A NONCOOPERATIVE DYNAMIC GAME MODEL OF OPINION DYNAMICS IN MULTILAYER SOCIAL NETWORKS

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By

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

A NONCOOPERATIVE DYNAMIC GAME MODEL OF OPINION DYNAMICS IN MULTILAYER SOCIAL NETWORKS

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How do people living in a society form their opinions on daily or prevalent topics? A noncooperative differential (dynamic) game model of opinion dynamics, where the agents' motives are shaped by how susceptible they are to others' influence, how stubborn they are, and how quick they are willing to change their opinions on socially prevalent issues is considered here. The agents connected through a multilayer network interact with each other on a set of issues (layers) for a finite time duration. They express their opinions, listen to others' and, hence, mutually influence each other. The tendency of agents to interact with people of similar traits, known as *homophily*, restricts them in their own localities, which may correspond to ethnicity but may as well be the ideological ones. This governs their interpersonal influences and is the cause of clustering in the network. As the agents build their biases, they also create conceptions about the correlation between the issues. As a result, antagonistic interactions arise if the agents see each other as holding inconsistent opinions on the issues according to their individual conceptions. This way the interpersonal influence becomes ineffective leading to conflict and disagreement between the agents. The dynamic game formulated here takes these subtle issues into account.

The game is proved to admit a unique Nash equilibrium under a mild necessary and sufficient condition. This condition is argued to be fulfilled if there is some harmony of views among the agents in the network. The harmony may be in the form of similarity in pairwise conceptions about the issues but may also be a collective agreement on the status of a leader in the network.

Since the agents do not seek any social motive in the game but their own individual motives, the existence of a Nash equilibrium can be interpreted as an

emergent collective behavior out of the noncooperative actions of the agents.

Keywords: Social networks, Dynamic game theory, Opinion dynamics, Optimal control.

ÖZET

ÇOK KATMANLI SOSYAL AĞLARDA OLUŞAN GÖRÜŞLERİN BİR İŞBİRLİKSİZ DİNAMİK OYUN OLARAK MODELLENMESİ

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Bir topluluk içinde yaşayan kişiler güncel veya gündemde olan konulardaki fikirlerini nasıl oluştururlar? Burada, diğer kişilerden ne miktarda etkilenecekleri, ne kadar inatçı oldukları ve kendi fikirlerini ne hızda değiştireceklerinden oluşan güdülerini dikkate alan, işbirliksiz türevsel bir oyun, sosyal ağlarda görüş (fikir) dinamiğinin bir modeli olarak sunulmaktadır. Bu topluluktaki bireyler çok katmanlı bir sosyal ağla birbirleriyle iletişimdedirler ve sonlu bir zaman aralığında fikir beyan eder, birbirlerinin fikirlerini dinler, görüş alışverişi yapar ve böylece birbirlerini karşılıklı etkilerler. Kişilerin kendi görüşlerine yakın insanlarla daha çok temasta bulunmaları, belki etnik kimlik belki de ideoloji nedeniyle oluşan *insani-yakınlık*, kendilerini bir miktar taşralaştırır. Bu onların karşılıklı etkileşiminde baskın öğe olabilir ve hizipleşmelere yol açabilir. Kişiler kendi yanlı görüşlerini bu şekilde oluştururken, aynı zamanda gündem konularının aralarında ne kadar örtüştüğü, bunların birbiriyle ilgisi, korelasyonu, hakkında da algılar edinirler. Eğer komşularında bu kişisel algılarının tersine bir yaklaşım tespit ederlerse, bu sefer komşularını kendilerine bir rakip olarak görmeye başlayabilirler. Bu şekilde, kişiler arası etkileşim sosyal ağ içinde anlaşmazlıklar ve çatışmalara yol açabilir. Bu tezde sunulan oyun, toplumsal etkileşimdeki tüm bu karmaşık noktaları da göz önüne almaktadır.

Sunulan dinamik oyunun tek bir Nash dengesi olması için gerek ve yeter bir koşul verilmektedir. Gereken koşulun makul durumlarda genellikle sağlandığı da gösterilmektedir ve esas olarak gereken, toplulukta bir tür ahenk bulunmasıdır. Bu ahenk karşılıklı fikir benzerliği olabileceği gibi lider olarak tanınan tek bir kişiyle uyuşma (itaat) şeklinde de oluşabilir.

Burada incelenen türden bir topluluk içinde toplumsal bir güdü olmadığı, her kişinin güdüsü bireysel olduğu, için ortaya çıktığı gösterilen bu Nash dengesi işbirliksiz bir ortamda kendiliğinden doğan bir ortak eylem gibi yorumlanabilir.

Anahtar sözcükler: Sosyal ağlar, Dinamik oyun teorisi, Fikir dinamiği, En iyi denetim.

In the name of God—the source of love, knowledge, and life.

To my family, for their love and support.

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Chapter 1

Introduction

The models describing opinion dynamics are quite helpful in explaining the situations of conflict and conformity in a society. These models, in addition to those in social sciences and economics, also find a variety of applications in the fields of engineering and computer science, [48] – [52]. Some of the applications include PageRank computation, [27], distributed optimization, [39], [55], formation control, [49], and power control, [47]. They have thus attracted a diverse community of researchers in different fields as evidenced by [1], [28], [34], and [35], in which several such models are surveyed. The earlier ones, like DeGroot's model, [13], describe how agents reach a consensus on some issue by averaging their opinions with others' via repeated interactions. But, a total consensus is hardly reached in reality and it may be claimed that the situations of disagreements are more common in a society, [31]. The social influence model by Friedkin and Johnsen (FJ), [19], and bounded confidence models like those of Hegselmann and Krause (HK), [22], and Deffuant et al. (DW), [12], explain disagreements in the form of network clustering and polarization. Several variations of these models with different convergence results have been presented in [7], [16], [37], [38], and [44]. When agents interact with each other on multiple issues, they change their opinions based on how they view each other's opinions on those issues, [54]. In the following, we brief describe some of these matters that arise during the interactions.

1.1 Consensus Models

The earliest models of opinion dynamics – like DeGroot's model [13] and Lehrer and Wagner's model [33], describe how a group of agents in a network reach an agreement and form a common subjective opinion on some issue. The agents iteratively interact with each other, express their individual opinions, and update them at the end of each interaction. They choose the opinion which is a weighted average of the initial opinions expressed at the beginning of each interaction by the group. The group reaches a consensus if the interaction graph is connected, i.e., every node (or agent) in the graph is reachable by every other node in finite steps. In other words, all the recurrent states of the Markov chain communicate with each other and are aperiodic, [13].

It is not possible to judge which agents are closer to the truth than others in these models. It is because of the assumption that no information, instruction, or observation from outside is available. Hence, the question that whether the agents converge to a true value is beyond the realm of these models.

Consensus formation is a desirable outcome in a kind of interactions described in these models. But if we follow a realistic approach and observe the world around us, we may arrive at a conclusion that the situations which lead to conflicts and disagreements are more common in a society. As pointed out in [26], any model of consensus formation must also be at the same time capable of providing a satisfactory conditions for disagreement. Therefore, the importance to study models of disagreements becomes coherent.

1.2 Social Influence Network Theory

One of the earliest models to describe disagreements along with consensus is Friedkin-Johnsen (FJ) model, [19]. When agents interact, they perform cognitive weighted averaging of opinions with their neighbors in the network; where the weights are determined by the influence of others and by their own susceptibilities to interpersonal influence. A social structure then emerges out of these interactions, which determines the pattern and strengths of interpersonal influences among the agents of the network.

In a social network, agents interact with each other on a set of prevalent issues for a finite time duration. They express their opinions, listen to others' and, hence, mutually influence each other. Sometimes the influence is one-sided when the interaction is based on passive observation. This interpersonal influence may arise due to the personal biases and prejudices of agents, who make their judgments based on their local information. The locality may refer to the geographical location of the agents in the network, but the role of this factor is reduced with the advent of online social networks, media, and telecommunication. Instead, it would be more appropriate to consider the ideological and ethnic closeness as important factors that incite interpersonal influences among agents. All these characteristic similarities forge a tendency in them to bond with each other, known as *homophily*, [28], which is the foundation of Axelrod's dissemination of culture model, [4], and the bounded confidence models, [12] and [22].

Due to homophily, the frequency of interaction is higher between the agents who are characteristically similar to each other. Hence, agents usually tend to appreciate and get influenced by those who are similar to them in ethnicity, religion, political ideology, etc. Moreover, the agents also assent to each other's opinions based on ideological similarity, which is characterized by their *bounded confidences*. When the agents are positively influenced by each other, they may reach a consensus. But in the case of negative influences, [23], they have antagonistic interactions, [3], [45], [53], where they try to move away from each other in opinions. By moving away from each other, they can defend their positions or mind set more robustly and receive lesser influence from each other.

1.3 Bounded Confidence Models

The models of opinion dynamics under bounded confidence have been presented by Deffuant et al., [12], and Hegselmann and Krause, [22]. In these models, an agent holds opinions on the issues from a certain opinion space and interacts with only those agents whose opinions are sufficiently close to his opinions. That is, the agent will consider making consensus with only those agents whose opinions are ε-close to his opinions. The threshold value ε_i of some agent i is known as his *bounded confidence*.

The DW and HK models are similar in a sense that whenever the agents interact they perform repeated averaging under their bounded confidences. However, the models follow a different approach on how agents communicate with each other. In DW model, which is partly inspired by Axelrod's model [4], agents meet in random pairwise encounters after which they decide whether to make consensus or not, based on their bounded confidences. On the other hand, agents move towards the average value of their bounded confidence neighborhoods in the HK model.

1.4 Consistency Theory and Cognitive Dissonance

We propose to study opinion dynamics on several prevalent issues that might be correlated with each other. This correlation affects the opinions of agents on the issues and may give rise to antagonistic interactions among the agents. In our model, agents interact antagonistically if they find each other to be unreasonable or inconsistent. The assumption we adopt is that an agent will boycott those who hold inconsistent opinions, as he perceives, on correlated issues. If two issues are positively correlated with each other according to the conception of some agent i, then he expects others to hold similar kind of opinions on the issues. For instance, in the case where the issues are such that one can either support or oppose the issues, agent i is willing to positively interact with only those who either support both issues or oppose both of them. In other scenarios, when some agent j supports one issue and opposes the other; agent i finds him unreasonable and interacts antagonistically, in which case he will move away from him. Hence, agents approve or boycott based on how they view the opinions of others, [54], and whether or not the opinions of others are consistent.

In [44], a multidimensional extension of FJ model, [19], along with the conditions of stability and convergence is presented. The agents are assumed to be consistent in their belief system, and hence avoid *cognitive dissonance*, [18]. People are sensitive to inconsistencies in their actions and beliefs, and the recognition of this will cause dissonance, which they will try to resolve is a basic assumption of the theory of cognitive dissonance. We recognize the role of cognitive dissonance in two places in forming the motive of a person. First, a person may be more prone to or less open to influence by people whom he believes are consistent in their beliefs. Second, he may make an effort to be consistent in his beliefs on correlated issues and this effort may be contra to his stubbornness. This way, stubborn agents are reluctant to change their opinions on the issues, [2], hence they try to avoid minimizing their own inconsistencies on the correlated issues. In this way, they play a role in shaping the opinions of the community, much like the *éminence grise* of [9].

1.5 Main Contributions of the Thesis

We present a noncooperative differential (dynamic) game model of opinion dynamics in a society, where the agents' motives are shaped by how susceptible they are to get influenced by others, how stubborn they are, and how quick they are willing to change their opinions on a set of issues in a prescribed time interval. We can think of the society envisaged here as a multilayer network, where each layer represents a prevalent issue. The layers representing two correlated issues are connected through interlayer edges. And, for uncorrelated issues, the layers of the network are disconnected. Each agent is assumed to control the rate of change of his opinion on each issue. In Chapter 3, we give a problem definition and propose a procedure to determine the weight matrices and incorporate them in a cost functional that constitutes the model of agents' motives. The motives depict their social and psychological dispositions – to compromise fairly on their personal biases in order to achieve conformity with their neighbors in the network. Also, we define the problem in the framework of linear quadratic (LQ) differential games. There is a wide literature on LQ games which can be used to analyze the game of opinion dynamics. But we find it easier to use our own framework to obtain the results in Chapter 4.

The main objective of the investigation is to determine under what conditions on the motives and network structure a Nash equilibrium of the game exists, and whether or not it is unique. There are of course many notions of equilibrium in games along with Nash equilibrium. An interpretation in [42] for static games suggests that if a game is played several times without any strategic links between consecutive plays, then a Nash equilibrium is most likely reached. Nash equilibrium is a useful construct whenever the objective is to probe under what conditions a pattern of collective behavior emerges from independent motives of agents.

We prove the existence of a unique Nash equilibrium under mild conditions on the social influence in the network. We investigate the situations where the agents disagree with each other, which is the case when they interact antagonistically, or where they cannot arrive at a certain decision when their opinions do not converge and undergo persistent oscillations. The results for finite time horizon can be extended to infinite time horizon and it can be seen that the control input always remains bounded. As a corollary, we also prove that when there is a single prevalent issue, then Nash equilibrium surely exists and is unique, and the opinion dynamics always converge to some value.

In the game of opinion dynamics, we argue that a unique Nash equilibrium always exists if there is some harmonious view among the agents in the network. The harmony may be in the form of similarity in pairwise conceptions about the issues but may also be a collective agreement on the status of a "leader" in the network. We determine the best response function of the Nash equilibrium and the resulting opinion trajectories for each agent, by making suitable assumptions on the network topology and the nature of the influence and stubbornness parameters. As mentioned above, the theory developed here for one-stage, finite time interval game can also be applied to infinite time interval as well as to games with multiple stages. As applications, we investigate two hybrid (continuous and discrete) games of multiple stages in Chapter 5, where the agents interact with each other randomly with some probability [15], and they change their opinions according to the influences of others that are dependent on their heterogeneous bounded confidences.

The construction of the motives of agents in our game is similar to that of [20], in which an optimal update scheme based on the best response dynamics of a *static single-issue game* is presented and the question of convergence is examined. Similar game theoretic models include [6], [17], and [21], in which repeated averaging is interpreted as a best-response dynamics and the convergence to equilibrium is studied. In [6], a bound for the cost of disagreement in the network relative to the social optimum is given and a design to reduce this cost by adding extra edges in the network is presented. The work in [17] is the extension of [6] in a sense that they also incorporate the rationality of the agents by assuming a noisy version of the best-response dynamics, called logit dynamics. The dynamic game here is also inspired by the foraging swarm models in [43], [56], [57]. We study a similar swarming behavior in the scenario of opinion dynamics in social networks.

1.6 Organization of the Thesis

In Chapter 2, we provide some preliminaries from graph theory, matrix analysis, and noncooperative differential games. Chapter 3 contains problem definition of a differential game of opinion dynamics. We also discuss a procedure to choose weight matrices which makes them suitable for the reasoning of social influence and consistency theories. The main results of existence and uniqueness of Nash equilibrium are given in Chapter 4. First, we investigate the game in a general multilayer network, and then we make some topological assumptions on the network to find the explicit Nash equilibrium trajectories. We investigate the examples of two hybrid games of multiple stages in Chapter 5, where the game is played repeatedly. Finally, we dedicate Chapter 6 for conclusions and possible future works.

1.7 Notation and Terminology

Vectors are denoted by bold lowercase letters and matrices, by uppercase letters. The set of real numbers and nonnegative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. The set $\mathbf{A} \setminus \mathbf{B}$ denotes those elements of set $\mathbf{A}$ that are not in set $\mathbf{B}$. For any set $\mathbf{A}$, $|\mathbf{A}|$ denotes its cardinality. The $n \times 1$ vector of all ones and the identity matrix of size $n \times n$ are $\mathbf{1}_n$ and I_n, respectively; whereas I is the identity matrix of appropriate dimension. The set of eigenvalues of matrix A is denoted by $\text{eig}(A)$. The transpose of a matrix A is denoted by A'. A symmetric positive definite (respectively, semidefinite) matrix A is denoted as $A \succ 0$ ($A \succeq 0$). A positive (respectively, nonnegative) matrix A, i.e., a matrix with all its entries positive (respectively, nonnegative), is written as $A > 0$ ($A \geq 0$). The operator $\otimes$ denotes the Kronecker product and $\mathscr{L}$, the Laplace transform. A (block) diagonal matrix with matrices $A, ..., Z$ at its diagonal is written as $\text{diag}[A, \ldots, Z]$.

Chapter 2

Preliminaries

This chapter briefly describes the basics of graph theory and some results from matrix analysis, which are used to obtain the results in Chapter 4. We also provide a general framework of noncooperative open loop differential (dynamic) games and state the necessary conditions for the existence of Nash equilibrium. We also discuss the existence and uniqueness of Nash equilibrium in the case of linear quadratic differential games.

2.1 Graph Theory and Multilayer Networks

A graph $\mathsf{G}(\mathbf{N}, \mathbf{E})$ consists of the set of nodes, $\mathbf{N} = \{1, \ldots, n\}$, and the edges, $\mathbf{E} = \{(i, j) : \ i, j \in \mathbf{N} \text{ and } w_{ij} > 0\}$; where w_{ij} is the weight of the edge from i to j. For undirected graphs, the set $\mathbf{E}$ consists of unordered pairs and $w_{ij} = w_{ji}$, for $i, j \in \mathbf{N}$. On the other hand, the weights w_{ij} and w_{ji} in directed graphs may not be equal. The adjacency matrix of the graph G is defined as

$$A := \begin{bmatrix} 0 & w_{12} & \ldots & w_{1n} \\ w_{21} & 0 & \ldots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \ldots & 0 \end{bmatrix},$$

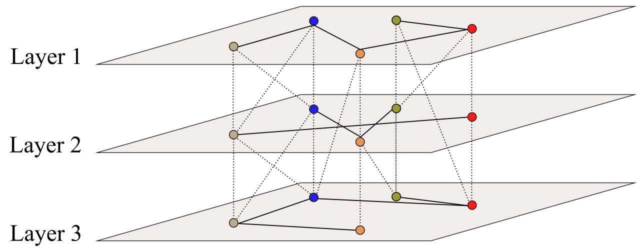


Figure 2.1: A multilayer network with three layers

where the entries $w_{ij} \geq 0$; and the degree matrix $D := \text{diag}[d_1, \ldots, d_n]$, where

$$d_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} w_{ij},$$

is the degree of node i. Then the Laplacian matrix is given by

$$L = \begin{bmatrix} d_1 & -w_{12} & \ldots & -w_{1n} \\ -w_{21} & d_2 & \ldots & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & \ldots & d_n \end{bmatrix}.$$

Consider a matrix $W = \text{diag}[w_{11}, \ldots, w_{nn}]$, which contains the self-weights of nodes at its diagonal. Then, we define a matrix $Q := L + W$

$$Q := \begin{bmatrix} m_1 & -w_{12} & \ldots & -w_{1n} \\ -w_{21} & m_2 & \ldots & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & \ldots & m_n \end{bmatrix},$$

where $m_i = d_i + w_{ii}$, $\forall\, i \in \mathbf{N}$.

A multilayer network, denoted by $\mathcal{M}(\mathcal{G}, \mathcal{C})$, consists of the family of graphs $\mathcal{G} = \{\mathsf{G}_{kk}(\mathbf{N}, \mathbf{E}_{kk});\ \mathbf{E}_{kk} \subset \mathbf{N} \times \mathbf{N},\ k \in \mathbf{D}\}$ at all the layers, and the set of crossed layer edges, $\mathcal{C} = \{\mathbf{E}_{kl} \subset \mathbf{N} \times \mathbf{N};\ k, l \in \mathbf{D},\ k \neq l\}$. Here, $\mathbf{D} = \{1, \ldots, d\}$ is the set of d layers, $\mathbf{E}_{kk}$ is the set of intralayer edges of G_{kk}, and $\mathbf{E}_{kl}$ is the set of interlayer edges between G_k and G_l, [8] and [29]. The Figure 2.1 shows a multilayer network with three layers; where the solid edges represent intralayer edges and dotted ones, interlayer edges. We define the matrix $Q \in \mathbb{R}^{nd}$ for multilayer networks as

$$Q := \begin{bmatrix} M_1 & -W_{12} & \dots & -W_{1n} \\ -W_{21} & M_2 & \dots & -W_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -W_{n1} & -W_{n2} & \dots & M_n \end{bmatrix}, \tag{2.1}$$

where $M_i = \sum_{j=1}^{n} W_{ij}$ and

$$W_{ij} = \begin{bmatrix} w_{ij,11} & w_{ij,12} & \dots & w_{ij,1d} \\ w_{ij,21} & w_{ij,22} & \dots & w_{ij,2d} \\ \vdots & \vdots & \ddots & \vdots \\ w_{ij,d1} & w_{ij,d2} & \dots & w_{ij,dd} \end{bmatrix}, \quad \forall\, i, j \in \mathbf{N},$$

with $w_{ij,kl}$ being the weight of the edge from node i at G_k to node j at G_l, for $k, l \in \mathbf{D}$ and $k \neq l$. Here, we assume that $w_{ij,kl} = w_{ij,lk}$ so that $W_{ij} = W_{ij}'$.

2.2 M-matrices and P-matrices

Let us denote the space of all real matrices of size $n \times n$ by $M_n(\mathbb{R})$, and a matrix $Q = [q_{ij}] \in M_n(\mathbb{R})$, where $i, j = 1, \dots, n$. Then, consider the following definitions (see [24] and [25]):

Definition 2.1: The set $Z_n \subset M_n(\mathbb{R})$ is defined as

$$Z_n = \{Q = [q_{ij}] \in M_n(\mathbb{R}) : q_{ij} \leq 0 \text{ if } i \neq j;\ i, j = 1, \dots, n\}.$$

Definition 2.2: A matrix $Q = [q_{ij}] \in Z_n$ is called M-matrix if $\mathrm{Re}(\lambda) > 0$, $\forall \lambda \in \mathrm{eig}(Q)$.

Definition 2.3: A matrix $Q \in M_n(\mathbb{R})$ is called a P-matrix if all its principal minors are positive.

Lemma 2.2.1: If Q is an M-matrix, then one can always write $Q = \bar{m}I - R$ with $\bar{m} = \max\{q_{ii} : i = 1, \dots, n\}$ and $R \geq 0$ (i.e., a nonnegative matrix).

Proof: For a detailed proof, see Section 2.5 of [25].

Lemma 2.2.2: If Q is a P-matrix, then every real eigenvalue of Q is positive.

Proof: For a detailed proof, see Section 2.5 of [25].

Theorem 2.1: (**Geršgorin Theorem**) Let $Q = [q_{ij}] \in M_n(\mathbb{R})$ and $r_i(Q) = \sum_{j \neq i} |q_{ij}|$, for $i = 1, \dots, n$, denote the absolute row sum of nondiagonal entries of i-th row of Q, and consider the n Geršgorin discs

$$\{z \in \mathbb{C} : |z - q_{ii}| \leq r_i(Q)\}, \quad i = 1, \dots, n.$$

The eigenvalues of Q are in the union of Geršgorin discs

$$G(Q) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - q_{ii}| \leq r_i(Q)\}. \tag{2.2}$$

Proof: Let $\lambda \in \text{eig}(Q)$ and $\mathbf{v} = [v_1 \dots v_n]' \in \mathbb{R}^n$ be the corresponding eigenvector such that $|v_m| = \max(|v_1|, \dots, |v_n|)$ for some $m \in \{1, \dots, n\}$. Consider the m-th row of Q. By the fact that $Q\mathbf{v} = \lambda\mathbf{v}$, we have

$$\sum_{j \neq m} q_{mj} v_j = \lambda v_m - q_{mm} v_m.$$

Taking the absolute value and using the triangle inequality,

$$|v_m||\lambda - q_{mm}| = \left| \sum_{j \neq m} q_{mj} v_m \right| \leq \sum_{j \neq m} |q_{mj} v_j| \leq |v_m| \sum_{j \neq m} |q_{mj}|.$$

Since $|v_m| \neq 0$, we have the inequality $|\lambda - q_{mm}| \leq r_m(Q)$ for all $\lambda \in \text{eig}(Q)$. $\square$

Definition 2.4: A matrix $Q \in M_n(\mathbb{R})$ is called *strictly row diagonally dominant* (respectively, *strictly column diagonally dominant*) if

$$|q_{ii}| > \sum_{j \neq i} |q_{ij}| \quad \text{(respectively, } |q_{ii}| > \sum_{j \neq i} |q_{ji}|\text{)},$$

for every $i, j \in \{1, \dots, n\}$ and $j \neq i$.

Theorem 2.2: (**Levy-Desplanques Theorem**) A strictly diagonally dominant matrix is nonsingular.

Proof: Let $Q \in M_n(\mathbb{R})$ be a strictly diagonally dominant matrix. Suppose det(Q)=0, then $Q\mathbf{v} = \mathbf{0}$ for some $\mathbf{v} = [v_1 \dots v_n]' \in \mathbb{R}^n$ such that $|v_m| = \max(|v_1|, \dots, |v_n|)$ for some $m \in \{1, \dots, n\}$. Consider the m-th row of Q and the fact that $Q\mathbf{v} = \mathbf{0}$, we have

$$\sum_{j=1}^{n} q_{mj} v_j = 0.$$

Take $q_{mm} v_m$ to the other side and consider the absolute value, we have

$$|q_{mm}||v_m| = \left| \sum_{j \neq m} q_{mj} v_j \right| \leq \sum_{j \neq m} |q_{mj} v_j| \leq |v_m| \sum_{j \neq m} |q_{mj}|.$$

Since $|v_m| \neq 0$, we have $|q_{mm}| \leq r_m(Q)$, which is a contradiction because Q is assumed to be a strictly diagonally dominant matrix. $\square$

2.3 Matrix Square Root and Hyperbolic Functions

Given a real matrix $Q \in \mathbb{R}^{m \times m}$, a (complex) square root of Q is $H \in \mathbb{C}^{m \times m}$ satisfying $Q = H^2$. The conditions for the existence of a square root are given in [25], Chapter 6. A square root always exists for a nonsingular Q and a *real* square root of a nonsingular Q exists if and only if Q has an even number of Jordan blocks of each size for every negative eigenvalue. It is also well known that a real positive (nonnegative) definite matrix has a unique real positive (nonnegative) definite square root (see [24], Theorem 7.2.6), which will be denoted by $Q^{\frac{1}{2}}$. Let

$$\begin{aligned} f(Qt) &:= \sum_{k=0}^{\infty} Q^k \frac{t^{2k}}{(2k)!}, \\ g(Qt) &:= \sum_{k=0}^{\infty} Q^k \frac{t^{2k+1}}{(2k+1)!}, \\ h(Qt) &:= \sum_{k=0}^{\infty} Q^k \frac{t^{2k+2}}{(2k+2)!}, \end{aligned}$$

which converge for all Q since they are the inverse Laplace transforms of rational matrices $s(s^2 I - Q)^{-1}$, $(s^2 I - Q)^{-1}$, and $s^{-1}(s^2 I - Q)^{-1}$, respectively. If H is any square root of Q, then, in (2.3), $f(Qt) = \cosh(Ht)$, $Hg(Qt) = \sinh(Ht)$, and $f(Qt) = I + Qh(Qt)$. Moreover, $\cosh(Ht)$ and $H\sinh(Ht)$ are functions of Q and are independent of the choice of the square root matrix H, whereas $\exp(\pm Ht) = \cosh(Ht) \pm \sinh(Ht)$ *is* dependent on its choice.

Lemma 2.3.1: The matrix $f(QT)$ is singular if and only if Q has a real negative eigenvalue $-r^2, r > 0$, and $T = \frac{(2k+1)\pi}{2r}$ for a nonnegative integer k.

Proof: Let $J_Q = P^{-1}QP$ be the (complex) Jordan normal form of Q for a nonsingular (complex) P. If $T = \frac{(2k+1)\pi}{2r}$ for some eigenvalue $-r^2$ of Q, then a Jordan block J_r of any size associated with this negative eigenvalue can be written as $-K_r^2$, where K_r has r at its diagonal, $-(2r)^{-1}$ at its upper diagonal, and zeros elsewhere. The corresponding Jordan block of $f(QT)$ is then $f(J_rT) = \sum_{i\geq 0}(-1)^i K_r^{2i}T^{2i}/(2i)! = \cos(K_rT)$, the diagonal entries of which are zero if and only if rT is an odd multiple of $\pi/2$. If Q is singular, then let J_0 be a Jordan block associated with the eigenvalue zero, and note by its series expression that $f(J_0T)$ has ones at its diagonal, i.e., nonsingular. Similarly, any other eigenvalue λ of Q is nonreal or positive so that a Jordan block J_λ has a nonsingular square root H_λ and $f(J_\lambda T) = \cosh(H_\lambda T)$ has eigenvalues all nonzero, i.e., nonsingular. $\square$

2.4 Noncooperative Open-loop Differential Game

Consider a set of agents (or players) represented by the nodes in $\mathbf{N} = \{1, \ldots, n\}$, each described by a dynamic state equation

$$\dot{\mathbf{x}}_i(t) = f_i(t, \mathbf{x}(t), \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)); \quad \mathbf{x}_i(0) = \mathbf{b}_i, \tag{2.3}$$

where $\mathbf{x}_i(t) \in \mathbb{R}^d$ is the state vector of agent i, $\mathbf{u}_i(t) \in \mathbb{R}^d$ is the control input, and $\mathbf{x}(t) := [\mathbf{x}_1'(t) \ldots \mathbf{x}_n'(t)]'$ is the state vector of the network. Let $\mathbf{b} := [\mathbf{b}_1' \ldots \mathbf{b}_n']'$ and the time $t \in [0, T]$, where T is the terminal time. The cost functional for some $i \in \mathbf{N}$ is given by

$$\mathcal{J}_i(\mathbf{u}_1, \ldots, \mathbf{u}_n) = q_i(\mathbf{x}(T)) + \int_0^T g_i(t, \mathbf{x}(t), \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t))dt. \tag{2.4}$$

The objective of each agent is to minimize this cost by choosing appropriate controls $\mathbf{u}_i$ in (2.3). This defines an n-agent dynamic game. The game is noncooperative since each agent minimizes his own cost and does not cooperate. Denote the information and strategy set of agent i by η_i and $\mathbf{S}_i$, respectively, then the actions (or controls) are determined by

$$\mathbf{u}_i = \gamma_i(\eta_i), \quad \text{where } i \in \mathbf{N}, \gamma_i \in \mathbf{S}_i.$$

In the case of open-loop games, the information set $\eta_i := \{(t, \mathbf{b}) : t \in [0,T], \mathbf{b} \in \mathbb{R}^{nd}\}$. Then, $\gamma_i : [0,T] \times \mathbb{R}^{nd} \to \mathbb{R}^d$. The following is based on Chapter 5 and 6 of [5].

Definition 2.5: A set of permissible control actions $\{\mathbf{u}_1^*, \ldots, \mathbf{u}_n^*\}$, where $\mathbf{u}_i^* := \gamma_i^*(\eta_i)$, $\forall\, i \in \mathbf{N}$, constitute a *Nash equilibrium* of the n-agent game if, for all permissible $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$, it holds that

$$J_i^* = \mathcal{J}_i(\mathbf{u}_1^*, \ldots, \mathbf{u}_{i-1}^*, \mathbf{u}_i^*, \mathbf{u}_{i+1}^*, \ldots, \mathbf{u}_n^*) \le \mathcal{J}_i(\mathbf{u}_1^*, \ldots, \mathbf{u}_{i-1}^*, \mathbf{u}_i, \mathbf{u}_{i+1}^*, \ldots, \mathbf{u}_n^*),$$

where $\mathbf{u}_i^*$ is the best response input of agent i and J_i^* is the optimal cost obtained with $\mathbf{u}_i^*$ when everyone else play their best strategies.

Assumption 2.1: Let $f_i(t, \cdot, \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t))$, $g_i(t, \cdot, \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t))$, and $q_i(\cdot)$ be continuously differentiable on $\mathbb{R}^d$.

Theorem 2.3: Consider a noncooperative dynamic game played by n agents in a prescribed duration $[0,T]$. Then, if $\{\gamma_i^*(\eta_i) =: \mathbf{u}_i^*;\ i \in \mathbf{N}\}$ provide an open-loop Nash equilibrium solution with $\{\mathbf{x}_i^*(t) : i \in \mathbf{N}, t \in [0,T]\}$ be the corresponding state trajectories, there exist n costate functions $\mathbf{p}_i : [0,T] \to \mathbb{R}^d$ such that the following relations hold:

$$\begin{aligned}
\dot{\mathbf{x}}_i^*(t) &= f_i(t, \mathbf{x}(t), \mathbf{u}_1^*(t), \ldots, \mathbf{u}_n^*(t)); \quad \mathbf{x}_i^*(0) = \mathbf{b}_i, \\
\mathbf{u}_i^*(t) &= \arg\min_{\mathbf{u}_i \in \mathbb{R}^d} \mathcal{H}_i(t, \mathbf{p}_i(t), \mathbf{x}^*(t), \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)), \\
\mathbf{p}_i(t) &= -\frac{\partial}{\partial \mathbf{x}_i} \mathcal{H}_i(t, \mathbf{p}_i(t), \mathbf{x}^*(t), \mathbf{u}_i^*(t)), \\
\mathbf{p}_i(T) &= \frac{\partial}{\partial \mathbf{x}_i} q_i(\mathbf{x}^*(T)),
\end{aligned} \tag{2.5}$$

where $t \in [0, T]$, $i \in \mathbf{N}$, and

$$\mathcal{H}_i(t, \mathbf{p}_i(t), \mathbf{x}(t), \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)) \triangleq g_i(t, \mathbf{x}(t), \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)) \\ + \mathbf{p}_i'(t) f_i(t, \mathbf{x}(t), \mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)).$$

Proof: The proof is given in Chapter 5 of [30]. □

2.5 Linear Quadratic (LQ) Differential Game

This section is based on Chapter 7 of [14]. Consider a linear differential equation

$$\dot{\mathbf{z}}(t) = A\mathbf{z}(t) + \sum_{j=1}^{n} B_j \mathbf{u}_j(t); \quad \mathbf{z}(0) = z_0, \tag{2.6}$$

where $\mathbf{z}(t) \in \mathbb{R}^m$ is the state vector, $\mathbf{u}_i(t) \in \mathbb{R}^d$ is the control input of agent i, $i \in \mathbf{N}$. The cost functional that each agent aims to minimize is

$$\mathcal{J}_i(\mathbf{u}_1, \ldots, \mathbf{u}_2) = \mathbf{z}'(T) G_{iT} \mathbf{z}(T) + \int_0^T [\mathbf{z}'(t) G_i \mathbf{z}(t) + \sum_{j=1}^{n} \mathbf{u}_j'(t) R_{ij} \mathbf{u}_j(t)] dt, \tag{2.7}$$

where the matrices G_i, R_{ij}, for $i \neq j$, are symmetric positive semi-definite and R_{ii} is symmetric positive definite. Then, the Hamiltonian in (2.5) becomes

$$\mathcal{H}_i = [\mathbf{z}(t)' G_i \mathbf{z}(t) + \sum_{j=1}^{n} \mathbf{u}_j'(t) R_{ij} \mathbf{u}_j(t)] + \hat{\mathbf{p}}_i'(t)[Az(t) + \sum_{j=1}^{n} B_j \mathbf{u}_j(t)], \tag{2.8}$$

where $\hat{\mathbf{p}}_i(t)$ are the costate functions. This gives

$$\mathbf{u}_i^*(t) = -R_{ii}^{-1} B_i' \hat{\mathbf{p}}_i(t), \tag{2.9}$$

where

$$\dot{\hat{\mathbf{p}}}_i(t) = -G_i \mathbf{z}(t) - A' \hat{\mathbf{p}}_i(t),$$

with $\hat{\mathbf{p}}_i(T) = G_{iT} \mathbf{z}(T)$. Moreover,

$$\dot{\mathbf{z}}(t) = A\mathbf{z}(t) - \sum_{j=1}^{n} S_j \hat{\mathbf{p}}_j(t),$$

with $\mathbf{z}(0) = \mathbf{z}_0$, where $S_j := B_j R_{jj}^{-1} B_j'$. Then, according to Theorem 2.3, the game defined by (2.6) and (2.7) admits a Nash equilibrium if the following differential equation has a solution:

$$\dot{\mathbf{y}}(t) = M\mathbf{y}(t); \quad X\mathbf{y}(0) + Y\mathbf{y}(T) = [\mathbf{z}_0' \; 0 \; \ldots \; 0]', \tag{2.10}$$

where $\mathbf{y}(t) = [\mathbf{z}'(t) \; \hat{\mathbf{p}}_1'(t) \; \ldots \; \hat{\mathbf{p}}_n'(t)]'$,

$$M = \begin{bmatrix} A & -S_1 & \ldots & -S_n \\ -G_1 & -A' & & \\ \vdots & & \ddots & \\ -G_n & & & -A' \end{bmatrix},$$

and

$$X = \begin{bmatrix} I & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ -G_{1T} & I & & \\ \vdots & & \ddots & \\ -G_{nT} & & & I \end{bmatrix}.$$

Theorem 2.4: Let the solutions $K_i(\cdot)$ of the n Riccati differential equations,

$$\dot{K}_i(t) = -A'K_i(t) - K_i(t)A + K_i(t)S_iK_i(t) - G_i; \; K_i(T) = G_{iT}, \; i \in \mathbf{N},$$

be symmetric on $[0, T]$. Then, the linear quadratic differential game defined in (2.6) and (2.7) has a unique Nash equilibrium if and only if

$$F(T) := [I \; 0 \; \ldots \; 0]e^{-MT}[I \; Q_{1T}' \; \ldots \; Q_{nT}']'$$

is invertible.

Proof: For a detailed proof, see Chapter 7 of [14]. $\square$

Chapter 3

A Game of Opinion Dynamics on Multiple Issues

A noncooperative open-loop dynamic game is defined below for a scenario where the agents form opinions on multiple prevailing issues. We discuss a procedure to determine weight matrices corresponding to the weights of a multilayer network and incorporate them in a cost functional. The cost functional describes the motive of an agent, which encapsulates his social and psychological dispositions. This choice of weight matrices is then linked with the consistency theory, [54]. We then mold our setup in the framework of linear quadratic (LQ) differential games.

3.1 Problem Definition

Consider a social network $\mathcal{M}(\mathcal{G},\mathcal{C})$ where n agents interact, discuss, and form opinions on d prevailing issues in a set $\mathbf{D}$. The agents are represented by the nodes $\mathbf{N} = \{1,\dots,n\}$ of the network. Here, $\mathcal{M}(\mathcal{G},\mathcal{C})$ is a multilayer network as defined in Section 2.1 of Chapter 2. Every agent in the network has some initial biases $\mathbf{b}_i = [b_{i,1} \dots b_{i,d}]' \in \mathbb{R}^d$, $i \in \mathbf{N}$, on the issues. Here, we let $b_{i,k}$ assume any number in $\mathbb{R} = (-\infty, \infty)$ ranging from strong refusal to complete support, with value zero indicating neutrality or lack of opinion. We define the *neighborhood* of agent i by $\mathbf{N}_i := \{j \in \mathbf{N} : \ w_{ij,kl} \neq 0 \text{ for some } k, l \in \mathbf{D}\}$, and let $|\mathbf{N}_i| = m_i$. Thus, any agent with whom agent i does not interact in any layer of the network is left out and others are collected in $\mathbf{N}_i$ as neighbors. Agents may update their initial opinions in a time interval $[0, T]$, where the terminal time $T > 0$ may also tend to infinity in which case the game played will be of infinite horizon.

Let $\mathbf{x}_i(t) = [x_{i,1}(t) \ldots x_{i,d}(t)]' \in \mathbb{R}^d$ denote the vector of opinions of agent i at time $t \in [0, T]$ on d issues. Thus, $\mathbf{x}_i(0) = \mathbf{b}_i$. The vector $\mathbf{x}(t) = [\mathbf{x}_1'(t) \ldots \mathbf{x}_n'(t)]' \in \mathbb{R}^{nd}$ is the opinion profile of the network at $t \in [0, T]$.

The motive that compels agent i to update his opinions can be described by postulating a cost functional that takes into account the cumulative costs of

(i) rapid changes in one's opinions,

(ii) the tendency to preserve his *mind set*, and

(iii) holding distinct opinions from one's neighbors.

The model here puts forward that every agent has a motive that may eventually dictate his opinion dynamics and that such a motive, if not consciously held, looms at the background in shaping his opinions. Thus, the cost functional of agent i given by

$$\mathcal{J}_i(\mathbf{x}, \mathbf{b}_i, \mathbf{u}_i) = \frac{1}{2} \int_0^T \{[\sum_{j \in \mathbf{N}_i} (\mathbf{x}_i - \mathbf{x}_j)' W_{ij} (\mathbf{x}_i - \mathbf{x}_j)] + (\mathbf{x}_i - \mathbf{b}_i)' W_{ii} (\mathbf{x}_i - \mathbf{b}_i) + \mathbf{u}_i' \mathbf{u}_i\} dt \tag{3.1}$$

represents his motive, in which the matrices W_{ij}, $W_{ii} \in \mathbb{R}^{d \times d}$, respectively, weigh the costs in (iii) and (ii) above. The third term penalizes (i) as $\mathbf{u}_i(t) := \dot{\mathbf{x}}_i(t), \forall\, i \in \mathbf{N}$, i.e., it is the instantaneous rate of change of opinions of agent i on d issues and represents the cumulative control effort of agent i when integrated in (3.1). It should be understood that $\mathbf{x}_i(t, T)$ and $\mathbf{u}_i(t, T)$ are functions of time t as well as the terminal time T, and that we suppress the dependence on T for the sake of brevity. We adopt the following throughout the paper.

Assumption 3.1: W_{ii} is symmetric positive definite and W_{ij}, for $i \neq j$, is symmetric nonnegative definite, $\forall\, i, j \in \mathbf{N}$. □

The first term in the cost functional requires agent i to cooperate with his neighbors, the second term to preserve his own biases and the third term to reduce the overall control effort. By Assumption 3.1, the weight associated with the control effort, normalized to identity in (3.1) rather than allowing it to be a positive definite weight matrix, is without loss of generality as we show in Remark 3.1.1 below. The agents interact with their neighbors for a finite duration $[0, T]$ and, due to an integral cost, every agent penalizes the cumulative effect of each of the three terms in the integrand during that duration. For instance, the first term does not penalize the instantaneous differences but the sum total of divergence from the opinions of the neighbors. Note that the opinions $\mathbf{x}_i(T)$ at terminal time T are not specified and left free. Thus, each agent minimizes his cost under *free terminal conditions*, [30]. A noncooperative, continuous-time, open-loop, infinite dynamic game (see [5], Section 5.3) is then played by n agents:

$$\min_{\mathbf{u}_i}\{\mathcal{J}_i\} \text{ subject to } \dot{\mathbf{x}}_i(t) = \mathbf{u}_i(t),\ \forall\, i \in \mathbf{N}. \tag{3.2}$$

This game is noncooperative because the agents seek their own individual motives and there is no prevailing “social motive” to be sought in the network. The main purpose of this setup is to examine under what circumstances, a pattern of collective behavior emerges. The information set of agent i is defined as $\eta_i := \{(t, \mathbf{b}_{\mathbf{N}_i}) : t \in [0, T],\ \mathbf{b}_{\mathbf{N}_i} \in \mathbb{R}^{m_i d}\}$, where $\mathbf{b}_{\mathbf{N}_i}$ is the vector containing the initial biases of agent i's neighbors. We assume that the control $\mathbf{u}_i(t) \in \mathbf{S}_i$, $\forall\, t \in [0, T]$, and define $\mathbf{S}_i$ to be a class of all permissible strategies of agent i, which are all continuously differentiable functions $\gamma_i : [0, T] \times \mathbb{R}^{m_i d} \to \mathbb{R}^d$.

Remark 3.1.1: The cost functional (3.1) is a simplified version of the following cost functional, in which $\hat{R}_i \in \mathbb{R}^{d\times d}$ is symmetric positive definite and $\hat{W}_{ij}$, $\hat{W}_{ii} \in \mathbb{R}^{d\times d}$ are symmetric nonnegative and positive definite matrices, respectively, for all $i, j \in \mathbf{N}$:

$$\mathcal{J}_i(\hat{\mathbf{x}}, \hat{\mathbf{b}}_i, \hat{\mathbf{u}}_i) = \frac{1}{2}\int_0^T \{[\sum_{j\in\mathbf{N}_i}(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j)'\hat{W}_{ij}(\hat{\mathbf{x}}_i - \hat{\mathbf{x}}_j)] + (\hat{\mathbf{x}}_i - \hat{\mathbf{b}}_i)'\hat{W}_{ii}(\hat{\mathbf{x}}_i - \hat{\mathbf{b}}_i) + \hat{\mathbf{u}}_i'\hat{R}_i\hat{\mathbf{u}}_i\}dt.$$

This is without any loss of generality. Notice that $\hat{R}_i = N_i'N_i$ for some nonsingular N_i and consider the transformation $\hat{\mathbf{u}}_i = N_i^{-1}\mathbf{u}_i$ and $\hat{\mathbf{x}}_i = N_i^{-1}\mathbf{x}_i$, where $\hat{W}_{ij} = N_i'W_{ij}N_i$, $\hat{W}_{ii} = N_i'W_{ii}N_i$. It is easy to see that the control input $\mathbf{u}_i^*$ is optimal for (3.1) with the trajectory $\mathbf{x}^*$ if and only if the control input $\hat{\mathbf{u}}_i^*$ is optimal with the trajectory $\hat{\mathbf{x}}^*$. $\triangle$

Definition 3.1: We will say that a *full consensus* is reached on some issue k at terminal time T if $x_{1,k}(T) = \cdots = x_{n,k}(T)$. On the other hand, if the opinions of agent i and agent j on some issue k are such that

$$|x_{i,k}(T) - x_{j,k}(T)| \leq \epsilon,$$

for some small $\epsilon > 0$. Then, we say that agent i and agent j have reached a *partial consensus* on issue k.

If there is only one issue ($d = 1$) under consideration, then (3.2) is the game considered in [40]. Similarly, if the matrices W_{ij} and W_{ii} are diagonal, then (3.1) will be minimum if and only if the d functionals obtained by its decomposition are minimum. In other words, the game then decomposes into d independently played games on every issue separately. We now make an attempt to justify the considerably more sophisticated game obtained in the non-diagonal W_{ij} cases.

3.2 On the Choice of Weight Matrices and Consistency Theory

The rationale in the choice of the weight matrices W_{ij}, $\forall\, i, j \in \mathbf{N}$, by agent i is encouraged by the role of attributes and by the notion of bounded confidence outlined in [35], [22], and [12], also being cognizant of the fact that in some cases similar attributes may lead to competition, [46]. We also take into account that a person's opinion on an issue is a function of the amount of information he has on that issue, [54]. The interpretation in terms of the reflexive property "consistency on two issues" for the off-diagonal entries of the influence and stubbornness weight matrices given below justifies the assumption of "symmetry" adopted by Assumption 3.1.

Let us first consider the influence weights and let $W_{ij} = V_{ij}^2$, $j \in \mathbf{N}_i$, where

$$V_{ij} = \begin{bmatrix} v_{ij,11} & v_{ij,12} & \cdots & v_{ij,1d} \\ v_{ij,21} & v_{ij,22} & \cdots & v_{ij,2d} \\ \vdots & \vdots & \ddots & \vdots \\ v_{ij,d1} & v_{ij,d2} & \cdots & v_{ij,dd} \end{bmatrix}; \quad \text{with } v_{ij,kl} = v_{ij,lk} \; \forall\, k, l \in \mathbf{D},$$

so that the nonnegative definiteness of W_{ij} is ensured. Suppose each agent i in the network possesses 'a' attributes such as social status, area of expertise, religious and ethnic identity, etc., which can be defined in some set $\mathbf{A}$. Then, a possible choice, for $i \in \mathbf{N}$ and $j \in \mathbf{N}_i$, is given by

$$v_{ij,kk} = \begin{cases} \phi_{ij,k}(\mathbf{a}_i, \mathbf{a}_j), & \text{if } |b_{i,k} - b_{j,k}| \le \varepsilon_i, \\ 0, & \text{otherwise;} \end{cases}$$

where $\phi_{ij,k} : \mathbf{A} \times \mathbf{A} \to \mathbb{R}_+$ and $\varepsilon_i > 0$ is the bounded confidence threshold of agent i. The off-diagonal entries of matrices V_{ij}, $\forall\, i, j \in \mathbf{N}$, can be chosen by

$$v_{ij,kl} = \begin{cases} c_i r_{i,kl}, & \text{if } v_{ij,kk} \neq 0 \text{ or } v_{ij,ll} \neq 0, \\ 0, & \text{otherwise;} \end{cases}$$

where $r_{i,kl} \in [-1, 1]$ is the correlation coefficient between issue k and issue l according to the conception of agent i, and $c_i \in \mathbb{R}_+$ is a proportionality constant that can be chosen dependent on the attributes $\mathbf{a}_i$ and $\mathbf{a}_j$. The positive (or negative) value of $r_{i,kl}$ indicates the positive (or negative) correlation between the issues k and l. Then, $\forall\, i, j \in \mathbf{N}$, the diagonal entries of W_{ij} are given as

$$w_{ij,kk} = \sum_{l=1}^{d} v_{ij,kl}^2,$$

where the symmetry $v_{ij,kl} = v_{ij,lk}$ is ensured; and the off-diagonal entries of W_{ij}, for $k \neq l$, are

$$w_{ij,kl} = v_{ij,kl}(v_{ij,kk} + v_{ij,ll}) + \sum_{\substack{m=1 \\ m \neq k,l}}^{d} v_{ij,mk} v_{ij,ml}.$$

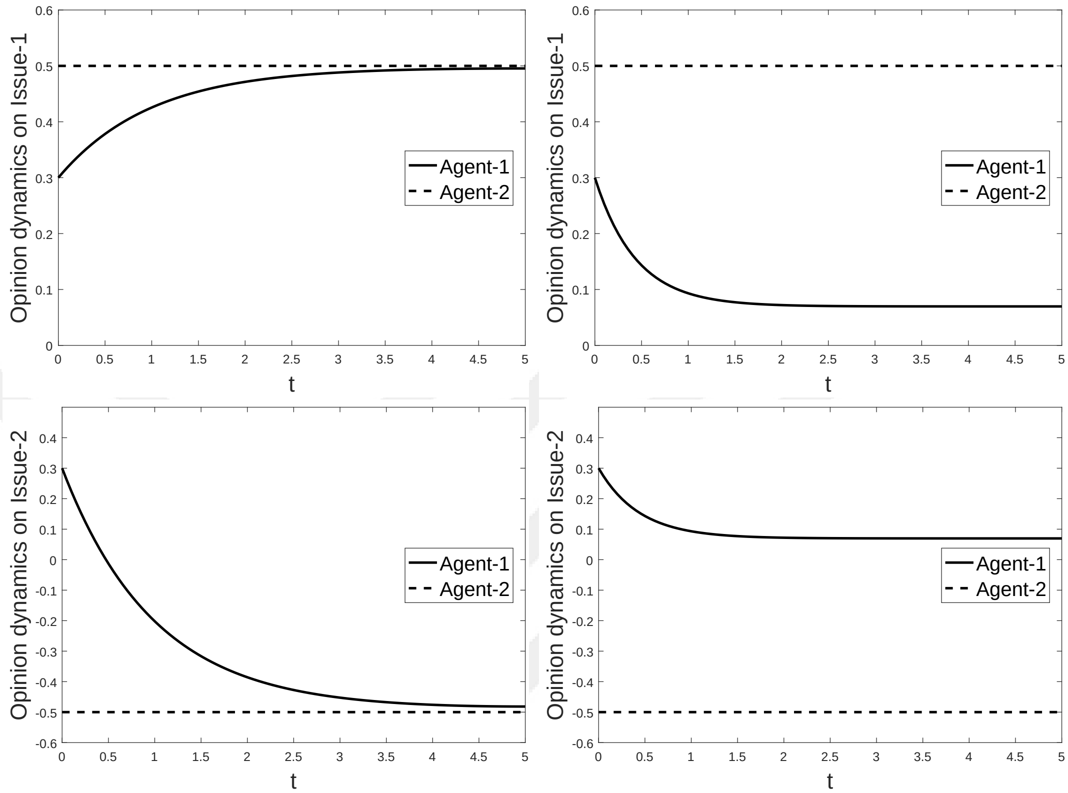
In order to see the effect of such choices in the cost function of agent i, consider the case of three issues, where the $(1,2)$-entry of W_{ij} in terms of $v_{ij,kl}$ is $w_{ij,12} = v_{ij,12}(v_{ij,11}+v_{ij,22})+v_{ij,13}v_{ij,23}$. The second term $v_{ij,13}v_{ij,23}$ shows that if issue 1 and issue 2 are separately correlated with issue 3, then opinions on issue 3 will also affect the opinions on issue 1 and issue 2. Also, if the product $(x_{i,1}-x_{j,1})(x_{i,2}-x_{j,2})$ and $w_{ij,12}$ have opposite signs, then the multiple of these terms will act as a repulsion term in (3.1). So agent i, in this case, will restrict himself to make consensus with agent j because, according to agent i, agent j does not hold reasonable opinions.

The choice of entries of a stubbornness weight matrix W_{ii} follows an entirely similar rationale, by simply replacing the agent j in the narrative above by the initial belief of the agent i. The stubbornness on an issue is penalized by the diagonal entries and the self-consistency on two issues by the off-diagonal entries.

Example 3.2.1: Consider a dyad, a network of two agents, interacting on two positively correlated issues, with initial biases $\mathbf{b}_1 = [0.3 \;\; 0.3]'$ and $\mathbf{b}_2 = [0.5 \; -0.5]'$. And suppose agent 1 has a one-way interaction with agent 2, while agent 2 does not interact with agent 1 (i.e., $W_{21} = 0, W_{22} = I$), such that $v_{11,kk} = 0.1$ and $v_{12,kk} = 1$, for $k = 1, 2$; and $v_{1j,12} = r_{1,12}$, for $j = 1, 2$, resulting in

$$W_{12} = \begin{bmatrix} 1 & r_{1,12} \\ r_{1,12} & 1 \end{bmatrix}^2, \quad W_{11} = \begin{bmatrix} 0.1 & r_{1,12} \\ r_{1,12} & 0.1 \end{bmatrix}^2.$$

Figure 3.1 (a) shows the opinion trajectories on issue 1 and issue 2 when $r_{1,12} = 0$ (no correlation) and Figure 3.1 (b), when $r_{1,12} = 1$. In this case, agent 1 makes consensus with agent 2 because of his influence. But in latter case, since issue 1 and issue 2 are positively correlated according to agent 1, and agent 2 holds contradictory beliefs as agent 1 is aware of $\mathbf{b}_2$, it can be seen that agent 1 moves away from agent 2 on issue 1. But as he changes his opinion on issue 1, he also



(a) (b)

Figure 3.1: Antagonistic interaction

needs to change his opinion on issue 2, towards that of agent 2, to minimize his own inconsistency. $\triangle$

3.3 The Framework of LQ Differential Game

Let us first observe that the cost functional (3.1) is a linear quadratic cost functional since it can be written in the form

$$\mathcal{J}_i(\mathbf{z}, \mathbf{u}_i) = \frac{1}{2} \int_0^T (\mathbf{z}' G_i \mathbf{z} + \mathbf{u}_i' \mathbf{u}_i)\, dt, \tag{3.3}$$

where $\mathbf{z} = [\mathbf{z}_1', \ldots, \mathbf{z}_n']' \in \mathbb{R}^{n^2 d}$; $\mathbf{z}_i = [\Delta_{i1}', \ldots, \Delta_{i(i-1)}', \Delta_{ii}', \Delta_{i(i+1)}', \ldots, \Delta_{in}']' \in \mathbb{R}^{nd}$ with $\Delta_{ij} := \mathbf{x}_i - \mathbf{x}_j$, for $j \neq i$, and $\Delta_{ii} := \mathbf{x}_i - \mathbf{b}_i$, $\forall i, j \in \mathbf{N}$. Also,

$$G_i = H_i \otimes K_i,$$

where $H_i \in \{0,1\}^{n\times n}$ is a zero matrix with (i,i)-entry equal to 1, and $K_i = \text{diag}[W_{i1},\ldots,W_{in}] \in \mathbb{R}^{nd\times nd}$. The differential equation of the network is given by

$$\dot{\mathbf{z}} = \sum_{j=1}^{n} B_j \mathbf{u}_j, \tag{3.4}$$

where $B_j \in \{0,1\}^{n^2 d\times d}$ is a matrix of ones and zeros,

$$B_j = \begin{bmatrix} -\mathbf{h}_j \otimes I_d \\ \vdots \\ \mathbf{1}_n \otimes I_d \\ \vdots \\ -\mathbf{h}_j \otimes I_d \end{bmatrix}.$$

Here, $\mathbf{h}_j \in \{0,1\}^n$ is a zero vector whose jth-entry is equal to 1, and note that the matrix $\mathbf{1}_n \otimes I_d$ is the jth block of B_j.

Defining the Hamiltonian as in (2.8) and $S_j := B_j B_j^T$, we obtain

$$\dot{\mathbf{y}}(t) = M\mathbf{y}(t);\;\; X\mathbf{y}(0) + Y\mathbf{y}(T) = [\mathbf{z}_0'\; 0\; \ldots\; 0]',$$

where $\mathbf{y}(t) = [\mathbf{z}'(t)\; \hat{\mathbf{p}}_1'(t)\; \ldots\; \hat{\mathbf{p}}_n'(t)]'$,

$$M = \begin{bmatrix} 0 & -S_1 & \ldots & -S_n \\ -G_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -G_n & 0 & \ldots & 0 \end{bmatrix},$$

and

$$X = \begin{bmatrix} I & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix},\;\; Y = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I \end{bmatrix}.$$

Then, for the existence of Nash equilibrium, as in Theorem 2.4, we need to check the invertibility of the following function of a matrix:

$$F(T) = f[(SG)T] := \sum_{k=0}^{\infty} (SG)^k \frac{T^{2k}}{(2k)!},$$

where $S = [S_1\; \ldots\; S_n]$ and $G = [G_1'\; \ldots\; G_n']'$.

Chapter 4

Existence and Uniqueness of a Nash Equilibrium

We first investigate the existence and uniqueness of Nash equilibrium in a general multilayer network. The game defined in Chapter 3 happens to admit a unique Nash equilibrium under mild conditions on the matrix Q in (2.1). In a single issue case, the Nash equilibrium always exists and the opinion trajectories always converge to some value. Also, in multi-issue case, a unique Nash equilibrium exists if agents hold some harmonious view among themselves. The harmony may refer to the similarity in pairwise conceptions about the issues but may also be an agreement on the status of some leader in the network. We then give explicit opinion trajectories at Nash equilibrium for these harmonious networks.

4.1 Nash Equilibrium in a General Multilayer Network

We present the existence and uniqueness results for the game (3.2) in this section. We first focus on the Nash equilibrium of the game in its most generality. In Proposition 4.1, we show, under mild conditions on weight matrices, that (3.2) has a unique Nash equilibrium.

Consider the cost functional (3.1), and let $W := \mathrm{diag}[W_{11}, \ldots, W_{nn}]$ and the matrix

$$Q := \begin{bmatrix} M_1 & -W_{12} & \ldots & -W_{1n} \\ -W_{21} & M_2 & \ldots & -W_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -W_{n1} & -W_{n2} & \ldots & M_n \end{bmatrix}, \tag{4.1}$$

with

$$M_i = \sum_{j \in \mathbf{N}} W_{ij}, \quad \forall\, i \in \mathbf{N}.$$

Note that $W_{ij} = 0$ in (4.1) whenever $j \notin \mathbf{N}_i$. Also, by Assumption 3.1, the block diagonal W is positive definite. Whether Q is symmetric, nonsingular, nonnegative, or positive definite, etc., depends on the choice of weight matrices as well as the network structure.

Proposition 4.1: Let Assumption 3.1 hold. Then,

(i) A unique Nash equilibrium $(\mathbf{u}_1^*, ..., \mathbf{u}_n^*)$ of the game (3.2) exists in the interval $[0, T]$ if and only if the matrix $f(QT)$ is nonsingular, in which case the Nash solution in $t \in [0, T]$ is given by

$$\mathbf{u}^*(t) = Q[h(Qt)f(QT) + g(Qt)g(QT)]f(QT)^{-1}(Q - W)\mathbf{b}, \tag{4.2}$$

where $\mathbf{b} := [\mathbf{b}_1' \ldots \mathbf{b}_n']'$, $\mathbf{u}^*(t) := [\mathbf{u}_1^*(t)' \ldots \mathbf{u}_n^*(t)']'$, and the resulting opinion profile in this Nash equilibrium is given by

$$\mathbf{x}^*(t) = \{I + [h(Qt)f(QT) + g(Qt)g(QT)]f(QT)^{-1}(Q - W)\}\mathbf{b}, \tag{4.3}$$

where $\mathbf{x}^*(t) := [\mathbf{x}_1^*(t)' \ldots \mathbf{x}_n^*(t)']'$.

(ii) If Q is nonsingular and H is a square root of Q, then the Nash solution and its opinion dynamics can be expressed as

$$\mathbf{u}^*(t) = H \sinh[H(T - t)] \cosh(HT)^{-1}(I - Q^{-1}W)\mathbf{b}, \tag{4.4}$$

$$\mathbf{x}^*(t) = \{Q^{-1}W + \cosh[H(T - t)] \cosh(HT)^{-1}(I - Q^{-1}W)\}\mathbf{b}. \tag{4.5}$$

(iii) If Q is nonsingular with a real negative eigenvalue $-r^2$, then $\mathbf{x}^*(t)$ has sustained oscillations for $T \neq \frac{(2k+1)\pi}{2r}$ and

$$\lim_{T \to \frac{(2k+1)\pi}{2r}} \mathbf{x}^*(t) \to \pm\infty$$

for every nonnegative integer k.

(iv) If Q is nonsingular with no real negative eigenvalue, then the opinion dynamics (4.5) in infinite horizon is given by

$$\lim_{T \to \infty} \mathbf{x}^*(t) = [Q^{-1}W + \exp(-H_p t)\,(I - Q^{-1}W)]\mathbf{b}, \tag{4.6}$$

where $H_p \in \mathbb{R}^{nd \times nd}$ is a positive stable matrix.

Proof: To show that there exists $\mathbf{u}_i^*(t)$ which provides a Nash equilibrium, we can write (3.1) as

$$\mathcal{J}_i = \int_0^T (\mathcal{H}_i - \mathbf{p}_i'\dot{\mathbf{x}}_i)dt, \tag{4.7}$$

where $\mathcal{H}_i$ and $\mathbf{p}_i$ are the Hamiltonian and the costate function, respectively, defined in (2.5). Notice that

$$\int_0^T \mathbf{p}_i'\dot{\mathbf{x}}_i dt = \mathbf{p}_i'(T)\mathbf{x}_i(T) - \mathbf{p}_i'(0)\mathbf{b}_i - \int_0^T \dot{\mathbf{p}}_i\mathbf{x}_i dt,$$

where $\mathbf{b}_i = \mathbf{x}_i(0)$. Then,

$$\mathcal{J}_i = \int_0^T (\mathcal{H}_i + \dot{\mathbf{p}}_i'\mathbf{x}_i)dt - \mathbf{p}_i'(T)\mathbf{x}_i(T) + \mathbf{p}_i'(0)\mathbf{b}_i. \tag{4.8}$$

Assuming that $\mathbf{u}_i^*$ is the optimal control path, we perturb it around its 'small' neighborhood with some continuous function $\mathbf{q}_i(t)$ so that we have its neighboring control paths

$$\mathbf{u}_i(t) = \mathbf{u}_i^*(t) + \epsilon\mathbf{q}_i(t),$$

where ϵ is a 'small' scalar. Then, $\mathbf{x}_i(t,\epsilon)$ are the neighboring state trajectories of $\mathbf{x}_i^*(t)$, and the cost functional $\mathcal{J}_i(\epsilon)$ is the corresponding cost. By the assumption that $\mathbf{u}_i^*$ is optimal, $\mathcal{J}_i(\epsilon)$ has a minimum at $\epsilon = 0$. And it can be easily verified that $\frac{d\mathcal{J}_i(\epsilon)}{d\epsilon} = 0$ when the necessary conditions (2.5) are satisfied and

$$\frac{d^2\mathcal{J}_i(\epsilon)}{d\epsilon^2} = \int_0^T \begin{bmatrix} \frac{d\mathbf{x}_i'}{d\epsilon} & \mathbf{q}_i' \end{bmatrix} \begin{bmatrix} (\sum_{j \in \mathsf{N}_i} W_{ij}) + W_{ii} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{x}_i}{d\epsilon} \\ \mathbf{q}_i \end{bmatrix} dt > 0$$

because $W_{ii} \succ 0$ and $W_{ij} \succeq 0$. Therefore, $\mathbf{u}_i^*(t)$ minimizes $\mathcal{J}_i$. And since $\mathcal{H}_i$ is strictly convex in $(\mathbf{x}, \mathbf{u}_i)$ $\forall t \in [0, T]$, we conclude that $\mathbf{u}_i^*$ is unique [36].

The combined state and costate equations from (2.5) give

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0 & -I \\ -Q & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ W & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{p}_0 \end{bmatrix},$$

which has the solution of the form

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix} = (\Phi(t) + \Psi(t)B) \begin{bmatrix} \mathbf{b} \\ \mathbf{p}_0 \end{bmatrix}, \tag{4.9}$$

where $\mathbf{p} = [\mathbf{p}_1' \ldots \mathbf{p}_n']'$, $\mathbf{p}(0) = \mathbf{p}_0$,

$$\Phi(t) = e^{At} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix},$$

$$\Psi(t) = \int_0^t e^{A(t-\tau)} d\tau = \begin{bmatrix} \psi_{11}(t) & \psi_{12}(t) \\ \psi_{21}(t) & \psi_{22}(t) \end{bmatrix},$$

$$A := \begin{bmatrix} 0 & -I \\ -Q & 0 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ W & 0 \end{bmatrix}.$$

Note that

$$\begin{aligned} \Phi(t) &= \mathscr{L}^{-1}\{(sI - A)^{-1}\} \\ &= \mathscr{L}^{-1}\left\{ \begin{bmatrix} s(s^2 I - Q)^{-1} & -(s^2 I - Q)^{-1} \\ -Q(s^2 I - Q)^{-1} & s(s^2 I - Q)^{-1} \end{bmatrix} \right\}. \end{aligned} \tag{4.10}$$

The state transition matrix $\Phi(t)$ and the matrix $\Psi(t)$ are calculated using the formal power series in s^{-1} of each block in (4.10) and, with (2.3) in view, are given by

$$\Phi(t) = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} = \begin{bmatrix} f(Qt) & -g(Qt) \\ -Qg(Qt) & f(Qt) \end{bmatrix},$$

$$\Psi(t) = \begin{bmatrix} \psi_{11}(t) & \psi_{12}(t) \\ \psi_{21}(t) & \psi_{22}(t) \end{bmatrix} = \begin{bmatrix} g(Qt) & -h(Qt) \\ -Qh(Qt) & g(Qt) \end{bmatrix}.$$

From (4.9), we have

$$\mathbf{x}(t) = [\phi_{11}(t) + \psi_{12}(t)W]\mathbf{b} + \phi_{12}(t)\mathbf{p}_0,$$

and,

$$\mathbf{p}(t) = \ [\phi_{21}(t) + \psi_{22}(t)W]\mathbf{b} + \phi_{22}(t)\mathbf{p}_0.$$

Evaluating at $t = T$ and employing the boundary condition $\mathbf{p}(T) = 0$, we have $\phi_{22}(T)\mathbf{p}(0) = -[\phi_{21}(T) + \psi_{22}(T)W]\mathbf{b}$. In order to be able to solve for $\mathbf{p}_0$ for any initial state $\mathbf{b}$, it is necessary and sufficient that $\phi_{22}(T) = f(QT)$ is invertible. We then obtain

$$\mathbf{p}_0 = -\phi_{22}(T)^{-1}[\phi_{21}(T) + \psi_{22}(T)W]\mathbf{b},$$

and

$$\mathbf{x}(t) = \{\phi_{11}(t) + \psi_{12}(t)W + \phi_{12}(t)\phi_{22}(T)^{-1}[Q\phi_{12}(T) - \phi_{12}(T)W]\}\mathbf{b}$$

by using that $\phi_{21} = Q\phi_{12}$ and $\psi_{22} = -\phi_{12}$. Substituting the expressions in terms of $f(Qt), g(Qt), h(Qt)$ above, and noting that functions of Q commute, it is not difficult to arrive at (4.3) and (4.2), where $\mathbf{u}(t) = \dot{\mathbf{x}}(t)$. This establishes the necessity and sufficiency of the condition that $f(QT)$ is nonsingular for a Nash equilibrium to exist and proves Proposition 4.1-(i).

Whenever Q is nonsingular, at least one square root H of Q exists. Then,

$$f(Qt) = Qh(Qt) + I = \cosh(Ht)$$

and

$$Hg(Qt) = \sinh(Ht),$$

which are independent of H, so that (4.2) and (4.3) result in (4.4) and (4.5), proving Proposition 4.1-(ii).

To prove Proposition 4.1-(iii), let H be any square root of Q and let J be its Jordan normal form so that $H = PJP^{-1}$ for P as in the proof of Lemma 1. If Q has a negative eigenvalue, say $-r^2$, then as in Fact 1, there are Jordan block(s) $J_r = \sqrt{-1}K_r$ associated with that eigenvalue for which $\cosh(J_r t) = \cos(K_r t)$ for $t \geq 0$. Then, the corresponding block in

$$P^{-1}\cosh[H(T-t)]\cosh(HT)^{-1}P = \cosh[J(T-t)]\cosh(JT)^{-1}$$

has the expression

$$\cos[K_r(T-t)]\cos(K_r T)^{-1} = \cos(K_r t) + \sin(K_r t)\tan(K_r T),$$

which is unbounded as t approaches $T = (2k+1)\pi/2r$, for every nonnegative integer k, which results in unbounded (4.5) for all initial values $\mathbf{b}$ (except when everyone starts at full consensus; where all $\mathbf{b}_i$, $\forall\, i \in \mathbf{N}$, are same). For any other value of T, $\tan(K_r T)$ will have a finite value and there will be oscillations of finite (but possibly large) amplitudes in all trajectories associated with the d issues. This proves the claims in Proposition 4.1-(iii).

If Q is free of any 'real' negative eigenvalue, then it has real square roots H, all of which have eigenvalues with nonzero real parts including a square root H_p with eigenvalues of all positive real parts. Note that any square root H of Q can be written as $H = H_+ + H_-$, where $H_+ = PJ_+P^{-1}$, $H_- = PJ_-P^{-1}$, and $J = J_+ + J_-$ is a decomposition of J into two matrices having blocks that correspond to eigenvalues of positive and negative real parts, respectively. It is now straightforward to show, using this decomposition, that if H is any square root of Q, then

$$\lim_{T\to\infty} \cosh[H(T-t)]\cosh(HT)^{-1} = \exp(-H_p t)$$

for every $t \geq 0$, where $H_p = H_+ - H_-$. Note that this limit remains bounded for every $t \in [0,\infty)$ and goes to zero as $t \to \infty$. This proves Proposition 4.1-(iv). $\square$

The existence of a Nash equilibrium is guaranteed only if the choices of influence and stubbornness matrices and terminal time T result in a nonsingular $f(QT)$. This requirement is always met when $d = 1$ since Q can be shown to have positive real eigenvalues in this single issue case. The weight matrices in this case are scalars, i.e., $W_{ij} = w_{ij} \geq 0$ for $i, j \in \mathbf{N}$. Then,

$$Q = \begin{bmatrix} m_1 & -w_{12} & \dots & -w_{1n} \\ -w_{21} & m_2 & \dots & -w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{n1} & -w_{n2} & \dots & m_n \end{bmatrix}; \quad m_i = \sum_{j=1}^{n} w_{ij}, \tag{4.11}$$

and note that $w_{ij} > 0$ for $i = j$.

Lemma 4.1.1: The matrix Q in (4.11) is an M-matrix.

Proof: First note that Q is a P-matrix since all its principal minors are positive.

That is, every real eigenvalue of Q is positive, see Lemma 2.2.2. Furthermore, let $r_i(Q) = \sum_{j\neq i} w_{ij}$. Then, from Geršgorin theorem we know that all the eigenvalues of Q lie in the union of all Geršgorin discs, $G(Q)$, defined in (2.2). Since Q is strictly diagonally dominant of its row entries, i.e., $m_i - r_i(Q) = w_{ii} > 0, \forall\, i \in \mathbf{N}$, therefore $G(Q)$ lies in the right half plane of $\mathbb{C}$. Hence, the matrix Q is positive stable, i.e., all its eigenvalues lie strictly in the right half plane.

Let $\bar{m} = \max\{m_i : i \in \mathbf{N}\}$, then $Q = \bar{m}I - R$, where R is a nonnegative matrix given by

$$R = \begin{bmatrix} \bar{m} - m_1 & w_{12} & \dots & w_{1n} \\ w_{21} & \bar{m} - m_2 & \dots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \dots & \bar{m} - m_n \end{bmatrix}.$$

The matrix R is nonnegative since $\bar{m} - m_i \geq 0, \forall\, i \in \mathbf{N}$. Hence, the result follows by Lemma 2.2.1. □

Corollary 4.1.1: For a single issue case, the following hold:

(i) The game (3.2) has a unique Nash equilibrium.

(ii) The opinions always converge to a partial consensus and there are no sustained oscillations.

Proof: The matrix Q is strictly row diagonally dominant, therefore its square root exists. By Lemma 4.1.1 above, the matrix Q in (4.11) is positive stable. Then by Proposition 4.1, a unique Nash equilibrium exists and the opinion trajectory of every agent i always converges to a partial consensus. □

Example 4.1.1: Consider a network with two leaders and one issue as shown in Figure 4.1 (a). Total population equals 10, and suppose that half of the followers (four each) support each leader. The followers of leader-1 can be named as followers-1, and of leader-10 as followers-10. The followers also have influence among themselves in a society, but a follower is assumed to receive more influence from his fellow followers than the opponents. We set $w_{ii} = 0.2$, $i = 1, \dots, 10$.

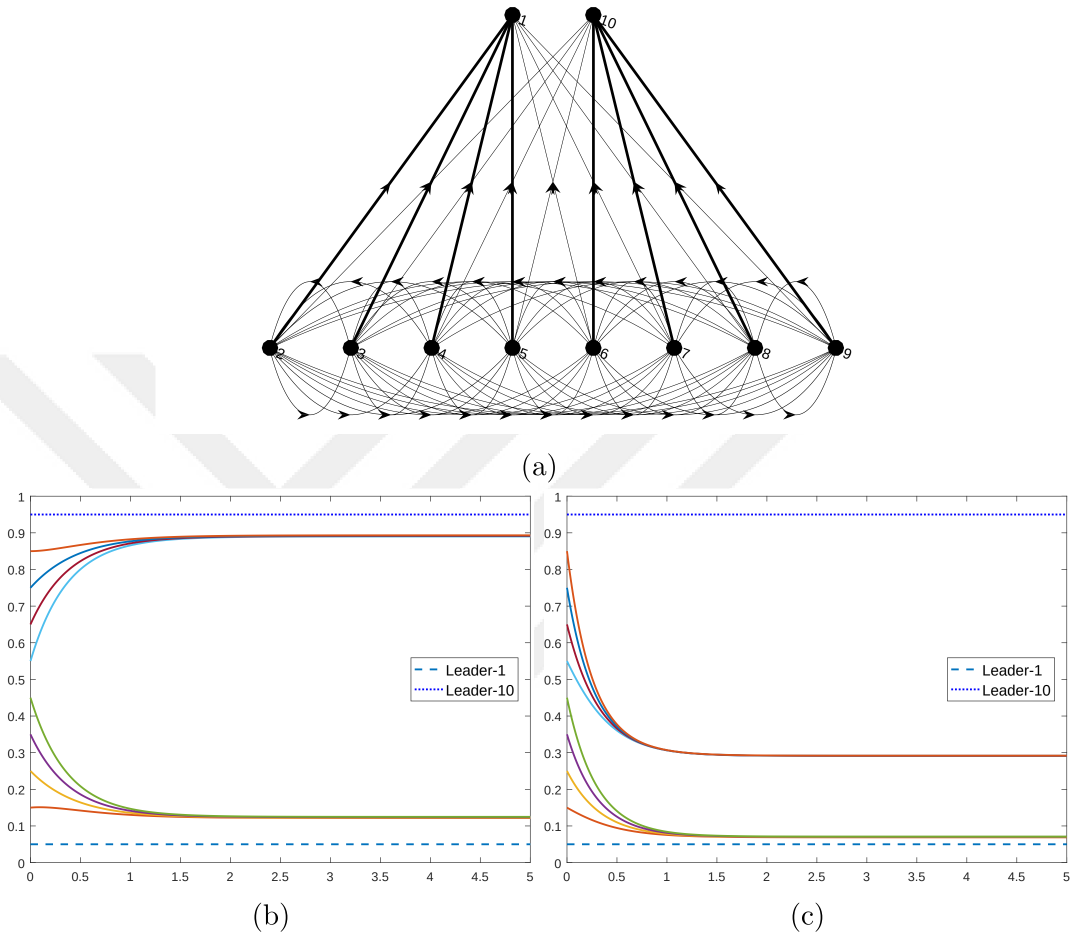
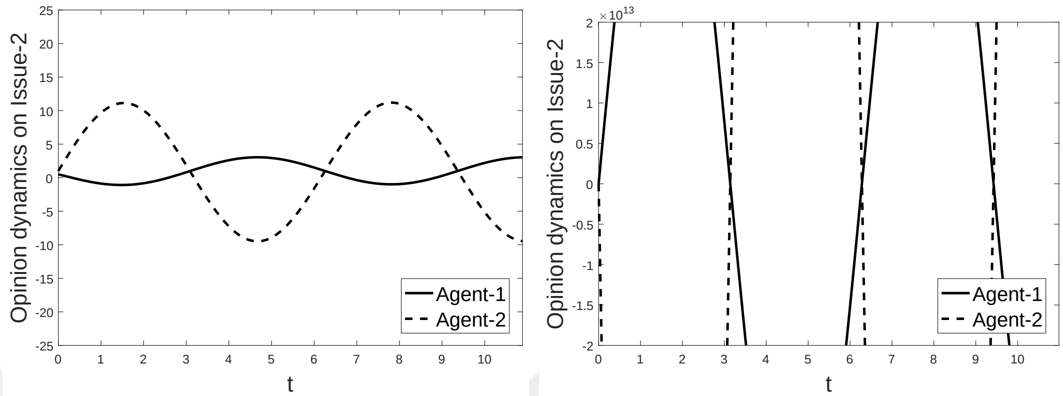


Figure 4.1: A network with two leaders (agent 1 and agent 10)

Suppose that $w_{1i} = w_{ni} = 0$, where $n = 10$ and $i = 2, \ldots, 9$. In Figure 4.1 (b), we assume that the influences of leader-1 and leader-10 on their followers is 10; the social impact of followers-1 and followers-10 among themselves is 2; the cross impact of followers-1 on followers-10, and vice versa, is 0.2; and the influence that followers take from other leader is 10 times less than their own leader's influence. In this case, agents are more likely going to follow their respective leaders. However, if we assume that followers-1 are more loyal to their leader and they run a good campaign in order to attract the followers-10, then they are able to steal followers-10 from their leader. For Figure 4.1 (c), we increase the influence of leader-1 to 20 and the cross impact of followers-1 on followers-10 to 10, while all other parameters are the same. It can be seen that rather than following leader-10, followers-10 tend to follow followers-1. This is due to social



(a) (b)

Figure 4.2: Oscillating and divergent opinion trajectories

impact of followers-1 on followers-10. △

For $d > 1$, the matrix Q may have negative real eigenvalues. In Example 4.1.2 below, a nonsingular Q having a negative eigenvalue results in a singular $f(QT)$ for a particular value of T and, hence, in a lack of Nash equilibrium. The expectation that a Nash equilibrium fails to exist whenever the agents have inconsistent views towards each other is only partly true. In fact, in Theorem 4.2-(i) and in Theorem 4.3 below, it exists in two somewhat diametrically opposite cases.

Example 4.1.2: By Proposition 4.1, if Q has a real negative eigenvalue, then a Nash equilibrium fails to exist for some critical values of T and has persistent oscillations for other values of T. Consider a dyad with

$$W_{11} = \begin{bmatrix} 0.5 & -1 \\ -1 & 2.5 \end{bmatrix}, \qquad W_{12} = \begin{bmatrix} 1.25 & -1 \\ -1 & 1.25 \end{bmatrix}$$

$$W_{22} = \begin{bmatrix} 2 & 2.5 \\ 2.5 & 3.25 \end{bmatrix}, \qquad W_{21} = \begin{bmatrix} 2 & 1.5 \\ 1.5 & 1.25 \end{bmatrix}$$

for which $\mathrm{eig}(Q) = \{8.3611, 4.9858, 0.8372, -0.1840\}$. Let

$$T = \frac{\pi}{2\sqrt{0.184}} = 3.6619$$

so that $(2k+1)3.6619$ are the critical values of T at which Nash equilibrium fails to exist. In fact with $\mathbf{b}_1 = [-0.5 \quad 0.5]'$, $\mathbf{b}_2 = [1 \quad 1]'$, and $k = 1$, the opinion trajectories diverge to infinity as $T \to 3(3.6619)$ as shown in Figure 4.2 (b) and has oscillations for a value less than but close to this critical one in Figure 4.2 (a). Figure 4.2 shows the trajectories for only issue 2 but those for issue 1 have the same features. Divergence will actually be obtained for all initial biases except that of full consensus. A main factor that causes Q to have a negative eigenvalue is that the off-diagonal entries of the weight matrices of agent 1 and agent 2 have different signs, i.e., the agents have opposite conceptions about the correlation of the two issues. $\triangle$

Remark 4.1.1: (a) The assumption that Q is nonsingular is convenient in obtaining compact expressions for $\mathbf{u}^*(t)$ and $\mathbf{x}^*(t)$. It is also practical, without loss of generality, since *almost all* matrices are nonsingular.

(b) Observe that the opinion profile in (4.5) consists of two parts. The constant part $Q^{-1}W\mathbf{b}$ can be viewed as a "weighted average opinion in the network." The time dependent part, on the other hand, represents the evolution of the dynamics dependent on the difference between the opinions of agents from that average. The evolution itself is dictated by the eigenvalues of Q in (4.3) and in (4.5), see [51].

(c) If Q is symmetric, then the matrix H_p in Proposition 4.1-(iv) is the unique positive definite square root of (positive definite) Q. $\triangle$

If the agents initially hold same opinions on the issues, then they don't change their opinions in $[0, T]$. This reinforces that the game (3.2) is indeed a model of consensus in opinion dynamics. On the other hand, as we will demonstrate, this constant dynamics is the only possible instance of full consensus because the motive of stubbornness is incorporated into the model.

Corollary 4.1.2: If $\mathbf{b}_i = \mathbf{b}_j$, $\forall\, i, j \in \mathbf{N}$, then $\mathbf{x}^*(t) = \mathbf{b}$, $\forall\, t \in [0, T]$.

Proof: In (4.3) and (4.5), if $\mathbf{b}_i = \mathbf{b}_j$ for all $i, j \in \mathbf{N}$, then $\mathbf{b}$ is in the null space of $Q - W$. This is because $Q - W$ is a matrix without self-weights of the nodes. In this case, the row sum of $Q - W$ is zero. Therefore, $\mathbf{x}_i^*(t) = \mathbf{b}_i$, $\forall\, t \in [0, T]$. Hence, an initial full consensus is preserved in the whole interval. $\square$

Corollary 4.1.3: In case of infinite time horizon, the opinion vector in the long run, converges to the weighted average of the initial opinions,, i.e.,

$$\lim_{t\to\infty}\lim_{T\to\infty}\mathbf{x}^*(t) = Q^{-1}W\mathbf{b}.$$

Proof: The result is a direct consequence of Proposition 4.1-(iv). □

Corollary 4.1.4: The control input $\mathbf{u}^*(t)$ in (4.4) remains bounded for $t \in [0, T]$ for all $T \geq 0$ and as $T \to \infty$.

Proof: Note that the $\cosh(HT)$ is invertible because $Q = H^2$ is assumed to have no negative real eigenvalues. Therefore, for any finite $T \geq 0$, the control input $\mathbf{u}^*(t)$ remains bounded. When $T \to \infty$,

$$\begin{aligned}\lim_{T\to\infty}\mathbf{u}(t) &= \lim_{T\to\infty} H\{\exp[H(T-t)] - \exp[-H(T-t)]\}\times \\ &\qquad [\exp(HT) + \exp(-HT)]^{-1}(I - Q^{-1}W)\mathbf{b}, \\ &= \lim_{T\to\infty} H\{\exp[2H(T-t)] - I\}\exp[-H(T-t)]\times \\ &\qquad \exp(HT)[\exp(2HT) + I]^{-1}(I - Q^{-1}W)\mathbf{b}, \\ &= -H\exp(Ht)(I - Q^{-1}W)\mathbf{b},\end{aligned}$$

where H is a matrix with eigenvalues in right half plane. □

Using the expression (4.5), one can also obtain the best response dynamics $\mathbf{x}_i(t)$ for every agent i. It is, however, possible to display the individual opinion dynamics of each agent more explicitly to allow easier interpretations. We now consider two specializations of the result of Proposition 4.1: First, where agents have pairwise similar views and second, where all the agents are only connected to one agent that will be called a leader. These two cases come from two extreme assumptions on the structure of the networks and lead to an explicit opinion trajectory expression for each agent.

4.2 Pairwise Similar Views

In the special case of agents having pairwise identical influence weights, i.e., undirected network, the existence of a Nash equilibrium is ensured. Since an influence weight matrix is allowed to be a zero matrix, pairwise similarity requires that if agent i leaves agent j out of $\mathbf{N}_i$, then agent j reciprocates and leaves him out of $\mathbf{N}_j$. This assumption may be justified for instance in a small community without much external influence. Since stubbornness weights may be distinct, the individuality of agents is still there. This section pursues the consequences of this type of a harmony existing in a network.

Lemma 4.2.1: Consider an undirected multilayer network $\mathcal{M}(\mathcal{G}, \mathcal{C})$. Then, the matrix Q is positive definite.

Proof: Here, the matrix Q is given by

$$Q = \begin{bmatrix} M_1 & -W_{12} & \dots & -W_{1n} \\ -W_{12} & M_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -W_{(n-1)n} \\ -W_{1n} & \dots & -W_{(n-1)n} & M_n \end{bmatrix}.$$

For any $\mathbf{y}_i \in \mathbb{R}^{nd}$, $i \in \mathbf{N}$, computing $[\mathbf{y}_1' \dots \mathbf{y}_n'] Q [\mathbf{y}_1' \dots \mathbf{y}_n']'$, we get

$$\mathbf{y}_1' W_{11} \mathbf{y}_1 + \dots + \mathbf{y}_n' W_{nn} \mathbf{y}_n + \sum_{\forall\, i,j \in \mathbf{N}, i \neq j} (\mathbf{y}_i' - \mathbf{y}_j') W_{ij} (\mathbf{y}_i - \mathbf{y}_j) > 0,$$

where we used the expression in (4.1) for M_i's and the hypothesis that W_{ij} is nonnegative definite for $i \neq j$ and positive definite for $i = j$. $\square$

Theorem 4.2: **(i)** If $W_{ij} = W_{ji}$, $\forall\, i, j \in \mathbf{N}$, then a Nash equilibrium exists and is unique. The resulting opinion profile is given by (4.5).

(ii) In a complete network, if $W_{ii} = F$ and $W_{ij} = G$, $\forall\, i, j \in \mathbf{N}$, $i \neq j$, then the unique Nash equilibrium opinion trajectory of agent i is given by

$$\mathbf{x}_i(t) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{b}_j + (F + nG)^{-1} \{F + \cosh[\hat{H}(T-t)] \cosh(\hat{H}T)^{-1} nG\} (\mathbf{b}_i - \frac{1}{n} \sum_{j=1}^{n} \mathbf{b}_j), \tag{4.12}$$

where $\hat{H} := (F + nG)^{\frac{1}{2}}$ is the real positive definite square root of $F + nG$.

(iii) The opinion dynamics (4.12) in infinite horizon satisfies

$$\lim_{T\to\infty} \mathbf{x}_i(t) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{b}_j + (F+nG)^{-1}[F + \exp(-\hat{H}t)nG](\mathbf{b}_i - \frac{1}{n}\sum_{j=1}^{n} \mathbf{b}_j) \quad (4.13)$$

with $\hat{H} = (F + nG)^{\frac{1}{2}}$.

Proof: Using the result from Lemma 4.2.1, it follows by Proposition 4.1-(i) that a unique Nash equilibrium exists, proving (i).

To see (ii), note by hypothesis that, the matrix Q simplifies to $Q = I_n \otimes (F + nG) - \mathcal{I} \otimes G$, where $\mathcal{I}$ is the $n \times n$ matrix of all 1's. It can be verified that $Q = P^{-1}DP$, where $D := \mathrm{diag}[F + nG, ..., F + nG, F]$, in which $F + nG$ is repeated $n - 1$ times and the matrices P, P^{-1} are

$$P = (\frac{1}{n}\mathcal{I} - K) \otimes I_d, \;\; P^{-1} = \begin{bmatrix} \mathbf{1}'_{n-1} & 1 \\ -I_{n-1} & \mathbf{1}_{n-1} \end{bmatrix} \otimes I_d,$$

where $K \in \{0,1\}^{n\times n}$ is a nilpotent matrix with ones at $(i, i+1)$ locations for $i = 1, \ldots, (n-1)$. It follows that $Q^{\frac{1}{2}} = P^{-1}D^{\frac{1}{2}}P$ and $\cosh(Q^{\frac{1}{2}}t) = P^{-1}\cosh(D^{\frac{1}{2}}t)P$. It is again a straightforward computation that (4.5) yields (4.12), which proves (ii). And the limit in (iii) follows by

$$\lim_{T\to\infty} \cosh[\hat{H}(T - t)] \cosh(\hat{H}T)^{-1} = \exp(-\hat{H}t). \qquad \square$$

The hypothesis in (ii) is that all agents have identical influence and identical stubbornness weight matrices and, in effect, means that the agents have "identical" views of influence from each other. If $G \neq 0$, then every agent is connected to every other agent at each layer of the network, i.e. complete multilayer network. These "unrealistic" assumptions portray an ideal situation against which the results obtained in more realistic situations can be compared.

Remark 4.2.1: (a) Observe from (4.12) that the two terms that comprise the opinion trajectory are the constant vector of average initial opinions in the network and the time dependent term which updates the difference of opinion of agent i on each issue against the average opinion.

(b) A full consensus is never reached on any issue for finite T. It is reached as $T \to \infty$ and $t \to \infty$ on all issues if and only if the stubbornness matrix tends to zero, in which case the consensus is on the average initial opinion in the network.

(c) In pairwise similar views, one expects a partial consensus. A full consensus is never reached due to the stubbornness matrix F. It is because, by (4.12), we have

$$\mathbf{x}_i(t) - \mathbf{x}_j(t) = (F + nG)^{-1}\{F + \cosh[\hat{H}(T-t)]\cosh(\hat{H}T)^{-1}nG\}(\mathbf{b}_i - \mathbf{b}_j), \quad (4.14)$$

for $i \neq j$,

$$\lim_{T\to\infty}[\mathbf{x}_i(t) - \mathbf{x}_j(t)] = (F + nG)^{-1}[F + \exp(-\hat{H}t)nG](\mathbf{b}_i - \mathbf{b}_j),$$

and

$$\lim_{t\to T}\lim_{T\to\infty}[\mathbf{x}_i(t) - \mathbf{x}_j(t)] = (F + nG)^{-1}F(\mathbf{b}_i - \mathbf{b}_j).$$

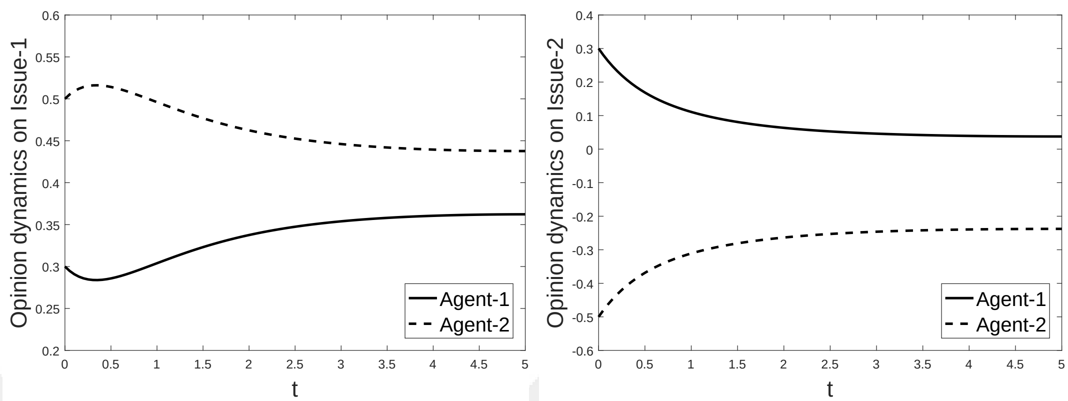
(d) From this one can conclude that faster convergence to a 'partial' consensus requires the real positive definite square root $\hat{H}$ having all its eigenvalues "large". This is same as the matrix $F + nG$ having large singular values.

(e) In the case of infinite horizon, we have

$$\begin{aligned}
\lim_{t\to\infty}\lim_{T\to\infty}\mathbf{x}_i(t) &= \frac{1}{n}\sum_{j=1}^{n}\mathbf{b}_j + (F + nG)^{-1}F(\mathbf{b}_i - \frac{1}{n}\sum_{j=1}^{n}\mathbf{b}_j),\\
&= (F + nG)^{-1}(F + nG)\frac{1}{n}\sum_{j=1}^{n}\mathbf{b}_j + (F + nG)^{-1}F(\mathbf{b}_i - \frac{1}{n}\sum_{j=1}^{n}\mathbf{b}_j),\\
&= (F + nG)^{-1}(F\mathbf{b}_i + G\sum_{j=1}^{n}\mathbf{b}_j).
\end{aligned}$$

That is, in the long run, agent i reaches at a "convex combination" of his own opinion and the average opinion in the network. $\triangle$

Example 4.2.1: The difference between opinions, (4.14), does not vary monotonically with time even in the uniform weights situation of Theorem 4.2-(ii). We illustrate this in the simple case of $n = 2$ and $d = 2$ with the two positively



(a) (b)

Figure 4.3: The effect of initial biases

correlated issues. Let the initial biases of agents be same as in Example 3.2.1, and suppose $v_{ij,kk} = 1$ and $v_{ij,12} = 0.5$, for $i, j \in \{1, 2\}$ and $k = 1, 2$ so that

$$F = G = \begin{bmatrix} 5/4 & 1 \\ 1 & 5/4 \end{bmatrix}, \ (F + 2G)^{\frac{1}{2}} = \begin{bmatrix} \sqrt{3} & \sqrt{3}/2 \\ \sqrt{3}/2 & \sqrt{3} \end{bmatrix}.$$

The derivative of the first component of (4.14) vanishes at $t^* = \frac{1}{\sqrt{3}} \ln(\frac{15}{8}) \approx 0.36$, which gives a maximum for the opinion distance on issue 1, i.e., $|x_{1,1}(t) - x_{2,1}(t)|$. From Figure 4.3 (a), it can be seen that agent 1 and agent 2 move away from each other on issue 1 at the start because of the antagonistic interaction, but eventually move towards each other. Also agent 2, by choosing opposite opinions on the issues initially, [9], manipulates agent 1 to hold contrary opinions even though agent 1 regards the issues to be positively correlated. $\triangle$

Corollary 4.2.1: If there is only one prevailing issue, i.e., $d = 1$, and $G = g$, $F = f$, $\mathbf{x}_i = x_i$, $\mathbf{b}_i = b_i$ are scalars, then for $i \in \mathbf{N}$ and $t \in [0, T]$ the opinion trajectory in a Nash equilibrium is given by

$$x_i(t) = \frac{1}{n}\sum_{j=1}^{n} b_j + \frac{1}{\lambda}\{f + ng\frac{\cosh[\sqrt{\lambda}(T-t)]}{\cosh(\sqrt{\lambda}T)}\}(b_i - \frac{1}{n}\sum_{j=1}^{n} b_j), \tag{4.15}$$

where $\lambda = f + ng$.

Proof: Note that the matrices in (4.1) can be written as: $W = fI$ and

$$Q = [f + (n-1)g]I - g(\mathcal{I} - I) = (f + ng)I - g\mathcal{I}.$$

The eigenvalues of Q are $\lambda_1 = f + ng$ with multiplicity $n-1$ and $\lambda_2 = f$ with multiplicity 1. Computing the corresponding eigenvectors, we obtain $Q = PDP^{-1}$, where $D = \text{diag}[\lambda_1, \ldots, \lambda_1, \lambda_2]$ and

$$P = [\hat{P}_1 \;\; \hat{P}_2], \quad P^{-1} = \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{bmatrix}.$$

Here, we need to note that $\hat{P}_2 = \mathbf{1}_n$ and $\tilde{P}_2 = \frac{1}{n}\mathbf{1}'_n$, because they are, respectively, right and left eigenvectors of Q associated with λ_2. Since $\hat{P}_1\tilde{P}_1 + \hat{P}_2\tilde{P}_2 = I$ and $\hat{P}_2\tilde{P}_2 = \frac{1}{n}\mathcal{I}$, we have $\hat{P}_1\tilde{P}_1 = I - \frac{1}{n}\mathcal{I}$. Then,

$$\begin{aligned} \cosh(Ht) &= [\hat{P}_1 \;\; \hat{P}_2] \begin{bmatrix} \cosh(\lambda_1^{\frac{1}{2}} t) I & 0 \\ 0 & \cosh(\lambda_2^{\frac{1}{2}} t) \end{bmatrix} \begin{bmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{bmatrix} \\ &= \cosh(\lambda_1^{\frac{1}{2}} t) I + \frac{1}{n}[\cosh(\lambda_2^{\frac{1}{2}} t) - \cosh(\lambda_1^{\frac{1}{2}} t)]\mathcal{I}, \end{aligned}$$

Substituting this in (4.5) and noting that $\lambda_1 = \lambda$, one arrives at (4.15). □

Remark 4.2.2: In an undirected complete network with one issue and identical weights, the opinion of agent i converges to a convex combination of his opinion and the initial average opinion in the network, which is given by

$$\lim_{T\to\infty} \lim_{t\to T} x_i(t) = \frac{fb_i + g\sum_{j=1}^{n} b_j}{f + ng}. \qquad \triangle$$

Example 4.2.2: Consider a complete network, Figure 4.4 (a), where every pair of distinct nodes is connected through a unique edge. We choose similar edge weights for all the pairs of agents (nodes in the network); i.e., $w_{ij} = 2$, $i \neq j$, and $w_{ii} = 0.2$, $\forall\, i, j = 1, \ldots, 10$. The opinion trajectories are shown in Figure 4.4 (b). We want to observe the rate of convergence to a partial consensus. Therefore, if we decrease the values of w_{ij} and w_{ii} by a same proportion; for instance, by a factor of 10, we obtain the trajectories shown in Figure 4.4 (c). Note that this parameter penalizes the control effort $\mathbf{u}(t)$ in the cost functional, which we removed without any loss of generality as mentioned in Remark 3.1.1. △

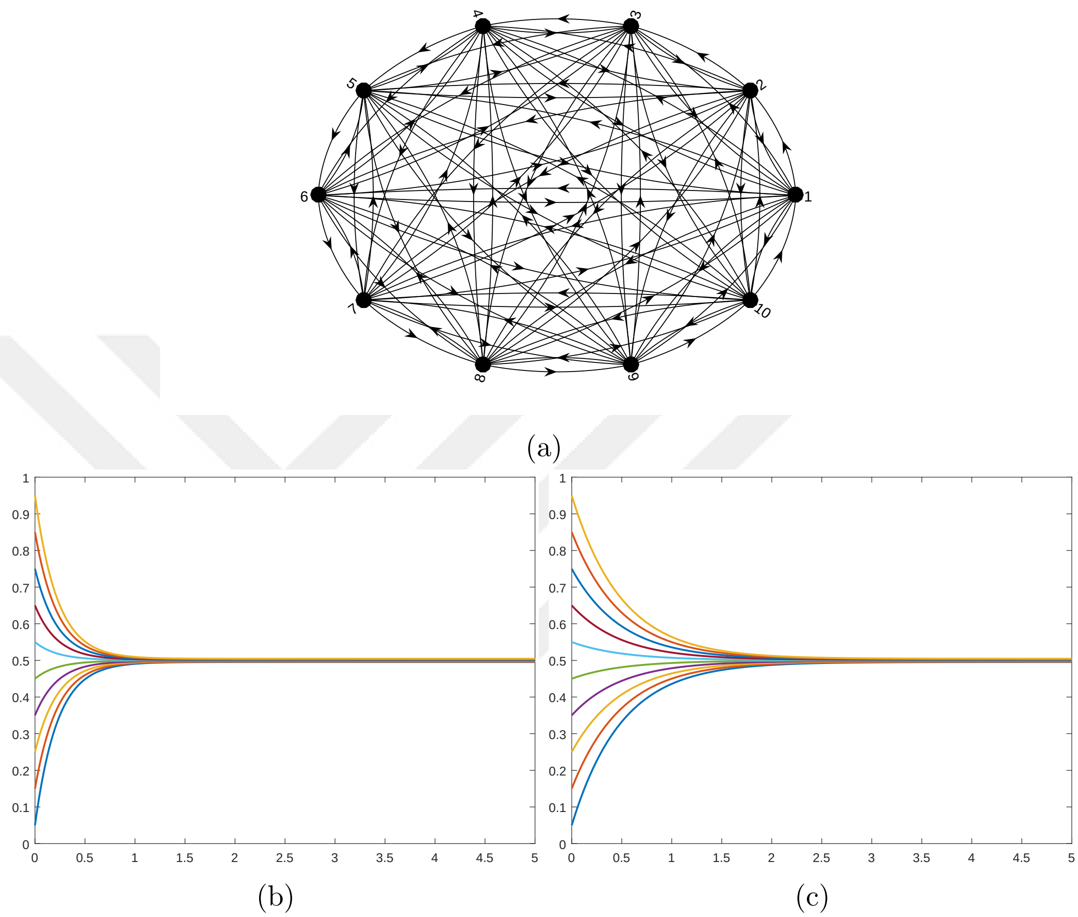


Figure 4.4: A complete network

4.3 A Network with One Leader

Consider a network where all the agents are only influenced by one agent, called the leader. The leader endures to his own opinions on the issues, which are not influenced by other agents throughout the interval $[0, T]$. The neighborhood of the leader (agent 1) is $\mathbf{N}_1 = \emptyset$ and the neighborhood of all other agents is $\mathbf{N}_i = \{1\}$, $\forall\, i \in \mathbf{N} \setminus \{1\}$ and $k \in \mathbf{D}$. It follows that, in this special case, we have

$$Q = \begin{bmatrix} W_{11} & & & \\ -W_{21} & M_2 & & \\ \vdots & & \ddots & \\ -W_{n1} & & & M_n \end{bmatrix}, \tag{4.16}$$

in which all entries left blank are zero matrices and $M_i = W_{ii} + W_{i1}, \forall\, i \in \mathbf{N}\backslash\{1\}$.

Definition 4.1: Let $A = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_n] \in \mathbb{R}^{m\times n}$, with $\mathbf{a}_i$ denoting its i-th column. The $\mathrm{vec}(A) \in \mathbb{R}^{mn}$ is defined as the vector formed by stacking the columns of A on top of each other, i.e., $\mathrm{vec}(A) := [\mathbf{a}_1' \ \ldots \ \mathbf{a}_n']'$.

Lemma 4.3.1: Consider a matrix

$$Q = \begin{bmatrix} \hat{W}_1 & 0 \\ -\hat{W}_3 & \hat{W}_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} \hat{W}_1 & 0 \\ 0 & \hat{W}_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix},$$

where $\hat{W}_1 = W_{11} \in \mathbb{R}^{d\times d}$, $\hat{W}_2 = \mathrm{diag}[W_{21} + W_{22}, \ldots, W_{n1} + W_{nn}] \in \mathbb{R}^{m\times m}$, $\hat{W}_3 = [W_{21}', \ldots, W_{n1}']' \in \mathbb{R}^{m\times d}$ with $m = (n-1)d$. And if $\hat{W}_1$ and $\hat{W}_2$ have no common eigenvalue, then $X \in \mathbb{R}^{m\times d}$ is determined by the following Sylvester equation:

$$X\hat{W}_1 - \hat{W}_2 X = \hat{W}_3. \tag{4.17}$$

Moreover, the positive definite square root of Q is given by

$$Q^{\frac{1}{2}} = R^{-1} \begin{bmatrix} \hat{W}_1^{\frac{1}{2}} & 0 \\ 0 & \hat{W}_2^{\frac{1}{2}} \end{bmatrix} R.$$

Proof: The proof follows from the fact that the lower block triangular Q is nonsingular with positive eigenvalues by positive definiteness of its diagonal blocks. Note that one can write (4.17) as

$$(\hat{W}_1 \otimes I_m - I_d \otimes \hat{W}_2)\mathrm{vec}(X) = \mathrm{vec}(\hat{W}_3).$$

The unique solution of this equation exists if and only if $\hat{W}_1 \otimes I_m - I_d \otimes \hat{W}_2$ is nonsingular, [32]. It is nonsingular if it doesn't have any zero eigenvalue. Its eigenvalues are in the form $\lambda_i - \mu_j$, where $\lambda_i \in \mathrm{eig}(\hat{W}_1)$, $i = 1, \ldots, d$, and $\mu_j \in \mathrm{eig}(\hat{W}_2)$, $j = 1, \ldots, m$. Hence, the result follows. $\square$

Theorem 4.3: (i) For a network where all the agents are only influenced by agent 1, a unique Nash equilibrium exists with the resulting opinion profile given by (4.5).

(ii) If W_{11} has no common eigenvalue with any $W_{i1} + W_{ii}$, for $i \in \mathbf{N}\backslash\{1\}$, then, in the unique Nash equilibrium that results, the opinion profile of each agent for

$t \in [0,T]$ is given by $\mathbf{x}_1(t) = \mathbf{b}_1$ and for $i \in \mathbf{N} \setminus \{1\}$

$$\mathbf{x}_i(t) = (W_{ii}+W_{i1})^{-1}\{W_{ii}\mathbf{b}_i + W_{i1}\mathbf{b}_1 + \cosh[\hat{H}_i(T-t)]\cosh(\hat{H}_iT)^{-1}W_{i1}(\mathbf{b}_i - \mathbf{b}_1)\}, \tag{4.18}$$

where $\hat{H}_i := (W_{ii} + W_{i1})^{\frac{1}{2}}$ is the real positive definite square root.

(iii) In the infinite horizon case, we have

$$\lim_{T\to\infty} \mathbf{x}_i(t) = (W_{ii} + W_{i1})^{-1}[W_{ii}\mathbf{b}_i + W_{i1}\mathbf{b}_1 + \exp(-\hat{H}_it)W_{i1}(\mathbf{b}_i - \mathbf{b}_1)],$$

in which

$$\lim_{t\to\infty}\lim_{T\to\infty} \mathbf{x}_i(t) = (W_{ii} + W_{i1})^{-1}(W_{ii}\mathbf{b}_i + W_{i1}\mathbf{b}_1).$$

Proof: From Lemma 4.3.1, $f(QT)$ is nonsingular and (i) holds by Proposition 4.1-(i). By the hypothesis in (ii), a solution X to the Sylvester equation (4.17) exists. It follows that

$$Q^{\frac{1}{2}} = R^{-1}\begin{bmatrix} \hat{W}_1^{\frac{1}{2}} & 0 \\ 0 & \hat{W}_2^{\frac{1}{2}} \end{bmatrix} R,$$

is a real positive definite square root, which gives an expression for $\cosh(Q^{\frac{1}{2}}t)$ in terms of $\hat{H}_i = (W_{ii}+W_{i1})^{\frac{1}{2}}$ for $i = 2, ..., n$. This proves (ii) by Proposition 4.1-(ii). Limits in (iii) easily follow by the expression (4.18). The square root $\hat{W}_1^{\frac{1}{2}} = W_{11}^{\frac{1}{2}}$, which appears in $Q^{\frac{1}{2}}$, cancels out in the final expressions (4.18). □

Remark 4.3.1: (a) The sufficient condition of Theorem 4.3-(ii) is satisfied for *almost all* choices of weight matrices. This condition ensures that a real positive definite square root of (4.16) exists.

(b) The expression (4.18) has again two parts. The constant first part $(W_{ii} + W_{i1})^{-1}(W_{ii}\mathbf{b}_i + W_{i1}\mathbf{b}_1)$ can be viewed as a weighted convex combination of the initial opinion vectors of the leader and agent i. The dynamic second part updates their difference in opinions in the interval $[0,T]$.

(c) By Theorem 4.3-(iii), the opinion reached in the limit by agent i thus turns out to be a convex combination of his own opinions (opinion vector) and those of the leader.

(d) We have, $\forall\, i \in \mathbf{N} \setminus \{1\}$,

$$\mathbf{x}_i(t) - \mathbf{x}_1(t) = (W_{ii} + W_{i1})^{-1}\{W_{ii} + \cosh[\hat{H}_i(T-t)]\cosh(\hat{H}_i T)^{-1} W_{i1}\}(\mathbf{b}_i - \mathbf{b}_1)$$

and

$$\lim_{T\to\infty}[\mathbf{x}_i(t) - \mathbf{x}_1(t)] = (W_{ii} + W_{i1})^{-1}\{W_{ii} + \exp(-\hat{H}_i t) W_{i1}\}(\mathbf{b}_i - \mathbf{b}_1).$$

Agent i will be of the same opinion as the leader in the long run if and only if he is not at all stubborn, i.e., $W_{ii} \to 0$. The larger all eigenvalues of $\hat{H}_i$ are, the faster will be the partial consensus that will be reached on the issues. $\triangle$

Corollary 4.3.1: If there is only one prevailing issue, i.e., $d = 1$, and $W_{i1} = w_{i1}$, $W_{ii} = w_{ii}$, $\mathbf{x}_i = x_i$ and $\mathbf{b}_i = b_i$ are scalars $\forall\, i \in \mathbf{N}$, then a Nash equilibrium for a network with one leader exists and is unique. The opinion dynamics of agents in this Nash equilibrium for $t \in [0, T]$ is given by $x_1(t) = b_1$ and for $i \in \mathbf{N} \setminus \{1\}$,

$$x_i(t) = \frac{w_{ii} b_i + w_{i1} b_1}{\lambda_i} + \frac{w_{i1}}{\lambda_i}\frac{\cosh[\lambda_i^{\frac{1}{2}}(T-t)]}{\cosh(\lambda_i^{\frac{1}{2}} T)}(b_i - b_1),$$

where $\lambda_i := w_{ii} + w_{i1}$.

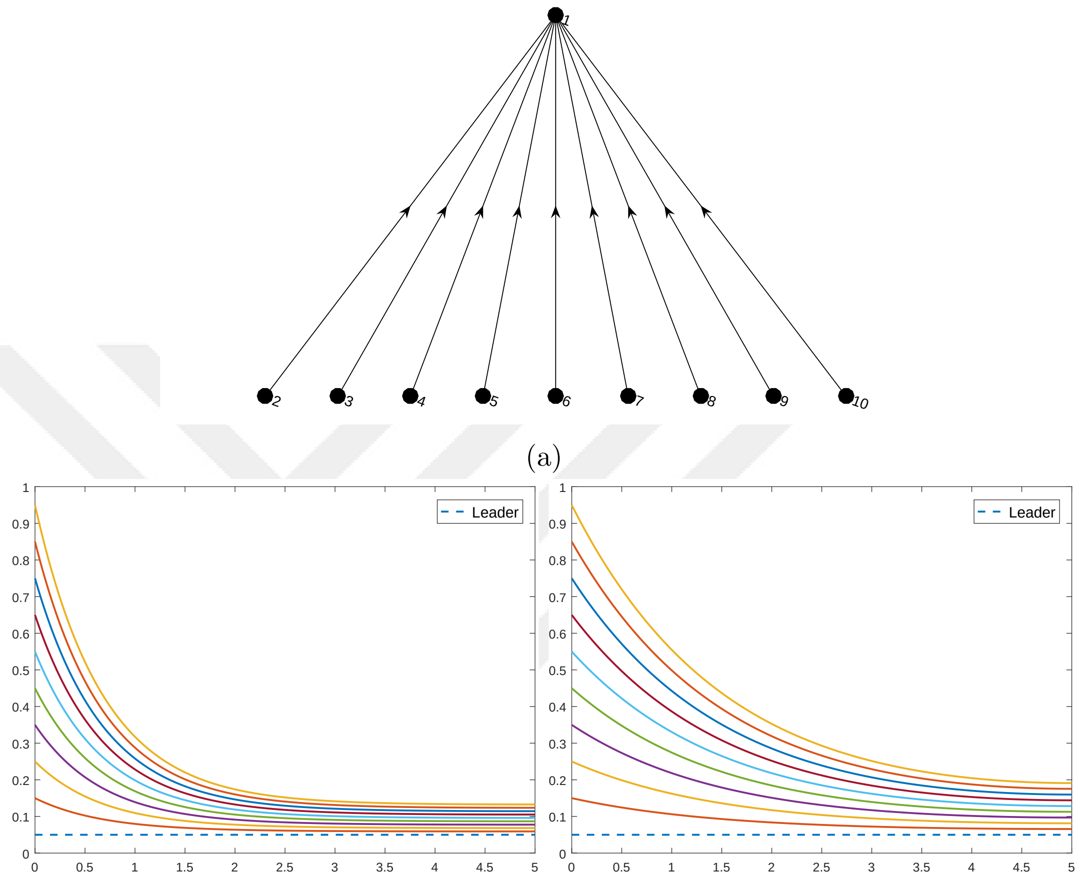
Proof: In a scalar case of one leader network, we have

$$Q = \begin{bmatrix} w_{11} & & & \\ -w_{21} & w_{21} + w_{22} & & \\ \vdots & & \ddots & \\ -w_{n1} & & & w_{n1} + w_{nn} \end{bmatrix}.$$

The eigenvalues are $\lambda_1 = w_{11}$ and for $i \in \mathbf{N} \setminus \{1\}$, $\lambda_i = w_{i1} + w_{ii}$. Then, we can write $Q = PDP^{-1}$, where $D = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$ and the matrix with corresponding eigenvectors is given as

$$P(v_{i1}) = \begin{bmatrix} 1 & & & \\ v_{21} & 1 & & \\ \vdots & & \ddots & \\ v_{n1} & & & 1 \end{bmatrix}.$$

Here, $P = P(v_{i1})$ and $P^{-1} = P(-v_{i1})$ with $v_{i1} = w_{i1}/(w_{i1} + w_{ii} - w_{11})$. Then, it is easy to calculate $\cosh(Q^{\frac{1}{2}} t)$. Also, since $w_{ii} > 0$, the eigenvalues of Q are positive and real. Using these in (4.5), we arrive at the result. $\square$



(a)

(b) (c)

Figure 4.5: A network with one leader (agent 1)

Remark 4.3.2: In a network with one leader and one issue, the opinion of agent i converges to a convex combination of his and the leader's initial opinions, i.e.,

$$\lim_{t \to T} \lim_{T \to \infty} x_i(t) = \frac{w_{ii} b_i + w_{i1} b_1}{\lambda_i}. \qquad \triangle$$

Example 4.3.1: Consider a network with one leader and one issue, Figure 4.5 (a). Selecting the weights as in Example 4.2.2, we obtain the opinion trajectories as shown in Figure 4.5 (b) and (c). Again we note the fact that if we increase the value of the parameter that penalizes control effort, we obtain the opinion trajectories with slower convergence rate. This parameter desensitizes the opinion dynamics in a sense that agents take more time to get convinced by the leader. $\triangle$

Chapter 5

Numerical Examples – Games with Multiple Stages

We now investigate two hybrid (continuous and discrete) games of multiple stages. In these networks, the agents interact with each other with some probability and the interaction network at each stage is generated accordingly, [15]. When the agents interact with each other, they determine their influence matrices depending on their bounded confidence, as given in Section 3.2.

Example 5.0.1: Consider a social network that contains two political parties (party-A and party-B), their supporters, and a group of neutral people. Suppose the size of the community is $n = 102$, including the two leaders of the parties. The leader of each party forms his opinions on issues according to the party's motto. The party-A claims to be "tough on crime" and party-B, confronting party-A, claims to be an advocate for "human rights."

The current debates are about two positively correlated issues. Issue 1 is eradication of death penalty and issue 2, private ownership of guns. Suppose $r_{i,12} = 0.5$, $\forall\, i \in \mathbf{N}$, so that people agree that the two issues are well-correlated. Party-A, since it is "tough on crime," holds negative opinions on both issues, i.e., death penalty should not be eradicated and private ownership of guns should

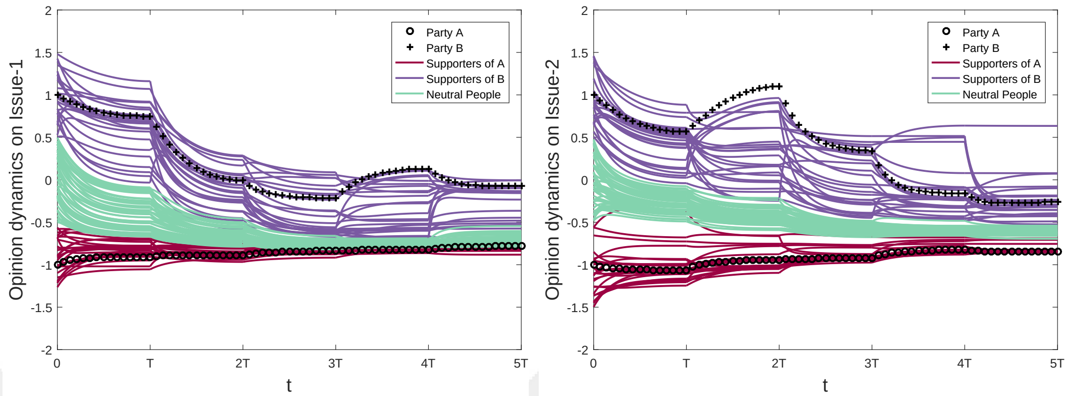


Figure 5.1: Opinion trajectories of the community with two political parties.

not be allowed. So, we can represent the initial opinion of party-A by a vector $\mathbf{b}_1 = [-1 \ \ -1]'$. However, party-B differs from party-A on both issues and holds the opinions that death penalty should be eradicated and private ownership of guns should be allowed for self-defensive purposes. Therefore, the initial opinions of party-B can be represented by $\mathbf{b}_2 = [1 \ \ 1]'$. We assume both parties propose reasonable arguments in support of their opinions so that they have almost same number of supporters. Then we have $\mathbf{N} = \mathbf{P}_A \cup \mathbf{P}_B \cup \mathbf{P}_N \cup \{1\} \cup \{2\}$; where $\mathbf{P}_A$, $\mathbf{P}_B$ and $\mathbf{P}_N$ are the sets of party-A supporters, party-B supporters and neutral people, respectively. And suppose $|\mathbf{P}_A| = |\mathbf{P}_B| = 25$ and $|\mathbf{P}_N| = 50$.

Members of the community discuss the issues with each other and form their opinions every day. During normal circumstances, the number of people that each member interacts with depends on how sociable the members of the community are, which can be roughly described by the probability ρ_{ij} that some agent i interacts with agent j. Suppose the game (3.2) is played five times and at the beginning of each game, the probabilities ρ_{ij} are used to generate a two-layered network, which is considered as an interaction network throughout the game. Suppose $\rho_{ij} = 0.2$, and the initial opinions of $\mathbf{P}_A$, $\mathbf{P}_B$ and $\mathbf{P}_N$ are uniformly distributed in the intervals $[-1.5, -0.5]$, $[0.5, 1.5]$ and $(-0.5, 0.5)$, respectively. The final opinions of the agents in each game are considered as their initial opinions for the next game, and suppose $T = 5$ is the duration of each game. Also, for simplicity, consider only one attribute, i.e., social status, for every agent. And say $\mathbf{a}_i = 1$ if agent i is a leader, otherwise $\mathbf{a}_i = 0$.

We assume that party-A supporters do a propaganda about the issues. For instance, they can increase their publicity by getting more media coverage or by starting a social media campaign. By doing so, it is almost certain that neutral people interact with (passively observe) party-A supporters, i.e., $\rho_{ij} \to 1$, for $i \in \mathbf{P}_N$, $j \in \mathbf{P}_A$.

When agent i interacts with agent j, the entries of V_{ij} are determined as

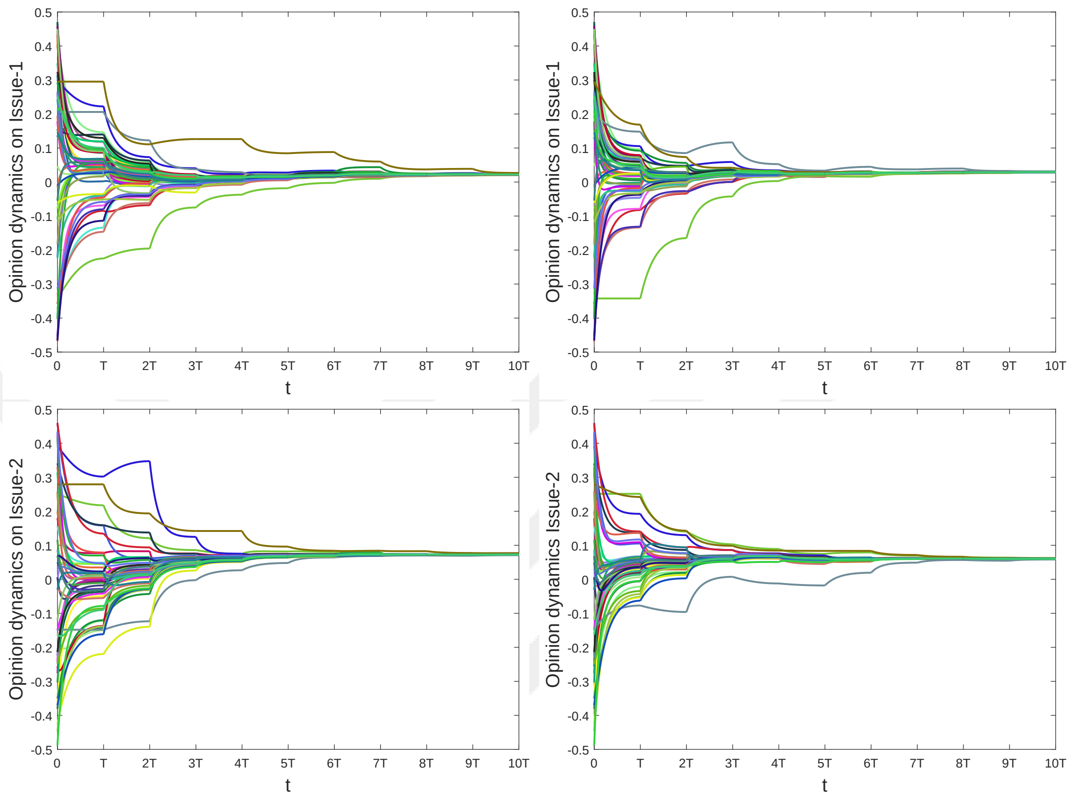
$$v_{ij,kk} = \begin{cases} 2\mathbf{a}_j + 0.5, & \text{if } |b_{i,k} - b_{j,k}| \leq \varepsilon_i; \\ 0, & \text{otherwise;} \end{cases}$$

where the bounded confidence $\varepsilon_i = 0.5$, $\forall\, i \in \mathbf{N} \setminus \{1,2\}$; and $\varepsilon_i = 0.1$, for $i \in \{1,2\}$. Also, suppose $v_{ii,kk} = \mathbf{a}_i + 0.1$, $\forall\, i \in \mathbf{N}$ and $k \in \{1,2\}$, so that the leaders are more stubborn. And $v_{ij,12} = r_{i,12} = 0.5$, $\forall\, i,j \in \mathbf{N}$. The opinion trajectories are obtained by (4.5), and are shown in Figure 5.1. Because of the propaganda, the agents in $\mathbf{P}_N$ mostly interact with those who are in $\mathbf{P}_A$, and seldom interact with those in $\mathbf{P}_B$. Therefore, they conform their opinions with $\mathbf{P}_A$. And as the agents in $\mathbf{P}_B$ interact with the agents in $\mathbf{P}_N$, they get influenced because some agents in $\mathbf{P}_N$ lie in the bounded confidence of agents in $\mathbf{P}_B$. Hence, they change their opinions by moving towards negative side on both issues. And with them moves along the party-B leader, since he cannot attract the followers if his opinions are dissimilar to the society's opinion.

Example 5.0.2: Again consider two prevailing issues with $n = 50$, $T = 5$, and suppose the game is played in 10 stages. At each stage, agent i interacts with agent j with some probability $\rho_{ij} \in [0.3, 0.7]$, and

$$v_{ij,kk} = \begin{cases} 0.8, & \text{if } |b_{i,k} - b_{j,k}| \leq \varepsilon_i; \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon_i \in [0,1]$ is a bounded confidence for agent i. Notice that the agents, unlike in Example 5.0.1, are heterogeneous since their bounded confidences are different. Also, suppose $v_{ii,kk} = 0.1$ and $v_{ij,kl} = r_{i,12}$, $\forall i,j \in \mathbf{N}$, $k,l \in \{1,2\}$, and $k \neq l$. The opinion trajectories, when $r_{i,12} = 1$, $\forall i \in \mathbf{N}$, are shown in Figure 5.2 (a), whereas the trajectories when $r_{i,12} = 1$, for $i \in \{1,\ldots,25\}$, and $r_{j,12} = -1$, for $j \in \{26,\ldots,50\}$, are shown in Figure 5.2 (b). The purpose of this example is



(a) (b)

Figure 5.2: Consensus formation

to show that when the game is repeated several times, the agents move towards (partial) consensus. That is, repetition of the interactions eventually leads to a consensus, [10], even when the agents have different conceptions about the issues, as in the case of Figure 5.2 (b).

Chapter 6

Conclusions

The model of opinion dynamics here has a number of features that puts it apart from those in the literature:

1. It is a game theoretic model played by the members in a society but a dynamic game as opposed to static.

2. It is a noncooperative game and our focus is on a Nash equilibrium as a solution that has the feature that if a player uses his best response strategy, then he is not worse off even when all the others play their best strategies.

3. In a multiple issue game, a Nash equilibrium may not exist in certain information structures and/or certain values for the weights used in forming the costs.

4. The model concludes that from individual motives of the agents, a collective behavior results. In a wide range of information structures, this collective behavior can be considered as a partial consensus in the network.

5. The existence condition for a Nash equilibrium turns out to be some kind of harmony in a society. But this harmony can arise from diametrically opposite characteristics of individuals – similarity in views or assenting to authorities in a society.

6. Games of multiple issues result in much more intricate dynamics than those on a single issue. The most prominent difference comes from being able to incorporate a penalization of inconsistency of an agent's neighbors on correlated issues.

The obvious future direction for research along similar lines would be to consider more sophisticated motives (cost functionals), also allowing the possibility of having non-uniform motives in a society, like, non-quadratic ones along with quadratic cost functionals. In several applications – for instance, in IoT (Internet-of-Things), the nature of entities (agents) of the network are different from each other and they may hold different motives. Also in social networks, not everyone is a passive observer but there are politicians, éminence grise, and activists who hold different motives and contend with each other to influence the masses. Another direction will be to consider the networks where the agents hold imperfect (or noisy) information about others.

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