2 + y_5 + y_6 + y_9 + y_{10} + y_{13} \\
x_2 &= x_{13} + y_7 + y_8 + y_{10} + y_{11} + y_{12} + y_{14} \\
x_3 &= x_{13} + y_5 + y_9 + y_{11} + y_{12} + y_{13} + y_{14} \\
x_4 &= x_{12} + y_4 + y_5 + y_6 + y_8 + y_9 + y_{10} \\
x_5 &= x_{12} + y_6 + y_7 + y_{12} + y_{13} \\
x_6 &= x_{12} + y_2 + y_4 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{14} \\
x_7 &= x_{12} + y_5 + y_6 + y_8 + y_{14} \\
x_8 &= x_{13} + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{12} + y_{13} + y_{14} \\
x_9 &= x_{12} + y_2 + y_5 + y_7 + y_{11} \\
x_{10} &= x_{12} + y_4 + y_7 + y_8 + y_{10} + y_{12} + y_{13} \\
x_{10} &= x_{12} + y_4 + y_7 + y_8 + y_{10} + y_{12} + y_{13} \\
x_{11} &= x_{13} + y_2 + y_4 + y_5 + y_9 + y_{10} + y_{12} + y_{13} + y_{14} \\
y_1 &= y_2 + y_5 + y_8 + y_9 + y_{10} \\
y_3 &= y_4 + y_6 + y_7 + y_{11} + y_{12} + y_{13} + y_{14}
\end{aligned}$$

By using the conditions above, we obtained 106 generators for the lifts of P_{54-3} to R_1 whose binary images $\phi_1(C_{54-3})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 33 The list of pure double-circulant self-dual codes of length 54-3

#	C_{54-3}	$ Aut \phi_1(C_{54-3}) $
1	220122232222312023321313311	$2^2 \cdot 3^3$

For $n = 54$;

Let $P_{54-4} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_4 = [y_1, y_2, y_3, y_4, x_1, y_5, y_6, y_7, y_8, y_9, x_2, x_3, y_{10}, x_4, x_5, y_{11}, y_{12}, x_6, y_{13}, y_{14}, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_4 , we obtain the following conditions for the lifts of P_{54-4} to R_k to be self dual:

$$x_1 = x_{13} + y_4 + y_5 + y_6 + y_{14}$$

$$x_2 = x_{13} + y_3 + y_6 + y_{11} + y_{12}$$

$$x_3 = x_{13} + y_3 + y_4 + y_5 + y_8 + y_9 + y_{11} + y_{12} + y_{14}$$

$$x_4 = x_{13} + y_4 + y_5 + y_9 + y_{11} + y_{13} + y_{14}$$

$$x_5 = x_{13} + y_2 + y_3 + y_4 + y_6 + y_7 + y_9 + y_{12} + y_{14}$$

$$x_6 = x_{13} + y_3 + y_4 + y_6 + y_7 + y_8 + y_{10} + y_{11} + y_{14}$$

$$x_7 = x_{13} + y_4 + y_6 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14}$$

$$x_8 = x_{13} + y_2 + y_3 + y_4 + y_7 + y_9 + y_{13}$$

$$x_9 = x_{13} + y_4 + y_7 + y_9 + y_{11}$$

$$x_{10} = x_{13} + y_3 + y_4 + y_5 + y_7 + y_9 + y_{10} + y_{13} + y_{14}$$

$$x_{11} = x_{13} + y_2 + y_3 + y_6 + y_7 + y_9 + y_{10} + y_{11} + y_{13}$$

$$x_{12} = x_{13} + y_2 + y_4 + y_5 + y_6 + y_8 + y_{13}$$

$$y_1 = y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14}$$

By using the conditions above, we obtained 109 generators for the lifts of P_{54-4} to R_1 whose binary images $\phi_1(C_{54-4})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 34 The list of pure double-circulant self-dual codes of length 54-4

#	C_{54-4}	$ Aut \phi_1(C_{54-4}) $
1	022232222213233223203113111	$2^2 \cdot 3^3$

For $n = 54$;

Let $P_{54-5} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_5 = [y_1, y_2, x_1, y_3, x_2, y_4, y_5, y_6, x_3, y_7, x_4, y_8, y_9, y_{10}, y_{11}, y_{12}, x_5, x_6, y_{13}, x_7, y_{14}, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_5 , we obtain the following conditions for the lifts of P_{54-5} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{12} + y_5 + y_6 + y_{12} + y_{14} \\
x_2 &= x_{12} + y_3 + y_4 + y_6 + y_9 + y_{10} + y_{11} \\
x_3 &= x_{13} + y_4 + y_7 + y_{11} + y_{13} \\
x_4 &= x_{12} + y_3 + y_4 + y_5 + y_6 + y_{13} + y_{14} \\
x_5 &= x_{13} + y_5 + y_6 + y_{10} + y_{12} + y_{13} + y_{14} \\
x_6 &= x_{12} + y_3 + y_8 + y_9 + y_{11} + y_{13} + y_{14} \\
x_7 &= x_{13} + y_3 + y_4 + y_9 + y_{10} \\
x_8 &= x_{12} + y_6 + y_7 + y_8 + y_{12} + y_{13} + y_{14} \\
x_9 &= x_{12} + y_5 + y_6 + y_8 + y_9 + y_{10} + y_{14} \\
x_{10} &= x_{13} + y_3 + y_5 + y_6 + y_{13} \\
x_{11} &= x_{12} + y_3 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{13} + y_{14} \\
y_1 &= y_6 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} \\
y_2 &= y_3 + y_4 + y_5 + y_7 + y_{14}
\end{aligned}$$

By using the conditions above, we obtained 109 generators for the lifts of P_{54-5} to R_1 whose binary images $\phi_1(C_{54-5})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 35 The list of pure double-circulant self-dual codes of length 54-5

#	C_{54-5}	$ Aut \phi_1(C_{54-5}) $
1	023212221212222031212311111	$2^2 \cdot 3^3$

For $n = 54$;

Let $P_{54-6} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_6 = [y_1, y_2, y_3, y_4, x_1, y_5, y_6, y_7, x_2, y_8, y_9, x_3, y_{10}, y_{11}, y_{12}, x_4, x_5, x_6, x_7, y_{13}, y_{14}, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_6 , we obtain the following conditions for the lifts of P_{54-6} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{13} + y_3 + y_5 + y_7 + y_{14} \\
x_2 &= x_{12} + y_2 + y_5 + y_6 + y_7 \\
x_3 &= x_{12} + y_5 + y_6 + y_7 + y_8 + y_{10} + y_{11} \\
x_4 &= x_{13} + y_5 + y_7 + y_9 + y_{12} + y_{13} + y_{14} \\
x_5 &= x_{12} + y_2 + y_3 + y_5 + y_6 + y_{11} + y_{13} \\
x_6 &= x_{12} + y_3 + y_5 + y_6 + y_7 + y_9 + y_{14} \\
x_7 &= x_{13} + y_3 + y_9 + y_{10} + y_{11} \\
x_8 &= x_{13} + y_2 + y_5 + y_6 + y_7 + y_{10} + y_{12} + y_{13} + y_{14} \\
x_9 &= x_{12} + y_3 + y_6 + y_8 + y_{10} + y_{13} + y_{14} \\
x_{10} &= x_{12} + y_5 + y_8 + y_9 + y_{10} + y_{12} + y_{14} \\
x_{11} &= x_{12} + y_3 + y_5 + y_6 + y_7 + y_8 + y_{10} + y_{12} + y_{14} \\
y_1 &= y_2 + y_3 + y_7 + y_{10} + y_{12} \\
y_4 &= y_5 + y_6 + y_8 + y_9 + y_{11} + y_{13} + y_{14}
\end{aligned}$$

By using the conditions above, we obtained 107 generators for the lifts of P_{54-6} to R_1 whose binary images $\phi_1(C_{54-6})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 36 The list of pure double-circulant self-dual codes of length 54-6

#	C_{54-6}	$ Aut \phi_1(C_{54-6}) $
1	222012221223202131322111111	$2^2 \cdot 3^3$

For $n = 54$;

Let $P_{54-7} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_7 = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, y_6, x_3, x_4, y_7, x_5, y_8, y_9, y_{10}, y_{11}, y_{12}, x_6, x_7, x_8, y_{13}, y_{14}, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_7 , we obtain the following conditions for the lifts of P_{54-7} to R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_{12} + y_3 + y_6 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14} \\
 x_2 &= x_{12} + y_3 + y_5 + y_9 + y_{11} + y_{12} + y_{13} \\
 x_3 &= x_{12} + y_5 + y_{11} + y_{13} + y_{14} \\
 x_4 &= x_{12} + y_7 + y_9 + y_{11} + y_{12} \\
 x_5 &= x_{13} + y_4 + y_5 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} \\
 x_6 &= x_{12} + y_3 + y_4 + y_7 + y_8 + y_9 + y_{10} + y_{12} + y_{14} \\
 x_7 &= x_{13} + y_3 + y_5 + y_6 + y_8 + y_{10} + y_{14} \\
 x_8 &= x_{13} + y_6 + y_7 + y_8 + y_{12} + y_{13} + y_{14} \\
 x_9 &= x_{12} + y_4 + y_6 + y_9 + y_{14} \\
 x_{10} &= x_{13} + y_3 + y_4 + y_5 + y_6 + y_9 + y_{10} + y_{12} + y_{14} \\
 x_{11} &= x_{12} + y_3 + y_8 + y_9 + y_{13} \\
 y_1 &= y_3 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{13} \\
 y_2 &= y_4 + y_{10} + y_{11} + y_{12} + y_{14}
 \end{aligned}$$

By using the conditions above, we obtained 107 generators for the lifts of P_{54-7} to R_1 whose binary images $\phi_1(C_{54-7})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 37 The list of pure double-circulant self-dual codes of length 54-7

#	C_{54-7}	$ Aut \phi_1(C_{54-7}) $
1	022223321123220221110031311	$2^2 \cdot 3^3$

For $n = 54$;

Let $P_{54-8} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_8 = [y_1, x_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, x_2, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, x_3, y_{15}, y_{16}, y_{17}, x_4, x_5, y_{18}, x_6, x_7, x_8, x_9]$. By using V_8 , we obtain the following conditions for the lifts of P_{54-8} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_9 + y_5 + y_8 + y_{10} + y_{13} + y_{15} + y_{16} + y_{17} + y_{18} \\
x_2 &= x_9 + y_4 + y_8 + y_{10} + y_{13} + y_{16} + y_{18} \\
x_3 &= x_9 + y_8 + y_{10} + y_{12} + y_{15} + y_{17} + y_{18} \\
x_4 &= x_9 + y_8 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{17} \\
x_5 &= x_9 + y_4 + y_5 + y_8 + y_{11} + y_{12} + y_{15} + y_{17} + y_{18} \\
x_6 &= x_9 + y_8 + y_{12} + y_{13} + y_{18} \\
x_7 &= x_9 + y_4 + y_8 + y_{10} + y_{12} + y_{14} + y_{15} + y_{16} + y_{18} \\
x_8 &= x_9 + y_8 + y_9 + y_{10} + y_{11} + y_{13} + y_{14} + y_{16} + y_{18} \\
y_1 &= y_5 + y_8 + y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{18} \\
y_2 &= y_4 + y_5 + y_8 + y_9 + y_{12} + y_{15} + y_{17} \\
y_3 &= y_5 + y_9 + y_{10} + y_{12} + y_{14} + y_{17} + y_{18} \\
y_6 &= y_9 + y_{12} + y_{14} + y_{16} + y_{18} \\
y_7 &= y_8 + y_{12} + y_{15} + y_{16} + y_{17}
\end{aligned}$$

By using the conditions above, we obtained 110 generators for the lifts of P_{54-8} to R_1 whose binary images $\phi_1(C_{54-8})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 38 The list of pure double-circulant self-dual codes of length 54-8

#	C_{54-8}	$ Aut \phi_1(C_{54-8}) $
1	230222222212222212003103111	$2^2 \cdot 3^3$

For $n = 54$;

Let $P_{54-9} = [I_{27} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_9 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, x_1, y_{11}, x_2, y_{12}, y_{13}, x_3, y_{14}, y_{15}, y_{16}, y_{17}, x_4, x_5, y_{18}, x_6, x_7, x_8, x_9]$. By using V_9 , we obtain the following conditions for the lifts of P_{54-9} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_9 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{15} \\
x_2 &= x_9 + y_5 + y_7 + y_8 + y_{14} + y_{15} + y_{16} \\
x_3 &= x_9 + y_5 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{13} + y_{17} \\
x_4 &= x_9 + y_5 + y_7 + y_9 + y_{10} + y_{11} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} \\
x_5 &= x_9 + y_5 + y_8 + y_{10} + y_{11} + y_{16} + y_{17} \\
x_6 &= x_9 + y_5 + y_9 + y_{12} + y_{13} \\
x_7 &= x_9 + y_5 + y_9 + y_{11} + y_{12} + y_{15} + y_{16} \\
x_8 &= x_9 + y_7 + y_8 + y_{11} + y_{12} + y_{16} + y_{18} \\
y_1 &= y_5 + y_7 + y_8 + y_{10} + y_{13} + y_{15} + y_{17} \\
y_2 &= y_8 + y_9 + y_{10} + y_{12} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18} \\
y_3 &= y_{12} + y_{13} + y_{14} + y_{16} + y_{17} \\
y_4 &= y_{11} + y_{14} + y_{15} + y_{17} + y_{18} \\
y_6 &= y_8 + y_{10} + y_{12} + y_{13} + y_{16} + y_{17} + y_{18}
\end{aligned}$$

By using the conditions above, we obtained 110 generators for the lifts of P_{54-9} to R_1 whose binary images $\phi_1(C_{54-9})$ are best-known Type I codes with parameters $[108,54,16]$. 1 of these generators is given below:

Table 39 The list of pure double-circulant self-dual codes of length 54-9

#	C_{54-9}	$ Aut \phi_1(C_{54-9}) $
1	202020222212122122221101131	$2^2.3^3$

For $n = 58$;

Let $P_{58-1} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, y_3, y_4, x_1, x_2, x_3, y_5, x_4, y_6, y_7, x_5, y_8, x_6, y_9, x_7, y_{10}, y_{11}, x_8, y_{12}, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}]$. By using V_1 , we obtain the following conditions for the lifts of P_{58-1} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{12} + x_{15} + x_{17} + y_4 + y_7 + y_8 + y_9 \\
x_2 &= x_{12} + x_{14} + x_{17} + y_5 + y_7 + y_8 + y_{12} \\
x_3 &= x_{14} + y_2 + y_3 + y_{10} + y_{12} \\
x_4 &= x_{15} + y_3 + y_4 + y_5 + y_7 + y_8 + y_{12} \\
x_5 &= x_{14} + y_3 + y_7 + y_9 + y_{11} \\
x_6 &= x_{14} + x_{15} + x_{17} + y_2 + y_3 + y_7 + y_8 + y_{10} + y_{11} \\
x_7 &= x_{14} + x_{15} + x_{17} + y_2 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} \\
x_8 &= x_{17} + y_3 + y_6 + y_7 + y_{10} \\
x_9 &= x_{12} + x_{14} + x_{15} + y_4 + y_5 + y_7 + y_{10} \\
x_{10} &= x_{12} + x_{15} + x_{17} + y_4 + y_5 + y_6 + y_8 + y_9 + y_{11} \\
x_{11} &= x_{15} + y_2 + y_3 + y_7 + y_8 + y_9 + y_{11} \\
x_{13} &= x_{14} + x_{15} + x_{17} + y_2 + y_3 + y_5 + y_6 + y_7 + y_8 + y_{11} + y_{12} \\
x_{16} &= x_{17} + y_2 + y_3 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{11} \\
y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-1} to R_1 whose binary images $\phi_1(C_{58-1})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 40 The list of pure double-circulant self-dual codes of length 58-1

#	C_{58-1}	$ Aut \phi_1(C_{58-1}) $
1	22221310300301212212131111111	$2^2 \cdot 29$
2	22203312102123210210111111131	$2^2 \cdot 29$
3	02021110320103212210113131111	$2^2 \cdot 29$
4	20201112302123010012113131111	$2^2 \cdot 29$
5	00201110322323032210131131131	$2^2 \cdot 29$
6	00001110120303010230113111111	$2^2 \cdot 29$
7	22223110300123030210311131133	$2^2 \cdot 29$
8	22023310300321010232313131113	$2^2 \cdot 29$
9	20221332122101230210311111133	$2^2 \cdot 29$
10	00203312322301212030313131113	$2^2 \cdot 29$
11	00003112322103232012311131133	$2^2 \cdot 29$
12	02223330120103210012111131311	$2^2 \cdot 29$
13	02203112122301210030313111311	$2^2 \cdot 29$
14	02201110122323210030111111311	$2^2 \cdot 29$
15	22201330322103032010311131311	$2^2 \cdot 29$
16	00021332300123230212311131311	$2^2 \cdot 29$
17	20021112100303012232111111311	$2^2 \cdot 29$
18	20023110100321012232313111311	$2^2 \cdot 29$
19	20003332102123012210111131311	$2^2 \cdot 29$
20	02021112322301010032133131333	$2^2 \cdot 29$
21	02003132120123012032333131333	$2^2 \cdot 29$
22	22001330300103030030311131313	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-2} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, x_1, y_4, x_2, y_5, x_3, y_6, y_7, x_4, y_8, x_5, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, x_6, y_{15}, y_{16}, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}]$. By using V_2 , we obtain the following conditions for the lifts of P_{58-2} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{13} + y_4 + y_5 + y_6 + y_9 + y_{11} + y_{14} + y_{15} + y_{16} \\
x_2 &= x_{13} + y_4 + y_7 + y_{10} + y_{13} + y_{14} + y_{16} \\
x_3 &= x_{13} + y_4 + y_5 + y_7 + y_8 + y_{11} + y_{12} + y_{14} + y_{15} \\
x_4 &= x_{12} + y_6 + y_7 + y_8 + y_9 + y_{11} + y_{14} \\
x_5 &= x_{12} + y_5 + y_{11} + y_{12} + y_{13} + y_{15} + y_{16} \\
x_6 &= x_{13} + y_6 + y_9 + y_{12} + y_{16} \\
x_7 &= x_{13} + y_4 + y_6 + y_7 + y_9 + y_{11} + y_{12} + y_{13} + y_{15} \\
x_8 &= x_{12} + y_4 + y_5 + y_6 + y_7 + y_8 + y_{11} + y_{14} + y_{16} \\
x_9 &= x_{12} + y_5 + y_6 + y_9 + y_{10} + y_{11} + y_{13} \\
x_{10} &= x_{13} + y_4 + y_5 + y_6 + y_8 + y_{10} + y_{13} \\
x_{11} &= x_{13} + y_4 + y_5 + y_7 + y_9 + y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} \\
y_1 &= y_6 + y_7 + y_8 + y_9 + y_{10} + y_{12} + y_{13} + y_{15} + y_{16} \\
y_2 &= y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{15} \\
y_3 &= y_4 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{14} + y_{15}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-2} to R_1 whose binary images $\phi_1(C_{58-2})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 41 The list of pure double-circulant self-dual codes of length 58-2

#	C_{58-2}	$ Aut \phi_1(C_{58-2}) $
1	02012323221230022003221331311	$2^2 \cdot 29$
2	20212323203230020201223113111	$2^2 \cdot 29$
3	00232123203012200221001133311	$2^2 \cdot 29$
4	00232121021030202221223331111	$2^2 \cdot 29$
5	20012323003230000001003333111	$2^2 \cdot 29$
6	02012321001012222201003331111	$2^2 \cdot 29$
7	22032301203232202001223131311	$2^2 \cdot 29$
8	20212103201012202003203133111	$2^2 \cdot 29$
9	20232103003210000003221311311	$2^2 \cdot 29$
10	22012103003012222023223131311	$2^2 \cdot 29$
11	00032101001012022003223313111	$2^2 \cdot 29$
12	22230121221210200223003313111	$2^2 \cdot 29$
13	00210123223210000203003131311	$2^2 \cdot 29$
14	02030123223012222223001311311	$2^2 \cdot 29$
15	02010123021210020223023133111	$2^2 \cdot 29$
16	00230321023030020221003131311	$2^2 \cdot 29$
17	22230321001230222201023111311	$2^2 \cdot 29$
18	02210303223032222221223333111	$2^2 \cdot 29$
19	22030101201232020001003331111	$2^2 \cdot 29$
20	22030103023210022001221133311	$2^2 \cdot 29$
21	02010303023032202021003113111	$2^2 \cdot 29$
22	00232123223212020203201131113	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-3} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_3 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, y_9, y_{10}, x_2, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, x_3, y_{18}, x_4, y_{19}, y_{20}, x_5, x_6, x_7, x_8, x_9]$. By using V_3 , we obtain the following conditions for the lifts of P_{58-3} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_9 + y_8 + y_{13} + y_{14} + y_{18} + y_{19} + y_{20} \\
x_2 &= x_9 + y_5 + y_8 + y_9 + y_{10} + y_{12} + y_{13} + y_{16} + y_{18} + y_{19} + y_{20} \\
x_3 &= x_9 + y_8 + y_{10} + y_{11} + y_{14} + y_{17} + y_{19} \\
x_4 &= x_9 + y_5 + y_8 + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{20} \\
x_5 &= x_9 + y_5 + y_{12} + y_{13} + y_{15} + y_{19} + y_{20} \\
x_6 &= x_9 + y_8 + y_{11} + y_{15} + y_{17} \\
x_7 &= x_9 + y_5 + y_{10} + y_{12} + y_{15} + y_{16} + y_{18} \\
x_8 &= x_9 + y_{11} + y_{12} + y_{16} + y_{18} \\
y_1 &= y_5 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{15} + y_{17} \\
y_2 &= y_5 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{19} + y_{20} \\
y_3 &= y_8 + y_9 + y_{13} + y_{16} + y_{17} + y_{18} + y_{20} \\
y_4 &= y_5 + y_8 + y_{13} + y_{15} + y_{16} + y_{18} + y_{19} \\
y_6 &= y_9 + y_{10} + y_{12} + y_{13} + y_{14} + y_{15} + y_{17} + y_{18} + y_{19} \\
y_7 &= y_{11} + y_{14} + y_{15} + y_{16} + y_{20}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-3} to R_1 whose binary images $\phi_1(C_{58-3})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 42 The list of pure double-circulant self-dual codes of length 58-3

#	C_{58-3}	$ Aut \phi_1(C_{58-3}) $
1	20022222122322200203230211311	$2^2 \cdot 29$
2	02002022322122020003212211131	$2^2 \cdot 29$
3	22202022122102002021032013131	$2^2 \cdot 29$
4	02002022122100220023032031111	$2^2 \cdot 29$
5	00022202320122220203210211111	$2^2 \cdot 29$
6	02222022120322202021032211311	$2^2 \cdot 29$
7	0220202230232222023030011111	$2^2 \cdot 29$
8	22222002302122022221032011331	$2^2 \cdot 29$
9	20002222302322020001030231311	$2^2 \cdot 29$
10	00002202102120222201230033331	$2^2 \cdot 29$
11	22202022302320202023232231111	$2^2 \cdot 29$
12	00222202302320022221012031111	$2^2 \cdot 29$
13	02022202302120000203030031311	$2^2 \cdot 29$
14	20202202100122200003032231111	$2^2 \cdot 29$
15	22222222100322022003030033311	$2^2 \cdot 29$
16	22002022300320200221232213331	$2^2 \cdot 29$
17	00002202300302022203212231331	$2^2 \cdot 29$
18	00222220122120020001212233311	$2^2 \cdot 29$
19	22222000322120002003012031331	$2^2 \cdot 29$
20	02022000122302202203210013131	$2^2 \cdot 29$
21	02222200122100022003032033311	$2^2 \cdot 29$
22	00222000122300002003210233131	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-4} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_4 = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, y_6, x_4, x_5, y_7, x_6, y_8, x_7, y_9, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, y_{10}, y_{11}, x_{14}, x_{15}, y_{12}, x_{16}, x_{17}]$. By using V_4 , we obtain the following conditions for the lifts of P_{58-4} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{12} + x_{15} + x_{16} + y_4 + y_6 + y_8 + y_9 + y_{11} + y_{12} \\
x_2 &= x_{16} + y_4 + y_5 + y_9 + y_{12} \\
x_3 &= x_{16} + y_3 + y_7 + y_8 + y_{10} + y_{11} + y_{12} \\
x_4 &= x_{12} + x_{16} + x_{17} + y_4 + y_6 + y_7 + y_9 + y_{10} + y_{12} \\
x_5 &= x_{15} + x_{16} + x_{17} + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_9 + y_{11} \\
x_6 &= x_{12} + x_{15} + x_{17} + y_4 + y_5 + y_8 + y_{10} \\
x_7 &= x_{15} + x_{16} + x_{17} + y_2 + y_3 + y_4 + y_6 + y_8 + y_{11} \\
x_8 &= x_{16} + y_2 + y_3 + y_5 + y_6 + y_7 + y_{10} \\
x_9 &= x_{12} + x_{16} + x_{17} + y_2 + y_6 + y_7 + y_9 \\
x_{10} &= x_{15} + y_2 + y_3 + y_4 + y_5 + y_6 + y_8 \\
x_{11} &= x_{15} + y_2 + y_3 + y_6 + y_{10} \\
x_{13} &= x_{15} + x_{16} + x_{17} + y_4 + y_5 + y_6 + y_7 + y_{11} + y_{12} \\
x_{14} &= x_{16} + y_2 + y_4 + y_5 + y_8 + y_{10} + y_{12} \\
y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-4} to R_1 whose binary images $\phi_1(C_{58-4})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 43 The list of pure double-circulant self-dual codes of length 58-4

#	C_{58-4}	$ Aut \phi_1(C_{58-4}) $
1	02222313211032321313110031211	$2^2 \cdot 29$
2	020221111033010321133110211211	$2^2 \cdot 29$
3	02002333031010123111132211211	$2^2 \cdot 29$
4	20220333231212103111130011011	$2^2 \cdot 29$
5	20200111233212301133112011011	$2^2 \cdot 29$
6	00020131033030121331130211211	$2^2 \cdot 29$
7	22222313033032121333132211213	$2^2 \cdot 29$
8	02220131013010301131130211013	$2^2 \cdot 29$
9	02202113213030103311112011213	$2^2 \cdot 29$
10	22202113011032103111130011213	$2^2 \cdot 29$
11	02022333231212121331110211013	$2^2 \cdot 29$
12	02022333013212103313130031213	$2^2 \cdot 29$
13	22022111031230103333130231013	$2^2 \cdot 29$
14	22020111033230323111132031013	$2^2 \cdot 29$
15	00202111233032303111130231213	$2^2 \cdot 29$
16	00200111231032123333132031213	$2^2 \cdot 29$
17	20200333213010123313132231013	$2^2 \cdot 29$
18	20200333031010101331112011213	$2^2 \cdot 29$
19	00020113211230123111132211013	$2^2 \cdot 29$
20	20020331233010123313130211213	$2^2 \cdot 29$
21	20020113013232123311110211013	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-5} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_5 = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, x_4, x_5, x_6, y_6, x_7, x_8, y_7, x_9, x_{10}, x_{11}, x_{12}, y_8, y_9, y_{10}, y_{11}, x_{13}, x_{14}, x_{15}, y_{12}, x_{16}, x_{17}]$. By using V_5 , we obtain the following conditions for the lifts of P_{58-5} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{13} + y_3 + y_5 + y_{11} + y_{12} \\
x_2 &= x_{14} + x_{15} + x_{17} + y_2 + y_3 + y_4 + y_6 + y_9 + y_{12} \\
x_3 &= x_{14} + y_3 + y_4 + y_7 + y_8 + y_9 + y_{11} \\
x_4 &= x_{14} + x_{15} + x_{17} + y_6 + y_8 + y_9 + y_{10} + y_{11} + y_{12} \\
x_5 &= x_{13} + x_{15} + x_{17} + y_3 + y_7 + y_8 + y_9 \\
x_6 &= x_{14} + y_3 + y_4 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{12} \\
x_7 &= x_{14} + y_2 + y_3 + y_4 + y_6 + y_8 + y_{12} \\
x_8 &= x_{15} + y_2 + y_4 + y_5 + y_8 + y_{10} + y_{12} \\
x_9 &= x_{13} + x_{15} + x_{17} + y_2 + y_3 + y_4 + y_5 + y_6 + y_8 + y_9 + y_{12} \\
x_{10} &= x_{17} + y_5 + y_6 + y_8 + y_{11} \\
x_{11} &= x_{17} + y_2 + y_9 + y_{11} + y_{12} \\
x_{12} &= x_{13} + x_{15} + x_{17} + y_3 + y_5 + y_6 + y_7 + y_{11} \\
x_{16} &= x_{17} + y_4 + y_6 + y_9 + y_{10} \\
y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-5} to R_1 whose binary images $\phi_1(C_{58-5})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 44 The list of pure double-circulant self-dual codes of length 58-5

#	C_{58-5}	$ Aut \phi_1(C_{58-5}) $
1	22222313313231031330200111231	$2^2 \cdot 29$
2	22220313111033011130220111031	$2^2 \cdot 29$
3	22202331113011031112000111211	$2^2 \cdot 29$
4	02022131113213231132022111031	$2^2 \cdot 29$
5	02020333113031031110020111211	$2^2 \cdot 29$
6	02000133333211033130220111031	$2^2 \cdot 29$
7	02000331313233033130020111211	$2^2 \cdot 29$
8	02000331131031233130022111031	$2^2 \cdot 29$
9	20222331131231033132200111231	$2^2 \cdot 29$
10	20222133333011233132002111231	$2^2 \cdot 29$
11	00022333113031013130000111031	$2^2 \cdot 29$
12	00020331113211231110222111011	$2^2 \cdot 29$
13	00002313111233211132002111231	$2^2 \cdot 29$
14	00000131111033013312200111031	$2^2 \cdot 29$
15	22220133113033231130222111013	$2^2 \cdot 29$
16	22220333331033033310220111013	$2^2 \cdot 29$
17	02000113331213231132002111213	$2^2 \cdot 29$
18	02000311133033031132200111213	$2^2 \cdot 29$
19	22000313311011013130200111213	$2^2 \cdot 29$
20	00222313311211213132022111013	$2^2 \cdot 29$
21	20222113331013031130220111013	$2^2 \cdot 29$
22	00002333331233233312002111213	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-6} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_6 = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, x_4, y_6, x_5, x_6, x_7, y_7, x_8, y_8, y_9, y_{10}, x_9, x_{10}, y_{11}, x_{11}, y_{12}, y_{13}, y_{14}, x_{12}, y_{15}, y_{16}, x_{17}]$. By using V_6 , we obtain the following conditions for the lifts of P_{58-6} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{13} + y_5 + y_7 + y_{12} + y_{16} \\
x_2 &= x_{13} + y_2 + y_4 + y_5 + y_7 + y_{11} + y_{12} \\
x_3 &= x_{13} + y_6 + y_8 + y_{10} + y_{11} + y_{14} + y_{16} \\
x_4 &= x_{13} + y_2 + y_4 + y_5 + y_8 + y_9 + y_{15} \\
x_5 &= x_{13} + y_2 + y_5 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{15} \\
x_6 &= x_{13} + y_2 + y_8 + y_{10} + y_{13} + y_{15} + y_{16} \\
x_7 &= x_{13} + y_5 + y_9 + y_{14} + y_{15} \\
x_8 &= x_{13} + y_7 + y_{10} + y_{11} + y_{12} + y_{15} + y_{16} \\
x_9 &= x_{13} + y_2 + y_4 + y_6 + y_7 + y_{11} + y_{12} + y_{15} + y_{16} \\
x_{10} &= x_{13} + y_4 + y_6 + y_7 + y_8 + y_9 + y_{12} + y_{13} + y_{16} \\
x_{11} &= x_{13} + y_4 + y_5 + y_7 + y_9 + y_{10} + y_{13} + y_{14} + y_{16} \\
x_{12} &= x_{13} + y_4 + y_9 + y_{10} + y_{14} + y_{15} + y_{16} \\
y_1 &= y_2 + y_4 + y_5 + y_{12} + y_{15} \\
y_3 &= y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{13} + y_{14} + y_{16}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-6} to R_1 whose binary images $\phi_1(C_{58-6})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 45 The list of pure double-circulant self-dual codes of length 58-6

#	C_{58-6}	$ Aut \phi_1(C_{58-6}) $
1	22022311321132120031232223201	$2^2.29$
2	22022313323312120033232001201	$2^2.29$
3	02022333323110122211232001021	$2^2.29$
4	22022131123310320011010023021	$2^2.29$
5	22222311321130300013230203001	$2^2.29$
6	22222313301112322213010001021	$2^2.29$
7	22222331301132300033210001021	$2^2.29$
8	02022113103310122231210021221	$2^2.29$
9	22022311301330120231032023221	$2^2.29$
10	22222131103310120231010221021	$2^2.29$
11	22222313303130320031012021221	$2^2.29$
12	22220313323132122213230023201	$2^2.29$
13	22220333321312302231212023021	$2^2.29$
14	02220313121330302013212221201	$2^2.29$
15	22020113103330322033232003021	$2^2.29$
16	22220311101330120011030203221	$2^2.29$
17	22220133303130100013230001201	$2^2.29$
18	22002111121112100233012223021	$2^2.29$
19	02002311321332300031012221201	$2^2.29$
20	22002333321330122013210001201	$2^2.29$
21	22202331101312322013212203001	$2^2.29$

For $n = 58$;

Let $P_{58-7} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_7 = [y_1, y_2, y_3, y_4, x_1, y_5, y_6, x_2, y_7, y_8, x_3, x_4, x_5, x_6, x_7, y_9, x_8, x_9, y_{10}, x_{10}, x_{11}, x_{12}, x_{13}, y_{11}, x_{14}, x_{15}, x_{16}, y_{12}, x_{17}]$. By using V_7 , we obtain the following conditions for the lifts of P_{58-7} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{13} + x_{14} + x_{15} + y_4 + y_5 + y_9 + y_{12} \\
x_2 &= x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + y_3 + y_4 + y_6 + y_7 \\
x_3 &= x_{13} + y_4 + y_6 + y_7 + y_8 + y_9 + y_{12} \\
x_4 &= x_{16} + y_4 + y_5 + y_8 + y_{11} \\
x_5 &= x_{13} + x_{14} + x_{15} + y_3 + y_6 + y_7 + y_{12} \\
x_6 &= x_{13} + x_{14} + x_{17} + y_3 + y_4 + y_7 + y_9 + y_{10} + y_{11} \\
x_7 &= x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + y_7 + y_8 + y_9 + y_{10} \\
x_8 &= x_{15} + y_3 + y_6 + y_7 + y_8 + y_9 + y_{11} \\
x_9 &= x_{13} + x_{14} + x_{17} + y_7 + y_9 + y_{10} + y_{12} \\
x_{10} &= x_{14} + y_3 + y_4 + y_5 + y_7 + y_8 + y_{10} + y_{11} + y_{12} \\
x_{11} &= x_{17} + y_6 + y_8 + y_9 + y_{10} + y_{11} + y_{12} \\
x_{12} &= x_{13} + x_{15} + x_{16} + y_4 + y_5 + y_{10} + y_{11} \\
y_1 &= y_4 + y_6 + y_8 + y_{10} + y_{12} \\
y_2 &= y_3 + y_5 + y_7 + y_9 + y_{11}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-7} to R_1 whose binary images $\phi_1(C_{58-7})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 46 The list of pure double-circulant self-dual codes of length 58-7

#	C_{58-7}	$ Aut \phi_1(C_{58-7}) $
1	20201203003131123121331211101	$2^2 \cdot 29$
2	02021023223131103101331011121	$2^2 \cdot 29$
3	20221023201133123123331211103	$2^2 \cdot 29$
4	02001203021133103103331011123	$2^2 \cdot 29$
5	00203223021131123321111211321	$2^2 \cdot 29$
6	22023003201131103301111011301	$2^2 \cdot 29$
7	00001221001313301303331211321	$2^2 \cdot 29$
8	02001201223333323321331011301	$2^2 \cdot 29$
9	00023001001313303301311013121	$2^2 \cdot 29$
10	20221223003311103301111013321	$2^2 \cdot 29$
11	02001003223311123321111213301	$2^2 \cdot 29$
12	22223021021111303321111031121	$2^2 \cdot 29$
13	00003201201111323301111231101	$2^2 \cdot 29$
14	20223003203311101321311031123	$2^2 \cdot 29$
15	22201021001333321303131231103	$2^2 \cdot 29$
16	00201001201113101323331031123	$2^2 \cdot 29$
17	22021221021113121303331231103	$2^2 \cdot 29$
18	02003021201313121301331231123	$2^2 \cdot 29$
19	20221003003313301101131231303	$2^2 \cdot 29$
20	00201221223113303301131231303	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-8} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_8 = [y_1, y_2, y_3, y_4, x_1, y_5, y_6, y_7, x_2, x_3, y_8, x_4, y_9, y_{10}, x_5, y_{11}, x_6, x_7, x_8, y_{12}, y_{13}, x_9, y_{14}, x_{10}, x_{11}, y_{15}, x_{12}, y_{16}, x_{13}]$. By using V_8 , we obtain the following conditions for the lifts of P_{58-8} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{12} + y_8 + y_{10} + y_{11} + y_{13} + y_{14} + y_{15} \\
x_2 &= x_{13} + y_6 + y_8 + y_9 + y_{15} \\
x_3 &= x_{13} + y_5 + y_{10} + y_{12} + y_{15} \\
x_4 &= x_{13} + y_4 + y_7 + y_9 + y_{10} + y_{14} + y_{16} \\
x_5 &= x_{13} + y_4 + y_5 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{13} + y_{14} + y_{15} \\
x_6 &= x_{13} + y_7 + y_8 + y_{12} + y_{13} \\
x_7 &= x_{12} + y_6 + y_7 + y_8 + y_{10} + y_{12} + y_{13} \\
x_8 &= x_{12} + y_4 + y_7 + y_9 + y_{10} + y_{13} + y_{15} \\
x_9 &= x_{13} + y_5 + y_8 + y_{10} + y_{11} + y_{12} + y_{13} \\
x_{10} &= x_{13} + y_5 + y_6 + y_7 + y_9 + y_{12} + y_{13} + y_{15} + y_{16} \\
x_{11} &= x_{12} + y_5 + y_6 + y_7 + y_8 + y_9 + y_{14} + y_{15} + y_{16} \\
y_1 &= y_6 + y_7 + y_8 + y_9 + y_{10} + y_{14} + y_{15} \\
y_2 &= y_4 + y_9 + y_{12} + y_{13} + y_{15} \\
y_3 &= y_5 + y_9 + y_{11} + y_{15} + y_{16}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-8} to R_1 whose binary images $\phi_1(C_{58-8})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 47 The list of pure double-circulant self-dual codes of length 58-8

#	C_{58-8}	$ Aut \phi_1(C_{58-8}) $
1	22221220112322303332230132101	$2^2 \cdot 29$
2	00021202332300303330230110121	$2^2 \cdot 29$
3	02223202330120301110210310121	$2^2 \cdot 29$
4	20021022312302121110030312101	$2^2 \cdot 29$
5	02223022110302301130210132101	$2^2 \cdot 29$
6	02221020332302321132010112121	$2^2 \cdot 29$
7	02223020130302103332012332101	$2^2 \cdot 29$
8	22023002332322121312230132121	$2^2 \cdot 29$
9	22021002312300101110032330101	$2^2 \cdot 29$
10	22021002330120321132012330101	$2^2 \cdot 29$
11	00221002130120321130212112101	$2^2 \cdot 29$
12	02001222112322323312030112101	$2^2 \cdot 29$
13	20003220112302123330010130101	$2^2 \cdot 29$
14	22001220132102301132010110121	$2^2 \cdot 29$
15	00201220332102301130210332121	$2^2 \cdot 29$
16	00201220310322121112230332121	$2^2 \cdot 29$
17	00203220330300101310032130101	$2^2 \cdot 29$
18	20003202132320123330210330121	$2^2 \cdot 29$
19	20001202330320301130212110101	$2^2 \cdot 29$
20	20003202130300323330010110101	$2^2 \cdot 29$
21	20003200112320321132012130121	$2^2 \cdot 29$
22	02201200310320101112232310121	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-9} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_9 = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, x_4, x_5, y_6, y_7, x_6, x_7, x_8, x_9, y_8, y_9, x_{10}, x_{11}, x_{12}, y_{10}, x_{13}, x_{14}, x_{15}, y_{11}, y_{12}, x_{16}, x_{17}]$. By using V_9 , we obtain the following conditions for the lifts of P_{58-9} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{16} + y_2 + y_3 + y_{11} + y_{12} \\
x_2 &= x_{17} + y_2 + y_4 + y_5 + y_6 + y_7 + y_9 + y_{10} + y_{11} \\
x_3 &= x_{14} + y_2 + y_4 + y_8 + y_{12} \\
x_4 &= x_{13} + x_{14} + x_{16} + y_3 + y_5 + y_6 + y_7 + y_9 + y_{10} + y_{11} + y_{12} \\
x_5 &= x_{13} + x_{14} + x_{17} + y_2 + y_3 + y_8 + y_{10} + y_{11} + y_{12} \\
x_6 &= x_{14} + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 \\
x_7 &= x_{13} + x_{14} + x_{16} + y_2 + y_6 + y_7 + y_{10} \\
x_8 &= x_{14} + x_{16} + x_{17} + y_2 + y_3 + y_6 + y_8 + y_9 + y_{12} \\
x_9 &= x_{14} + x_{16} + x_{17} + y_2 + y_4 + y_5 + y_6 + y_7 + y_{10} + y_{11} + y_{12} \\
x_{10} &= x_{13} + y_5 + y_6 + y_8 + y_9 + y_{11} + y_{12} \\
x_{11} &= x_{14} + y_4 + y_5 + y_9 + y_{10} \\
x_{12} &= x_{13} + x_{14} + x_{17} + y_6 + y_{10} + y_{11} + y_{12} \\
x_{15} &= x_{16} + y_2 + y_4 + y_5 + y_8 + y_{11} + y_{12} \\
y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-9} to R_1 whose binary images $\phi_1(C_{58-9})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 48 The list of pure double-circulant self-dual codes of length 58-9

#	C_{58-9}	$ Aut \phi_1(C_{58-9}) $
1	22222111112013332033121112211	$2^2 \cdot 29$
2	22022131330233110231121112011	$2^2 \cdot 29$
3	22022113330031130013121110211	$2^2 \cdot 29$
4	22002111312033132233321112011	$2^2 \cdot 29$
5	22002133110013330013301132011	$2^2 \cdot 29$
6	00220133112213332213321130211	$2^2 \cdot 29$
7	00220111310233130033301110211	$2^2 \cdot 29$
8	00200113332231132213101112011	$2^2 \cdot 29$
9	00000111110213330233101110011	$2^2 \cdot 29$
10	22002113130013110013101132013	$2^2 \cdot 29$
11	22000113132213130233301112013	$2^2 \cdot 29$
12	02000133110033332233101130213	$2^2 \cdot 29$
13	20222133112233330033121132013	$2^2 \cdot 29$
14	00222113130013132033321110213	$2^2 \cdot 29$
15	00220113132213112213121130213	$2^2 \cdot 29$
16	20202131332211330033301112013	$2^2 \cdot 29$
17	02222331112031312211121132231	$2^2 \cdot 29$
18	22202313310031330033101130031	$2^2 \cdot 29$
19	22022131332011112211121132231	$2^2 \cdot 29$
20	22020113330211332031321110031	$2^2 \cdot 29$
21	00220133112231130213121110031	$2^2 \cdot 29$
22	00202113332011330231301112231	$2^2 \cdot 29$
23	00020313312231332233121132231	$2^2 \cdot 29$

For $n = 58$;

Let $P_{58-10} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_{10} = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, x_4, y_6, y_7, x_5, y_8, y_9, x_6, x_7, x_8, x_9, y_{10}, y_{11}, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, y_{12}, x_{17}]$. By using V_{10} , we obtain the following conditions for the lifts of P_{58-10} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{15} + x_{16} + x_{17} + y_5 + y_6 + y_9 + y_{11} \\
x_2 &= x_{13} + x_{15} + x_{16} + y_3 + y_4 + y_5 + y_6 + y_7 + y_{12} \\
x_3 &= x_{16} + y_2 + y_4 + y_7 + y_8 + y_9 + y_{12} \\
x_4 &= x_{17} + y_2 + y_4 + y_8 + y_{10} + y_{11} + y_{12} \\
x_5 &= x_{13} + x_{15} + x_{17} + y_2 + y_3 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{11} \\
x_6 &= x_{13} + y_4 + y_5 + y_6 + y_8 \\
x_7 &= x_{17} + y_3 + y_4 + y_6 + y_7 \\
x_8 &= x_{15} + y_3 + y_7 + y_{11} + y_{12} \\
x_9 &= x_{13} + x_{16} + x_{17} + y_4 + y_6 + y_7 + y_8 + y_9 + y_{12} \\
x_{10} &= x_{15} + x_{16} + x_{17} + y_3 + y_6 + y_7 + y_8 + y_{10} + y_{12} \\
x_{11} &= x_{13} + x_{15} + x_{16} + y_2 + y_4 + y_7 + y_9 + y_{11} + y_{12} \\
x_{12} &= x_{13} + x_{16} + x_{17} + y_7 + y_8 + y_{10} + y_{12} \\
x_{14} &= x_{17} + y_3 + y_4 + y_5 + y_6 + y_8 + y_{11} \\
y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12}
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-10} to R_1 whose binary images $\phi_1(C_{58-10})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 49 The list of pure double-circulant self-dual codes of length 58-10

#	C_{58-10}	$ Aut \phi_1(C_{58-10}) $
1	02000111302320331300333131121	$2^2.29$
2	20222111320302331322333131101	$2^2.29$
3	02202113322120313122131111103	$2^2.29$
4	22202131320320331322313111103	$2^2.29$
5	00020131302302331300313111123	$2^2.29$
6	20020113300102313100131111123	$2^2.29$
7	02220111100122111300133131321	$2^2.29$
8	22020313102100311120131131321	$2^2.29$
9	22002113102302333122111111321	$2^2.29$
10	00220113120320333100111111301	$2^2.29$
11	00202313120122311102131131301	$2^2.29$
12	00022313122302333122311111321	$2^2.29$
13	20002111122100111322133131301	$2^2.29$
14	20002111302320131120333111321	$2^2.29$
15	02220331122300131320111111303	$2^2.29$
16	02220313320100113100131111303	$2^2.29$
17	02220113302320111320113111303	$2^2.29$
18	00022111302320331322313131303	$2^2.29$
19	20002113320302111302113111323	$2^2.29$
20	20002313302122113122131111323	$2^2.29$
21	22220113322302113322133113121	$2^2.29$
22	02220311320122331100113113101	$2^2.29$

For $n = 58$;

Let $P_{58-11} = [I_{29} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V_{11} = [y_1, y_2, y_3, x_1, x_2, y_4, x_3, x_4, x_5, y_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, y_6, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, y_7, y_8, x_{19}, x_{20}, x_{21}]$. By using V_{11} , we obtain the following conditions for the lifts of P_{58-11} to R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{16} + x_{17} + x_{18} + x_{20} + x_{21} + y_3 + y_4 \\
x_2 &= x_{16} + x_{18} + x_{20} + y_2 + y_4 \\
x_3 &= x_{15} + x_{17} + x_{19} + y_3 + y_5 + y_6 + y_8 \\
x_4 &= x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{21} + y_2 + y_3 + y_4 + y_7 \\
x_5 &= x_{14} + x_{17} + x_{21} + y_6 + y_7 \\
x_6 &= x_{15} + x_{19} + x_{20} + y_4 + y_6 + y_7 + y_8 \\
x_7 &= x_{15} + x_{18} + x_{20} + y_2 + y_3 + y_4 + y_6 + y_7 + y_8 \\
x_8 &= x_{16} + x_{19} + x_{21} + y_2 + y_6 \\
x_9 &= x_{14} + x_{19} + x_{21} + y_2 + y_4 + y_6 + y_7 \\
x_{10} &= x_{14} + x_{18} + x_{19} + y_4 + y_5 + y_6 + y_8 \\
x_{11} &= x_{14} + x_{15} + x_{16} + x_{19} + x_{21} + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 \\
x_{12} &= x_{16} + x_{17} + x_{18} + x_{19} + x_{21} + y_2 + y_3 + y_4 + y_5 \\
x_{13} &= x_{17} + x_{18} + x_{19} + x_{20} + x_{21} \\
y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{58-11} to R_1 whose binary images $\phi_1(C_{58-11})$ are best-known Type I codes with parameters $[116,58,18]$ which are given below:

Table 50 The list of pure double-circulant self-dual codes of length 58-11

#	C_{58-11}	$ Aut \phi_1(C_{58-11}) $
1	22011233101331133321111100113	$2^2 \cdot 29$
2	02033013303131111321111122113	$2^2 \cdot 29$
3	02033031103133311321111100113	$2^2 \cdot 29$
4	20233231123133311301111122113	$2^2 \cdot 29$
5	20233211121113313321111302133	$2^2 \cdot 29$
6	02231211103133133301113320131	$2^2 \cdot 29$
7	20031011123133133321113302131	$2^2 \cdot 29$

For $n = 60$;

Let $P_{60} = [I_{30} | A]$ be the generator of pure double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, x_1, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, x_2, y_{14}, y_{15}, x_3, x_4, x_5, x_6, x_7, x_8, x_9, y_{16}, x_{10}, x_{11}, x_{12}, y_{17}, x_{13}]$. By using V , we obtain the following conditions for the lifts of P_{60} to R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_{13} + y_2 + y_9 + y_{13} + y_{14} \\
 x_2 &= x_{11} + y_7 + y_8 + y_{10} + y_{16} \\
 x_3 &= x_{13} + y_2 + y_6 + y_9 + y_{14} \\
 x_4 &= x_{11} + y_{10} + y_{12} \\
 x_5 &= x_{13} + y_2 + y_4 + y_6 + y_{11} + y_{13} + y_{14} \\
 x_6 &= x_{11} + y_7 + y_{12} + y_{15} + y_{17} \\
 x_7 &= x_{13} + y_4 + y_6 + y_9 + y_{11} + y_{13} + y_{14} \\
 x_8 &= x_{11} + y_{10} + y_{17} \\
 x_9 &= x_{13} + y_2 + y_4 + y_6 + y_9 + y_{11} + y_{13} \\
 x_{10} &= x_{13} + y_2 + y_9 + y_{11} + y_{14} \\
 x_{12} &= x_{13} + y_2 + y_4 + y_9 + y_{14} \\
 y_1 &= y_7 + y_8 + y_{15} \\
 y_3 &= y_7 + y_{10} + y_{12} + y_{15} + y_{17} \\
 y_5 &= y_7 + y_{15} + y_{16}
 \end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of P_{60} to R_1 whose binary images $\phi_1(C_{60})$ are best-known Type I codes with parameters $[120, 60, 18]$ which are given below:

Table 51.1 The list of pure double-circulant self-dual codes of length 60

#	C_{60}	$ Aut \phi_1(C_{60}) $
1	222202212200021221133333031121	$2^3 \cdot 3 \cdot 5$
2	220202032022221203113333031301	$2^3 \cdot 3 \cdot 5$
3	020222010220221201133333231101	$2^3 \cdot 3 \cdot 5$
4	222002210022003203311313231121	$2^3 \cdot 3 \cdot 5$
5	020002010222223201133333011301	$2^3 \cdot 3 \cdot 5$
6	020002010202201003331331031121	$2^3 \cdot 3 \cdot 5$
7	222002010022201223111333231101	$2^3 \cdot 3 \cdot 5$

By using the same conditions above, we obtained the generators of the lifts of P_{60} to R_1 whose binary images $\phi_1(C_{60})$ are best-known Type II codes with parameters $[120,60,20]$ which are given below:

Table 51.2 The list of pure double-circulant self-dual codes of length 60

#	C_{60}	$ Aut \phi_1(C_{60}) $
1	2202020300000203021331113231101	$2^3 \cdot 3 \cdot 5$
2	020220210200221223311131231121	$2^3 \cdot 3 \cdot 5$
3	020200030202201203311131011121	$2^3 \cdot 3 \cdot 5$
4	220022212220201023113133211101	$2^3 \cdot 3 \cdot 5$
5	220020032020003003333113231321	$2^3 \cdot 3 \cdot 5$
6	022000030000001003133133031321	$2^3 \cdot 3 \cdot 5$
7	000220212200021003111333031121	$2^3 \cdot 3 \cdot 5$
8	202220212020021223331331231101	$2^3 \cdot 3 \cdot 5$
9	202220212000003021133333211321	$2^3 \cdot 3 \cdot 5$

5.2 BORDERED DOUBLE-CIRCULANT SELF-DUAL CODES OVER R_k

For $n = 12$;

Let $B_{12} = [I_6 | A]$ be the generator of bordered double-circulant self-dual code where the first row of A is $V = [x_1, x_2, x_3, y_1, x_4]$. By using V , we obtain the following conditions for the lifts of B_{12} on R_k to be self dual:

$$x_1 = x_2$$

$$x_3 = x_4$$

$$y_1 = z * x * y$$

By using the conditions above, we obtained the generator of the lift of B_{12} to R_1 whose binary image $\phi_1(C_{12})$ is an extremal Type I code with parameters $[24, 12, 6]$ which is given below:

Table 52.1 The list of bordered double-circulant self-dual codes for length 12

#	C_{12}	Type	$ Aut \phi_1(C_{12}) $
1	11303	I	$2^{10} \cdot 3^3 \cdot 5$

By using the same conditions above, we obtained the generator of the lift of B_{12} to R_1 whose binary image $\phi_1(C_{12})$ is an extremal Type II code with parameters $[24, 12, 8]$ which is given below:

Table 52.2 The list of bordered double-circulant self-dual codes for length 12

#	C_{12}	Type	$ Aut \phi_1(C_{12}) $
1	11323	II	$2^{10}.3^3.5.7.11.23$

For $n = 20$;

Let $B_{20-1} = [I_{10} | A]$ be the generator of bordered double-circulant self-dual code where the first row of A is $V_1 = [y_1, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$. By using V_1 , we obtain the following conditions for the lifts of B_{20-1} on R_k to be self dual:

$$x_1 = x_8$$

$$x_2 = x_7$$

$$x_3 = x_6$$

$$x_4 = x_5$$

$$y_1 = z * x * y$$

By using the conditions above, we obtained the generators of the lifts of B_{20-1} to R_1 whose binary images $\phi_1(C_{20-1})$ are extremal with parameters $[40,20,8]$ which are given below:

Table 53 The list of bordered double-circulant self-dual codes of length 20-1

#	C_{20-1}	Type	β	$ Aut \phi_1(C_{20-1}) $
1	013111131	I	10	$2^{11}.3^2$
2	231111113	II		$2^{11}.3^2$

For $n = 20$;

Let $B_{20-2} = [I_{10} | A]$ be the generator of bordered double-circulant self-dual code where the first row of A is $V_2 = [x_1, x_2, y_1, x_3, y_2, y_3, x_4, y_4, y_5]$. By using V_2 , we obtain the following conditions for the lifts of B_{20-2} on R_k to be self dual:

$$x_1 = x_4 + y_5 + z * x * y$$

$$x_2 = x_4 + y_3 + z * x * y$$

$$x_3 = x_4 + y_3 + y_5$$

$$y_1 = y_3 + y_5 + z * x * y$$

$$y_2 = y_4$$

By using the conditions above, we obtained the generators of the lifts of B_{20-2} to R_1 whose binary images $\phi_1(C_{20-2})$ are extremal with parameters $[40,20,8]$ which are given below:

Table 54.1 The list of bordered double-circulant self-dual codes of length 20-2

#	C_{20-2}	Type	β	$ Aut \phi_1(C_{20-2}) $
1	132302100	I	1	$2^2 \cdot 3^2$
2	130320122	II		$2^2 \cdot 3^2$

Additionally, by using the same conditions above, we obtained the generators of the lifts of B_{20-2} to R_2 whose binary images $\phi_2(C_{20-2})$ are best-known Type I codes with parameters $[80,40,14]$ which are given below:

Table 54.2 The list of bordered double-circulant self-dual codes for length 20-2

#	C_{20-2}	$ Aut \phi_2(C_{20-2}) $
1	B5CFA61A8	$2^3 \cdot 3^2$
2	B5076296C	$2^3 \cdot 3^2$
3	1D6B2072C	$2^3 \cdot 3^2$

For $n = 24$;

Let $B_{24} = [I_{12} | A]$ be the generator of bordered double-circulant self-dual code where the first row of A is $V = [x_1, x_2, y_1, x_3, y_2, y_3, y_4, x_4, x_5, x_6, y_5]$. By using V , we obtain the following conditions for the lifts of B_{24} on R_k to be self dual:

$$\begin{aligned} x_1 &= x_6 + y_2 + y_4 + y_5 + z * x * y \\ x_2 &= x_6 + y_2 + y_3 + y_4 + y_5 \\ x_3 &= x_6 + y_3 + y_4 + y_5 + z * x * y \\ x_4 &= x_6 + y_2 + y_3 + y_4 + z * x * y \\ x_5 &= x_6 + y_2 + y_3 + y_5 + z * x * y \\ y_1 &= y_2 + y_3 + y_4 + y_5 + z * x * y \end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of B_{24} to R_1 whose binary images $\phi_1(C_{24})$ with parameters $[48, 24, 8]$ which are given below:

Table 55.1 The list of bordered double-circulant self-dual codes of length 24

#	C_{24}	Type	$ Aut \phi_1(C_{24}) $
1	112122211112	I	$2^{17} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11^2$
2	33032221310	I	$2^2 \cdot 11$
3	11212203310	I	$2^2 \cdot 11$
4	13032001110	I	$2^2 \cdot 11$
5	31032223312	II	$2^2 \cdot 11$
6	31032201110	II	$2^2 \cdot 11$
7	11032003112	II	$2^2 \cdot 11$
8	11010001110	II	$2^{21} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 23^2$

Additionally, by using the same conditions above, we obtained the generators of the lifts of B_{24} to R_2 whose binary images $\phi_2(C_{24})$ are best-known codes with parameters $[96,48,16]$ which are given below:

Table 55.2 The list of bordered double-circulant self-dual codes of length 24

#	C_{24}	Type	$ Aut \phi_2(C_{24}) $
1	33A92847F1C	I	$2^2.11$
2	BB21288BB18	I	$2^4.11$
3	9903288B91A	I	$2^4.11$
4	990328A9B18	I	$2^4.11$
5	77ED28A9516	I	$2^3.11$
6	BB2128A991A	I	$2^4.11$
7	3303228B91A	II	$2^4.11$
8	7F49226D318	II	$2^3.11$
9	5D65226D11A	II	$2^3.11$
10	BB8B2265F1C	II	$2^3.11$
11	330322A9B18	II	$2^4.11$
12	5D6522A1D16	II	$2^3.11$
13	112122A991A	II	$2^4.11$
14	D58B248311A	II	$2^3.11$
15	5D03246D71C	II	$2^3.11$
16	D58B24A1318	II	$2^3.11$
17	772124A991A	II	$2^3.11$
18	91CF24A171C	II	$2^3.11$
19	336524CFB18	II	$2^3.11$
20	5D0324C7D16	II	$2^3.11$
21	91CF24C711A	II	$2^3.11$
22	772124CFF1C	II	$2^3.11$
23	BB212803310	II	$2^4.11$
24	118B2847D1E	II	$2^3.11$

For $n = 28$;

Let $B_{28} = [I_{14} | A]$ be the generator of bordered double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, y_4, y_5, x_1, y_6, y_7, y_8, x_2, x_3, y_9, x_3]$. By using V , we obtain the following conditions for the lifts of B_{28} on R_k to be self dual:

$$\begin{aligned}
x_1 &= x_4 + y_4 + y_7 \\
x_2 &= x_4 + y_4 + y_5 \\
x_3 &= x_4 + y_5 + y_7 \\
y_1 &= y_4 + y_5 + y_7 + y_9 + z * x * y \\
x_5 &= x_6 + y_2 + y_3 + y_5 + z * x * y \\
y_1 &= y_2 + y_3 + y_4 + y_5 + z * x * y \\
y_2 &= y_4 + y_5 + y_7 + y_8 + y_9 \\
y_3 &= y_4 + y_5 + y_7 + y_8 + z * x * y \\
y_6 &= y_8 + y_9 + z * x * y
\end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of B_{28} to R_1 whose binary images $\phi_1(C_{28})$ are extremal Type II codes with parameters $[56,28,12]$ which are given below:

Table 56.1 The list of bordered double-circulant self-dual codes of length 28

#	C_{28}	$ Aut \phi_1(C_{28}) $
1	2202230021301	$2^2.13$
2	2222032023121	$2^2.13$
3	0022030023101	$2^2.13$
4	0000012021121	$2^2.13$
5	0220010001121	$2^3.3^4.7.13$

By using the same conditions above, we obtained the generators of the lifts of B_{28} to R_1 whose binary images $\phi_1(C_{28})$ are best known Type I codes with parameters $[56,28,10]$ which are given below:

Table 56.2 The list of bordered-double circulant self-dual codes of length 28

#	C_{28}	$ Aut \phi_1(C_{28}) $
1	0202210221121	$2^2.13$
2	2002212221101	$2^2.3.13$
3	0222232021301	$2^2.13$
4	0002230001301	$2^2.13$
5	2222030003101	$2^2.13$

For $n = 36$;

Let $B_{36} = [I_{18} | A]$ be the generator of bordered double-circulant self-dual code where the first row of A is $V = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, x_1, x_2, x_3, y_8, x_4, y_9, x_5, x_6, x_7, x_8]$. By using V , we obtain the following conditions for the lifts of B_{36} on R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_8 + y_3 + y_5 + y_6 + y_9 \\
 x_2 &= x_8 + y_4 + y_6 + y_7 + y_8 \\
 x_3 &= x_8 + y_3 + y_5 + y_6 + z * x * y \\
 x_4 &= x_8 + y_3 + y_4 + y_6 + y_7 + y_9 + z * x * y \\
 x_5 &= x_8 + y_3 + y_4 + y_6 + y_8 + y_9 + z * x * y \\
 x_6 &= x_8 + y_3 + y_4 + y_7 + y_8 \\
 x_7 &= x_8 + y_3 + y_6 + y_7 + y_8 + y_9 + z * x * y \\
 y_1 &= y_3 + y_4 + y_6 + y_7 + y_8 \\
 y_2 &= y_5 + y_9 + z * x * y
 \end{aligned}$$

By using the conditions above, we obtained the generators of the lifts of B_{36} to R_1 whose binary images $\phi_1(C_{36})$ are best known Type I codes with parameters $[72,36,12]$ which are given below:

Table 57.1 The list of bordered double-circulant self-dual codes of length 36

#	C_{36}	$ Aut \phi_1(C_{36}) $
1	20222221132323131	$2^2 \cdot 17$
2	22222223132101111	$2^2 \cdot 17$
3	02222223330103331	$2^2 \cdot 17$
4	00222201332123311	$2^2 \cdot 17$
5	02222203332301331	$2^2 \cdot 17$
6	20222201130121131	$2^2 \cdot 17$
7	00222023312121111	$2^2 \cdot 17$
8	22222021110301311	$2^2 \cdot 17$
9	20222003112321331	$2^2 \cdot 17$
10	22222001112103311	$2^2 \cdot 17$
11	00222003310323111	$2^2 \cdot 17$
12	02222001310101131	$2^2 \cdot 17$
13	22220223112323131	$2^2 \cdot 17$
14	20220221112101111	$2^2 \cdot 17$
15	02220223310321311	$2^2 \cdot 17$
16	00220221310103331	$2^2 \cdot 17$
17	02220203312123311	$2^2 \cdot 17$
18	00220201312301331	$2^2 \cdot 17$
19	20220201110303111	$2^2 \cdot 17$
20	02220021332121111	$2^2 \cdot 17$
21	22220021130123331	$2^2 \cdot 17$
22	20220023130301311	$2^2 \cdot 17$
23	22220001132321331	$2^2 \cdot 17$

By using the same conditions above, we obtained the generators of the lifts of B_{36} to R_1 whose binary image $\phi_1(C_{36})$ are best known Type II codes with parameters $[72,36,12]$ which are given below:

Table 57.2 The list of bordered double-circulant self-dual codes of length 36

#	C_{36}	$ Aut \phi_1(C_{36}) $
1	20222223112303131	$2^2 \cdot 17$
2	02222221310123331	$2^2 \cdot 17$
3	00222223310301311	$2^2 \cdot 17$
4	00222203312103311	$2^2 \cdot 17$
5	22222201110323111	$2^2 \cdot 17$
6	20222203110101131	$2^2 \cdot 17$
7	02222023332323131	$2^2 \cdot 17$
8	00222021332101111	$2^2 \cdot 17$
9	22222023130321311	$2^2 \cdot 17$
10	20222021130103331	$2^2 \cdot 17$
11	22222003132123311	$2^2 \cdot 17$
12	20222001132301331	$2^2 \cdot 17$
13	02222003330121131	$2^2 \cdot 17$
14	00222001330303111	$2^2 \cdot 17$
15	20220223132121111	$2^2 \cdot 17$
16	22220221132303131	$2^2 \cdot 17$
17	00220223330123331	$2^2 \cdot 17$
18	02220221330301311	$2^2 \cdot 17$
19	00220203332321331	$2^2 \cdot 17$
20	02220201332103311	$2^2 \cdot 17$
21	20220203130323111	$2^2 \cdot 17$
22	22220201130101131	$2^2 \cdot 17$
23	00220021312323131	$2^2 \cdot 17$
24	02220023312101111	$2^2 \cdot 17$
25	20220021110321311	$2^2 \cdot 17$
26	20220001112123311	$2^2 \cdot 17$
27	22220003112301331	$2^2 \cdot 17$

For $n = 40$;

Let $B_{40-1} = [I_{20} | A]$ be the generator of bordered-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, y_9, y_{10}, x_2, y_{11}, x_3, x_4, y_{12}, x_5, y_{13}, x_6]$. By using V_1 , we obtain the following conditions for the lifts of B_{40-1} on R_k to be self dual:

$$x_1 = x_6 + y_5 + y_7 + y_8 + y_{10} + y_{11} + z * x * y$$

$$x_2 = x_6 + y_8 + y_{10} + y_{12} + z * x * y$$

$$x_3 = x_6 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13}$$

$$x_4 = x_6 + y_7 + y_8 + y_{11} + y_{13}$$

$$x_5 = x_6 + y_5 + y_9 + y_{10} + y_{11}$$

$$y_1 = y_7 + y_8 + y_{12}$$

$$y_2 = y_8 + y_{10} + y_{11} + y_{12} + z * x * y$$

$$y_3 = y_9 + y_{10} + z * x * y$$

$$y_4 = y_5 + y_7 + y_8 + y_9 + z * x * y$$

$$y_6 = y_7 + y_9 + y_{10} + y_{12} + y_{13}$$

By using the conditions above, we obtained the generator of the lift of B_{40-1} to R_1 whose binary image $\phi_1(C_{40-1})$ is best known Type I code with parameters $[80,40,14]$ which is given below:

Table 58 The list of bordered double-circulant self-dual codes of length 40-1

#	C_{40-1}	$ Aut \phi_1(C_{40-1}) $
1	2002200012010112101	$2^2 \cdot 19$

For $n = 40$;

Let $B_{40-2} = [I_{20} | A]$ be the generator of bordered-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, y_4, y_5, x_1, y_6, y_7, x_2, x_3, x_4, x_5, x_6, x_7, y_8, y_9, x_8, x_9, x_{10}]$. By using V_2 , we obtain the following conditions for the lifts of B_{40-2} on R_k to be self dual:

$$\begin{aligned}
x_1 &= x_{10} + y_4 + y_5 + y_7 + y_8 \\
x_2 &= x_{10} + y_8 + y_9 \\
x_3 &= x_9 + y_5 + y_6 + y_8 + y_9 \\
x_4 &= x_9 + y_3 + y_5 + y_7 + z * x * y \\
x_5 &= x_{10} + y_4 + y_5 + y_7 + y_9 \\
x_6 &= x_9 + y_3 + y_4 + y_5 + y_7 + y_8 + y_9 \\
x_7 &= x_9 + y_3 + y_4 + y_5 + z * x * y \\
x_8 &= x_9 + y_3 + y_6 \\
y_1 &= y_4 + y_7 + z * x * y \\
y_2 &= y_3 + y_5 + y_6 + y_8 + y_9
\end{aligned}$$

By using the conditions above, we obtained the generator of the lift of B_{40-2} to R_1 whose binary image $\phi_1(C_{40-2})$ is extremal Type II code with parameters $[80,40,16]$ which is given below:

Table 59.1 The list of bordered double-circulant self-dual codes of length 40-2

#	C_{40-2}	$ Aut \phi_1(C_{40-2}) $
1	2022010013333102313	$2^2 \cdot 19$

By using the same conditions above, we obtained the generators of the lifts of B_{40-2} to R_1 whose binary image $\phi_1(C_{40-2})$ are best known Type I codes with parameters $[80,40,14]$ which are given below:

Table 59.2 The list of bordered double-circulant self-dual codes of length 40-2

#	C_{40-2}	$ Aut \phi_1(C_{40-2}) $
1	2002032211113102313	$2^2 \cdot 19$
2	0200212213333120313	$2^2 \cdot 19$

For $n = 40$;

Let $B_{40-3} = [I_{20} | A]$ be the generator of bordered-circulant self-dual code where the first row of A is $V_3 = [y_1, y_2, y_3, y_4, x_1, y_5, x_2, y_6, x_3, x_4, x_5, x_6, y_7, y_8, x_7, x_8, y_9, x_9, x_{10}]$. By using V_3 , we obtain the following conditions for the lifts of B_{40-3} on R_k to be self dual:

$$\begin{aligned} x_1 &= x_{10} + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 \\ x_2 &= x_{10} + y_4 + y_5 + y_9 + z * x * y \\ x_3 &= x_{10} + y_2 + y_4 + y_5 + y_6 + y_7 + y_9 \\ x_4 &= x_{10} + y_4 + y_5 + y_6 + z * x * y \\ x_5 &= x_{10} + y_2 + y_3 + y_4 + y_6 + y_7 + y_8 + y_9 + z * x * y \\ x_6 &= x_{10} + y_5 + y_6 + y_9 + z * x * y \\ x_7 &= x_{10} + y_2 + y_4 + y_5 + y_6 + y_8 + y_9 \\ x_8 &= x_{10} + y_2 + y_5 + y_7 + y_8 \\ x_9 &= x_{10} + y_3 + y_5 \\ x_{10} &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + z * x * y \end{aligned}$$

By using the conditions above, we obtained 54 generators for the lifts of B_{40-3} to R_1 whose binary images $\phi_1(C_{40-3})$ have parameters $[80,40,12]$. 2 of these generators are given below:

Table 60 The list of bordered double-circulant self-dual codes of length 40-3

#	C_{40-3}	Type	$ Aut \phi_1(C_{40-3}) $
1	0200103033130033011	I	$2^2 \cdot 19$
2	02003010111112031011	II	$2^2 \cdot 19$

For $n = 44$;

Let $B_{44-1} = [I_{22} | A]$ be the generator of bordered-circulant self-dual code where the first row of A is $V_1 = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, y_6, y_7, y_8, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, y_9, x_{11}, x_{12}]$. By using V_1 , we obtain the following conditions for the lifts of B_{44-1} on R_k to be self dual:

$$\begin{aligned}x_1 &= x_9 + y_2 + y_4 + y_5 + z * x * y \\x_2 &= x_9 + y_2 + y_4 + y_6 + y_7 + y_9 + z * x * y \\x_3 &= x_{12} + y_2 + y_3 + y_6 + y_7 + y_8 + z * x * y \\x_4 &= x_9 + y_3 + y_5 + y_8 + y_9 \\x_5 &= x_9 + y_2 + y_3 \\x_6 &= x_9 + y_2 + y_6 + y_7 + y_8 \\x_7 &= x_{12} + y_2 + y_3 + y_5 + y_7 \\x_8 &= x_{12} + y_4 + y_6 + y_7 + z * x * y \\x_{10} &= x_{12} + y_2 + y_3 + y_4 + y_6 + y_7 + y_8 \\x_{11} &= x_{12} + y_2 + y_3 + y_5 + y_6 \\y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + z * x * y\end{aligned}$$

By using the conditions above, we obtained 42 generators for the lifts of B_{44-1} to R_1 whose binary images $\phi_1(C_{44-1})$ are extremal Type II codes with parameters $[88,44,16]$. 1 of these generators is given below:

Table 61.1 The list of bordered double-circulant self-dual codes of length 44-1

#	C_{44-1}	$ Aut \phi_1(C_{44-1}) $
1	222221320013113311211	$2^2.3.7$

By using the same conditions above, we obtained 48 generators of the lifts of B_{44-1} to R_1 whose binary image $\phi_1(C_{44-1})$ is a Type I code with parameters $[88,44,14]$ which are given below:

Table 61.2 The list of bordered double-circulant self-dual codes of length 44-1

#	C_{44-1}	$ Aut \phi_1(C_{44-1}) $
1	222223102211131113231	$2^2.3.7$

For $n = 44$;

Let $B_{44-2} = [I_{22} | A]$ be the generator of bordered-circulant self-dual code where the first row of A is $V_2 = [y_1, y_2, y_3, y_4, y_5, x_1, x_2, x_3, x_4, y_6, y_7, y_8, x_5, x_6, x_7, y_9, x_8, x_9, x_{10}, x_{11}, x_{12}]$. By using V_2 , we obtain the following conditions for the lifts of B_{44-2} on R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_{11} + y_2 + y_5 + y_7 + y_9 \\
 x_2 &= x_{12} + y_4 + y_8 \\
 x_3 &= x_{12} + y_2 + y_7 + y_8 + z * x * y \\
 x_4 &= x_{11} + y_2 + y_3 + y_5 + y_7 \\
 x_5 &= x_{12} + y_2 + y_4 + y_5 + y_6 \\
 x_6 &= x_{11} + y_3 + y_4 + y_5 + y_7 + y_8 + y_9 \\
 x_7 &= x_{12} + y_3 + y_4 + y_5 + y_6 + y_9 + z * x * y \\
 x_8 &= x_{12} + y_3 + y_4 + y_7 + y_9 \\
 x_9 &= x_{11} + y_5 + y_6 \\
 x_{10} &= x_{11} + y_4 + y_6 + y_7 + y_8 \\
 y_1 &= y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + z * x * y
 \end{aligned}$$

By using the conditions above, we obtained 39 generators for the lifts of B_{44-2} to R_1 whose binary images $\phi_1(C_{44-2})$ are extremal Type II codes with parameters $[88,44,16]$. 1 of these generators is given below:

Table 62.1 The list of bordered double-circulant self-dual codes of length 44-2

#	C_{44-2}	$ Aut \phi_1(C_{44-2}) $
1	022221111022313213311	$2^2.3.7$

By using the same conditions above, we obtained 49 generators of the lifts of B_{44-2} to R_1 whose binary image $\phi_1(C_{44-2})$ is a Type I code with parameters $[88,44,14]$. 1 of these generators is given below:

Table 62.2 The list of bordered double-circulant self-dual codes of length 44-2

#	C_{44-2}	$ Aut \phi_1(C_{44-2}) $
1	022223311220111031311	$2^2.3.7$

For $n = 44$;

Let $B_{44-3} = [I_{22} | A]$ be the generator of bordered-circulant self-dual code where the first row of A is $V_3 = [y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, y_9, y_{10}, x_2, x_3, y_{11}, x_4, y_{12}, x_5, x_6, y_{13}, x_7, x_8]$. By using V_3 , we obtain the following conditions for the lifts of B_{44-3} on R_k to be self dual:

$$\begin{aligned}
 x_1 &= x_8 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{13} + z * x * y \\
 x_2 &= x_8 + y_5 + y_7 + y_{11} + y_{13} \\
 x_3 &= x_8 + y_{11} + z * x * y \\
 x_4 &= x_8 + y_7 + y_8 + y_{11} + y_{13} \\
 x_5 &= x_8 + y_6 + y_{12} + y_{13} + z * x * y \\
 x_6 &= x_8 + y_6 + y_7 + y_9 + y_{10} + y_{12} + z * x * y \\
 x_7 &= x_8 + y_6 + y_{11} \\
 y_1 &= y_5 + y_6 + y_8 + y_{11} + z * x * y \\
 y_2 &= y_5 + y_8 + y_{10} + y_{12} + y_{13} \\
 y_3 &= y_7 + y_{12} + y_{13} \\
 y_4 &= y_5 + y_8 + y_9 + y_{12} + y_{13}
 \end{aligned}$$

By using the conditions above, we obtained 32 generators for the lifts of B_{44-3} to R_1 whose binary images $\phi_1(C_{44-3})$ are extremal Type II codes with parameters $[88,44,16]$. 1 of these generators is given below:

Table 63.1 The list of bordered double-circulant self-dual codes of length 44-3

#	C_{44-3}	$ Aut \phi_1(C_{44-3}) $
1	200200023023121031031	$2^2.3.7$

By using the same conditions above, we obtained 33 generators of the lifts of B_{44-3} to R_1 whose binary image $\phi_1(C_{44-3})$ is a Type I code with parameters $[88,44,14]$. 1 of these generators is given below:

Table 63.2 The list of bordered double-circulant self-dual codes of length 44-3

#	C_{44-3}	$ Aut \phi_1(C_{44-3}) $
1	200000001003323011031	$2^2.3.7$

For $n = 44$;

Let $B_{44-4} = [I_{22} | A]$ be the generator of bordered-circulant self-dual code where the first row of A is $V_4 = [y_1, y_2, y_3, y_4, y_5, y_6, x_1, y_7, y_8, x_2, y_9, y_{10}, y_{11}, x_3, x_4, x_5, y_{12}, x_6, y_{13}, x_7, x_8]$. By using V_4 , we obtain the following conditions for the lifts of B_{44-4} on R_k to be self dual:

$$\begin{aligned}
x_1 &= x_8 + y_3 + y_4 + y_7 + y_8 + y_{11} + y_{12} \\
x_2 &= x_8 + y_3 + y_4 + y_7 + y_8 + y_9 + y_{12} + y_{13} + z * x * y \\
x_3 &= x_8 + y_3 + y_9 + y_{11} + y_{12} + y_{13} + z * x * y \\
x_4 &= x_8 + y_7 + y_9 + y_{11} + y_{12} + y_{13} + z * x * y \\
x_5 &= x_8 + y_3 + y_4 + y_7 + y_9 + y_{10} + y_{13} \\
x_6 &= x_8 + y_9 + y_{10} + y_{12} + y_{13} \\
x_7 &= x_8 + y_4 + y_9 + y_{11} + y_{12} + y_{13} + z * x * y \\
y_1 &= y_8 + y_9 + y_{11} + y_{13} + z * x * y \\
y_2 &= y_3 + y_4 + y_7 + y_9 + y_{10} + y_{12} + y_{13} \\
y_5 &= y_{10} + y_{13} + z * x * y \\
y_6 &= y_9 + y_{10} + z * x * y
\end{aligned}$$

By using the conditions above, we obtained 42 generators for the lifts of B_{44-4} to R_1 whose binary images $\phi_1(C_{44-4})$ are extremal Type II codes with parameters $[88,44,16]$. 1 of these generators is given below:

Table 64.1 The list of bordered double-circulant self-dual codes of length 44-4

#	C_{44-4}	$ Aut \phi_1(C_{44-4}) $
1	022200322102033121031	$2^2.3.7$

By using the same conditions above, we obtained 43 generators of the lifts of B_{44-4} to R_1 whose binary image $\phi_1(C_{44-4})$ is a Type I code with parameters $[88,44,14]$. 1 of these generators is given below:

Table 64.2 The list of bordered double-circulant self-dual codes of length 44-4

#	C_{44-4}	$ Aut \phi_1(C_{44-4}) $
1	222200322322011121211	$2^2.3.7$

5.3 CONCLUSION

In this thesis, we intended to search the conditions to construct binary self-dual codes of length $2^k \cdot n$, as images of self-dual codes over R_k of length n , starting with known binary self-dual codes.

In chapter 2, some fundamental properties of linear codes especially self-dual codes are given. An upper bound for the minimum distance d of an $[n, n/2]$ self-dual code which was given by Rains is examined. We completed the chapter with a few examples including extended binary Hamming code and extended binary Golay code.

In chapter 3, we introduced the ring R_k and defined self-dual codes over the ring. We investigated the rings R_1 and R_2 in detail. The definitions of some well-known weight functions such as Hamming weight and the Lee weight on \mathbb{Z}_4 are given. The Gray map which is a linear weight preserving map from R_k^n to $\mathbb{F}_2^{2^k \cdot n}$ is described. We gave some examples after we observed the theorems about the existence of self-dual codes over R_k . At the end of the chapter we investigated projections and lifts in R_k and also we observed that the projections of a self-dual code are also self-dual but not all lifts are self-dual.

In chapter 4, we studied circulant and double-circulant matrices. We examined special constructions of double-circulant self-dual codes over the ring R_1 . By using the known self-dual codes which were obtained by Gulliver and Harada, we developed a new construction method for self-dual codes via double-circulant matrices.

As a result, we obtained new self-dual codes over the rings R_1 and R_2 for different values of n . We classified double-circulant self-dual codes over R_k .

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