FRACTIONAL MODELING OF OSCILLATING DYNAMIC SYSTEMS

A THESIS SUBMITTED TO

THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

OF

ATILIM UNIVERSITY

BY

ADEL AGILA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MODELING AND DESIGN OF ENGINEERING SYSTEMS (MODES) PhD PROGRAM

(Main Field of Study: Mechatronics Engineering)

JUNE 2015

Approval of the Graduate School of Natural and Applied Sciences, Atılım University.

Prof. Dr. İbrahim Akman
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Doctor of Philosophy.

Prof. Dr. Abdulkadir Erden
Program Chair

This is to certify that we have read the thesis " Fractional Modeling of Oscillating Dynamic Systems " submitted by " Adel Agila " and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Assoc. Prof. Dr. Rajeh Eid	Assist. Prof. Dr. Bülent İrfanoğlu
Co-Supervisor	Supervisor

Examining Committee Members:

Assoc. Prof. Dr. Rajeh EID

Assist. Prof. Dr. Dumitru BALEANU

Assoc. Prof. Dr. Yiğit YAZICIOĞLU

Assist. Prof. Dr. Kutluk Bilge ARIKAN

Assist. Prof. Dr. Bülent İRFANOĞLU

Date: June 17, 2015

I declare and guarantee that all data, knowledge and information in this document has been obtained, processed and presented in accordance with academic rules and ethical conduct. Based on these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name : Adel Agila

Signature :

ABSTRACT

Fractional Modeling of Oscillating Dynamic Systems

Agila, Adel

PhD in Modeling and Design of Engineering Systems (MODES)

Supervisor : Assist. Prof. Dr. Bülent İrfanoğlu

Co-Supervisor : Assoc. Prof. Dr. Rajeh Eid

June 2015, 141 pages

In recent years, a special attention is given to model fractional dynamical systems. These systems include fractional oscillating dynamical systems. Many methods are used to model the fractional oscillating dynamical systems. The responses of some systems are obtained by means of fractional calculus and calculus of variations. In this thesis, fractional representations based on fractional calculus, calculus of variations are classified into two types:

The first type is the fractional Euler-Lagrange equations representations of free oscillating fractional systems. The fractional representation appears in the coefficients of damping terms of variable coefficient second order homogeneous differential equations.

In the second type, the differential operators are subjected to fractional orders. The considered case studies are models given by second order homogeneous and non-homogeneous three-term fractional order differential equations with fractional damping terms.

The two types are combined to produce extended fractional Euler-Lagrange equations models. In these models the differential operators are subjected to fractional orders in the damping term of the system. Additionally, the time varying coefficients of the damping terms contain a fractional integral order.

A hybrid method is introduced to obtain the responses of fractional oscillating systems. These systems are modeled by means of second order homogeneous three-term fractional order differential equations with fractional damping terms. The responses are compared with Wright function based solutions.

Keywords: Fractional calculus, fractional oscillating systems, extended fractional Euler-Lagrange equations, Wright function.

ÖZ

Osilasyon Yapan Dinamik Sistemlerin Kesirli Dereceli Modellenmesi

Agila, Adel

Doktora, Mühendislik Sistemlerinin Modellenmesi ve Tasarımı

Tez Yöneticisi : Yrd. Doç. Dr. Bülent İrfanoğlu

Ortak Tez Yöneticisi : Doç. Dr. Rajeh Eid

Haziran 2015, 141 sayfa

Son yıllarda, dinamik sistemlerin kesirli dereceli modellenmeleri ile ilgili çalışmalara özel önem verilmektedir. Kesirli dereceli modellenen osilasyon yapan dinamik sistemler bu sistemlerdendir ve modelleme için çeşitli yöntemler kullanılmaktadır. Bu tezde, kesirli gösterimler kesirli dereceli matematik ve değişimlere dayanarak iki gruba ayrılmıştır:

Birinci grup, kesirli Euler-Lagrange denklemleriyle gösterilen serbest olarak osilasyon yapan sistemlerdir. Kesirli gösterim değişken katsayılı, homojen ikinci dereceden diferansiyel denklemlerin sönümleme katsayında bulunmaktadır.

ikinci grupta, diferansiyel operatörler kesirli üslüdürler. Ele alınan örnek çalışmalar, kesirli sönümleme terimlerine sahip, ikinci dereceden homojen veya homojen olmayan, üç terimli kesirli dereceli diferansiyel denklemlerdir.

Belirtilen iki grup genişletilmiş kesirli Euler-Lagrange denklemleriyle ifade edilen modelleri oluşturmak için birleştirilmiştir. Bu modellerde, kesirli diferansiyel operatörler yanlız sistemin sönümleme teriminde bulunmaktadır. Ek olarak, sönümleme terimlerinin zamana bağlı değişebilen katsayıları kesirli derecelidir.

Kesirli modellenmiş ve osilasyon yapan sistemlerin davranışlarını elde etmek için hibrid bir yöntem aktarılacaktır. Bu sistemler, ikincinci dereceden, homojen, üç-terimli ve kesirli sönümleme terimi olan diferansiyel denklemler ile modellenmiştir. Bu sistemlerin davranışları, Wright fonksiyonları tabanlı sonuçlarla karşılaştırılmıştır.

Anahtar Kelimeler: Kesirli matematik, kesirli modelli osilasyonlu sistemler, genişletilmiş Euler-Lagrange denklemleri, Wright fonksiyonu.

To spirit of my father. To my wife and kids.

ACKNOWLEDGMENTS

The praise is to the God, who is the lord of the world. I wish to express my thanks to my supervisor Assist. Professor Dr. Bulent Irfanoglu and my co-supervisor Assoc. Professor Dr. Rajeh Eid. Your encouragement, guidance, and advice during my research study inspire me to work hard, confident, and continual. I would like to thank my committee members for offering me their comments and suggestions to accomplish my work in appropriate way.

I take this opportunity to express gratitude, sincere, and show appreciation to Assist. Professor Dr. Dumitru Baleanu. By helping of his recommendations, instructions, and guidance, which were really invaluable, I managed to publish a paper in record time. I'm looking forward to working with him in the future.

TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Mechanical vibrations and oscillatory motions are associated with most of the various artificial and natural activities. For instance, living creatures' activities such as walking, breathing, and heart beat in our bodies or movements of brunches of trees due to breeze or wind blowing are particular examples which involve natural oscillatory motions. Artificially, oscillatory motions of machines parts in order to accomplish certain jobs are considered as one of the useful forms of vibration [1]. On the contrary, there may be many dreadful or unfavorable vibrations in both of these activities. Earthquakes and undesirable vibrations occurring in some parts of industrial machines are examples to natural and artificial destructive vibration forms, respectively.

Many researches exist in the modeling of these above mentioned different vibration forms, such as models obtained by using finite element methods [7, 13] or by means of Euler-Lagrange equations (ELEs) [19]. Another active research area is in the models used in vibration control studies. Moreover, recently many studies have focused on the identification of the model parameters [25]. Additionally, there are many recent studies that are interested in applications of fractional calculus in modeling and controlling vibrating systems. Examples to these studies can be given as, the mathematical model of a fractionally-damped single degree-of-freedom spring-mass-damper system [31], the model of the two electric pendulum consisting of two planar pendula [37] and a fractional order PID controller that is designed for fractional order system models [43].

Based on the fractional calculus and the variational calculus fundamentals this thesis research involves two main types of representations. These types are the fractional

Euler-Lagrange equations (FELEs) modeling, and the fractional mathematical modeling by means of three-term fractional order differential equations (FODEs).

First the FELEs representations are introduced. In these models the fractional order which represents the order of the Riemann-Liouville fractional integral [49] appears in the coefficient of damping term. The mathematical models of these representations are expressed as variable coefficients second order homogeneous linear differential equations, or variable coefficients second order homogeneous non-linear differential equations. Additionally, hypothetical case studies are introduced for free oscillating systems.

In the second type of representation the differential operators are subjected to fractional orders. In this category, the integer orders of the integer derivatives which are interpreted as natural numbers are reinterpreted to be real numbers. The considered case studies are homogeneous and non-homogeneous three-term fractional order differential equations with fractional damping terms. These differential equations represent free and forced oscillating fractional vibrating systems, respectively.

The combination of the modeling approaches of these two representation types is considered to be the major contribution of this thesis.

Some fundamentals of fractional calculus are introduced in the second chapter. The fundamentals include some special functions that are used in the rest of thesis work. These functions involve the Gamma function, the Mittag-Leffler functions, the Wright function, and the Bessel functions. The Gamma function is considered as a main function that is used to define most of the fractional derivatives and integrals formulas [57, 61, 67], such as the Grunwald-Letnikov fractional integrals and derivatives, the Riemann-Liouville fractional integrals and derivatives, and the Caputo fractional integrals and derivatives. The Mittag-Leffler function is exploited through the Laplace transformation to obtain the solutions of the FODEs [73, 78, 83]. The Wright function [87] is explicitly used to obtain the solutions of some kinds of the FODEs [66, 93]. The Bessel function [66, 100] is exploited to obtain the solutions of linearized second order differential equations with time variant coefficients. Other important fractional calculus fundamentals are the fractional derivatives and integrals methods. The Riemann-Liouville, the Caputo, and the direct method [77, 82] are fractional derivative and integral techniques used in this work. These methods are used to obtain the fractional derivative of functions. Moreover, they are used to derive some methods and techniques in fractional representations. The derivation includes the FELEs derivation and the derivation of a hybrid method to obtain the responses of free oscillating fractional order systems. The later is introduced in chapter 6. A part of this method is introduced as a contribution to this work.

Some properties of fractional derivatives such as the distributive property, Leibniz rule obeying, and the property of the composition fractional derivatives are introduced in chapter 2. The properties are applied to the rest of the thesis chapters. Additionally, some applications of fractional integrodifferential forms are applied, in chapter 2, to some functions. The aim of these applications is to compare the well known fractional derivative forms with the direct method of fractional derivatives of certain functions.

Homogeneous and non-homogeneous three-term FODEs are studied in chapter 3. Some techniques are exploited to solve them such as the Laplace transform of R-L fractional derivative. This type of derivative is used to obtain the solution of the commensurate three-term non-homogeneous FODEs. The Bagley-Trovik equation is used to represent the solution of non-homogeneous three-term FODEs with range of fractional derivative order. Additionally, in chapter 3, the Laplace transforms of Mittag-Leffler function is used to obtain the solution of homogeneous three-term FODEs. Moreover, a comparison is done between the fractional order representations solutions and the integer order representations solutions of the FODEs that are solved via Mittag-Leffler functions.

The calculus of variations, the Riemann-Liouville fractional integral, Caputo fractional derivative, and the derivations of the traditional Euler-Lagrange equations are used, in chapter 4, to derive the FELEs. In the FELEs model the coefficient of the dissipation term appears as a function of time. This coefficient depends also on the Riemann-Liouville fractional integral order α. The FELEs, as illustrated in chapter 5, can be used to identify the parameters of damping term coefficients of some systems. The dissipation force behaviors of these systems are different from the behaviors of the systems that have constant damping force coefficients.

The FELEs are applied to some hypothetical systems in chapter 5. The applications are used to investigate the responses of systems that are modeled by the FELEs, and compare them with integer order representations. Some techniques are considered to obtain non-linear and linearized responses of the integer order representations. These techniques include obtaining the responses by means of Jacobian Elliptic function and the Runge-Kutta numerical method. The two techniques are introduced in section 5.2. The responses of the non-linear systems that are obtained by both techniques and the linearized responses that are obtained by using only the Runge-Kutta numerical technique are studied in chapter 5.

FELEs models of hypothetical free oscillating systems are investigated and their responses are obtained by using the Runge-Kutta numerical technique and obtained by means of the Bessel functions. These methods are used to obtain the solutions of the second order homogeneous variant coefficients linear differential equations. A comparison is done between the two methods for a particular case study. The linearization of the FELEs modeling of the case study is considered for different values of initial conditions. This is done to illustrate the effect of the oscillation range on the non-linear and the linearized oscillating systems.

The effect of the Riemann-Liouville fractional integrals order α on the responses of the systems that are modeled by FELEs are explained in section 5.4.2, and section 5.4.3. The phase space trajectories of the system responses are depicted to show the effect of the Riemann-Liouville fractional integrals order α. Additionally, the phase space trajectories are considered to investigate the effect of the initial angular velocity on the system behaviors.

The physical interpretation of the Riemann-Liouville fractional integrals order α is discussed in chapter 5. The order α appears in the damping term coefficient of the systems that are modeled by means of FELEs. Some previous experimental studies are considered to explain the physical meaning of α.

Free oscillating fractional damped systems are considered in chapter 6. Homogeneous three-term FODEs with fractional damping term represent the mathematical models of these systems. Some systems can be considered as real cases to represent this type of model [48]. The Kelvin-Voigt model [24] depicts the schematic diagram of such systems. By means of the Wright function [45] an analytical method is introduced to obtain the responses of the free oscillating fractional damped systems. The order of fractional derivatives of the damping terms is given as $\mathbf{(0 < \alpha < 2)}$.

Based on the graphical comparisons between Riemann-Liouville and the fractional derivatives Direct method and an analytical evaluating proof of the Direct method a hybrid method is introduced in chapter 6. The method is used to obtain the responses of free oscillating fractional damped systems. A comparison is done between the solutions that are obtained by means of the Wright function and those obtained by the introduced method. A case study is considered to illustrate the feasibility of the two methods.

The two main fractional representations approaches that are aforementioned above are combined to produce extended fractional Euler-Lagrange equations models. The models are introduced as the essential contribution to this work. The reality of these models comes from some real models [48, 71]. In these models the differential operators are subjected to fractional orders in the damping terms of the systems. Moreover, the coefficients of the damping terms are given as functions of time. That matches the behaviors of the introduced models. Some types of fractional derivatives, discretization, and numerical techniques are considered to obtain the responses of these systems.

CHAPTER 2

SOME FUNDAMENTALS of FRACTIONAL CALCULUS

2.1 Introduction

The fractional calculus is defined as the extension of a derivative from being an integer order to a fractional order [2]. Even though all of them give the same meaning, the fractional calculus is defined in many different ways. These include the growing of the traditional definitions of calculus operators of integral and derivative in the way of the fractional exponents growing from being integer exponents [8]. Another definition states that the fractional calculus gives an efficient representation of the nature behaviors [14]. The fractional calculus is considered as an emerging field in the area of applied mathematics and mathematical physics and it has many applications in many areas of science and engineering [53]. The FODEs are considered as one of the fractional calculus branches, in spite of some references define the fractional calculus as a generalization of integral and derivative operators. In this section some special functions such as Euler's Gamma function, Mittag-Leffler function, Wright function, and Bessel functions are introduced. The aforementioned functions are not the only functions that are used in fractional calculus, but they are defined in this section, in order to be used in later sections of the thesis. Moreover, the fractional derivatives and integral formulas such as, the Riemann-Liouville, and the Caputo are defined. Some applications of these formulas are illustrated, and investigated.

2.2 Special Functions

In order to solve the fractional integrals, fractional derivatives, and FODEs, or to obtain the responses of fractional order systems, certain special functions must be used. These special functions are exploited in some computations and verifications in this thesis. The functions include the Gamma function [20], the Bessel functions, the Mittag-Leffler, and the Wright function. The Gamma function is used to find the factorial value of real numbers [26]. The follwoing are some examples of these functions' usage:

The Gamma function is used to obtain the factorial of real numbers in Riemann-Liouville fractional derivatives and integrals.

The Bessel functions are used to obtain the responses of continuous fractional systems models [32].

The Mittag-Leffler functions are used to solve the FODEs or systems after taking the Laplace transform, thus any fractional explicit transfer function can be represented in terms of Mittag-Leffler functions [38]. The Wright function is used to obtain the responses of fractional order systems as well [44]. The following subsections give some details about these special functions.

2.2.1 The Euler's Gamma Function

The Euler's gamma function is one of the important functions that is used in the fractional calculus. This function generalizes the factorial $n!$ by allowing n to take non-integer values [14]. The Gamma function is defined as [50],

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \ , \ n \geq 0 \ . \tag{2.1}$$

For integer values, it can be given as follows :

$$\Gamma(n) = (n-1)! \ , \ n \geq 0 \ . \tag{2.2}$$

2.2.2 The Mittag-Leffler Functions

The Mittag-Leffler functions (M-L functions) play a significant role in the fractional calculus. The generalized M-L function that is used right here is the two-parameter M-L function which is defined as [6, 57, 62],

$$E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \beta)} \; , \alpha \geq 0 \; , \; \beta \geq 0 \; , \; \alpha \;\; \beta \in \mathbb{R} \; , \tag{2.3}$$

where z can be represented as a variable or a function. For instance, $z = at^{\alpha}$; a is a constant, α; and β are constants and represent the first and the second parameters of the M-L function, respectively, and t is a variable [6].

The M-L function which is represented by Eq.(2.3) can be written in terms of the first parameter α as $E_{\alpha}(z) = E_{\alpha,1}(z)$, if the second parameter β equals one [68, 74]. Some particular cases of M-L functions can be defined as follows [14]:

The two-parameter M-L function, where both parameters equal one, is expressed as,

$$E_{1,1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j+1)} \; . \tag{2.4}$$

Equation (2.4) represents a M-L function with $\alpha = 1$, and $\beta = 1$. This function can be expanded as,

$$\begin{aligned} E_{1,1}(z) &= \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j+1)} \\ &= \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j)!} \; . \end{aligned} \tag{2.5}$$

On other hand, the Taylor series expansion of the exponential function e^z around $z_0 = 0$ is expressed as the right hand side of Eq.(2.5). So, by equality, the right hand side of Eq.(2.4) can be expressed in terms of exponential function as following [84]:

$$E_{1,1}(z) = e^{z}\ . \tag{2.6}$$

Similarly, the two-parameter M-L function with $\alpha = 1$, and $\beta = 2$ can be represented by [14]

$$E_{1,2}(Z) = \frac{e^{z}-1}{z}, \qquad z \neq 0, \tag{2.7}$$

and for $\alpha = 2$, and $\beta = 1$ the M-L function is given by [14]

$$\begin{aligned} E_{2,1}\left(-z^{2}\right) &= \sum_{j=0}^{\infty} \frac{\left(-z^{2}\right)^{j}}{\Gamma(2j+1)} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^{j}\left(z^{2}\right)^{j}}{(2j)!} \\ &= \cos(z)\ . \end{aligned} \tag{2.8}$$

The M-L functions represented in Eq.(2.7) and Eq.(2.8), and their corresponding known functions $\frac{e^{z}-1}{z}$ and $\cos(z)$, respectively, are illustrated in Fig. 2.1. It's noticeable from the figure that the Mittag-Leffler of $z = f(t) = t$, with $\alpha = 1$, and $\beta = 2$ for a certain truncation $\jmath = 50$ matches exactly the exponential function $f(t) = \frac{e^{t}-1}{t}$, where $t \neq 0$. Additionally, the Mittag-Leffler of $f(t) = -(3t)^{2}$, with $\alpha = 2$, and $\beta = 1$ for a truncation of the index $\jmath = 50$ matches exactly the trigonometric function $f(t) = \cos(3t)$.

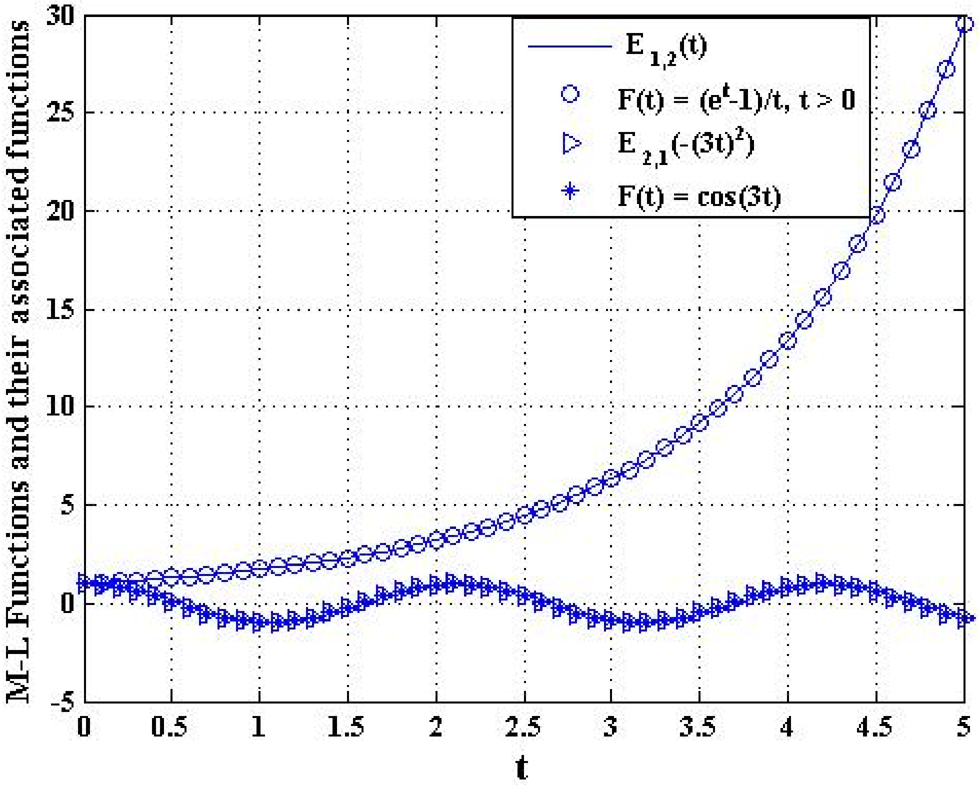


Figure 2.1: M-L functions vs. the corresponding known functions.

The general formula of the k^{th} derivative of M-L functions is defined as [90],

$$E_{1,1}^{k}(z) = \sum_{j=0}^{\infty} \frac{(j+k)!z^{j}}{j!\Gamma(\alpha j + \alpha k + \beta)}, k = 1, 2, ... \quad (2.9)$$

where the M-L function derivative order $k \in \mathbb{N}$.

2.2.3 The Wright Function

The Wright function is considered as one of the important functions that are used in fractional calculus. This function gives a numerical solution of three-term FODEs [66, 93]. The general formula of the Wright function is expressed as follows [66]:

$$
\begin{aligned}
{}_p\Psi_q(z) &= {}_p\Psi_q\left[\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\ \left(b_j,\beta_j\right)_{1,q}\end{matrix}\right| z\right] \\
&= \sum_{k=0}^{\infty}\frac{\Pi_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\Pi_{j=1}^{q}\Gamma\left(b_j+\beta_j k\right)}\frac{z^k}{k!}\,, \qquad (2.10)
\end{aligned}
$$

where $z, a_i, b_j \in \mathbb{C}, \alpha_i, \beta_j \in \mathbb{R}, k \in \mathbb{N}, i = 1, 2, ..., p,\ and\ j = 1, 2, ..., q.$

2.2.4 The Bessel Functions

The type $J_v(z)$, and the type $Y_v(z)$ are two kinds of Bessel functions defined in this section; both of them are utilized in this research. The definition of the first kind is given by [66, 100]

$$
J_v(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k+v}}{k!\,\Gamma(v+k+1)}\,,\quad k, v \in \mathbb{C}\,. \qquad (2.11)
$$

The formula of the second kind [100] $Y_v(z)$ is expressed via the Bessel function of the first kind $J_v(z)$, as follows:

$$
Y_v(z) = \frac{1}{sin(v\pi)}\left[cos(v\pi)\,J_v(z) - J_{-v}(z)\right]\,. \qquad (2.12)
$$

2.3 Fractional Derivatives and Integrals

The generalization of integration and differentiation to non-integer-order with the fundamental operator ${}_aD_t^{\alpha}$ [79, 103] is the main definition of the fractional integrodifferentiation, where α is the order of the integral or derivative operation, and the variables a and t are the bounds of the operation. The extension of the integer derivative operator, D^{α} to include positive or negative real values of α gives fractional derivatives or fractional integrals operators, respectively [26].

The order of fractional derivative and integral α defines the type of fractional integrodifferential whether it's integer or non integer, and whether it's fractional integral or fractional derivative. The following definition illustrates the fractional derivative of order α of a function $f(t)$ [103]:

$$\frac{d^{\alpha}}{dt^{\alpha}} f(t) = \begin{cases} f^{(n)}(t), & \text{if } \alpha = n \in \mathbb{N}, \\ \dfrac{t^{n-\alpha-1}}{\Gamma(n-\alpha)} * f^{(n)}(t), & \text{if } n-1 < \alpha < n,\ n \in \mathbb{N}, \end{cases} \tag{2.13}$$

where the asterisk * denotes the time convolution between two functions. The continuous fractional integrodifferential operator is defined, according to the fractional order, as follows [79]:

$$ {}_aD_t^{\alpha} f(t) = \begin{cases} \dfrac{d^{\alpha}}{dt^{\alpha}} f(t), & \text{if } \alpha \in \mathbb{R}^{+} \\ 1, & \text{if } \alpha = 0, \\ \displaystyle\int_a^t [f(\tau)\, d\tau]^{(\alpha)}, & \text{if } \alpha \in \mathbb{R}^{-} \end{cases} . \tag{2.14}$$

2.3.1 The Fractional Derivative and Integral Formulas

There are many definitions of the fractional derivative and integral formulations. The most famous definitions are the Grunwald-Letnikov (G-L) fractional integrals and derivatives, the Riemann-Liouville (R-L), the Caputo, the Hadamard, and the Marchaud definitions [110]. Aditionally, to these definitions the Coimbra fractional derivative [143] is considered to be one of the widely applied fractional derivative formulas. These definitions are equivalent under some conditions [79]. The basic definition of the G-L fractional integrals is described as [45]

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \longrightarrow 0} \frac{1}{\Gamma(\alpha) h^{\alpha}} \sum_{k=0}^{\frac{(t-a)}{h}} \frac{\Gamma(\alpha+k)}{\Gamma(k+1)} f(t+kh) . \tag{2.15}$$

The approximation of G-L fractional integrals definition is given by [14]

$$
{}_aD_t^{\alpha} f(t) = \lim_{h \longrightarrow 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{(t-a)}{h}\right]} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(t-kh) \ , \tag{2.16}
$$

where [.] means the integer part, h is the sampling period, and the binomial coefficient $\begin{pmatrix} \alpha \\ k \end{pmatrix}$ [14, 117] is represented as $\begin{pmatrix} \alpha \\ k \end{pmatrix} = \dfrac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-k+1)}{k!}$.

The definition of the second formula of the fractional derivatives and integrals is the R-L. Because of its simplicity, this definition is widely used in many applications and is given in left and right forms. If $f(t)$ is an absolutely continuous function in the closed time interval $[a, b]$ and $n-1 \leq \alpha < n, n \in \mathbb{N}$, then the left and the right R-L fractional derivatives are, respectively, given by [66, 115]

$$
{}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \ \text{ left } \ \alpha > 0, \tag{2.17}
$$

and

$$
{}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d^n}{dt^n}\right) \int_t^b \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \ \text{ right } \ \alpha > 0. \tag{2.18}
$$

If the order of fractional derivative $\alpha < 0$ [6], then the fractional derivative represents fractional integral with $\alpha > 0$. The left and right R-L fractional integrals are given, respectively, by [66]

$$
{}_aD_t^{-\alpha} f(t) = {}_a I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \ \text{ left } , \tag{2.19}
$$

and

$$
{}_tD_b^{-\alpha} f(t) = {}_t I_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} d\tau \ \text{ right } . \tag{2.20}
$$

Another important type of the fractional derivatives and integrals is the Caputo fractional derivative. Like the R-L fractional integrodifferential, it is given into two forms (left and right), which are illustrated in the following [123]:

$$
{}_{a}^{c}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}(t-\tau)^{n-\alpha-1}\frac{d^{n}f(\tau)}{d\tau^{n}}d\tau\ ,\ \text{left}\ , \tag{2.21}
$$

and

$$
{}_{t}^{c}D_{b}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}(t-\tau)^{n-\alpha-1}(-1)^{n}\frac{d^{n}f(\tau)}{d\tau^{n}}d\tau\ ,\ \text{right}\ , \tag{2.22}
$$

where $f(t)$ is a continuous function of t in the closed interval $[a, b]$, and α is the order of the derivative such that $n-1 \leq \alpha < n,\ n \in \mathbb{N},$ and $\alpha \in \mathbb{R}$.

The advantage of using the Caputo fractional derivative over the R-L fractional derivative is that the Caputo derivative of a constant is zero, which matches the integer order derivative of the constant, while the R-L derivative of a constant C, as a function of t, $D_{t}^{\alpha}[C]$ is $\frac{C(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}$. It follows that, in mathematical modeling of some applications of various areas, there is a difficulty to interpret the initial conditions that are required for the initial value problems of FODEs [126]. However, there is a relationship between the R-L and Caputo fractional derivatives. This relationship can be given by

$$
{}_{t_0}^{c}D_{t}^{\alpha}f(t) = {}_{t_0}D_{t}^{\alpha}\left[f(t)-f(t_0)\right]\ . \tag{2.23}
$$

So that the Caputo fractional derivative at any time can be expressed as [14]

$$
{}_{t_0}^{c}D_{t}^{\alpha}f(t) = {}_{t_0}D_{t}^{\alpha}f(t) - \frac{f(t_0)}{\Gamma(1-\alpha)}(t-t_0)^{-\alpha}\ . \tag{2.24}
$$

As a particular case, for $f(t_0) = 0$, the equality of the R-L and the Caputo fractional derivatives be held as follows:

$$
{}_{t_0}^{c}D_{t}^{\alpha}f(t) = {}_{t_0}D_{t}^{\alpha}f(t)\ . \tag{2.25}
$$

2.3.2 Some Properties of Fractional Derivatives

The considered properties in this section are not applied only to the integer order derivatives, but also they are valid in fractional order derivatives [6, 128, 141].

The first property is the association property, which can be given by

$$D_t^{\alpha}\left[Cf(t)\right] = CD_t^{\alpha}\left[f(t)\right] \ , \tag{2.26}$$

where C is a constant.

The second property we would like to incorporate into the fractional calculus is the distribution, which can be given as follows:

$$D_t^{\alpha}\left[f(t) \pm g(t)\right] = D_t^{\alpha}\left[f(t)\right] \pm D_t^{\alpha}\left[g(t)\right] \ . \tag{2.27}$$

The third property is the operator obeys Leibniz rule for taking the derivative of the product of two functions, which can be expressed as

$$D_t^{\alpha}\left[f(t)\, g(t)\right] = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D_t^{\alpha-k}\left[f(t)\right] D_t^{k}\left[g(t)\right] \ , \tag{2.28}$$

where the binomial coefficient is given by

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha-1)}{\Gamma(k+1)\,\Gamma(\alpha+1-k)} \ .$$

The fourth property is the composition fractional derivative, which states that; if the function $f(t)$ can be represented by the power series and the following fractional derivatives: $D_t^{\alpha}[f(t)]$, $D_t^{u}[f(t)]$, and $D_t^{v}[f(t)]$, where $\alpha = u + v$, and u and $v \in \mathbb{R}$, can be taken for the function $f(t)$, then, under some constraints [6], the following composition property is valid [128, 131]:

$$D_t^{\alpha}\left[f(t)\right] = D_t^{u+v}\left[f(t)\right] = D_t^{u}\left\{D_t^{v}\left[f(t)\right]\right\} = D_t^{v}\left\{D_t^{u}\left[f(t)\right]\right\} \ . \tag{2.29}$$

2.4 Applications of Fractional Integrodifferential Forms

The fractional order modeling is applied to many fields of sciences and engineering including fluid flow, diffusive transport, electrical networks, electromagnetic theory and probability [133]. Moreover, the fractional order representation is applied to other sciences such as, biology, ecology, etc. For example the Caputo fractional derivatives are used to establish a fractional equation model to represent the allometric scaling laws that exist in biology [135]. In this section, some of fractional order forms that are introduced in the previous section are applied to obtain the derivatives and the integrals of some functions.

2.4.1 R-L Fractional Derivatives of Some Functions

The R-L fractional integrodifferentiation is classified into two types, which are R-L fractional derivatives and R-L fractional integrals. There are many applications of the later type. For instance, the derivation of the fractional Euler Lagrange Equations which is explained in chapter (4) is done by using the R-L fractional integral, Furthermore, the R-L fractional derivative is exploited to proof the Direct method of fractional derivative. Since both types of R-L integrodifferentiations have the same steps when the fractional derivatives or fractional integrals are taken for any function, so let's consider one of them in the following:

As illustrating examples let's consider the left R-L fractional derivative, and for simplicity let the lower limit a of the fractional derivative equals zero.

i) The R-L Fractional Derivative of $f(t) = 1$, with $n = 1$, and $n - 1 \leq \alpha < n$. Applying the left R-L fractional derivative formula using Eq.(2.17), for $a = 0$, as following:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \; . \tag{2.30}$$

Substituting the function $f(t) = 1$, with $n = 1$, and $n - 1 \leq \alpha < n$ into Eq.(2.30) yields

$$ {}_0D_t^{\alpha}\left(1\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \frac{d}{dt} \int_0^t \frac{1}{\left(t-\tau\right)^{\alpha}} d\tau \ . \tag{2.31} $$

Solving the last integrodifferentiation in Eq.(2.31) to obtain the R-L fractional derivative of $f\left(t\right) = 1$ with given n and as a function of the fractional order α as following:

$$ {}_0D_t^{\alpha}\left(1\right) = \frac{1}{\Gamma\left(1-\alpha\right)} t^{-\alpha} \ , \quad 0 \leq \alpha < 1 \ . \tag{2.32} $$

ii) The R-L fractional derivative of $f\left(t\right) = t^m$, with $n-1 \leq \alpha < n, \ m \in \mathbb{N}$. Applying the left R-L fractional derivative that is given by Eq.(2.17) to the function $f\left(t\right) = t^m$ yields

$$ {}_0D_t^{\alpha}\left(t^m\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \frac{d^n}{dt^n} \int_a^t \frac{\tau^m}{\left(t-\tau\right)^{\alpha+1-n}} d\tau \ . \tag{2.33} $$

As a particular case, let the lower bound $a = 0$ to obtain

$$ {}_0D_t^{\alpha}\left(t^m\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \frac{d^n}{dt^n} \int_0^t \frac{\tau^m}{\left(t-\tau\right)^{\alpha+1-n}} d\tau \ . \tag{2.34} $$

Solving integrodifferential in Eq.(2.34) to obtain the general solution formula of the R-L fractional derivative of the function $f\left(t\right) = t^m$, with $n-1 \leq \alpha < n$ as follows:

$$ {}_0D_t^{\alpha}\left(t^m\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \left[\prod_{p=n}^{m} \left(\frac{p}{p-\alpha} \right) t^{m-\alpha} \right], \ \ m \geq n, \ p \in \mathbb{N} \ . \tag{2.35} $$

Considering the particular case, when $f\left(t\right) = t^m$, with $n = 1$, where $n-1 \leq \alpha < n$, then the solution becomes

$$ {}_0D_t^{\alpha}\left(t^m\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \left[\prod_{p=1}^{m} \left(\frac{p}{p-\alpha} \right) t^{m-\alpha} \right], \ \ m \geq 1, \ p \in \mathbb{N} \ . \tag{2.36} $$

For more specific illustrating example let $m = 4$, and $\alpha = \frac{1}{2}$, so the solution of the R-L fractional derivative of $f\left(t\right) = t^4$, for $\alpha = \frac{1}{2}$, is given by

$$ {}_0D_t^{\frac{1}{2}}\left(t^4\right) = 2.0632\, t^{\frac{7}{2}} \; . \tag{2.37} $$

Figure 2.2 illustrates extended R-L fractional derivative of $f(t) = t^4$, which is given by Eq.(2.35) through Eq.(2.37), for $m = 4$, and $\alpha = 0 : 0.2 : 0.8$, where α represents the order of the fractional derivative and 0.2 is the increment of α.

Figure 2.3 illustrates the R-L fractional derivative of $f(t) = t^4$, with $n = 1$, and $\alpha = 1 : 0.05 : 1.95$, in three dimensions, where α represents the order of the fractional derivative, and 0.05 is the increment of α.

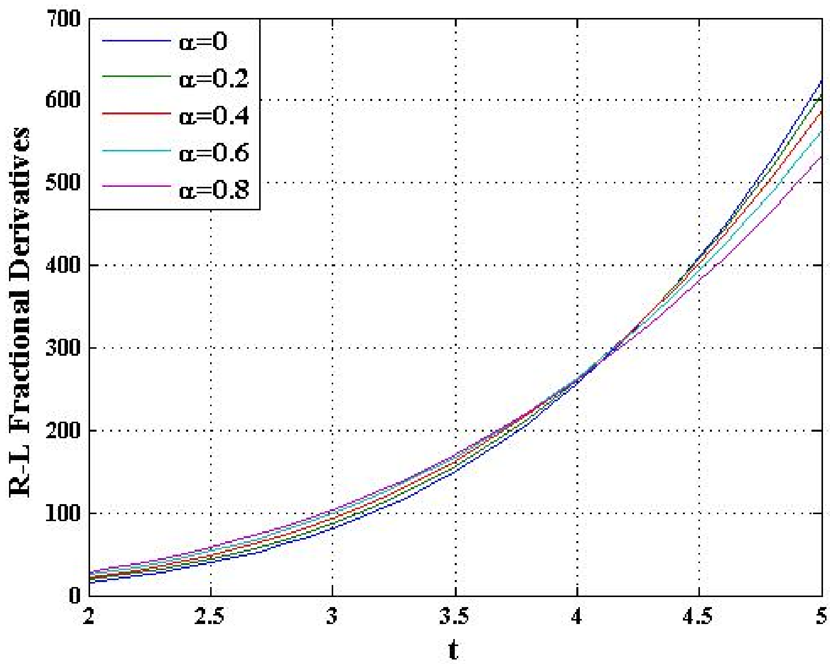


Figure 2.2: R-L fractional derivative of $f(t) = t^4$, with $n = 1$, $\alpha = 0 : 0.2 : 0.8$.

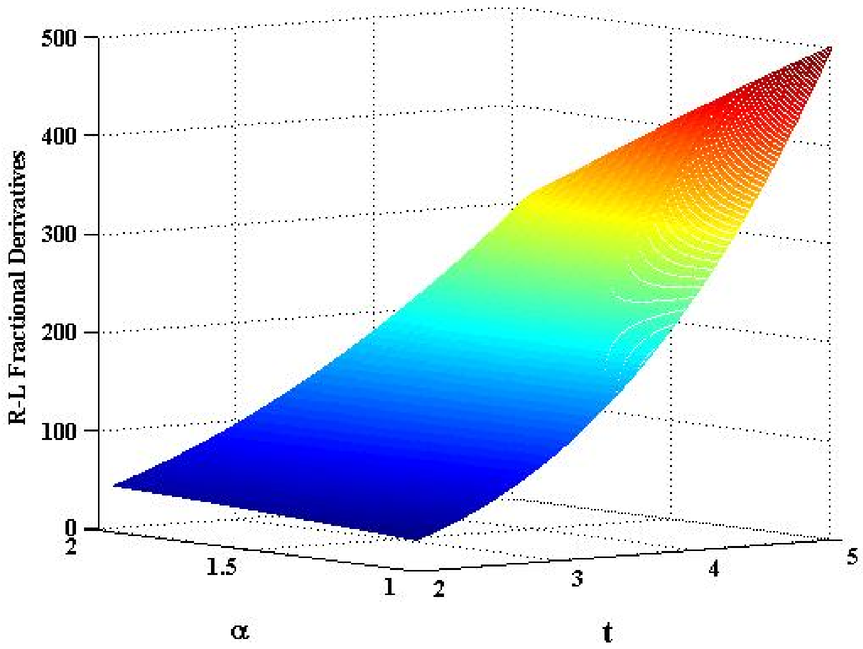


Figure 2.3: R-L derivative of $f(t) = t^4$, with $n = 2$, $\alpha = 1 : 0.05 : 1.95$.

iii) The R-L fractional derivative of the exponential function $f(t) = e^t$, for the derivative order $n - 1 \leq \alpha < n$.

Since the exponential function e^t can be expressed as follows:

$$\begin{aligned} f(t) &= e^t \\ &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^k}{k!} + \ldots \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!} . \end{aligned} \tag{2.38}$$

Taking the R-L fractional derivative of the given function, with $n = 1$ yields

$$ {}_0D_t^{\alpha}\left(e^t\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{e^{\tau}}{(t-\tau)^{\alpha}} d\tau . \tag{2.39}$$

Substituting the series of the exponential function, that is represented in Eq.(2.38) into Eq.(2.30), assigning a certain truncation k, and solving the integrodifferentiations, according to the solution of the previous example, obtain the following general solution of R-L fractional derivative of the exponential function $f(t) = e^t$, with $n = 1$:

$$ {}_0D_t^{\alpha}\left(e^t\right) = \frac{1}{\Gamma(1-\alpha)} \left\{ \begin{array}{c} x^{-\alpha} + \left[\prod_{p=1}^{1}\left(\frac{p}{p-\alpha}\right)t^{1-\alpha}\right] + \left[\frac{1}{2!}\prod_{p=1}^{2}\left(\frac{p}{p-\alpha}\right)t^{2-\alpha}\right] \\ + \left[\frac{1}{3!}\prod_{p=1}^{3}\left(\frac{p}{p-\alpha}\right)t^{3-\alpha}\right] + \ldots + \left[\frac{1}{k!}\prod_{p=1}^{k}\left(\frac{p}{p-\alpha}\right)t^{k-\alpha}\right] + \ldots \end{array} \right\}. \quad (2.40) $$

The R-L fractional derivative of the above exponential function, in the first stage of the R-L fractional derivative $n-1 \leq \alpha < n$, is depicted in Fig. 2.4. It is clear from the figure that the curves depict the behaviors of exponential functions.

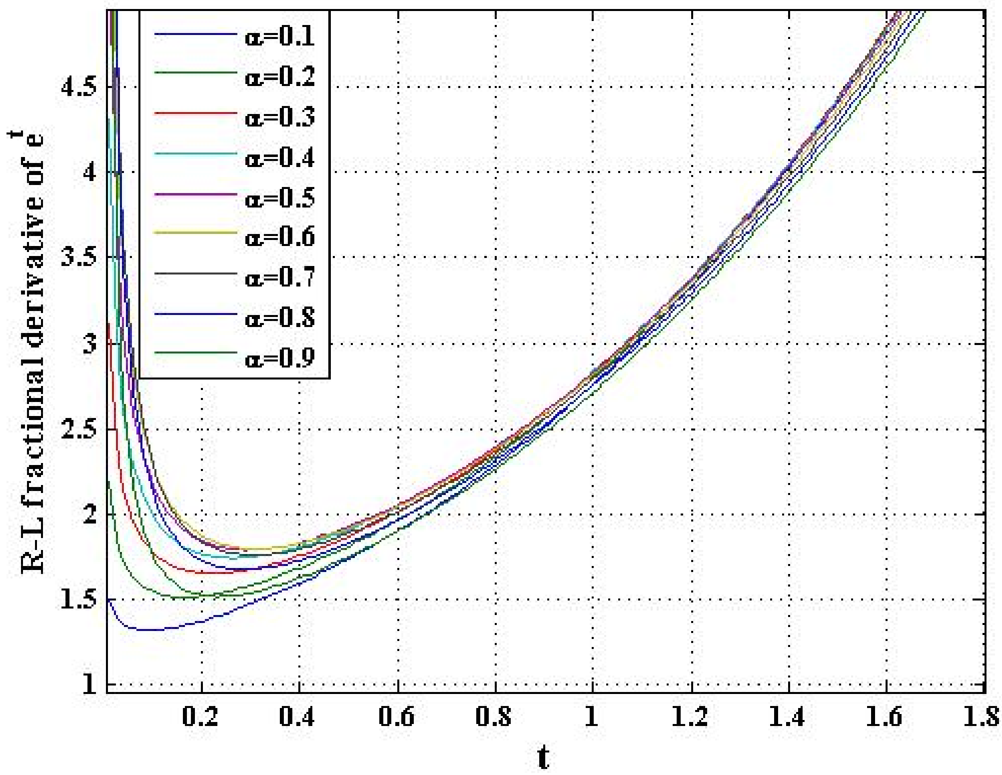


Figure 2.4: R-L fractional derivative of $f(t) = e^t$, with $n = 1$, $\alpha = 0 : 0.1 : 0.9$.

A comparison between the R-L fractional derivatives and the integer order derivatives of the same exponential function that is given in the above application $f(t) = e^t$ is shown in Fig. 2.5. It's perceptible that the 1^{st}, and 2^{nd}, integer derivatives of the exponential function $f(t) = e^t$ match the R-L fractional derivatives at the order of fractional derivatives $\alpha = 0$, and $\alpha = 1$. There are small deviations at the end of the matching lines. That lines represented by circles for 1^{st} integer derivative, asterisks for 2^{nd} integer derivative, and dashed lines and solid lines for R-L fractional derivative with $n = 1$, and $n = 2$,, respectively. The deviations are due to the values of truncation, which is given by 5 in Fig. 2.5. So if the truncation increases, for a certain interval of t, these deviations will decrease.

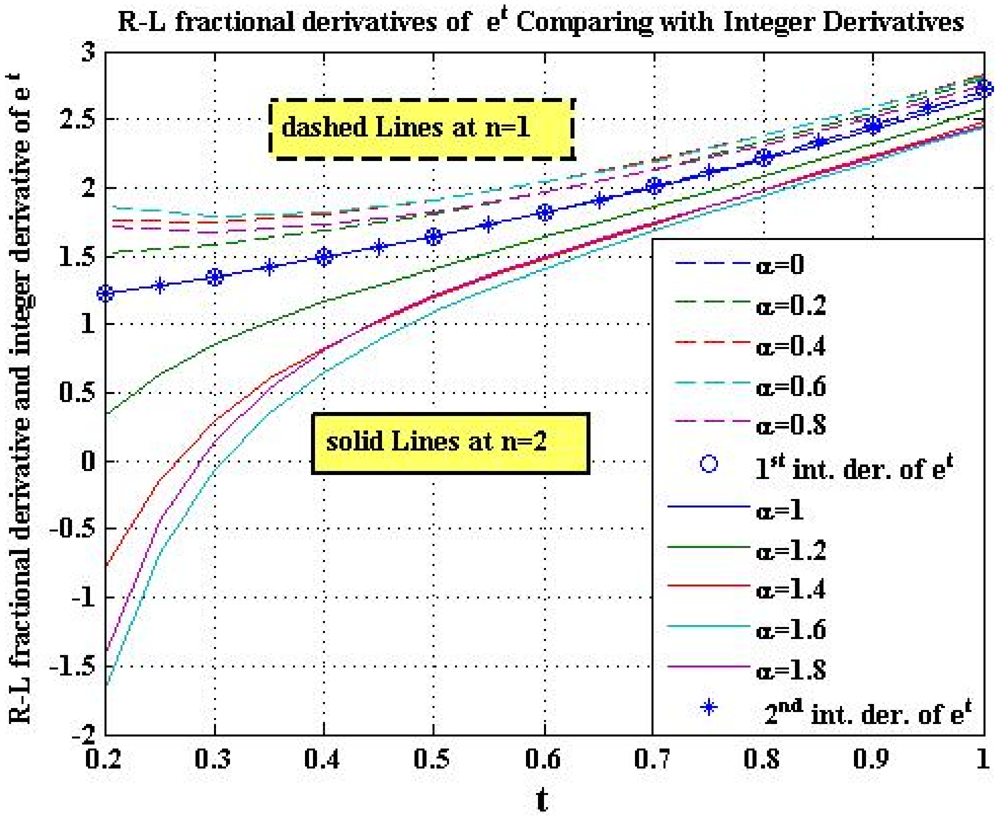


Figure 2.5: A comparison between R-L fractional derivatives and integer order derivatives of the exponential function $f(t) = e^t$.

vi) The R-L fractional derivative of the trigonometric function $f(t) = \sin(t)$, with $n - 1 \leq \alpha < n$, is given by the following formula:

$$
{}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{\sin(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \, . \tag{2.41}
$$

Since the trigonometric function $f(t) = \sin(t)$ can be expressed in terms of the following series [137]:

$$\sin(t) = \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j+1}}{(2j+1)!} \, . \tag{2.42}$$

Substitute Eq.(2.42) into Eq.(2.41) to obtain the R-L fractional derivative of the function $f(t) = \sin(t)$ as following:

$${}_0D_t^{\alpha} \sin(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \sum_{j=0}^{\infty} (-1)^j \frac{t^{2j+1}}{(2j+1)!} \frac{1}{(t-\tau)^{\alpha+1-n}} d\tau \, . \tag{2.43}$$

Open the series of the integrand in Eq.(2.43), with a certain truncation (k) for j, and exploit the following formula that is used in the previous applications (application I, and application II), which is illustrated by Eq.(2.44). As a particular case set $n = 1$ to obtain

$$\frac{d^n}{dt^n} \int_0^t \frac{(\tau)^m}{(t-\tau)^{\alpha+1-n}} d\tau = \prod_{p=n}^{m} \left(\frac{p}{p-\alpha} \right) t^{m-\alpha} \, , \quad m \geq n \, . \tag{2.44}$$

The R-L fractional derivative of the trigonometric function $f(t) = \sin(t)$, with fractional order $n - 1 \leq \alpha < n$, can be given as

$${}_0D_t^{\alpha} [\sin(t)] = \frac{1}{\Gamma(1-\alpha)} \left\{ \begin{array}{c} \left[\prod_{p=1}^{1} \left(\frac{p}{p-\alpha} \right) t^{1-\alpha} \right] - \left[\frac{1}{3!} \prod_{p=1}^{3} \left(\frac{p}{p-\alpha} \right) t^{3-\alpha} \right] \\ + \left[\frac{1}{5!} \prod_{p=1}^{5} \left(\frac{p}{p-\alpha} \right) t^{5-\alpha} \right] + \ldots \\ + \left[\frac{(-1)^k}{(2k+1)!} \right] \left[\prod_{p=1}^{2k+1} \left(\frac{p}{p-\alpha} \right) t^{2k-\alpha+1} \right] + \ldots \end{array} \right\} . \tag{2.45}$$

Figure 2.6 shows the R-L fractional derivatives of $f(t) = \sin(t)$, with fractional order $0 \leq \alpha < 1$. The figure also illustrates the comparison between the R-L fractional derivative at $\alpha = 1$ and the first order integer derivative of $\sin(t)$.

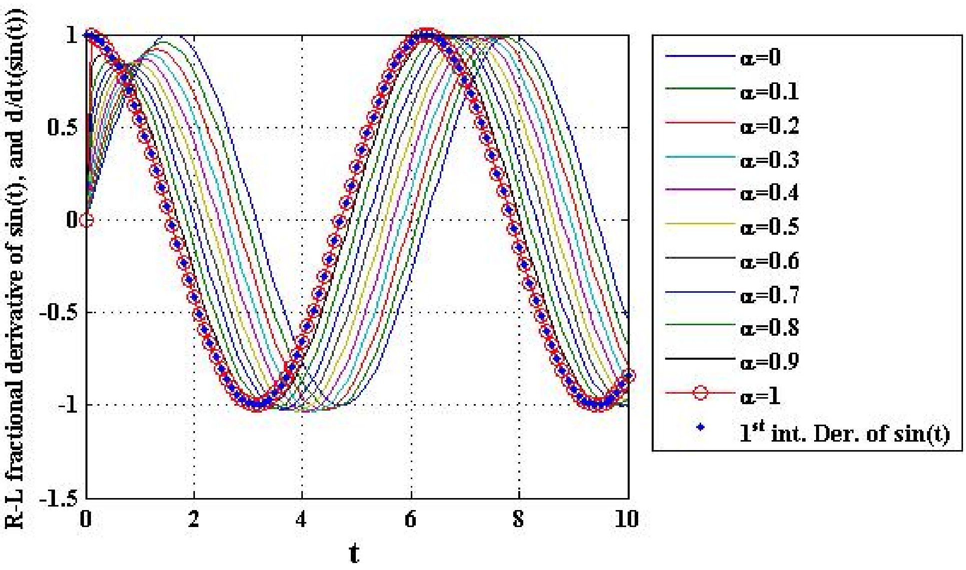


Figure 2.6: R-L fractional derivatives of $f(t) = \sin(t)$, with $\alpha = 0 : 0.1 : 1$, comparing with the first integer order of $\sin(t)$.

It's concluded from Fig. 2.6 that the R-L fractional derivative of $f(t) = \sin(t)$ at fractional order $\alpha = 1$, and the first integer order of the same function $\left(\frac{d}{dt}\sin(t)\right)$ match each other. However, the initial value of the R-L fractional derivatives of $f(t) = \sin(t)$ at $t = 0$ equals zero, whereas the first integer order of $f(t) = \sin(t)$ at $t = 0$ is $\frac{d}{dt}\sin(t) = 0 = \cos(0) = 1$. This error is discussed in the next section.

2.4.2 Direct Method to Obtain the Fractional Derivatives

The Direct method is a logical way by which the fractional derivative of some functions can be directly obtained [77, 82]. This is done by comparing the fractional order derivative with the logic of integer order derivatives of the same functions. In this section, the fractional derivatives, by using the Direct Method, of the trigonometric function $f(t) = \sin(t)$ is discussed, and compared with the R-L fractional derivatives of the same function. Moreover, this method is exploited to obtain the fractional derivatives of FODEs, and this is investigated in chapter (6).

To explain the logic of the Direct method let's start with the derivatives of the sine function which given in following well known sequence:

$$
\begin{aligned}
\tfrac{d^0}{dt^0}\sin(t) &= \sin(t), \\
\tfrac{d}{dt}\sin(t) &= \cos(t), \\
\tfrac{d^2}{dt^2}\sin(t) &= -\sin(t), \\
\tfrac{d^3}{dt^3}\sin(t) &= -\cos(t), \\
\tfrac{d^4}{dt^4}\sin(t) &= \sin(t), \ldots
\end{aligned}
\tag{2.46}
$$

From graphs of the functions in the above sequence, as shown in Fig. 2.7, there is a shift between each consecutive integer order derivatives of the trigonometric function $f(t) = \sin(t)$. This shift is given by $\dfrac{\pi}{2}$.

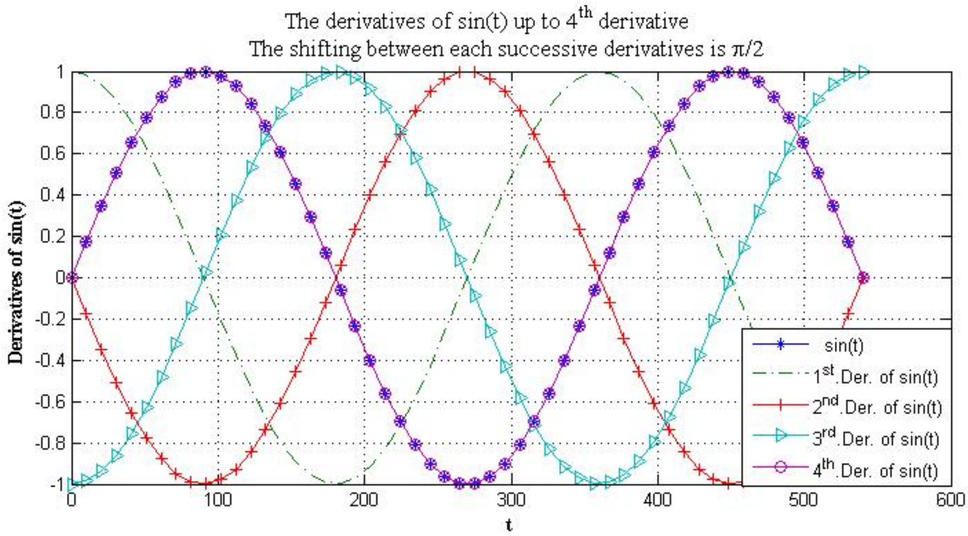


Figure 2.7: The shift between each two successive derivatives of $\sin(t)$.

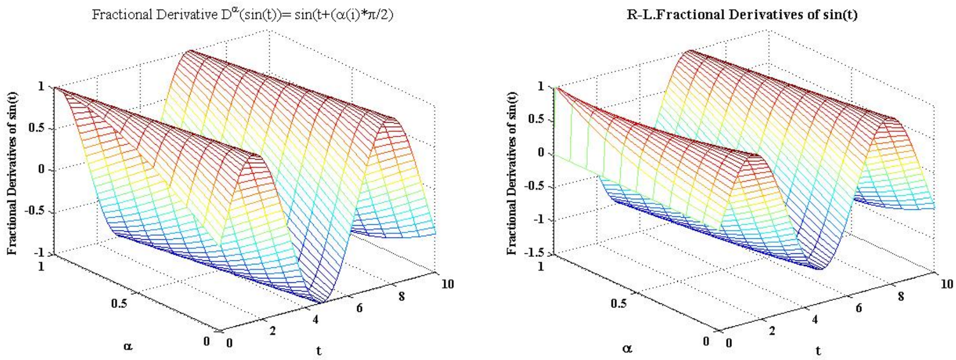
So the following expressions of the derivatives of the trigonometric function $f(t) = \sin(t)$ can be considered:

$$
\begin{aligned}
&D^0 \sin(t) = \sin(t), \\
&D^1 \sin(t) = \sin\left(t + \tfrac{\pi}{2}\right), \\
&D^2 \sin(t) = \sin\left(t + \tfrac{2\pi}{2}\right), \\
&\ldots, \\
&D^n \sin(t) = \sin\left(t + \tfrac{n\pi}{2}\right),
\end{aligned}
\tag{2.47}
$$

where $D^k = \dfrac{d^k}{dt^k}$. By similarity for an arbitrary non-integer order, $\alpha \geq 0,\ \alpha \in \mathbb{R}$, the following fractional order derivative of trigonometric function $f(t) = \sin(t)$ is valid as well:

$$D^{\alpha} \sin(t) = \sin\left(t + \frac{\alpha\pi}{2}\right) . \tag{2.48}$$

The last formula gives the Direct fractional derivatives method of $\sin(t)$ of order α. Figure 2.8 shows the fractional order derivatives of $\sin(t)$ by using the R-L fractional derivatives and the Direct fractional derivatives method, where fractional derivative order is given as $\alpha = 0 : 0.1 : 1$.



(a) R-L fractional derivatives of sin(t) (b) F. D. of sin(t) given by Direct method

Figure 2.8: Fractional derivatives of sin(t) using (a) the R-L, and (b) the Direct method

A comparison between the R-L fractional derivatives of $\sin(t)$ and the fractional derivatives of $\sin(t)$ given by the Direct method can be inferred from Fig. 2.8. The figure shows, for specific value of α, the similarity between the two methods' solutions. However, there is a noticeable error appears at $t = 0$. This error is clearly illustrated in Fig. 2.9 where fractional derivative order $\alpha = 0.4$. This error is perceptible at $t = 0$ and gradually decreases as t increases.

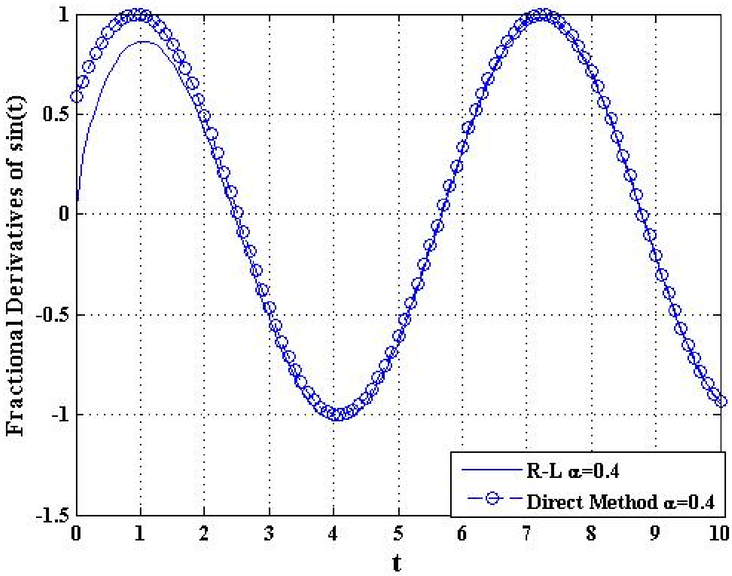


Figure 2.9: Comparison between the R-L fractional derivative, and the Direct fractional derivatives method of $\sin(t)$, where the fractional derivative order $\alpha = 0.4$

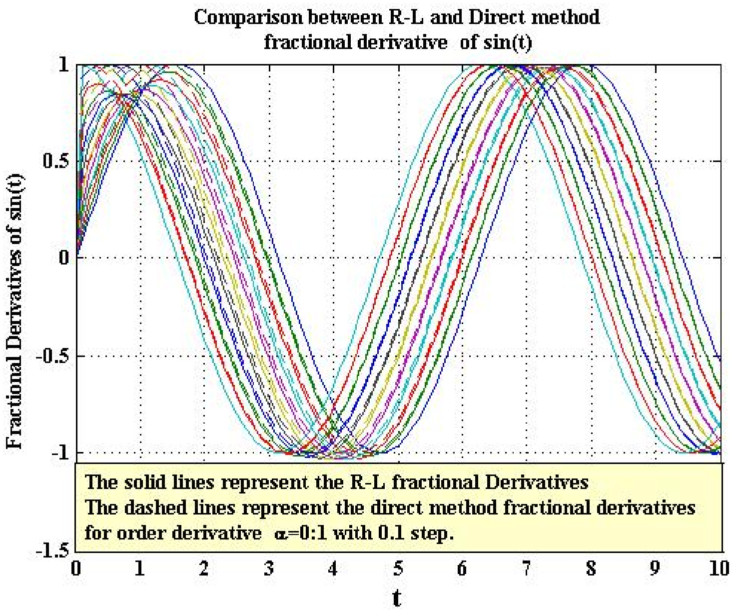
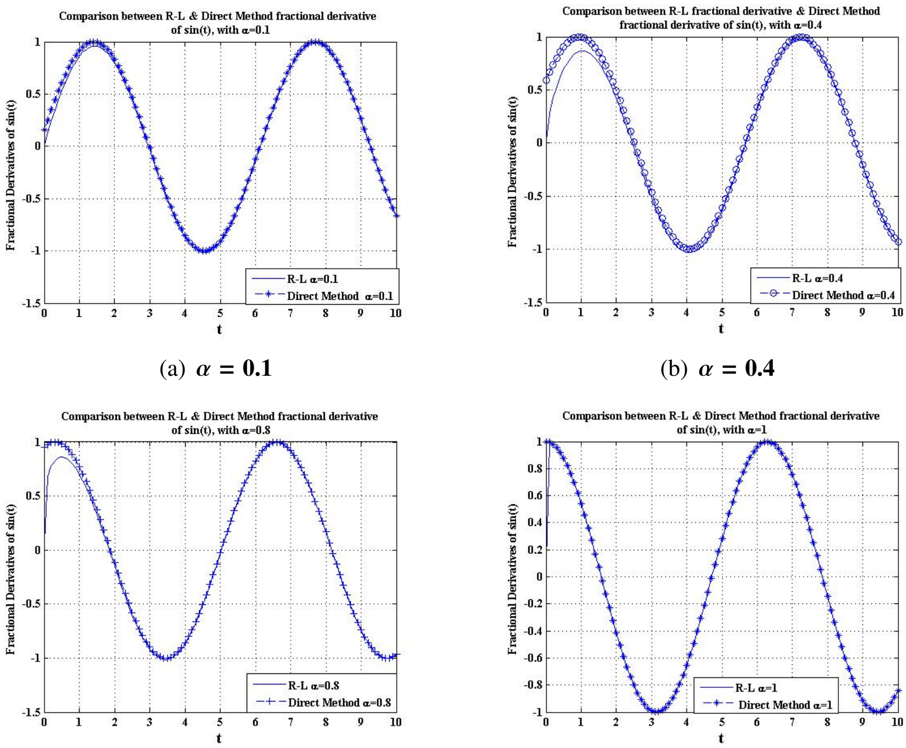


Figure 2.10: Fractional derivative of $\sin(t)$ using the R-L and the Direct method.

More focus, on the errors between the R-L fractional derivatives and the Direct method of fractional derivatives, is illustrated in Fig. 2.11 in which certain values are selected among the fractional order derivatives (α's), in order to show the effect of the order derivatives, and the variable t on the errors between the two methods. It's concluded from the Fig. 2.10, and Fig. 2.11 that the errors get the maximum values at the middle and vanish at the extremities $\alpha = 0$, and $\alpha = 1$.



(a) α = **0.1** (b) α = **0.4**

(c) α = **0.8** (d) α = **1**

Figure 2.11: A comparison between the R-L, and the Direct fractional derivatives methods of $\sin(t)$ for certain α's .

Three-dimensional representation is depicted in Fig. 2.12, where the fractional derivative order α is chosen to be in the range from zero to one with 0.1 step. Both methods' solutions are plotted in the figure in which the error between them is gradually vanishes as t increases.

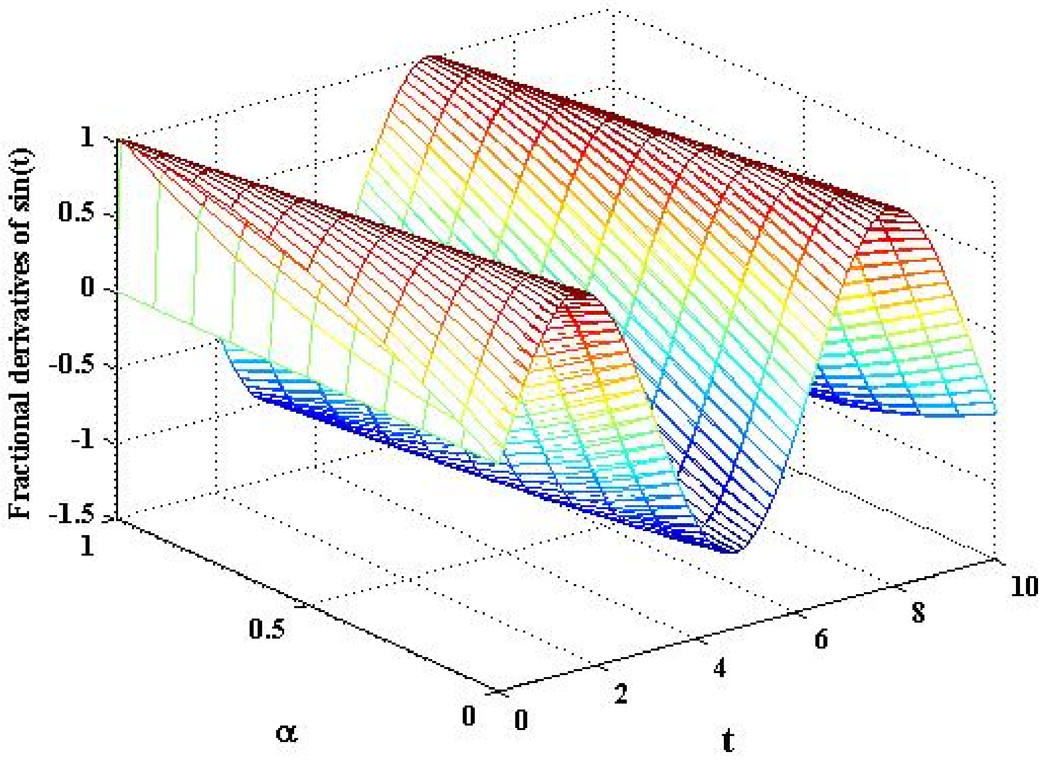
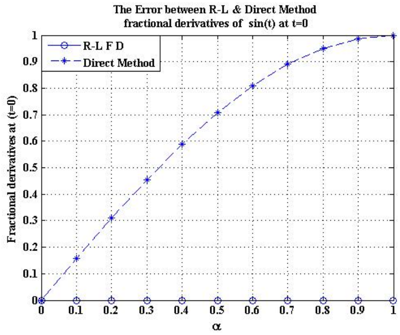
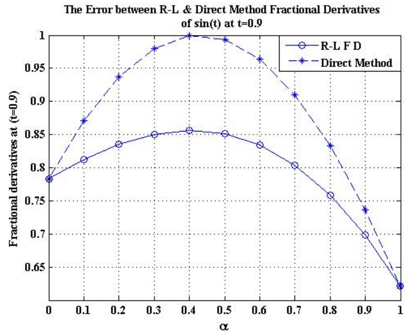


Figure 2.12: Three-dimensional representation of the comparison between the R-L, and the Direct fractional derivatives methods of $\sin(t)$.

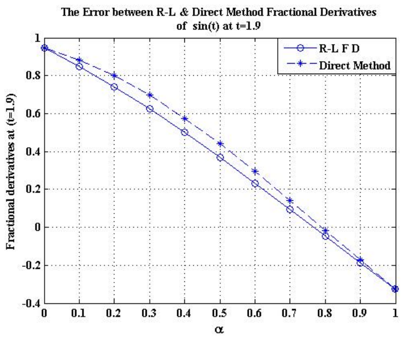
The errors between the two methods are examined at certain points in the interval from $t = 0$ to $t = 20$. This examination is illustrated in Fig. 2.13. It is concluded from this figure that the error reaches its maximum values in the middle value and vanishes at extremes (according to α's values); however, the error between the two methods decreases as t increases. For instance, the maximum value of error is about 0.15 at $\alpha = 0.4$ and $t = 0.9$, while the maximum value of error is about 0.0105 at $\alpha = 0.4$ and $t = 9.9$ as depicted in the sub-figures: Fig. 2.13-b and Fig. 2.13-e, respectively.



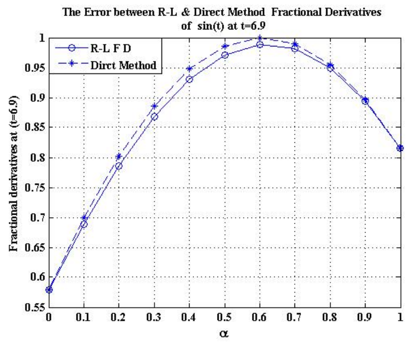
(a) The error at $t = 0$



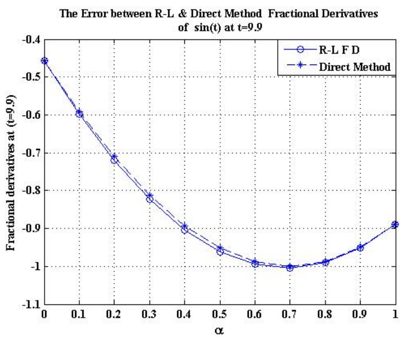
(b) The error at $t = 0.9$



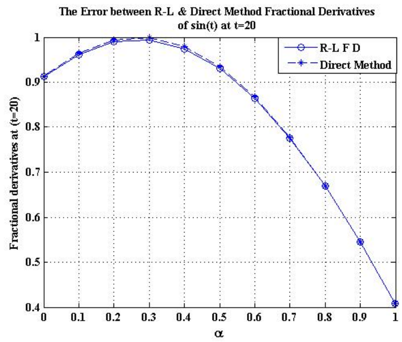
(c) The error at $t = 1.9$



(d) The error at $t = 6.9$



(e) The error at $t = 9.9$



(f) The error at $t = 20$

Figure 2.13: The errors between the R-L and the Direct fractional derivatives methods of $\sin(t)$ at different points of t .

It's well known that the zero derivative of $\sin(t)$ is $\sin(t)$, and the first derivative of $\sin(t)$ is $\cos(t)$. It's logical to say that the fractional derivative of $\sin(t)$ at $t = 0$ between $\alpha = 0$, and $\alpha = 1$ is gradually increased from 0 *to* 1 as α increases from $\alpha = 0$ to $\alpha = 1$. This is verified by using the fractional derivative direct method. However, when the R-L fractional derivative method is applied at $t = 0$ all the fractional derivatives of $\sin(t)_{t=0}$ are *zeros* regardless the fractional derivative order α values.

CHAPTER 3

FRACTIONAL ORDER DIFFERENTIAL EQUATIONS and M-L FUNCTIONS

3.1 Introduction

Any differential equation that contains at least one fractional order term is a FODE. The FODEs can be linear, non-linear, homogenous, non-homogenous, and can be also partial fractional differential equations [3]. The difference in the classification between the integer order and fractional order differential equations is given by means of order. While the integer order differential equations are classified as first, second, third, and so on order differential equation, the FODEs are classified by one, two, three, and so on term FODE [6, 9], in which the fractional derivatives order can be real number. The solutions of FODEs can be obtained analytically [6, 15] by using the M-L functions [21, 27], and the Green function [33], or numerically [39] such as Oustaloup's approximation method [45], and Bagley-Torvik Equation solution [6]. The later is considered in this chapter as an example of numerical technique to solve FODEs. The analytical solution that is represented in this chapter is the direct Laplace transform of FODE and then taking the inverse Laplace transform of the resulted fractional transfer function using the M-L function and its pairs. That is why the Laplace transform and the inverse Laplace transform of M-L functions are studied, and investigated in this chapter. Moreover, an analytical solution by using M-L function is illustrated, in the next section, in order to obtain a direct solution of ordinary FODEs.

3.2 Laplace Transform of Fractional Order Differential Equations

The general formula of the fractional-order system can be described by a FODE of the form [18, 39, 79]

$$ {}_{a_n}D_t^{\alpha_n} y(t) + {}_{a_{n-1}} D_t^{\alpha_{n-1}} y(t) + ... + {}_{a_0} D_t^{\alpha_0} y(t) = {}_{b_m} D_t^{\beta_m} u(t) + {}_{b_{m-1}} D_t^{\beta_{m-1}} u(t) + ... + {}_{b_0} D_t^{\beta_0} u(t) \ , \tag{3.1} $$

where $\alpha_n > \alpha_{n-1} > ... > \alpha_0$, and $\beta_m > \beta_{m-1} > ... > \beta_0$, which are real or rational numbers, are the fractional orders of the derivatives of the functions $y(t)$, and $u(t)$, respectively, and $a_q : q = 0, 1, ..., n, and\ b_p : p = 0, 1, ..., m$, are constant coefficients. The transfer function of the general FODE that represented in Eq.(3.1) is obtained by taking the Laplace transform of each term, which can be accomplished by using he Laplace transform of the R-L fractional derivative as explained in the next subsection.

3.2.1 The Laplace Transform of R-L Fractional Derivative

The general expression for the Laplace transform of R-L fractional derivative is given by [6, 54, 58, 63, 69]

$$ L\left[{}_aD_t^{\alpha} f(t)\right] = s^{\alpha} F(s) - \sum_{k=0}^{\infty} s^k \left[{}_aD_t^{\alpha-k-1} f(t)\right]_{t=0} \ , \tag{3.2} $$

where the fractional order derivative $n-1 \leq \alpha < n;\ \alpha \in \mathbb{R},\ n \in \mathbb{N}$; and a is the lower limit of the t interval.

Considering a particular case where $n = 1$ and the lower limit of t interval is $a = 0$, the Laplace transform of R-L fractional derivative of the function $f(t)$ is given by

$$ L\left[{}_aD_t^{\alpha} f(t)\right] = s^{\alpha} F(s) \ . \tag{3.3} $$

It's concluded from the above particular case, according to R-L fractional derivative, that the second term of the right hand side of Eq.(3.2) represents the initial conditions of $f(t)$.

3.2.2 The Transfer Functions of FODEs

Considering the Laplace of R-L fractional derivatives of each term, of Eq.(3.1), with zero initial conditions yields the corresponding transfer function of incommensurate-real orders FODE that given in Eq.(3.1) as the following form:

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + ... + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + ... + a_0 s^{\alpha_0}} \ . \tag{3.4}$$

where $\alpha_n > \alpha_{n-1} > ... > \alpha_0$, and $\beta_m > \beta_{m-1} > ... > \beta_0$, are real or rational numbers, and $a_q : q = 0, 1, ..., n$, and $b_p : p = 0, 1, ..., m$, are constants.

The solutions of the incommensurate-real orders FODEs can be obtained by taking the inverse Laplace transform of the transfer function that is given in Eq.(3.4). In the next section, for simplicity, a particular form of commensurate FODE is considered to obtain its solution.

3.2.3 Solution of Commensurate FODEs

In the particular case of incommensurate order systems it holds that [79] $\alpha_k = \alpha k$, and $\beta_k = \alpha k$, where $0 < \alpha < 1$, $\forall k \in \mathbb{Z}$, which represents a commensurate FODE [75]. So the transfer function of the system can be written as follows:

$$\begin{aligned} G(s) &= K_0 \frac{\sum_{k=0}^{M} \ b_k (s^\alpha)^k}{\sum_{k=0}^{N} \ a_k (s^\alpha)^k} \\ &= K_0 \frac{b_0 s^0 + b_1 s^\alpha + b_2 s^{2\alpha} + ... + b_M s^{M\alpha}}{a_0 s^0 + a_1 s^\alpha + a_2 s^{2\alpha} + ... + a_N s^{N\alpha}} \ . \end{aligned} \tag{3.5}$$

The last equation can be expressed by means of two functions, $Q(s^\alpha)$, and $P(s^\alpha)$. These functions represent the numerator and denominator of the fraction in the right hand side of Eq.(3.5), respectively.

$$
\begin{aligned}
G(s) &= K_0 \frac{\sum_{k=0}^{M} \; b_k \left(s^\alpha\right)^k}{\sum_{k=0}^{N} \; a_k \left(s^\alpha\right)^k} \\
&= K_0 \frac{Q\left(s^\alpha\right)}{P\left(s^\alpha\right)} \; . \qquad (3.6)
\end{aligned}
$$

For $N > M$, the transfer function $G(s)$ that is expressed in Eq.(3.6) can be represented by means of a rational function. This function can be represented by M-L Functions, and can be also expanded in the following partial fractions:

$$
G(s) = K_0 \sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \; , \qquad (3.7)
$$

where $\lambda_i \;\; : i = 1, 2, ..., N$ are the roots of the polynomial or the system poles, A_i represent the residue coefficients [80]. The M-L functions can be used to obtain the analytical solution of the system that is given by Eq.(3.7), as following [79]:

$$
\begin{aligned}
g(t) &= \mathcal{L}^{-1} \left(K_0 \sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right) \\
&= K_0 \sum_{i=1}^{N} A_i t^\alpha E_{\alpha,\alpha} \left(-\lambda_i t^\alpha\right) \; . \qquad (3.8)
\end{aligned}
$$

Since the two-parameter M-L function in the last equation can be expressed as:

$$
E_{\alpha,\alpha} \left(-\lambda_i t^\alpha\right) = \sum_{k=0}^{\infty} \frac{\left(-\lambda_i t^\alpha\right)^k}{\Gamma\left(\alpha k + \alpha\right)} \; . \qquad (3.9)
$$

The analytical solution in Eq.(3.8) can be rewritten as following:

$$
g(t) = K_0 \sum_{i=0}^{\infty} \left[A_i t^\alpha \sum_{k=0}^{\infty} \frac{\left(-\lambda_i t^\alpha\right)^k}{\Gamma\left(\alpha k + \alpha\right)} \right] \; . \qquad (3.10)
$$

3.3 Solutions of Non-Homogenous Three-Term FODEs

Bagley-Trovik equation is considered as one of the applications of FODE models. This equation is used to represent the motion of a thin plate in fluid, [6] in which the shear stress due to the velocity profile of the thin plate in the fluid gives a fractional order damping force which represents the damping force in the Bagley-Trovik equation as following [6, 66, 75, 142]:

$$m\, \ddot{y}(t) + c\ {}_0D_t^{\alpha} y(t) + k\, y(t) = u(t)\ , \tag{3.11}$$

where the fractional order of the derivative in the second term (damping force) is determined to be $\alpha = \dfrac{2}{3}$ to represent the mathematical model of the thin plate motion in the fluid, $u(t)$ represents an applied force on the thin plate, k is a stiffness of a massless spring connected to the thin plate with mass m, c is the damping coefficient depends on the viscosity of the fluid and the shear stress, and $y(t)$ is the position of the thin plate, as a function of time t, in the fluid. The initial conditions of this model are given at equilibrium as following:

$$y(0) = y_0\ , \tag{3.12a}$$

$$\dot{y}(0) = \dot{y}_0\ . \tag{3.12b}$$

The above case study is considered as a particular case of the general form of the three-term FODEs, in which the second term depends on the value of the fractional order α and the damping coefficient c.

$$F_{damping} = c\ {}_0D_t^{\alpha} y(t)\ . \tag{3.13}$$

So, in general the numerical solution of Bagley-Trovik equation can be written as follows [6]:

$$y_m(t) = \frac{h^2\left(u_m - ky_{m-1}\right) + m\left(2y_{m-1} - y_{m-2}\right) - c\sqrt{h}\sum_{j=1}^{m} w_j^{\alpha} y_{m-j}}{m + c\sqrt{h}}\ , m = 2, 3, ... \tag{3.14}$$

where h is the time increment and the initial conditions

$$y(0) = y_0 \ , \tag{3.15a}$$

$$y(1) = y_0 \ , \tag{3.15b}$$

$$\dot{y}(0) = \dot{y}_0 \ , \tag{3.15c}$$

and

$$w_j^{\alpha} = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix}, \quad j = 0, 1, 2, \dots \ . \tag{3.16}$$

where the binomial coefficients is given by

$$\begin{pmatrix} \alpha \\ j \end{pmatrix} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-j+1)}{j!} \ . \tag{3.17}$$

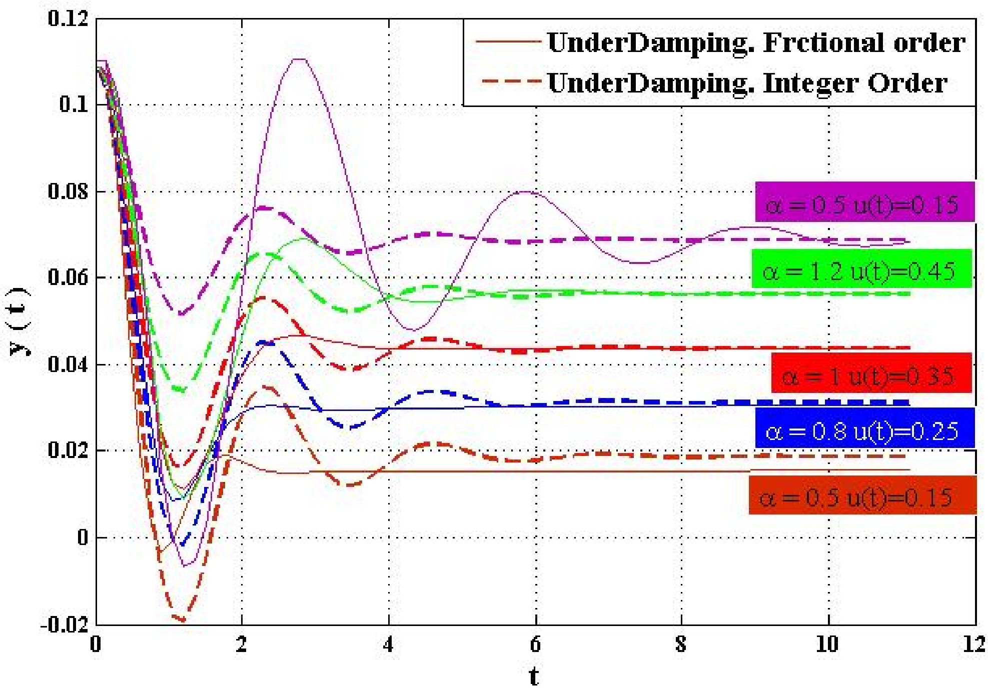


Figure 3.1: Comparison between fractional order model and integer order model of thin-plate spring viscous damping system for different applied force $u(t)$, where α is the fractional order.

Figure 3.1 shows a comparison between extended fractional order representations that given by Bagley-Trovik equation, and the integer order representation $\alpha = 1$

of the thin plate spring viscous-damping model for different applied force $u(t)$. It's inferred from Fig. 3.1 that the fractional order representation and integer order representation of the system at $\alpha = 1$ are same, while the integer order model $\alpha = 1$ reaches the steady state faster than the fractional order for $\alpha > 1,\ or\ \alpha < 1$. Additionally, the fractional order α values depend on the properties of damping system.

3.4 Laplace Transform of Mittag-Leffler Function

The M-L functions that are given in section (2.2.2) represent some forms of M-L functions. The M-L functions play a significant role in solving the FODEs [94]. The solutions of FODEs can be obtained by using M-L Functions and their Laplace transformations. In this section, the Laplace transforms of M-L Functions are discussed and investigated in order to be used in this purpose.

From Eq.(2.5) the exponential function is considered to be a particular case of M-L functions. So the Laplace transform of M-L function can be obtained with the help of the analogy between this function and the exponential function that is given in Eq.(2.6). This analogy states that the Laplace transform of the exponential function e^{at} which can be expressed in M-L function form as $E_{1,1}\left(e^{at}\right)$, is given by

$$\begin{aligned} \int_0^\infty e^{-st} e^{at} dt &= \int_0^\infty e^{-st} E_{1,1}(at)\, dt \\ &= \frac{1}{s-\alpha}\ ,\ For\ convergence\ s > a \end{aligned} \tag{3.18}$$

The M-L functions can be used as a pair with other functions. The Laplace transform of these pairs might match the Laplace transform of a certain FODE or a certain fractional order system transfer function. The following technique is used to obtain the Laplace transform of M-L function as a pair with other functions.

Take the first derivative with respect to a [6], for both sides of Eq.(3.18) as follows:

$$\frac{d}{da} \int_0^\infty e^{-st} e^{at} dt = \frac{d}{da} \left(\frac{1}{s-a} \right) . \tag{3.19}$$

Since the function e^{-st} is considered to be constant with respect to a, the last equation can be rewritten as

$$\int_0^\infty e^{-st} \frac{d}{da} e^{at} dt = \frac{d}{da} \left(\frac{1}{s-a} \right) . \tag{3.20}$$

Take the derivative of the exponential functions e^{at}, and $\dfrac{1}{s-a}$ with respect to a to obtain

$$\int_0^\infty e^{-st} t e^{at} dt = \frac{1}{(s-a)^2} . \tag{3.21}$$

Considering the analogy between M-L function $E_{1,1}(at)$ and the exponential function e^{at} yields the resulting formula of the above derivation, in Eq.(3.19). This represents the Laplace transform of the pair $tE_{1,1}(at)$, and can be rewritten as

$$\int_0^\infty e^{-st} t E_{1,1}(at)\, dt = \frac{1}{(s-a)^2} . \tag{3.22}$$

Let's take another derivative with respect to a for Eq.(3.18). The second derivative of both side of this equation is given by

$$\frac{d^2}{da^2} \int_0^\infty e^{-st} e^{at} dt = \frac{d^2}{da^2} \left(\frac{1}{s-a} \right) . \tag{3.23}$$

The last formula can be rearranged as

$$\int_0^\infty e^{-st} \frac{d^2}{da^2} e^{at} dt = \frac{d^2}{da^2} \left(\frac{1}{s-a} \right) . \tag{3.24}$$

Solving the derivatives of both side of the above equation yields the following:

$$\int_0^\infty e^{-st} t^2 e^{at} dt = \frac{2}{(s-a)^3} . \tag{3.25}$$

The resulting formula of the above derivation represents the Laplace transform of the pair $t^2 E_{1,1}(at)$, and can be rewritten as

$$\int_0^{\infty} e^{-st} t^2 E_{1,1}(at)\, dt = \frac{2}{(s-a)^3} \;. \tag{3.26}$$

Taking the derivatives of Eq.(3.18) up to the k^{th} derivative to obtain the following general formula of the Laplace transform of the pair $t^k E_{1,1}(at)$ [6]

$$\int_0^{\infty} e^{-st} t^k E_{1,1}(at)\, dt = \frac{k!}{(s-a)^{k+1}} \;, \quad for\ convergence\ s > a. \tag{3.27}$$

The Laplace transform of some pairs of M-L functions $E_{\alpha,\beta}(at^{\alpha})$ and t^x, where x is a function of M-L function parameters α; and β, plays an important role in solving the FODEs [91]. Another important pair is $t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})$, where whose Laplace transform can be obtained as follows:

$$\mathcal{L}\left[t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})\right] = \int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})\, dt \;. \tag{3.28}$$

The two-parameter M-L function in the right hand side of Eq.(3.28) can be expressed in the summation form, as illustrated in section (2.2.2), so the last formula can be rewritten as following:

$$\mathcal{L}\left[t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})\right] = \int_0^{\infty} e^{-st} t^{\beta-1} \sum_{k=0}^{\infty} \frac{(at^{\alpha})^k}{\Gamma(\alpha k+\beta)} dt \;. \tag{3.29}$$

Exchange the position order of the integral operator and the summation symbol (sigma notation), and take the constants out of the integral to obtain

$$\mathcal{L}\left[t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})\right] = \sum_{k=0}^{\infty} \frac{a^k}{\Gamma(\alpha k+\beta)} \int_0^{\infty} t^{\alpha k+\beta-1} e^{-st} dt \;. \tag{3.30}$$

The solution of the integral in the right hand side of the above equation is give by

$$\int_0^{\infty} t^{\alpha k+\beta-1} e^{-st} dt = \frac{(\alpha k+\beta-1)!}{s^{\alpha k+\beta-1+1}} \tag{3.31}$$

Substitute Eq.(3.31) into Eq.(3.30) and consider the gamma function integral formula that given in Eq.(2.1) to obtain

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right)\right] = \sum_{k=0}^{\infty}\left[\frac{a^{k}}{(\alpha k+\beta-1)!}\frac{(\alpha k+\beta-1)!}{s^{\alpha k+\beta-1+1}}\right] . \tag{3.32}$$

So the Laplace transform of the pair $t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right)$ can be given as

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right)\right] = \sum_{k=0}^{\infty}\frac{a^{k}}{s^{\alpha k+\beta-1+1}} . \tag{3.33}$$

The Infinite Geometric Series in the right hand side of the last equation is given by

$$\sum_{k=0}^{\infty}\frac{a^{k}}{s^{\alpha k+\beta-1+1}} = \frac{s^{\alpha-\beta}}{s^{\alpha}-a} , s^{\alpha>a} . \tag{3.34}$$

Substituting Eq.(3.34) into Eq.(3.33) to obtain the last form of the Laplace transform of the given pair $t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right)$ as follows:

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right)\right] = \frac{s^{\alpha-\beta}}{s^{\alpha}-a} , \mathbb{R}\left(s\right) > |a|^{\frac{1}{\alpha}} . \tag{3.35}$$

Similarly, using the same derivation of the Laplace transform of $t^{\beta-1}E_{\alpha,\beta}\left(at^{\alpha}\right)$ that derived above, the Laplace transform of the pair $t^{\alpha-1}E_{\alpha,\alpha}\left(-at^{\alpha}\right)$ can be given as following [94]:

$$\mathcal{L}\left[t^{\alpha-1}E_{\alpha,\alpha}\left(-at^{\alpha}\right)\right] = \frac{1}{s^{\alpha}+a} , \quad \left|\frac{a}{s^{\alpha}}\right| < 1 . \tag{3.36}$$

In general the Laplace transforms of the M-L Functions pair $t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}\left(\pm at^{\alpha}\right)$ is given by [6]

$$\mathcal{L}\left[t^{\alpha k+\beta-1}E^{(k)}_{\alpha,\beta}\left(\pm at^{\alpha}\right)\right] = \frac{k!\,s^{\alpha-\beta}}{\left(s^{\alpha}\mp a\right)^{k+1}} , \quad \mathbb{R}\left(s\right) > |a|^{\frac{1}{\alpha}} . \tag{3.37}$$

where the k^{th} derivative of M-L Function $E^{(k)}_{\alpha,\beta}\left(\pm at^{\alpha}\right)$ is expressed as follows:

$$E_{\alpha,\beta}^{(k)}(z) = \frac{d^k}{dz^k} E_{\alpha,\beta}(z) \ . \tag{3.38}$$

Example 3.1

The pair of M-L Functions and their Laplace transforms can be used to obtain the solutions or the responses of FDEs or the fractional modeled systems such as fractional oscillator [97]. As an example, let's consider the three-term FDEs represented as

$$a_0 \, D_t^\alpha \left[D_t^\alpha y(t) \right] + a_1 \, {}_0D_t^\alpha y(t) + a_2 \, y(t) = u(t) \ , \tag{3.39}$$

Assume that the fractional derivatives property that is given in section (2.3.3) is applicable to the function $y(t)$, and the input signal is given as unit impulse $u(t) = \delta(t)$, with zero initial conditions. Furthermore, as a particular case assume the coefficients a_0, a_1, and a_2 to be ones. Apply the Laplace transform of R-L fractional derivative to the function $y(t)$, such that the fractional order derivative α is in the range $0 \le \alpha < 1$. The corresponding fractional transfer function of the given FDE in Eq.(3.39) can be obtained as following:

$$s^{2\alpha} Y(s) + s^\alpha Y(s) + Y(s) = U(s) \ . \tag{3.40}$$

The fractional transfer function of the system in Eq.(3.40) is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^{2\alpha} + s^\alpha + 1} \ . \tag{3.41}$$

Since the input signal is a unit impulse where its Laplace transform equals one [80], then the fractional transfer function above can be rewritten as:

$$\begin{aligned} G(s) &= Y(s) \\ &= \frac{1}{s^{2\alpha} + s^\alpha + 1} \ , \end{aligned} \tag{3.42}$$

The solution of the given FDE, as a function of t , can be obtained by taking the inverse Laplace transform of the transfer function in Eq.(3.42) as following:

$$y(t) = \mathcal{L}^{-1} \left[\frac{1}{s^{2\alpha} + s^\alpha + 1} \right] \ . \tag{3.43}$$

The transfer function that is given in Eq.(3.42) can be rewritten as

$$G(s) = \frac{1}{s^{2\alpha} + s^\alpha + \frac{1}{4} + \frac{3}{4}} \ . \tag{3.44}$$

Multiply and divide the right hand side of Eq.(2.44) by $\frac{\sqrt{3}}{2}$ to obtain

$$G(s) = \frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(s^{\alpha} + \frac{1}{2}\right)^2 + \frac{3}{4}} \ . \tag{3.45}$$

Let $w = \frac{\sqrt{3}}{2}$, and $a = \frac{1}{2}$, then the last fractional transfer function that is given in Eq.(3.45) can be expressed as

$$G(s) = \frac{1}{w} \frac{w}{(s^{\alpha} + a)^2 + w^2} \ . \tag{3.46}$$

The integer order representation or the non-homogenous differential equation that is corresponded to the given FODE in Eq.(3.39) is given, at $\alpha = 1$, as following:

$$\ddot{y}(t) + \dot{y}(t) + y(t) = u(t) \ , \tag{3.47}$$

where $u(t) = \delta(t)$, the corresponding transfer function of the integer order representation is given by

$$G(s) = \frac{2}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} \ . \tag{3.48}$$

Consider the assumptions $w = \frac{\sqrt{3}}{2}$, and $a = \frac{1}{2}$. The transfer function of the integer order system that is given by Eq.(3.48) can be rewritten as

$$G(s) = \frac{1}{w} \frac{w}{(s + a)^2 + w^2} \ . \tag{3.49}$$

Since the Laplace transform of $\frac{w}{(s+a)^2 + w^2}$ is given as $e^{-at} \sin(wt)$ [80], then the solution of the integer order differential equation that given in Eq.(3.47) in time domain, is obtained as follows:

$$y(t) = \frac{1}{w} \left[e^{-at} \sin(wt)\right] \ , \tag{3.50}$$

where the trigonometric function $\sin(wt)$ can be represented in terms of exponential function as following [104]:

$$\sin(wt) = \frac{e^{iwt} - e^{-iwt}}{2i} \ . \tag{3.51}$$

Moreover, from Eq(2.3) through Eq.(2.6) the exponential function e^{-at} can be rewritten in terms of M-L Function as

$$\begin{aligned} e^{-at} &= E_{1,1}(-at) \\ &= \sum_{k=0}^{\infty} \frac{(-at)^k}{\Gamma(k+1)} \, . \end{aligned} \tag{3.52}$$

From Eq.(3.18), the Laplace transform of M-L Function $E_{1,1}(-at)$ which is represented by e^{-at} is given by

$$\mathcal{L}\left[E_{1,1}(-at)\right] = \frac{1}{s+a} \, . \tag{3.53}$$

The M-L Function $E_{1,1}(-at)$ is a particular case of the two parameter M-L Function, that can be written as following:

$$E_{\alpha,\alpha}(-at) = \sum_{k=0}^{\infty} \frac{(-at)^k}{\Gamma(\alpha k+\alpha)} \, . \tag{3.54}$$

Similarly, from equation (3.51), the trigonometric function, with fractional power angle, $\sin(wt^\alpha)$ can be given as following:

$$\sin(wt^\alpha) = \frac{e^{iwt^\alpha} - e^{-iwt^\alpha}}{2i} \, . \tag{3.55}$$

So, the multiplication of the functions e^{-at^α}, and $\sin(wt^\alpha)$ can be represented as follows:

$$e^{-at^\alpha}\sin(wt^\alpha) = e^{-at^\alpha}\left(\frac{e^{iwt^\alpha} - e^{-iwt^\alpha}}{2i}\right) \, . \tag{3.56}$$

The desired transfer function of the given system in Eq.(3.39), $G_{desired}(s)$, can be obtained by taking the Laplace transform of the last expression as follows:

$$\mathcal{L}\left[e^{-at^\alpha}\sin(wt^\alpha)\right] = \mathcal{L}\left[e^{-at^\alpha}\left(\frac{e^{iwt^\alpha} - e^{-iwt^\alpha}}{2i}\right)\right] \, . \tag{3.57}$$

Using the distributive property of multiplication in the last equation, and taking $\frac{1}{2i}$ out of the square brackets yields

$$\mathcal{L}\left[e^{-at^\alpha}\sin(wt^\alpha)\right] = \frac{1}{2i}\mathcal{L}\left[e^{(-a+iw)t^\alpha} - e^{(-a-iw)t^\alpha}\right]. \tag{3.58}$$

From Eq.(3.36) the following Laplace transform of $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_i t^\alpha)$ can be expressed:

$$\mathcal{L}\left[t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_i t^{\alpha}\right)\right]=\frac{1}{s^{\alpha}-\lambda_i}\ , \qquad \left|\frac{\lambda_i}{s^{\alpha}}\right|<1,\ \ i=1,2 \tag{3.59}$$

where the roots of the system λ_i, in integer representation, are given by $\lambda_1 = (-a + iw)$, and $\lambda_2 = (-a - iw)$. So, the desired Laplace transform of the system can be close to the following Laplace transform:

$$G_{desired}\left(s\right)=\mathcal{L}\left\{t^{\alpha-1}E_{\alpha,\alpha}\left[\left(-a+iw\right)t^{\alpha}\right]-t^{\alpha-1}E_{\alpha,\alpha}\left[\left(-a-iw\right)t^{\alpha}\right]\right\}\ . \tag{3.60}$$

Take the Laplace transform of the last equation that given by Eq.(3.60) to obtain

$$G_{desired}\left(s\right)=\left[\frac{1}{s^{\alpha}-\left(-a+iw\right)}-\frac{1}{s^{\alpha}-\left(-a-iw\right)}\right]\ . \tag{3.61}$$

Multiply the last transfer function by $\frac{1}{2iw}$, and manipulate it in order to match the transfer function of the given FDE

$$G_{desired}\left(s\right)=\frac{1}{2iw}\left[\frac{1}{s^{\alpha}-\left(-a+iw\right)}-\frac{1}{s^{\alpha}-\left(-a-iw\right)}\right]\ . \tag{3.62}$$

Manipulating the last formula to obtain

$$G_{desired}\left(s\right)=\frac{1}{2iw}\left[\frac{s^{\alpha}+a+iw-s^{\alpha}-a+iw}{s^{2\alpha}+as^{\alpha}+iws^{\alpha}+as^{\alpha}-iws^{\alpha}+a^2+w^2}\right]\ . \tag{3.63}$$

Some terms in the Eq.(3.63) may cancel each other to obtain the following transfer function:

$$G_{desired}\left(s\right)=\frac{1}{2iw}\left[\frac{2iw}{s^{2\alpha}+2as^{\alpha}+a^2+w^2}\right]\ . \tag{3.64}$$

The last formula that obtained in Eq.(3.64) of the desired transfer function, can be expressed as

$$G_{desired}\left(s\right)=\frac{1}{w}\left[\frac{w}{\left(s^{\alpha}+a\right)^2+w^2}\right]\ . \tag{3.65}$$

The last transfer function exactly matches the transfer function of the given system in Eq.(3.39). From Eq.(3.60), and Eq.(3.61). The inverse Laplace transform of the of the last transfer function Eq.(3.65), which equals the time domain response of the given fractional order system, in Eq.(3.39), due to impulse input, is given by

$$y_{fractional}(t) = \frac{1}{2iw} t^{\alpha-1} \left\{ E_{\alpha,\alpha} \left[(-a+iw) t^{\alpha} \right] - E_{\alpha,\alpha} \left[(-a-iw) t^{\alpha} \right] \right\} . \tag{3.66}$$

Figure 3.2 illustrates the solutions that is obtained in Eq.(3.66) of the non-homogeneous three-term FDE that is given by Eq.(3.39), for different values of the fractional order α. The figure also shows the solution which is obtained by Eq.(3.50), of the 2nd order non-homogeneous differential equation that is represented by Eq.(3.47).

It's concluded from Fig. 3.2 that the fractional order representation exactly identical to the integer representation when the fractional order α of the FDE equal one. That is verified analytically, later on as well. Additionally, the response of the fractional model is getting the steady state faster as the fractional order α goes to zero. Also the overshoot of the response decreases as the fractional order decreases.

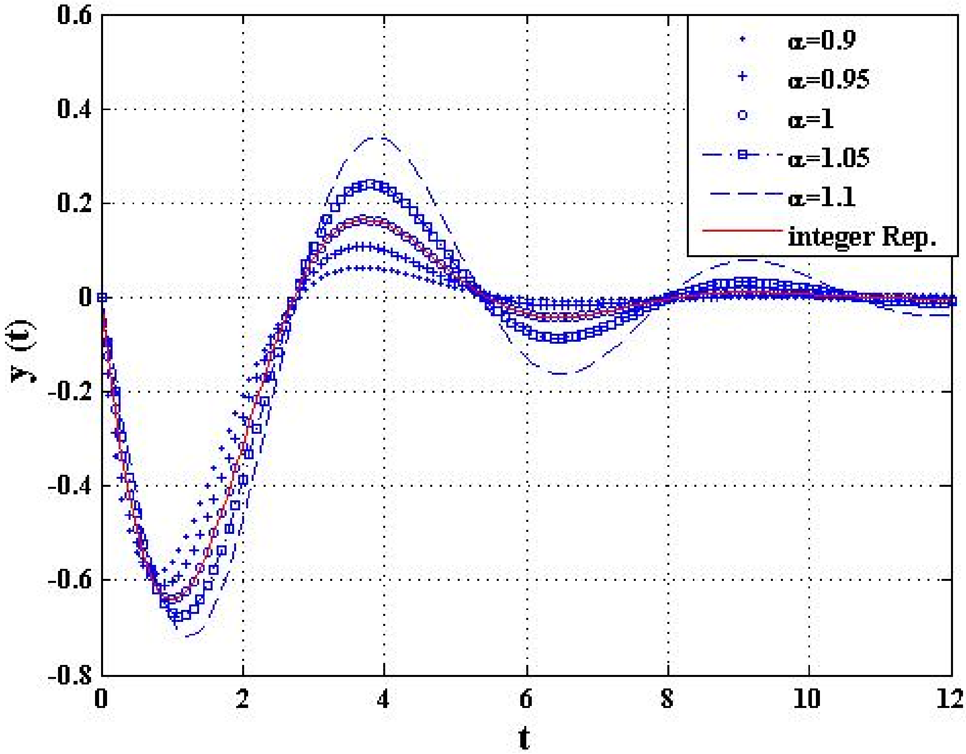


Figure 3.2: The solution of a non-homogeneous three-term FDE for different values of α compared with the solution of 2nd order non-homogeneous differential equation.
.

Verification of the Fractional Order Solution

In order to verify the fractional response, that is given by Eq.(3.66), consider the

FODE that given in Eq.(3.39) and its solution that is given by Eq.(3.66).

Let's investigate the FDE and its solution at fractional order $\alpha = 1$. Due to this the FDE is converted to the second order differential equation that given in Eq.(3.47) whose solution is given by Eq.(3.50)

Examine the following fractional order solution that is given in Eq.(3.66) at fractional order α equals one

$$\underset{\alpha=1}{y_{fractional}}(t) = \frac{1}{2iw} t^{0} \left\{ E_{1,1}\left[(-a + iw)\, t\right] - E_{1,1}\left[(-a - iw)\, t\right] \right\} . \tag{3.67}$$

From Eq.(3.52), the two-parameter M-L function with parameters $\alpha = 1$, and $\beta = 1$ can be written as following:

$$\begin{aligned} E_{1,1}(-\lambda_i t) &= \sum_{k=0}^{\infty} \frac{(-\lambda_i t)^k}{\Gamma k + 1} \\ &= e^{-at} \quad i = 1, 2. \end{aligned} \tag{3.68}$$

where λ_1, and λ_2 represent the roots $(-a + iw)$, and $(-a - iw)$, respectively, so the fractional order solution in(3.67) can be rewritten as

$$\underset{\alpha=1}{y_{fractional}}(t) = \frac{1}{2iw} \left[e^{(-a+iw)t} - e^{(-a-iw)t} \right] . \tag{3.69}$$

Manipulating Eq.(3.69) yields

$$\underset{\alpha=1}{y_{fractional}}(t) = \frac{1}{2iw} \left(e^{-at} e^{iwt} - e^{-at} e^{-iwt} \right) . \tag{3.70}$$

Take e^{-at} out of the parentheses and use the distributive property of multiplication by distributing $\frac{1}{2i}$ to the parentheses to obtain

$$\underset{\alpha=1}{y_{fractional}}(t) = \frac{1}{w} e_{-at} \left(\frac{e^{iwt} - e^{-iwt}}{2i} \right) . \tag{3.71}$$

According to the above, the fractional order solution of the given FODE, at the order $\alpha = 1$, is given by

$$y_{\substack{fractional \\ \alpha=1}}(t) = \frac{1}{w} e^{-at} \sin(wt) \ . \tag{3.72}$$

The last solution that is given by Eq.(3.72) exactly equals the solution of integer order differential equation, that given in Eq.(3.50). It's concluded from the above verification that the solution of the given FODE in Eq.(3.39) is given by Eq.(3.66).

3.5 Non-Homogenous Three-Term FDE with Time Delay

The time delay is transportation lags or apparent time delays may be present in a process due to actuator limitations or process measurements [107], or in the identification exercise when a higher-order process is approximated by a lower order model. From Eq.(3.6) a commensurate transfer function of order α for fractional order time-delay system is given as [111, 114, 118]

$$G(s) = K_0 \frac{\sum_{k=0}^{M} \ b_k (s^{\alpha})^k}{\sum_{k=0}^{N} \ a_k (s^{\alpha})^k} \exp^{-\tau s} \ , \tag{3.73}$$

where τ represents the time delay of the system.

The $\mathbf{2}^{nd}$ Order Non-Homogenous three-term FDE with time delay is represented as following [111]:

$$m \ \ddot{y}(t) + c \ {}_0D_t^{\alpha} y(t) + k \ y(t) = u(t - \tau) \ . \tag{3.74}$$

The solution of Eq.(3.74) using the M-L function and Laplace transform with zero initial conditions can be obtained as following: The Laplace transform of the fractional system that is given in Eq.(3.74) is [6, 111, 114]

$$Y(s) = \frac{1}{ms^2 + cs^{\alpha} + k} e^{\tau s} \ , \tag{3.75}$$

Manipulating the system transfer function that given in Eq.(3.75) yields [6]

$$Y(s) = \frac{1}{k} \frac{ks^{\alpha}}{ms^{2-\alpha} + c} \frac{1}{1 + \frac{ks^{-\alpha}}{ms^{2-\alpha}+c}} e^{\tau s} \ . \tag{3.76}$$

Equation (3.76) can be rewritten as following [6]:

$$Y(s) = \frac{1}{k}\sum_{n=1}^{\infty} (-1)^n \left(\frac{k}{m}\right)^{n+1} \frac{s^{-\alpha n+\alpha}}{\left(s^{2-\alpha} + \frac{c}{m}\right)^{n+1}} e^{\tau s} \ . \tag{3.77}$$

The time domain response of the system can be written as follows:

$$y(t) = \frac{1}{m}\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{k}{m}\right)^n (t-\tau)^{2(n+1)-1} E^{(n)}_{2-\alpha,2=\alpha n}\left(-\frac{c}{m}(t-\tau)^{2-\alpha}\right) \ . \tag{3.78}$$

Using Eq.(2.9) to open the derivative of M-L. function to obtain the last form of the time domain response as following [6]:

$$y(t) = \frac{1}{k}\sum_{j=0}^{\infty} \left(-\frac{c}{m}\right)^j \sum_{n=0}^{\infty} (-1)^n \left(\frac{k}{m}\right)^{n+1} \frac{(j+n)!\,(t-\tau)^{2(j+n)+2-1-\alpha j}}{n!\ j!\ \Gamma\left[2(j+n+1)-\alpha j\right]} \ . \tag{3.79}$$

The response of $\mathbf{2}^{nd}$ order Non-Homogenous three-term FDE with different values of time delay $\boldsymbol{\tau}$ according to the solution that is given in Eq.(3.79) is illustrated in Fig. 3.3 in which the damping fractional derivative order $\boldsymbol{\alpha = 1.2}$. It's concluded from the figure that as the time delay increases, the overshot and the steady state time of the response increase. This observation support the reasons of the lag that occurred due to the capability of the system and the process measurements effects.

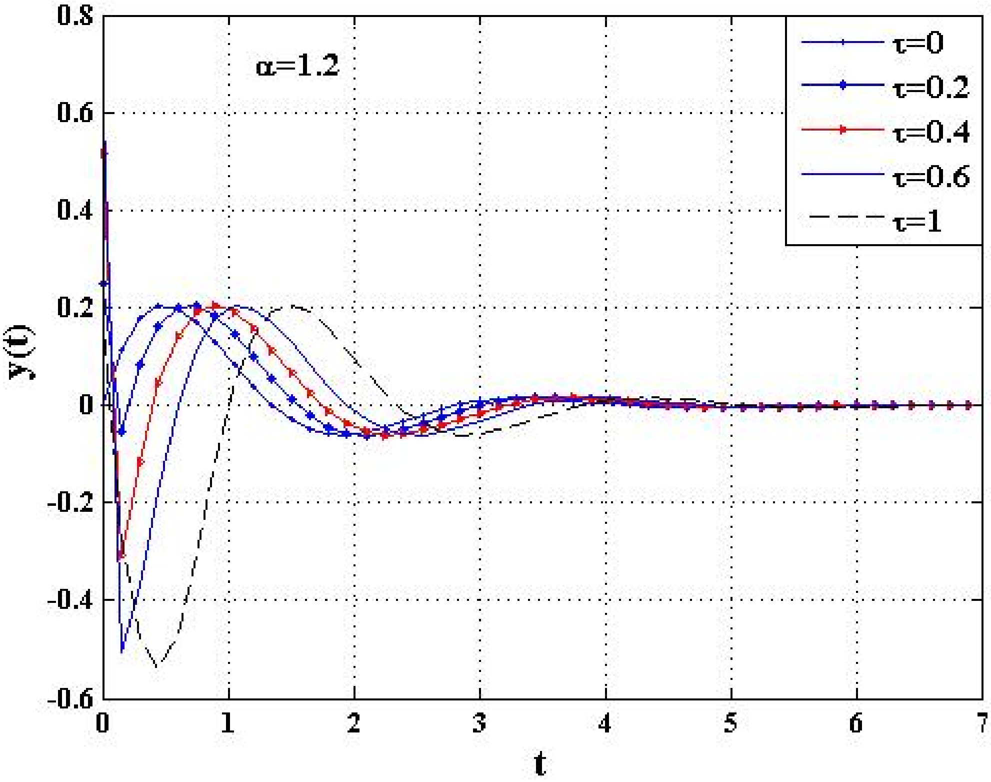


Figure 3.3: Responses of 2^{nd} order Non-Homogenous three-term FDE with different values of time delay τ.

CHAPTER 4

FRACTIONAL EULER-LAGRANGE EQUATIONS

4.1 Introduction

The calculus of variations plays an important role in some disciplines like engineering, and applied and pure mathematics. Moreover, recent researches have investigated that some physical systems can be modeled more accurately using fractional representations [4]. In this chapter, the derivation of FELEs is investigated, and applied in the following chapter, in order to interpret its parameters. This includes the interpretation of the mathematical and physical meaning of the R-L integral order α, and the intrinsic time τ, which appear in the damping part of the EELEs. Moreover, an alternative formula of FELEs is introduced in this chapter, and compared with the approved formula.

4.2 The Euler-Lagrange Equations

Most of the mechanical systems motions are governed by the Principle of Least Action, or Hamilton's Principle. By means of this principle, any mechanical system can be characterized by a variational function $f\left(x, y, y^{'}\right)$ [10]. The principle of Least Action states that the appropriate track of motion of any dynamic system with holonomic constraints is a stationary solution of the action [16]

$$f\left(x, y_1, y_2, ..., y_n, y_1^{'}, y_2^{'}, ..., y_n^{'}\right) = f\left(x, y, y^{'}\right) \ , \tag{4.1}$$

where the system dynamic motion satisfies certain conditions [10]. Additionally, the Hamilton's Principle governs many physical and mathematical problems, in which some quantities need to be minimized (or maximized) and expressed as an integral. These quantities obey the Hamilton's Principle [16]. The classical problem in calculus of variations is exploited to find the function $y(x)$ that minimizes or maximizes the integral

$$I = \int_{x_1}^{x_2} F\left(x, y, y^{'}\right) dx \, , \tag{4.2}$$

The Calculus of Variations methods transform the problem of minimizing the integral that is given by Eq.(4.2), into a solution of differential equation. That solution is expressed as $y(x)$ in terms of derivatives of the integrand $F\left(x, y, y^{'}\right)$. That solution is treated as a function of $y(x)$ and $y^{'}(x)$ [22].

The Euler-Lagrange formula is used to terminate the integral I. To do so, the functions $y(x)$, and $y^{'}(x)$ must be defined such that I is exterimaized [28]. The function $y^{'}(x)$ obeys a few necessary constraints:

i) $y^{'}(x_1, \epsilon) = y(x_1)$, and $y^{'}(x_2, \epsilon) = y(x_2)$, where ϵ is a varying parameter.

ii) $y^{'}(x, 0) = y(x)$.

ii) $y^{'}(x, \epsilon)$ and all of its second order derivatives are continuous derivatives of x, and ϵ.

According to Fermat's principle [34] which states that "The path of a light ray traveling between two points is the shortest path", and for small variations in the taken path, the optical path length stays the same". This shortest path can be determined.

In order to find that path let us consider two paths $y(x)$, and $y_1(x) = y(x) + \epsilon\eta(x)$, where $\eta(x)$ is arbitrary smooth function of x. Since we are looking for specific value of the variable x, by which the system reaches the end point via the two paths, the end points of the two paths must be same as illustrated in Fig. 4.1. The two paths can be expressed as

$$y(x_1) = y_1(x_1) = y_a \ , \tag{4.3a}$$

$$y(x_2) = y_2(x_2) = y_b . \tag{4.3b}$$

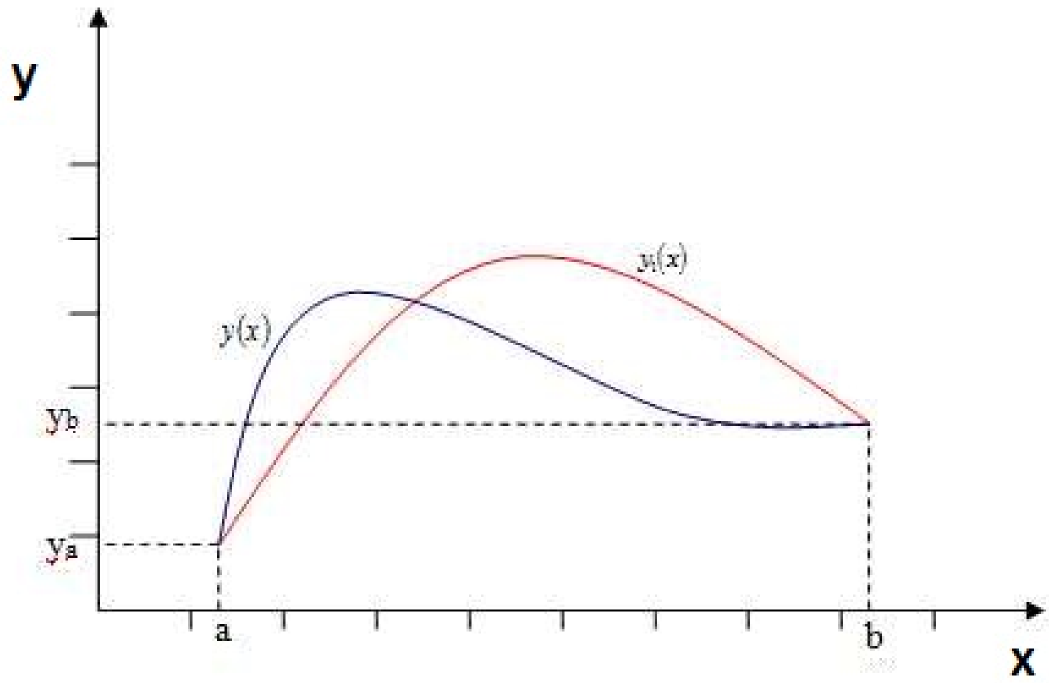


Figure 4.1: The paths of the functions $y(x)$, and $y_1(x)$.

Therefore, the arbitrary smooth function $\eta(x)$ must vanish at the end points as illustrated in the following formula

$$\eta(x_1) = \eta(x_2) = 0 \ . \tag{4.4}$$

The second path $y_1(x) = y(x) + \epsilon\eta(x)$ is considered to be small variation of the path $y(x)$, where $\epsilon \ll 1$, so one can say that the functions $y(x)$, and $y_1(x)$ satisfy the boundary conditions in Eq.(4.4). The corresponding Taylor expansion can be applied [40, 46] to the integrand as a function of $y(x)$, which can be expressed by the following:

$$
\begin{aligned}
F\left(x, y_1, y_1^{'}\right) &= F\left(x + y + \epsilon\eta,\ y^{'} + \epsilon\eta^{'}\right) \\
&= F\left(x, y, y^{'}\right) + \epsilon\eta\frac{\partial F}{\partial y} + \epsilon\eta^{'}\frac{\partial F}{\partial y^{'}} + O\left(\epsilon^2\right)\ .
\end{aligned} \tag{4.5}
$$

Since it is assumed that the integral I in Eq.(4.2) has a minimum or maximum value contingent upon $y\left(x\right)$, and consider Eq.(4.5), the following formula can be written

$$
\int_{x_1}^{x_2} F\left(x, y_1, y_1^{'}\right) dx = \int_{x_1}^{x_2} \left(x, y, y^{'}\right) dx + \int_{x_1}^{x_2} F\left(\epsilon\eta\frac{\partial F}{\partial y} + \epsilon\eta^{'}\frac{\partial F}{\partial y^{'}}\right) dx + O\left(\epsilon^2\right)\ . \tag{4.6}
$$

The new integral that is illustrated in Eq.(4.6) is considered to be minimized or maximized and can be expressed as a summation of the old integral in Eq.(4.2) and the integral δI. The later is resulted from the variation of the function $y\left(x\right)$. So it can be obtained from the subtraction of the two integrals as follows:

$$
\delta I = \int_{x_1}^{x_2} F\left(x, y, y^{'}\right) dx + \int_{x_1}^{x_2} \left(\epsilon\eta\frac{\partial F}{\partial y} + \epsilon\eta^{'}\frac{\partial F}{\partial y^{'}}\right) dx + O\left(\epsilon^2\right) - \int_{x_1}^{x_2} F\left(x, y, y^{'}\right) dx\ . \tag{4.7}
$$

The first and the third terms of the right hand side of Eq.(4.7) cancel each other, so the result from the above equation is just the integral that is resulted from applying the variations followed by Taylor series. This can be expressed as following:

$$
\delta I = \int_{x_1}^{x_2} \left(\epsilon\eta\frac{\partial F}{\partial y} + \epsilon\eta^{'}\frac{\partial F}{\partial y^{'}}\right) dx\ . \tag{4.8}
$$

Consider Eq.(4.8), integrate the second term by parts, and apply the boundary conditions to $\eta\left(x\right)$, that is given in Eq.(4.4) [46].

Firstly, the integration by parts of the second term yields

$$
\delta I = \epsilon\left[\int_{x_1}^{x_2} \eta\left(x\right)\frac{\partial F}{\partial y} dx + \frac{\partial F}{\partial y^{'}}\eta\left(x\right)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta\left(x\right)\frac{d}{dx}\frac{\partial F}{\partial y^{'}} dx\right]\ . \tag{4.9}
$$

Secondly, applying the boundary conditions, $\eta(x_1) = \eta(x_2) = 0$ yields the vanishing of the second term of Eq.(4.9) to become

$$\delta I = \epsilon \left[\int_{x_1}^{x_2} \eta(x) \frac{\partial F}{\partial y} dx - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \frac{\partial F}{\partial y'} dx \right] . \tag{4.10}$$

Rearrange Eq.(4.10) to obtain the following:

$$\delta I = \epsilon \int_{x_1}^{x_2} \eta(x) \left[\frac{\partial F}{\partial y} dx - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] dx . \tag{4.11}$$

Due to the boundary conditions and for every ϵ, the integral δI becomes zero [28], so Eq.(4.11) can be rewritten as follows:

$$\delta I = \epsilon \int_{x_1}^{x_2} \eta(x) \left[\frac{\partial F}{\partial y} dx - \frac{d}{dx} \frac{\partial F}{\partial y'} \right] dx = 0 . \tag{4.12}$$

Considering the lemma states that: If $\delta I = \epsilon \int_{x_1}^{x_2} M(x) h(x) dx = 0$, and $h(x)$ is continuous function with second partial derivatives and is not identically zero, then $M(x) = 0$ on $[x_1, \ x_2]$ interval [28]. As aresult the second part of the integrand in Eq.(4.12) equals zero and that can be expressed as following:

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 . \tag{4.13}$$

Equation (4.13) is known as The ELE.

4.2.1 Generalized Euler-Lagrange Equations

Equations of motion of a dynamic system such as the double pendulum system can be derived using ELE which is described in the following formula [51]:

$$\frac{\partial L}{\partial q_r(t)} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_r(t)} = f_r(t) \ , \tag{4.14}$$

where L represents the Lagrangian function and defines as the difference between the kinetic energy T and potential energy V of the system as following:

$$L = T - V \ . \tag{4.15}$$

The variable q in Eq.(2.14) represents the generalized coordinate, f is the external applied force, r is the system element number, and t represents the observer time.

Another term can be added to Eq.(2.14), in case the dynamic system contains a dissipation energy. The addition of the dissipative term to ELE is not a part of the the ELE [55], however, it is considered as a resistance force to the external force. By adding this force to ELE the generalized form of the ELE can be depicted as follows:

$$\frac{\partial L}{\partial q_r(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r(t)} = -\frac{\partial D}{\partial \dot{q}_r(t)} + f_r(t) \ , \tag{4.16}$$

where D is the dissipation energy of the system [55].

4.2.2 Expanded Form of Euler-Lagrange Equations

Note that the variables x, y, and $y^{'}$ in the integrand of Eq.(4.2) have been treated as independent variables. Consequently, the derivatives of the second term of the ELE in Eq.(4.13) are reevaluated. The Chain rule is used to express $\frac{d}{dx}$ in terms of $\frac{\partial}{\partial y'}, \frac{\partial}{\partial y}, \ and \ \frac{\partial}{\partial x}$ as follows [59]:

For the original ELE in Eq.(4.13), the following derivative operators of the function F, and its independent variables can be defined as

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \left[\frac{d}{dx}y^{'}\frac{\partial}{\partial y'} + \frac{d}{dx}y\frac{\partial}{\partial y} + \frac{\partial}{\partial x}\right]\frac{\partial F}{\partial y'} \ . \tag{4.17}$$

Substitute Eq.(4.17) into the original ELE that is given by Eq.(3.14) to obtain the following new formula:

$$\frac{\partial F}{\partial y}-\left[\frac{d}{dx}\frac{dy}{dx}\frac{\partial}{\partial y'}\frac{\partial}{\partial y'}F+\frac{d}{dx}y\frac{\partial}{\partial y}\frac{\partial}{\partial y'}F+\frac{\partial}{\partial x}\frac{\partial}{\partial y'}F\right]=0\ . \tag{4.18}$$

Manipulate Eq.(4.18) to obtain the following Equation [59]

$$\frac{\partial}{\partial y}F-\left(\frac{\partial^2}{\partial y'^2}F\right)\frac{d^2y}{dx^2}-\left(\frac{\partial^2}{\partial x\partial y'}F\right)\frac{dy}{dx}-\frac{\partial^2}{\partial x\partial y'}F=0\ . \tag{4.19}$$

Multiply Eq.(4.19) by -1 to obtain the last formula of the expanded ELE as following:

$$\left(\frac{\partial^2}{\partial y'^2}F\right)\frac{d^2y}{dx^2}+\left(\frac{\partial^2}{\partial y\partial y'}F\right)\frac{dy}{dx}+\frac{\partial^2}{\partial x\partial y'}F-\frac{\partial}{\partial y}F=0\ . \tag{4.20}$$

The ELEs can be expressed in terms of integer representation as investigated in the previous section and fractional representations. By exploiting same technique that is used to derive the ELE, the fractional representations of the ELE can be obtained.

4.3 The General Formula of Fractional Euler Lagrange Equations

The ELE and the expanded ELE are derived by using the calculus of variation as explained in the previous sections. Additionally, the FELE can be derived by mean of fractional calculus and calculus of variation [51, 64, 70, 76, 81]. In some dynamic applications such as the fractional modeling of a free oscillating double pendulum system, the governing equations of system motion are obtained through fractional variational principles [51]. Fundamental concepts of a variational calculus and transversality conditions is introduced to derive the FELE [85]. In order to derive the FELE, let's define the Lagrangian in a sub interval $[a, b]$ as follows [88, 92, 95]:

$$J\left[q\left(t\right)\right]=\int_a^b L\Big(t,\ q\left(t\right),\ {}_aD_t^{\beta}q\left(t\right),\ {}_tD_b^{\gamma}q\left(t\right)\Big)\,dt\ , \tag{4.21}$$

where J is a function to be minimized (or maximized), q is an absolutely continuous function in the interval $[a, b]$ and satisfies the boundary conditions $q\left(a\right)=q_a$

and $q(b) = q_b$ [98]. Since the function $q(t)$ is subjected to traditional boundary conditions, the Caputo derivative seems to be more natural for the Fractional Variation Problems (FVPs) [101]. Consequently, the fractional derivatives of the function $q(t)$, which are defined in Eq.(4.21) rerepresented by the left and the right Caputo fractional derivatives [105] as

$$ {}_aD_t^{\beta} q(t) = {}_a^C D_t^{\beta} q(t) \ , \tag{4.22a}$$

$$ {}_aD_t^{\gamma} q(t) = {}_a^C D_t^{\gamma} q(t) \, . \tag{4.22b}$$

Substituting Eq.(4.22) into Eq.(4.21) yields

$$ J[q(t)] = \int_a^b L\left[t,\ q(t),\ {}_a^C D_t^{\beta} q(t),\ {}_t^C D_b^{\gamma} q(t)\right] dt \ . \tag{4.23}$$

The Caputo fractional derivatives are considered right here, because they have advantages compared with other methods of fractional derivatives. These advantages include its derivative of a constant is zero, whereas the R-L fractional derivative of a constant is not zero, and Laplace transform of the Caputo fractional derivative is written in terms of the integer derivatives evaluated at the origin [108, 112]. But the R-L fractional integral can be exploited in order to derive the FELE. Before starting the derivation let's define the R-L fractional integral of the Lagrangian L.

If the function $L(t) \in \mathbb{C}\ [a, b]\ : a \leq t \leq b$, then the left and right R-L fractional integral of $L(t)$, of order α, are defined, respectively, as follows [108, 115]:

$$ {}_aI_t^{\alpha} L(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{L(\tau)}{(t-\tau)^{1-\alpha}} d\tau \ \ , t > a \ \text{ left} \ , \tag{4.24}$$

$$ {}_tI_b^{\alpha} L(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{L(\tau)}{(t-\tau)^{1-\alpha}} d\tau \ \ , t < b \ \text{ right} \ , \tag{4.25}$$

where $\alpha > 0, \ \ \alpha \in \mathbb{R}$.

Take the R-L fractional integral of the Lagrangian L to define the following action function [119]:

$$
\begin{aligned}
S &= {}_aI_b^{\alpha} L(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b L\left[q(\tau),\ {}_a^C D_t^{\beta} q(\tau),\ {}_t^C D_b^{\gamma} q(\tau)\right] (t-\tau)^{\alpha-1}\, d\tau\ .
\end{aligned} \tag{4.26}
$$

Since the fractional derivatives ${}_a^C D_t^{\beta} q(\tau)$, ${}_t^C D_b^{\gamma} q(\tau)$ are considered to be Caputo fractional derivatives, the fractional orders β and γ must be taken as real numbers in the following range $n-1 \leq \beta < n$, and $n-1 \leq \gamma < n$, where n is a natural number represents the upper integer order limit of the derivatives.

Because of the advantage of using of R-L fractional derivatives within the variational principles, the integral in Eq.(4.26) is chosen to be R-L fractional integral. This advantage is that the integration by parts can be defined [121] in this type of fractional integral. That is exploited in the FELE derivation.

In order to define an admissible varied functions [124] satisfy the boundary conditions $q(a) = q_a$, and $q(b) = q_b$, and the Lagrangian in Eq.(4.21) as well. Let's express the variation of the function $q(t)$ in terms of finite number ϵ [127] which is represented as a finite variations of the function S [119]. Therefore, a small variation of the function $q(t)$ can be written as follows:

$$
Q(t) = q(t) + \epsilon \delta q(t)\ . \tag{4.27}
$$

Since $\epsilon \ll 1$, and the variation of the function q, $(\delta q \in \mathbb{C})$, satisfies the boundary conditions $\delta q(a) = \delta q(b) = 0$, then the functions Q, and q satisfy the same boundary conditions [4, 105]. Consequently, the Lagrangian L can be expressed in terms of the new variation function $Q(t) = q(t) + \epsilon \delta q(t)$ as following:

$$
L(t, Q) = L\left(q + \epsilon\delta q,\ {}_a^C D_t^{\beta} q + \epsilon\, {}_a^C D_t^{\beta} \delta q,\ {}_t^C D_b^{\gamma} q + \epsilon\, {}_t^C D_b^{\gamma} \delta q\right)\ . \tag{4.28}
$$

By means of the new continuous function $Q(t)$, and the R-L fractional integral of the Lagrangian L, the action function S that is given by Eq.(4.26) can be defined as follows:

$$S + \Delta_\epsilon S = \frac{1}{\Gamma(\alpha)} \int_a^b L \begin{pmatrix} q + \epsilon\delta q,\ {}_a^C D_t^\beta q + \\ \epsilon\ {}_a^C D_t^\beta \delta q,\ {}_t^C D_b^\gamma q + \epsilon\ {}_t^C D_b^\gamma \delta q \end{pmatrix} (t-\tau)^{\alpha-1}\, d\tau\ , \tag{4.29}$$

where $\Delta_\epsilon S$ represents the variation of the action function S. The variation can be extracted from Eq.(4.29) by manipulating the variation of the Lagrangian in Eq.(4.28), and comparing the result with Eq.(4.26). To achieve this and for simplicity let's start with the following derivation:

To derive the traditional ELE the following integral is considered, as mentioned in section (4.2), to find the function q that minimizes or maximizes the integral

$$j = \int L(t,\ q,\ \dot{q})\, dt\ . \tag{4.30}$$

For small constant variation δ, and considering the Eqs.(4.5 through 4.8), the variation of the integral in Eq.(4.30) is given by

$$\delta j = \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right) dt\ . \tag{4.31}$$

Integrate the second term of the integrand in Eq.(4.31) by parts. To do so consider the rule

$$\int u dv = uv - \int v du\ . \tag{4.32}$$

let

$$u = \frac{\partial L}{\partial \dot{q}}\ . \tag{4.33}$$

Take the derivatives of both sides of Eq.(4.33) to obtain

$$du = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) dt\ , \tag{4.34}$$

and let

$$dv = \frac{\partial}{\partial t}\delta q dt \ . \tag{4.35}$$

Take the integrals of both sides of of Eq.(4.35) to obtain

$$v = \delta q \ . \tag{4.36}$$

Substitute Eqs.(4.33 through 4.36) into Eq.(4.32) to obtain the following:

$$\begin{aligned} \int \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}\delta q dt &= \int \frac{\partial L}{\partial \dot{q}} d\left(\delta q\right) \\ &= \frac{\partial L}{\partial \dot{q}}\delta q - \int \delta q \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) dt\right] \ . \end{aligned} \tag{4.37}$$

Similarly, the same steps can be done for the Lagrangian in Eq.(4.28) to obtain two main terms, as follows:

$$L\left(t, Q\right) = L\left(q,\ {}_a^C D_t^{\beta} q,\ {}_t^C D_b^{\gamma} q\right) + L\begin{pmatrix} \epsilon \delta q \frac{\partial L}{\partial\, q} + \epsilon \frac{\partial L}{\partial\, {}_a^C D_t^{\beta} q} \\ {}_a^C D_t^{\beta} \delta q + \epsilon \frac{\partial L}{\partial\, {}_t^C D_b^{\gamma} q}\ {}_t^C D_b^{\gamma} \delta q \end{pmatrix} + O\left(\epsilon^2\right). \tag{4.38}$$

where the second main term represents the variation of the Lagrangian, so the variation of the action function $\Delta_\epsilon S$ can be expressed in terms of the variation of the Lagrangian as following [119]:

$$\Delta_\epsilon S \ = \int_a^b \begin{bmatrix} \delta q \frac{\partial L}{\partial q}\left(t-\tau\right)^{\alpha-1} + \frac{\partial L}{\partial\, {}_a^C D_t^{\beta} q}\left(t-\tau\right)^{\alpha-1}\ {}_a^C D_t^{\beta} \delta q \\ + \frac{\partial L}{\partial\, {}_t^C D_b^{\gamma} q}\left(t-\tau\right)^{\alpha-1}\ {}_t^C D_b^{\gamma} \delta q \end{bmatrix} + \epsilon d\tau + O\left(\epsilon^2\right) \ . \tag{4.39}$$

Integrate, by parts, the second and the third terms of the integrand of Eq.(4.39) as following:

$$I_{2^{nd} term} = \int_a^b \left[\frac{\partial L}{\partial\, {}_a^C D_t^{\beta} q}\left(t-\tau\right)^{\alpha-1}\ {}_a^C D_t^{\beta} \delta q\right] d\tau \ . \tag{4.40}$$

Apply the integration by parts on Eq.(4.40), by considering the formula that given by Eq.(4.32), and assigning the variable u, and v as follows:

$$u = \frac{\partial L}{\partial\, {}_a^C D_t^\beta q} (t-\tau)^{\alpha-1} \ , \tag{4.41a}$$

$$du = \left[\frac{\partial L}{\partial\, {}_a^C D_t^\beta q}\, {}_a^C D_t^\beta (t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_a^C D_t^\beta \frac{\partial L}{\partial\, {}_a^C D_t^\beta q} \right] d\tau \ . \tag{4.41b}$$

$${}_a^C D_t^\beta v = {}_a^C D_t^\beta \delta q \ , \tag{4.41c}$$

$$v = \delta q \ . \tag{4.41d}$$

Substitute Eq.(4.41) into Eq.(4.32) to obtain

$$\int_a^b \left[\frac{\partial L}{\partial\, {}_a^C D_t^\beta q} (t-\tau)^{\alpha-1}\ {}_a^C D_t^\beta \delta q \right] d\tau = \frac{\partial L\,(t-\tau)^{\alpha-1}}{\partial\, {}_a^C D_t^\beta q} \delta q\,(\tau)\,|_a^b - \int_a^b \delta q\,(\tau) \left[\frac{\partial L}{\partial\, {}_a^C D_t^\beta q}\, {}_a^C D_t^\beta (t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_a^C D_t^\beta \frac{\partial L}{\partial\, {}_a^C D_t^\beta q} \right] d\tau \ . \tag{4.42}$$

Since the variation of the function $q\,(t)$ obeys the boundary conditions $\delta q\,(a) = \delta q\,(b) = 0$, then the first term in the right hand side of Eq.(4.42) vanishes, and the integral of the second term $I_{2^{nd}term}$ of Eq.(4.39) can be expressed as follows:

$$\int_a^b \left[\frac{\partial L}{\partial\, {}_a^C D_t^\beta q} (t-\tau)^{\alpha-1}\ {}_a^C D_t^\beta \delta q \right] d\tau = - \int_a^b \delta q\,(\tau) \left[\frac{\partial L}{\partial\, {}_a^C D_t^\beta q}\, {}_a^C D_t^\beta (t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_a^C D_t^\beta \frac{\partial L}{\partial\, {}_a^C D_t^\beta q} \right] d\tau \ . \tag{4.43}$$

In order to integrate, by parts, the third term of Eq.(4.39), let's follow the same steps that used to integrate the second term by parts, by considering the following:

$$u = \frac{\partial L}{\partial\, {}_t^C D_b^\gamma q} (t-\tau)^{\alpha-1} \ , \tag{4.44a}$$

$$du = \left[\frac{\partial L}{\partial\, {}_t^C D_b^\gamma q}\, {}_t^C D_b^\gamma (t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_t^C D_b^\gamma \frac{\partial L}{\partial\, {}_t^C D_b^\gamma q} \right] d\tau \ . \tag{4.44b}$$

$${}_t^C D_b^\gamma v = {}_t^C D_b^\gamma \delta q \ , \tag{4.44c}$$

$$v = \delta q \ . \tag{4.44d}$$

So in order to obtain the new form of the third term integral $I_{3^{rd}term}$ of Eq.(4.39) substitute Eq.(4.44) into Eq. (4.32) to obtain

$$\int_a^b \left[\frac{\partial L}{\partial\, {}_t^C D_b^\gamma q} (t-\tau)^{\alpha-1}\ {}_t^C D_b^\gamma \delta q \right] d\tau = \frac{\partial L\,(t-\tau)^{\alpha-1}}{\partial\, {}_t^C D_b^\gamma q} \delta q\,(\tau) \Big|_a^b - \int_a^b \delta q\,(\tau)$$
$$\left[\frac{\partial L}{\partial\, {}_t^C D_b^\gamma q}\, {}_t^C D_b^\gamma (t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_t^C D_b^\gamma \frac{\partial L}{\partial\, {}_t^C D_b^\gamma q} \right] d\tau \ . \tag{4.45}$$

Similarly, since the variation of the function $q\,(t)$ obeys the boundary conditions $\delta q\,(a) = \delta q\,(b) = 0$, the formula of the third term integral $I_{3^{rd}term}$ of Eq.(4.39) can be expressed as follows:

$$\int_a^b \left[\frac{\partial L}{\partial\, {}_t^C D_b^\gamma q} (t-\tau)^{\alpha-1}\ {}_t^C D_b^\gamma \delta q \right] d\tau =$$
$$-\int_a^b \delta q\,(\tau) \left[\frac{\partial L}{\partial\, {}_t^C D_b^\gamma q}\, {}_t^C D_b^\gamma (t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_t^C D_b^\gamma \frac{\partial L}{\partial\, {}_t^C D_b^\gamma q} \right] d\tau \ . \tag{4.46}$$

The second and the third parts of the integrands in the right hand sides of the Eq.(4.43),and Eq.(4.46), respectively, are just derivatives of multiplication of two functions. This is illustrated in the following two equations for the second and the third terms, respectively.

$$\frac{\partial L}{\partial\, {}_a^C D_t^\beta q}\, {}_a^C D_t^\beta (t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_a^C D_t^\beta \frac{\partial L}{\partial\, {}_a^C D_t^\beta q} = {}_a^C D_t^\beta \left[\frac{\partial L}{\partial\, {}_a^C D_t^\beta q} (t-\tau)^{\alpha-1} \right] \tag{4.47}$$

$$\frac{\partial L}{\partial\, {}_{t}^{C}D_{b}^{\gamma}q}\, {}_{t}^{C}D_{b}^{\gamma}(t-\tau)^{\alpha-1} + (t-\tau)^{\alpha-1}\ {}_{t}^{C}D_{b}^{\gamma}\frac{\partial L}{\partial\, {}_{t}^{C}D_{b}^{\gamma}q} = {}_{t}^{C}D_{b}^{\gamma}\left[\frac{\partial L}{\partial\, {}_{t}^{C}D_{b}^{\gamma}q}(t-\tau)^{\alpha-1}\right] \quad (4.48)$$

Substitute Eq.(4.47) into Eq.(4.43) to obtain the second term integral $I_{2^{nd}term}$ final formula as follows:

$$\int_{a}^{b}\left[\frac{\partial L}{\partial\, {}_{a}^{C}D_{t}^{\beta}q}(t-\tau)^{\alpha-1}\ {}_{a}^{C}D_{t}^{\beta}\delta q(\tau)\right]d\tau = \\ -\int_{a}^{b}\delta q(\tau)\ {}_{a}^{C}D_{t}^{\beta}\left[\frac{\partial L}{\partial\, {}_{a}^{C}D_{t}^{\beta}q}(t-\tau)^{\alpha-1}\right]d\tau\ . \quad (4.49)$$

Substitute Eq.(4.48) into Eq.(4.46) to obtain the third term integral $I_{3^{rd}term}$ final formula as follows:

$$\int_{a}^{b}\left[\frac{\partial L}{\partial\, {}_{t}^{C}D_{b}^{\gamma}q}(t-\tau)^{\alpha-1}\ {}_{t}^{C}D_{b}^{\gamma}\delta q(\tau)\right]d\tau = \\ -\int_{a}^{b}\delta q(\tau)\ {}_{t}^{C}D_{b}^{\gamma}\left[\frac{\partial L}{\partial\, {}_{t}^{C}D_{b}^{\gamma}q}(t-\tau)^{\alpha-1}\right]d\tau\ . \quad (4.50)$$

The formula of the action function variation that is expressed in Eq.(4.39) can be written [119] by substituting Eq.(4.49) and Eq.(4.50) into Eq.(4.39):

$$\Delta_{\epsilon}S = \int_{a}^{b}\left\{\begin{matrix}\delta q(\tau)\frac{\partial L}{\partial q}(t-\tau)^{\alpha-1} - \delta q(\tau)\ {}_{a}^{C}D_{t}^{\beta}\left[\frac{\partial L}{\partial\, {}_{a}^{C}D_{t}^{\beta}q}(t-\tau)^{\alpha-1}\right] \\ -\delta q(\tau)\ {}_{t}^{C}D_{b}^{\gamma}\left[\frac{\partial L}{\partial\, {}_{t}^{C}D_{b}^{\gamma}q}(t-\tau)^{\alpha-1}\right]\end{matrix}\right\}\epsilon d\tau + O\left(\epsilon^{2}\right)\ . \quad (4.51)$$

Take δq out of the parentheses, and according to the following rule [4, 129, 130, 132]

$${}_{a}D_{t}^{\alpha}f(t) = \left(\frac{\partial f}{\partial t}\right)^{\alpha}, \quad (4.52a)$$

$${}_{t}D_{b}^{\alpha}f(t) = \left(-\frac{\partial f}{\partial t}\right)^{\alpha}. \quad (4.52b)$$

Follows that, the following relations between the left and the right fractional derivative operators

$$ {}_aD_t^\alpha = -{}_tD_b^\alpha \ , \tag{4.53a} $$

$$ {}_tD_b^\alpha = -{}_aD_t^\alpha \ . \tag{4.53b} $$

Substitute Eq.(4.53) into the finite variation equation in Eq.(4.51), and take δq out of the parenthesis to obtain the following equation:

$$ \Delta_\epsilon S = \int_a^b \left\{ \begin{array}{c} \frac{\partial L}{\partial q}(t-\tau)^{\alpha-1} + {}_t^CD_b^\beta\left[\frac{\partial L}{\partial\, {}_a^CD_t^\beta q}(t-\tau)^{\alpha-1}\right] + \\ {}_a^CD_t^\gamma\left[\frac{\partial L}{\partial\, {}_t^CD_b^\gamma q}(t-\tau)^{\alpha-1}\right] \end{array} \right\} \delta q(\tau)\,\epsilon d\tau + O\left(\epsilon^2\right) \ . \tag{4.54} $$

By considering the boundary conditions at the action function S extremities the finite variation equation ΔS becomes zero [119], or the corresponding variations vanishes [134]. As a result the integrand of the last equation must equal zero [119], as illustrated in the following formula:

$$ \frac{\partial L}{\partial q}(t-\tau)^{\alpha-1} + {}_t^CD_b^\beta\left[\frac{\partial L}{\partial\, {}_a^CD_t^\beta q}(t-\tau)^{\alpha-1}\right] + {}_a^CD_t^\gamma\left[\frac{\partial L}{\partial\, {}_t^CD_b^\gamma q}(t-\tau)^{\alpha-1}\right] = 0 \ . \tag{4.55} $$

For simplicity let the orders of the fractional derivatives $\beta = 1$, and $\gamma = 1$ [119], open the fractional derivatives in the last formula, and consider Eq.(4.55) depends only on ${}_tD_b^\beta$ or ${}_aD_t^\gamma$ [119, 136] to obtain

$$ \frac{\partial L}{\partial q} + \frac{1}{(t-\tau)^{\alpha-1}}\left\{-\frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{q}}(t-\tau)^{\alpha-1}\right]\right\} = 0 \ . \tag{4.56} $$

Take the partial derivative with respect to time of the two functions multiplication $\frac{\partial L}{\partial \dot{q}}(t-\tau)^{\alpha-1}$ in Eq.(4.56) to get the following:

$$ \frac{\partial L}{\partial q} + \frac{1}{(t-\tau)^{\alpha-1}}\left\{-\left[\frac{\partial L}{\partial \dot{q}}(\alpha-1)(t-\tau)^{\alpha-2} + (t-\tau)^{\alpha-1}\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}}\right]\right\} = 0 \ . \tag{4.57} $$

Open the square brackets of the Eq.(4.57), and rearrange it to obtain the FELE as following:

$$\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}} - \frac{(\alpha-1)}{(t-\tau)}\frac{\partial L}{\partial \dot{q}} = 0 \quad t \neq \tau \;, \tag{4.58}$$

where τ represents the intrinsic time and t the observer time [81]. The FELE model can be applied to a system in which harmonic oscillator with time-dependent mass and frequency w obeys merely the standard harmonic oscillator dynamical equation $\ddot{x} + w^2 x = 0$ [81].

4.3.1 An Alternative Form of FELEs.

The extended fractional integral of a continues function $L(t)$ in the interval $[a,\ b]$ of an derivative order $\alpha > 0$ are defined by [138]

$$\begin{aligned} S &= {}_aI_b^{\alpha} L(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b L[\tau,\ q(\tau),\ \dot{q}(\tau))(\cosh(t) - \cosh(\tau)]^{\alpha-1}\, d\tau \;, \end{aligned} \tag{4.59}$$

where $\dot{q} = \dfrac{dq}{d\tau}$; τ is the intrinsic time, and t is the observer time.

Under the boundary conditions $q(a) = q_a$, and $q(b) = q_b$ of the continuous function $q(.) \in \mathbb{C}$ in the interval $[a,\ b] \in \mathbb{R}$, if the functions $q(.)$ are considered to be solutions to the above problem, then $q(.)$ satisfy the following FELEs [81]:

$$\frac{\partial L}{\partial q} - \frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{q}} - \frac{(\alpha-1)\sinh(\tau)}{[\cosh(t) - \cosh(\tau)]}\frac{\partial L}{\partial \dot{q}} = 0 \;. \tag{4.60}$$

The extended weak dissipative parameter $D(\tau) = \dfrac{(\alpha-1)\sinh(\tau)}{[\cosh(t) - \cosh(\tau)]}$ limitations can be written as follows:

$$D(\tau) = (\alpha - 1)\, \frac{\exp^{\tau} - \exp^{-\tau}}{\exp^{t} + \exp^{-t} - (\exp^{\tau} + \exp^{-\tau})} \;. \tag{4.61}$$

so,

$$\begin{cases} \text{if } \tau \longrightarrow \infty, & then \;\; D(\tau) \longrightarrow 1-\alpha, \\ \text{if } \tau \longrightarrow 0, & then \;\; D(\tau) \longrightarrow 0. \end{cases}$$

Harmonic oscillators that have time-depended mass and frequency may be an excellent picture of non-conservative or weak dissipative dynamical systems [81].

4.4 Conclusion

The dissipative terms of FELEs in Eq.(4.58), and Eq.(4.60) contain some parameters such as the order of the R-L fractional integrals α, and the intrinsic time τ which can be a plant delay time due to physical transportation delay intrinsic to the system under control [139] in some dynamic systems. Additionally, the fractional order α, represents one of the system damping force parameters [48], in the damping term, with time-decreasing coefficient [140]. Moreover, there is a contrast between damping force that given in the FELE which is illustrated in Eq.(4.58), and the damping force given by the alternative form of FELE that is given by Eq.(4.60). Since in the later one, as the intrinsic time τ goes to zero, the damping force goes to zero as well, and this is illustrated in Eq(4.61), while in the first form of FELE that is illustrated in Eq.(4.58), as the intrinsic time τ goes to zero, the damping force becomes time-decreasing damping force. But the intrinsic time τ is a delay time and generally in most applications very close to zero, so the damping force of the dynamic system that represented by the FELE that illustrated in Eq.(4.58), vanishes as the observer time t increases. Case studies of the FELEs applications are discussed and investigated in the next chapter.

CHAPTER 5

APPLICATIONS of the FRACTIONAL EULER-LAGRANGE EQUATIONS

5.1 Introduction

The FELE that is introduced in chapter 4 is applied in this chapter to obtain the weak dissipative force that is generated from two case studies modeled by FELE. The responses of the system are compared with the traditional ELE model in terms of the effects of linearity, the fractional integral order α, and the initial conditions. A numerical technique and an analytical technique are used to obtain the responses and they are investigated and compared. Additionally, the phase space trajectories of the systems are discussed in order to given more interpretation to the behaviors of the system responses. Moreover, the physical meaning of the R-L fractional integral order α is discussed. This discussion is done by comparing with previous experimental studies whose damping forces affects gave the same behaviors of the damping force generated by FELE modeling.

5.2 A Single Pendulum Integer Order Modeling

Consider a free oscillating conservative single pendulum (uniform rod) system with initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$ as illustrated in Fig. 5.1. The ELE [5] of the system can be given as follows:

$$\frac{\partial L}{\partial \theta(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}(t)} = 0 \,, \tag{5.1}$$

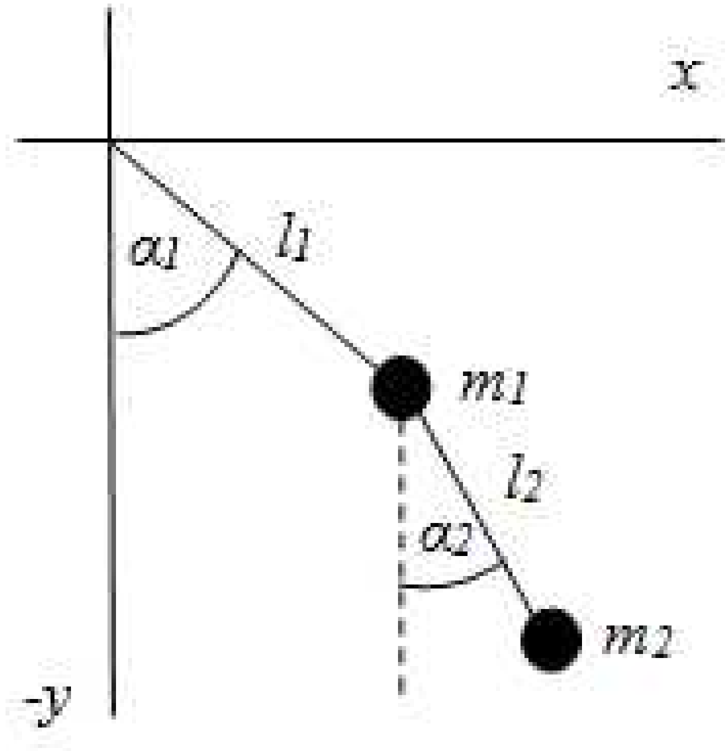


Figure 5.1: A free oscillating conservative single pendulum (uniform rod).

To obtain the system equation of motion, where L which is given as a function of general coordinate $\theta(t)$ is the Lagrangian of the system as demonstrated in Eq.(4.15).

The kinetic energy of the oscillating single pendulum system that is illustrated in Fig. 5.1 can be given by

$$T = \frac{1}{8} m l^2 \dot{\theta}(t) \,, \tag{5.2}$$

where the center of gravity of the pendulum is at the center, l represents the length of the rod, and m is the mass of the pendulum.

The potential energy of the pendulum can be expressed as,

$$V = -\frac{1}{2} mgl \cos\theta(t) \ , \tag{5.3}$$

where g is the gravitational acceleration. Substitute Eq.(5.2) and Eq.(5.3) into Eq.(4.15) to obtain the Lagrangian of the conservative single pendulum system, that is shown in Fig. 5.1, as follows:

$$L = \frac{1}{8} ml^2 \dot{\theta}(t) + \frac{1}{2} mgl \cos\theta(t) \ , \tag{5.4}$$

then, the ELE model of the system is obtained by substituting the Lagrangian L that is given by Eq.(5.4) into Eq.(5.1) as following:

$$\frac{\partial}{\partial\theta(t)}\left[\frac{1}{8} ml^2 \dot{\theta}(t) + \frac{1}{2} mgl \cos\theta(t)\right] - \frac{d}{dt}\frac{\partial}{\partial\dot{\theta}(t)}\left[\frac{1}{8} ml^2 \dot{\theta}(t) + \frac{1}{2} mgl \cos\theta(t)\right] = 0 \ . \tag{5.5}$$

Solve the derivatives in the Eq.(5.5) to obtain the following free oscillating undamped single pendulum equation of motion:

$$\ddot{\theta}(t) + 2\frac{g}{l} \sin\theta(t) = 0 \ . \tag{5.6}$$

Equation (5.6) is a second order non-linear homogeneous differential equation, This equation can be solved by means of Jacobian Elliptic Function [11].

5.2.1 Jacobian Elliptic Function

The exact solution of the non-linear ELE model of single pendulum that is represented by Eq.(5.6) can be given by [11],

$$\theta(t) = 2\sin^{-1}\left[k \ sn\left(\sqrt{\frac{g}{l}}(t - t_0) ; \ k\right)\right] \ , \tag{5.7}$$

Where the function $sn(x; \ k)$ is the Jacobian Elliptic function, k represents an associated modulus with the Jacobian Elliptic function; and can be expressed as $k = \sin^2\left(\frac{\theta_0}{2}\right)$, and the time t_0 is the initial time [17]. The function $sn(x; \ k)$ is defined as a kind of inversion of an integral given by the following function [23],

$$x(y,\ k) = \int_0^y \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} dt\ , \tag{5.8}$$

so, the Jacobian elliptic function is defined as the inverse of the x function

$$y = sn(x;\ k)\ . \tag{5.9}$$

5.2.2 Runge-Kutta Numerical Method

The non-linear equation of motion that is given by Eq.(5.6) can be solved by using Runge-Kutta numerical method which is a weight average slope method to obtain numerical solutions of differential equations. The Runge-Kutta numerical method computational cost is comparable to that of the trapezoidal method [29]. That makes it a more accurate method, but it requires more equations to be solved and computed [35].

The solution of the single pendulum system can be obtained using the 4^{th} order time step Runge-Kutta numerical method as following:

Consider the system equation of motion that is given by Eq.(5.6), with the following initial conditions:

$$\theta(t_0) = \theta_0\ , \tag{5.10a}$$

$$\omega(t_0) = \omega_0\ , \tag{5.10b}$$

where t represents the observed time, and $\omega(t)$ is the angular velocity of the pendulum that can be given by

$$\omega(t) = \frac{d}{dt}\theta(t)\ . \tag{5.11}$$

Let's define the function $F(t,\ \theta)$ as follows:

$$
\begin{aligned}
F(x,\ \theta(t)) &= \dot{\omega}(t) \\
&= -\omega_n^2 \sin\theta(t)\ , \qquad (5.12)
\end{aligned}
$$

where $\omega_n = \sqrt{2\frac{g}{l}}$ is the natural frequency of the system. A according to the method definition, the function $F(t,\ \theta)$ represents the first slope. By means of the first slope the successive slopes can be obtained, and the position of the pendulum at any time can be expressed as:

$$
\theta(t+h) = \theta(t) + \Delta\theta\ , \qquad (5.13)
$$

where $\Delta\theta$ is the average of the four slopes generated by Runge-Kutta numerical method.

Case Study 5.1

The case study is considered to investigate the behaviours of a free oscillating single pendulum (rod) system responses. The system is modeled by means of ELE and FELE. The Effects of the linearty, the R-L fractional integral α, and the initial conditions on the system responses are studied in this case study. Additionaly, some methods are used to obtain the system responses such as Jacoban Elliptic function, Runge-Kutta numerical technique, and the bassel function based solution.

Consider the following parameters of the single pendulum (rod) that is depicted in Fig. 5.1 with pendulum mass $m = 0.230\ kg$, and pendulum full length $l = 0.6413\ m$ [41].

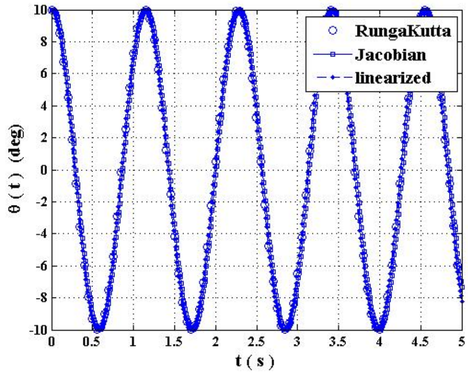
The responses of the non-linear system, for different initial conditions $\omega(0)$ and $\theta(0)$ can be obtained. The non-linear and linear representations of the system are compared by means of the above two methods for small angle of oscillation as $\sin\theta(t) \simeq \theta(t)$.

The responses of the system oscillations according to the given parameters and different conditions are illustrated in Fig. 5.2.

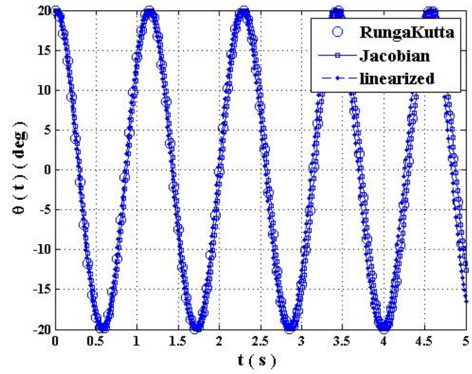
The response of the non-linear representation and the response the linearized representation of the system for certain system parameters must match each other for small oscillations $\theta(t) < \dfrac{\pi}{6}$. In order to investigate that let's consider the following:

The subfigures Fig. 5.2-a through Fig. 5.2-e illustrate a comparison among three representations of the system solutions. These representations are a nonlinear response obtained the Jacobian Elliptic function, a nonlinear response obtained by using the Runge-Kutta numerical technique, and the linearized response of the system.

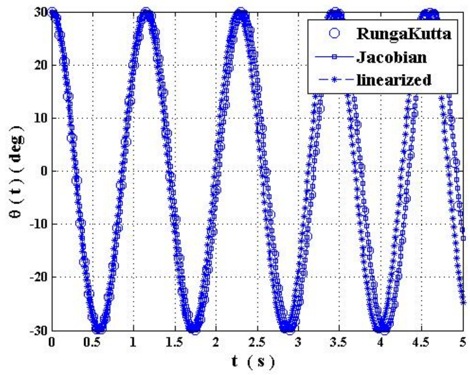
The initial angular velocity can be introduced when the Runge-Kutta numerical technique is applied. The effect of initial angular velocity on the system response is noticeable, even-though the initial angular displacement is small. That is illustrated by Fig. 5.2-f. The Runge-Kutta numerical technique solutions of the non-linear representations matches the solutions of the linearized representations, as the initial values of the oscillating $\theta(0)$ gets smaller.



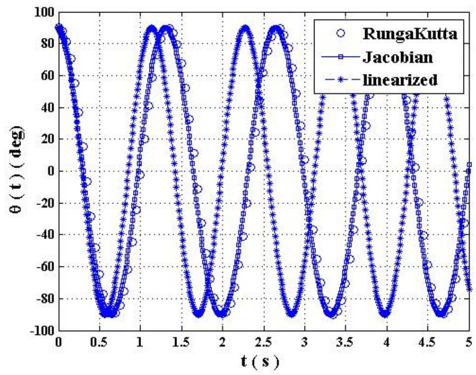
(a) $\theta(0) = \frac{\pi}{18},\ and\ \omega(0) = 0$



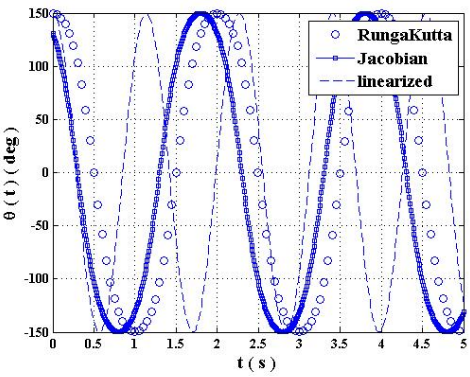
(b) $\theta(0) = \frac{\pi}{9},\ and\ \omega(0) = 0$



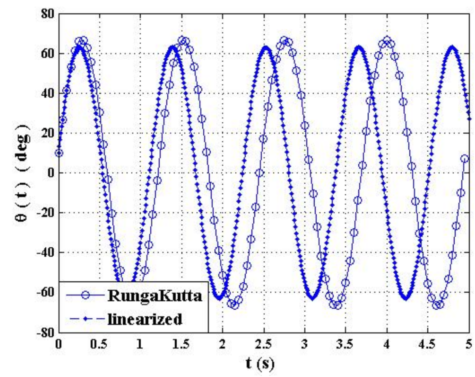
(c) $\theta(0) = \frac{\pi}{6},\ and\ \omega(0) = 0$



(d) $\theta(0) = \frac{\pi}{2},\ and\ \omega(0) = 0$



(e) $\theta(0) = \frac{5\pi}{6},\ and\ \omega(0) = 0$



(f) $\theta(0) = \frac{\pi}{6},\ and\ \omega(0) = 6\ \frac{rad}{s}$

Figure 5.2: The responses of non-linear and linearized free oscillating undamped single pendulum system using Jacobian function solution and Runge-Kutta technique.

5.3 FELEs Modeling of a Free Oscillating Single Pendulum

The FELE can be applied hypothetically (the assumption states that there is a dissipation force, decreases with time, applied to the system) to obtain the response of the single pendulum system. Considered system is introduced in the previous section. The dissipation of the system is assumed to be gradually decreased as a function of time. This dissipation force may be due to dry friction or magnetic force [47].

Consider the system illustrated in Fig. 5.1, where the Lagrangian L of the system is represented in Eq.(5.4).

Introducing a gradually decreased damping force to the system yields a function of time dissipating force coefficient. That coefficient depends also in the properties of the damper. One of them is the order fractional derivative α as mentioned in the application of the fractional order modeling of the forced oscillating double pendulum system [51]. Consequently, the system can be modeled by means of the FELE. This equation is given as follows:

$$\frac{\partial L}{\partial \theta(t)} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}(t)} - \frac{\alpha - 1}{t - \tau}\frac{\partial L}{\partial \dot{\theta}(t)} = 0\ . \tag{5.14}$$

Substituting the Lagrangian in Eq.(5.4) into Eq.(5.14) yields

$$\begin{aligned}\frac{\partial}{\partial \theta(t)}\left[\frac{1}{8}ml^2\dot{\theta}(t) + \frac{1}{2}mgl\cos\theta(t)\right] &- \frac{d}{dt}\frac{\partial}{\partial \dot{\theta}(t)}\left[\frac{1}{8}ml^2\dot{\theta}(t) + \frac{1}{2}mgl\cos\theta(t)\right] \\ &- \frac{\alpha - 1}{t - \tau}\frac{\partial}{\partial \dot{\theta}(t)}\left[\frac{1}{8}ml^2\dot{\theta}(t) + \frac{1}{2}mgl\cos\theta(t)\right] = 0\ . \\ &.\end{aligned} \tag{5.15}$$

Solving the derivatives in Eq.(5.15) yields

$$-mg\frac{l}{2}\sin\theta(t) - \frac{1}{4}ml^2\ddot{\theta}(t) - \frac{1}{4}\frac{\alpha - 1}{t - \tau}ml^2\dot{\theta}(t) = 0\ . \tag{5.16}$$

Manipulate and rearrange Eq.(5.16) to obtain

$$\ddot{\theta}(t) + \frac{\alpha - 1}{t - \tau}\dot{\theta}(t) + \omega_n^2 \sin\theta(t) = 0 \;, \tag{5.17}$$

where $\omega_n = \sqrt{2\frac{g}{l}}$ is the system natural frequency. For the same parameters and with the same initial conditions, the mathematical modeling of the system in the case study (5.1) can be obtained by introducing the FELE. The solutions of the system model can be obtained using the Runge-Kutta method, the Bessel function solution (using Matlab software), or the power series methods to solve variable coefficients second order linear differential equations [52].

The system derived by FELE model is represented by non-linear equations of motion with variable coefficients, where the responses of this system can be obtained by using Runge-Kutta numerical method that introduced in section 5.2.2.

5.3.1 The Solutions of the FELEs Modeling of Single Pendulum System

Before getting the responses of the nonlinear system, let's linearized the system then obtain the response using different techniques. Those techniques include the Runge-Kutta numerical techniques, and the solutions by using Bessel Functions.

The linear formula of the system can be represented, for small oscillation, as follows:

$$\ddot{\theta}(t) + \frac{\alpha - 1}{t - \tau}\dot{\theta}(t) + 2\frac{g}{l}\theta(t) = 0 \;, \tag{5.18}$$

with given initial conditions $\theta(t_0) = \theta_0, \;\; \omega(t_0) = \omega_0$.

The responses of the system that is given by Eq.(5.18), with different initial conditions $\theta(t_0) = \theta_0, \;\; \omega(t_0) = \omega_0$ can be obtained by using the Bessel function solution (using Matlab software) , or the Runge-Kutta numerical method as following:

I) **Solutions based on the Bassel Functions**

The Matlab program gives the solution of Eq.(5.18),using the function ***dsolve***, which is a numerical technique exploits the Bessel functions to obtain the solutions of second-order linear homogeneous differential equations with variable coefficients. The responses of the single pendulum FELEs model that is given by Eq.(5.18) are obtained by mean of Bessel function. The responses are compared, as illustrated in Fig. 5.3, with Runge-Kutta numerical technique solutions which are explained in the following subsection.

II) **The Runge-Kutta Numerical Technique Solution**

The responses of the linearized single pendulum system that is represented by Eq.(5.18) can be obtained using the Runge-Kutta numerical technique with same initial conditions $\theta(t_0) = \theta_0, \quad \omega(t_0) = \omega_0$. These conditions are used to obtain the responses of the system by usage of Bessel function. Moreover, the angular velocity $\omega(t) = \dot{\theta}(t)$ of the system can be derived and evaluated. In order to apply the Runge-Kutta, let's assign the function $F(x,\ \theta(t),\ \omega(t))$ that corresponds to the linear system as following:

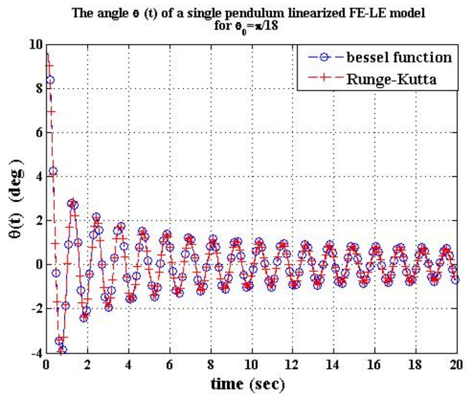
$$\begin{aligned} F(x,\ \theta(t),\ \omega(t)) &= \dot{\omega}(t) \\ &= -\frac{\alpha-1}{t-\tau}\omega(t) - \omega_n^2\theta(t)\ . \end{aligned} \tag{5.19}$$

According to the Runge-Kutta numerical method the position of the pendulum at any time can expressed as

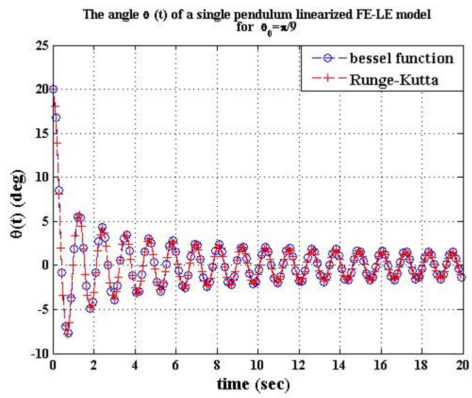
$$\theta(t+h) = \theta(t) + \Delta\theta\ , \tag{5.20}$$

where $\Delta\theta$ is the average of the angle of oscillation $\theta(t)$ slopes generated by Runge-Kutta numerical method, and h is the time step.

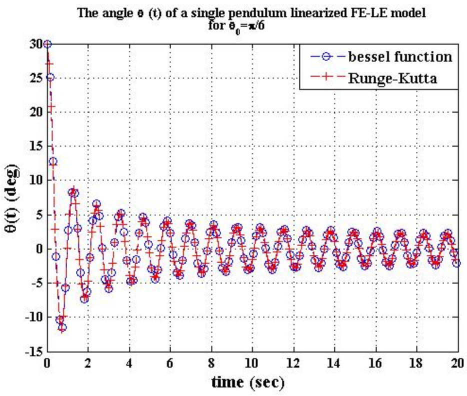
Figure 5.3 illustrates the responses of the linearized FELE model of free oscillating single pendulum for fractional order integral $\alpha = 2$, with different values of initial angular displacement $\theta(t_0) = \theta_0$ and zero initial angular velocity $\omega(t_0) = 0$. These responses are obtained by using the Runge-Kutta numerical technique and the Bessel function solution. It's concluded from the Fig. 5.3 that the responses of the linearized system, which are obtained by using the two methods, are identical to each other for same initial conditions. Moreover, the dissipation force is gradually decreases, as the time increases. That is because of the variant dissipative coefficient that appears in Eq.(5.18).



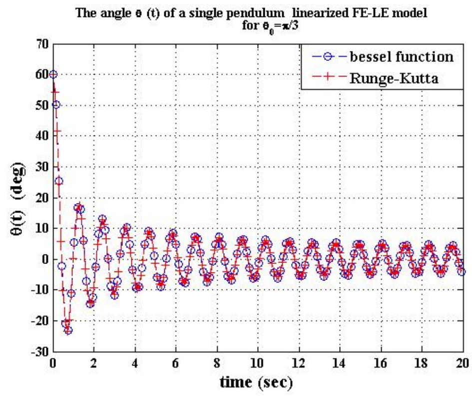
(a) $\theta(0) = \dfrac{\pi}{18}$, *and* $\omega(0) = 0$



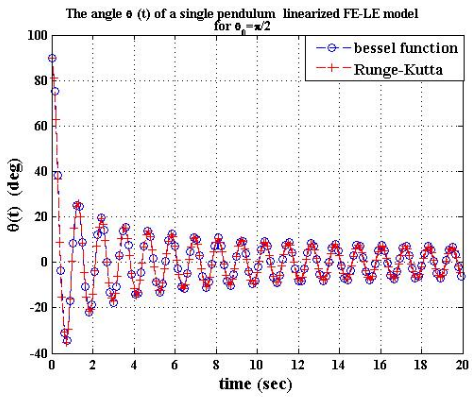
(b) $\theta(0) = \dfrac{\pi}{9}$, *and* $\omega(0) = 0$



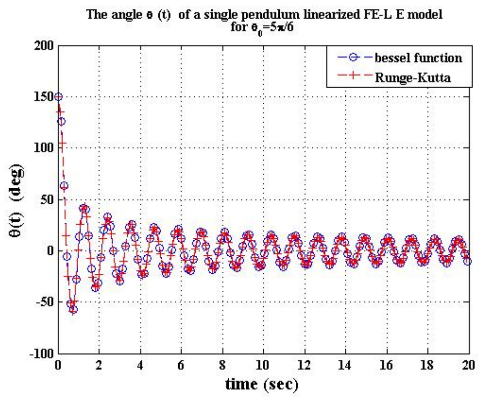
(c) $\theta(0) = \dfrac{\pi}{6}$, *and* $\omega(0) = 0$



(d) $\theta(0) = \dfrac{\pi}{3}$, *and* $\omega(0) = 0$



(e) $\theta(0) = \dfrac{\pi}{2}$, *and* $\omega(0) = 0$



(f) $\theta(0) = \dfrac{5\pi}{6}$, *and* $\omega(0) = 0$

Figure 5.3: Responses of linearized free oscillating single pendulum modeled by FELE solved by using Runge-Kutta and Bessel function, with different initial conditions and, constant $\alpha = 2$.

5.4 Non-Linear and Linearized System Representations

The linearization of the free oscillating un-damped single pendulum system modeled by FELE, for different values of initial angular displacement $\theta(t_0) = \theta_0$ and zero initial angular velocity $\omega(t_0) = 0$, is accomplished by considering $\sin\theta(t) \simeq \theta(t)$ for small oscillations approximately $\theta_0 \leq \dfrac{\pi}{6}$. The responses of the nonlinear system compared with the linearized responses are illustrated in Fig. 5.4. In this figure the subfigures Fig. 5.4-a, and Fig. 5.4-b, illustrate that the linear and nonlinear system give the same responses due to small oscillations. The linearized representations of the system due to large oscillations for initial oscillating angle approximately $\theta_0 \geq \dfrac{\pi}{4}$ is depicted in Fig. 5.4-c through Fig. 5.4-f, where the solutions of the system are unlike the exact solutions for the nonlinear system.

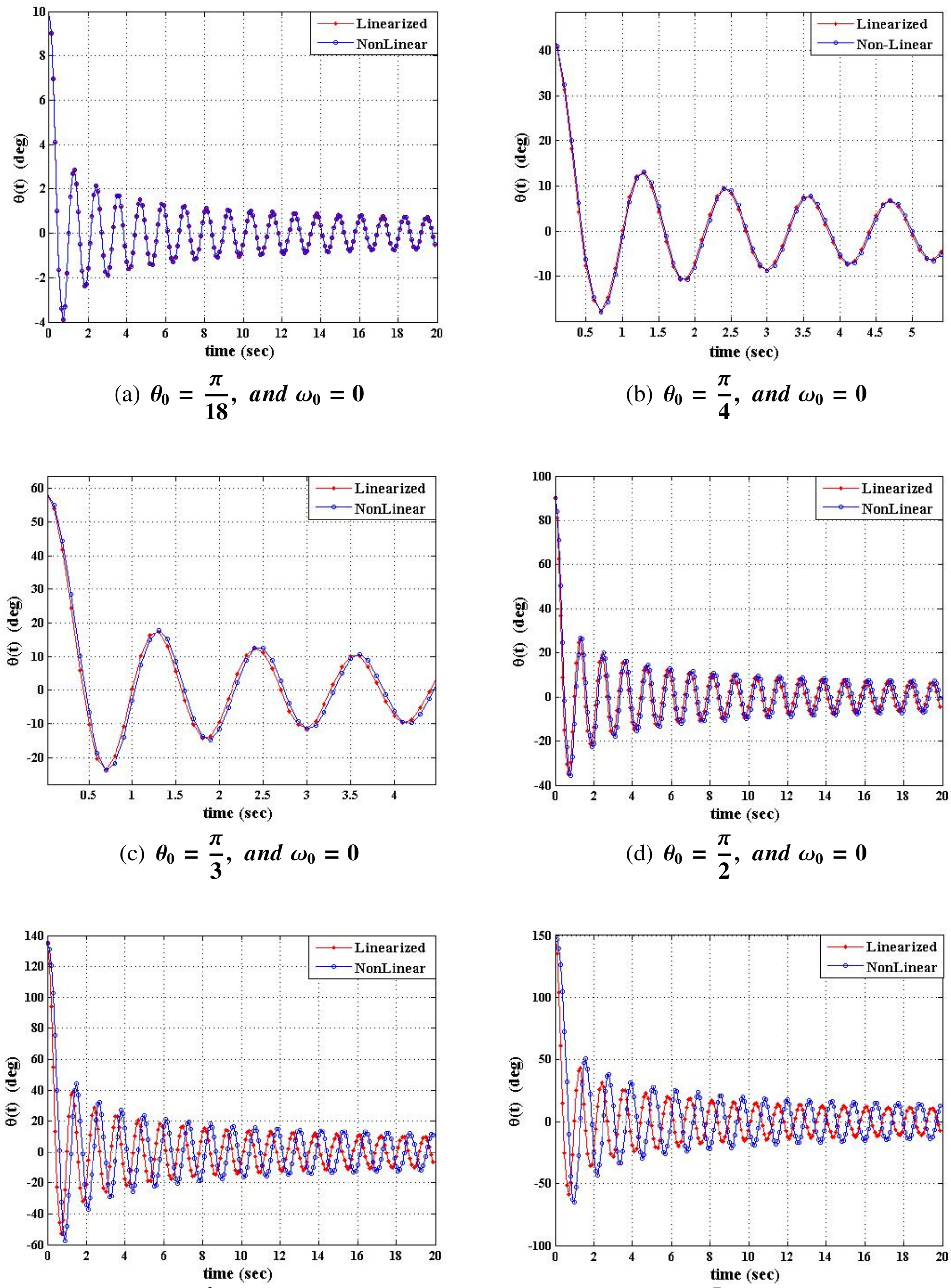
5.4.1 The Angular Velocity Representations

According to Runge-Kutta numerical method, the angular velocity of the free oscillating single pendulum system that is modeled by FELE, at any time, is given by

$$\omega(t+h) = \omega(t) + \Delta\omega \,, \tag{5.21}$$

where $\Delta\omega$ is the average of the four slopes of the oscillation angular velocity $\omega(t)$, and h is the time step.

The angular velocity representations that are obtained by usage of Runge-Kutta numerical method can be plotted as illustrated in Fig. 5.5. The figure shows the oscillations angular velocities for various initial conditions and for non-linear and linearized system representations as well. In the subfigure Fig. 5.5-a, since the initial angular displacement is very small, the angular velocity representations of the nonlinear and linearized system representations are exactly same. On the contrary, the representations of the system (linearized and non-Linear) that are illustrated in subfigures Fig. 5.5-c and Fig. 5.5-d are different from each other. That is because the linearization is valid only for small oscillation. Figure 5.5-b. illustrates the angular velocities representations for linearized system with different initial angular velocities $\omega_0 = -0.1,\ 0,$ and $0.01\ \dfrac{rad}{sec}$, where the initial angular displacement is $\theta_0 = \dfrac{\pi}{6}$ for all of them. It's concluded from this subfigure that as the initial angular velocity decreases, the dissipation of the system increases for same initial displacements.

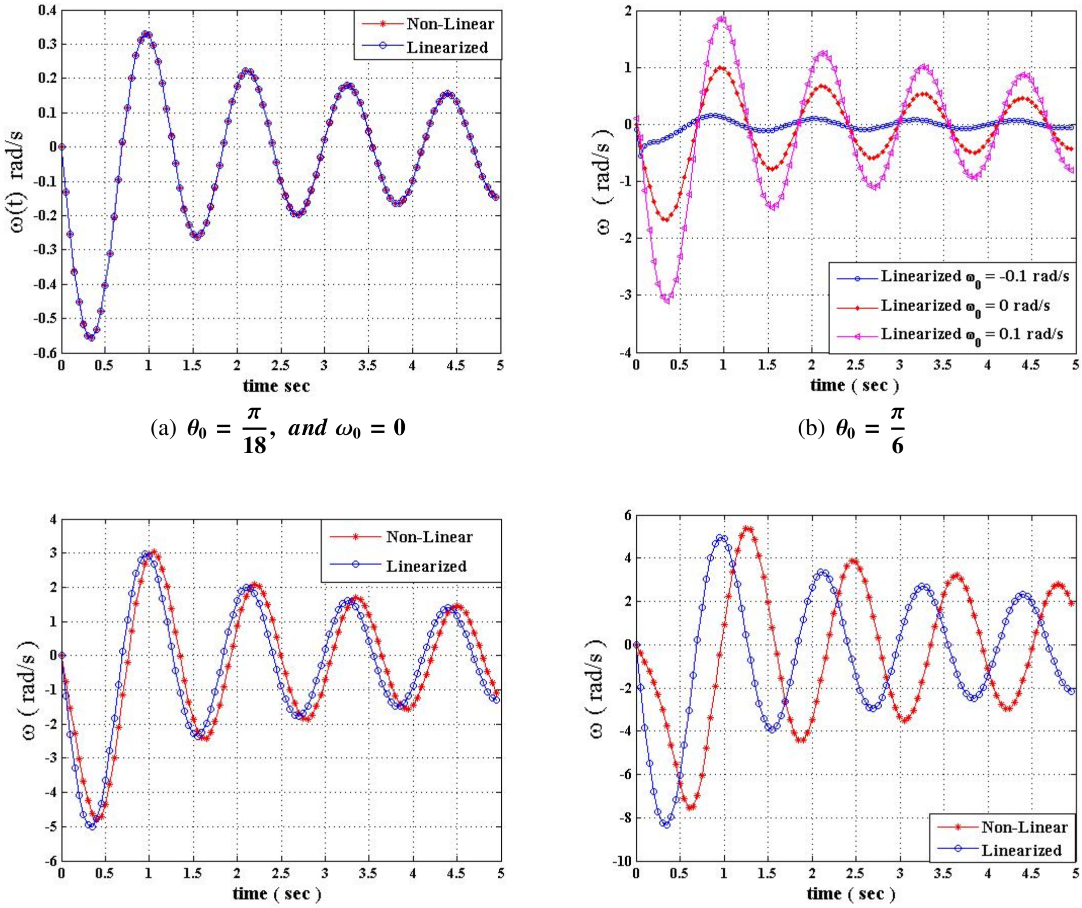


(a) $\boldsymbol{\theta_0 = \dfrac{\pi}{18},\ and\ \omega_0 = 0}$ (b) $\boldsymbol{\theta_0 = \dfrac{\pi}{4},\ and\ \omega_0 = 0}$

(c) $\boldsymbol{\theta_0 = \dfrac{\pi}{3},\ and\ \omega_0 = 0}$ (d) $\boldsymbol{\theta_0 = \dfrac{\pi}{2},\ and\ \omega_0 = 0}$

(e) $\boldsymbol{\theta_0 = \dfrac{3\pi}{4},\ and\ \omega_0 = 0}$ (f) $\boldsymbol{\theta_0 = \dfrac{5\pi}{6},\ and\ \omega_0 = 0}$

Figure 5.4: Non-Linear vs. linearized FELE model of a single pendulum for different oscillating ranges.



(a) $\boldsymbol{\theta_0 = \frac{\pi}{18}, \; and \; \omega_0 = 0}$

(b) $\boldsymbol{\theta_0 = \frac{\pi}{6}}$

(c) $\boldsymbol{\theta_0 = \frac{\pi}{2}, \; and \; \omega_0 = 0}$

(d) $\boldsymbol{\theta_0 = \frac{5\pi}{6}, \; and \; \omega_0 = 0}$

Figure 5.5: The non-linear and linearized angular velocity representations of a single pendulum modeled by FELE for different initial conditions.

5.4.2 The Effect of Fractional Order Integral α on the System Response

The fractional integral order α appears in the numerator of the time variant coefficient of the damping force of the system FELE model. Thus, as the fractional integral order α increases, the damping force increases at relatively small observed time t. Additionally, the damping force gradually decreases as the observed time increases. Fig. 5.6 and Fig. 5.7 illustrate the effect of t on the free oscillating FELE modeled single pendulum system. Effects of three values of the order α are compared in Fig. 5.6, which are $\alpha = 1.1$, $\alpha = 1.5$, and $\alpha = 2$. Since the first value of α is comparatively small, the response of the system, under this dissipation is gradually decreases slower than the responses in the other cases $\alpha = 1.5$, and $\alpha = 2$ where the responses get faster closes to equilibrium. The observed time t is considered as one of the factors that affect the dissipation force. This effect depends on α values, as shown in Fig. 5.6, the effect of the observed time decreases, as α increases. More general the system depicted in Fig. 5.7 is tested for successive values of α in the range from 1.1 to 2, where the response amplitude decreases, as α increases. Moreover, since the system is stable, for each certain value of α the response amplitude decreases, as t increases.

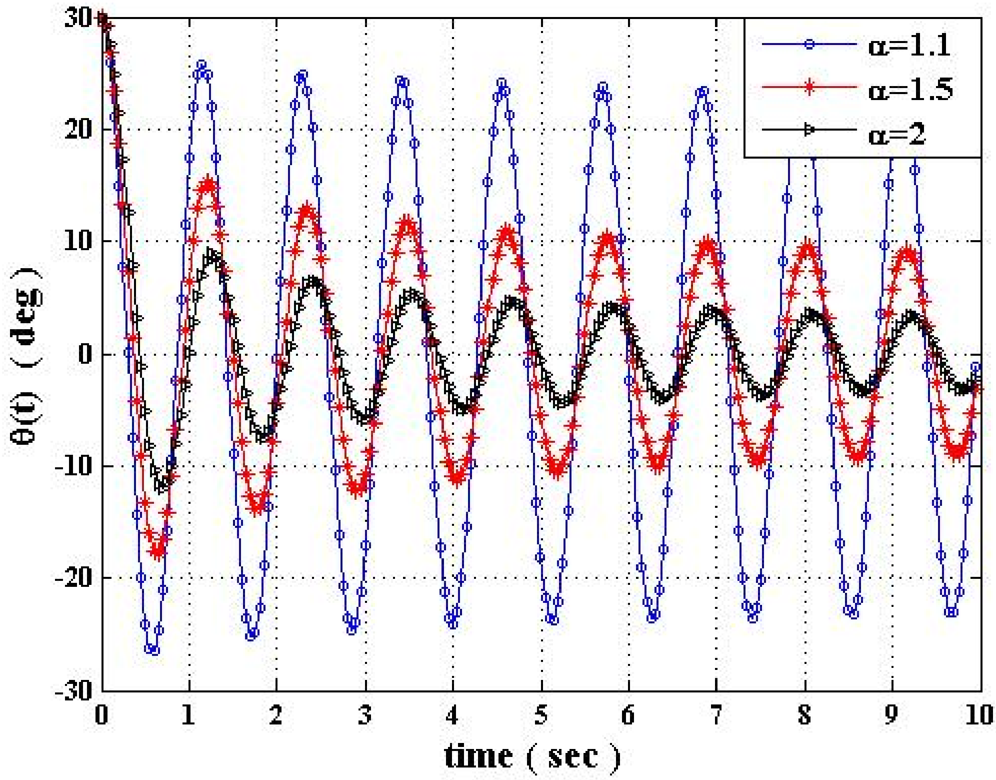


Figure 5.6: The free oscillating single pendulum system responses for different values of fractional integral order α.

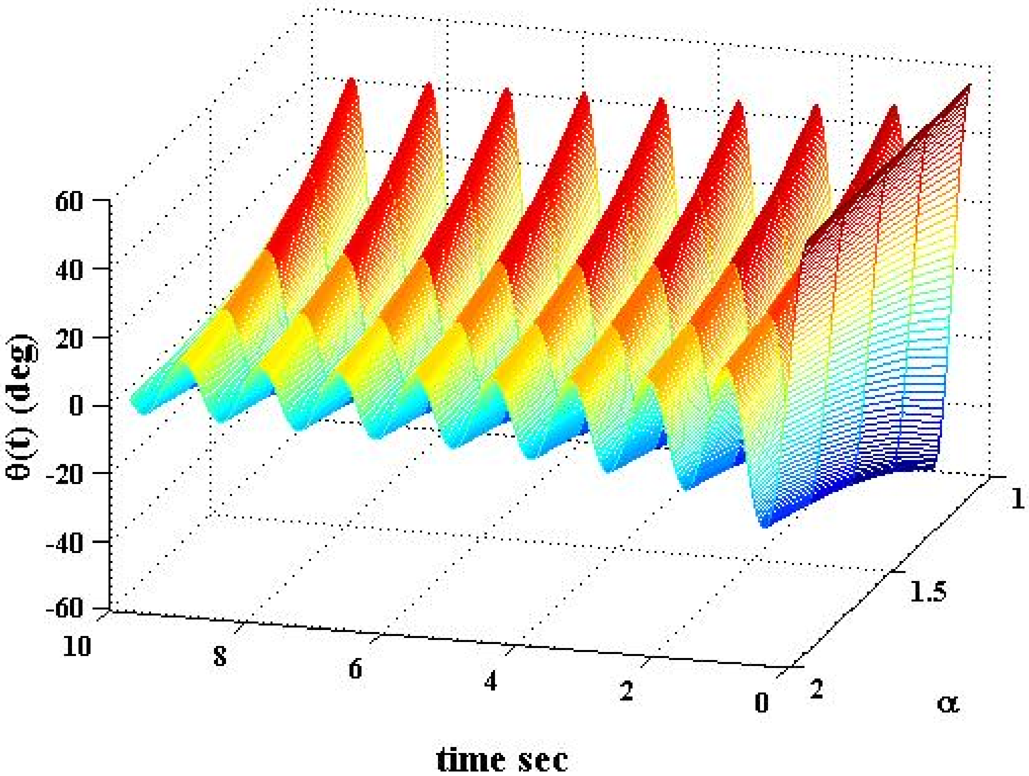


Figure 5.7: Responses of free oscillating single pendulum modeled by FELE for multi values of α.

5.4.3 Phase Space Trajectories

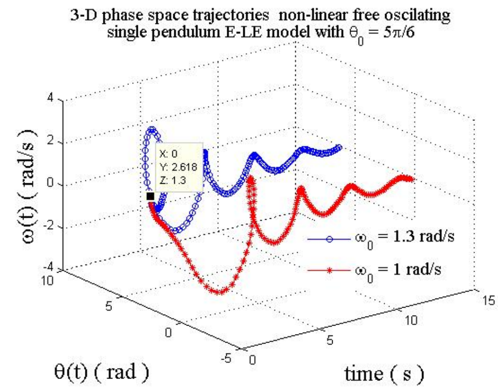
The phase space trajectories give more interpretation for the system behavior, which can be accomplished by studying the relationship between the angular displacement and the angular velocity. The damped-driven pendulum can exhibit both periodic and chaotic motions [56]. The attraction of these motion is generally at the origin (down equilibrium point at $\theta = 0$. The damping parameter strength indicates how long the solutions in phase space take to reach the steady state. In the free oscillating single pendulum system modeled by FELE, the dissipation force is gradually decreased with observed time. That is because the coefficient of the damping force is a function of time. The trajectories in the phase space of angular velocity vs. angular displacement are strongly depending on the degree of the non-linearity, the damping factor, and the initial conditions [60]. A set of examples are depicted in the following figures

Fig. 5.8 through Fig. 5.11 in order to represent the effects of these parameters on the system behavior. The effect of the initial angular velocity is illustrated in Fig. 5.8 as following:

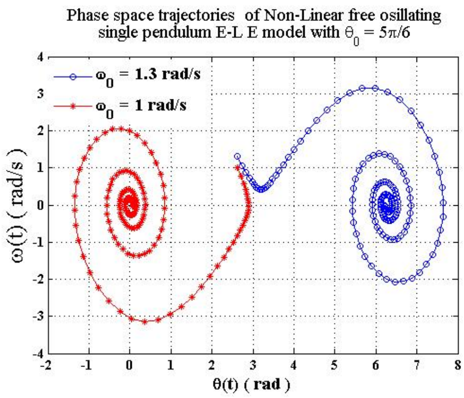
The phase space trajectories of a free oscillating single pendulum system modeled by using ELE, and FELE are illustrated in Fig. 5.8. In order to investigate the effect of the initial angular velocity on the non-linear system behavior, the responses of the system must be analyzed for different values of initial angular velocity ω_0. Three-dimension phase space trajectories of a non-linear free oscillating single pendulum ELE model is plotted in Fig. 5.8-a. In this subfigure the initial angular displacement $\theta_0 = 5\pi/6\ rad$ is constant while the angular velocity ω_0 is varied. The trajectories are attracted to *zero* and 2π for initial angular velocities $\omega_0 = 1\ rad/s$, and $1.3\ rad/s$, respectively. This is depicted in Fig. 5.8-b which represents phase space trajectories of a non-linear free oscillating single pendulum ELE model in two-dimension. According to the phase space trajectories of the system in the subfigures, there is a certain initial angular velocity (*critical angular velocity* $c\omega_0$) in which the attractive point is changed from being *zero* to be 2π. The critical angular velocity ($c\omega_0$) can be computed by using a computer program. The $c\omega_0$ of the system modeled by ELE in the subfigures Fig. 5.8-a and Fig. 5.8-b is $c\omega_0 = 1.1\ rad/s$.

The FELE modeled free oscillating non-linear single pendulum system whose phase space trajectories are represented in the subfigures Fig. 5.8-c and Fig. 5.8-d has a low $c\omega_0 = 0.12\ rad/s$ due to a very weak dissipative force. Therefore, the $c\omega_0$ of the system modeled by FELE is smaller than the $c\omega_0$ of the same system modeled by ELE with introduced damping term.

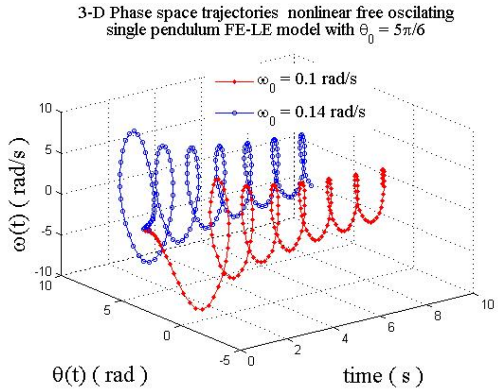
Due to higher value of the initial angular velocity ω_0 the attractive point can be $n\pi$ instead of *zero* where $n \in 2\mathbb{N}$. This is illustrated in Fig. 5.8-e, where the attractive point increases, as the initial angular velocity increases. For instant, the phase space trajectory of a non-linear free oscillating single pendulum system modeled by ELE with initial angular velocity $\omega_0 = 5.5\ rad/s$ is attracted at 4π, whereas the attractive point 2π is associated with $\omega_0 = 1.2\ rad/s$ and $\omega_0 = 3\ rad/s$ for the same system parameters.



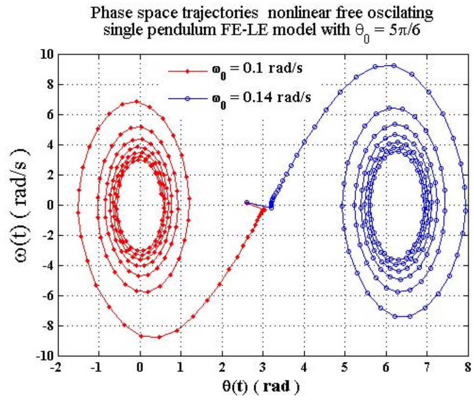
(a) $\omega_0 = 1$ $vs.$ $\omega_0 = 1.3$ rad/s for $\theta_0 = 5\frac{\pi}{6}$ rad



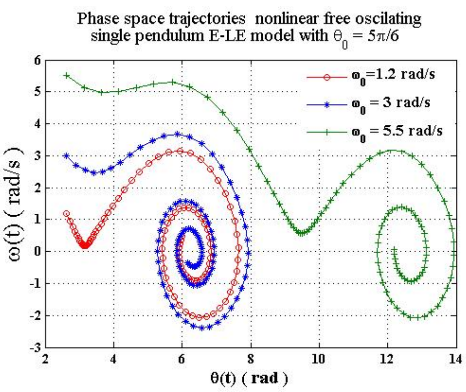
(b) $\omega_0 = 1$ $vs.$ $\omega_0 = 1.3$ rad/s for $\theta_0 = 5\frac{\pi}{6}$ rad



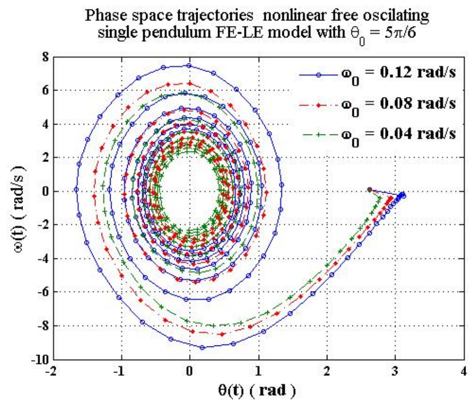
(c) $\omega_0 = 0.1$ $vs.$ $\omega_0 = 0.14$ rad/s for $\theta_0 = 5\frac{\pi}{6}$ rad



(d) $\omega_0 = 0.1$ $vs.$ $\omega_0 = 0.14$ rad/s for $\theta_0 = 5\frac{\pi}{6}$ rad



(e) $\omega_0 = 1.2$, $\omega_0 = 3$, $\omega_0 = 3$ rad/s and ,$\theta_0 = 5\frac{\pi}{6}$ rad



(f) $\theta_0 = \frac{5\pi}{6}$ $rad,$ and $\omega_0 = 0$ rad/s

Figure 5.8: The effect of the initial angular velocity on the system trajectory.

Figure 5.8-f shows the effect of the initial angular velocity on the attraction speed of the phase space trajectory of a non-linear free oscillating single pendulum modeled

by FELE. In this model, the initial displacement $\theta_0 = 5\pi/6$ *rad* is constant, whereas the initial angular velocity is changed. It's concluded that as the initial angular velocity ω_0 decreases, the attraction to the origin increases.

The phase space trajectories of the linearized FELE model of a free oscillating single pendulum system are affected by the R-L fractional integral order α as well. Fig. 5.9 shows the effect of the integral order α. It's illustrated, in the figure, that as α increases, the time of attraction to the origin decreases.

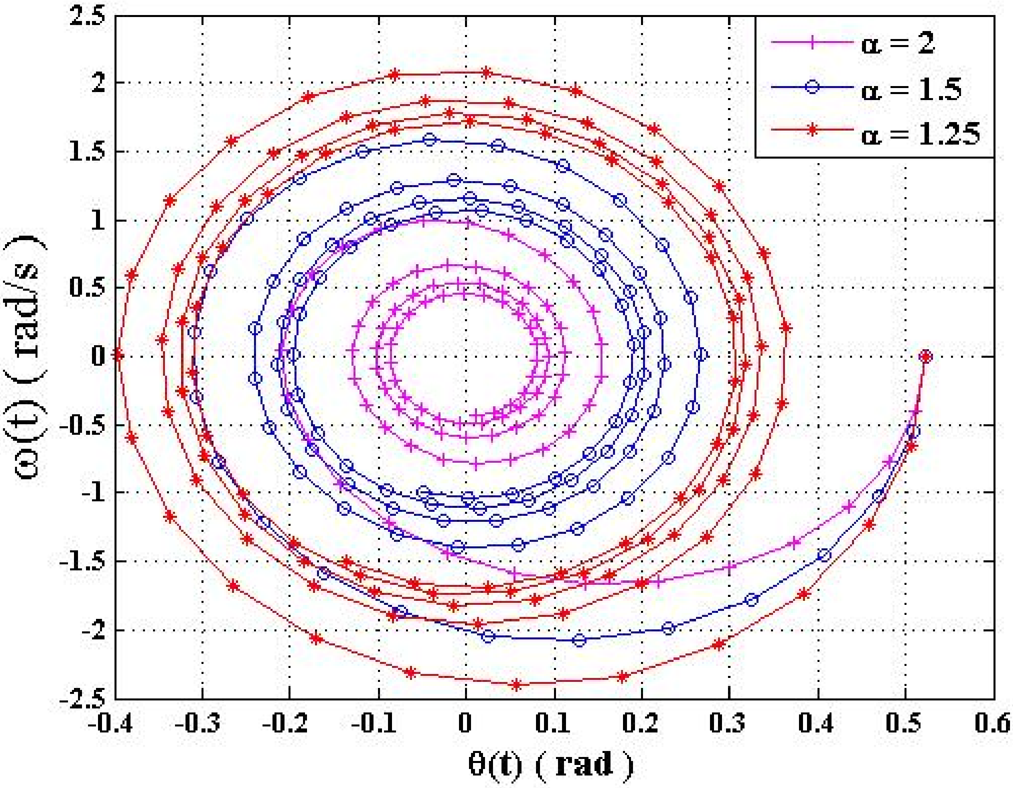


Figure 5.9: The effect of the R-L fractional integral order α on the phase space trajectories of of linearized free osillating single pendulum FELE model for $\theta_0 = \pi/6\ rad,\ and\ \omega_0 = 0\ rad/s$.

More general the effects of successive values of R-L fractional integral order α on the phase space trajectories of the linearized FELE model of a free oscillating single pendulum system are shown in Fig. 5.10. It's clear from the figure that the time of the attraction to the origin decreases, as α increases. This can be interpreted from the coefficient of the dissipation force which is given as a function of time. That functions proportional to α and inversely proportional to the time $t - \tau$. Thus, the effect of the dissipation force increases, as α increases at certain interval of observed time t.

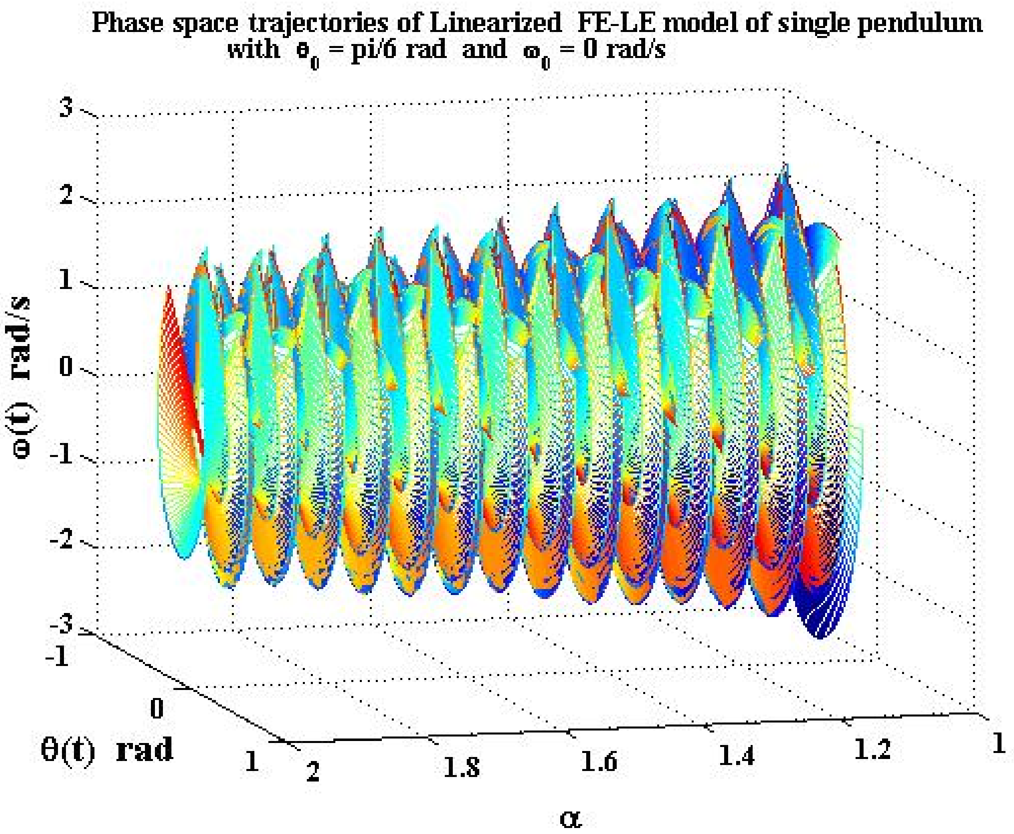


Figure 5.10: Three-dimensional representation of the effect of the R-L fractional integral order α on the phase space trajectories.

5.5 FELEs Modeling of Free Oscillating Double Pendulum System

In this section a free oscillating double pendulum system is considered with massless rods l_1 and l_2 connecting at the ends to masses m_1 and m_2, respectively. The angles of oscillating are given by α_1 and α_2 as illustrated in Fig. 5.11. Additionally, a case study of particular parameters of this system is studied and investigated as well.

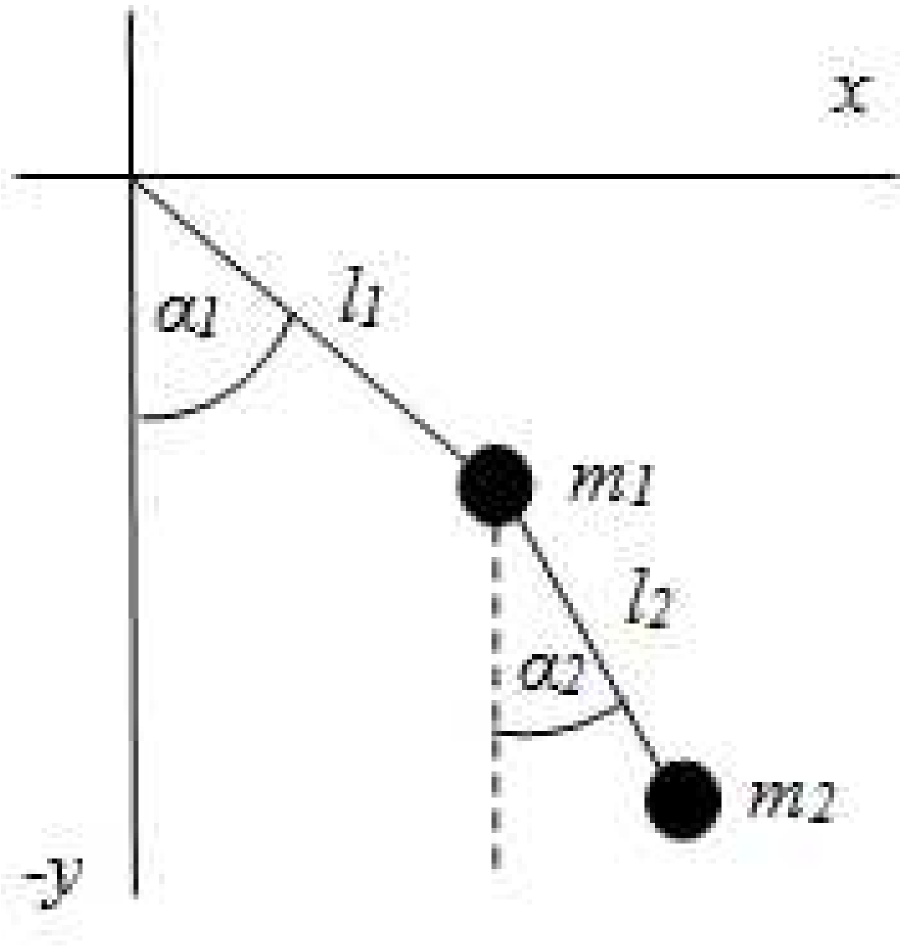


Figure 5.11: A free oscillating double pendulum system.

The fractional mathematical modeling of the given system can be obtained by using the FELEs that is given in Eq.(4.58). The Lagrangian L, in this model, is represented in terms of the kinetic and potential energies of the system.

The total kinetic energy and the total potential energy of the system can be given, respectively, as following:

$$T = \frac{1}{2}m_1\left[\dot{x}_1^2(t) + \dot{y}_1^2(t)\right] + \frac{1}{2}m_2\left[\dot{x}_2^2(t) + \dot{y}_2^2(t)\right] , \tag{5.22}$$

$$P = m_1 g y_1(t) + m_2 g y_2(t) \ , \tag{5.23}$$

where g is the gravitation, and $x_1(t)$, $x_2(t)$, $y_1(t)$, *and* $y_2(t)$ are the positions of the masses m_1 and m_2 according to the Cartesian coordinates X, and Y, respectively. These positions can be given as a function of $\alpha_1(t)$ and $\alpha_2(t)$, at any time, as following:

$$x_1(t) = \ l_1 \sin\alpha_1(t) \ , \tag{5.24a}$$

$$x_2(t) = \ l_1 \sin\alpha_1(t) + l_2 \sin\alpha_2(t) \ , \tag{5.24b}$$

$$y_1(t) = -l_1 \cos\alpha_1(t) \ , \tag{5.24c}$$

$$y_2(t) = -l_1 \cos\alpha_1(t) - l_2 \cos\alpha_2(t) \ . \tag{5.24d}$$

Substitute Eq.(5.24a) through Eq.(5.24d) into Eq.(5.22) and Eq.(3.23). The resultant equations of the total kinetic energy and the total potential energy, as a functions of rotation angles $\alpha_1(t)$ and $\alpha_2(t)$, are substituted into Eq.(4.15) to obtain the system Lagrangian L as follows:

$$\begin{aligned} L = \left(\frac{m_1}{2} + \frac{m_2}{2}\right) l_1^2 \dot{\alpha}_1^2 \ &+ \ \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) \\ &+ \ (m_1 + m_2) g l_1 \cos(\alpha_1) + m_2 g l_2 \cos(\alpha_2) \ , \end{aligned} \tag{5.25}$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are general coordinates of the system, Substitute the system Lagrangian L that is represented in the last equation into the FELE that is given in Eq.(4.58) to obtain the fractional mathematical model of the system as following:

$$-m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - (m_1 + m_2) g l_1 \sin(\alpha_1) - \frac{\alpha - 1}{t - \tau}(m_1 + m_2) l_1^2 \dot{\alpha}_1$$

$$\frac{\alpha - 1}{t - \tau} m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) - (m_1 + m_2) l_1^2 \ddot{\alpha}_1 - m_2 l_1 l_2$$

$$[\cos(\alpha_1 - \alpha_2)\ddot{\alpha}_2 - \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2)(\dot{\alpha}_1 - \dot{\alpha}_2)] = 0 \ , \tag{5.26a}$$

$$m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - m_2 g l_1 \sin(\alpha_2) - \frac{\alpha - 1}{t - \tau} m_2 l_2^2 \dot{\alpha}_2 - \frac{\alpha - 1}{t - \tau} m_2 l_1 l_2 \dot{\alpha}_2$$

$$\cos(\alpha_1 - \alpha_2) - m_2 l_2^2 \ddot{\alpha}_2 - m_2 l_1 l_2$$

$$[\cos(\alpha_1 - \alpha_2)\ddot{\alpha}_1 - \dot{\alpha}_1 \sin(\alpha_1 - \alpha_2)(\dot{\alpha}_1 - \dot{\alpha}_2)] = 0 \ . \tag{5.26b}$$

The system that is expressed by Eq.(5.26a) and Eq.(5.26b) represents the nonlinear mathematical modeling of free oscillating double pendulum system.

5.5.1 The Linearization of Double Pendulum FELEs Model

Consider small oscillations of the pendulums by choosing the oscillating of the general coordinates $\alpha_1(t)$ and $\alpha_2(t)$ about the zero equilibrium points, where both angels are measured with respect to the negative direction of y-axis. The nonlinear Lagrangian L that is given by Eq.(5.25) can be linearized by using Maclaurin series expansion [145], so the trigonometric functions can be replaced by the following approximated expansions:

$$\cos(\alpha_i) \simeq 1 - \frac{\alpha_i^2}{2} \ , i = 1, 2/; , \tag{5.27a}$$

$$\cos(\alpha_1 - \alpha_2) \simeq 1 \ . \tag{5.27b}$$

Substitute Eq.(5.27a) and Eq.(5.27b) into Eq.(5.25) to obtain the linearized Lagrangian of the system as following:

$$
\begin{aligned}
L &= \left(\frac{m_1}{2}+\frac{m_2}{2}\right) l_1^2\dot{\alpha}_1^2 + m_2 l_1 l_2 \dot{\alpha}_1\dot{\alpha}_2 \\
&+ \left(m_1+m_2\right) g l_1 \left(1-\frac{\alpha_1^2}{2}\right) + m_2 g l_2 \left(1-\frac{\alpha_1^2}{2}\right) .
\end{aligned}
\tag{5.28}
$$

Equation (5.28) represents the linearized Lagrangian L of the free oscillating double pendulum system. Substitute the system Lagrangian L that is obtained in the Eq.(5.28) into the FELE that is given by Eq.(4.58) to obtain the fractional mathematical modeling of the Linearized system as following:

$$
\begin{aligned}
&\begin{bmatrix} -\left(m_1+m_2\right) l_1^2 & -m_2 l_1 l_2 \\ -m_2 l_1 l_2 & -m_2 l_2^2 \end{bmatrix}
\begin{bmatrix} \ddot{\alpha}_1 \\ \ddot{\alpha}_2 \end{bmatrix}
+ \begin{bmatrix} -\frac{\alpha-1}{t-\tau}\left(m_1+m_2\right) l_1^2 & -\frac{\alpha-1}{t-\tau} m_2 l_1 l_2 \\ -\frac{\alpha-1}{t-\tau} m_2 l_1 l_2 & -\frac{\alpha-1}{t-\tau} - m_2 l_2^2 \end{bmatrix}
\begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{bmatrix} \\
&+ \begin{bmatrix} -\left(m_1+m_2\right) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}
\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix} .
\end{aligned}
\tag{5.29}
$$

***CaseStudy* 5.2**

Consider the given double pendulum that is shown in Fig. 5.11, with the following particular parameters: $l_1 = 0.335\ m,\ l_2 = 0.225\ m,\ m_1 = 0.125\ kg,\ and\ m_2 = 0.085\ kg$, and with initial conditions,

$$
\alpha_{1_0} = \frac{\pi}{6}, \quad \dot{\alpha_{1_0}} = 0 \ , \tag{5.30a}
$$

$$
\alpha_{2_0} = \frac{\pi}{18}, \quad \dot{\alpha_{2_0}} = 0 \ . \tag{5.30b}
$$

The modal Matrix of the system can be obtained as,

$$
\phi = \begin{bmatrix} 4.467i & -7.1645i \\ 6.2846i & 18.7328i \end{bmatrix} . \tag{5.31}
$$

Consider the initial conditions, according the principle coordinates, the equations of motion of the system can be given by the following equations:

$$\ddot{\eta}_1(t) + \frac{\alpha - 1}{t - \tau}\dot{\eta}_1(t) + 21.167\eta_1(t) = 0 \,, \tag{5.32a}$$

$$\ddot{\eta}_2(t) + \frac{\alpha - 1}{t - \tau}\dot{\eta}_2(t) + 101.1589\eta_2(t) = 0 \,. \tag{5.32b}$$

The responses of the system according to the general coordinates $\alpha_1(t)$, and $\alpha_2(t)$ can be obtained as follows:

$$[\alpha] = [\phi][\eta] \,, \tag{5.33}$$

where $[\alpha]$ represents the vector of the general coordinates $\alpha_1(t)$ and $\alpha_2(t)$ of the given system, $[\phi]$ is the modal matrix of the system, and $[\eta]$ represents the vector of the system principle coordinates $\eta_1(t)$ and $\eta_2(t)$. The responses $\alpha_1(t)$ and $\alpha_2(t)$ of the linearized free oscillating double pendulum system that modeled by FELE are illustrated in Fig. 5.12, and Fig. 5.13.

Figure 5.12 shows the responses of the system with R-L fractional order integral $\alpha = 1$, that means, according to Eqs.(5.32), no dissipation force applied to the system. This also concluded from the figure where there is no gradual fade out of the system oscillations. In Fig. 5.13 the R-L fractional order integral $\alpha = 2$, so the dissipative force is exist as a function of time. It's inferred from the figure that the dissipative force is gradually decreased, as the observed time increases. That is because the dissipative force coefficients in Eq.(5.32) is inversely proportional to the time $t - \tau$.

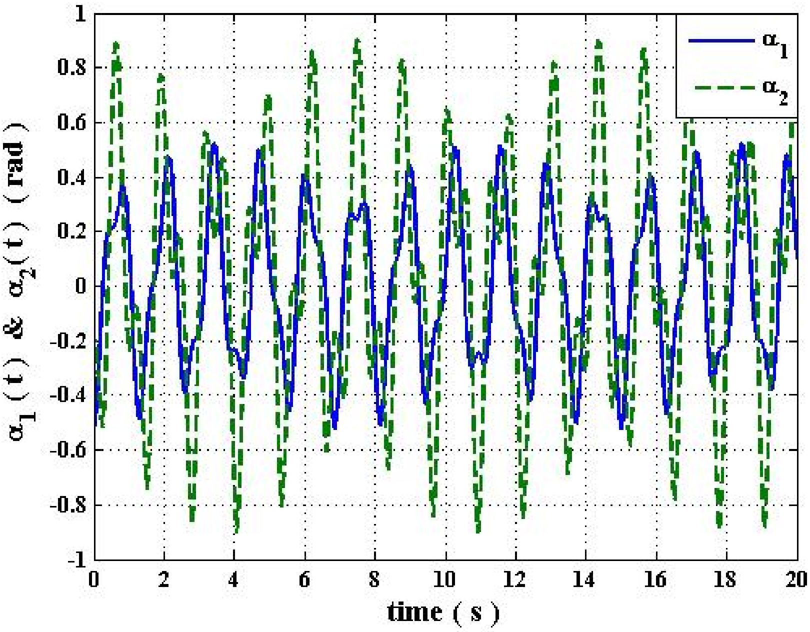


Figure 5.12: A free oscillating linearized double pendulum system modeled by FELE for R-L fractional order integral $\alpha = 1$.

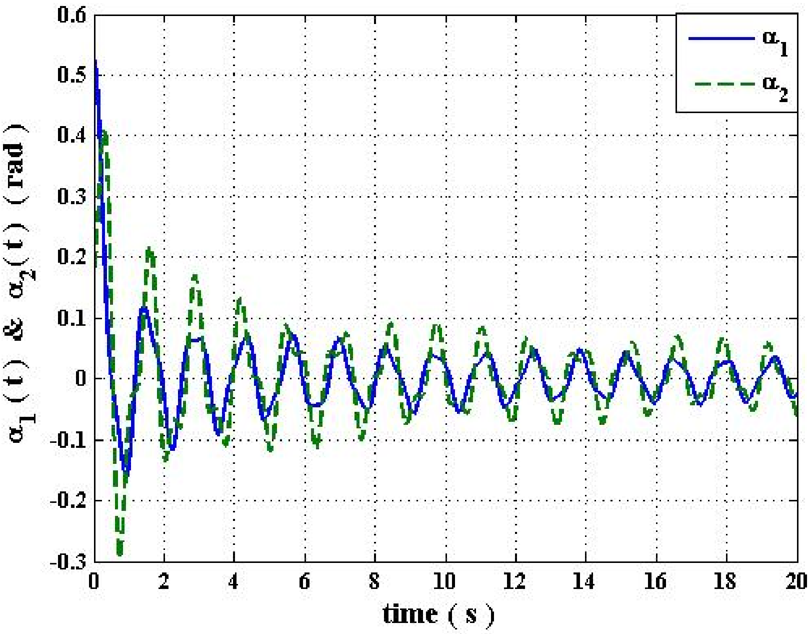
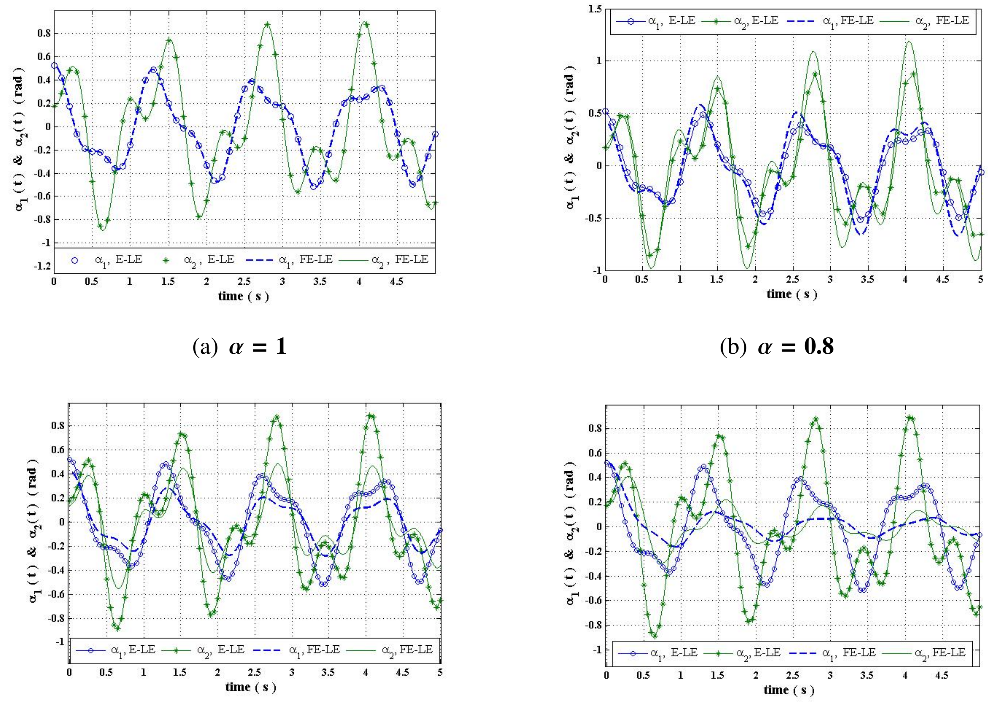


Figure 5.13: A free oscillating linearized double pendulum system modeled by FELE with R-L fractional order integral $\alpha = 2$.

A comparison between the non-dissipative traditional ELE model and FELE model of the given free oscillating double pendulum system is depicted in Fig. 5.14. The two models are considered for different values of the R-L fractional integral order α. It's concluded from Fig. 5.14-a that the responses of the corresponding general coordinates of the system are identical to each other for $\alpha = 1$. That means there are no dissipation forces in both models. The other subfigures Fig. 5.14-b through Fig. 5.14-d show that as the R-L fractional order integral α is far from one, the deviations between the two models increases. That is due to the effect of the damping coefficient $\dfrac{\alpha - 1}{t - \tau}$ on the FELE model of the system.



(a) $\boldsymbol{\alpha = 1}$

(b) $\boldsymbol{\alpha = 0.8}$

(c) $\boldsymbol{\alpha = 1.2}$

(d) $\boldsymbol{\alpha = 2}$

Figure 5.14: A comparison between the non-dissipative traditional ELE model and FELE model of a free oscillating double pendulum system for different values of α.

5.6 The physical Interpretation of the Fractional Integral Order α

The coefficient of the damping force generated by applying the FELEs is a function of the R-L fractional integral order α, and the time which consists of intrinsic time τ and observing time t. In a particular case, when α and τ are constants, the damping force coefficient can be given as a function of time t as following:

$$C(t) = \frac{\alpha - 1}{t - \tau}, \qquad t \neq \tau, \tag{5.34}$$

Figure 5.15 shows the damping force coefficient as a function of time for small value of intrinsic time τ and a certain value of the order α ($\alpha = 2$). It's noticeable that the damping force gradually decreases, as the time increases. This is obviously noticeable in responses of the FELE models of the single and double pendulum systems, which are studied section 5.4, and section 5.5, respectively.

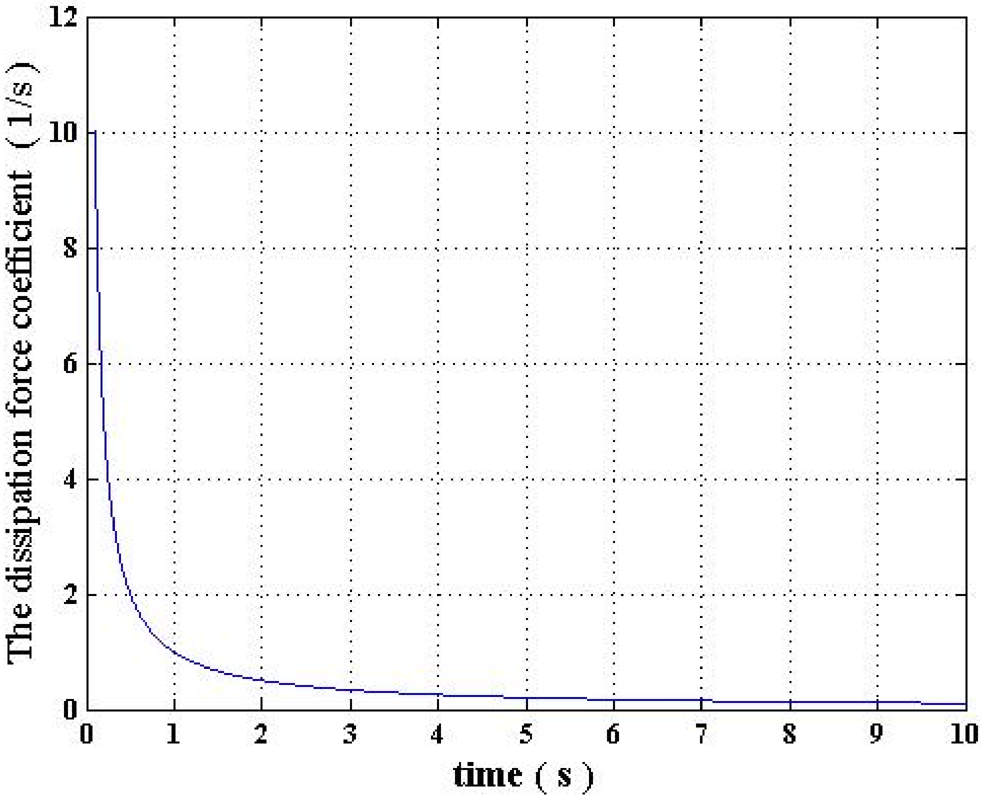


Figure 5.15: The damping force coefficient of a system modeled by FELE as a function of time for $\alpha = 2$.

In the brush disk sliding friction [47], the friction damping force is inversely proportional to the distance between the brush and the disk. Consider the results of this experimental study and the FELE modeling of the system. The R-L fractional integral order α can be identified as a friction force coefficient between the brush and the disk. Another experimental study was done [65]. A variable stiffness Magneto-Rheological MR fluid-based damper for vibration suppression was considered. In this experiment the damping force coefficient is given as a function of the magnetic field that is supplied to the MR Valve. The magnetic field produced by a controllable DC power. The viscosity of the liquid can be changed depending on the magnetic field intensity. Thus the viscosity of the MR fluid is proportional to the amount of current applied to the damper [71]. therefore if the DC power supply is given as a function of time then the damping force be a function of time as well. Since the viscosity of the MR fluid is affected by the magnetic field, then the R-L fractional order α in Eq.(5.34), can be identified as a variable. This variable is contingent upon the viscosity of the MR fluid.

CHAPTER 6

THE RESPONSES of a FREELY OSCILLATING FRACTIONAL DYNAMIC SYSTEM

6.1 Homogeneous Fractional Order Differential Equations

The two-term FODEs are considered as starting of the fractional order ordinary differential equations study, since the fractional calculus definitions have been founded. These equations include the relaxation and oscillation, then continuum to more term equations. The fractional calculus definitions are considered as beginning of the fractional-order linear time invariant systems study [18].

The non-homogeneous FODEs are discussed in chapter three. The solutions of these equations were obtained and evaluated by two methods. These techniques are the Bagley-Trovik Equation, and the Laplace Transform of M-L function. In this chapter the three-term homogenous FODEs are considered. The general form of these equations which is given by Eq.(6.1) includes the fractional order in all their terms operators [18, 12]. In this chapter two main cases are considered.

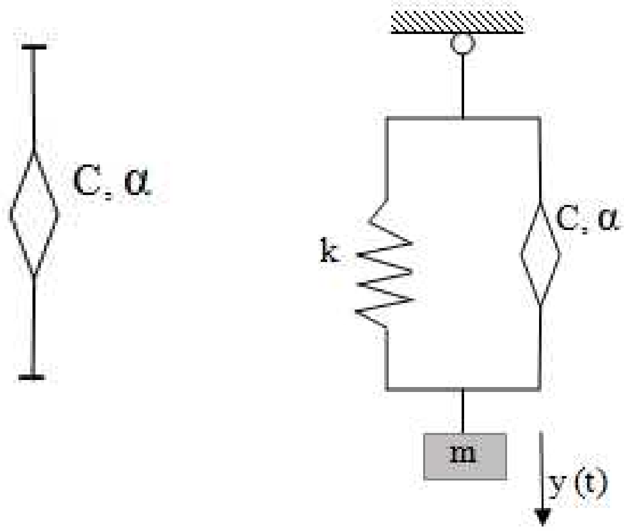
In the first case the fractional order is introduced only in the damping term, where the fractional order of the first term $\beta = 2$, the fractional order of the third term $\gamma = 0$, and all coefficients are constants.

$$a_2\left(t\right)\ {}_0D_t^{\beta} y\left(t\right) + a_1\left(t\right)\ {}_0D_t^{\alpha} y\left(t\right) + a_0\left(t\right)\ {}_0D_t^{\gamma} y\left(t\right) = 0\ , \tag{6.1}$$

where, t is the observer time, $a_i\left(t\right), i = 0, 1, 2$ are coefficients, and α; β; and γ are the orders of fractional derivative operator D.

The second case is the extended FELEs model in which the the first and the second terms of a three-term FODE are subject to fractional order.

For $\beta = 2$ and the fractional order of the third term $\gamma = 0$ the system defined by Eq.(6.1) represents a freely oscillating fractional order dissipating system. In this system the damping force is proportional to the fractional derivative of the displacement. The element that represents the damping force is a springpot (rheological element for viscoelasticity) [24]. This element is illustrated in Fig. 6.1-a. The schematic diagram of the whole fractional oscillating system is given in the Kelvin-Voigt model, [24, 30, 36, 42] as depicted in Fig. 6.1-b. The system generates a spring mass springpot model. This system is considered as a good representation for real models, such as an earthquake model [48].



(a) ***Springpot element*** (b) ***Kelvin – Voigt model***

Figure 6.1: The schematic diagram of springpot element and the Kelvin-Voigt model [36, 42, 48].

The free oscillating spring mass springpot system can be represented by a three-term homogenous fractional damping differential equation as declared in the following equation:

$$m\ {}_0D_t^2 y(t) + c\ {}_0D_t^{\alpha} y(t) + k\ y(t) = 0\ , \tag{6.2}$$

where $t \geq 0,\quad 0 \leq \alpha \leq 2$, m is the mass, c is the springpot coefficient, and k is the stiffness coefficient of the spring. Because a vibrating system is considered, the first term derivative order must be 2, so the fractional order of the first term of Eq.(6.2) $\beta = 2$.

An analytical method is introduced to obtain the solution of Eq.(6.1). The obtained solution is compared with the solution of one of previous methods that obtain the solution of such equation. Those methods give explicit solutions to Eq.(6.1) based on the R-L fractional derivatives, the M-L functions, and the Wright functions [21, 44, 66, 72]. An investigation of the analytical solution that is obtained by using the Wright function is done, and its solution is compared with the solution of the introduced method.

6.2 An Analytical Method to Obtain the Response of a Free Oscillating Fractional Damping Vibrating System

The Wright function or the nth derivative of the M-L function [44, 93], which is given in Eq.(6.4) is used to obtain the solution of the system given by Eq.(6.2). The solution of the system, based on these functions, is represented in Eq.(6.3). This solution is valid for $t \geq 0,\ 0 \leq \alpha \leq 2$. By means of these functions, the time domain response of the above system can be obtained as follows [66]:

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} \frac{\left(-\frac{k}{m}\right)^n}{n!} t^{2n}\ {}_1\Psi_1\left[\left\| \begin{matrix} (n+1,1) \\ (2n+1, 2-\alpha) \end{matrix} \right\| \left(-\frac{c}{m} t^{2-\alpha}\right)\right] + \frac{c}{m} \\ & \sum_{n=0}^{\infty} \frac{\left(-\frac{k}{m}\right)^n}{n!} t^{(2n+2-\alpha)}\ {}_1\Psi_1\left[\left\| \begin{matrix} (n+1,1) \\ (2n+3-\alpha, 2-\alpha) \end{matrix} \right\| \left(-\frac{c}{m} t^{2-\alpha}\right)\right] \end{aligned} \tag{6.3}$$

The general formula of the Wright function ${}_p\Psi_q$ is given by Eq.(2.10). As a particular case for $p = q = 1$ the Wright function ${}_1\Psi_1$ can be declared as follows [44]:

$$
{}_1\Psi_1\left[\left|\begin{matrix}(n+1,1)\\(\alpha n+\beta,\alpha)\end{matrix}\right|(z)\right] = \sum_{j=0}^{\infty}\frac{\Gamma(n+j+1)}{\Gamma(\alpha n+\beta+\alpha j)}\frac{z^j}{j!}
$$

$$
= \frac{\partial^n}{\partial z^n}E_{\alpha,\beta}(z)\ . \tag{6.4}
$$

where α, and β (in our study $\beta = 2$) are the derivative operator orders, and the function z can be a function of time. In our system, this function is define as $z = -\frac{c}{m}t^{2-\alpha}$. The Wright function can be given as $\frac{\partial^n}{\partial z^n}E_{\alpha',\beta'}(z)$. This expression is the n^{th} partial derivative of the two-term M-L function of the function z. The Wright function that is introduced in Eq.(6.3) can be written as following:

$$
{}_1\Psi_1\left[\left|\begin{matrix}(n+1,1)\\(2n+1,2-\alpha)\end{matrix}\right|\left(-\frac{c}{m}t^{2-\alpha}\right)\right] = \sum_{j=0}^{\infty}\frac{\Gamma(n+j+1)}{\Gamma(2n+1+(2-\alpha)j)}\frac{\left(-\frac{c}{m}t^{2-\alpha}\right)^j}{j!} \tag{6.5a}
$$

$$
{}_1\Psi_1\left[\left|\begin{matrix}(n+1,1)\\(2n+3-\alpha,2-\alpha)\end{matrix}\right|\left(-\frac{c}{m}t^{2-\alpha}\right)\right] = \sum_{j=0}^{\infty}\frac{\Gamma(n+j+1)}{\Gamma(2n+3-\alpha+(2-\alpha)j)}\frac{\left(-\frac{c}{m}t^{2-\alpha}\right)^j}{j!} \tag{6.5b}
$$

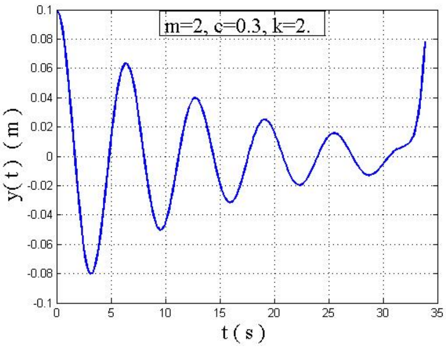
Example 6.1 Consider a free fractional oscillating system with a mass $m = 2\ kg$, a dissipative coefficient $c = 0.3\ \frac{N.s^{\alpha}}{m^{\alpha}}$, a stiffness $k = 2\ \frac{N}{m}$, and a derivative order of the dissipative force $\alpha = 1.1$. This system can be represented by the following equation:

$$
2\ {}_0D_t^2 y(t) + 0.3\ {}_0D_t^{1.1} y(t) + 2\ y(t) = 0\ . \tag{6.6}
$$

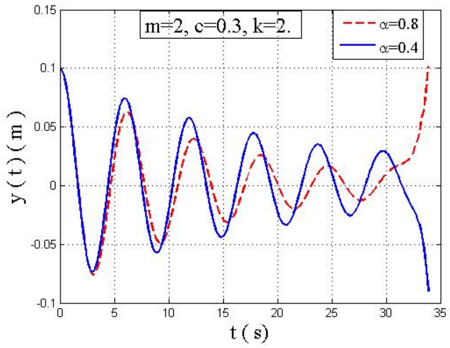
The response of the system, according to the solution that is given by Eq.(6.3), is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}\ {}_1\Psi_1\left[\left|\begin{matrix}(n+1,1)\\(2n+1,0.9)\end{matrix}\right|\left(-\frac{0.3}{2}t^{0.9}\right)\right] + \frac{0.3}{2}\\
&\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{(2n+0.9)}\ {}_1\Psi_1\left[\left|\begin{matrix}(n+1,1)\\(2n+1.9,0.9)\end{matrix}\right|\left(-\frac{0.3}{2}t^{0.9}\right)\right]
\end{aligned} \tag{6.7}
$$

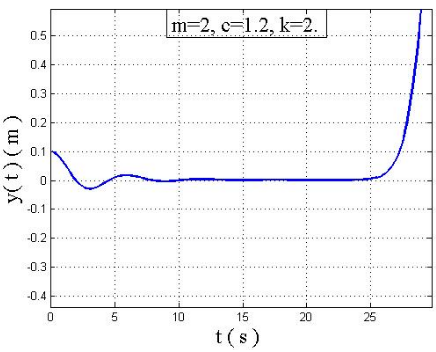
The responses of the system, with the given system parameters values, are depicted in Fig. 6.2.a, whereas the other subfigures represent the responses of the same system for different system parameters.



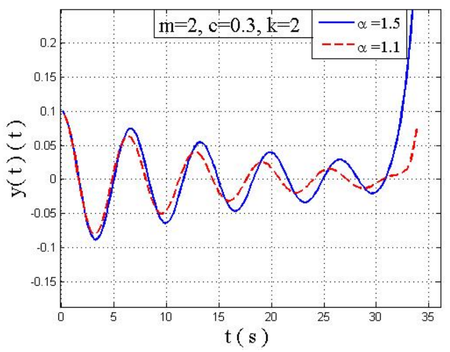
(a) α = **1.1**



(b) α = **0.8** ***versus*** α = **0.4**



(c) α = **0.8,** ***and*** C = **1.2**



(d) α = **1.5** ***versus***. α = **1.1** ***and*** C = **0.3**

Figure 6.2: Responses of a mass spring springpot system by using Wright function for different system parameters values.

Figure 6.2 illustrates the responses of the system that is given by Eq.(6.6). The responses which are expressed in time domain are obtained by Eq.(6.3). The indexes of the solution n, and j are truncated at high values; however, both indexes n, and j go to infinity. The values of indexes truncations are just about eighty. That is because of the capabilities of the computation facilities that are used to obtain the responses. Therefore, the truncations cannot be increased more. If the truncations increase more than eighty, the responses go to infinity before reach the steady state.

The subfigure Fig. 6.2-a is plotted for a fractional damping force coefficient $c = 0.3\ \dfrac{N.s^{1.1}}{m^{1.1}}$, a mass $m = 2\ kg$, a stiffness $k = 2\ \dfrac{N}{m}$, and a fractional derivative order of damping force $\alpha = 1.1$. For the other subfigures, the parameters values are changed and the responses changed as well according to the given system parameters. In comparison between the subfigures Fig. 6.2-b and Fig. 6.2-c, the effect of the damping force coefficient is clear. It's obviously from the comparison of the subfigures that the system response with damping force coefficient $c = 0.3\ \dfrac{N.s^{0.8}}{m^{0.8}}$ represents under damped system response, whereas the system response with damping force coefficient $c = 1.2\ \dfrac{N.s^{0.8}}{m^{0.8}}$ is close to the critical damped response. Moreover the subfigures Fig. 6.2-a and Fig. 6.2-d illustrate the effect of the fractional order of the damping force derivative operator α in the second term of the system. It's concluded from the subfigures that as the fractional order α is taken far from one, the overshoot of the system response and the settling time increase.

6.3 Analytical and Graphical Comparisons between R-L and Direct Method Fractional Derivative of the Exponential Function e^{at}

There are many methods to express the fractional derivative of the exponential functions such as Caputo fractional derivative, R-L fractional derivative, etc. This section discuses the comparison between the R-L fractional derivatives and the Direct method solution, ${}_0D_t^{\alpha} e^{at} = a^{\alpha} e^{at}$, of the fractional derivative of exponential functions.

The fractional derivative of the exponential functions can be done, just by, identical to the integer order derivative such as the following:

$$\frac{d^n}{dt^n} e^{at} = a^n e^{at} \ , \tag{6.8}$$

where n is the derivative integer order, $n \in \mathbb{N}$. Comparing with fractional order derivative for the same function, one can write the following [77, 82, 86]:

$$\begin{aligned} \frac{d^n}{dt^n} e^{at} &= {}_0D_t^{\alpha} e^{at} \\ &= a^{\alpha} e^{at}, \end{aligned} \tag{6.9}$$

where α is the fractional order of the derivative, $\alpha \in \mathbb{R}$.

Proof.

Additional to the investigations that were introduced in chapter 2 more emphasize is done in this chapter in order to evaluate the two methods of fractional derivatives. Since all the range of the R-L fractional derivatives have the same logic, let's start with the R-L fractional derivative, for $n = 2$, of the exponential function as follows:

$$ {}_0D_t^{\alpha}e^{at} = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dt^2}\int_0^t \frac{e^{a\tau}}{(t-\tau)^{\alpha+1-2}}d\tau, \quad \alpha > 0. \tag{6.10} $$

Consider $\Gamma(n) = (n-1)!$, and derive the R-L fractional derivative of the exponential function that given in the last equation to obtain

$$ {}_0D_t^{\alpha}e^{at} = \frac{t^{-\alpha}}{(-\alpha)!} + \frac{at^{1-\alpha}}{(1-\alpha)!} + \frac{a^2t^{2-\alpha}}{(2-\alpha)!} + \frac{a^3t^{3-\alpha}}{(3-\alpha)!} + ... + \frac{a^kt^{k-\alpha}}{(k-\alpha)!} + ... + \frac{a^pt^{p-\alpha}}{(p-\alpha)!}. \tag{6.11} $$

The above equation can be rewritten as a series with respect to p. It is noticeable from Eq.(6.11) that as $p \to \infty$, α can be neglected. Equation (6.11) represents the end result of the R-L fractional derivative of the exponential function, where $p \to \infty$. A variable k can be introduced as a certain truncation of the index p. On other hand, from the right hand side of Eq.(6.9) the following series can be expressed:

$$ a^{\alpha}\sum_{p=0}^{\infty}\frac{(at)^{p-\alpha}}{(p-\alpha)!} = \frac{t^{-\alpha}}{(-\alpha)!} + \frac{at^{1-\alpha}}{(1-\alpha)!} + \frac{a^2t^{2-\alpha}}{(2-\alpha)!} + \frac{a^3t^{3-\alpha}}{(3-\alpha)!} + ... \tag{6.12} $$

It is concluded from Eq.(6.11) and Eq.(6.12) that when a truncation k is taken for p in Eq.(6.12), both equations are equal to each other, so the following can be written:

$$ a^{\alpha}\sum_{p=0}^{\infty}\frac{(at)^{p-\alpha}}{(p-\alpha)!} = {}_0D_t^{\alpha}e^{at} \tag{6.13} $$

In Eq.(6.13) as $p \to \infty$, α can be neglected in the summation, and the last equation can be rewritten as follows:

$$ a^{\alpha}\sum_{p=0}^{\infty}\frac{(at)^{p}}{(p)!} = {}_0D_t^{\alpha}e^{at} \tag{6.14} $$

Because the summation is just a Taylor series expansion of the exponential function [89], so the above equation can be rewritten again as follows:

$$
\begin{aligned}
{}_0D_t^{\alpha} e^{at} &= a^{\alpha} \sum_{p=0}^{\infty} \frac{(at)^p}{(p)!} \\
&= a^{\alpha} e^{at}, \qquad (6.15)
\end{aligned}
$$

In conclusion, from the above proof, the R-L fractional Derivative of exponential function and the fractional derivative that is obtained by the Direct method ${}_0D_t^{\alpha} e^{at} = a^{\alpha} e^{at}$ are equal to each other, when the truncation of p is high with respect to α. A graphical investigation for certain values of the coefficient a, and the order α can verify the equality between the R-L fractional derivative, and the Direct fractional derivative ${}_0D_t^{\alpha} e^{at} = a^{\alpha} e^{at}$ as illustrated in Fig. 6.3.

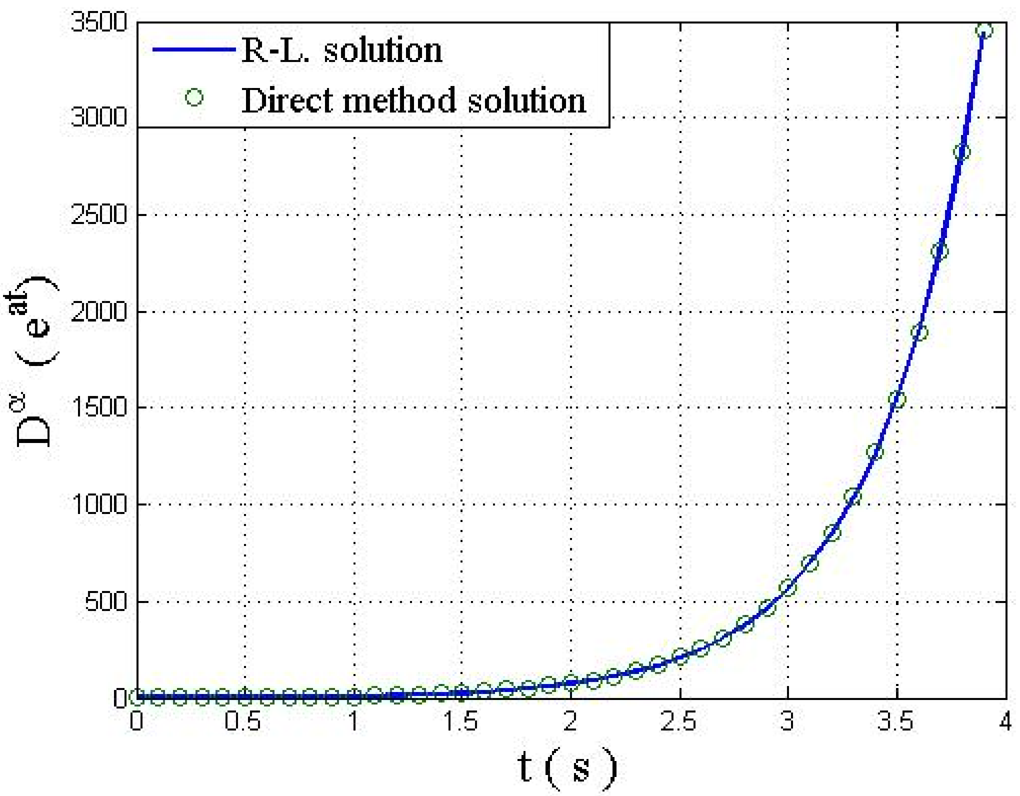


Figure 6.3: comparison between R-L and Direct method fractional derivatives of an exponential function for $\alpha = 0.5$.

6.4 A Hybrid Method to Obtain the Response of a Freely Oscillating Fractional Dynamical System

Second order homogenous differential equations with fractional order dissipation term can be easily and fast solved by using the fractional derivative direct method of the exponential function. By exploiting the relationship between the trigonometric functions and the exponential functions the following can be satisfactory:

$$ {}_0D_t^{\alpha} \sin(t) = \sin\left(t + \frac{\alpha\pi}{2}\right). \tag{6.16} $$

By exploiting Eq.(6.15), and Eq.(6.16) the solutions of second order homogenous differential equations with fractional order dissipation term can be obtained. This is done by determining the characteristic equation of the fractional order system, evaluating the stability of the system, and then obtaining the corresponding responses of the system.

6.4.1 Characteristic Equation of the Fractional Order System

Starting with the free oscillating fractional damping vibrating system that is represented by Eq.(6.2), with initial conditions

$$ y(0) = y_0 \tag{6.17a} $$

$$ {}_0D_t^{\alpha} y(0) = \left(\frac{d^{\alpha}y}{dt^{\alpha}}\right)_0 \tag{6.17b} $$

The fractional characteristic equation of the system can be written as follows [79, 96]:

$$ ms^2 + cs^{\alpha} + k = 0\ , \tag{6.18} $$

where α is a rational number, and can be expressed as $\left(\frac{v}{u}\right)$ [96, 99]. In order to obtain the roots of the characteristic equation of the system, we have to determine the Least Common Multiple (ℓ) of the characteristic equation exponents [96, 102]. Therefore, the above characteristic equation is rewritten in terms of ℓ [102] as follows:

$$ms^{\frac{v_1}{\ell}} + cs^{\frac{v_2}{\ell}} + k = 0 \ , \tag{6.19}$$

where v_1 and $v_2 \in \mathbb{N}$.

Map the s-domain into w-domain, which is also a complex domain, [18, 106, 109] by means of the following relationship:

$$w = s^{\frac{1}{\ell}} \ . \tag{6.20}$$

Substitute Eq.(6.20) into Eq.(6.19) to obtain the w-domain characteristic equation [96, 113] as following:

$$mw^{v_1} + cw^{v_2} + k = 0 \ , \tag{6.21}$$

The order of the above equation is max $[v_i]\ :\ i = 1, 2, 3, \ldots$, [113] which usually is high. Thus the number of the roots of w-domain characteristic equation is high as well. A computer program can be used to obtain them.

6.4.2 Stability of the Fractional Order System

Stability is one of the important properties of a dynamic system, which can be analyzed by means of different methods [116]. The external stability, in time domain, can be measured by using the concept bounded input-bounded output (BIBO) [120]. The system is BIBO-stable if any bounded input results with bounded output [122]. A linear time invariant (LTI) system with impulse response is stable, if the following necessary and sufficient condition is satisfied [122]:

$$\int_0^{\infty} |h(\tau)|\, d\tau < \infty, \tag{6.22}$$

where $h(\tau)$ is impulse response, and the output of the system is defined as follows [96]:

$$\begin{aligned} y(t) &= h(t) * u(t) \\ &= \int_0^{\infty} h(t)\, u(t-\tau)\, d\tau \,. \end{aligned} \tag{6.23}$$

The stability can be also analyzed via s-domain by taking the transfer function of the open loop system. This can be done by obtaining the characteristic equation of the system by means of s-domain, and transforming the s-plane into w-plane [116]. This technique of stability analysis is used when dealing with fractional order system [96], such as the system that is given by Eq.(6.2). This analysis can be accomplished by mapping the s-domain into w-domain system characteristic equation. Once the roots of Eq.(6.21) are obtained, the system stability can be analyzed by considering the w-plane, and the Riemann surface of the system. The Riemann surface of the system can be plotted and whose sheets' number is defined to be equal to ℓ [125]. For instance, The Riemann surface of a system characteristic equation with $\ell = 10$ is illustrated in Fig. 6.4. As the figure shows there are ten sheets in this Riemann surface.

Only the first sheet is a significance sheet, in which the stability of the system is analyzed according to the w-plane. The other sheets are located in the Non-physical region. The location of the first sheet is defined as $-\dfrac{\pi}{\ell} < \phi < \dfrac{\pi}{\ell}$ [106]. The unstable region is defined inside the first sheet of the Riemann surface [106], and located in the region $-\dfrac{\pi}{2\ell} < \phi < \dfrac{\pi}{2\ell}$ as illustrated in Fig. 6.5. This figure represents the w-plane of the characteristic equation of the fractional order system.

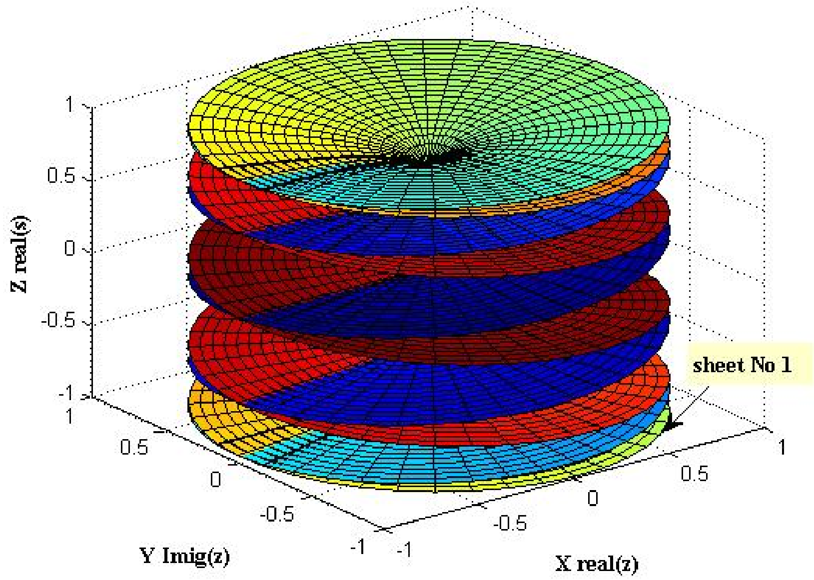


Figure 6.4: Ten-sheet Riemann surface of characteristic equation with $\ell = 10$.

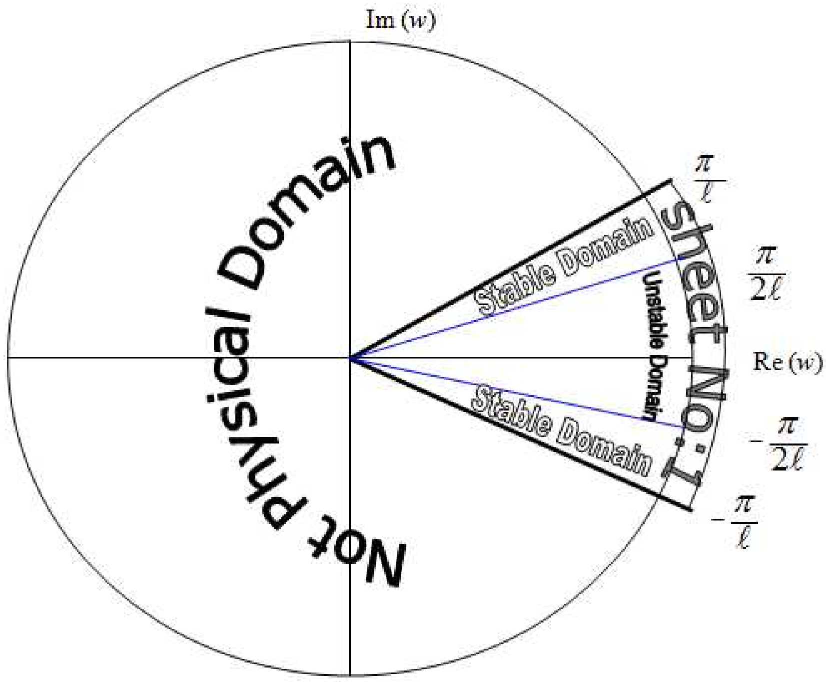


Figure 6.5: w-plane of fractional order system characteristic equation.

6.4.3 General Solution of the Fractional Order System

To check the stability, map the roots again into the s-domain by using the relationship $s = w^{\ell}$. The obtained roots are s-domain roots of the system open loop characteristic equation. This s-domain roots will be considered in the system solution. If the system is stable, then only one root, in w-plane, locates in the physical stable region. The corresponding s-domain root to that root is selected to be the root $((\alpha_1 \pm \beta_1 i)$ that is substituted in the solution of second order homogeneous differential equation with given conditions. This solution can be given as follows:

$$y(t) = e^{\alpha_1 t}\left(c_1 \cos(\beta_1 t) + c_2 \sin(\beta_1 t)\right) , \tag{6.24}$$

where the roots $(\alpha_1 \pm \beta_1 i)$, represents the roots of the characteristic equation of the open loop system in s-domain. The following example explains the introduced method with stability analysis to obtain the response of free oscillating fractional damping system.

Example 6.2
Consider the free oscillating fractional system that is modeled by Eq.(6.2), with the following system parameters: mass $m = 2\ kg$, $c = 0.4\ \dfrac{N.s^{0.8}}{m^{0.8}}$, $k = 2\ \dfrac{N}{m}$, and $\alpha = 0.8$. The system is subjected to initial conditions

$$y(0) = y_0 \tag{6.25a}$$

$${}_0D_t^{\alpha} y(0) = \left(\frac{d^{\alpha}y}{dt^{\alpha}}\right)_0 . \tag{6.25b}$$

The characteristic equation of the system is expressed as follows:

$$2s^2 + 0.4s^{0.8} + 2 = 0. \tag{6.26}$$

The fractional derivative order α can be written as rational number $\dfrac{v}{u} = \dfrac{8}{10}$. Therefore, ℓ of the denominators of exponents 2, and $\dfrac{8}{10}$ is $\ell = 5$. Consider ℓ to rewrite the above characteristic equation that given by Eq.(6.26) as following:

$$2s^{\frac{10}{5}} + 0.4s^{\frac{4}{5}} + 2 = 0. \tag{6.27}$$

The s-plane is mapped into w-plane, where $w = s^{\frac{1}{5}}$, to obtain

$$2w^{10} + 0.4w^4 + 2 = 0. \tag{6.28}$$

Now, let's obtain the roots of w-domain characteristic equation. The roots in w-plane and the absolute values of corresponding arguments (angles in rad) are illustrated in the following table:

Table 6.1: The w-domain roots and the corresponding absolute values of arguments of w-domain characteristic equation.

i	w_i	$\lvert arg(w_i) \rvert$
1	$-0.9513 + 0.3290i$	2.8086
2	$-0.9513 - 0.3290i$	2.8086
3	$-0.5879 + 0.7889i$	2.2112
4	$-0.5879 - 0.7889i$	2.2112
5	$0.0000 + 1.0198i$	1.5708
6	$0.0000 - 1.0198i$	1.5708
7	***0.9513 + 0.3290i***	***0.3330***
8	***0.9513 - 0.3290i***	***0.3330***
9	$0.5879 + 0.7889i$	0.9304
10	$0.5879 - 0.7889i$	0.9304

The stability of the system can be analyzed by using the Riemann surface and the w-plane, as explained in the following:

Consider the w-plane roots of the system, among them, the physical significance roots lie in the primary sheet of the Riemann surface, which is illustrated in Fig. 6.6. Since the Least Common Multiple of the s-domain characteristic equation that is given in Eq.(6.27) is 5, then the Riemann surface has five sheets. The condition of the system stability states that none of the w-domain roots lies on the unstable region of the w-Plane. This plane is illustrated in Fig. 6.7. The unstable region locates in the first sheet of the Riemann surface. The first sheet is demonstrated, in the w-plane, by $-\frac{\pi}{5} < arg\,(w_i) < \frac{\pi}{5}$. The conjugate seventh and eighth w-roots locate in the physical region (the first sheet of the Riemann surface), so they must be checked, whether they are located in the unstable region or not. The unstable region is demonstrated by $-\frac{\pi}{10} < arg\,(w_i) < \frac{\pi}{10}$, as illustrated in Fig. 6.7.

Now, our aim is to check whether the seventh and eighth w-roots locate in the stable or unstable region of the physical domain. Since their argument is $0.3330 = \frac{\pi}{9.434}$, then the they locate in the stable part of the first sheet region of the Riemann surface. That means the system is stable, because all the absolute values of the other w-roots arguments are greater than the absolute value of the arguments of the seventh and eighth w-roots.

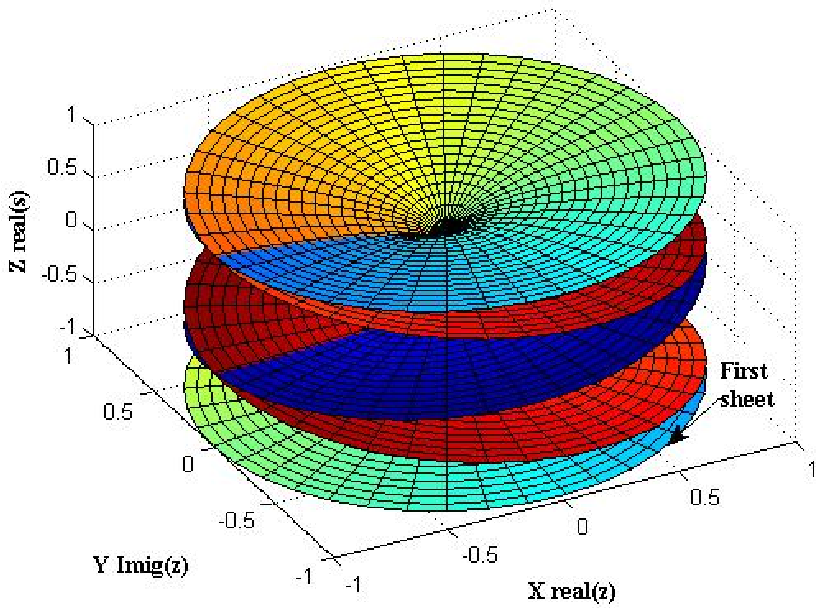


Figure 6.6: Five-sheet Riemann surface of characteristic equation with $\ell = 5$.

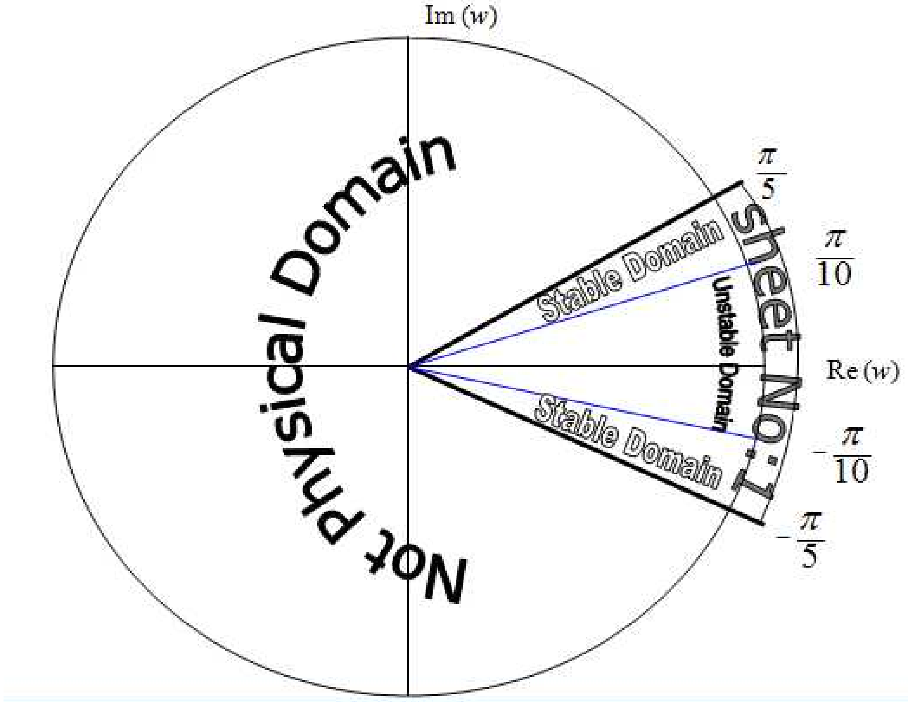


Figure 6.7: w-plane of fractional order system characteristic equation that given by Eq.(6.28).

Map w-domain roots again to s-domain. This can be done by the expression $s = w^{\ell}$. The s-domain roots of the system characteristic equation are illustrated in the following table:

Table 6.2: The s-domain roots of the fractional order system characteristic equation that is given by Eq.(6.27).

i	s_i
1	$0.0970 + 1.0286i$
2	$0.0970 - 1.0286i$
3	$0.0558 + 0.9200i$
4	$0.0558 - 0.9200i$
5	$0.0000 + 1.1029i$
6	$0.0000 - 1.1029i$
7	***-0.0970 + 1.0286i***
8	***-0.0970 - 1.0286i***
9	$-0.0558 + 0.9200i$
10	$-0.0558 - 0.9200i$

The system is stable, then there is only one conjugate root lies on the physical stable region of the w-plane. So in order to obtain the response of the fractional order system choose the root, in s-domain, that corresponds to the w-domain root that locates in the physical stable region. This conjugate s-domain root is $s = -0.0970 \pm 1.0286i$. Substitute this root into the solution of the second order homogenous ordinary differential equation of under-damped system. The solution is given by Eq(6.24).

The aim, now, is to obtain the arbitrary constants c_1, and c_2. The first arbitrary constant can be determined by substituting the value of the initial time, $t_0 = 0$, into Eq.(6.24), and using the first initial condition as follows:

$$y(0) = e^0 \left(c_1 \cos(0) + c_2 \sin(0)\right) . \tag{6.29}$$

From the last equation, the arbitrary constant $c_1 = y_0$. The second arbitrary constant c_2, with the assumption ${}_0D_t^{\alpha} y(0) = 0$, can be obtained for $t = t_0$ as follows:

$$\begin{aligned} {}_0D_t^{\alpha} y(0) &= 0D_t^{\alpha}\left[e^{\alpha_1 t_0}\left(c_1 \cos(\beta_1 t_0) + c_2 \sin(\beta_1 t_0)\right)\right] \\ &= 0 . \end{aligned} \tag{6.30}$$

For $t_0 = 0$, the arbitrary constants c_1, and c_2 are given by

$$c_1 = \quad y_0, \tag{6.31a}$$

$$c_2 = -\frac{c_1}{\sin\left(\frac{\alpha\pi}{2}\right)}\left[\left(\frac{\alpha_1}{\beta_1}\right)^{\alpha} + \cos\left(\frac{\alpha\pi}{2}\right)\right] . \tag{6.31b}$$

Substituting the values of the arbitrary constants that obtained in Eq.(6.31a) and Eq.(6.31b) into Eq.(6.24) to obtain the response of the fractional order system as follows:

$$y(t) = e^{\alpha_1 t} y_0 \left\{ \cos(\beta_1 t) - \frac{1}{\sin\left(\frac{\alpha\pi}{2}\right)} \left[\left(\frac{\alpha_1}{\beta_1}\right)^{\alpha} + \cos\left(\frac{\alpha\pi}{2}\right) \right] \sin(\beta_1 t) \right\}. \tag{6.32}$$

The last equation represents the general formula of the solutions of Eq.(6.2) with the given parameters and initial conditions regardless the value of derivative fractional order α. Substituting the given fractional order value $\alpha = 0.8$ of the fractional order system into Eq.(6.31) to obtain the arbitrary constants values followed by the system time domain response.

The first arbitrary constant $c_1 = y_0$, and the second arbitrary constant for given system parameters is given by

$$c_2 = -\frac{y_0}{\sin\left(\frac{0.8\pi}{2}\right)} \left[\left(\frac{-0.0970}{1.0286}\right)^{0.8} + \cos\left(\frac{0.8\pi}{2}\right) \right]. \tag{6.33}$$

The system time domain response for the given fractional order $\alpha = 0.8$, of the system with its given parameters, is expressed as following:

$$y(t) = e^{-0.0970t} y_0 \left\{ \cos(1.0286t) - \left[\frac{\left(\frac{-0.0970}{1.0286}\right)^{0.8}}{\sin\left(\frac{0.8\pi}{2}\right)} + \frac{\cos\left(\frac{0.8\pi}{2}\right)}{\sin\left(\frac{0.8\pi}{2}\right)} \right] \sin(1.0286t) \right\}. \tag{6.34}$$

The following figure depicts the above response of the system, that is obtained in Eq.(6.34).

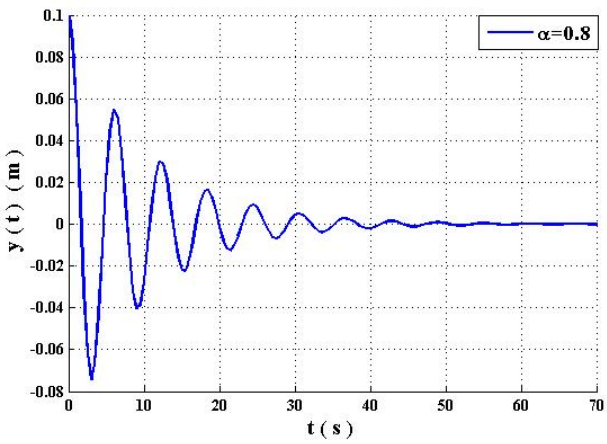


Figure 6.8: The response of the fractional order system that is given by *Eq.*(6.26) using the introduced method for $\alpha = 0.8$.

The responses of the system can be obtained using the same method for different system parameters. A comparison between the integer order and fractional order representations for order $\alpha = 1$ is depicted in Fig. 6.9. The behaviors of the system responses for different fractional order α values are illustrated in Fig. 6.10.

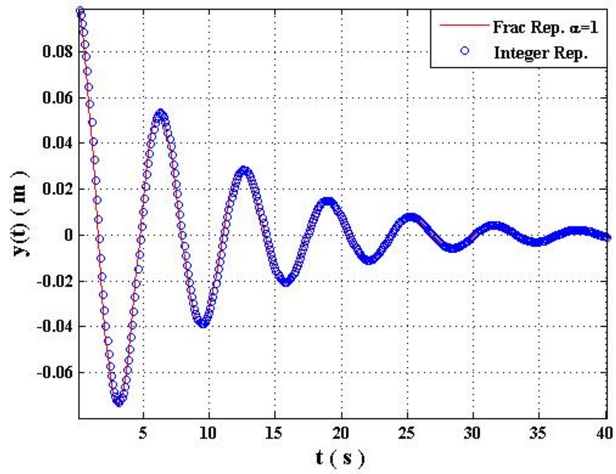


Figure 6.9: Integer versus fractional representation for $\alpha = 1$ of the system that is given by *Eq*.(6.26).

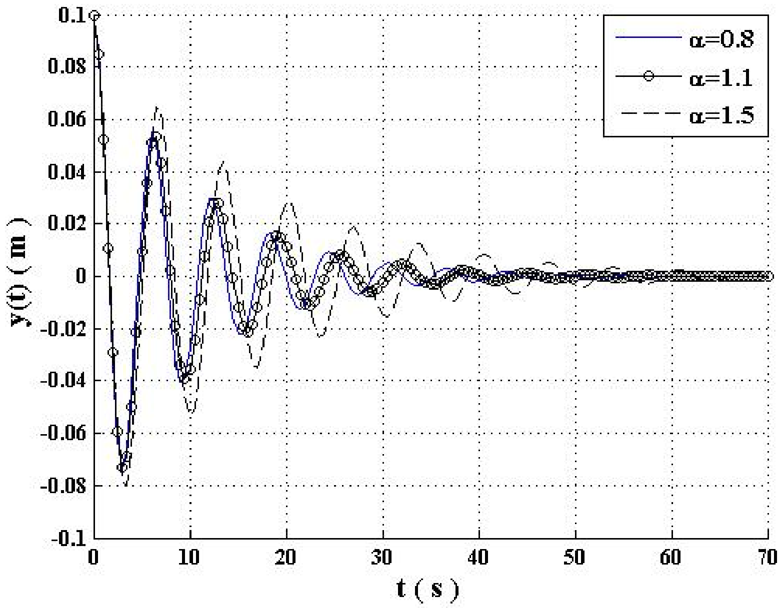


Figure 6.10: Responses of the system obtained by using the introduced method for different α values.

It's concluded from Fig. 6.9 that the integer order representation and the fractional representation for $\alpha = 1$ exactly match each other. Moreover Fig. 6.9 and Fig. 6.10 show that the integer representation and the fractional representation for $\alpha = 1$ give the minimum settling time. On contrary, there are some systems represented in fractional order models due to their behaviors [48].

If the initial fractional derivative of $y(t)$ is considered as ${}_0D_t y(0) = v_0$, then c_2 is given by

$$c_2 = \frac{v_0 - y_0\,\beta_1^\alpha \cos\left(\frac{\alpha\pi}{2}\right) - y_0\alpha_1^\alpha}{\beta_1^\alpha \sin\left(\frac{\alpha\pi}{2}\right)}. \tag{6.35}$$

Figure 6.11 illustrates the responses of the fractional representation of the system for $\alpha = 1$ and the integer representation when the initial fractional derivative of $y(t)$ is considered.

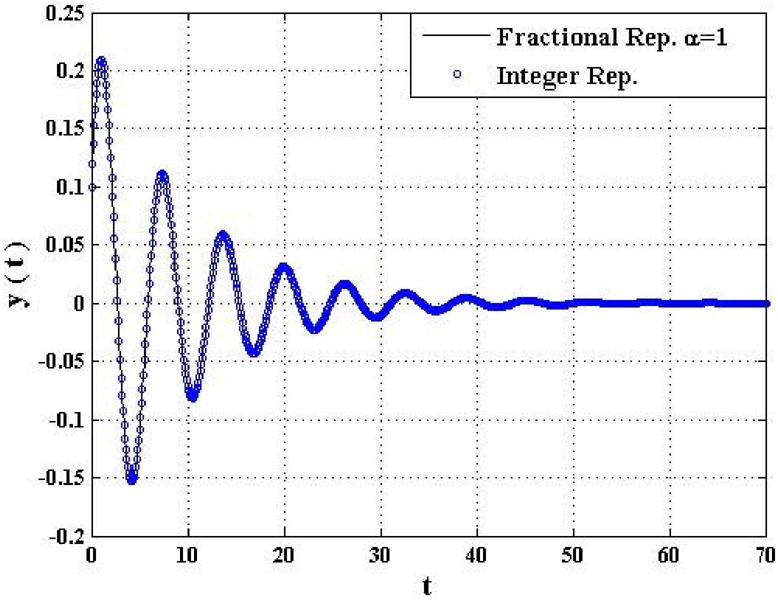


Figure 6.11: Responses of the system for fractional representation with $\alpha = 1$ and the integer representation when the initial fractional derivative of $y(t)$ is considered where $y_0 = 0.1$ *and* $v_0 = 0.2$.

Let's consider a certain free oscillating fractional system. For this system, table 6.3 shows the w-domain roots that are located in the Physical stable region. These system roots is obtained for different α values. The table shows also the corresponding absolute arguments. Moreover it shows the limits of the Physical region and the unstable region of each case. Furthermore the table illustrates corresponding s-domain roots. These roots are used to the obtain the responses of the system for each case.

Table 6.3: The w-domain roots and the corresponding absolute values; physical, and unstable regions limits, and s-domain roots for a free oscillating fractional order system.

α	w	$\lvert arg(w) \rvert$	$\pi/2\ell$	π/ℓ	s
0.2	$\mathbf{0.9665 \pm 0.3198}\boldsymbol{i}$	**0.3195**	**0.3142**	**0.6283**	$\mathbf{-0.0293 \pm 1.0928}\boldsymbol{i}$
0.5	$\mathbf{0.7080 \pm 0.7562}\boldsymbol{i}$	**0.8183**	**0.7854**	**1.5708**	$\mathbf{-0.0706 \pm 1.0708}\boldsymbol{i}$
0.8	$\mathbf{0.9513 \pm 0.3290}\boldsymbol{i}$	**0.3330**	**0.3142**	**0.6238**	$\mathbf{-0.0970 \pm 1.0286}\boldsymbol{i}$
1.1	$\mathbf{0.9845 \pm 0.1659}\boldsymbol{i}$	**0.1669**	**0.1571**	**0.3142**	$\mathbf{-0.0968 \pm 0.9787}\boldsymbol{i}$
1.5	$\mathbf{0.6598 \pm 0.7047}\boldsymbol{i}$	**0.8183**	**0.7854**	**1.5708**	$\mathbf{-0.0613 \pm 0.9298}\boldsymbol{i}$
1.8	$\mathbf{0.9326 \pm 0.3086}\boldsymbol{i}$	**0.3195**	**0.3142**	**0.6283**	$\mathbf{-0.0245 \pm 0.9144}\boldsymbol{i}$

The whole w-domain roots of the system are shown in table 6.4 through table 6.9 for different α values. The w-domain roots that are located in the physical stable region are shown in highlight font. The corresponding absolute arguments, and the s-domain roots are demonstrated in the tables as well.

Table 6.4: The roots of a free oscillating fractional order system, and their absolute arguments for $\alpha = 0.2$.

i	w_i	$\lvert arg(w_i) \rvert$	s_i
1	$\mathbf{0.9665 + 0.3198}\boldsymbol{i}$	**0.3195**	$\mathbf{-0.0293 + 1.0928}\boldsymbol{i}$
2	$\mathbf{0.9665 - 0.3198}\boldsymbol{i}$	**0.3195**	$\mathbf{-0.0293 - 1.0928}\boldsymbol{i}$
3	$0.5828 + 0.8275i$	0.9572	$0.0781 - 1.0596i$
4	$0.5828 - 0.8275i$	0.9572	$0.0781 + 1.0596i$
5	$-0.0199 + 1.0014i$	1.5906	$-0.0998 + 1.0030i$
6	$-0.0199 - 1.0014i$	1.5906	$-0.0998 - 1.0030i$
7	$-0.5954 + 0.7897i$	2.2169	$0.0838 - 0.9423i$
8	$-0.5954 - 0.7897i$	2.2169	$0.0838 + 0.9423i$
9	$-0.9340 + 0.2960i$	2.8347	$-0.0328 + 0.9024i$
10	$-0.9340 - 0.2960i$	2.8347	$-0.0328 - 0.9024i$

Table 6.5: The roots of a free oscillating fractional order system, and their absolute arguments for $\alpha = 0.5$.

i	w_i	$\lvert arg(w_i) \rvert$	s_i
1	$\mathbf{0.7080 + 0.7562}\boldsymbol{i}$	**0.8183**	$\mathbf{-0.0706 + 1.0708}\boldsymbol{i}$
2	$\mathbf{0.7080 - 0.7562}\boldsymbol{i}$	**0.8183**	$\mathbf{-0.0706 - 1.0708}\boldsymbol{i}$
3	$-0.7080 + 0.6562i$	2.3941	$0.0706 - 0.9292i$
4	$-0.7080 - 0.6562i$	2.3941	$0.0706 + 0.9292i$

Table 6.6: The roots of a free oscillating fractional order system, and their absolute arguments for $\alpha = 0.8$.

i	w_i	$\lvert arg(w_i) \rvert$	s_i
1	$-0.9513 + 0.3290i$	2.8086	$0.0970 + 1.0286i$
2	$-0.9513 - 0.3290i$	2.8086	$0.0970 - 1.0286i$
3	$-0.5879 + 0.7889i$	2.2112	$0.0558 - 0.9200i$
4	$-0.5879 - 0.7889i$	2.2112	$0.0558 + 0.9200i$
5	$0.0000 + 1.0198i$	1.5708	$0.0000 + 1.1029i$
6	$0.0000 - 1.0198i$	1.5708	$0.0000 - 1.1029i$
7	$\mathbf{0.9513 + 0.3290}\boldsymbol{i}$	**0.3330**	$\mathbf{-0.0970 + 1.0286}\boldsymbol{i}$
8	$\mathbf{0.9513 - 0.3290}\boldsymbol{i}$	**0.3330**	$\mathbf{-0.0970 - 1.0286}\boldsymbol{i}$
9	$0.5879 + 0.7889i$	0.9304	$-0.0558 - 0.9200i$
10	$0.5879 - 0.7889i$	0.9304	$-0.0558 + 0.9200i$

Table 6.7: The roots of a free oscillating fractional order system, and their absolute arguments for $\alpha = 1.1$.

i	w_i	$\lvert arg(w_i) \rvert$	s_i
1	$-0.9907 + 0.1468i$	2.9944	$0.1005 - 1.0099i$
2	$-0.9907 - 0.1468i$	2.9944	$0.1005 + 1.0099i$
3	$-0.8829 + 0.4597i$	2.6615	$0.0842 + 0.9512i$
4	$-0.8829 - 0.4597i$	2.6615	$0.0842 - 0.9512i$
5	$-0.7172 + 0.7070i$	2.3634	$0.0768 - 1.0706i$
6	$-0.7172 - 0.7070i$	2.3634	$0.0768 + 1.0706i$
7	$-0.4461 + 0.8851i$	2.0376	$0.0407 + 0.9144i$
8	$-0.4461 - 0.8851i$	2.0376	$0.0407 - 0.9144i$
9	$-0.1596 + 0.9973i$	1.7295	$0.0175 - 1.1046i$
10	$-0.1596 - 0.9973i$	1.7295	$0.0175 + 1.1046i$
11	$\mathbf{0.9845 + 0.1659}\boldsymbol{i}$	**0.1669**	$\mathbf{-0.0968 + 0.9787}\boldsymbol{i}$
12	$\mathbf{0.9845 - 0.1659}\boldsymbol{i}$	**0.1669**	$\mathbf{-0.0968 - 0.9787}\boldsymbol{i}$
13	$0.8991 + 0.4480i$	0.4622	$-0.0939 - 1.0418i$
14	$0.8991 - 0.4480i$	0.4622	$-0.0939 + 1.0418i$
15	$0.6972 + 0.7070i$	0.7924	$-0.0648 + 0.9294i$
16	$0.6972 - 0.7070i$	0.7924	$-0.0648 - 0.9294i$
17	$0.1534 + 0.9783i$	1.4152	$-0.0138 + 0.9068i$
18	$0.1534 - 0.9783i$	1.4152	$-0.0138 - 0.9068i$
19	$0.4622 + 0.8969i$	1.0950	$-0.0504 - 1.0926i$
20	$0.4622 - 0.8969i$	1.0950	$-0.0504 + 1.0926i$

Table 6.8: The roots of a free oscillating fractional order system, and their absolute arguments for $\alpha = 1.5$.

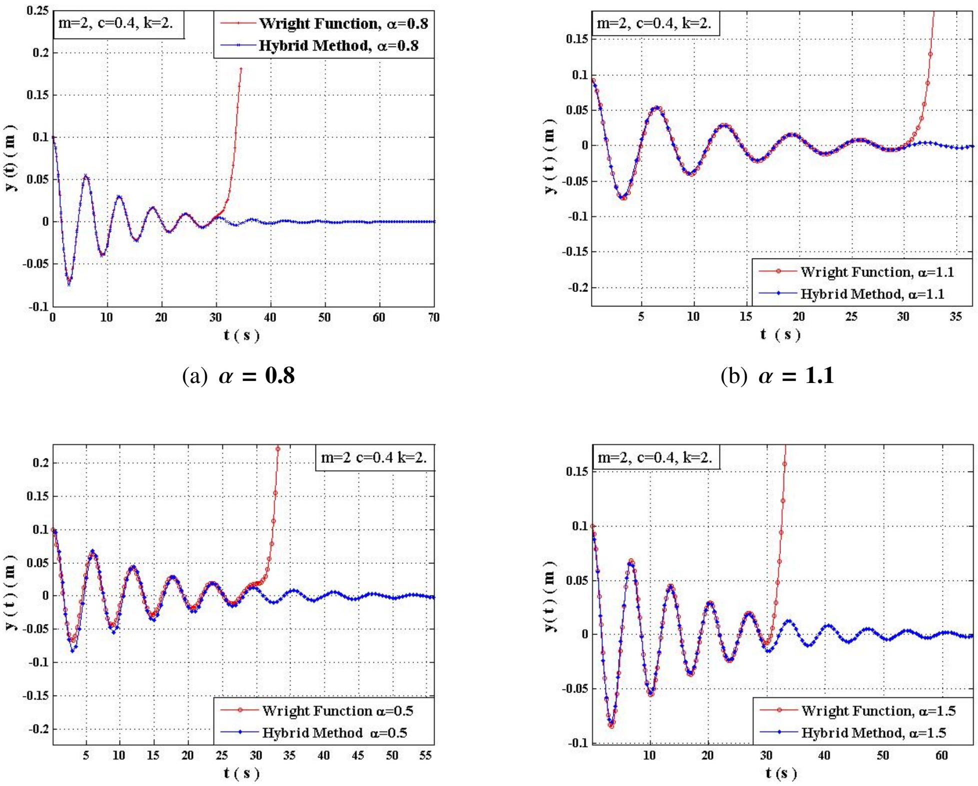
i	w_i	$\lvert arg(w_i) \rvert$	s_i
1	$-0.7598 + 0.7042i$	2.3941	$0.0813 - 1.0700i$
2	$-0.7598 - 0.7042i$	2.3941	$0.0813 + 1.0700i$
3	$\mathbf{0.6598 + 0.7047}\boldsymbol{i}$	**0.8183**	$\mathbf{-0.0613 + 0.9298}\boldsymbol{i}$
4	$\mathbf{0.6598 - 0.7047}\boldsymbol{i}$	**0.8183**	$\mathbf{-0.0613 - 0.9298}\boldsymbol{i}$

Table 6.9: The roots of a free oscillating fractional order system, and their absolute arguments for $\alpha = 1.8$.

i	w_i	$\lvert arg(w_i) \rvert$	s_i
1	$-0.9729 + 0.3083i$	2.8347	$-0.0402 + 1.1067i$
2	$-0.9729 - 0.3083i$	2.8347	$-0.0402 - 1.1067i$
3	$-0.6088 + 0.8074i$	2.2169	$0.0936 - 1.0529i$
4	$-0.6088 - 0.8074i$	2.2169	$0.0936 + 1.0529i$
5	$-0.0198 + 0.9982i$	1.5906	$-0.0983 + 0.9872i$
6	$-0.0198 - 0.9982i$	1.5906	$-0.0983 - 0.9872i$
7	$0.5689 + 0.8077i$	0.9572	$0.0692 - 0.9387i$
8	$0.5689 - 0.8077i$	0.9572	$0.0692 + 0.9387i$
9	$\mathbf{0.9326 + 0.3086}\boldsymbol{i}$	**0.3195**	$\mathbf{-0.0245 + 0.9144}\boldsymbol{i}$
10	$\mathbf{0.9326 - 0.3086}\boldsymbol{i}$	**0.3195**	$\mathbf{-0.0245 - 0.9144}\boldsymbol{i}$

6.5 Comparison between the Wright Function and the Hybrid Method Solutions

There are many methods to obtain the response of the fractional order system. Using the Wright function is one of analytical methods that are used to obtain the response of free oscillating fractional order system [66]. Comparisons between the analytical method based on the Wright function and the introduced method are illustrated in Fig. 6.12.



(a) α = **0.8**

(b) α = **1.1**

(c) α = **0.5**

(d) α = **1.5**

Figure 6.12: A comparison between the Wright function and the introduced method solutions.

In comparison between the two methods, as Fig. 6.12 illustrates, the introduced method has some advantages over the using of the Wright function, such as whatever the parameters are, by using the introduced method the response of the system can reach the steady state, if there is. In the solution that is obtained by using the Wright function the steady state, in some system, can't be reached. Moreover, the computational cost is very high of the solution obtained by using Wright function comparing with the usage of the introduced method. Furthermore, due to the indexes truncation the accuracy of the hybrid method is higher than the accuracy when the Wright function is used.

6.6 Modeling of a Freely Oscillating Dynamical Systems by Using Extended FELEs

The freely oscillating dynamical system can be modeled by using the FELEs as introduced in chapter 5. The FELEs model of free oscillating single pendulum system is given by Eq.(5.17). In this model, the damping force is proportional to the velocity with time varying coefficient. For different damping behaviour the free oscillating system can be modeled by three-term FODEs. This model is given by Eq.(6.2), in which the damping force is proportional to the fractional derivative of the displacement with time invariant coefficient. Both models can be combined, depending on the damping behaviour of a freely oscillating dynamical system, to represent the system model as following:

Consider a free oscillating single pendulum with length l and uniform mass distribution. The equation of motion of the system represented by the combined model is given by

$$\ddot{\theta}\,(t) + \frac{\alpha - 1}{t - \tau}\, {}_0D_t^{\beta(t)}\theta\,(t) + \omega_n^2 \sin\theta\,(t) = 0\ , \tag{6.36}$$

where $\beta\,(t)$ represents the variable order of fractional derivative of the angular dispacement θ, $\omega_n = \sqrt{2g/l}$ is the natural frequaency of the system, α is the R-L fractional integral order, τ is an intrinsic time, and t represents the observing time .

Now our aim is to obtain the solution of the linearized system that is given by Eq.(6.36).

6.6.1 Methods and Methodologies

Numerical techniques are used to approximate the derivatives solutions of Eq.(6.36). The finite difference method is used to approximate the first term and the second term is approximated by the Coimbra fractional derivative [143] and the finite difference technique. The third term is linearized such that the oscillation angle is small.

Applying the finite difference method to $\ddot{\theta}(t)$ to obtain

$$\theta(t_{n+1}) = \theta(t_n + h) = \theta(t_n) + h\dot{\theta}(t_n) + \frac{h^2}{2!}\ddot{\theta}(t_n) + \frac{h^3}{3!}\dddot{\theta}(t_n) + \ldots, \tag{6.37a}$$

$$\theta(t_{n-1}) = \theta(t_n - h) = \theta(t_n) - h\dot{\theta}(t_n) + \frac{h^2}{2!}\ddot{\theta}(t_n) - \frac{h^3}{3!}\dddot{\theta}(t_n) + \ldots, \tag{6.37b}$$

$$\theta(t_n) = \theta(t_n + 0) = \theta(t_n), \tag{6.37c}$$

where h is the time increment. Subtraction of Eq.(6.37b) from Eq.(6.37a) yields

$$\dot{\theta}(t) = \frac{\theta_{n+1} - \theta_{n-1}}{2h} + O\left(h^2\right). \tag{6.38}$$

Add Eq.(6.37a) to Eq.(6.37b) and subtrate twice Eq.(6.37c) to obtain

$$\ddot{\theta}(t) = \frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{h^2} + O\left(h^2\right). \tag{6.39}$$

The fractional derivative ${}_0D_t^{\beta(t)}\theta(t)$ can be approximated by using Coimbra fractional derivative as following [143]:

$${}_0D_t^{\beta(t)}\theta(t_n) = \frac{1}{\Gamma[1-\beta(t_n)]}\int_0^{t_n}(t_n - \zeta)^{-\beta(t_n)}\frac{\partial\theta(\zeta)}{\partial\zeta}d\zeta + \frac{\theta(0^+) - \theta(0^-)}{\Gamma[1-\beta(t_n)]}. \tag{6.40}$$

The discretization of the integral that is given by Eq.(6.40) can be done as following for $\theta\left(0^+\right) = \theta\left(0^-\right)$ [144]:

$${}_0D_t^{\beta(t)}\theta(t_n) = \frac{1}{\Gamma[1-\beta(t_n)]}\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(t_n - \zeta)^{-\beta(t_n)}\frac{\partial\theta(\zeta)}{\partial\zeta}d\zeta. \tag{6.41}$$

Applying forward difference approximation to $\dfrac{\partial\theta(\zeta)}{\partial\zeta}$ yields

$$
\begin{aligned}
{}_0D_t^{\beta(t)}\theta\left(t_n\right) &= \frac{1}{\Gamma\left[1-\beta\left(t_n\right)\right]} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(t_n-\zeta\right)^{-\beta(t_n)} \left(\frac{\theta\left(t_{j+1}\right)-\theta\left(t_j\right)}{h} + h.E_1 \right) d\zeta \\
&= \frac{1}{\Gamma\left[1-\beta\left(t_n\right)\right]} \sum_{j=0}^{n-1} \frac{\theta\left(t_{j+1}\right)-\theta\left(t_j\right)}{h} \int_{t_j}^{t_{j+1}} \left(t_n-\zeta\right)^{-\beta(t_n)} d\zeta + E_{1n} \; .
\end{aligned} \tag{6.42}
$$

where the error E_{1n} can be given as

$$
E_{1n} = \frac{1}{\Gamma\left[1-\beta\left(t_n\right)\right]} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(t_n-\zeta\right)^{-\beta(t_n)} h.E_1 d\zeta \; . \tag{6.43}
$$

Solving the integral of the first term of the right hand side of Eq.(6.42) yields

$$
{}_0D_t^{\beta(t)}\theta\left(t_n\right) = \frac{1}{\Gamma\left[2-\beta\left(t_n\right)\right]} \sum_{j=0}^{n-1} \left\{ \frac{\theta\left(t_{j+1}\right)-\theta\left(t_j\right)}{h} \left[-\left(t_n-t_{j+1}\right)^{1-\beta(t_n)} + \left(t_n-t_j\right)^{1-\beta(t_n)} \right] \right\} \; . \tag{6.44}
$$

For $j = 0, 1, 2, ..., n-1$ the following can be considered:

$$
-\left[\left(t_n-t_{j+1}\right)^a - \left(t_n-t_j\right)^a \right] = h^a \left[(n-j)^a - (n-j-1)^a \right] \text{ where } a \in \mathbb{R}.
$$

Then Eq.(6.44) can be written as

$$
\begin{aligned}
0D_t^{\beta(t)}\theta\left(t_n\right) &= \frac{h^{1-\theta(t_n)}}{\Gamma\left[2-\beta\left(t_n\right)\right]} \sum_{j=0}^{n-1} \left\{ \frac{\theta\left(t_{j+1}\right)-\theta\left(t_j\right)}{h} \left[(n-j)^{1-\beta(t_n)} - (n-j-1)^{1-\beta(t_n)} \right] \right\} \\
&= \frac{h^{-\theta(t_n)}}{\Gamma\left[2-\beta\left(t_n\right)\right]} \sum_{j=0}^{n-1} \left\{ \left[(n-j)^{1-\beta(t_n)} - (n-j-1)^{1-\beta(t_n)} \right] \left[\theta\left(t_{j+1}\right)-\theta\left(t_j\right) \right] \right\} \; .
\end{aligned} \tag{6.45}
$$

Substitute Eq.(6.39) and Eq.(6.45) into Eq.(6.36) and consider the linearization to obtain

$$
\frac{\theta(t_{n+1}) - 2\theta(t_n) + \theta(t_{n-1})}{h^2} + \frac{\alpha - 1}{t_n - \tau} \frac{h^{-\beta(t_n)}}{\Gamma[2 - \beta(t_n)]}
\sum_{j=0}^{n-1} \left\{ \left[(n-j)^{1-\beta(t_n)} - (n-j-1)^{1-\beta(t_n)} \right] \left[\theta\left(t_{j+1}\right) - \theta\left(t_j\right) \right] \right\} + \omega_n^2 \theta(t_n) = 0 \,. \tag{6.46}
$$

Open the summation for $j = n - 1$ and rewrite Eq.(6.46)as follows:

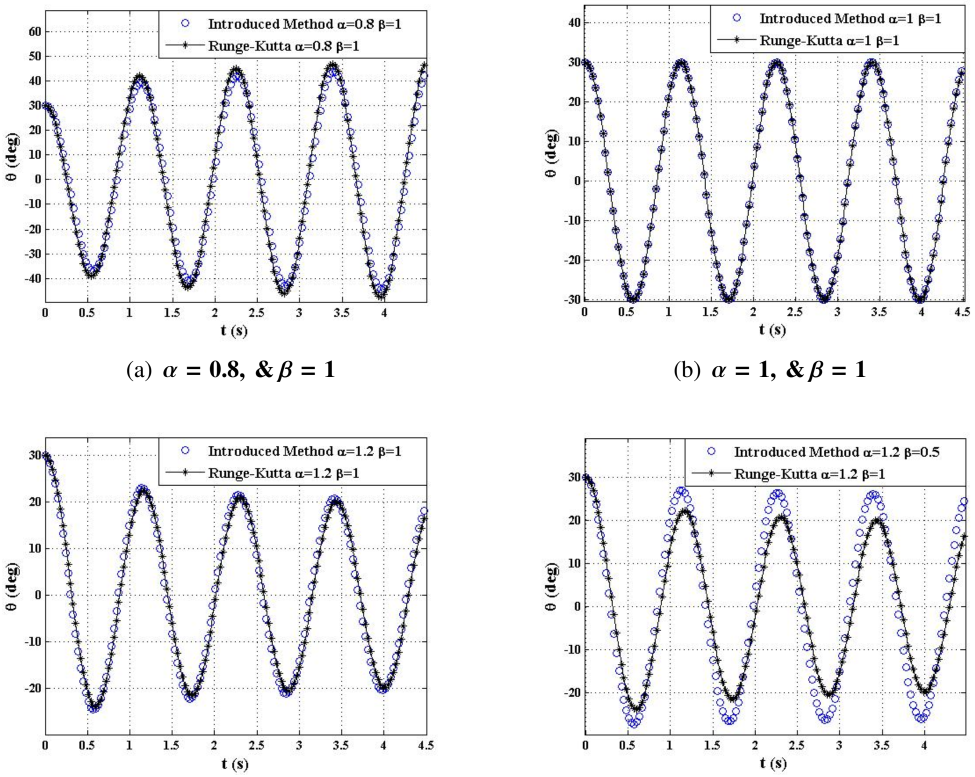
$$
\begin{aligned}
&\frac{\theta(t_{n+1}) - 2\theta(t_n) + \theta(t_{n-1})}{h^2} + \frac{\alpha - 1}{t_n - \tau} \frac{h^{-\theta(t_n)}}{\Gamma[2 - \beta(t_n)]} \\
&\sum_{j=0}^{n-2} \left\{ \left[(n-j)^{1-\beta(t_n)} - (n-j-1)^{1-\beta(t_n)} \right] \left[\theta\left(t_{j+1}\right) - \theta\left(t_j\right) \right] \right\} + \frac{\alpha - 1}{t_n - \tau} \frac{h^{-\theta(t_n)}}{\Gamma[2 - \beta(t_n)]} \\
&\left[(n-(n-1))^{1-\beta(t_n)} - (n-(n-1)-1)^{1-\beta(t_n)} \right] [\theta(t_n) - \theta(t_{n-1})] + \omega_n^2 \theta(t_n) = 0 \,.
\end{aligned} \tag{6.47}
$$

Multiply both sides of Eq.(6.47) by h^2, consider the initial conditions, and take into account $\left[(n-(n-1))^{1-\beta(t_n)} - (n-(n-1)-1)^{1-\beta(t_n)} \right] = 1$ to obtain

$$
\begin{aligned}
&\theta(t_{n+1}) - 2\theta(t_n) + \theta(t_{n-1}) + \frac{h^{2-\beta(t_n)}(\alpha - 1)}{(t_n - \tau)\left\{\Gamma[2 - \beta(t_n)]\right\}} \\
&\sum_{j=0}^{n-2} \left\{ \left[(n-j)^{1-\beta(t_n)} - (n-j-1)^{1-\beta(t_n)} \right] \left[\theta\left(t_{j+1}\right) - \theta\left(t_j\right) \right] \right\} \\
&+ \frac{h^{2-\beta(t_n)}(\alpha - 1)}{(t_n - \tau)\left\{\Gamma[2 - \beta(t_n)]\right\}} [\theta(t_n) - \theta(t_{n-1})] + h^2 \omega_n^2 \theta(t_n) = 0 \,.
\end{aligned} \tag{6.48}
$$

6.6.2 Verification and Numerical Results

The second order homogenous variable coefficients differential equation is considered as a particular case of Eq.(6.36). This can be accomplished by taking $\beta(t) = 1$. Consider the linearity of $sin\theta(t)$ for small oscillation, the solutions of the integer representation and the fractional representation must be identical for $\beta(t) = 1$. This is shown in Fig. 6.13 in which the integer and fractional representations solutions are obtained by using Runge-Kutta numerical technique and the introduced approximation, respectively. The figure illustrates that both solutions are identical to each other for $\beta = 1$.



(a) $\alpha = 0.8, \ \& \beta = 1$

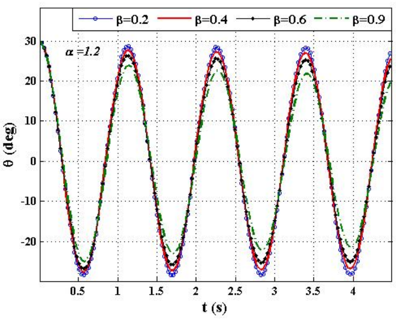
(b) $\alpha = 1, \ \& \beta = 1$

(c) $\alpha = 1.2, \ \& \beta = 1$

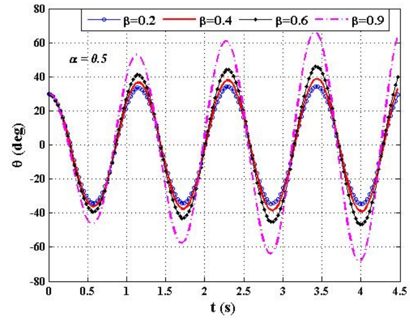
(d) $\alpha = 1.2, \ \& \beta = 0.5 \, versus. \beta = 1$

Figure 6.13: A comparison between the integer and fractional representation solutions obtained by using Runge-Kutta numerical technique and the introduced approximation, respectively, for free oscillating system modeled by extended FELEs.

For $0 \leq \beta \leq 1$, selected values of β are tested for $\alpha > 1$ and $\alpha < 1$ as shown in Fig. 6.14-a and Fig. 6.14-b, respectively. It's concluded from Fig. 6.14 that the response of the system reaches the steady state earlier, as β be close to one for $\alpha > 1$. For $\alpha < 1$ the divergence of the system increases, as β increases.



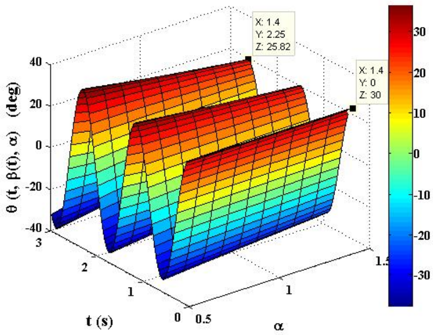
(a) $\boldsymbol{\alpha > 1}$



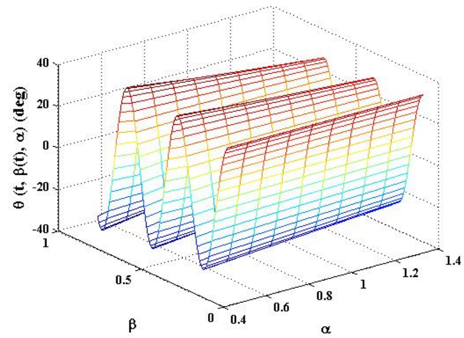
(b) $\boldsymbol{\alpha < 1}$

Figure 6.14: The effect of β value on the system responses, $a : \alpha > 1,\ b : \alpha < 1$.

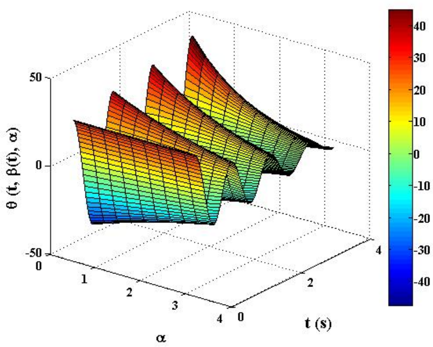
The effect of α is investigated in Fig. 6.15 in which the following can be concluded: For $\alpha < 1$, as α decreases, the divergence of the system increases, while for $\alpha > 1$, as α increases, the convergence of the system increases for the same values of β. For $\alpha = 1$ there is no dissipation force acting on the system.



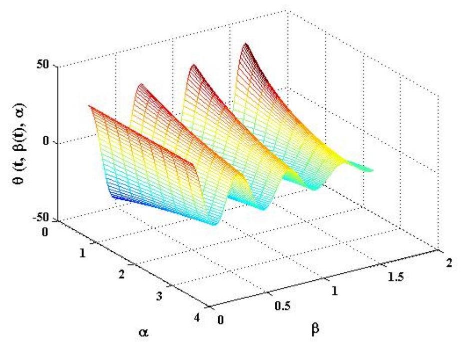
(a) $\boldsymbol{\alpha = 0.5 : 1.5}$



(b) $\boldsymbol{\alpha = 0.5 : 1.5}$.



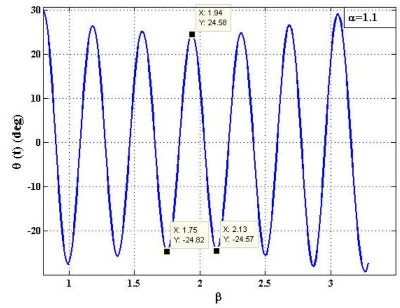
(c) $\boldsymbol{\alpha = 0 : 4}$.



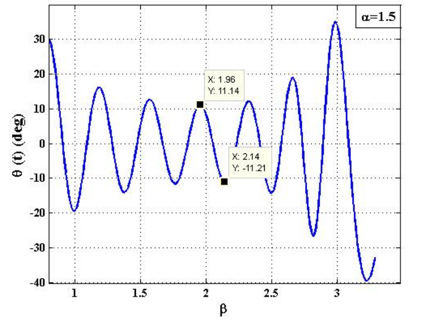
(d) $\boldsymbol{\alpha = 0 : 4}$.

Figure 6.15: Illustration of the effect of α value on the system responses.

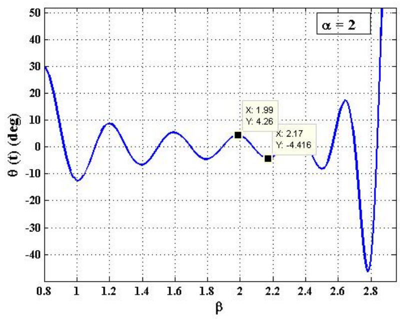
It's inferred from Fig. 6.15 that the system response is convergent for $\alpha > 1$. That is because of the positive damping force effect. The convergence of the system responses due to varying β values is demonstrated in Fig. 6.16. It's noticeable from the figure that for different values of $\alpha > 1$ the system is convergent, if $0 < \beta < 2$.



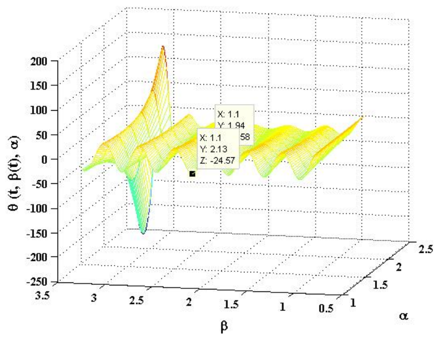
(a) α **= 1.1**.



(b) α **= 1.5**.



(c) α **= 2**.



(d) α **= 1.1 : 2.4**.

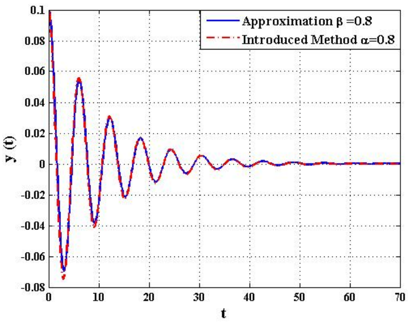
Figure 6.16: Illustration of the upper limit of β value.

In conclusion the introduced model represents an extended version of FELEs. This version is given as a combination of two types of damped system model. In the first model, the damping force is proportional to the fractional derivative of the displacement [24, 30, 36, 42], and in the second one the coefficient of the damping force is time varying [47]. In addition, the fractional order of the damping term is given as a variable $\beta(t)$.

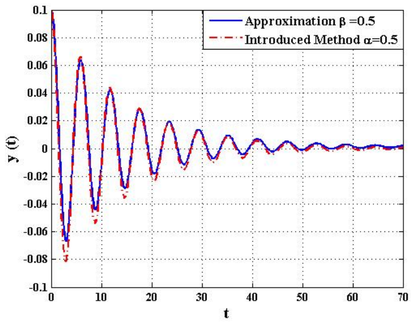
The Coimbra fractional derivative and the finite difference technique are used to approximate the solution of the model. The verification of the approximation shows the accuracy of the approximation as illustrated in Fig. 6.13, where the approximated solutions are compared with the solution of the Runge-Kutta numerical technique for $\beta(t) = 1$. The convergence of the model is affected by the values of the R-L fractional integral order α and the variable order of the damping term β. That is shown in Fig. 6.14 through Fig. 6.16.

6.7 Comparison between the Hybrid Method and the Approximation Technique to Obtain the Solution of a Freely Oscillating Fractional Dynamical System.

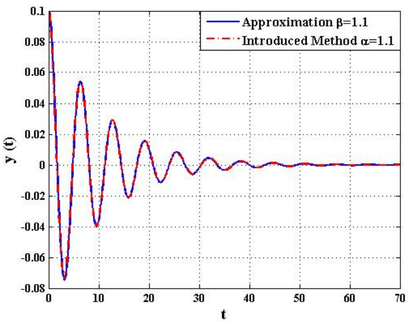
The approximation technique that is used to solve the model introduced in Eq.(6.36) can be compared with the introduced analytical method. The later is discussed in section 6.4. The comparison between the two techniques can be done for time invariant damping coefficient. The responses of the system obtained by both methods must be identical for the same system parameters. Fig. 6.17 illustrates the comparison between the two techniques.



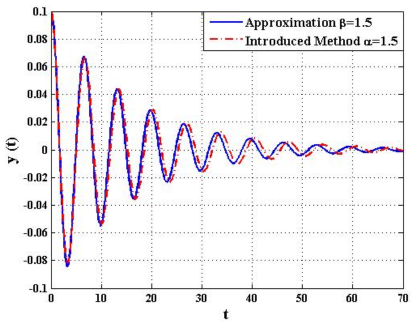
(a) $\alpha = \beta = \mathbf{0.8}$.



(b) $\alpha = \beta = \mathbf{0.5}$.



(c) $\alpha = \beta = \mathbf{1.1}$.



(d) $\alpha = \beta = \mathbf{1.5}$.

Figure 6.17: Illustration of a comparison between the introduced method and the approximation to solve a freely oscillating fractional dynamical system.

CHAPTER 7

DISCUSSION and CONCLUSION

oscillating dynamical systems that are studied in the research are modeled by means of the fractional calculus and calculus of variations. The responses of the systems are investigated according to some considerations such as system parameters and initial conditions. The considered fractional models of the oscillating systems are classified into two main types. The first formula is represented by homogeneous or non-homogeneous three-term FODEs. The second formula is contingent upon the FELEs modeling of oscillating systems. These models are represented by non-linear and linearized second order homogeneous differential equations with variable coefficients.

The FELEs derivation is based on the calculus of variations and the fractional calculus. Specifically, it is based on the Fermat's principle, the R-L fractional integrals, and the Caputo fractional derivatives. The systems that are modeled by FELEs in our research are hypothetically assumed to involve dissipation forces, however the parameters of these dissipation forces can be experimentally identified. One of these parameters is the R-L fractional integral order α. The second parameter is the intrinsic time which is a time delay due to system performance.

The dissipation force, in this type of fractional modeling, is increased, as the R-L fractional integral α increases, whereas this dissipation force is gradually decreased due to the observed time effect. Additionally, the FELEs model gives a free oscillating undamped model, if the R-L fractional integral α equals one. Moreover, even though the mathematical representations of these models are given by second order homogeneous nonlinear differential equations with variables damping coefficients, the FELEs are introduced as fractional models. This is because of the existence of the R-

L fractional integral α in the damping term coefficients, and also it's because of the derivation of the FELEs is based on the fractional calculus. The non-linear systems that are modeled by means of FELEs can be linearized by the same approaches that are used in the linearization of integer order modeled systems.

The other formula of the fractional models, that is considered in the thesis contains derivative operators subject to fractional order in the dissipation terms of homogeneous or non-homogeneous three-term FODEs.

A comparison between extended fractional order representations that are given by Bagley-Trovik equations of non-homogeneous three-term FODEs is done. The comparison shows the effects of the fractional order operator α, and the external applied forces. It's concluded from the comparison that for α equals one the fractional order representation and the integer order representation approximately match each other.

The M-L functions, the M-L functions pairs, with some other functions, and their Laplace transformations are studied and investigated. They are also exploited to obtain the solutions of non-homogeneous three-term FODEs with two-fractional terms for different values of α. Analytical and graphical comparisons between the fractional representations solutions and the solution of second order order non-homogeneous differential equation for same system parameters are done. It's deduced from the comparisons that when the fractional order α equals one the fractional representation and the integer representation solutions exactly identical.

Some real systems are modeled, as mentioned in section 6.1, by means of homogeneous three-term FODEs. The analytical method that is based on the Wright function and a hybrid analytical method based on integer and fractional calculus fundamentals are introduced. These methods are used to obtain the responses of fractional order free oscillating systems. The systems are modeled by second order homogeneous three-term FODEs for different system parameters as particular cases. The method that is based on the Wright function gives a general solution of the homogeneous three-term FODEs, whereas the introduced method gives the solution of a particular case. This particular case can represent a real system. Even-though the Wright function-based method gives a general solution, it's inferred from the comparison between the two methods that there are some advantages of using the hybrid method over the usage of the Wright function-based method. These advantages include the following: Whatever the parameters are, the responses of the system, which are obtained by using the hybrid method reach the steady state, if there is. On the contrary, in the solution using the Wright function-based method the steady state, in some system, can't be reached. The program running time of the solution using the Wright function-based method is too long. This long running time increases the computational cost, whereas the running time to obtain the response of the same system is couple of seconds, when the hybrid method is used. The stability of the system can be analyzed before obtaining the response, when the hybrid method is used. The accuracy of the hybrid method is better than the accuracy of the Wright function-based method. That is because of the effect of the truncations that the Wright function-based method has.

The aforementioned two main sections that are considered to represent the fractional models of the oscillating dynamical systems are combined together in sections 6.6 and section 6.7. The result of this combination is an extended version of FELEs model. This model is represented by homogeneous varying coefficients three-term FDE.

A comparison is done between this model for $\boldsymbol{\beta = 1}$ and the second order homogeneous variable coefficients differential equation. It's concluded from the comparison that the responses of both representations are identical to each other for different $\boldsymbol{\alpha(t)}$ values. This is illustrated in Fig. 6.13.

For $0 \leq \beta \leq 1$, selected values of β are tested for $\alpha > 1$ and $\alpha < 1$ as shown in Fig. 6.14-a and Fig. 6.14-b, respectively. It's concluded from Fig. 6.14 that the response of the system reaches the steady state earlier, as β be close to one for $\alpha > 1$. For $\alpha < 1$ the divergence of the system increases, as β increases. The effects of the R-L fractional integral order of the model are investigated in Fig. 6.15. It's deduced from the investigation that the system response is convergent for $\alpha > 1$. Additionally, it's inferred from Fig. 6.16 that the approximation is valid for $0 < \beta < 2$. Moreover, the convergence of the model is affected by the values of the R-L fractional integral order α and the variable order of the damping term β.

A comparison is done between the Hybrid method and the approximation technique to obtain the solution of a freely oscillating fractional dynamical systems . It's concluded from the comparison, for constant coefficients, that the system responses that are obtained by both techniques match each other for same values of fractional orders, α, and β, of the dissipation terms.

As future work, it's recommended to concentrate on the experimental studies. These studies may include the FELEs models, and the extended FELEs models of oscillating dynamical systems. The main objectives of these experimental studies are to identify the models parameters such as the R-L fractional integral $\boldsymbol{\alpha}$, and the fractional derivative order $\boldsymbol{\beta}$. Additionally, integer and fractional controllers can be applied to these models. Furthermore, the application of the external force and the time delay to these models can be a feasible future work in the research area.

REFERENCES

[1] Singiresu S. Rao, ”*Mechanical Vibrations*,” Fourth Edition. ISBN 0-13-120768-7, Prentice Hall, 2004.

[2] Yongqiang Ye, Jianhong Wang, Xiang Pan and Xudong Gao, ”*Fractional 90-Degree Phase-Shift Filtering Based on Liouville-Weyl Differintegral*,” 978-1-4799-1501-9/13/ 2013 IEEE.

[3] Sergei V. Rogosin, ”*Advances in Harmonic Analysis and Operator Theory: The Role of S.G. Samko in the Establishing and Development of the Theory of Fractional Differential Equations and Related Integral Operators*,” pp 49-63, Online ISBN 978-3-0348-0516-2, Springer Basel, 2013.

[4] Om P. Agrawal, ”*Formulation of Euler Lagrange Equations for Fractional Variational Problems*,” Journal of Mathematical Analysis & Applications, 272 (2002), pp. 368-379, ACADEMIC PRESS.

[5] Katsuhisa Furuta, Masaki Yamakita, Seiichi Kobayashi, ”*Swing Up Control of Inverted Pendulum*,” I ECON ’91, CH2976-9/91/0000-2193 1.00 0 1991 IEEE.

[6] Igor Podlubny, ”*Fractional Differential Equations*,” Mathematics in Science & Engineering. Vol. 198, 1999.

[7] A. Srikantha Phani, and J.Woodhouse, ”*Experimental Identification of Viscous Damping in Linear Vibration*,” Journal of Sound & Vibration, ELSEVIER, 2008.

[8] Mehdi Dalir, and Majid Bashour, ”*Applications of Fractional Calculus*,” Applied Mathematical Sciences. Vol. 4, No. 21, 2010, PP. 1021 - 1032.

[9] Kai Diethelm, and Neville J. Ford, ”*Multi-order fractional differential equations and their numerical solution*,” Applied Mathematics & Computation. Vol. 154 2004, PP 621-640, ELSEVIER.

[10] W. He et al., ”*Dynamics and Control of Mechanical Systems in Offshore Engineering*,” DOI 10.1007978-1-4471-5337-5-2, Springer-Verlag London 2014.

[11] Ljupco Kocarev, Shiguo Lian, ”*Chaos-based Cryptography: Theory, Algorithms and Applications*,” ISBN 978-3-642-20541-5 2011 Springer.

[12] Marc Weilbeer, ”*Eficient Numerical Methods for Fractional Differential Equations and Their Analytical Background,*” PhD Thesis, ISBN 978-3-89720-846, 2005.

[13] Doudou Badiane, Alain Gasser, and Eric Blond, ”*Vibrating Beam in Viscous Fluid for Viscosity Sensing: Application to an Industrial Vibrating Viscometer*,” 12th Pan-American Congress of Applied Mechanics - PACAM XII January 02-06, 2012, Port of Spain, Trinidad.

[14] Shantanu Das, "*Functional Fractional Calculus*," ISBN 978-3-642-20544-6, 2011 Springer.

[15] Fitiavana Anjara, and Joelson Solofoniaina, "*Solution of General Fractional Oscillation Relaxation Equation by Adomian's Method*," Gen. Math. Notes. Vol. 20, No. 2, February 2014, pp. 1-11 ISSN 2219-7184.

[16] Simon J.A. Malham, "*An Introduction to Lagrangian and Hamiltonian Mechanics*," Maxwell Institute for Mathematical Sciences & School of Mathematical and Computer Sciences Heriot-Watt University, Edinburgh EH14 4AS, UK, 2014.

[17] A. Belendez, C. Pascual, D.I. Mendez, T. Belendez and C. Neipp, "*Exact Solution for the Nonlinear Pendulum*," Revista Brasileira de Ensino de Fsica, Vol. 29, No. 4, 2007, pp. 645-648.

[18] Concepcion A. Monje, YangQuan Chen, Blas M. Vinagre, Dingyu Xue, and Vicente Feliu, "*Fractional-order Systems and Controls Fundamentals and Applications*," ISBN 978-1-84996-334-3, Springer-Verlag London Limited, 2010.

[19] Rao V. Dukkipati, "*Solving Vibration Analysis Problems Using MATLAB*," ISBN 978-81-224-2427-0, NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHERS, 4835/24, New Delhi - 110002, 2007.

[20] Raoelina Andriambololona, Rakotoson Hanitriarivo, Tokiniaina Ranaivoson, Roland Raboanary, "*Two Definitions of Fractional Derivatives of Powers Functions*" , Pure and Applied Mathematics Journal. Vol. 2, No. 1, 2013, pp. 10-19. DOI 10.11648/j. pamj.2013 02 01.12.

[21] Igor Podlubny, "*The Laplace Transform Method for Linear Differential Equations of the Fractional Order*," Department of Control Engineering Faculty of Mining, University of Technology B.Nemcovej 3, 042 00 Kosice, Slovakia, 1994.

[22] Frederick W. Byron Jr., Robert W. Fuller, "*Mathematics of Classical and Quantum Physics*," Dover Publications, 1992.

[23] B.C. Carlson, "*Jacobian Elliptic Functions as Inverses of an Integral*," Journal of Computational & Applied Mathematics. Vol. 174, Issue 2, 15 February 2005, PP. 355-359.

[24] Lawrence S. Taylor, Amy L. Lerner, Deborah J. Rubens, and Kevin J. Parker, "*A Kelvin-Voight Fractional Derivative Model for Viscoelastic Characterization of Liver Tissue*," ASME International Mechanical Engineering Congress & Exposition. November 17-22, 2002, New Orleans, Louisiana.

[25] R. Gorenflo, and F. Mainardi, "*Fractional Oscillations and Mittag-Leffler Functions*," pre-print A-14/96, Department of Mathematics & Computer Science, Freie University, Berlin, 1996.

[26] J. L. Lavoie, T. J. Osler, and R. Tremblay, "*Fractional Derivatives and Special Functions*, SIAM review. Vol. 18 Issue 2(1976), PP. 240-268, JSTOR.

[27] Ying Luo, HongSheng Li, and YangQuan Chen, "*Fractional Order Proportional and Derivative Controller Synthesis for A Class of Fractional Order Systems: Tuning Rule and Hardware-in-the-loop Experiment*," Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference Shanghai, P.R. China, December 16-18, 2009.

[28] Stuart E. Dreyfus, "*Dynamic Programming and the Calculus of Variations*," Journal of Mathematical Analysis & Applications. Vol. 1, 1960, PP. 228-239.

[29] F. Iavernaro and B. Pace, "*S-Stage Trapezoidal Methods for the Conservation of Hamiltonian Functions of Polynomial Type*," American Institute of Phisycs, T. Simos et al. eds. 2007, pp.603-606.

[30] Richard Magin, Manuel D. Ortigueira, Igor Podlubny, and Juan Trujillo, "*On the Fractional Signals and Systems*," Signal Processing. Vol. 91, 2011, PP.350-371, Science Direct, Elsevier B.V.

[31] Lixia Yuan, and Om P. Agrawal, "*A Numrical Scheme for Dynamic Systems Containing Fractional Derivatives*," Proceedings of DETC'98, 1998 ASME Design Engineering Technical Conferences, September 13-16, 1998, Atlanta, Georgia.

[32] Mario Di Paola, Francesco Paolo Pinnola, Massimiliano Zingales, "*Fractional Differential Equations and Related Exact Mechanical Models*," Computers and Mathematics with Applications, in press, DOI. org/10.1016/j. camwa.2013.03.012, 2013.

[33] Yan Li, Dingyu Xue, and YangQuan Chen, "*Sensitivity Function of LTI Fractional Order Dynamic Systems with Respect to the Orders*," 2010 American Control Conference Marriott Waterfront, Baltimore, MD, USA June 30-July 02, 2010.

[34] Jan W. Gooch, "*Encyclopedic Dictionary of Polymers*," ISBN 978-0-387-31021-3 (Print) 978-0-387-30160-0 (Online), 2007.

[35] Jens Niegemann, Richard Diehl, Kurt Busch, "*Efficient Low-Storage Runge-Kutta Schemes with Optimized Stability Regions*," Journal of Computational Physics. Vol. 231, 2012, PP. 364-372.

[36] Katerina D. Papoulia, Vassilis P. Panoskaltsis, Nishu V. Kurup, Igor Korovajchuk, "*Rheological Representation of Fractional Order Viscoelastic Material Models*," Rheol Acta. Vol. 49, 2010, PP. 381-400, DOI 10.1007/s00397-010-0436-y, Springer.

[37] Dumitru Baleanu, Jihad H. Asad, and Ivo Petras, "*Fractional-Order Two-Electric Pendulum*," Romanian Reports in Physics. Vol. 64, No. 4, 2012, PP. 907-914.

[38] Rachid Malti, Mohamed Aoun, Olivier Cois, Alain Oustaloup. Asad, and Francois Levron, "*H_2 Norm of Fractional Differential Systems*," ASME 2003 Design Engineering Technical Conferences & Computers & Information in Engineering Conference, September 2-6, 2003, Chicago, Illinois, USA.

[39] LI Dazi, Fan Weiguang, Jin Qibing, and Tan Tianwei, "*An IMC-$PI^{\lambda}D^{\mu}$ Controller Design for Fractional Calculus System*," Proceedings of the 29th Chinese Control Conference, July 29-31, 2010, Beijing, China.

[40] E.Harrier, S.P.Norsett, G. Wanner "*Solving Ordinary Differential Equations Nonstiff Problems*," Second Edition ISBN 978-3-540-56670-0, 1993, Springer.

[41] Quanser Innovate Educate, "*IP01 and IP02 Single Inverted Pendulum*," Lab. Document No. 511, Linear Motion Servo Plants: IP01 and IP02, User manual.

[42] L. Deseri, M. Di Paola, M. Zingales, and P. Pollaci, "*Power-Law Hereditariness of Hierarchical Fractal Bones*," International Journal for Numerical Methods in Biomedical Engineering, 29.12 2013, PP. 1338-1360.

[43] L. Chunna Zhao, Dingyu Xue, and YangQuan Chen, "*A Fractional Order PID Tuning Algorithm for A Class of Fractional Order Plants*," Proceedings of the IEEE International Conference on Mechatronics & Automation Niagara Falls, CANADA, July 2005.

[44] Vasily E. Tarasov, "*Fractional Dynamics Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*," Nonlinear Physical Science, ISBN 978-7-04-029473-6, Higher Education Press, Beijing, Springer.

[45] Dingyu Xue, Chunna Zhao, and YangQuan Chen, "*A Modified Approximation Method of Fractional Order System*," Proceedings of the 2006 IEEE International Conference on Mechatronics & Automation June 25 - 28, 2006, Luoyang, China.

[46] Agnieszka B. Malinowska, Delfim F.M. Torres "*Generalized Natural Boundary Conditions for Fractional Variational Problems in Terms of the Caputo Derivative*," Computers & Mathematics with Applications. Vol. 59, 2010, PP. 3110-3116.

[47] Giacomo Torzo, Elisabeta Ana Ur, "*Study of the Pendulum with Computer Based Laboratory (CBL) and Optical Encoder,*" ANALELE University DIN BUCURESTI, FIZICa - LVI 2007, ITALY.

[48] R. Lewandowski, and B. Chorazyczewski, "*Identification of the Parameters of the Kelvin-Voigt and the Maxwell Fractional Models, Used to Modeling of Viscoelastic Dampers*," Computers & Structures 2009, ELSEVIER.

[49] M. Klimek, "*Solutions of Euler-Lagrange Equations in Fractional Mechanics*," Proceedings of the XXVI Workshop on Geometrical Methods in Physics, Bialowieza 2006, Eds. P. Kielanowski, A. Odzijewicz, T. Voronov, 2007, Vol. 956 of AIP Conference Proceedings, American Institute of Physics, PP. 73-78.

[50] Arsalan Shokooh and Luis Suarez, "*A Comparison of Numerical Methods Applied to a Fractional Model of Damping Materials*," Journal of Vibration & Control 1999 5: 331, DOI 10.1177/107754639900500301.

[51] Elmas Anli, and Ibrahim Ozkol, "*Classical and Fractional-Order Analysis of the Free and Forced Double Pendulum*," Scientific Research Engineering. Vol. 2, 2010, PP. 935-949, dOI 10.4236.

[52] Francis B. Hildebrand, "*Advanced Calculus for Applications*," PRENTICE-HALL, INC. Englewood Cliffs, New Jersey 1962.

[53] M. Merdan, "*On the Solutions of Time-Fractional Generalized Hirota-Satsuma Coupled-kdv Equation with Modified Riemann-Liouville Derivative by an Analytical Technique*," Proceeding of the Romanian Academy, Series A. Vol. 16, Number 1/2015, PP. 3-10.

[54] Zaid M. Odibat, "*Adaptive Feedback Control and Synchronization of Non-Identical Chaotic Fractional Order Systems*," Nonlinear Dyn 2010, 60: 479-487, DOI 10.1007/s11071-009-9609-6, Springer Science & Business Media B.V. 2009.

[55] Romeo Ortega, Antonio Loria, Per Johan Niklasson, and Herbertt Sira-Ramirez, "*Passivity-based Control of Euler-Lagrange Systems Mechanical, Electrical, and Electromechanical Applications*," Communications & Control Engineering Series ISSN 0178-5354, ISBN 1-85233-016-3, Springer-Verlag London Limited 1998.

[56] G. L. Baker J.P.Gollub, "*Chaotic Dynamics: An Introduction*," Published by the Press Syndicate of the University of Cambridge, 1990. Available from the British Library, ISBN 0-521-47685-2 paperback.

[57] Susumu Sakakibara, "*Fractional Derivative Models of Damped Oscillations*," 'n Partial Differential Equations and Time-Frequency Analysis, Suuken Kokyuroku, RIMS Kyoto University, 2004.

[58] C. H. Eab, and S. C. Lim, "*Fractional Langevin Equation of Distributed Order*," 52, Jalan Desa Maju, Taman Desa, 58100 Kuala Lumpur, Malaysia, 2010.

[59] j. H. Heinbockel, "*Introduction to the Variational Calculus*," ISBN-13 978-1425103521, 2006, Trafford on Demand Pub.

[60] Masatsugu Suzuki and Itsuko S. Suzuki, "*Physics of Simple Pendulum a Case Study of Nonlinear Dynamics*," Department of Physics, State University of New York at Binghamton, New York 3902-6000,2008, U.S.A.

[61] Om P. Agrawal, "*A Formulation and Numerical Scheme for Fractional Optimal Control Problems*," Journal of Vibration & Control 2008 14: 1291, DOI 10.1177/1077546307087451.

[62] Changpin Li, and Yutian Ma, "*Fractional Dynamical System and its Linearization Theorem*," Nonlinear Dyn 2013 Vol. 71, 2010, PP. 621-633, DOI 10.1007/s11071-012-0601-1, Springer Science & Business Media B.V.

[63] Indranil Pan, Saptarshi Das, "*Chaotic Multi-Objective Optimization Based Design of Fractional Order $PI^{\lambda}D^{\mu}$ Controller in AVR System*," Electrical Power & Energy Systems 43 (2012) 393-407, ELSEVIER.

[64] Dumitru Baleanu, and Juan I. Trujillo, "*A New Method of Finding the Fractional Euler-Lagrange and Hamilton Equations within Caputo Fractional Derivatives*," Commun Nonlinear Sci Numer Simulat. Vol. 15 2010, PP. 1111-1115, ELSEVIER.

[65] Xianzhou Zhang, Weihua Li, and Yang Zhou, "*A Variable Stiffness MR Damper for Vibration Suppression*," 2009 IEEE/ASME International Conference on Advanced Intelligent Mechatronics, AIM, IEEE, Singapore, Singapore, pp. 106-111.

[66] Anatoly A. Kilbas, Hari M Srivastava, and Juan J. Trujillo, "*Theory and Applications of Fractional Differential equations*," ISBN-13 978-0-444-51832-3, 2006 Elsevier B.V.

[67] M.P. Singh, T. S. Chang, and H. Nandana, "*Algorithms for Seismic Analysis of MDOF Systems with Fractional Derivatives*," Engineering Structures. Vol. 33 2011, PP. 2371-2381, Elsevier.

[68] R. Hilfer, "*Threefold Introduction to Fractional Derivatives*," ISBN 978-3-527-40722-4, Anomalous Transport: Foundations & Applications, R. Klages et al. (eds.),Wiley-VCH,Weinheim, 2008, page 17.

[69] YangQuan Chen, Dingyu Xue, and Huifang Dou, "*Fractional Calculus and Biomimetic Control*," Proceedings of the 2004 IEEE International Conference on Robotics & Biomimetics August 22 - 26, 2004, Shenyang, China.

[70] Tatiana Odzijewicz, and Delfim F. M. Torres, "*Calculus of Variations with Classical and Fractional Derivatives*," Math. Balkanica 26 (2012), No. 1-2, PP. 191-202.

[71] Karla A. Villarreal, "*Effects of MR Damper Placement on Structure Vibration Parameters*," REUJAT Symposium, Tokyo, Japan, 2004.

[72] Francesco Mainardi, Antonio Mura, and Gianni Pagnini, "*The M-Wright Function in Time-Fractional Diffusion Processes: A Tutorial Survey*," Hindawi Publishing Corporation International Journal of Differential Equations, 2010, Article ID 104505, 29 pages DOI 10.1155/2010/104505.

[73] 73 Chunna Zhao and Liming Luo, Yu Zhao, "*Fractional Modeling Approach with Mittag-Leffler Functions for Linear Fractional-order System*," Fifth International Conference on Intelligent Computation Technology & Automation, 2012.

[74] Francesco Mainardia, Rudolf Goren, "*On Mittag-Leffler-Type Functions in Fractional Evolution Processes*," Journal of Computational & Applied Mathematics. Vol. 118 2000, PP. 283-299, Elsevier.

[75] A. Charef, D. Idiou, A. Djouambi, and A. Voda, "*Identification of Linear Fractional Systems of Commensurate Order*," Proceedings of the 3rd International Conference on Systems and Control, Algiers, Algeria, October 29-31, 2013.

[76] Ricardo Almeida, and Delfim F.M. Torres, "*Calculus of Variations with Fractional Derivatives and Fractional Integrals*," Applied Mathematics Letters. Vol. 22 2009, PP. 1816-1820, Elsevier.

[77] David A. Miller, and Stephen J. Sugden, "*Insight into the Fractional Calculus via a Spreadsheet*," Spreadsheets in Education (eJSiE): Vol. 3: Iss. 2, Article 4, 2009.

[78] B.N. Narahari Achar, J.W. Hanneken, T. Enck, and T. Clarke, "*Dynamics of the Fractional Oscillator*," Physica A, Vol. 297, 2001, PP. 361-367, Elsevier.

[79] Ivo Petras, " *A Note on the Fractional-Order Volta's System*," Communications in Nonlinear Science & Numerical Simulation 15 (2010) 384-393, Elsevier.

[80] Katsuhiko Ogata, "*Modern Control Engineering*," Fourth Edition, Prentice Hall.

[81] El-Nabulsi Ahmad Rami, "*Fractional Variational Problems from Extended Exponentially Fractional Integral*," Applied Mathematics and Computation. Vol. 217 2011, PP. 9492-9496, Elsevier.

[82] Marcia Kleinz, and Thomas J. Osler, "*A Child's Garden of Fractional Derivatives*," The College Mathematics Journal. Vol. 31, No. 2, 2000, pp. 82-887.

[83] Rudolf Gorenflo, and Francesco Mainardi, "*FRACTIONAL CALCULUS : Integral and Differential Equations of Fractional Order*," International Centre for Mechanical Sciences Palazzo del Torso, Piazza Garibaldi, Udine, Italy.

[84] James Stewart, "*Multivariable Calculus: Consepts and Contexts*," ISBN-13 978-0-495-56054-8, Cengage Learning Inc, 2009.

[85] Tatiana Odzijewicz , Agnieszka B. Malinowska, and Delfim F.M. Torres, "*Fractional Variational Calculus with Classical and Combined Caputo Derivatives*," Nonlinear Analysis 75 (2012) 1507-1515, Elsevier Ltd.

[86] Ivee K. Guce, "*On Fractional Derivatives: The Non-integer Order of the Derivative*," International Journal of Scientific & Engineering, Research. Vol. 4, Issue 3, March-2013 1, ISSN 2229-5518.

[87] A. M. Mathai, Ram Kishore Saxena, and Hans J. Haubold, "*H-Function Theory and Applications*," Springer Science & Business Media, 2009.

[88] Mohamed A.E. Herzallah, and Dumitru Baleanu, "*Fractional-Order Euler-Lagrange Equations and Formulation of Hamiltonian Equations*," Nonlinear Dyn 2009, PP. 58, PP. 385-391, DOI 10.1007/s11071-009-9486-z.

[89] A. Hanslmeier, and R. Dvorak, "*Numerical Integration with Lie Series*," Astronomy and Astrophysics, Vol. 132, 1984, No. 1, PP. 203.

[90] Igor podlubny, "*Fractional-Order Systems and Fractional-Order Controllers*," Slovak Academy of Sciences, Institute of Experimental Physics, UEF-03-94, 1994.

[91] Francesco Mainardia, Marco Raberto, Rudolf Gorenflo, and Enrico Scalas, "*Fractional Calculus and Continuous-Time Fnance II: the Waiting-Time Distribution*," Physica A Vol. 287 2000, PP. 468-481, ELSEVIER.

[92] Tansel Avkar, and Dumitru Baleanu, "*Fractional Euler-Lagrange Equations for Constrained Systems*," Gp729, Global Analysis and Applied Mathematics: International Workshop on Global Analysis, 2004 American Institute of Physics 0-7354-0209-4/04/22.00.

[93] Yu. Luchko, R. Gorenflo, "*Scale-Invariant Solutions of a Partial Differential Equation of Fractional Order*," Fractional Calculus and Applied Analysis. Vol. 1, 1998, PP. 63-78.

[94] A. M. Mathai, Hans J. Haubold, "*Special Functions for Applied Scientists*," ISBN 978-0-387-75893-0 (Print) 978-0-387-75894-7 (Online) Springer, 2008.

[95] S.A. Yousefi, Mehdi Dehghan, and A. Lotfi, "*Generalized Euler-Lagrange Equations for Fractional Variational Problems with Free Boundary Conditions*," Computers & Mathematics with Applications. Vol. 62 2011, PP. 987-995, ELSEVIER.

[96] Ivo Petras, "*Fractional-Order Nonlinear Systems Modeling, Analysis and Simulation*," ISBN 978-3-642-18100-9, e-ISBN 978-3-642-18101-6, Springer Science & Business Media, 2011.

[97] B.N. Narahari Achar, John W. Hanneken, and T. Clarke, "*Damping Characteristics of a Fractional Oscillator*," Physica A. Vol. 339 2004, PP. 311-319, ELSEVIER.

[98] El-Nabulsi Ahmad Rami, "*Fractional Euler-Lagrange Equations of Order* (α,β) *for Lie Algebroids*," Studies in Mathematical Sciences. Vol. 1, No. 1, 2010, pp. 13-20, ISSN 1923-8452.

[99] Riccardo Caponetto, Giovanni Dongola, Luigi Fortuna, and Ivo Petras, "*Fractional Order Systems Modeling and Control Applications*," World Scientific Publishing Co. Pte. Ltd. 5 Toh Tuck Link, Singapore 596224, ISBN-13 978-981-4304-19-1.

[100] D. Valerio, J. J. Trujillo, M. Rivero, J. T. Machado, and D. Baleanu, "*Fractional Calculus: A Survey of Useful Formulas*," The European Physical Journal Special Topics. Vol. 222, 2013, No. 8, pp. 1827-1846.

[101] Duarte Valerio1, Juan J. Trujillo, Dongling Wang, and Aiguo Xiao, "*Fractional Variational Integrators for Fractional Euler-Lagrange Equations with Holonomic Constraints*," Commun Nonlinear Sci Numer. Simulat. Vol. 18 2013, PP. 905-914, ELSEVIER.

[102] Margarita Rivero, Sergei V. Rogosin, Jose A. TenreiroMachado, and Juan J. Trujillo, "*Stability of Fractional Order Systems*," Hindawi Publishing Corporation Mathematical Problems in Engineering. 2013, Article ID 356215, 14 pages.

[103] Varsha Bhambhani, and YangQuan Chen, "*Experimental Study of Fractional Order Proportional Integral (FOPI) Controller for Water Level Control*," Proceedings of the 47th IEEE Conference on Decision and Control Cancun, Mexico, Dec. 9-11, 2008.

[104] Debra Anne Ross, "*Master Math Trigonometry*," Library of Congress Control Number: 2009924622, ISBN-10 1-59863-985-4, Course Technology PTR 2009.

[105] Om P. Agrawal, "*Generalized Euler-Lagrange Equations and Transversality Conditions for FVPs in terms of the Caputo Derivative*," Journal of Vibration & Control 2007 13: 1217 DOI 10.1177/1077546307077472.

[106] Ivo Petras, "*Stability of Fractional-Order Systems with Rational Orders: A Survey*," Fractional Calculus & Applied Analysis. Vol. 12, 2009, No. 3, pp. 269-298.

[107] Anuj Narang, Sirish L. Shah, and Tongwen Chen, "*Tuning of Fractional PI Controllers for Fractional Order System Models with and without Time Delays*," 2010 American Control Conference, Marriott Waterfront, Baltimore, MD, USA June 30-July 02, 2010.

[108] Ricardo Almeida, Hassan Khosravian-Arab and Mostafa Shamsi, "*A Generalized Fractional Variational Problem Depending on Indefinite Integrals: Euler-Lagrange Equation and Numerical Solution*," Journal of Vibration & Control 2013 19: 2177 originally published online 4 September 2012, DOI 10.1177/1077546312458818.

[109] YangQuan Chen, Ivo Petras and Dingyu Xue, "*Fractional Order Control - A Tutorial*," 2009 American Control Conference, Hyatt Regency Riverfront, St. Louis, MO, USA. June 10-12, 2009.

[110] Dumitru Baleanu, Kai Diethlem, Enrico Scalas, and juan J. Trujillo, "*Fractional Calculus Models and Numerical Methods*," published by World Scientific Publishing Co. Singapore 596224, 2012.

[111] A. Narang, S.L. Shah, and T. Chen, "*Continuous-time Model Identification of Fractional-Order Models with Time Delays*," Published in IET Control Theory & Applications, 2010, DOI 10.1049/iet-cta.2010.0718, ISSN 1751-8644.

[112] Hans J. Haubold, "*Understanding Nature Through Reaction and Diffusion*," Special Functions and Functions of Matrix Argument: Recent Advances and Applications in Stochastic Processes, Statistics and Astrophysics 1.27 (2006): 1-A58.

[113] Farshad Merrikh-Bayat, and Mahdi Afshar, "*Extending the Root-Locus Method to Fractional-Order Systems*," Hindawi Publishing Corporation Journal of Applied Mathematics Volume 2008, Article ID 528934, 13 pages, DOI 10.1155/2008/528934.

[114] Dingyu Xue,YangQuan Chen, and Derek P. Atherton, "*Linear Feedback Control Analysis and Design with MATLAB*," SIAM Press, ISBN 978-0-898716-38-2. (348 pages) Chapter-8: Fractional-order Controller - An Introduction, 2007.

[115] Om Prakash Agrawal, "*Generalized Variational Problems and Euler-Lagrange Equations*," Computers & Mathematics with Applications 59 (2010) 1852-1864, ScienceDirect.

[116] Richard Bellman, "*Stability Theory of Differential Equations*," McGRAW-HILL BOOK COMPANY, New York, N.Y. 1953.

[117] T. Poinot, J. C. Trigeassou, "*A Method for Modelling and Simulation of Fractional Systems*," Signal Processing. Vol. 83 2003, PP. 2319 - 2333, ELSEVIER.

[118] Prashanth Ramachandran,and Y.M.Ram, "*Stability Boundaries of Mechanical Controlled System with Time Delay*," Mechanical Systems & Signal Processing. Vol. 27 2012, PP. 523-533, ELSEVIER.

[119] Sami I. Muslih, and Dumitru Baleanu, "*Fractional Euler-Lagrange Equations of Motion in Fractional Space*," Journal of Vibration & Control 2007, 13, 1209, DOI 10.1177/1077546307077473.

[120] Rachid Malti, Mohamed Aoun, Jocelyn Sabatier, and Alain Oustaloup, "*Tutorial on System Identification Using fractional Differentiation Models*," 14th IFAC Symposium on System Identification, Newcastle, Australia, 2006.

[121] Dumitru Baleanu, and sami I Muslih, "*Fractional Euler-Lagrange and Fractional Hamilton Equations for Super Symmetric Classical Model*," Fractals. Vol. 15, No. 4, 2007, PP. 379-383, World Scientific Publishing Company.

[122] K. M. Hangos, J. Bokor, and G. Szederkenyi, "*Analysis and Control of Nonlinear Process Systems*," Advanced textbook in control and signal processing, ISSN 1439-2232, ISBN 1-85233-600-5, Springer Science & Business Media, 2004.

[123] Yan Li, and YangQuan Chen, "*Fractional Order Linear Quadratic Regulator*," 1-4244-2368-2/08/20.OO 2008 IEEE.

[124] L. T. Lee, and D. C. Lee, "*Free Vibration of Rectangular Plates Elastic Point Supports with the Application of a New Type of Admissible Function*," Computer & Structures. Vol. 65, 1997, No. 2, pp. 149-156. Elsevier Science Ltd Printed in Great Britain PII: SO045-7949(96)00426-9.

[125] Manisha K.Bhole, Mukesh D. Patil, and Vishwesh A. Vyawahare, "*Stability Analysis of Fractional-order Systems*," International Conference & Workshop on Recent Trends in Technology, (TCET)2012 Proceedings published in International Journal of Computer Applications (IJCA).

[126] Dumitru Baleanu, Jose Antonio Tenreiro Machado, and Albert C. J. Luo, " *Fractional Dynamics and Control*," A book based on the 3rd Conference on Nonlinear Science & Complexity (NSC), Ankara, Turkey, July 27-31, 2010.

[127] Francis B. Hildebrand, "*Methods of Applied Mathematics*," Prentice-Hall, Inc, 1965.

[128] Mauro Bologna, "*Short Introduction to Fractional Calculus*," Lecture Notes, PP. 41-54, Univ. Tarpeca Chile, 2005.

[129] Dumitru Baleanu, and Juan J. Trujillo, "*On Exact Solutions of a Class of Fractional Euler-Lagrange Equations*," Nonlinear Dyn, Vol. 52, 2008, PP. 331-335, DOI 10.1007/s11071-007-9281-7, Springer Science & Business Media B.V. 2007.

[130] Tansel Avkar, and Dumitru Baleanu, "*Fractional Euler-Lagrange Equations for Constrained Systems*," Cp729, Global Analysis & Applied Mathematics: International Workshop on Global Analysis, 2004 American Institute of Physics 0-7354-0209-4/04/22.00.

[131] Faycal Ben Adda, and Jacky Cresson, "*About Non-differentiable Functions*," Journal of Mathematical Analysis & Applications. Vol. 263, 2001, PP. 721-737, DOI 10.1006/jmaa.2001.7656.

[132] Jacky Cresson, "*Fractional Embedding of Differential Operators and Lagrangian Systems*," Journal of Mathematical Physics. Vol. 48 No. 3 2007, 033504.

[133] Mehdi Rahimy, "*Applications of Fractional Differential Equations*," Applied Mathematical Sciences. Vol. 4, 2010, No. 50, PP. 2453 - 2461.

[134] Abolhassan Razminia, Dumitru Baleanu, and Vahid Johari Majd, "*Conditional Optimization Problems: Fractional Order Case*," J. Optim. Theory Appl. 2013, Vol. 156, PP. 45-55, DOI 10.1007/s10957-012-0211-6, Published online at Springer Science & Business Media New York 2012.

[135] Zhengang Zhao, Qian Guo, and Chan gpin Li, "*A Fractional Model for the Allometric Scaling Laws*," The Open Applied Mathematics Journal, 2008, 2, 26-30.

[136] T M Atanackovic, S Konjik, and S. Pilipovic, "*Variational Problems with Fractional Derivatives: Euler-Lagrange Equations*," Journal of Phsics A: Mathematical & Theoretical, DOI 10.1088/1751-8113/41/9/095201, 2008.

[137] Erwin Kreyszig, "*Advanced Engineering mathematics*," Tenth edition, ISBN 978-0-470-45836-5 John Wiley & Sons, INC, 1988.

[138] Emine Ozergin, "*Some Properties of Hypergeometric Functions*," Ph.D. Thesis, Eastern Mediterranean University, North Cyprus, Turkey, 2011.

[139] Tetsuya Suemitsu, Kenji Siojima, Takashi Makimura, and Naoteru Shigekawa, " *Intrinsic Transit Delay and Effective Electron Velocity of AlGaN/GaN High Electron Mobility Transistors*," Japanese Journal of Applied Physics. Vol. 44, No. 6, 2005, pp. L 211-L 213, 2005 The Japan Society of Applied Physics.

[140] El-Nabulsi Ahmed Rami, "*A Fractional Approach to Nonconservative Lagrangian Dynamical Systems*," FIZIKA A 14. Vol. 4, 2005, PP. 289-298.

[141] M. El-Kady, Amaal El-Sayed, "*A Fractional Differentiation Matrices for Solving Fractional Orders Differential Equations* ," International Journal of Pure & Applied Mathematics. Vol. 84 No. 2 2013, PP. 1-13, ISSN 1311-8080 (printed version); ISSN 1314-3395 (on-line version).

[142] K. Diethelm, and N. J. Ford, "*Numerical Solution of The Bagley-Torvik Equation*," Swets & Zeitlinger, BIT Numerical Mathematics, 0006-3835/02/4203-0490, 16.00. Vol. 42, 2002, No. 3, pp. 490-507.

[143] G. Diaz, C. F. M. Coimbra, "*Nonlinear Dynamics and Control of a Variable Order Oscillator with Application to the Van Der Pol Equation*," Nonlinear Dynamics. Vol. 56, April 2009, Issue 1-2, pp 145-157, Springer.

[144] S. Shen, F. Liu, J. Chen, I. Turner, V. Anh, "*Numerical Techniques for the Variable Order Time Fractional Diffusion Equation*," Applied Mathematics & Computation. Vol. 218, 2012, PP. 2012 10861-10870, Elsevier.

[145] S. Hemamalini, Sishaj P. Simon, "*Dynamic Economic Dispatch Using Maclaurin Series Based Lagrangian Method*," Energy Conversion & Management. Vol. 51, 2010, PP. 2212-2219, Elsevier.