BOUNDARY ELEMENT SOLUTION OF MAGNETOHYDRODYNAMIC FLOW

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

JUNE 2001

Approval of the Graduate School of Natural and Applied Sciences

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ABSTRACT

BOUNDARY ELEMENT SOLUTION OF MAGNETOHYDRODYNAMIC FLOW

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June 2001, 69 pages

In this thesis, first off all, interpolation using Radial Basis Functions (RBF) are explained. Several types of radial basis functions are tested in an example. The Dual Reciprocity Boundary Element Method (DRBEM) using RBF interpolation is given for Poisson's equations of the forms $\nabla^2 u = b(x, y, u)$ and $\nabla^2 u = b(x, y, u_x)$. Then the DRBEM is applied to Magnetohydrodynamic (MHD) Channel Flow problem with homogeneous boundary conditions which can be put in these two types of Poisson's equations. Boundary Element Method is used with constant and linear elements.

Computations are carried out for several values of Hartmann number, for different types of radial basis functions which are used in the interpolation of right hand side functions. The two forms of right hand side function are compared and we conclude that the derivative form $\nabla^2 u = b(x, y, u_x)$ with constant boundary elements gives very well agreement with the exact solution.

The velocity and the induced magnetic field values which are obtained show the well known characteristics of MHD Flow.

Keywords: Interpolation, Radial basis functions, MHD flow, Dual reciprocity boundary element method.

ÖZ

MAGNETOHİDRODİNAMİK AKIMIN SINIR DEĞER ÇÖZÜMÜ

Erdönmez, Cengiz
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi: Prof. Dr. Münevver Tezer

Haziran 2001, 69 sayfa

Bu tezde öncelikle, Radyal Baz Fonksiyonları (RBF) kullanılarak yapılan enterpolasyon anlatılmaktadır. Bir örnek üzerinde, bir kaç tip radyal baz fonksiyonu test edilmektedir. Radyal baz fonksiyonları kullanılarak yapılan karşılıklı sınır elemanı metodu (DRBEM) Poisson denkleminin $\nabla^2 u = b(x, y, u)$ ve $\nabla^2 u = b(x, y, u_x)$ formları için verilmiştir. Daha sonra karşılıklı sınır elemanı metodu, Poisson denkleminin her iki formuna da konulabilinen magnetohidrodinamik kanal akım problemine homojen sınır değerleri için uygulanmaktadır. Sınır elemanı metodu sabit ve lineer elemanlar ile kullanılmıştır.

Hesaplamalar, Hartmann sayısının bir kaç değeri ve sağ taraf fonksiyonunun enterpolasyonunda kullanılan radyal baz fonksiyonlarının değişik formları için yapılmıştır. Sağ taraf fonksiyonunun her iki formu için elde edilen sonuçlar karşılaştırıldığında, türevli formun, sabit sınır elemanları ile kullanımının teorik çözüm ile mukayese anlamında, daha iyi olduğu görülmektedir.

Elde edilen hız ve indüklenmiş manyetik alan değerleri, magnetohidrodinamik akımın iyi bilinen karakteristik özelliklerini göstermektedir.

Anahtar Kelimeler : Enterpolasyon, Radyal baz fonksiyonları, MHD akım, Karşılıklı sınır elemanı metodu.

ACKNOWLEDGMENTS

I would like to express my sincere appreciation to the members of the Mathematics Department of Middle East Technical University, especially to Prof. Dr. Münevver TEZER for her valuable guidance and helpful suggestions in every aspect from the very beginning onwards.

Best regards to my friend Selçuk Han AYDIN who helped and supported me during the thesis period. Also thanks to my colleagues for their understandings.

Finally, deepest regards and love to my mother for her existence, the biggest part of my success is depended on my wife's, Dr. Dilek ERDÖNMEZ, incredible support, love and help for easing my life in every area.

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INTRODUCTION

The method used in this thesis which is now a well-established numerical technique for solving boundary-value problems is called the Boundary Element Method (BEM). Boundary Element Method has emerged as a powerful alternative to Finite Element Method, particularly in cases where better accuracy is required (Brebbia, Dominguez (1989)). The main aim of the Boundary Element Method is to transform the given differential equation to an equivalent integral equation and solve it on the boundary using discretization procedures. Boundary discretization is the only necessity for homogeneous Partial Differential Equations (PDE). This provides reduction in the dimension of the problem which permits accurate solutions to be obtained very efficiently. This is the main advantage of the BEM approach (Yamada, Wrobel(1993)).

The first book on Boundary Element Method was published by Brebbia(1978) and it was proven that the BEM, like Finite Element Method (FEM) and many other numerical methods, can be obtained as a special case of the general weighted residual statement.

The textbook of Banerjee and Butterfield(1981), represents the first comprehensive work on the BEM and its applications in the various fields of engineering science. The more recent textbook of Brebbia, Telles and Wrobel(1984) is another comprehensive and general work on the same subject. The boundary elements literature is recently becoming richer and richer in textbooks devoted to specialized subjects, such as creep and fracture problems on solid mechanics, fluid mechanics, inelastic problems, the potential theory, electromagnetics, and magnetohydrodynamic problems.

There are basically two kinds of approachs in BEM, the indirect and the direct ones. In the indirect approach the discretized integral equations are first solved for the density of the singular solutions over the boundary surface and then the remaining boundary quantities are computed in terms of these densities which have no physical significance. In the direct approach the discretized integral equations are formulated with the help of certain fundamental integral theorems and connect directly the unknown with the known boundary quantities. Even though it has been shown by Brebbia and Butterfield(1978) that the indirect and direct BEMs are formally equivalent, more emphasis is usually given to direct BEM because it is more appealing to scientists and engineers. In addition, direct BEM appears to be, at least so far, more easily amenable to improvements of the FEM with indirect form of BEM (Beskos (1987)).

Applications of the BEM abound in many areas of engineering such as general potential theory, potential fluid flow, acoustics, torsion of shafts, electric and magnetic field theory, elastostatics, elastodynamics, plates and shells, transient heat conduction, viscoelasticity, fracture, plasticity, water waves, viscous fluid flow, ground water flow, Navier-Stokes equations, wave propagation, thermoelasticity and other time dependent problems (Brebbia (1981)).

In the application of the BEM on an inhomogeneous partial differential equation, the integral equation involves a domain integral, and the reduction in dimension therefore the advantage of BEM is apparently lost. Among various methods which have been proposed to deal with this problem, the most successful one is the so-called Dual Reciprocity Method(DRM) (Partridge, Brebbia and Wrobel (1992), Wrobel and Brebbia (1987), Partridge (1997)). This method divides the solution into two parts: a known particular solution of the inhomogeneous partial differential equation plus a complementary solution of its homogeneous counterpart. Since particular solutions to complex problems are very difficult or impossible to obtain, the inhomogeneity is normally represented by a series expansion in terms of simpler functions for which particular solutions can be easily determined.

The crucial point in the DRM formulation is the choice of the approximating functions in the series, since this will dictate the accuracy and efficiency of the solution. The most widely used functions are of the type $r+c$, with r representing the distance between fixed and generic points and c a constant. These functions have been empirically chosen and demonstrated to work well for a variety of problems. Recently, a proof of convergence has been derived by Yamada(1993) where the fairly local behaviour of the interpolation functions resulting radial basis function approximation is demonstrated.

The above function forms part of a more general class known as Radial Basis Functions (RBF) which have some established properties and are popular for multivariate interpolation. The RBF interpolation was first introduced by Hardy(1971), and Harder and Desmarais(1972), Powell(1987) and the procedure leads to the DRM for the solution of Poisson-type equations and was first suggested by Nardini and Brebbia(1983).

In this thesis we present an interpolation scheme using radial basis functions. Several radial basis functions (polynomials, exponentials, rationals and logarithmic functions) are used and tested on the interpolation of a function. Then we will use this interpolation technique in the solution of Poisson's equation with Dual Reciprocity Boundary Element Method(DRBEM). Applications are carried out to nonlinear Poisson's equation of the form $\nabla^2 u = b(x, y, u, u_x, u_y)$ which permits to solve magnetohydrodynamic channel flow equations.

The magnetohydrodynamic(MHD) flow finds a lot of applications in various branches of science and technology such as MHD generator, pumps, accelerators, flowmeters, space propulsion, astrophysics and geophysics. Hartmann(1937) investigated for the first time the MHD flow of a viscous, incompressible, electrically conducting fluid between two parallel planes in the presence of a transverse magnetic field. Since then a number of researchers have investigated the flow of an electrically conducting fluid through channels. Due to the coupling of the equations of fluid mechanics and electrodynamics, analytical solutions are not available or not practical to use. Therefore, it is desirable to explore more efficient numerical methods for accurate numerical solutions.

Singh and Lal(1982,1984) presented finite element method(FEM) solution to solve MHD channel flow problems for arbitrary wall conductivity and for small Hartmann number $M \leq 5$. Later Tezer-Sezgin and Köksal(1989) improved the FEM results for high Hartmann numbers. Gardner and Gardner(1995) employed bi-cubic B-spline elements in FEM. Sezgin, Aggarwala and Ariel(1988) solved MHD channel flow problems taking into account various electomagnetic boundary conditions. The application of the boundary element method(BEM) to the MHD flow in a cylindrical duct for small Hartmann numbers was given by Carabineanu, Dinu and Oprea(1995). Tezer-Sezgin and Dost(1994) , Singh and Agarwal(1984) applied BEM to solve MHD flow problems by transforming the problem into integral equations.

This thesis gives dual reciprocity boundary element method (DRBEM) application to solve MHD flow problem in a rectangular channel. The coupled velocity and magnetic field equations are firstly transformed into decoupled nonlinear Poisson equations where the right-hand side functions are appoximated by a series of radial basis functions.

The decoupled nonlinear Poisson equations obtained are of the two types, i.e. $\nabla^2 u = b(x, y, u)$ and $\nabla^2 u = b(x, y, u_x)$. The computations are carried out for Hartmann number $M \leq 15$ using both constant and linear (discontinuous) boundary elements. Comparisons for two types of formulations of Poisson equation are made and it is found that $\nabla^2 u = b(x, y, u_x)$ formulations gives better results. All the results obtained are also in good agreement with the exact solution of the MHD Channel Flow problem.

1.1 Plan of the Thesis

In Chapter 2, first we introduce Radial Basis Functions and then interpolation using Radial Basis Functions is given with a test function using several radial basis functions.

In Chapter 3, the theory of the Boundary Element Method together with Dual Reciprocity procedure is given. The DRBEM using Radial Basis Functions for the cases $\nabla^2 u = b(x, y)$, $\nabla^2 u = b(x, y, u)$ and for the general case $\nabla^2 u = b(x, y, u, u_x, u_y)$ are respectively presented. Also the computational algorithms for all the cases are derived. In Section 3.5 magnetohydrodynamic channel flow problem is introduced and the Dual Reciprocity Boundary Element Method application for this problem is given.

Chapter 4 presents applications of DRBEM on Poisson's equations of the types $\nabla^2 u = b(x, y, u)$ and $\nabla^2 u = b(x, y, u_x)$ which are the basic equations in MHD flow. These two forms are used in the solution of MHD channel flow problem. The solutions are compared with the exact solution and the efficiency of the solutions are indicated in terms of graphics. Solutions are obtained for several values of Hartmann number with constant and linear boundary elements.

Finally, conclusion provides some comparisons for the solution of the MHD channel flow problems written in the form of $\nabla^2 u = b(x, y, u)$ or $\nabla^2 u = b(x, y, u_x)$.

CHAPTER 2

INTERPOLATION USING RADIAL BASIS FUNCTIONS

This Chapter contains interpolation of functions defined in two dimensions. The interpolating functions used are taken as radial basis functions.

2.1 Radial Basis Functions

Radial basis functions belong to a class of functions where the distance from a fixed point to a variable field point is used as the independent variable. For this purpose, the Euclidean distance function r_k is defined in 2D as

$$r_k = r_k(x, y; x_k, y_k) = \sqrt{(x - x_k)^2 + (y - y_k)^2} \tag{2.1}$$

where x and y are the positional coordinates of a variable field point, and x_k and y_k correspond to the coordinates of the k^{th} interpolation point. Similar definitions hold for higher dimensions. Functions using r_k as the independent variable are then referred as the Radial Basis Functions(RBF). Several researchers (Franke(1982), Powell(1987), Kansa(1990), Carlson and Foley(1991)) have used them in the context of multivariate interpolation. Powell(1987) has written a special chapter about radial basis function interpolation in his book for approximation. Zheng, Coleman and Phan-Thien(1991) used radial basis functions to find approximate solutions for nonhomogeneous potential problems.

Thus, the interpolation of a function b is represented as

$$b(P) = \sum_{k=1}^{N+L} \Phi_k(P)\alpha_k \tag{2.2}$$

where Φ_k belongs to a class of RBF's. In equation (2.2), P represents the location of any variable field point. The number of interpolation points are denoted by N+L, where N usually represents the number of points on the boundary of the domain and L the number of internal interpolation points. The interpolating coefficients α_k are computed by satisfying the interpolation condition at N+L points,

$$b(P_i) = b_i = \sum_{k=1}^{N+L} \Phi_{ik}\alpha_k \ , \ i = 1, \ldots, N+L \tag{2.3}$$

where Φ_{ik} represents the value of Φ_k at the point (x_i, y_i) and b_i is the function value at this point.

The interpolation using RBF involves a single independent variable r_k as opposed to the use of multiple spatial variables, regardless of the dimension of the problem. Hence, it is clear that the RBF interpolation is most useful when the domain of the function can not be expressed as simple product domains of lower dimensions. The RBF interpolation was first introduced by Hardy(1971) and Harder and Desmarais(1972). Subsequently, a wide variety of RBF have been used for the interpolation and some of the common types of RBF as follows (Ramachandran and Karur (1998)).

1. Linear function : $\Phi_k = 1 + r_k$
2. Duchon radial cubics : $\Phi_k = r_k^3$
3. Radial quadratic plus cubic : $\Phi_k = 1 + r_k^2 + r_k^3$
4. Thin plate splines : $\Phi_k = r_k^2 ln r_k$

5. Hardy multiquadrics : $\Phi_k = (r_k^2 + C^2)^{n/2}$ with C being a user specified constant and n being a positive integer.

6. Inverse multiquadrics : $\Phi_k = 1/(\sqrt{r_k^2 + C^2})$

7. Gaussian : $\Phi_k = exp(-r_k^2/C^2)$

8. $\Phi_k = (1 + r_k^2)^{-3/2}$.

The radial basis functions are now being used increasingly in many areas. In neural networks (Pottmann and Henson(1997)), kinetic modeling (Venkatesh(1997)), Scattered data interpolation (Dubal(1992)), solution of differential equations (Kansa(1990)), solution of integral equations (Makraoğlu(1992)). A bibliography of many applications of RBF in various fields of engineering and physics has been recently published by Golberg and Chen(1994).

2.2 Radial Basis Function Interpolation

Assuming the function $b(x, y)$ is given in a domain Ω, it can be interpolated as

$$b(x, y) = \sum_{k=1}^{N+L} \Phi_k(P)\alpha_k \tag{2.4}$$

where the function $b(x, y)$ is defined at every point in the domain and the functional values can be calculated exactly. N represents the number of interpolation points chosen on the boundary of the domain and L represents the number of interpolation points chosen in the domain. Φ_k is the arbitrarily chosen function from the class of radial basis function which is defined at interpolation points. P represents the coordinates of interpolation points. And α_k's are unknown interpolation coefficients.

In order to calculate α_k's, we use the functional values at interpolation points and the values of the chosen radial basis function at these points in the following equation:

$$b_i = b(x_i, y_i) = \sum_{k=1}^{N+L} \Phi_{ik}\alpha_k \ , \ i = 1, \ldots, N+L \tag{2.5}$$

where (x_i, y_i) is the i^{th} interpolation point used as variable field point and

$$\Phi_{ik} = \Phi(x_i, y_i; x_k, y_k).$$

Open form of the equations are written explicitly as follows:

$$\begin{array}{lllllll}
b_1 & = & \Phi_{1,1}\alpha_1 & + \cdots + & \Phi_{1,N}\alpha_N & + \cdots + & \Phi_{1,N+L}\alpha_{N+L} \\
\vdots & & \vdots & \vdots \quad \vdots & & \vdots \quad \vdots & \\
b_N & = & \Phi_{N,1}\alpha_1 & + \cdots + & \Phi_{N,N}\alpha_N & + \cdots + & \Phi_{N,N+L}\alpha_{N+L} \\
b_{N+1} & = & \Phi_{N+1,1}\alpha_1 & + \cdots + & \Phi_{N+1,N}\alpha_N & + \cdots + & \Phi_{N+1,N+L}\alpha_{N+L} \\
\vdots & & \vdots & \vdots \quad \vdots & & \vdots \quad \vdots & \\
b_{N+L} & = & \Phi_{N+L,1}\alpha_1 & + \cdots + & \Phi_{N+L,N}\alpha_N & + \cdots + & \Phi_{N+L,N+L}\alpha_{N+L}.
\end{array}$$

The corresponding system of equations can be written in the matrix form as follows:

$$\begin{bmatrix} \Phi_{1,1} & \cdots & \Phi_{1,N} & \cdots & \Phi_{1,N+L} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \Phi_{N,1} & \cdots & \Phi_{N,N} & \cdots & \Phi_{N,N+L} \\ \Phi_{N+1,1} & \cdots & \Phi_{N+1,N} & \cdots & \Phi_{N+1,N+L} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \Phi_{N+L,1} & \cdots & \Phi_{N+L,N} & \cdots & \Phi_{N+L,N+L} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ \alpha_{N+1} \\ \vdots \\ \alpha_{N+L} \end{Bmatrix} = \begin{Bmatrix} b_1 \\ \vdots \\ b_N \\ b_{N+1} \\ \vdots \\ b_{N+L} \end{Bmatrix} \tag{2.6}$$

which can be represented as $[\Phi]\{\alpha\}=\{b\}$. When the interpolation coefficients are calculated then the functional value at any point (x, y) in the domain can be calculated as

$$b(x, y) = \sum_{k=1}^{N+L} \Phi_k(x, y; x_k, y_k)\alpha_k. \tag{2.7}$$

The interpolation points are selected with the distance r_k such that $\Phi_k(x_i, y_i; x_k, y_k)$ will form a nonsingular matrix Φ therefore it can be inverted.

The effect of the radial basis functions used in the approximations will be analyzed on the test function b(x,y)=sin(πx)sin(πy). The following radial basis functions are used:

1. $\Phi_k^{(1)} = 1 + r_k^2 + r_k^3$
2. $\Phi_k^{(2)} = r_k^2 ln r_k$
3. $\Phi_k^{(3)} = exp(-r_k^2)$
4. $\Phi_k^{(4)} = (r_k^2 + 1)^{1/2}$
5. $\Phi_k^{(5)} = (r_k^2 + 1)^{-1/2}$

6. $\Phi_k^{(6)} = (r_k^2 + 1)^{-3/2}$

The results are given in the following table comparing with the exact values at several points to seven digits accuracy.

Interpolation of b(x,y)=sin(πx)sin(πy) with different radial basis functions

x	y	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Exact
.500	.000	.0000003	-.0013681	.0000000	-.0002882	-.0000250	.0000085	.0000000
.600	.100	.2923740	.2907650	.2892651	.2935859	.2939028	.2938865	.2938926
.800	.500	.5906507	.5869771	.5876026	.5886373	.5879930	.5883617	.5877849
.300	.700	.6545056	.6511268	.6545084	.6550757	.6543935	.6545206	.6545084
.800	.800	.3445975	.3495130	.3454702	.3447119	.3457959	.3459712	.3454911
1.000	1.000	-.0490655	.0019755	-.0884074	-.0000757	-.0015637	-.0028692	.0000000

From the table, it is seen that even the polynomial choice gives better results. Rational or exponential radial basis functions give quite accurate values too.

CHAPTER 3

DUAL RECIPROCITY BOUNDARY ELEMENT METHOD FOR POISSON TYPE EQUATIONS

In this thesis, our interest in Radial Basis Function(RBF) interpolation is in the context of its application in the Dual Reciprocity (DR) boundary element method (BEM) for the solution of the equations of the type

$$\nabla^2 u = b(x, y, u, u_x, u_y) \tag{3.1}$$

where (x,y) represents the spatial coordinates and u is the dependent variable. Also the thesis aims to solve magnetohydrodynamic (MHD) channel flow problem in a rectangular duct with insulating walls. This problem can be transformed to the Poisson's type equations of the forms

$$\begin{aligned} \nabla^2 u &= b(x, y, u) \\ \nabla^2 u &= b(x, y, u_x), \end{aligned} \tag{3.2}$$

therefore the dual reciprocity boundary element method solution of these equations will automatically solve the MHD channel flow problem. This solution will be given in Section 3.5.

The direct boundary element method is difficult for the problem on the equation (3.1) due to the presence of the nonhomogeneous term b in the right hand side of equation, and involves the evaluation of a domain integral corresponding to the right hand side. One way to circumvent the problem is to find an approximate particular solution to the nonhomogeneous term. For this purpose, the nonhomogeneous term b can be expanded in terms of a set of a basis functions Φ_k as in equation (2.2). If additional auxiliary functions f_k are defined such that

$$\nabla^2 f_k = \Phi_k \tag{3.3}$$

then it is possible to transfer the domain integrals to the boundary integrals. This procedure leads to the Dual Reciprocity Method (DRM) for the solution of Poisson type equations and was first suggested by Nardini and Brebbia (1983) and then established as a book by Partridge, Brebbia and Wrobel(1992).

3.1 Boundary Element Method for $\nabla^2 u = 0$ and $\nabla^2 u = b(x, y)$

Consider the Laplace equation first

$$\nabla^2 u = 0 \quad in \quad \Omega \tag{3.4}$$

with the following boundary conditions:

(i) "Essential" conditions of type $\quad u = \bar{u} \quad on \quad \Gamma_1$

(ii) "Natural" conditions such as $\quad q = \partial u / \partial n = \bar{q} \quad on \quad \Gamma_2$

where n is the outward normal to the boundary $\Gamma = \Gamma_1 + \Gamma_2$ and the bars indicate known values.

In principle, the errors introduced in the above equations if the exact (but unknown) values of u and q are replaced by an approximate solution can be minimized by orthogonalizing them with respect to a weight function $u^\star$, the normal derivative of which along the boundary is $q^\star = \partial u^\star/\partial n$.

In other words, if ϵ defines the residuals, the following equations are obtained:

$$\begin{aligned} \epsilon_\Omega &= \nabla^2 u \neq 0 \quad in \quad \Omega \\ \epsilon_1 &= u - \bar{u} \neq 0 \quad on \quad \Gamma_1 \\ \epsilon_2 &= q - \bar{q} \neq 0 \quad on \quad \Gamma_2 \end{aligned} \tag{3.5}$$

where u and q are approximate values.

These residuals can be weighted by using weight functions $u^\star$ and $q^\star$ on the equations in (3.5) respectively for obtaining method of weighted residual statement (Brebbia and Dominguez (1989))

$$\int_\Omega \epsilon_\Omega u^\star d\Omega + \int_{\Gamma_1} \epsilon_1 q^\star d\Gamma_1 - \int_{\Gamma_2} \epsilon_2 u^\star d\Gamma_2 = 0. \tag{3.6}$$

The objective of this procedure is to force the residuals to be zero in an average sense. The sign of the different terms will become clear during the process of integration by parts.

$$\int_\Omega \nabla^2 u u^\star d\Omega + \int_{\Gamma_1} (u - \bar{u}) q^\star d\Gamma_1 - \int_{\Gamma_2} (q - \bar{q}) u^\star d\Gamma_2 = 0. \tag{3.7}$$

According to the Green's Identity

$$\int_\Omega \Phi \nabla^2 \psi d\Omega = \int_{\partial\Omega} \Phi \frac{\partial \psi}{\partial n} d\partial\Omega - \int_\Omega \nabla\Phi \nabla\psi d\Omega \tag{3.8}$$

we rewrite the equation (3.7) as

$$\int_{\Gamma_1+\Gamma_2} \frac{\partial u}{\partial n} u^\star d\Gamma - \int_\Omega \nabla u \nabla u^\star d\Omega + \int_{\Gamma_1} (u - \bar{u}) q^\star d\Gamma_1 - \int_{\Gamma_2} (q - \bar{q}) u^\star d\Gamma_2 = 0. \tag{3.9}$$

Applying the Green's Identity to the second integral, we obtain

$$\int_{\Gamma_1+\Gamma_2} qu^\star d\Gamma + \int_\Omega u\nabla^2 u^\star d\Omega - \int_{\Gamma_1+\Gamma_2} uq^\star d\Gamma + \int_{\Gamma_1} (u-\bar{u})q^\star d\Gamma_1 - \int_{\Gamma_2} (q-\bar{q})u^\star d\Gamma_2 = 0. \tag{3.10}$$

In equation (3.10) we choose weight function $u^\star$ such that

$$\int_\Omega u\nabla^2 u^\star d\Omega = -c_i u_i$$

where $c_i = c(x_i, y_i)$ is a coefficient whose value depends on the position of the point $P_i = (x_i, y_i)$ and $u_i = u(x_i, y_i)$. Substitution of the above equation into the equation (3.10) and making use of the boundary conditions in equation (3.4) we finally obtain

$$c_i u_i - \int_{\Gamma_1} (qu^\star - \bar{u}q^\star) d\Gamma_1 - \int_{\Gamma_2} (\bar{q}u^\star - uq^\star) d\Gamma_2 = 0. \tag{3.11}$$

The weight function $u^\star$ is also called the fundamental solution of the given differential equation which is the Laplace equation in this thesis.

The equation (3.11) applies to a concentrated source at the point $P_i = (x_i, y_i)$ and consequently the values of $u^\star$ and $q^\star$ are those corresponding to that particular point. For each different point a new integral equation is obtained.

3.1.1 Fundamental Solution

The fundamental solution $u^\star$ represents the field generated by a concentrated unit source acting at a point P_i. The effect of this source is propagated from $P_i = (x_i, y_i)$ to infinity without any consideration of boundary conditions. Because of this $u^\star$ satisfies the following Poisson equation,

$$\nabla^2 u^\star + \Delta_i = 0 \tag{3.12}$$

where Δ_i represents a Dirac delta function which becomes infinity at the point $P_i = (x_i, y_i)$ and is equal to zero elsewhere. The integral of Δ_i over the domain is equal to one. The use of Dirac delta function is an elegant way of representing unit concentrated sources and forces when dealing with differential equations. Integral property of Dirac delta function gives

$$\int_\Omega u\nabla^2 u^\star d\Omega = \int_\Omega u(-\Delta_i)d\Omega = -c_i u_i$$

where $u = u(x_i, y_i)$ and $c_i = \begin{cases} \frac{1}{2} & (x_i, y_i) \in \Gamma \\ 1 & (x_i, y_i) \in \Omega. \end{cases}$

For an isotropic two-dimensional medium the fundamental solution of equation (3.12) is (Federico and José, (1997))

$$u^\star = \frac{1}{2\pi}\ln(\frac{1}{r}) \tag{3.13}$$

and for three-dimensional isotropic medium

$$u^\star = \frac{1}{4\pi r}. \tag{3.14}$$

In order to make some simplification on the equation (3.11) introduce the following notations:

$$\tilde{u} = \begin{cases} u & \text{on} & \Gamma_2 \\ \bar{u} & \text{on} & \Gamma_1 \end{cases}$$

and

$$\tilde{q} = \begin{cases} q & \text{on} \quad \Gamma_1 \\ \bar{q} & \text{on} \quad \Gamma_2 \end{cases}$$

where $\bar{u}$ is the known value of the u and $\bar{q}$ is the known value of q.

Therefore final form of the equation (3.11) becomes

$$c_i\tilde{u}_i - \frac{1}{2\pi}\int_\Gamma (\tilde{q}\ln|\vec{r}-\vec{r_i}| - \tilde{u}\frac{1}{|\vec{r}-\vec{r_i}|^2}(\vec{r}-\vec{r_i}).\vec{n})d\Gamma = 0 \tag{3.15}$$

where $\vec{r} = (x, y)$ and $\vec{r_i} = (x_i, y_i)$ are the free and fixed(source) points.

3.1.2 Discretization of the Boundary

We consider now how the equation (3.15) can be discretized to find the system of equations from which the boundary values are calculated. The boundary of our two dimensional region is divided into N segments or elements. The points where the unknown values are considered are called "nodes". The number of nodes on an element is equal to the number of unknowns of the approximate solution for this element.

In general case we can write discretization form,

$$c_i\tilde{u}_i - \frac{1}{2\pi}\sum_{e=1}^{E}\int_{\Gamma_e}(\sum_{m=1}^{\#e}\tilde{q}_m N_m)\ln|\vec{r}-\vec{r_i}|d\Gamma_e + \frac{1}{2\pi}\sum_{e=1}^{E}\int_{\Gamma_e}(\sum_{m=1}^{\#e}\tilde{u}_m N_m)\frac{(\vec{r}-\vec{r_i})\vec{n}}{|\vec{r}-\vec{r_i}|^2}d\Gamma_e = 0 \tag{3.16}$$

where $\tilde{u}_m$ and $\tilde{q}_m$ are the function and normal derivative values at node m respectively, N_m is the trial(shape) function for element e, E is the number of elements on the boundary and #e is the number of nodes on element e.

3.1.3 Constant Element Formulation

The boundary is assumed to be divided into N elements. In this case the values of u and q are assumed to be constant over each element and equal to the value at the mid-element node. The points at the extremes of the elements are used only for defining the geometry of the problem. Note that for this type of element (constant) the boundary is always "smooth" at the nodes as these are located at the center of the elements, hence the c_i constant is $\frac{1}{2}$. Also the shape functions N_m's for each element is equal to 1 in constant element method. Hence, the equation (3.16) for constant element method becomes:

$$\frac{1}{2}\tilde{u}_i + \sum_{j=1}^{N} \bar{H}_{ij}\tilde{u}_j - \sum_{j=1}^{N} G_{ij}\tilde{q}_j = 0 \tag{3.17}$$

where

$$\bar{H}_{ij} = \frac{1}{2\pi}\int_{\Gamma_j} \frac{(\vec{r}-\vec{r_i}).\vec{n}}{|\vec{r}-\vec{r_i}|^2} d\Gamma_j \tag{3.18}$$

$$G_{ij} = \frac{1}{2\pi}\int_{\Gamma_j} \ln|\vec{r}-\vec{r_i}| d\Gamma_j \tag{3.19}$$

and $r_i = (x_i, y_i)$, $r = (x, y)$ are both varying on the boundary nodes, $r(x, y)$ being on the j^{th} element.

If we define

$$H_{ij} = \bar{H}_{ij} + \frac{1}{2}\delta_{ij} \tag{3.20}$$

where δ is the Kronecker delta, in such a way that the $\frac{1}{2}$ value is summed to $\bar{H}$

when $i = j$, then the equation (3.17) can now be written as

$$\sum_{j=1}^{N} H_{ij}\tilde{u}_j = \sum_{j=1}^{N} G_{ij}\tilde{q}_j. \tag{3.21}$$

If it is now assumed that the position of node i also varies from 1 to N, *i.e.* one assumes that the fundamental solution is applied at each node successively, a system of equations is obtained resulting from the application of equation (3.21) to each boundary point in turn.

This set of equations can be expressed in matrix form as

$$[H]\{u\} = [G]\{q\} \tag{3.22}$$

where H and G are two NxN matrices and u and q are vectors of length N.

Notice that N_1 values of u and N_2 values of q are known on Γ_1 and Γ_2 respectively ($N_1 + N_2 = N$), hence there are only N unknowns in the system of equations (3.22). To introduce these boundary conditions into equation (3.22) one has to rearrange the system by moving columns of H and G from one side to the other. When all unknowns are passed to the left-hand side, we can write

$$[A]\{x\} = \{y\} \tag{3.23}$$

where x is a vector of unknown boundary values of u and q, and y is found by multiplying the corresponding columns of H or G by the known values of u or q. It is interesting to point out that the unknowns are now a mixture of the potential and its normal derivative, rather than the potential only as in finite element method. This is a consequence of the boundary element method being a "mixed" formulation, and constitutes an important advantage over finite element method.

Equation (3.23) can now be solved and all the boundary values will then be known. Once this is done it is possible to calculate internal values of u. The values of u are calculated at any internal point i using the following formula:

$$\tilde{u}_i = \int_\Gamma \tilde{q} u^\star d\Gamma - \int_\Gamma \tilde{u} q^\star d\Gamma. \tag{3.24}$$

Notice that now the fundamental solution is considered to be acting on an internal point i and that all values of u and q are already known. The process is then one of direct integration. The same discretization is used for the boundary integrals, *i.e.*

$$\tilde{u}_i = \sum_{j=1}^{N} G_{ij}\tilde{q}_j - \sum_{j=1}^{N} \bar{H}_{ij}\tilde{u}_j. \tag{3.25}$$

The coefficients G_{ij} and $\bar{H}_{ij}$ have to be calculated now for each different internal point.

The coefficients G_{ij} and $\bar{H}_{ij}$ can be calculated using numerical integration such as Gauss quadrature formula for the case $i \neq j$. In the case $i = j$ the singularity of the fundamental solution requires a more accurate integration scheme. For these integrals it is recommended to use higher-order integration rules or a special formula such as logarithmic integration. For constant element case G_{ij} and $\bar{H}_{ij}$ can be calculated analytically. $\bar{H}_{ii} = 0$ since $\vec{n}.\vec{r} = 0$. The G_{ii} integrals are

$$G_{ii} = \frac{l}{2\pi}\{ln(\frac{2}{l}) + 1\} \tag{3.26}$$

where l is the length of the element.

3.1.4 Linear Element Formulation

Now we consider linear variation of u and q over an element for which the nodes are located at the ends of the element.

After discretizing the boundary into series of N elements, equation can be written as

$$c_i\tilde{u}_i - \frac{1}{2\pi}\sum_{e=1}^{N}\int_{\Gamma_j}(\sum_{j=1}^{2}\tilde{q}_j N_j) ln|\vec{r}-\vec{r_i}|d\Gamma_j + \frac{1}{2\pi}\sum_{e=1}^{N}\int_{\Gamma_j}(\sum_{j=1}^{2}\tilde{u}_j N_j)\frac{(\vec{r}-\vec{r_i})\vec{n}}{|\vec{r}-\vec{r_i}|^2}d\Gamma_j = 0. \tag{3.27}$$

The integrals in this equation are more difficult to evaluate than those for the constant element as u and q vary linearly over each element Γ_j and hence it is not possible to take them out of the integrals.

The values of u and q at any point on the element can be defined in terms of their nodal values and two linear interpolation functions N_1 and N_2, which are given in terms of the homogeneous coordinate ξ as:

$$\begin{aligned} u(\xi) &= N_1u_1 + N_2u_2 = \begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ q(\xi) &= N_1q_1 + N_2q_2 = \begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}. \end{aligned} \tag{3.28}$$

The dimensionless coordinate ξ varies from -1 to 1 for an element of length l and the two interpolating functions are (Beskos(1987), Brebbia(1981))

$$\begin{aligned} N_1(\xi) &= \tfrac{1}{2}(1-\xi) \\ N_2(\xi) &= \tfrac{1}{2}(1+\xi). \end{aligned} \tag{3.29}$$

Let us now consider the integrals over an element j. The second integral in equation (3.27) can be written as:

$$\int_{\Gamma_j} \tilde{u} q^\star d\Gamma = \int_{\Gamma_j} \begin{bmatrix} N_1 & N_2 \end{bmatrix} q^\star d\Gamma \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} h_{ij}^1 & h_{ij}^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{3.30}$$

where, for each element j, we have the two terms

$$h_{ij}^1 = \int_{\Gamma_j} N_1 q^\star d\Gamma \tag{3.31}$$

and

$$h_{ij}^2 = \int_{\Gamma_j} N_2 q^\star d\Gamma \tag{3.32}$$

Similarly, the first integral in equation (3.27) gives

$$\int_{\Gamma_j} \tilde{q} u^\star d\Gamma = \int_{\Gamma_j} \begin{bmatrix} N_1 & N_2 \end{bmatrix} u^\star d\Gamma \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} g_{ij}^1 & g_{ij}^2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3.33}$$

where

$$g_{ij}^1 = \int_{\Gamma_j} N_1 u^\star d\Gamma \tag{3.34}$$

and

$$g_{ij}^2 = \int_{\Gamma_j} N_2 u^\star d\Gamma. \tag{3.35}$$

When the boundary of the region is discretized into linear elements, node 2 of element j is the same point as node 1 of element $j+1$. So, the entries $\tilde{H}_{ij}$ are equal to the h_{ij}^1 term of element j plus the $h_{i,j-1}^2$ term of element $j-1$. To take into account the possibility of the flux at node 1 of the next element, the fluxes can be arranged in a $2N$ array. Hence the assembled equation for node i is

$$c_i u_i + \sum_{j=1}^{N} \bar{H}_{ij} u_j = \sum_{j=1}^{2N} G_{ij} q_j$$

which becomes

$$\sum_{j=1}^{N} H_{ij} u_j = \sum_{j=1}^{2N} G_{ij} q_j$$

and the whole set in matrix form becomes

$$[H]\{u\} - [G]\{q\} = \{0\}.$$

We next consider the Poisson's equation $\nabla^2 u = b(x,y)$ where the BEM formulation differs from $\nabla^2 u = 0$ because of the term $b(x,y)$. BEM formulation leaves domain integral of $b(x,y)$ which is not wanted, thus we deal with the methods which transform this domain integral to a boundary integral.

3.1.5 Boundary Element Formulation for $\nabla^2 u = b(x,y)$

Consider the problem

$$\nabla^2 u = b(x,y) \quad in \quad \Omega \tag{3.36}$$

where $b(x,y)$ is assumed to be a known function of position. We can apply weighted residual formulation similar to Laplace equation in order to deduce the basic integral equations. Similar to equation (3.15) we get the following equation

$$c_i \tilde{u}_i - \frac{1}{2\pi} \int_\Gamma (\tilde{q} \ln |\vec{r} - \vec{r_i}| - \tilde{u} \frac{1}{|\vec{r} - \vec{r_i}|^2} (\vec{r} - \vec{r_i}).\vec{n}) d\Gamma = \frac{1}{2\pi} \int_\Omega b(x,y) \ln |\vec{r} - \vec{r_i}| d\Omega. \tag{3.37}$$

Notice that although the b function is known and consequently the integral in the domain do not introduce any new unknown, the problem has been changed in character as we need now to carry out a domain integral as well as the boundary integrals.

There are different methods for computation of this domain integral. One may subdivide the region into subregions on which a numerical integration can be applied (Cell integration approach); By taking random integration points, Monte Carlo method can also be applied; If $b(x, y)$ is a harmonic function, application of Green's theorem transforms the domain integral to boundary integral (The Galerkin vector approach); The use of particular solutions which split the solution into the particular solution and the homogeneous solution can also be used in which BEM is applied again to Laplace equation only; Finally, the dual reciprocity method can be used to get rid of the domain integral by computing a series of approximate particular solutions. We give the details of the dual reciprocity method since we are going to use in this thesis.

3.2 Dual Reciprocity Boundary Element Method for $\nabla^2 u = b(x, y)$

In early BEM applications there was always a requirement that a fundamental solution for the problem under consideration had to be available. This fundamental solution had to take into account all the terms in the governing equation in order to avoid domain integrals in the formulation of the boundary integral equation, otherwise internal cells had to be defined and evaluation of these integrals increases the amount of data needed to run the program and hence the method loses some of its attraction to other techniques.

The Dual Reciprocity Method (DRM) was proposed in 1982 and may be used with any type of fundamental solution. It does not need internal cells, however it permits the definition of internal nodes if needed.

The Dual Reciprocity Method (DRM) is explained with reference to the Poisson equation

$$\nabla^2 u = b \tag{3.38}$$

where $b = b(x, y)$ is considered to be a known function of position.

The solution to equation (3.38) can be expressed as the sum of the solution of a homogeneous Laplace's equation and a particular solution $\hat{u}$ such that

$$\nabla^2 \hat{u} = b. \tag{3.39}$$

It is generally difficult to find a solution $\hat{u}$ that satisfies the above equation. The Dual Reciprocity Method proposes the use of a series of particular solutions $\hat{u}_j$ instead of a single function $\hat{u}$. The number of $\hat{u}_j$ used is equal to the total number of nodes in the problem. If there are N boundary nodes and L internal nodes, there will be $N + L$ values of $\hat{u}_j$.

The following approximation for b is proposed

$$b \approx \sum_{j=1}^{N+L} \alpha_j f_j \tag{3.40}$$

where α_j are a set of initially unknown coefficients and f_j are approximating functions. The particular solutions $\hat{u}_j$, and the approximating functions f_j are linked through the relation

$$\nabla^2 \hat{u}_j = f_j. \tag{3.41}$$

Substituting the equation (3.41) into the equation (3.40) gives

$$b = \sum_{j=1}^{N+L} \alpha_j (\nabla^2 \hat{u}_j). \tag{3.42}$$

Then equation (3.42) can be substituted in the original equation (3.38) to get

$$\nabla^2 \tilde{u} = \sum_{j=1}^{N+L} \alpha_j (\nabla^2 \hat{u}_j). \tag{3.43}$$

Now we apply boundary element method to this equation by multiplying the equation (3.43) by fundamental solution and integrate over the domain

$$\int_\Omega (\nabla^2 \tilde{u}) u^\star d\Omega = \sum_{j=1}^{N+L} \alpha_j \int_\Omega (\nabla^2 \hat{u}_j) u^\star d\Omega. \tag{3.44}$$

Integrating terms similar to previous section produces the following integral equation for each source node i,

$$c_i \tilde{u}_i + \int_\Gamma \tilde{u} q^\star d\Gamma - \int_\Gamma \tilde{q} u^\star d\Gamma = \sum_{j=1}^{N+L} \alpha_j \Big(c_i \hat{u}_{ij} + \int_\Gamma \hat{u}_j q^\star d\Gamma - \int_\Gamma \hat{q}_j u^\star d\Gamma \Big). \tag{3.45}$$

The term $\hat{q}$ in equation (3.45) is defined as $\hat{q}_j = \partial \hat{u}_j / \partial n$ where n is the unit outward normal to Γ and can be expanded to

$$\hat{q}_j = \frac{\partial \hat{u}_j}{\partial x} \frac{\partial x}{\partial n} + \frac{\partial \hat{u}_j}{\partial y} \frac{\partial y}{\partial n}. \tag{3.46}$$

Note that equation (3.45) involves no domain integral. The source term b has been substituted by equivalent boundary integrals.

Now writing the equation (3.45) in discretized form, with summations over the boundary elements replacing the integrals gives for a source node i the expression

$$c_i\tilde{u}_i + \sum_{k=1}^{N}\int_{\Gamma_k}\tilde{u}q^\star d\Gamma_k - \sum_{k=1}^{N}\int_{\Gamma_k}\tilde{q}u^\star d\Gamma_k =$$
$$\sum_{j=1}^{N+L}\alpha_j\Big(c_i\hat{u}_{ij} + \sum_{k=1}^{N}\int_{\Gamma_k}\hat{u}_j q^\star d\Gamma_k - \sum_{k=1}^{N}\int_{\Gamma_k}\hat{q}_j u^\star d\Gamma_k\Big). \quad (3.47)$$

When f is defined, $\hat{u}_j$ and $\hat{q}_j$ will be known. Performing the integrations over each segment Γ_k with the substitution of fundamental solution $u^\star$ and its normal derivative $q^\star$ from equation (3.13) we get

$$c_i\tilde{u}_i + \sum_{k=1}^{N} H_{ik}\tilde{u}_k - \sum_{k=1}^{N} G_{ik}\tilde{q}_k = \sum_{j=1}^{N+L}\alpha_j\Big(c_i\hat{u}_{ij} + \sum_{k=1}^{N} H_{ik}\hat{u}_{kj} - \sum_{k=1}^{N} G_{ik}\hat{q}_{kj}\Big) \quad (3.48)$$

and this equation can be written in the matrix form as

$$[H]\{\tilde{u}\} - [G]\{\tilde{q}\} = \sum_{j=1}^{N+L}\alpha_j([H]\{\hat{u}_j\} - [G]\{\hat{q}_j\}) \quad (3.49)$$

where H and G matrices are the same as obtained in Section 3.1. (equations (3.18) and (3.19)).

If each of vectors $\hat{u}_j$ and $\hat{q}_j$ is considered to be one column of the matrices $\hat{U}$ and $\hat{Q}$ respectively then equation (3.49) may be written without the summation in the final form as

$$[H]\{\tilde{u}\} - [G]\{\tilde{q}\} = ([H][\hat{U}] - [G][\hat{Q}])\{\alpha\} \quad (3.50)$$

where α is the vector containing the unknown coefficients α_j.

3.2.1 Interior Computation

Equation (3.50) contains only boundary nodes. When the system is arranged in $Ax = b$ form and is solved, all the values $\tilde{u}$ and $\tilde{q}$ will be known on the boundary. Then redefining the H and G matrices in the equation (3.48) to contain the point P_i as an interior point one can calculate $\tilde{u}$ at internal nodes.

When interior nodes are defined, each one is independently placed, and they do not form part of any element or cell, thus only the coordinates are needed as input data. Hence, these nodes may be defined in any order.

3.2.2 The Vector α

According to the equation (3.40) b is approximated as

$$b \approx \sum_{j=1}^{N+L} \alpha_j f_j \tag{3.51}$$

by taking the value of b at $(N+L)$ different points , a set of equations is obtained and may be expressed in matrix form as

$$\{b\} = [F]\{\alpha\} \tag{3.52}$$

where each column of F consists of a vector f_j containing the values of the function f_j.

Therefore α may be obtained as

$$\{\alpha\} = [F^{-1}]\{b\} \tag{3.53}$$

and when it is substituted back into equation (3.50) we get the system

$$[H]\{u\} - [G]\{q\} = ([H][\hat{U}] - [G][\hat{Q}])[F^{-1}]\{b\}.$$

This system now can be solved for unknown values of u and q with the procedure mentioned in Section 3.1.

3.3 Dual Reciprocity Boundary Element Method for $\nabla^2 u = b(x, y, u)$

We consider the problem

$$\nabla^2 u = b(x, y, u) \tag{3.54}$$

and split the right hand side as

$$b(x, y, u) = b_1(x, y) + b_2 u \tag{3.55}$$

where b_2 is a constant.

Approximation of $b(x, y, u)$ in terms of basis functions f_j's as

$$b(x, y, u) = b_1(x, y) + b_2 u = \sum_{j=1}^{N+L} \alpha_j f_j \tag{3.56}$$

will give

$$\alpha = F^{-1} b_1 + b_2 F^{-1} u$$

where F is the same $(N + L) \times (N + L)$ position matrix consisting of the values of f_j at the $N + L$ points and b_1 is the column vector containing $b_1(x, y)$ at the $N + L$ points. This equation can be written again in vector notation taking the unknown as $\widetilde{u}$

$$\{\alpha\} = \{S_b\} + [S_u]\{\widetilde{u}\} \tag{3.57}$$

where $\{S_b\} = \{F^{-1} b_1\}$ and $[S_u] = [b_2 F^{-1}]$.

Application of DRBEM will give now

$$[H]\{\widetilde{u}\} - [G]\{\widetilde{q}\} = ([H][\hat{U}] - [G][\hat{Q}])\{\alpha\} \tag{3.58}$$

where

$$\{\alpha\} = [F^{-1}]\{b_1\} + [b_2F^{-1}]\{\tilde{u}\}. \tag{3.59}$$

Since the function b contains unknown values of $\tilde{u}$ at boundary and internal nodes, α vector can not be calculated explicitly. Therefore during the solution procedure we have to carry α vector (at least some part) as unknown.

For the solution of the boundary values we have

$$H^b_{ij}\tilde{u}_j - G^b_{ij}\tilde{q}_j = (H^b_{ij}\hat{U}^b - G^b_{ij}\hat{Q}^b)\alpha \tag{3.60}$$

and for internal nodes

$$H^i_{ij}\tilde{u}_j - G^i_{ij}\tilde{q}_j = (I\hat{U}^i + H^i_{ij}\hat{U}^b - G^i_{ij}\hat{Q}^b)\alpha \tag{3.61}$$

where superscript b refers to the boundary nodes and superscript i to the internal nodes.

Combining these equations together at one step one can get the following system of equations

$$\begin{bmatrix} H^b & 0 \\ H^i & I \end{bmatrix} \left\{ \begin{matrix} \tilde{u}_b \\ \tilde{u}_i \end{matrix} \right\} - \begin{bmatrix} G^b & 0 \\ G^i & 0 \end{bmatrix} \left\{ \begin{matrix} \tilde{q}_b \\ 0 \end{matrix} \right\} =$$

$$\left(\begin{bmatrix} H^b & 0 \\ H^i & I \end{bmatrix} \begin{bmatrix} \hat{U}^b \\ \hat{U}^i \end{bmatrix} - \begin{bmatrix} G^b & 0 \\ G^i & 0 \end{bmatrix} \begin{bmatrix} \hat{Q}^b \\ 0 \end{bmatrix} \right) \left\{ \alpha \right\} \tag{3.62}$$

in short

$$[H]\{\tilde{u}\} - [G]\{\tilde{q}\} = [M']\{\alpha\}. \tag{3.63}$$

where

$$M' = \begin{bmatrix} H^b & 0 \\ H^i & I \end{bmatrix} \begin{bmatrix} \hat{U}^b \\ \hat{U}^i \end{bmatrix} - \begin{bmatrix} G^b & 0 \\ G^i & 0 \end{bmatrix} \begin{bmatrix} \hat{Q}^b \\ 0 \end{bmatrix}.$$

Therefore equation (3.63) is rewritten as using equation (3.50)

$$\big([H] - [M'][S_u]\big)\{\tilde{u}\} - [G]\{\tilde{q}\} = [M']\{S_b\}, \tag{3.64}$$

when this system is arranged to $Ax = y$ form after applying boundary conditions and solved, all the unknowns on the boundary and internal values of $\tilde{u}$ are calculated. This $Ax = y$ form is an $(N + L) \times (N + L)$ system, x contains N boundary values of u or q plus L interior values of u.

3.4 Dual Reciprocity Boundary Element Method for $\nabla^2 u = b(x, y, u, u_x, u_y)$

When the function b contains derivatives of the unknown function u, the function $u(x, y)$ is also approximated in the form

$$u = F\beta \tag{3.65}$$

for some unknown vector $\beta \neq \alpha$ (containing unknown coefficients β_j.)

Differentiating both sides of the equation with respect to x we get

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial x}\beta \tag{3.66}$$

and in equation (3.65) leaving β alone, substituting in above equation we get

$$\frac{\partial u}{\partial x} = \frac{\partial F}{\partial x}F^{-1}u. \tag{3.67}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial F}{\partial y} F^{-1} u. \tag{3.68}$$

Writing the function b as (assuming b is a linear function of u, u_x, u_y)

$$b(x, y, u, u_x, u_y) = b_1(x, y) + b_2 u + b_3 u_x + b_4 u_y \tag{3.69}$$

where b_2, b_3, b_4 are constants and approximating in the form

$$b(x, y, u, u_x, u_y) = \sum_{j=1}^{N+L} \alpha_j f_j,$$

α vector becomes

$$\alpha = F^{-1} b_1 + b_2 F^{-1} \tilde{u} + b_3 F^{-1} \frac{\partial F}{\partial x} F^{-1} \tilde{u} + b_4 F^{-1} \frac{\partial F}{\partial y} F^{-1} \tilde{u} \tag{3.70}$$

in matrix-vector form

$$\{\alpha\} = \{S_b\} + \Big([S_u] + [S_x] + [S_y]\Big)\{\tilde{u}\} \tag{3.71}$$

where

$$\{S_b\} = [F^{-1}]\{b_1\}, \ [S_u] = [b_2 F^{-1}], \ [S_x] = [b_3 F^{-1} \frac{\partial F}{\partial x} F^{-1}], \ [S_y] = [b_4 F^{-1} \frac{\partial F}{\partial y} F^{-1}].$$

When we substitute $\{\alpha\}$ in equation (3.63) we obtain

$$\Big([H] - [M']([S_u] + [S_x] + [S_y])\Big)\{\tilde{u}\} - [G]\{\tilde{q}\} = [M']\{S_b\} \tag{3.72}$$

which can be solved for $\{\tilde{u}\}$ and $\{\tilde{q}\}$.

3.5 Dual Reciprocity Boundary Element Method for Magnetohydrodynamic Channel Flow (MHD Flow)

In this example we will deal with the well known Maxwell equations of electromagnetism and the basic equations of fluid mechanics which lead to the coupled system of equations in the velocity and magnetic field. These equations of steady laminar, fully developed flow of viscous, incompressible and electrically conducting fluid in a rectangular duct Ω, subjected to a constant and uniform applied magnetic field B_0, can be put in the following non-dimensional form

$$\begin{array}{l} \nabla^2 V + M\frac{\partial B}{\partial x} = -1 \\ \nabla^2 B + M\frac{\partial V}{\partial x} = 0 \end{array} \quad \text{in} \quad \Omega \tag{3.73}$$

with the boundary conditions

$$V = B = 0 \quad \text{on} \quad \partial\Omega \tag{3.74}$$

where the boundaries of the duct are assumed to be insulating.

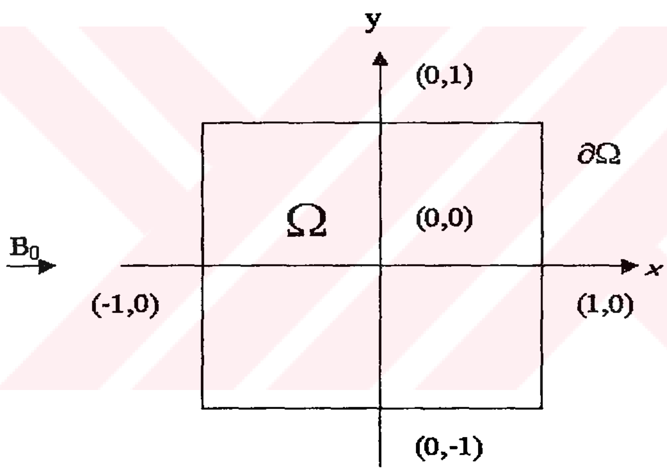

Figure 3.1: MHD Flow problem

$V(x, y)$, $B(x, y)$ are the velocity and the induced magnetic field respectively, M is the Hartmann number. Here it is assumed that the applied magnetic field B_0 is parallel to x-axis, $V(x, y)$, $B(x, y)$ are in the z-direction which is the axis of the duct, and the fluid is driven down the duct by means of a constant pressure gradient.

Equations (3.73) may be decoupled by the change of variables

$$u_1 = V + B \ , \ u_2 = V - B \tag{3.75}$$

as

$$\left.\begin{aligned} \nabla^2 u_1 + M\tfrac{\partial u_1}{\partial x} &= -1 \\ \nabla^2 u_2 - M\tfrac{\partial u_2}{\partial x} &= -1 \end{aligned}\right. \qquad \text{in} \quad \Omega \tag{3.76}$$

with the boundary conditions

$$u_1 = u_2 = 0 \qquad \text{on} \qquad \partial\Omega. \tag{3.77}$$

Both of the equations in (3.76) are of the type

$$\nabla^2 u = b(x, y, \frac{\partial u}{\partial x}) \qquad \text{in} \qquad \Omega \tag{3.78}$$

with

$$u = 0 \qquad \text{on} \qquad \partial\Omega \tag{3.79}$$

where

$$b(x, y, \frac{\partial u}{\partial x}) = -1 - M\frac{\partial u}{\partial x}$$

and

$$b(x, y, \frac{\partial u}{\partial x}) = -1 + M\frac{\partial u}{\partial x} \tag{3.80}$$

respectively for the first and second equations. So, we can solve the equation (3.78) with boundary condition (3.79) for u_1 first and then $u_2(x, y; M) = u_1(x, y; -M)$.

Furthermore, if we define a transformation

$$\begin{aligned} U_1 &= e^{\frac{M}{2}x} u_1 \\ U_2 &= e^{-\frac{M}{2}x} u_2 \end{aligned} \tag{3.81}$$

equations (3.76) become

$$\begin{aligned} \nabla^2 U_1 &= \tfrac{M^2}{4} U_1 - e^{\frac{M}{2}x} \\ \nabla^2 U_2 &= \tfrac{M^2}{4} U_2 - e^{-\frac{M}{2}x} \end{aligned} \qquad \text{in } \Omega \tag{3.82}$$

with

$$U_1 = U_2 = 0 \quad \text{on} \quad \partial\Omega. \tag{3.83}$$

Again both of the equations in (3.82) are of the form

$$\nabla^2 U = b(x, y, U) \quad \text{in} \quad \Omega \tag{3.84}$$

with

$$U = 0 \quad \text{on} \quad \partial\Omega \tag{3.85}$$

where

$$b(x, y, U) = \frac{M^2}{4} U - e^{\frac{M}{2}x}$$

and

$$b(x, y, U) = \frac{M^2}{4} U - e^{-\frac{M}{2}x} \quad \text{in} \quad \Omega \tag{3.86}$$

for the first and second equations respectively. Thus, solution of equation (3.84) - (3.85) with $+M$ will lead $U_1(x, y)$ and $U_2(x, y; M) = U_1(x, y; -M)$.

We will consider the dual reciprocity boundary element method solution of the Poisson type equations

$$\nabla^2 U = b(x, y, U) \tag{3.87}$$

and

$$\nabla^2 U = b(x, y, \frac{\partial U}{\partial x}) \tag{3.88}$$

for the solution of MHD flow equations of these forms with homogeneous boundary conditions. In either case, it is possible to go back to the original unknowns $V(x,y)$ and $B(x,y)$, through the equations (3.81) and (3.75) from the solution of equation (3.84) and through the equation (3.75) from the solution of equation (3.78). Calculated results of $V(x,y)$ and $B(x,y)$ with DRBEM using Radial Basis function interpolation are compared with the Shercliff's(1953) exact solution which may be written as

$$V = \frac{1}{2}(m^2 - y^2) - \sum_{k=1}^{\infty} A_k \frac{\sinh m_1 \cosh m_2 x + \sinh m_2 \cosh m_1 x}{\sinh(m_1 + m_2)} \cos \omega_k y \quad (3.89)$$

$$B = \sum_{k=1}^{\infty} A_k \frac{\sinh m_1 \sinh m_2 x - \sinh m_2 \sinh m_1 x}{\sinh(m_1 + m_2)} \cos \omega_k y \quad (3.90)$$

where

$$\alpha = M/2$$

and

$$m_1 = -\alpha + \mu_k \quad , \quad m_2 = \alpha + \mu_k \quad (3.91)$$

$$\omega_k = \frac{(2k-1)\pi}{2m} \quad , \quad \mu_k = (\alpha^2 + \omega^2)^{1/2} \quad (3.92)$$

$$A_k = \frac{16m^2}{\pi^3} \frac{(-1)^{k+1}}{(2k-1)^3}.$$

CHAPTER 4

RESULTS AND DISCUSSIONS

In this Chapter, the solution of magnetohydrodynamic channel flow problem which is introduced in Section 3.5 is given for several values of Hartmann numbers, different forms of Poisson's equations and constant and linear elements. The problem in which the equations can be transformed either to the form $\nabla^2 u = b(x, y, u)$ or $\nabla^2 u = b(x, y, u_x)$ will be solved using DRBEM. Solutions which are obtained using constant boundary elements for both of the forms $\nabla^2 u = b(x, y, u)$ and $\nabla^2 u = b(x, y, u_x)$ will be given for several values of Hartmann number and f-expansions and then compared. Later the solution is obtained by using linear boundary elements for the derivative form $\nabla^2 u = b(x, y, u_x)$. Computer programs are written in Fortran language (Brebbia (1978)), and run in PC platform. All the graphics are obtained by using MATLAB package.

4.1 Solutions using constant boundary elements for $\nabla^2 u = b(x, y, u)$ form

In this formulation MHD equations are solved which are given in equation (3.82) with homogeneous boundary conditions. The Poisson's equation here is in the form of $\nabla^2 u = b(x, y, u)$. MHD flow solution then can be obtained going back through equations (3.81) and (3.75).

In the computations, domain is defined by $|x| \leq 1$, $|y| \leq 1$ taking the origin at the center of the section and axes parallel to the sides. In the DRBEM we choose 11×11 uniform mesh ($h = 0.2$) i.e. 40 constant boundary elements and 81 internal collocation points are used. The values of the velocity $V(x,y)$ and the induced magnetic field $B(x,y)$ are obtained as a result of DRBEM at the interior points. Radial basis function interpolations are used for the right hand side function $b(x,y,u)$ and $b(x,y,u_x)$. In general polynomial radial basis functions are used since they give quite good approximations and easy to use.

In Figures (4.1) and (4.2) we see the velocity V(0,y) with f-expansions $f = 1 + r$, $f = 1 + r + r^2 + r^3$ respectively for Hartmann numbers M=2, 5 and 10. Since the induced magnetic field B(0,y)=0 (see equation (3.90)) there is no need to present B(0,y) curves. Figures (4.3) and (4.4) give V(x,0) curves again for linear and cubic f-expansions and for Hartmann numbers M=2 and 5. Corresponding B(x,0) curves for the same f-expansions and Hartmann numbers are presented in Figures (4.5) and (4.6).

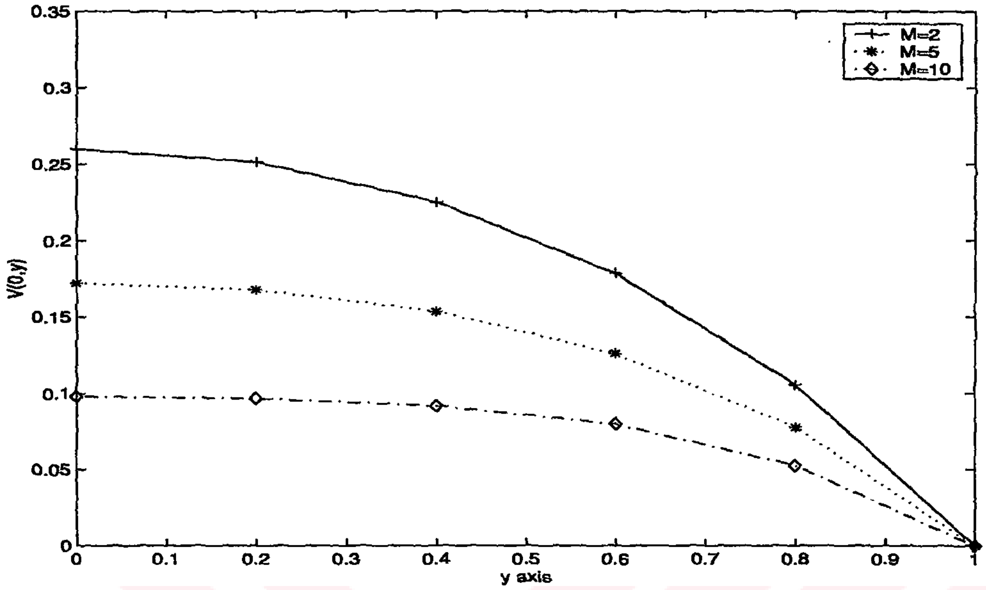

Figure 4.1: V(0,y) for $f = 1 + r$, $0 \leq y \leq 1$, $\nabla^2 u = b(x, y, u)$

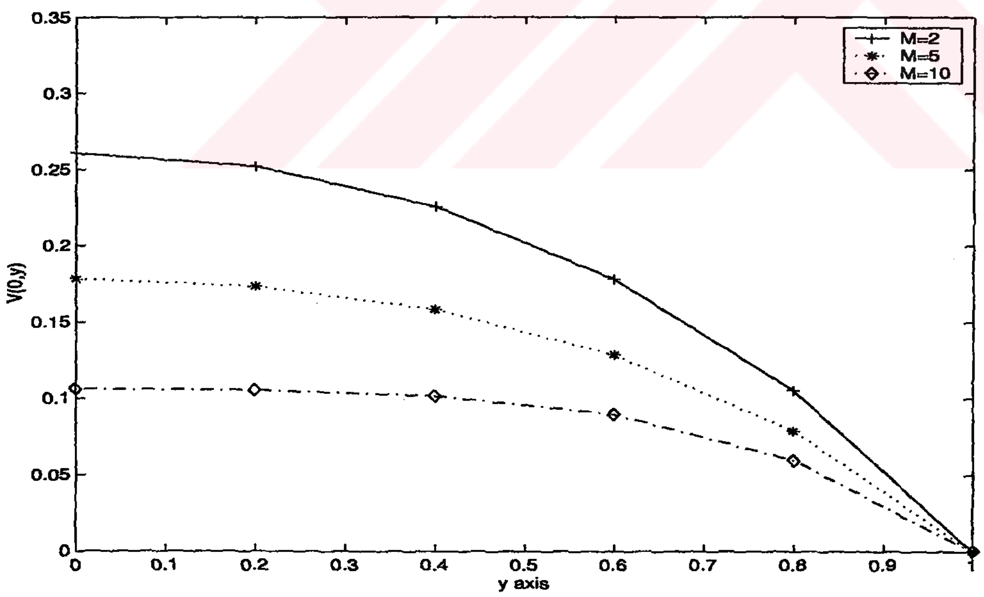

Figure 4.2: V(0,y) for $f = 1 + r + r^2 + r^3$, $0 \leq y \leq 1$, $\nabla^2 u = b(x, y, u)$

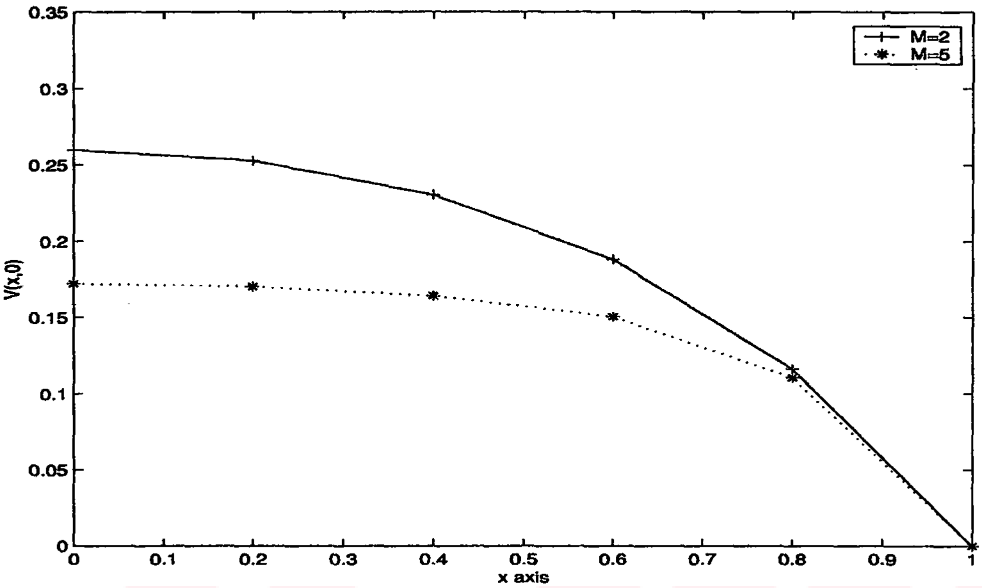

Figure 4.3: V(x,0) for $f = 1 + r$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u)$

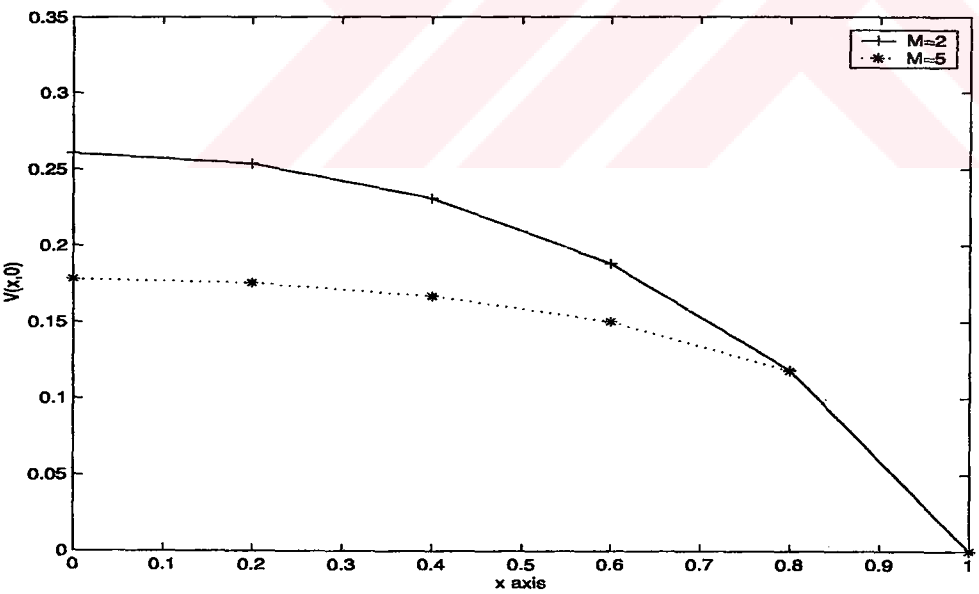

Figure 4.4: V(x,0) for $f = 1 + r + r^2 + r^3$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u)$

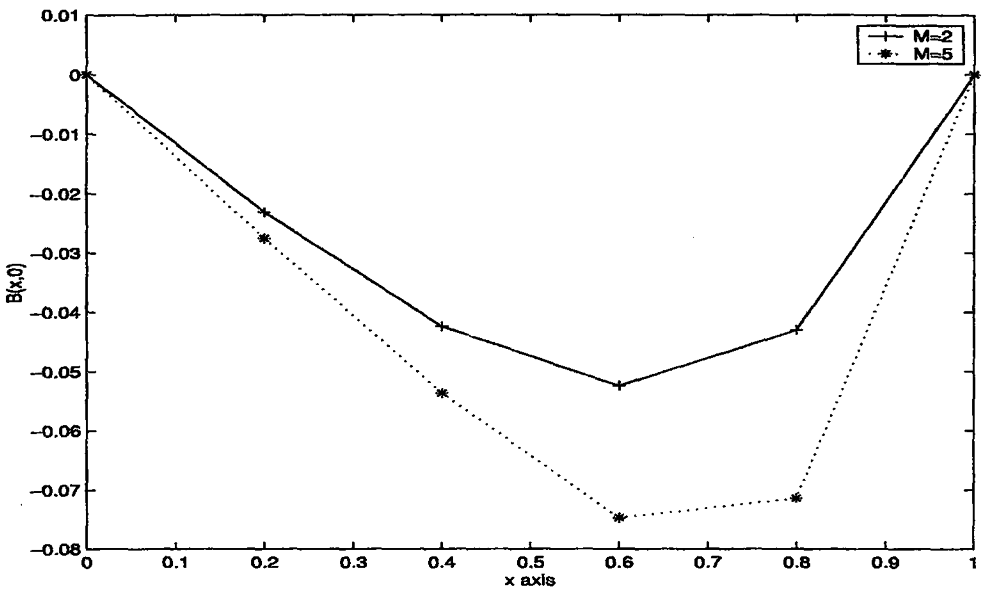

Figure 4.5: B(x,0) for $f = 1 + r$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u)$

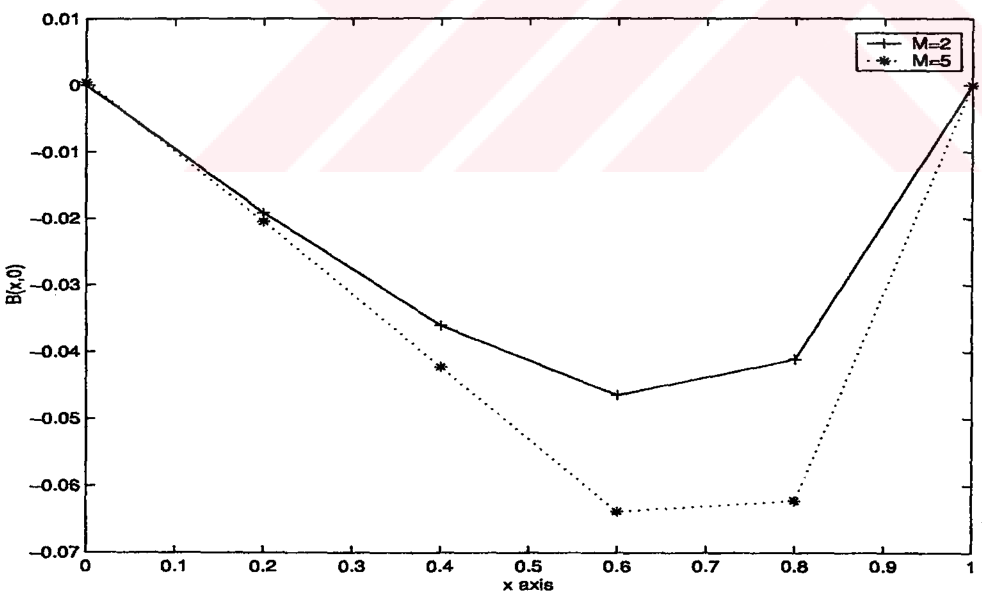

Figure 4.6: B(x,0) for $f = 1 + r + r^2 + r^3$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u)$

Different degree of polynomials in radial basis function (f-expansions) are carried for Hartmann number M=2. The velocity V(x,0) - induced magnetic field B(x,0) curves are plotted in Figures (4.7) and (4.8) respectively.

In Figures (4.9)-(4.10) and (4.11)-(4.12) we give velocity and magnetic field contours (equal velocity lines and current lines) with different f-expansions and exact solution for Hartmann numbers M=2 and M=5 respectively. From these contours we notice the flattening tendency for velocity values and boundary layer formation for both velocity and induced magnetic field as the Hartmann number increases. This is the well-known behaviour of MHD channel flow problems.

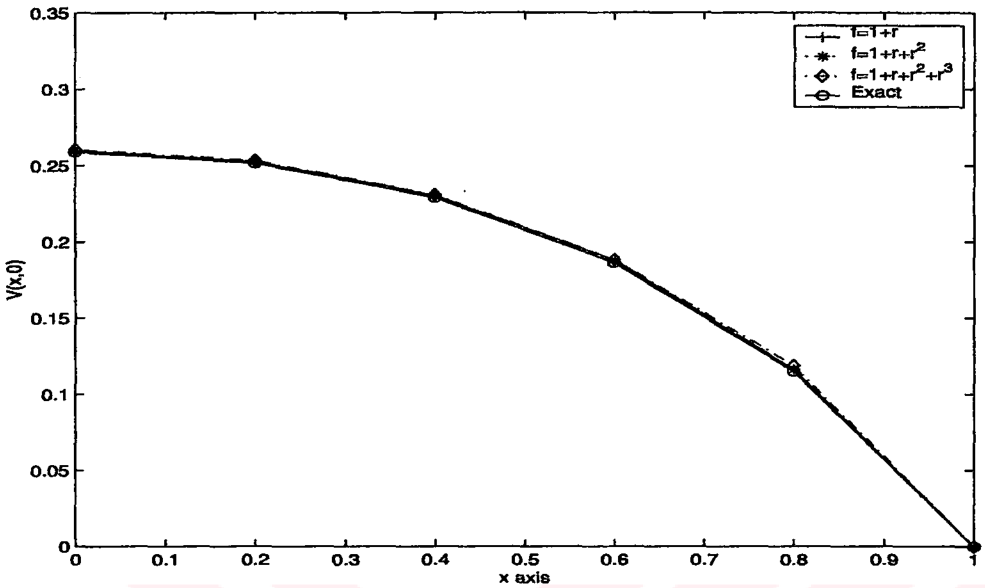

Figure 4.7: V(x,0) for M=2, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u)$

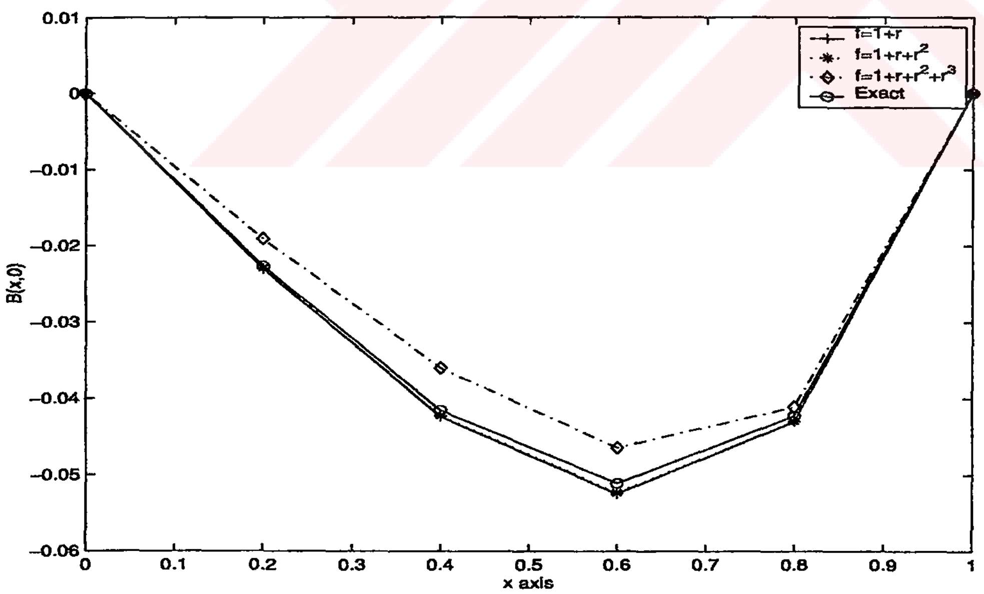

Figure 4.8: B(x,0) for M=2, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u)$

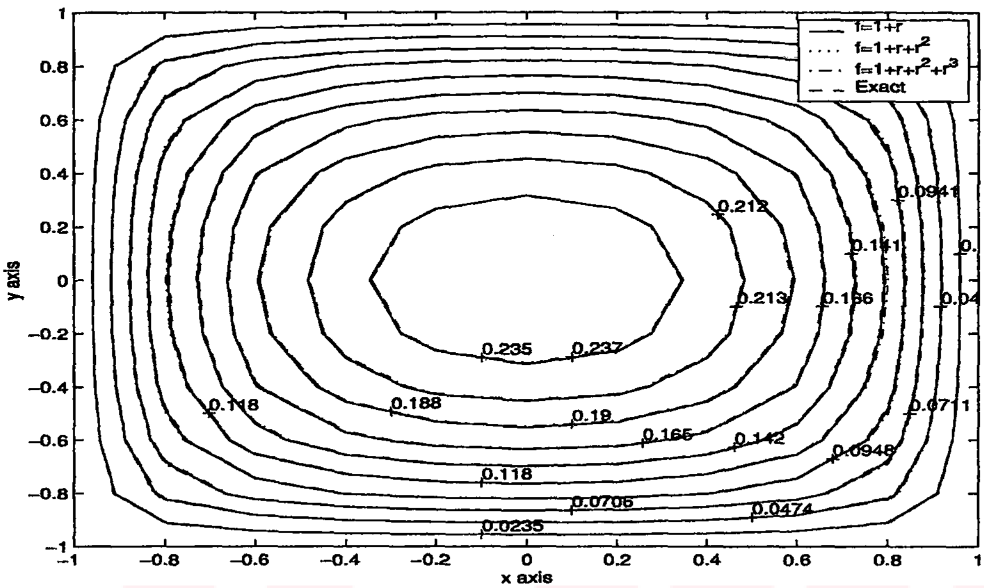

Figure 4.9: V(x,y) for M=2, $\nabla^2 u = b(x, y, u)$

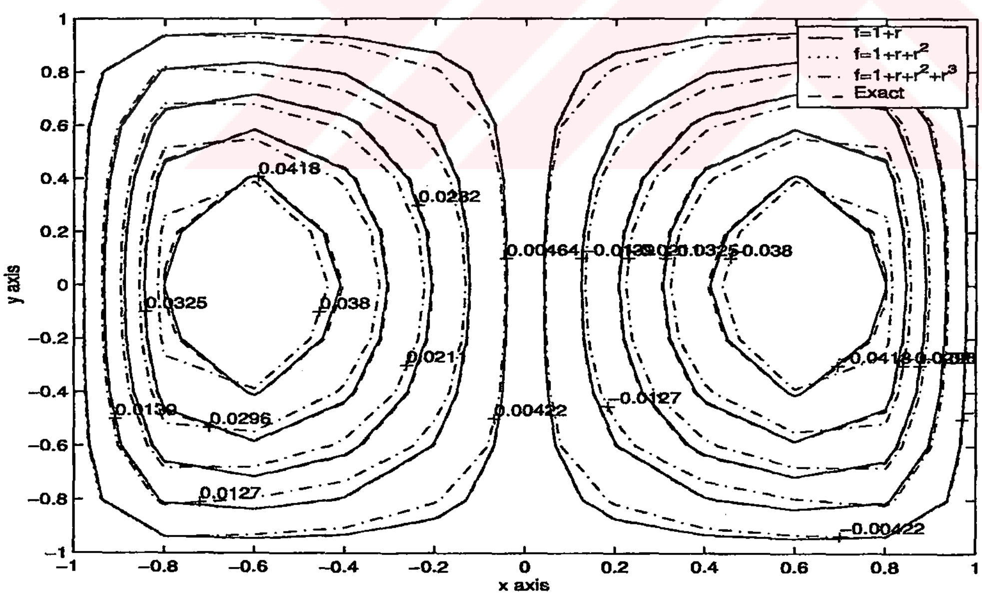

Figure 4.10: B(x,y) for M=2, $\nabla^2 u = b(x, y, u)$

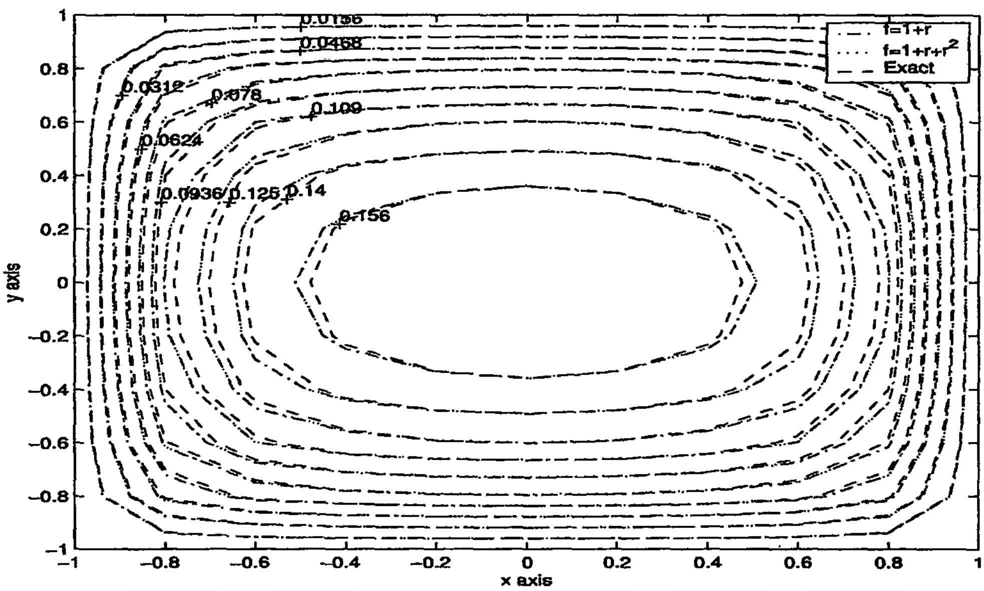

Figure 4.11: V(x,y) for M=5, $\nabla^2 u = b(x, y, u)$

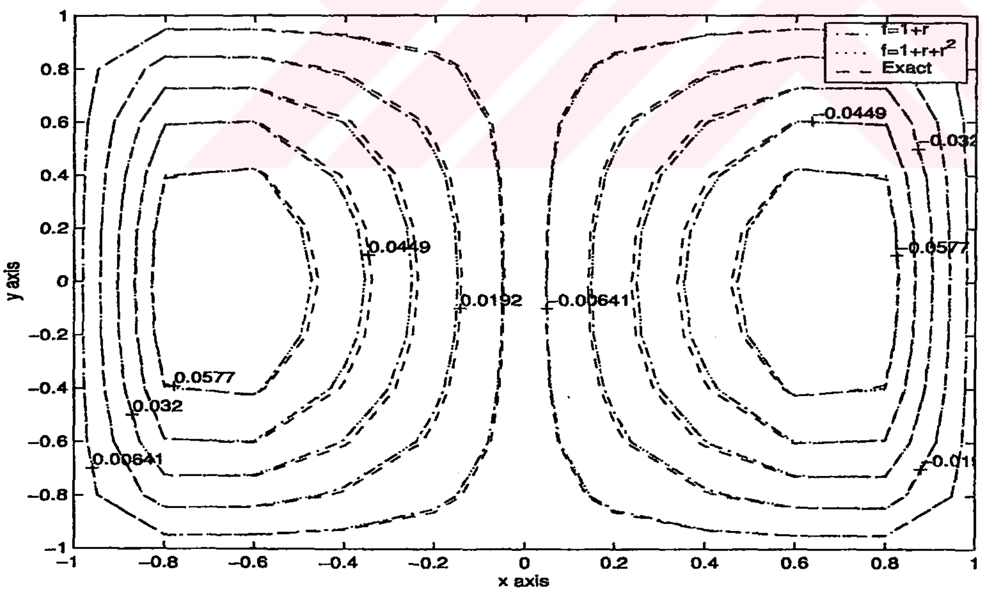

Figure 4.12: B(x,y) for M=5, $\nabla^2 u = b(x, y, u)$

4.2 Solutions using constant boundary elements for $\nabla^2 u = b(x, y, u_x)$ form

The equations to be solved here are given in equation (3.76) with homogeneous boundary conditions. The Poisson's equation is in the form of $\nabla^2 u = b(x, y, u_x)$. Again velocity V and induced magnetic field B can be obtained back by using equation (3.75).

Computations are carried out with the same domain and discretization for the boundary and the interior of the domain as in Section 4.1.

Figures (4.13) and (4.14) show the velocity V(0,y) for Hartmann numbers M=2, 5, 10 and 15 with $f = 1+r$ and $f = 1+r+r^2$ respectively. Again B(0,y)=0. V(x,0) curves are given for the same f-expansions (linear and quadratic) and for M=2, 5 and 10 in Figures (4.15) and (4.16). Induced magnetic field B(x,0) curves are presented in Figures (4.17) and (4.18) with the same f-expansions.

Again in f-expansions polynomial radial basis functions with different orders are tested in terms of V(x,0) and B(x,0) for Hartmann number M=2 in Figures (4.19) and (4.20).

Figures (4.21)-(4.22) and (4.23)-(4.24) present equal velocity lines and current lines for Hartmann numbers M=2 and M=5 respectively. Boundary layer formation is quite remarkable for Hartmann number M=5 for both V(x,y) and B(x,y).

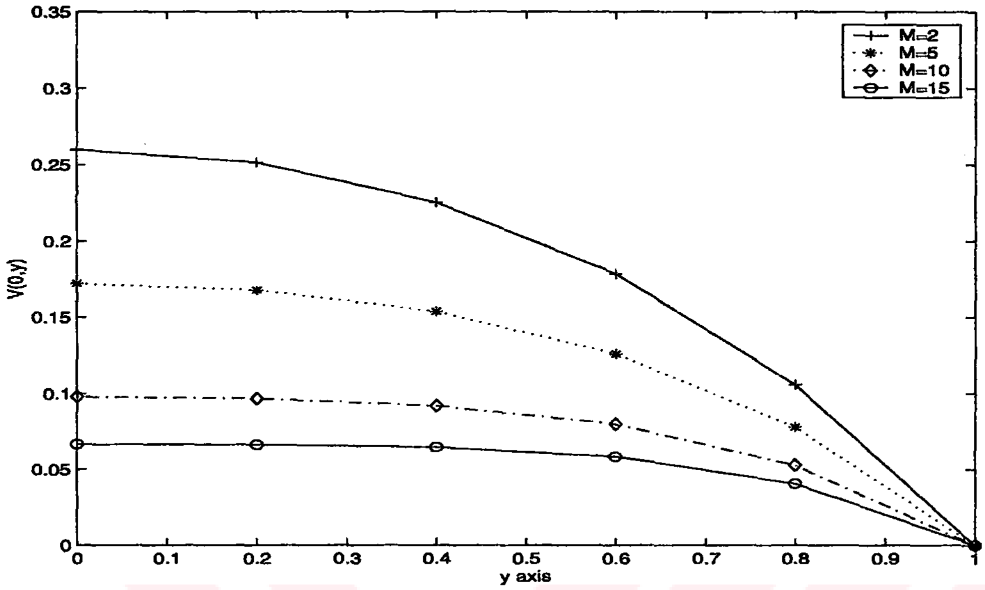

Figure 4.13: V(0,y) for $f = 1 + r$, $0 \leq y \leq 1$, $\nabla^2 u = b(x, y, u_x)$

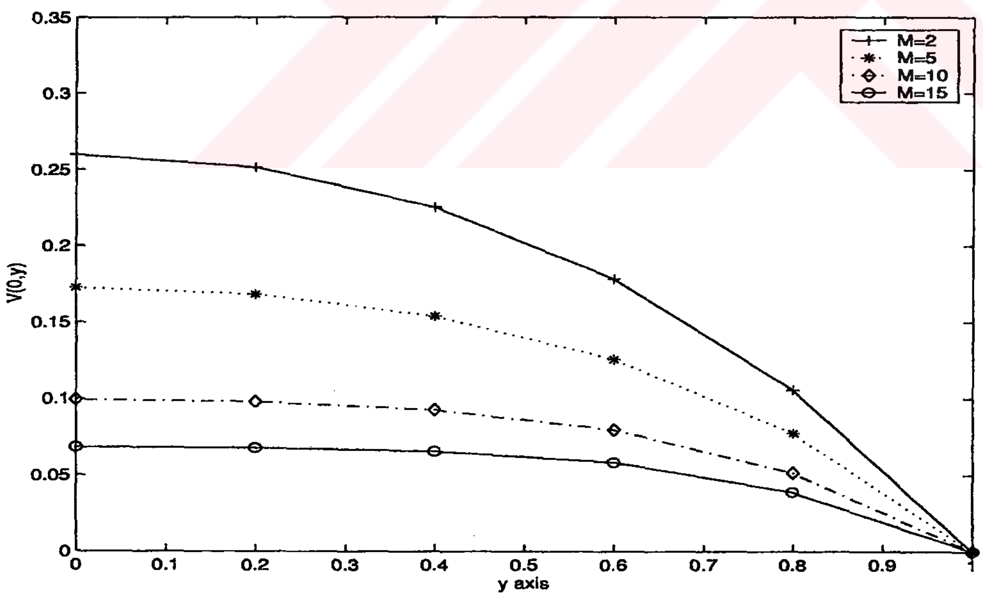

Figure 4.14: V(0,y) for $f = 1 + r + r^2$, $0 \leq y \leq 1$, $\nabla^2 u = b(x, y, u_x)$

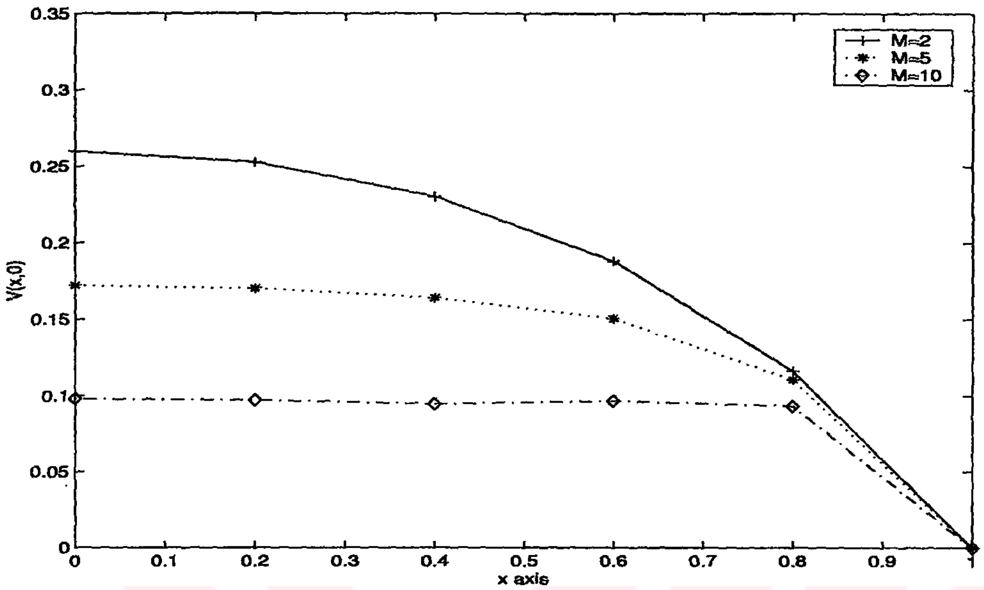

Figure 4.15: V(x,0) for $f = 1 + r$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

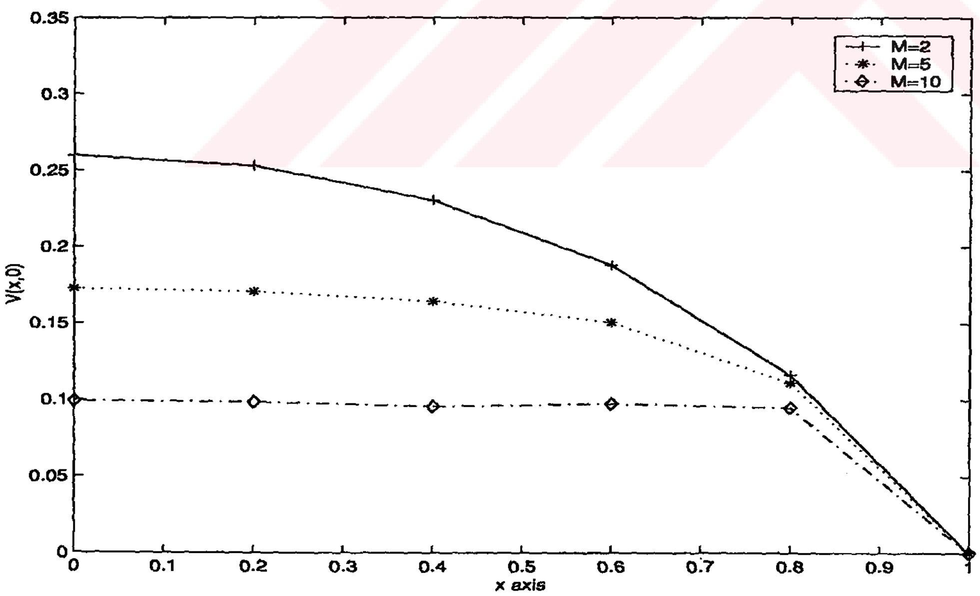

Figure 4.16: V(x,0) for $f = 1 + r + r^2$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

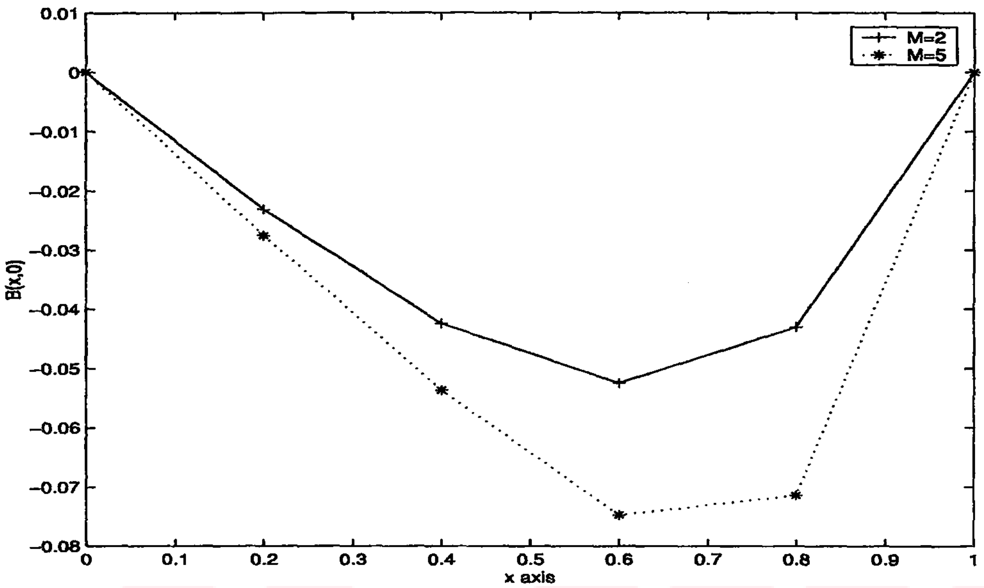

Figure 4.17: B(x,0) for $f = 1 + r$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

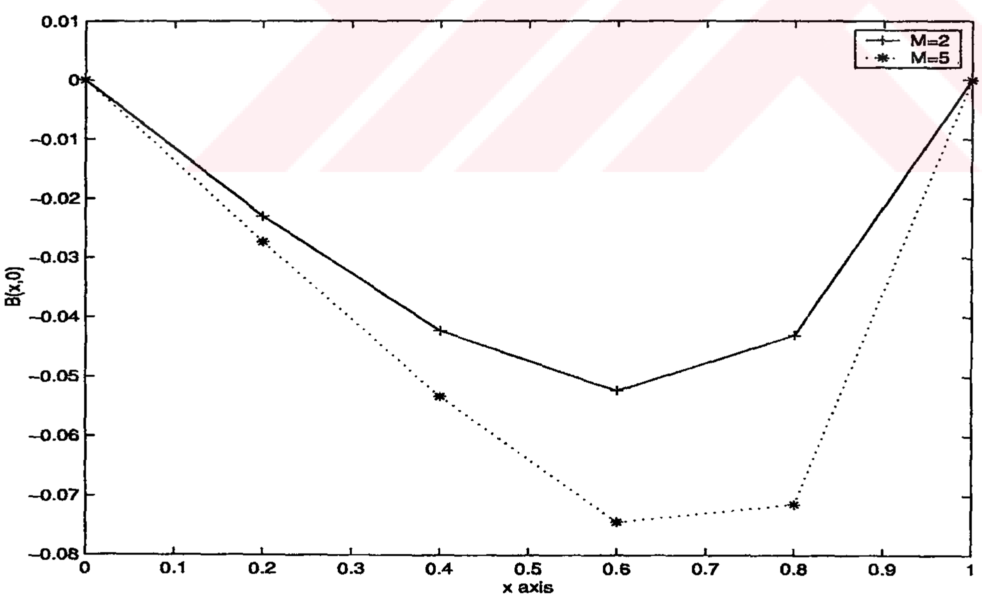

Figure 4.18: B(x,0) for $f = 1 + r + r^2$, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

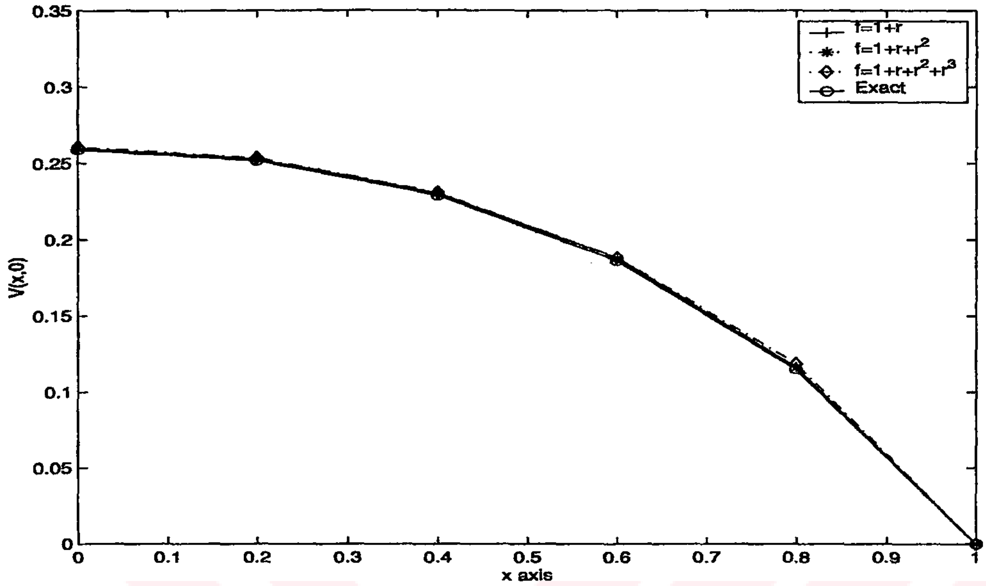

Figure 4.19: V(x,0) for M=2, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

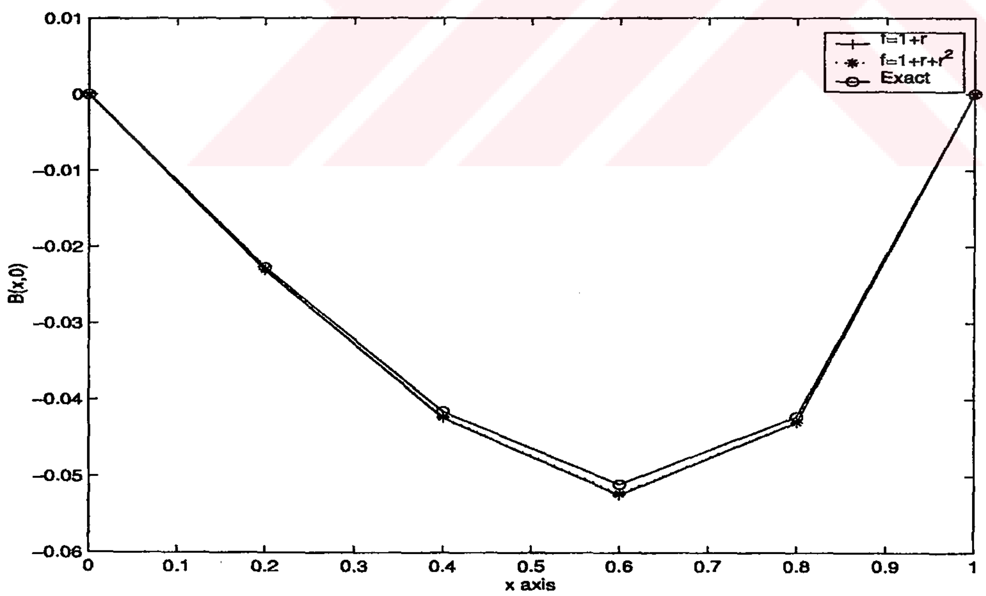

Figure 4.20: B(x,0) for M=2, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

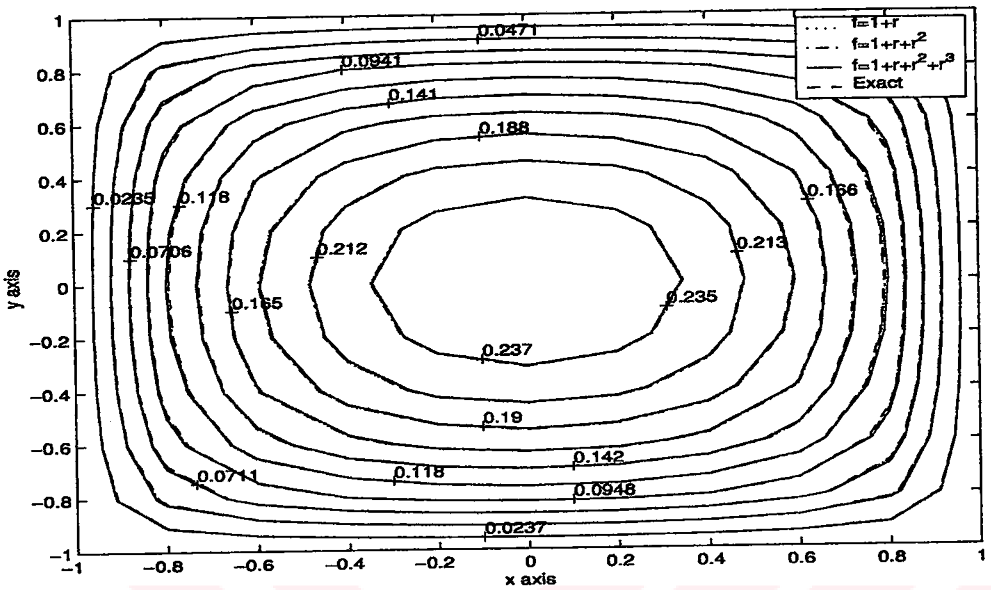

Figure 4.21: V(x,y) for M=2, $\nabla^2 u = b(x, y, u_x)$

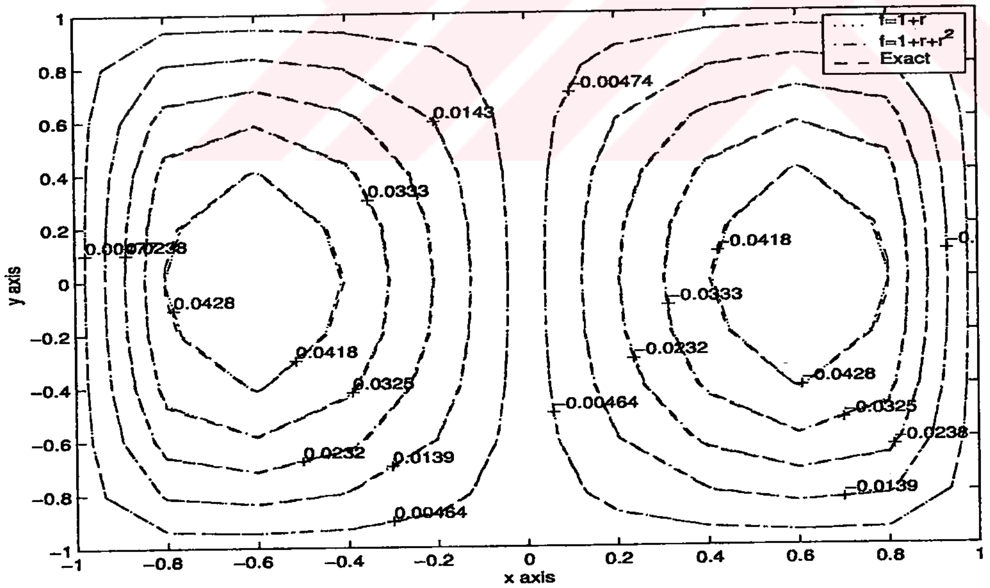

Figure 4.22: B(x,y) for M=2, $\nabla^2 u = b(x, y, u_x)$

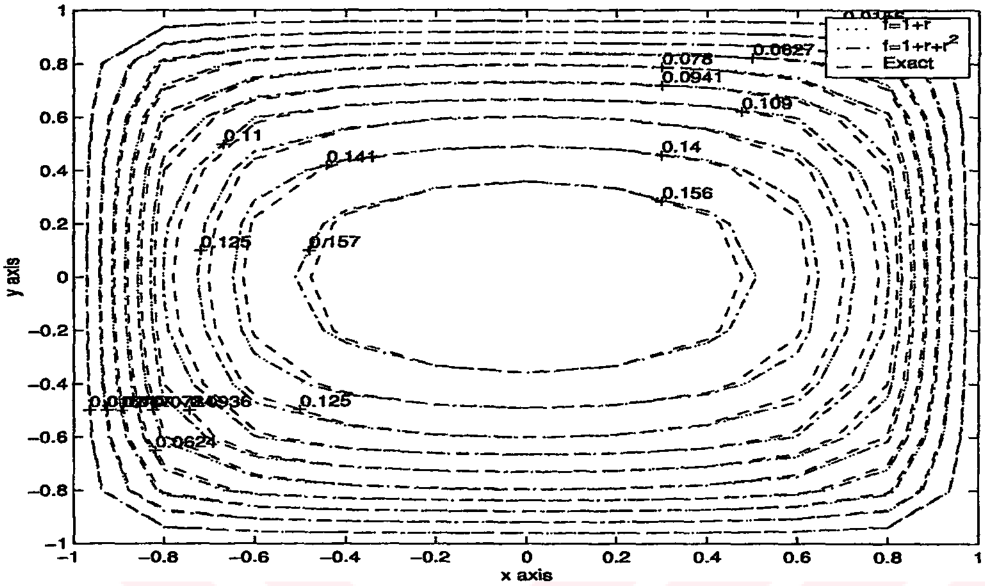

Figure 4.23: V(x,y) for M=5, $\nabla^2 u = b(x, y, u_x)$

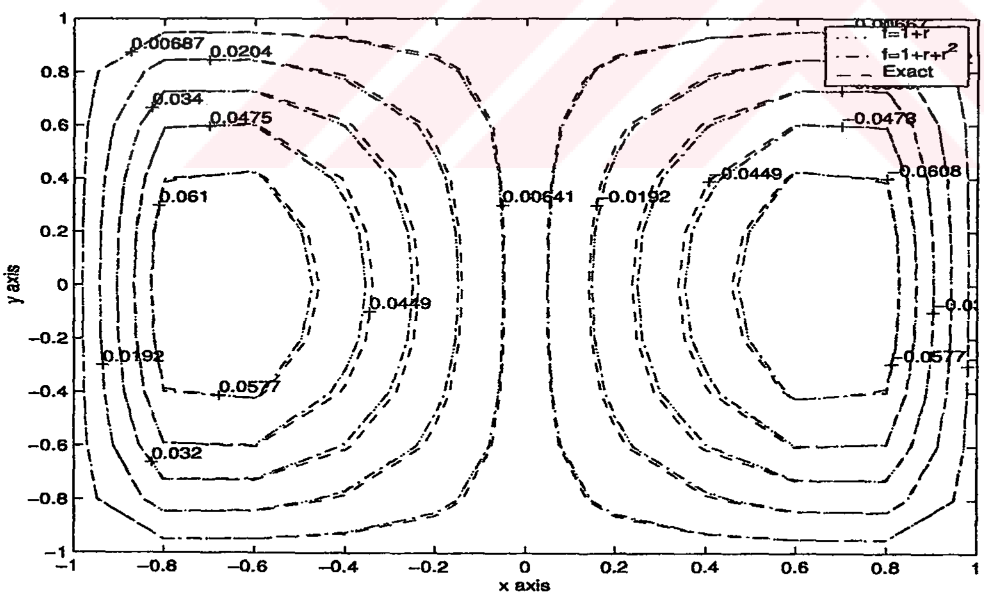

Figure 4.24: B(x,y) for M=5, $\nabla^2 u = b(x, y, u_x)$

4.3 Comparisons of the solutions for $\nabla^2 u = b(x,y,u)$ with $\nabla^2 u = b(x,y,u_x)$ using constant boundary elements

In this Section, solutions of the MHD channel flow problem are compared in the two forms as $\nabla^2 u = b(x,y,u)$ and $\nabla^2 u = b(x,y,u_x)$. We can see that the solutions with $\nabla^2 u = b(x,y,u_x)$ form give better results. Since the derivative form contains more information this result was expected mathematically too.

We compare the two formulations for MHD flow equations in terms of V(x,0), B(x,0) curves and V-B contours. From the previous Sections 4.1 and 4.2 we notice that f-expansion using quadratic radial basis function was suitable for Hartmann numbers at most M=5. This is why we don't continue with the cubic radial basis functions. Figures (4.25)-(4.26) give the comparison for M=2 and Figures (4.27)-(4.28) for M=5 respectively. Especially for M=5 we can see that derivative formulation $\nabla^2 u = b(x,y,u_x)$ gives better results when the curves are compared with exact solution.

In this Section all the graphics which compare the two forms $\nabla^2 u = b(x,y,u)$ and $\nabla^2 u = b(x,y,u_x)$ contain legends for the Laplace operator as $V^2 u$, since MATLAB graphic package doesn't contain $\nabla^2 u$ operator.

The equal velocity lines and current lines are given for the two formulations in Figures (4.29)-(4.30) and (4.31)-(4.32) for M=2 and M=5 respectively. One can easily see that the derivative formulation $\nabla^2 u = b(x,y,u_x)$ is better when the contours are compared with the exact solution.

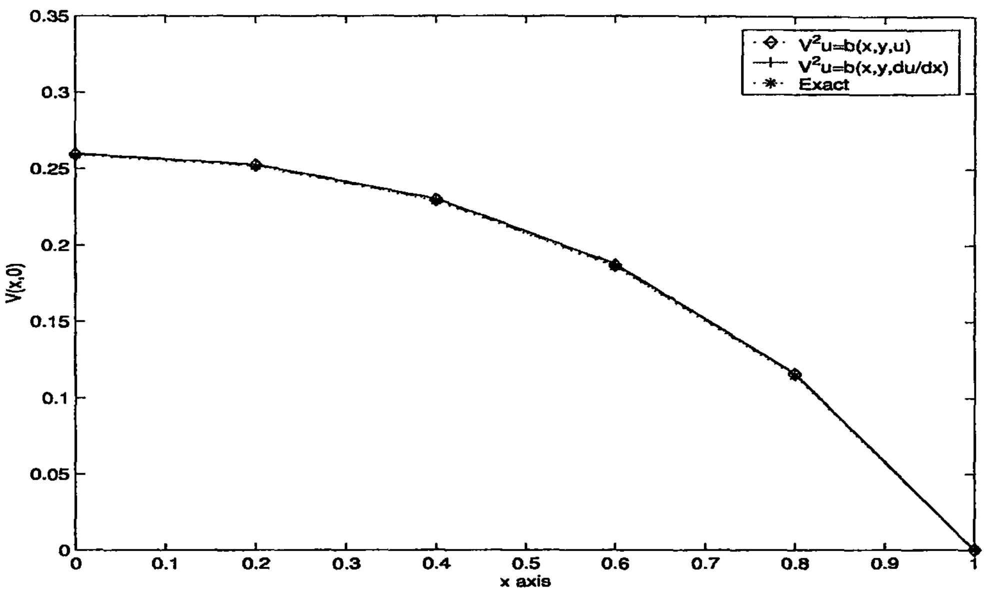

Figure 4.25: V(x,0) for $f = 1 + r + r^2$, M=2, $0 \leq x \leq 1$

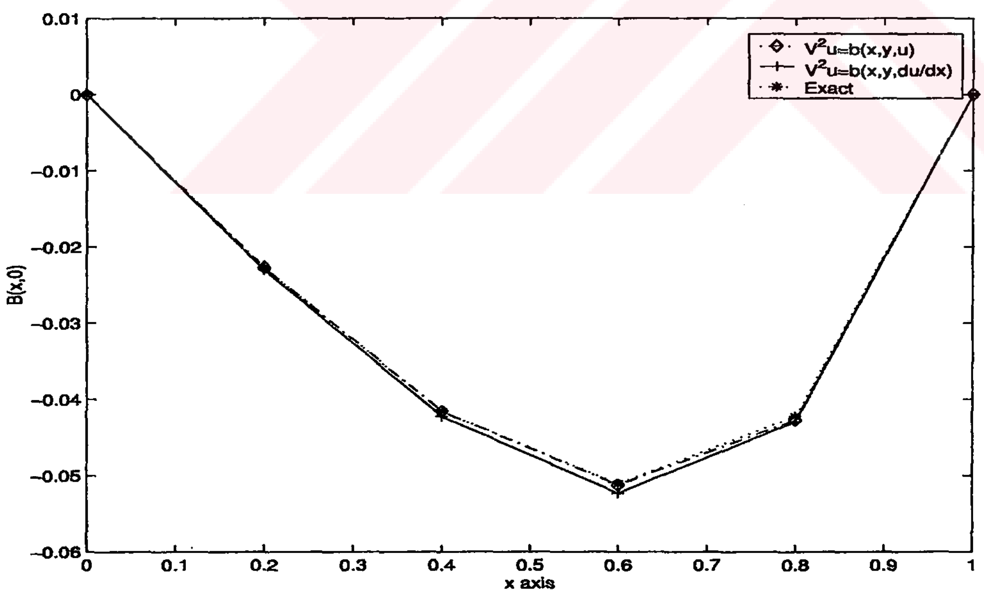

Figure 4.26: B(x,0) for $f = 1 + r + r^2$, M=2, $0 \leq x \leq 1$

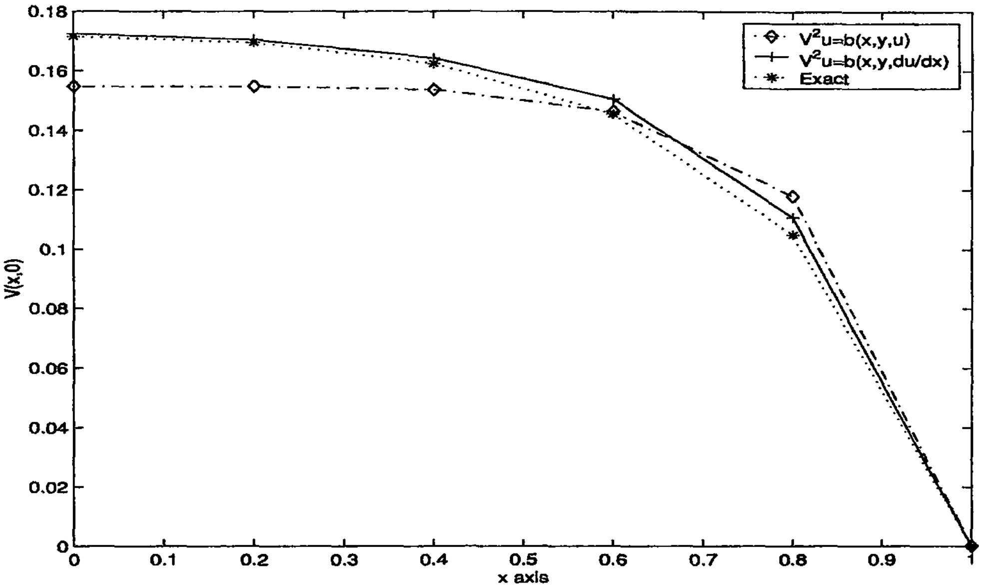

Figure 4.27: V(x,0) for $f = 1 + r + r^2$, M=5, $0 \leq x \leq 1$

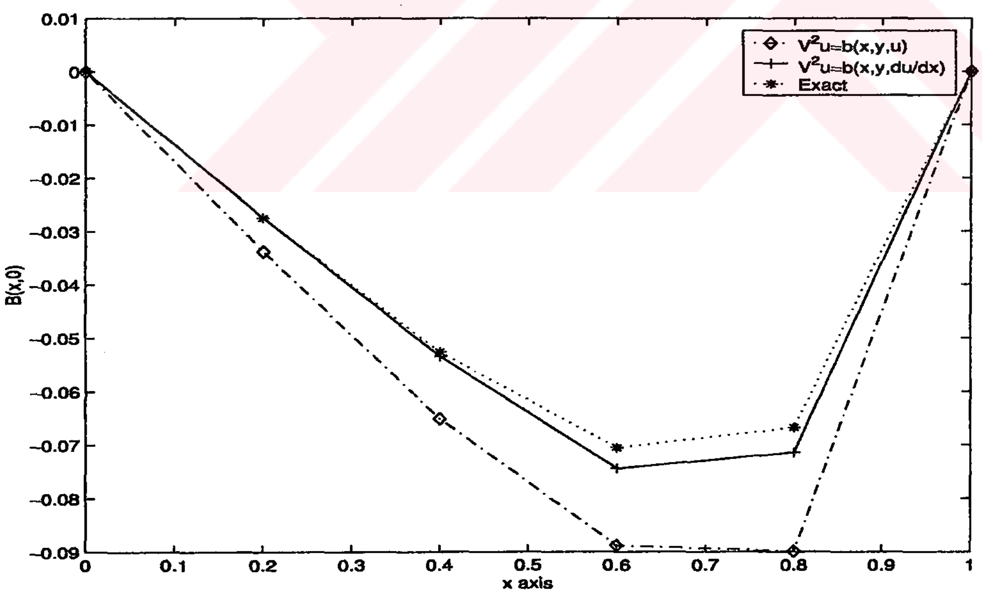

Figure 4.28: B(x,0) for $f = 1 + r + r^2$, M=5, $0 \leq x \leq 1$

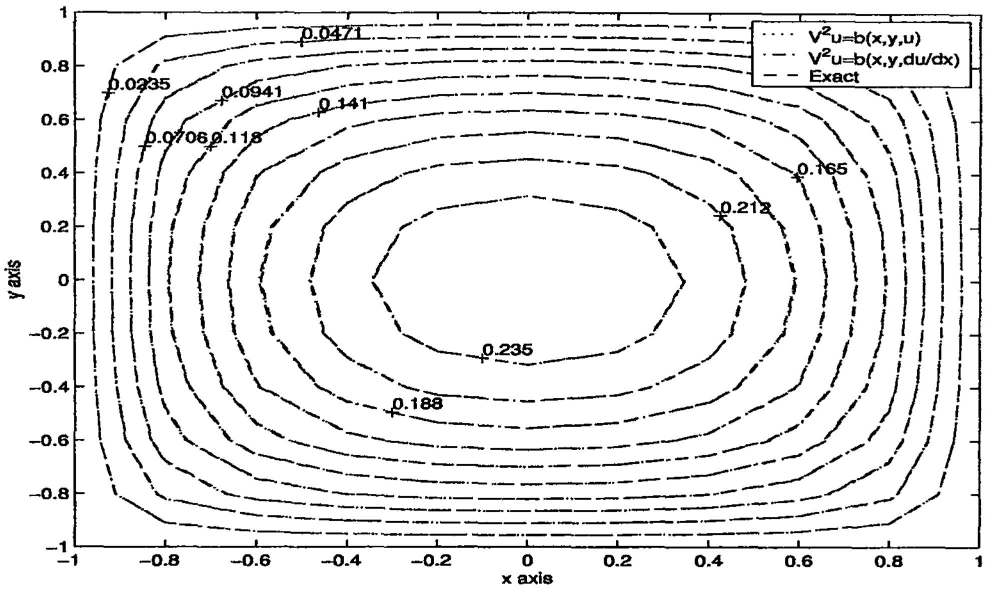

Figure 4.29: V(x,y) for $f = 1 + r + r^2$, M=2

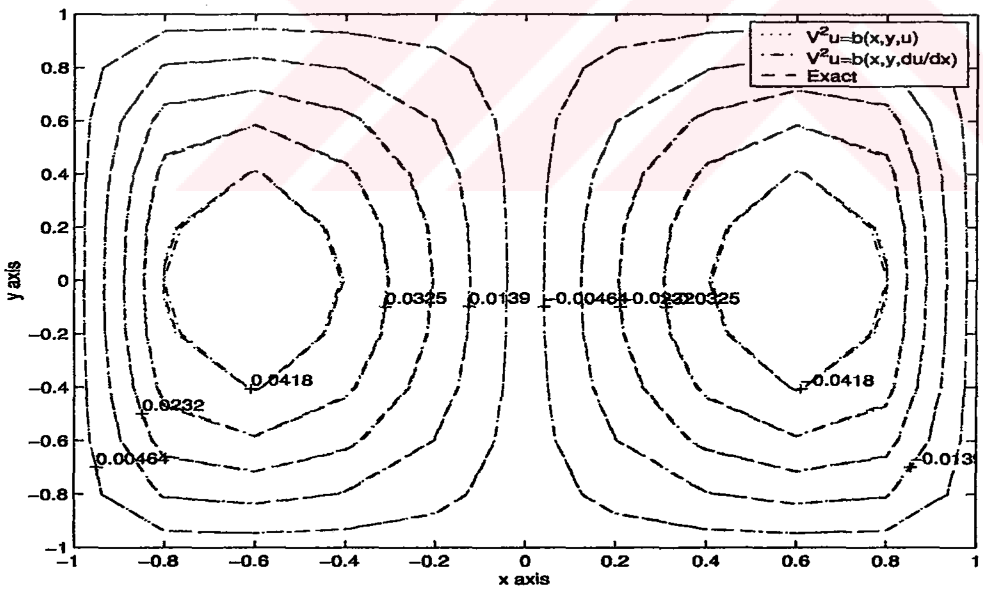

Figure 4.30: B(x,y) for $f = 1 + r + r^2$, M=2

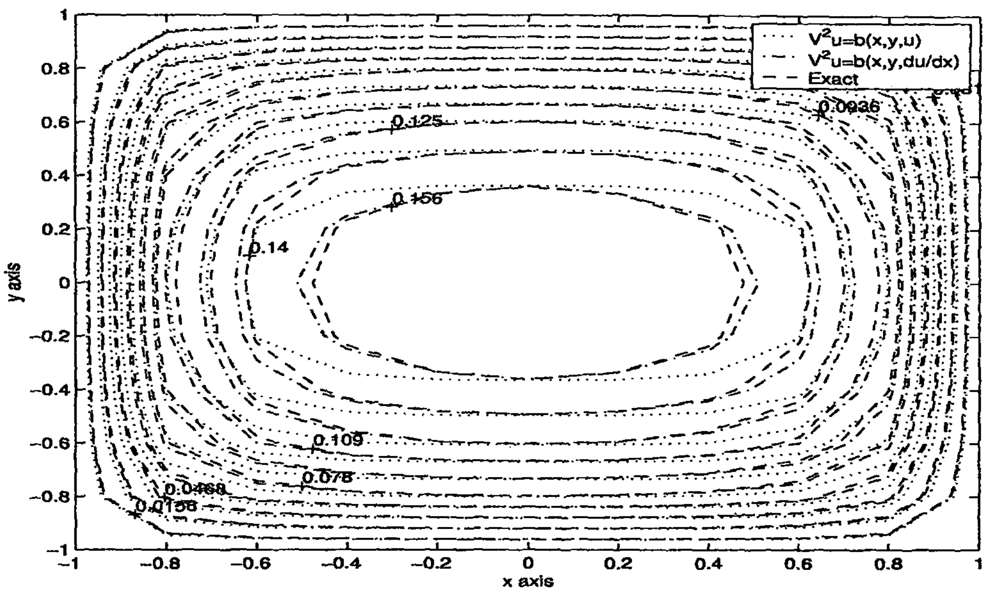

Figure 4.31: V(x,y) for $f = 1 + r + r^2$, M=5

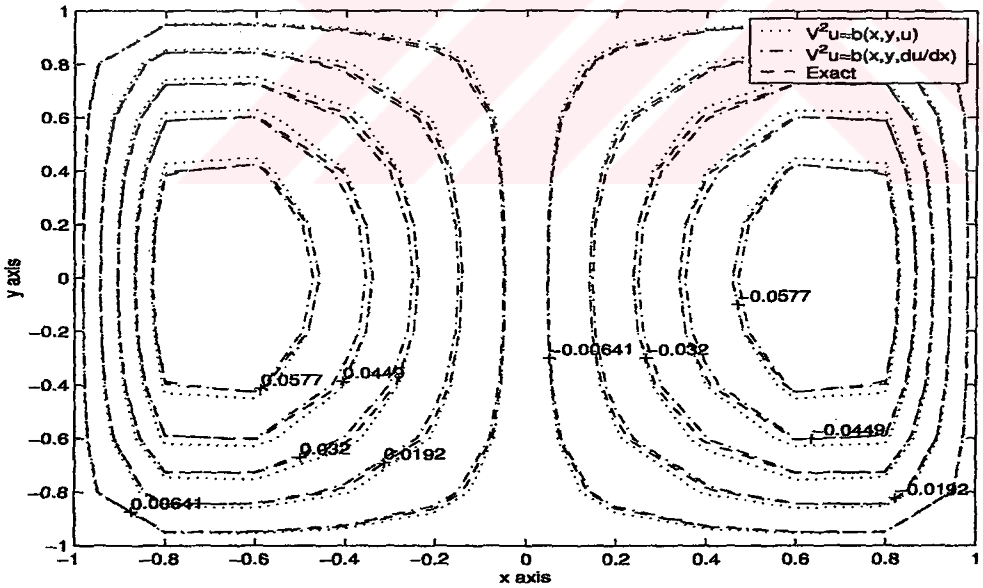

Figure 4.32: B(x,y) for $f = 1 + r + r^2$, M=5

4.4 Solutions using linear boundary elements for $\nabla^2 u = b(x, y, u_x)$ form

This Section represents solutions obtained from $\nabla^2 u = b(x, y, u_x)$ form by using linear boundary elements. We choose 11×11 uniform mesh (h=0.2) for the same domain $|x| \leq 1$, $|y| \leq 1$ using 44 linear boundary elements and 81 interior nodes. Close to the corners of the channel two nodes are placed as the end points of the elements on both boundaries. This way of handling corner points is called discontinuous linear elements. Since from the constant element graphs it is understood that the derivative form $\nabla^2 u = b(x, y, u_x)$ gives better results so, we continue to the linear case only with the derivative formulation.

Figure (4.33) gives variation of Hartmann numbers for V(0,y) curves with quadratic f-expansion. Figures (4.34) and (4.35) present V(x,0)-B(x,0) curves for M=2 and for linear and quadratic f-expansions. In Figures (4.36)-(4.37) we compare constant and linear BEM formulations for M=2 with quadratic f-expansion. From B(x,0) curves we see that linear approximation results are much closer to the exact solution.

Finally we present comparisons of constant-linear BEM approximations for Hartmann numbers M=2 and M=5 in Figures (4.38)-(4.39) and (4.40)-(4.41) respectively using linear boundary elements with quadratic f-expansion. For equal velocity lines linear approximation looks like giving better results in the sense of closeness to the exact solution. But this improvement is not observed in the current lines. As a result, we conclude that even constant approximation can be used all over the computations.

Again V^2 represents ∇^2 in Figures (4.33)-(4.41), because of the legend problem in MATLAB graphic package for ∇^2-operator.

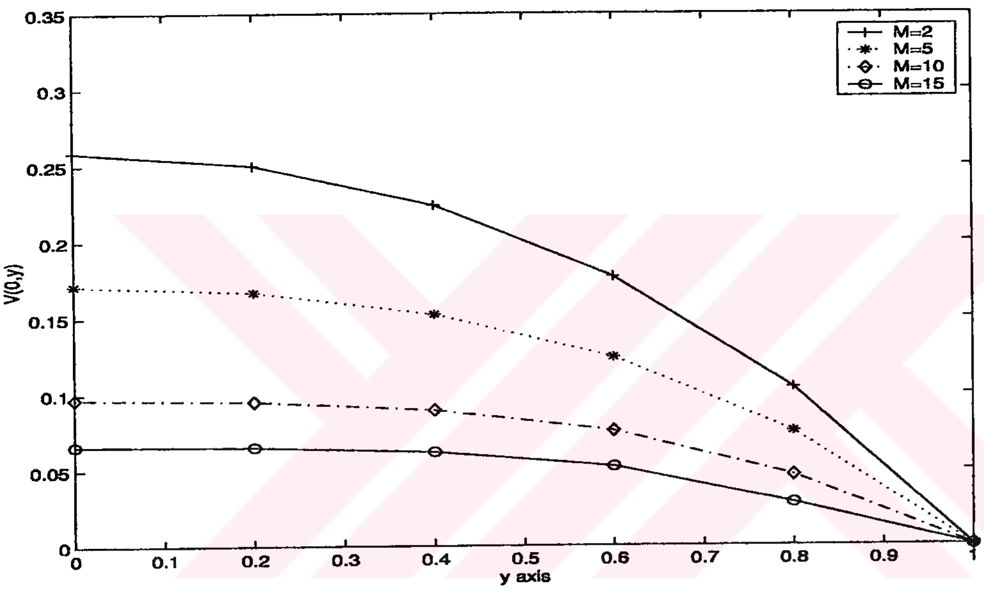

Figure 4.33: V(0,y) for $f = 1 + r + r^2$, $0 \leq y \leq 1$, $\nabla^2 u = b(x, y, u_x)$

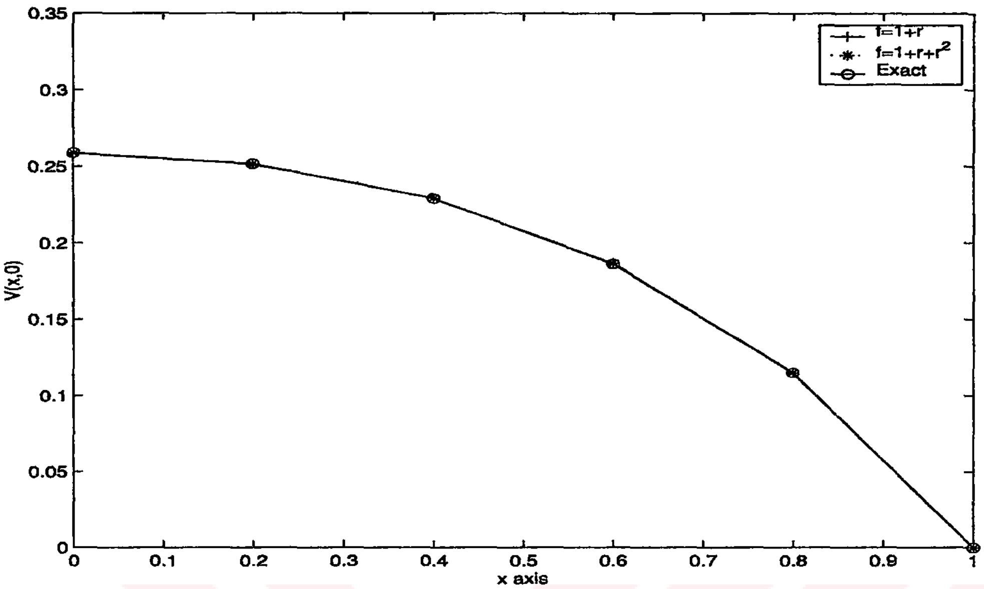

Figure 4.34: V(x,0) for M=2, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

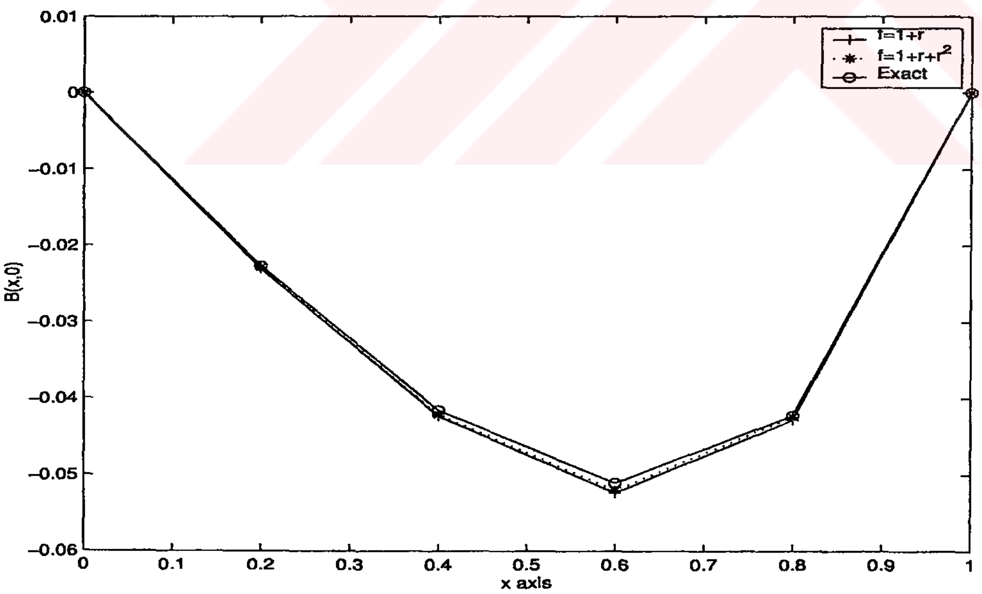

Figure 4.35: B(x,0) for M=2, $0 \leq x \leq 1$, $\nabla^2 u = b(x, y, u_x)$

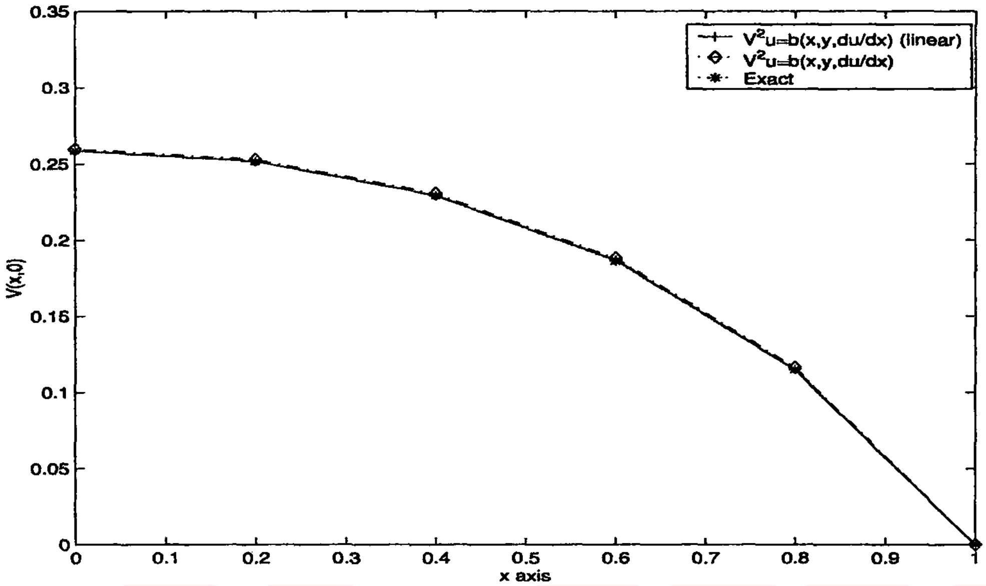

Figure 4.36: V(x,0) for $f = 1 + r + r^2$, M=2, $0 \leq x \leq 1$

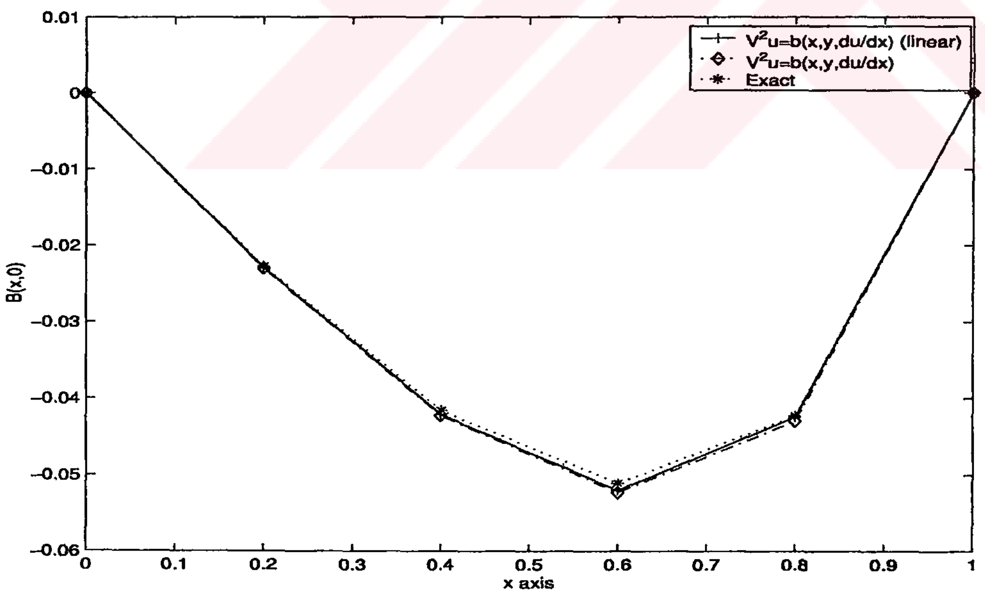

Figure 4.37: B(x,0) for $f = 1 + r + r^2$, M=2, $0 \leq x \leq 1$

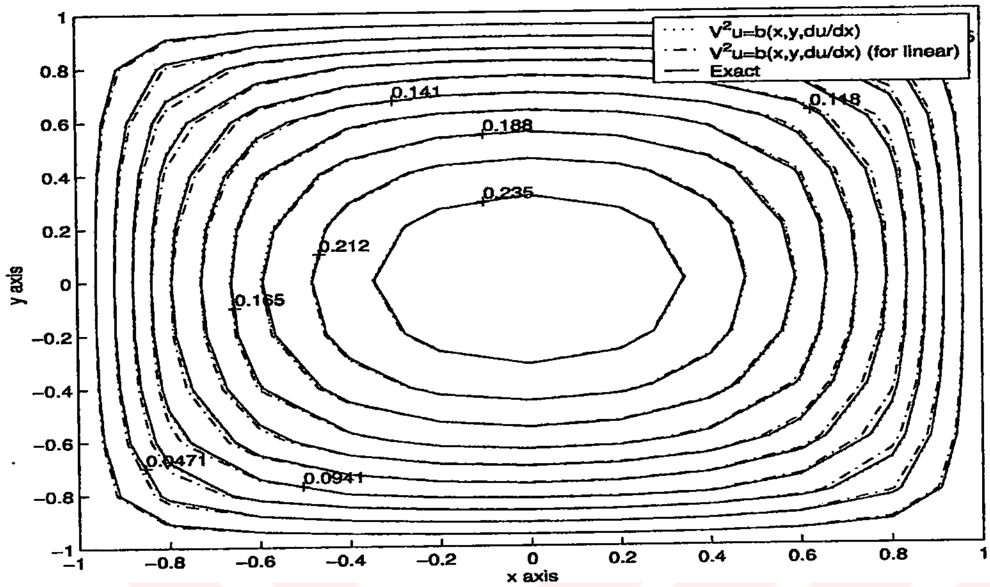

Figure 4.38: V(x,y) for $f = 1 + r + r^2$, M=2

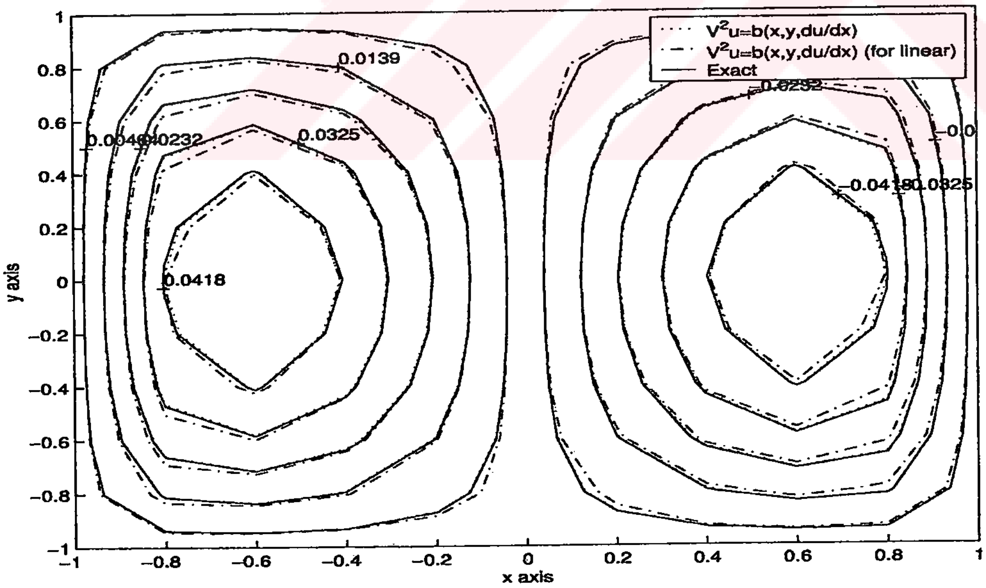

Figure 4.39: B(x,y) for $f = 1 + r + r^2$, M=2

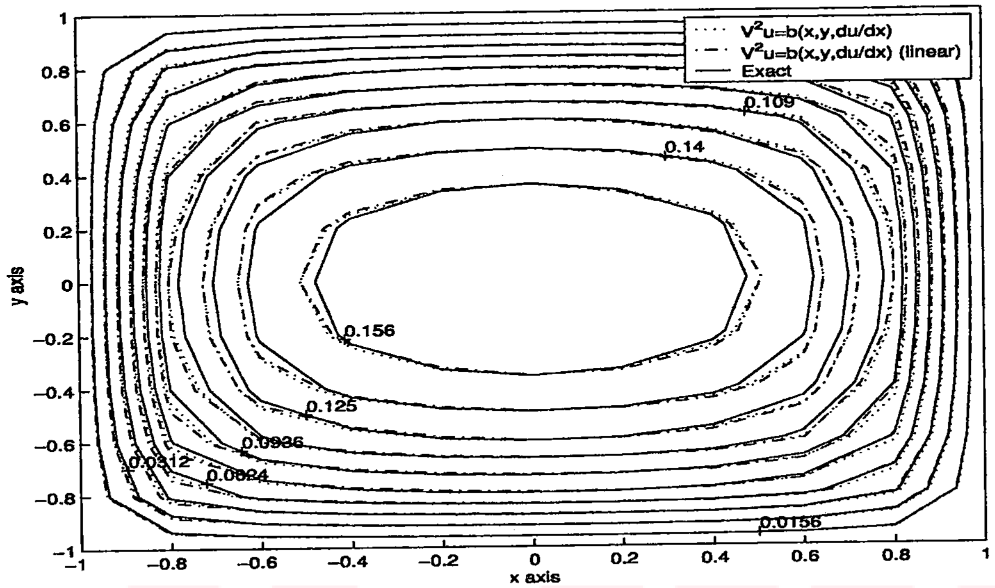

Figure 4.40: V(x,y) for $f = 1 + r$, M=5

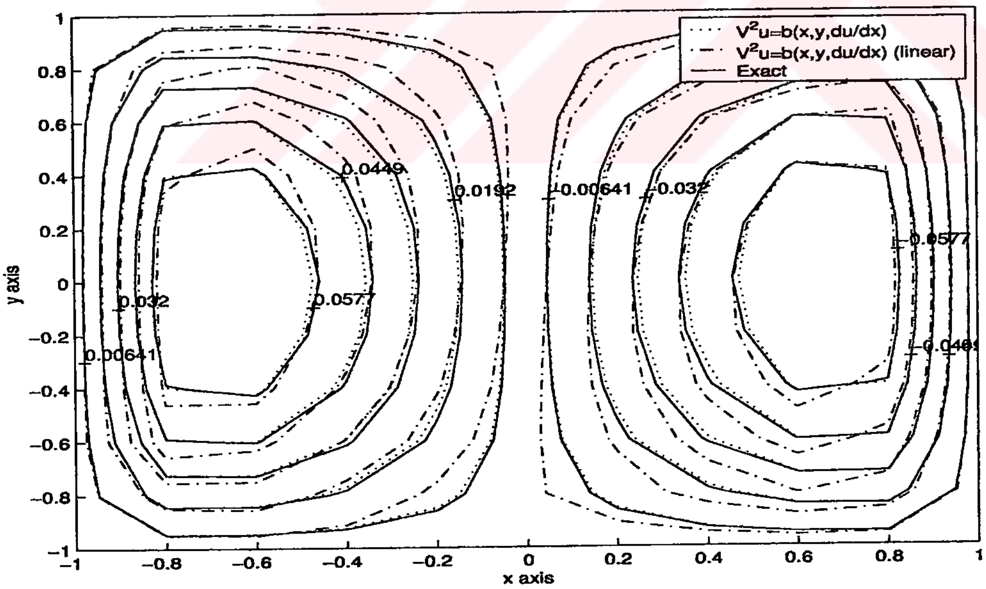

Figure 4.41: B(x,y) for $f = 1 + r$, M=5

CHAPTER 5

CONCLUSION

In this thesis, we deal with the Dual Reciprocity Boundary Element Method (DRBEM) solution of Magnetohydrodynamic Channel Flow problem using Radial Basis Function (RBF) interpolation. We interpolated a function which is b(x,y)=sin(πx)sin(πy) using several radial basis functions and therefore we introduced them first. While solving MHD Channel Flow problem the right hand side function is either of the form, $\nabla^2 u = b(x, y, u)$ or $\nabla^2 u = b(x, y, u_x)$, therefore we apply the radial basis function interpolation to these two forms. Since the Boundary Element Method is connected to the interpolation of the right hand side functions of these two forms, in the interpolation N values from the boundary and L values from the interior part of the domain are used.

The theory of DRBEM for Poisson type equation of the forms, $\nabla^2 u = b(x, y, u)$ and $\nabla^2 u = b(x, y, u_x)$, using the RBF interpolation for b function is given. Then the computations are carried out for these two forms and for linear and constant boundary elements.

We deal with the well known Maxwell equations of electromagnetism and the basic equations of fluid mechanics which lead to the coupled system of equations in the velocity and magnetic field. They can be put in the forms of $\nabla^2 u = b(x, y, u)$ and $\nabla^2 u = b(x, y, u_x)$. Solution of the MHD Channel Flow problem for several Hartmann numbers and different f-expansions are presented by using constant and linear boundary element methods separately.

We present solutions of MHD Channel Flow problem in terms of graphics by using constant boundary elements for $\nabla^2 u = b(x, y, u)$ and $\nabla^2 u = b(x, y, u_x)$ forms. From the results of these two forms, we conclude that $\nabla^2 u = b(x, y, u_x)$ form gives better results than $\nabla^2 u = b(x, y, u)$ form. The results agree well with the exact solution even by using polynomial RBF interpolations for $b(x, y, u_x)$ function. Also the well known behaviour of MHD flow which is the boundary layer formation for increasing Hartmann number, and very well agreement with the exact solution is obtained.

We continued with linear boundary elements for the derivative $\nabla^2 u = b(x, y, u_x)$ form. Solutions of both constant and linear BEM approximations for several Hartmann numbers are compared. For equal velocity lines, linear boundary approximation looks like giving better results in the sense of closeness to the exact solution. But this improvement is not observed in the current lines. As a result, we conclude that constant BEM approximation for $\nabla^2 u = b(x, y, u_x)$ form can be used all over the computations.

Further studies should be carried out for large values of Hartmann number and for different combinations of boundary values of the channel.

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