RELATIVE ENDO-TRIVIAL MODULES

by

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ABSTRACT

RELATIVE ENDO-TRIVIAL MODULES

We first develop the necessary tools from the theory of G-algebras and endomorphism rings. Then the basic results of Green's theory of vertices and sources for kG-modules is given. Building on these, we retrace the steps of Lassueur and introduce the group of relative endo-trivial modules, which gives a generalization of the Dade group. Lastly, we introduce a related group in terms of a specific type of algebras. We describe the Jacobson radical of such algebras.

ÖZET

GÖRECELİ KENDİNE-AŞİKAR MODÜLLER

Öncelikle G-cebirleri ve endomorfizm halkalarının teorisinden gerekli aletleri geliştiriyoruz. Sonrasında Green'in kG-modülleri için geliştirdiği köşeler ve kaynaklar teorisinin temel sonuçlarını veriyoruz. Bunların üzerine, Lassueur'ün adımlarını tekrardan atarak görece kendine-aşikar modüllerin grubunu takdim ediyoruz. Bu Dade grubun bir genellemesini veriyor. Son olarak, spesifik bazı cebirler tarafından tarif edilen başka bir grubu inşa ediyoruz ve söz konusu cebirlerin Jacobson radikalini tarif ediyoruz.

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LIST OF SYMBOLS

A	An algebra
G, H, K	Finite groups
k	A field
L, M, N	Modules
p	A prime number
S	Sylow p-subgroup
$mod(kG)$	The category of kG-modules
G/H	Coset space
$K \setminus G/H$	Double coset space
$M_n(k)$	Full matrix algebra over k
$\leq$	Subgroup
$\cong$	Isomorphic to
$[\ \ :\ \ ]$	Group index
$\mid$	Direct summand of

1. INTRODUCTION

In his papers [1, 2], Dade introduced the notion of an endo-permutation module and what is now called Dade group for p-groups. In the same paper, he intoduced endo-trivial modules which have been crucial for the understanding of endo-permutation modules. Spanning more than 30 years, the classification of endo-trivial modules in the p-group case and endo-permutation modules have been completed. For a detailed account of this, we refer to [3]. While the definition of an endo-trivial module has no difficulty handling the case of a general finite group, the generalization of endo-permutation modules and Dade group to non-p-groups have proven to be non-trivial. Two such generalizations were given by Urfer in [4] and by Lassueur in [5]. The theory first introduced by Lassueur is also developed further by Gelvin and Yalçın in their paper [6]. We trace the steps taken by Lassueur and attempt to take it a bit further also.

The thesis is organized as follows: In Chapter 2, the necessary tools involving kG-modules and G-algebras are developed. In Chapter 3, the notion of relative projectivity and the basics of Green's theory of vertices and sources are given. The notion of a p-permutation module is also introduced. These two sections contain standard results and we make no attempt to be exhaustive. For a much more involved account, one can refer to the book of Nagao and Tsushima [7] or Thevenaz's book [8]. Perhaps one non-standard result here that deserves special mention is Proposition 3.0.11.

The main results of the thesis are in Chapter 4. We first develop the notion of relative endo-trivial modules by constructing a category we call $\mathcal{S}$-stable category. It is then possible to define the group of endo-$\mathcal{S}$-trivial modules and identify a subgroup for the maximal choice of $\mathcal{S}$ as the generalized Dade group, which reduces to the usual Dade group when we consider p-groups. This is a special case of the theory developed by Lassueur in [5], who considered V-projectivity for any module V. We stick to the classic notion of H-projectivity for a subgroup H and hopefully give a more accessible exposition of the subject. At the end of Chapter 4, a group is given in terms of what we call $T_{\mathcal{S}}$-algebras, which turn out to give a quotient of the generalized Dade group.

The Jacobson radical of G-fixed point algebra of $T_{\mathcal{S}}$-algebras is also described. These two last results appear for the first time. We list the assumptions we make throughout the paper. The field k is assumed to be algebraically closed. We only consider finite dimensional modules and algebras over this field k and finite groups. When we consider the coset space or the double coset space for some groups, we always implicitly fix a set of representatives.

2. G-ALGEBRAS AND ENDOMORPHISM RINGS

In this chapter, we develop the necessary definitions and results from the standard theory of kG-modules and G-algebras. For a full account, we refer to [7] and [8].

Definition 2.0.1. *A k-algebra A is called a G-algebra if there a group homomorphism*

$$\phi : G \to \mathrm{Aut}(A)$$

or equivalently, if there's a G-action on A compatible with the k-algebra structure on A. If the action of G on A is given by invertible elements of A, or if ϕ can be modified to be a map

$$\phi : G \to A^*$$

where A^* is the multiplicative group of A then A is called an interior G-algebra.
If $g \in G$ and $a \in A$, we write ${}^g a$ for $\phi(g)(a)$.

Definition 2.0.2. *For any G-algebra A and $H \leq G$, denote by A^H the subalgebra of H-fixed points of A.*
Given $K \leq H \leq G$ and a G-algebra A, the map $tr_K^H : A^K \to A^H$ defined by

$$tr_K^H(a) = \sum_{x \in H/K} {}^x a$$

is called the trace map. The trace map tr_K^H lands in A^H since elements of H permute the coset space H/K.

We list some properties of the trace maps which will be used throughout the paper without reference.

Proposition 2.0.3. *(i) tr_K^H is k-linear for $K \leq H$.*

(ii) $tr_K^G = tr_H^G \circ tr_K^H$ *for* $K \leq H \leq G$.

(iii) $tr_K^H(A^K)$ *is an ideal of* A^H *for* $K \leq H$.

Proof. (i) Clear.

(ii) If $a \in A^K$, then

$$\begin{aligned} tr_H^G \circ tr_K^H(a) &= tr_H^G(\sum_{x \in H/K} {}^x a) = \sum_{x \in H/K} tr_H^G({}^x a) \\ &= \sum_{x \in H/K} \sum_{y \in G/H} ({}^{yx} a) = \sum_{z \in G/K} {}^z a = tr_K^G(a). \end{aligned}$$

(iii) Let $a \in A^H$ and $b \in A^K$, then

$$a tr_K^H(b) = a \sum_{x \in H/K} {}^x b = \sum_{x \in H/K} a \, {}^x b = \sum_{x \in H/K} {}^x a \, {}^x b = \sum_{x \in H/K} {}^x (ab) = tr_K^H(ab).$$

By a similar calculation for $tr_K^H(b)a$, the result follows. □

Remark 2.0.4. *By the above proposition,*

$$\sum_{H \in \mathcal{S}} tr_H^G(A^H)$$

is an ideal of A^G, *for any set* $\mathcal{S}$ *of subgroups of* G.

Corollary 2.0.5. *If the index* $[G : H]$ *is not divisible by* p, *then* $tr_H^G(A^H) = A^G$.

Proof. The ideal $tr_H^G(A^H)$ contains 1 since

$$\frac{1}{[G:H]} tr_H^G(1) = 1.$$

□

Definition 2.0.6. *If A is a k-algebra, then an element e of A is called an idempotent if $e^2 = e$. Two idempotents e and f are called orthogonal if $ef = fe = 0$. An idempotent is called primitive if it cannot be written as a sum of two orthogonal idempotents. A decomposition*

$$1 = e_1 + e_2 + ... + e_n$$

is called an idempotent decomposition of identity, if e_i's are mutually orthogonal idempotents.

Remark 2.0.7. *Given an idempotent decomposition*

$$1 = e_1 + e_2 + ... + e_n$$

in A, we have a decomposition of A

$$A = \bigoplus_{i,j} e_i A e_j.$$

If each e_i is an element of A^G, the decomposition is G-invariant.

Remark 2.0.8. *Since we only consider finite dimensional algebras, the above remark also shows that we always have an idempotent decomposition of identity composed of primitive idempotents.*

Definition 2.0.9. *If the identity is a primitive idempotent in A^G, then the G-algebra A is called primitive.*

Let M be a kG-module. Then $\mathrm{End}_k(M)$ is a k-algebra with a G-action defined by

$$(g\phi)(m) = g\phi(g^{-1}m),$$

where $g \in G$ and $m \in M$.

Moreover, given an element $g \in G$, there's an invertible element ϕ_g of $\mathrm{End}_k(M)$ defined in the obvious way and this defines the same G-action on $\mathrm{End}_k(M)$, which shows that $\mathrm{End}_k(M)$ is an interior G-algebra. This will be our primary example of a G-algebra.

Remark 2.0.10. *If* $A = \mathrm{End}_k(M)$, *then* $\mathrm{End}_k(M)^H = \mathrm{End}_{kH}(M)$.

Definition 2.0.11. *Let M and N be kG-modules. Then we can define a G-action on $M \otimes_k N$ by*

$$g(m \otimes n) = gm \otimes gn,$$

where $g \in G$, $m \in M$ and $n \in N$. We suppress the subscript k in this tensor product notation.

Remark 2.0.12. *There's an isomorphism of kG-modules $M \otimes N \cong N \otimes M$ given by $m \otimes n \to n \otimes m$.*

Definition 2.0.13. *If M and N are kG-modules, we can define a G-action on* $\mathrm{Hom}_k(M, N)$ *by*

$$(g\phi)(m) = g\phi(g^{-1}m),$$

where $g \in G$ and $m \in M$. If $N = k$, the kG-module $\mathrm{Hom}_k(M, k)$ *is called the dual of M and is denoted by M^*.*

Remark 2.0.14. *Fix a basis and a dual basis of M and M^*. If the action of g^{-1} on M is given by the matrix (α_{ij}), then the action of g on M^* is given by the matrix (α_{ji}). This follows because the action of G on k is trivial.*

Proposition 2.0.15. $(M \otimes N)^* \cong M^* \otimes N^*$.

Proof. Let $\{m_i\}$ and $\{n_j\}$ be bases of M and N, respectively. Let $\{m_i^*\}$ and $\{n_j^*\}$ be the respective dual bases. Then the desired isomophism is given by $\theta : (M \otimes N)^* \to M^* \otimes N^*$

$$\theta((m_i \otimes n_j)^*) = m_i^* \otimes n_j^*.$$

□

In order to understand $\mathrm{End}_k(M)$, we link tensor products, duals and Hom.

Proposition 2.0.16. *If M and N are kG-module then*

$$M^* \otimes N \cong \mathrm{Hom}_k(M, N).$$

Proof. Fix bases $\{m_i\}$ and $\{n_j\}$ of M and N, respectively. Write $\phi_{ij} : M \to N$ for the linear map mapping m_i to n_j and every other basis element to zero. Write $\phi_i : M \to k$ for the linear map taking m_i to 1 and everything else to zero. Define $\theta : \mathrm{Hom}_k(M, N) \to M^* \otimes N$ by

$$\theta(\phi_{ij}) = \phi_i \otimes n_j,$$

which is a k-isomorphism as it takes a basis to a basis. For $g \in G$, write

$$g^{-1}m_r = \sum_s \alpha_{rs} m_s$$

and

$$gn_j = \sum_t \beta_{jt} n_t.$$

We have

$$(g\phi_{ij})(m_r) = g\phi_{ij}(g^{-1}m_r) = g\phi_{ij}(\sum_s \alpha_{rs} m_s) = g\alpha_{ri} n_j = \sum_t \alpha_{ri}\beta_{jt} n_t,$$

thus

$$g\phi_{ij} = \sum_r \sum_t \alpha_{ri}\beta_{jt}\phi_{rt}.$$

Also

$$(g\phi_i)(m_r) = \phi_i(g^{-1}m_r) = \phi_i(\sum_s \alpha_{rs}m_s) = \alpha_{ri}$$

so

$$g\phi_i = \sum_r \alpha_{ri}\phi_r,$$

which shows that θ is G-invariant. □

Corollary 2.0.17. $\mathrm{End}_k(M) \cong M \otimes M^*$.

The idempotents of $\mathrm{End}_{kG}(M)$ have the following interpretation:

Proposition 2.0.18. *There is a correspondence between idempotent decompositions of identity in* $\mathrm{End}_{kG}(M)$ *and* G*-invariant decompositions of* M*.*

Proof. Let

$$1 = e_1 + e_2 + ... + e_n$$

be an idempotent decomposition of identity in $\mathrm{End}_{kG}(M)$. Set

$$M_i = e_i(M)$$

so that

$$M = M_1 + M_2 + ... + M_n,$$

since the e_i's sum up to 1. The sum is direct since the e_i's are mutually orthogonal.

Conversely, given a G-invariant decomposition

$$M = \bigoplus_{i=1}^{n} M_i,$$

leting e_i denote the composition of the projection to M_i and the embedding of M_i into M, we get the desired idempotent decomposition. □

Finally, we list some results without proofs.

Definition 2.0.19. *Let R be a ring. The Jacobson radical $J(R)$ of R is the intersection of all its maximal left ideals.*

The proof for the following theorem can be found in Theorem 3.3 of Chapter 1 in [7].

Theorem 2.0.20. *The following are equivalent for a k-algebra A:*
(i) Letting J(A) denote the Jacobson radical of A, we have $A/J(A) \cong k$.
(ii) There's a unique left ideal of A.
(iii) In A, $1 \neq 0$ and for any $a \in A$, either a or $1 - a$ is invertible.
In the case that these equivalent conditions hold, A is called a local algebra.

The proof for the following theorem can be found in Theorem 5.7 of Chapter 1 in [7].

Theorem 2.0.21. *The following for a kG-module M are equivalent:*
(i) M is indecomposable.
(ii) $\mathrm{End}_{kG}(M)$ *is a local algebra.*
(iii) $\mathrm{End}_k(M)$ *is a primitive G-algebra.*

The proof for the following theorem can be found in Theorem 6.1 of Chapter 1 in [7].

Theorem 2.0.22. *The category mod(kG) has the Krull-Schmidt property, that is, given decompositions into indecomposable modules*

$$M = N_1 \oplus N_2 \oplus ... \oplus N_n$$

and

$$M = L_1 \oplus L_2 \oplus .... \oplus L_m$$

in mod(kG), we have $n = m$ and after possibly a renumbering, $N_i \cong L_i$.

The following is called Rosenberg's lemma. A proof can be found in Proposition 4.9 of Chapter 1 in [8].

Lemma 2.0.23. *If e is a primitive idempotent of a k-algebra A and I_1 and I_2 are ideals of A with $e \in I_1 + I_2$, then e is contained either in I_1 or I_2.*

3. RELATIVE PROJECTIVITY

In this chapter, we develop the necessary definitions and results from the theory of relative projectivity. Unless otherwise stated, the definitions and results are standard and we refer to [7] for a full account.

Definition 3.0.1. *Let $H \leq G$. A kG-module M is called relatively H-projective if given a diagram*

$$\begin{array}{ccccc} M & & & & \\ \vdots & \searrow & & & \\ N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

in $mod(kG)$ with exact row such that diagram can be completed to a commutative triangle in $mod(kH)$, then the diagram can be completed in $mod(kG)$ to a commutative triangle. We just say H-projective instead of relatively H-projective.

Remark 3.0.2. *Since k is a field, every k-module is projective and the above diagram can always be completed in k-mod. Thus 1-projectivity is the same as projectivity.*

In order to study relatively projective modules, we need to introduce induction and restriction functors. If M is a kH-module with $H \leq G$, then we denote the tensor product of kG with M over kH or $kG \otimes_{kH} M$, by $\mathrm{Ind}_H^G(M)$, which gives a functor:

$$\mathrm{Ind}_H^G : mod(kH) \to mod(kG).$$

We also have a functor called restriction in the reverse direction:

$$\mathrm{Res}_H^G : mod(kG) \to mod(kH),$$

which just forgets the action of elements of G outside of H.

Setting $g \otimes M = \{g \otimes m \in \mathrm{Ind}_H^G(M) : m \in M\}$, we have an identification:

$$\mathrm{Ind}_H^G(M) = \bigoplus_{g \in G/H} g \otimes M.$$

It should not be forgotten that in the tensor product notation $g \otimes M$, the subscript H is suppressed. Given $K \leq H \leq G$, it is clear that Res_K^G is isomorphic to the functor $\mathrm{Res}_K^H \circ \mathrm{Res}_H^G$. We also have an (kG, kK)-bimodule isomorphism

$$kG \cong (kG \otimes_{kH} kH)$$

which shows that the functors Ind_K^G and $\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H$ are also isomorphic. If $H \leq G$, $x \in G$ and M is a kH-module, the formula:

$${}^x h \cdot m = h \cdot m$$

defines a ${}^x H$-action on M. The resulting $k[{}^x H]$-module is denoted by ${}^x M$. This operation is called conjugation and its interaction with induction and restriction is easy to describe. We have

$${}^x(\mathrm{Res}_K^H(M)) \cong \mathrm{Res}_{{}^x K}^{{}^x H}({}^x M)$$

and

$${}^x(\mathrm{Ind}_K^H(M)) \cong \mathrm{Ind}_{{}^x K}^{{}^x H}({}^x M).$$

If $x \in G$, then there's a $k[{}^x H]$-isomorphism

$$x \otimes M \cong {}^x M.$$

Hence a kG-isomorphism

$$\mathrm{Ind}_H^G(M) \cong \mathrm{Ind}_{{}^xH}^G({}^xM).$$

The interaction between induction and restriction is given by the following proposition, which is known as Mackey's formula.

Proposition 3.0.3. *If $H, K \leq G$ and M is a kH-module, there's an isomorphism*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G(M) \cong \bigoplus_{x \in K\backslash G/H} \mathrm{Ind}_{K\cap {}^xH}^K \mathrm{Res}_{K\cap {}^xH}^{{}^xH}({}^xM)$$

where x runs over a set of double coset representatives of K, H in G.

Proof. Note that we have a decomposition:

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G(M) = \bigoplus_{x \in G/H} x \otimes M$$

and there's a $k[{}^xH]$-isomorphism

$$x \otimes M \cong {}^xM.$$

Also there's a decomposition

$$\mathrm{Ind}_{K\cap {}^xH}^K \mathrm{Res}_{K\cap {}^xH}^{{}^xH}({}^xM) = \bigoplus_{y \in K/(K\cap {}^xH)} y \otimes {}^xM,$$

so using the identification

$$y \otimes {}^xM \cong yx \otimes M,$$

we can see $\mathrm{Ind}_{K\cap {}^xH}^K \mathrm{Res}_{K\cap {}^xH}^{{}^xH}({}^xM)$ as a direct summand of $\mathrm{Res}_K^G \mathrm{Ind}_H^G(M)$. Let $z \in G$. We would like to get rid of repetitions in our decomposition.

The condition that $z \otimes M$ appears in $\operatorname{Ind}_{K\cap {}^xH}^{K} \operatorname{Res}_{K\cap {}^xH}^{{}^xH}({}^xM)$ is equivalent to there being an element $k \in K/(K \cap {}^xH)$ and $h \in H$ such that $zh = kx$, which in turn is equivalent to z being in the double coset KxH. The result follows. □

Using the induction and restriction functors, we get the following characterizations for H-projectivity:

Theorem 3.0.4. *The following are equivalent for a kG-module M:*
(i) M is H-projective.
(ii) M is a direct summand of $\operatorname{Ind}_H^G \operatorname{Res}_H^G(M)$.
(iii) M is a direct summand of $\operatorname{Ind}_H^G(N)$ *for some kH-module N.*
(iv)(Higman's Criterion) $id_M = tr_H^G(\psi)$ for some $\psi \in \operatorname{End}_{kH}(M)$.
(v) $\operatorname{End}_{kG}(M) = tr_H^G(\operatorname{End}_{kH}(M))$.

Proof. $(i) \implies (ii)$: Consider the map $\phi : \operatorname{Ind}_H^G \operatorname{Res}_H^G(M) \to M$ defined by

$$\phi(g \otimes m) = gm$$

which is G-invariant since for $x \in G$,

$$x\phi(g \otimes m) = xgm = g'hm = \phi(g' \otimes hm) = \phi(g'h \otimes m) = \phi(xg \otimes m),$$

where $xg = g'h$ with $h \in H$. The map ϕ is clearly surjective. So there's a diagram

$$\begin{array}{ccccc} M & & & & \\ \downarrow & \searrow^{id_M} & & & \\ \operatorname{Ind}_H^G \operatorname{Res}_H^G(M) & \xrightarrow[\phi]{} & M & \longrightarrow & 0 \end{array}$$

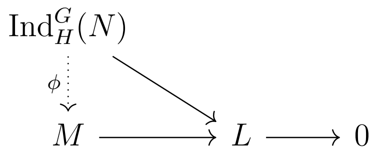
which can be completed in kH-mod by the map $\theta : M \to \operatorname{Ind}_H^G \operatorname{Res}_H^G(M)$ defined by

$$\theta(m) = 1 \otimes m.$$

Since M is H-projective, the diagram can be completed in kG-mod also, which shows that M is a direct summand of $\operatorname{Ind}_H^G \operatorname{Res}_H^G(M)$.

$(ii) \implies (iii)$: Clear.

$(iii) \implies (i)$: Suppose we have a diagram in kG-mod with exact row and ϕ being an H-invariant map



Define $\tilde{\phi} : \operatorname{Ind}_H^G(N) \to M$ by

$$\tilde{\phi}(g \otimes n) = g\phi(1 \otimes n)$$

which is G-invariant and completes the diagram in kG-mod, which shows that $\operatorname{Ind}_H^G(N)$ is H-projective. Using the diagram for $\operatorname{Ind}_H^G(N)$, it is easy to show that any direct summand of $\operatorname{Ind}_H^G(N)$ is also H-projective.

$(iv) \iff (v)$: Clear, since $tr_H^G(\operatorname{End}_{kH}(M))$ is an ideal of $\operatorname{End}_{kG}(M)$.

$(iv) \implies (ii)$: Let $id_M = tr_H^G(\psi)$ and the maps ϕ be as in the proof of $(i) \implies (ii)$. Then define $\iota : M \to \operatorname{Ind}_H^G \operatorname{Res}_H^G(M)$ by

$$\iota(m) = \sum_{x \in G/H} x \otimes \psi(x^{-1}m)$$

which is G-invariant since if $g \in G$

$$g\iota(m) = \sum_{x \in G/H} gx \otimes \psi(x^{-1}m) = \sum_{x' \in G/H} x' \otimes \psi((x')^{-1}gm) = \iota(gm)$$

and $\phi \circ \iota = tr_H^G(\psi) = id_M$, so the result follows.

$(ii) \implies (iv)$: Let $\iota : M \to \operatorname{Ind}_H^G(M)$ denote the canonical embedding map.

Let $\pi : \mathrm{Ind}_H^G(M) \to M$ denote the canonical projection map. Write

$$\iota(m) = \sum_{x \in G/H} x \otimes m_x.$$

Define $\tilde{\iota} : M \to \mathrm{Ind}_H^G(M)$ by

$$\tilde{\iota}(m) = 1 \otimes m_1$$

which is an H-invariant map. Letting $m \in M$, we have

$$\sum_{x \in G/H} x\tilde{\iota}(x^{-1}m) = \sum_{x \in G/H} x \otimes (x^{-1}m)_1 = \sum_{x \in G/H} x \otimes m_x = \iota(m).$$

So that

$$tr_H^G(\tilde{\iota}) = \iota$$

and hence

$$id_M = \pi \circ \iota = \pi \circ tr_H^G(\tilde{\iota}) = tr_H^G(\pi \circ \tilde{\iota}).$$

□

Corollary 3.0.5. *If p does not divide $[G : H]$, then every kG-module M is H-projective.*

Proof. This follows by Higman's criterion and Corollary 2.0.5. □

Remark 3.0.6. *By the above corollary, only p-subgroups of G are of interest for relative projectivity. We'll write P-projective instead of H-projective from now on for p-subgroups P of G.*

Corollary 3.0.7. *If M is P-projective, then $M \otimes N$ is P-projective for any N.*

Proof. By Higman's criterion, write

$$id_M = tr_P^G(\psi)$$

for some $\psi \in \mathrm{End}_{kP}(M)$. The map $\psi \otimes id_N$ is P-invariant and

$$\begin{aligned} tr_P^G(\psi \otimes id_N)(m \otimes n) &= \sum_{g \in G/P} g(\psi \otimes id_N)(g^{-1}m \otimes g^{-1}n) = \sum_{g \in G/P} g(\psi(g^{-1}m) \otimes g^{-1}n) \\ &= \sum_{g \in G/P} (g\psi(g^{-1}m)) \otimes n = tr_P^G(\psi)(m) \otimes n = m \otimes n, \end{aligned}$$

thus

$$tr_P^G(\psi \otimes id_N) = id_{M \otimes N}$$

□

The following is an important result, the proof follows the one in Proposition 4.8 of [9]. For an alternative proof, one can also use Proposition 2.3 of [10].

Proposition 3.0.8. *M is P-projective if and only if $M \otimes M^*$ is P-projective.*

Proof. Let $\{m_i\}$ denote a basis of M and $\{m_i^*\}$ the dual basis of M^* associated to it. The only if part of the proposition is clear by Corollary 3.0.7. For the reverse direction, we define two maps, $\iota : M \to M \otimes M^* \otimes M$ and $\pi : M \otimes M^* \otimes M \to M$ by

$$\iota(m) = \sum_i m \otimes m_i^* \otimes m_i$$

and

$$\pi(m \otimes n^* \otimes m') = n^*(m)m'.$$

Then we have $\pi \circ \iota = id_M$. The two maps are also G-invariant since if we write

$$gm_i = \sum_j \alpha_{ij} m_j,$$

and

$$gm_i^* = \sum_j \beta_{ij} m_j^*,$$

then

$$\begin{aligned} g\iota(m) = \sum_i gm \otimes gm_i^* \otimes gm_i &= \sum_i (gm \otimes (\sum_j \beta_{ij} m_j^*) \otimes (\sum_k \alpha_{ik} m_j)) \\ &= \sum_{i,j,k} gm \otimes \beta_{ij} m_j^* \otimes \alpha_{ik} m_k = \sum_i gm \otimes m_i^* \otimes m_i = \iota(gm) \end{aligned}$$

where the second to last equality follow by Remark 2.0.14. As for the map π, we have:

$$\begin{aligned} \pi(gm \otimes gn^* \otimes gm') = (gn^*)(gm)gm' = (g(n^*(g^{-1}gm)))gm' &= n^*(m)gm' \\ &= g\pi(m \otimes n^* \otimes m'). \end{aligned}$$

Thus, we see that M is a direct summand of $M \otimes M^* \otimes M$. Since $M \otimes M^* \otimes M$ is P-projective if $M \otimes M^*$ is P-projective by Corollary 3.0.7, we get the desired result. □

The following theorem is the first main result in Green's theory of vertices and sources. While we give a full proof, we also refer to Theorem 3.3 and Theorem 3.6 of Chapter 4 in [7].

Theorem 3.0.9. *Given an indecomposable kG-module M, consider the pairs (P, N) such that $M | \operatorname{Ind}_P^G(N)$. Then there is a unique G-conjugacy class of such pairs with P minimal with respect to the subgroup relation and N indecomposable. In this case, P is called a vertex and N a source of M.*

Proof. Let (P, N) and (Q, L) be pairs as above with P and Q minimal, N and L indecomposable. Then we have

$$\mathrm{Ind}_Q^G \mathrm{Res}_Q^G(M) \quad | \bigoplus_{x \in Q \backslash G/P} \mathrm{Ind}_{Q \cap {}^xP}^G \mathrm{Res}_{Q \cap {}^xP}^{{}^xP}({}^xN)$$

and hence

$$M \quad | \quad \mathrm{Ind}_{Q \cap {}^xP}^G \mathrm{Res}_{Q \cap {}^xP}^{{}^xP}({}^xN)$$

for some x. But P is minimal, so we get ${}^xP \leq Q$. Carrying out the argument the other way around we see that ${}^xP = Q$. Now, let M' be an indecomposable direct summand of $\mathrm{Res}_Q^G(M)$ so that M is a direct summand of $\mathrm{Ind}_Q^G(M')$. We have

$$\mathrm{Res}_Q^G(M) \quad | \bigoplus_{x \in Q \backslash G/P} \mathrm{Ind}_{Q \cap {}^xP}^Q \mathrm{Res}_{Q \cap {}^xP}^{{}^xP}({}^xN)$$

and M' can only be a direct summand of $\mathrm{Ind}_{Q \cap {}^xP}^Q \mathrm{Res}_{Q \cap {}^xP}^{{}^xP}({}^xN)$ with ${}^xP \leq Q$ since Q is minimal. Thus M' is a direct summand of ${}^xN$, but since ${}^xN$ is indecomposable, $M' = {}^xN$. Following the argument the other way around, we see that M' is conjugate to L as well. Thus by an appropriate choice of z, we get that ${}^z(P, N) = (Q, L)$. □

We will now give some basic properties and examples of an important class of modules, called p-permutation modules, which turn out to have the simplest possible source.

Definition 3.0.10. *A kG-module M is called a permutation module if M has a basis permuted by G, which we call a permutation basis. If M is a direct summand of a permutation module, it is called a p-permutation module.*

Remark 3.0.11. *Let M be a permutation module with permutation basis T.*

Let $T_1, ..., T_n$ denote the the G-orbits of T. Then we have a decomposition:

$$M = \bigoplus_{i=1}^{n} kT_i$$

where kT_i denotes the k-span of T_i. Letting H_i denote the stabilizer of T_i and identifying T_i with the G-set G/H_i we have a kG-isomorphism

$$\mathrm{Ind}_{H_i}^{G}(k) \cong kT_i$$

given by the map which takes $x \otimes 1$ to xH_i

Remark 3.0.12. *Let M_1 and M_2 be permutation modules with permutation bases T_1 and T_2. The tensor product $M_1 \otimes M_2$ is a permutation module with permutation basis $\{t_1 \otimes t_2 : t_1 \in T_1, t_2 \in T_2\}$. The dual module M_1^* is also a permutation module with permutation basis dual to the basis T_1. So the class of permutation modules is closed under tensor products and taking duals.*

Proposition 3.0.13. *The class of p-permutation modules is closed under tensor products, taking duals and direct summands.*

Proof. The first two properties follows from the corresponding properties for permutation modules. The last one follows from the definition of p-permutation modules. □

Remark 3.0.14. *In the literature, p-permutation module is usually defined to be a module which becomes a permutation module after restriction to a Sylow p-subgroup, which turns out to be an equivalent condition. The notion of permutation module and p-permutation module coincides for p-groups. For a proof, see 5.27 of [8].*

Definition 3.0.15. *A kG-module which is a direct sum of indecomposable kG-modules with source k is called a trivial source module.*

Proposition 3.0.16. *For a kG-module M, the following are equivalent:*
(i) M is a trivial source module.
(ii) M is a p-permutation module.

Proof. $(i) \implies (ii)$: Clear, since $\operatorname{Ind}_P^G(k)$ is a permutation module with basis given by G/P.
$(ii) \implies (i)$: Suppose M is a direct summand of $\operatorname{Ind}_H^G(k)$ and let P denote the vertex of M. Consider

$$\operatorname{Res}_P^G \operatorname{Ind}_H^G(k) \cong \bigoplus_{x \in P\backslash G/H} \operatorname{Ind}_{P\cap {}^xH}^P \operatorname{Res}_{P\cap {}^xH}^{{}^xH}({}^xk).$$

Then M can only be a direct summand of $\operatorname{Ind}_{P\cap {}^xH}^P \operatorname{Res}_{P\cap {}^xH}^{{}^xH}({}^xk)$ for which we have $P \cap {}^xH = P$. Finally, for any $x \in G$ $\operatorname{Res}_{P\cap {}^xH}^{{}^xH}({}^xk) \cong k$, so the result follows. □

Remark 3.0.17. *We call any module which satisfies one of the above two equivalent conditions a p-permutation module.*

Proposition 3.0.18. *Every one-dimensional kG-module M has vertex-source pair (S, k) where S is a Sylow p-subgroup of G.*

Proof. If M is a one-dimensional kG-module with generator m, the formula

$$gm = \phi(g)m$$

where $\phi(g) \in k^*$ defines a homomorphism $\phi \in \operatorname{Hom}(G, k^*)$. Since the map ϕ determines M, we'll denote M by k_ϕ and the generator of k_ϕ by 1. In particular, this shows that the only one-dimensional module for p-groups is the trivial module, since the field k has no p-th roots of unity.
Consider the map $\iota : k_\phi \to \operatorname{Ind}_S^G(k)$ defined by

$$\iota(1) = \sum_{x \in G/S} \phi(x^{-1})x \otimes 1.$$

The map ι is G-invariant since for $g \in G$

$$g\iota(1) = \sum_{x \in G/S} \phi(x^{-1}) gx \otimes 1 = \sum_{x' \in G/S} \phi(x^{-1}) x' \otimes 1 = \sum_{x' \in G/S} \phi(x^{-1}x')\phi((x')^{-1})x' \otimes 1$$
$$= \sum_{x' \in G/S} \phi(g)\phi((x')^{-1})x' \otimes 1 = \iota(g1)$$

where $gx = x's$ with $s \in S$ and $\phi(g) = \phi(x'x^{-1}) = \phi(x^{-1}x')$. Now consider the map $\pi : \mathrm{Ind}_S^G(k) \to k_\phi$ defined by:

$$\pi(\sum_{x \in G/S} \alpha_x x \otimes 1) = \frac{\sum_{x \in G/S} \alpha_x \phi(x))}{[G:S]}$$

which is G-invariant since for $g \in G$:

$$\pi(g(\sum_{x \in G/S} \alpha_x x \otimes 1)) = \pi((\sum_{x' \in G/S} \alpha_x x' \otimes 1)) = \frac{\sum_{x' \in G/S} \alpha_x \phi(x')}{[G:S]} = \frac{\sum_{x \in G/S} \alpha_x \phi(g)\phi(x))}{[G:S]}$$
$$= g\pi(\sum_{x \in G/S} \alpha_x x \otimes 1),$$

where $gx = x's$ for some $s \in S$. Note that the map π is not defined if S is replaced by a non-Sylow p-subroup of G. As $\pi \circ \iota = id_{k_\phi}$, we see that k_ϕ is a direct summand of $\mathrm{Ind}_S^G(k)$, so k_ϕ is a p-permutation module. So the vertex source pair of k_ϕ should be (P, k) for some P. However, one can easily see that $\mathrm{Ind}_P^G(k)$ has only one choice for a submodule isomorphic to k_ϕ, which is essentially the same as the one considered above for $P = S$ and the embedding of this submodule has only one candidate for a splitting, which is essentially the same as the map π considered above, which is not possible to define unless $P = S$. □

One important property of one-dimensional modules is the following, which says that they are "invertible" in the usual category of kG-modules. It is clear by dimension considerations that they are the only class of modules with this property.

Proposition 3.0.19. *If M is a one-dimensional kG-module, then $M \otimes M^* \cong k$.*

Proof. First, it is clear that M^* is also one-dimensional and if the action of g on M is multiplication by α, then its action on M^* is by α^{-1}, by Remark 2.0.14. So the isomorphism is defined by taking any generator of $M \otimes M^*$ to 1. $\square$

4. THE CATEGORY OF $\mathcal{S}$-STABLE MODULES

In Chapter 2, we have only considered the trace maps for G-algebras. However, it is possible to define them for arbitrary modules, in particular for $\mathrm{Hom}_k(M,N)$. The analogous results of Proposition 2.0.3 hold in this case, with similar proofs. We'll define a category using these trace maps. To this end, let $\mathcal{S}$ be a set of subgroups of S. For $P \in \mathcal{S}$ and $M, N \in mod(kG)$, define $\mathrm{PHom}_{kG}(M,N)$ to be the set of all homomorphisms which factor through a P-projective module.

Proposition 4.0.1. $\mathrm{PHom}_{kG}(M,N) = tr_P^G(\mathrm{Hom}_{kP}(M,N))$.

Proof. Suppose $\phi : M \to N$, $\phi_1 : M \to L$ and $\phi_2 : L \to N$ with $\phi = \phi_2 \circ \phi_1$ and L being P-projective. Then $id_L = tr_P^G(\psi)$ for some $\psi \in \mathrm{End}_{kP}(L)$ by Higman's criterion. Thus,

$$\phi = \phi_2 \circ \phi_1 = \phi_2 \circ id_L \circ \phi_1 = \phi_2 \circ (\sum_{g \in G/P} {}^g\psi) \circ \phi_1 = \sum_{g \in G/P} \phi_2 \circ {}^g\psi \circ \phi_1$$
$$= \sum_{g \in G/P} {}^g(\phi_2 \circ \psi \circ \phi_1) = tr_P^G(\phi_2 \circ \psi \circ \phi_1),$$

which shows that $\mathrm{PHom}_{kG}(M,N) \subset tr_P^G(\mathrm{Hom}_{kP}(M,N)$. Next, suppose $\phi = tr_P^G(\psi)$ for some $\psi \in \mathrm{Hom}_{kP}(M,N)$. Set

$$L = \mathrm{Ind}_P^G \mathrm{Res}_P^G(M) = \bigoplus_{g \in G/P} (g \otimes M),$$

which is P-projective. Define $\phi_1 : M \to L$ by

$$\phi_1(m) = \sum_{g \in G/P} g \otimes g^{-1}m.$$

The map ϕ_1 is G-invariant. This follows since letting $x \in G$, we have

$$x\phi_1(m) = x(\sum_{g_i \in G/P} g_i \otimes g_i^{-1}m) = \sum_{g_i \in G/P} xg_i \otimes g_i^{-1}m = \sum_{g_i \in G/P} g_j p_{ij} \otimes g_i^{-1}m$$
$$= \sum_{g_i \in G/P} g_j \otimes p_{ij} g_i^{-1}m = \sum_{g_j \in G/P} g_j \otimes g_j^{-1}xm = \phi_1(xm),$$

where we set $xg_i = g_j p_{ij}$ with $p_{ij} \in P$.
Define also $\phi_2 : L \to N$ by

$$\phi_2(\sum_{g \in G/P} g \otimes m) = \sum_{g \in G/P} g\psi(m),$$

which is G-invariant since for $x \in G$, we have

$$x\phi_2(\sum_{g_i \in G/P} g_i \otimes m_i) = x(\sum_{g_i \in G/P} g_i\psi(m_i)) = \sum_{g_i \in G/P} xg_i\psi(m_i) = \sum_{g_j \in G/P} g_j p_{ij}\psi(m_i)$$
$$= \sum_{g_j \in G/P} g_j\psi(p_{ij}m_i) = \phi_2(\sum_{g_j \in G/P} g_j \otimes p_{ij}m_i) = \phi_2(\sum_{g_j \in G/P} g_j p_{ij} \otimes m_i)$$
$$= \phi_2(\sum_{g_i \in G/P} xg_i \otimes m_i) = \phi_2(x(\sum_{g_i \in G/P} g_i \otimes m_i)).$$

Lastly, we see that $\phi = \phi_2 \circ \phi_1$ since

$$\phi_2(\phi_1(m)) = \phi_2(\sum_{g \in G/P} g \otimes g^{-1}m) = \sum_{g \in G/P} g\psi(g^{-1}m) = \sum_{g \in G/P} {}^g\psi(m) = \phi(m).$$

□

The above propositon makes it possible to define a category, called the $\mathcal{S}$-stable category of G, denoted by $stab_{\mathcal{S}}(G)$, where we declare the objects to be kG-modules and morphisms, called $\mathcal{S}'$-homomorphisms, are given by

$$\operatorname{Hom}_{\mathcal{S}'}(M, N) := \operatorname{Hom}_{kG}(M, N)/\mathcal{S}\operatorname{Hom}_{kG}(M, N).$$

We set

$$\mathcal{S}\,\mathrm{Hom}_{kG}(M,N) = \sum_{P\in\mathcal{S}} \mathrm{PHom}_{kG}(M,N).$$

Remark 4.0.2. *If $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ with respect to taking subgroups and conjugates, we have $\overline{\mathcal{S}}\,\mathrm{Hom}_{kG}(M,N) = \mathcal{S}\,\mathrm{Hom}_{kG}(M,N)$ so we always assume that $\overline{\mathcal{S}} = \mathcal{S}$.*

Before describing the isomorphism classes of objects in $stab_{\mathcal{S}}(G)$, we make some observations. Write $M \cong_{\mathcal{S}'} N$ if M and N are isomorphic in $stab_{\mathcal{S}}(G)$. First, note that an $\mathcal{S}$-projective module M becomes isomorphic to zero in $stab_{\mathcal{S}}(G)$. Also if M is indecomposable with vertex $P \notin \mathcal{S}$ then $M \ncong_{\mathcal{S}'} 0$ since if $id_M \in \mathcal{S}\,\mathrm{End}_{kG}(M)$ then $id_M \in P\,\mathrm{End}_{kG}(M)$ for some $P \in \mathcal{S}$ by Rosenberg's Lemma, which is not possible by Higman's criterion. In particular, we see that $stab_{\mathcal{S}}(G)$ is trivial if and only if $S \in \mathcal{S}$. Finally note that, there is a decomposition $M = M_{\mathcal{S}'} \oplus M_{\mathcal{S}}$ where $M_{\mathcal{S}}$ is $\mathcal{S}$-projective and $M_{\mathcal{S}'}$ has no $\mathcal{S}$-projective summands. With this notation, there is the following result:

Proposition 4.0.3. *$M \cong_{\mathcal{S}'} N$ if and only if $M_{\mathcal{S}'} \cong N_{\mathcal{S}'}$.*

Proof. First, suppose that M is indecomposable with vertex $P \notin \mathcal{S}$ and there are maps $\phi : M \to N$ and $\psi : N \to M$ such that $\psi \circ \phi$ is the identity in $stab_{\mathcal{S}}(G)$. Then

$$(id_M - \psi \circ \phi) \in \mathcal{S}\,\mathrm{End}_{kG}(M).$$

Since $\mathrm{End}_{kG}(M)$ is local and $\mathcal{S}\,\mathrm{End}_{kG}(M) \neq \mathrm{End}_{kG}(M)$, $(id_M - \psi \circ \phi)$ cannot be invertible and hence $\psi \circ \phi$ is invertible. Setting, $\theta = \psi \circ \phi$, $\theta^{-1} \circ \psi$ gives a splitting of ϕ showing that M is a direct summand of N.
Next, suppose $M \cong_{\mathcal{S}'} N$ with isomorphism given by $\phi : M \to N$ and $\psi : N \to M$. Since $M \cong_{\mathcal{S}'} M_{\mathcal{S}'}$ and $N \cong_{\mathcal{S}'} N_{\mathcal{S}'}$, we assume $M = M_{\mathcal{S}'}$ and $N = N_{\mathcal{S}'}$. Let M' be an indecomposable direct summand of M.

Let $\pi : M \to M'$ be the canonical projection and $\iota : M' \to M$ be the canonical embedding. Then $\phi \circ \iota : M' \to N$ and $\pi \circ \psi : N \to M'$ satisfies the assumptions of the above paragraph so M' is also a direct summand of N. Conversely, we see that any direct summand of N is a direct summand of M, thus $M \cong N$ by the Krull-Schmidt property. □

Remark 4.0.4. *Thus we can think of isomorphism classes of objects of $stab_{\mathcal{S}}(G)$ as direct sum of indecomposable modules with vertex lying outside $\mathcal{S}$. Denote by $[M]$ the isomorphism class of M in $stab_{\mathcal{S}}(G)$.*

There is a tensor product in $stab_{\mathcal{S}}(G)$ coming from $mod(kG)$, which is well defined on isomorphism classes in $stab_{\mathcal{S}}(G)$, since tensoring $\mathcal{S}$-projective modules produces an $\mathcal{S}$-projective module.

Definition 4.0.5. *Call a kG-module M relative endo-$\mathcal{S}$-trivial, or just endo-$\mathcal{S}$-trivial for short, if $[M \otimes M^*] = [k]$. Denote by $T_{\mathcal{S}}(G)$ the (abelian) group of isomorphism classes of endo-$\mathcal{S}$-trivial modules with the group operation given by the tensor product, the identity element $[k]$ and inverses given by taking duals.*

Proposition 4.0.6. *If M is endo-$\mathcal{S}$-trivial then $M_{\mathcal{S}'}$ is indecomposable with vertex S.*

Proof. Assume $M = M_{\mathcal{S}'}$. Set $M = M_1 \oplus M_2$. Then both $M_1 \otimes M_1^*$ and $M_2 \otimes M_2^*$ are direct summands of $(M \otimes M^*)_{\mathcal{S}'}$. But $(M \otimes M^*)_{\mathcal{S}'} = k$ so M is indecomposable.
Next, if M is P-projective, then $M \otimes M^*$ and consequently k is P-projective. But we know that k has vertex S. Thus M also has vertex S. □

From now on, we always take an indecomposable endo-$\mathcal{S}$-trivial module as a representative of an element in $T_{\mathcal{S}}(G)$.

Remark 4.0.7. *If $\mathcal{S}_1 \subset \mathcal{S}_2$, we have an embedding $T_{\mathcal{S}_1}(G) \leq T_{\mathcal{S}_2}(G)$ since an indecomposable module M with vertex S satisfying $M \otimes M^* \cong_{\mathcal{S}_1'} k$ also satisfies $M \otimes M^* \cong_{\mathcal{S}_2'} k$. Furhtermore, non-isomorphic indecomposable modules with vertex S remain non-isomorphic in both $stab_{\mathcal{S}_1}(G)$ and $stab_{\mathcal{S}_2}(G)$.*

Thus, we have a minimal choice for $\mathcal{S}$, namely $\mathcal{S} = \{1\}$. In this case, endo-$\mathcal{S}$-trivial modules are just the usual endo-trivial modules. We also have a maximal choice for $\mathcal{S}$, which is $\mathcal{S} = \{P : P < S\}$. We write $T_{\mathcal{S}}(G) = T_{<S}(G)$ in this case.

Definition 4.0.8. *Let $D(G)$ denote the subgroup of $T_{<S}(G)$ generated by isomorphism classes of modules such that $M \otimes M^*$ is a p-permutation module. We call $D(G)$ the generalized Dade group of G.*

Remark 4.0.9. *When G is a p-group, $D(G)$ reduces to the cap group considered in [1], which is usually called the Dade group in the literature.*

In [11], Puig gave a description of the Dade group in terms of algebras, which turned out to be crucial for the biset functors constructed using the Dade groups. For details of this construction, refer to Chapter 12 of [3]. We'll now give a similar definition of what we call a $T_{\mathcal{S}}$-algebra, which yields a quotient group of $T_{\mathcal{S}}(G)$. To this end, first observe that the class of one-dimensional modules generate a subgroup of $T_{\mathcal{S}}(G)$, which we denote by $X(G)$.

Definition 4.0.10. *A k-algebra A is called a $T_{\mathcal{S}}$-algebra if A satisfies the three conditions below:*
(i) $A \cong M_n(k)$ for some natural number n.
(ii) A has a primitive interior G-algebra structure.
(iii) $A \cong_{\mathcal{S}'} k$.

Remark 4.0.11. *If G is a p-group, any G-algebra is an interior G-algebra, so we can relax condition (ii) in the above definition. For this result, we refer to Theorem 12.3.6 of [3].*

We'll now define an operation on $T_{\mathcal{S}}$-algebras. Given two $T_{\mathcal{S}}$-algebras A_1, A_2, consider the usual tensor product of algebras, $A_1 \otimes A_2$. If $A_1 \cong M_{n_1}(k)$ and $A_2 \cong M_{n_2}(k)$, then

$$A_1 \otimes A_2 \cong M_{n_1}(k) \otimes M_{n_2}(k) \cong M_{n_1 n_2}(k).$$

Clearly, we also have $A_1 \otimes A_2 \cong_{\mathcal{S}'} k$. However, even though $A_1 \otimes A_2$ has an interior G-algebra structure, it is not primitive. To rectify this, set $A_1 \otimes A_2 = A$ and let $1 = e_1 + e_2 + ... + e_n$ be a pritimive idempotent decomposition in A^G. There's a decomposition

$$A = \bigoplus_{i,j} e_i A e_j$$

which is G-invariant and since $A \cong_{\mathcal{S}'} k$, there are unique i and j such that k is a G-invariant direct summand of $e_i A e_j$. But then k is a G-invariant direct summand of $e_j A e_i$ so $i = j$ and we set $e = e_i = e_j$. Then eAe inherits an interior G-algebra structure from A and is primitive by construction. Letting $e = f_1 + f_2 + ... + f_m$ be a primitive idempotent decomposition in A, we see that $eAe \cong M_m(k)$ since A is a full matrix algebra over k. Thus eAe is a $T_{\mathcal{S}}$-algebra. The isomorphism class of eAe depends only on the isomorphism classes of A_1 and A_2 since any fixed isomorphism takes a primitive idempotent decomposition of the identity to a primitive idempotent decomposition of the identity. Thus this defines an operation on the set of isomorphism classes of $T_{\mathcal{S}}$-algebras, denoted by $\widetilde{T_{\mathcal{S}}}(G)$. This operation is associative. We can see this as follows. Write $[A]$ for the isomorphism class of A. Let A_1, A_2 and A_3 be $T_{\mathcal{S}}$-algebras. Then carrying out the above procedure for $([A_1]+[A_2])+[A_3]$ and $[A_1]+([A_2]+[A_3])$, we produce idempotents of $(A_1 \otimes A_2) \otimes A_3$ and $A_1 \otimes (A_2 \otimes A_3)$, respectively. Identifying $(A_1 \otimes A_2) \otimes A_3 \cong A_1 \otimes (A_2 \otimes A_3)$, we see that $A_1 \otimes A_2 \otimes A_3$ has only one idempotent e such that the trivial module is a direct summand of $e(A_1 \otimes A_2 \otimes A_3)e$, which gives the same result as the idempotents produced beforehand. We write $T_{<S}$-algebra for $T_{\mathcal{S}}$-algebra with $\mathcal{S} = \{P : P < S\}$.

Proposition 4.0.12. *$\widetilde{T_{\mathcal{S}}}(G)$ is a group isomorphic to $T_{\mathcal{S}}(G)/X(G)$.*

Proof. Given $[M] \in T_{\mathcal{S}}(G)$, consider $\mathrm{End}_k(M)$. Clearly, we have $\mathrm{End}_k(M) \cong M_n(k)$ with $n = dim(M)$ and

$$\mathrm{End}_k(M) \cong_{\mathcal{S}'} M \otimes M^* \cong_{\mathcal{S}'} k.$$

Also $\mathrm{End}_k(M)$ is a primitive interior G-algebra since M is indecomposable. Thus $\mathrm{End}_k(M)$ is a $T_{\mathcal{S}}$-algebra. So define $\theta : T_{\mathcal{S}}(G) \to \widetilde{T_{\mathcal{S}}}(G)$ by

$$\theta([M]) = [\mathrm{End}_k(M)].$$

Given a $T_{\mathcal{S}}$-algebra A, let M be a simple module of A ,unique up to isomorphism, which is possible since A is a full matrix algebra. The action of G on A defines an action of G on M through multiplication and the isomorphism of k-algebras $A \cong \mathrm{End}_k(M)$ becomes a G-algebra isomorphism since the action of G on A is given by conjugation with invertible elements of A and passing through the isomorphism, we see that the action of G on $\mathrm{End}_k(M)$ is precisely the same. Also M is indecomposable as a kG-module since A is primitive. So θ is surjective. And $\mathrm{End}_k(M) \cong k$ holds only if M is a one-dimensional module so if we show that θ preserves the product, the result follows. Now, let $[M], [N] \in T_{\mathcal{S}}(G)$ and $M \otimes N \cong_{\mathcal{S}'} L$, where L is indecomposable with vertex S. Let e be the idempotent in $\mathrm{End}_k(M \otimes N)$ associated to L. Identify the isomorphic algebras

$$\mathrm{End}_k(M \otimes N) \cong \mathrm{End}_k(M) \otimes \mathrm{End}_k(N).$$

Since $e\,\mathrm{End}_k(M \otimes N)e \cong \mathrm{End}_k(L)$ and $\mathrm{End}_k(L)$ has k as a direct summand we see that

$$\begin{aligned}\theta([M]) + \theta([N]) = [\mathrm{End}_k(M)] + [\mathrm{End}_k(N)] = [e\,\mathrm{End}_k(M \otimes N)e] = [\mathrm{End}_k(L)] = \theta([L]) \\ = \theta([M \otimes N]) = \theta([M] + [N]).\end{aligned}$$

Finally, since $\mathrm{End}_k(M^*) \cong \mathrm{End}_k(M)^{op}$, we see that the inverses in $\widetilde{T_{\mathcal{S}}}(G)$ are given by taking opposites. □

Definition 4.0.13. *A $T_{<S}$-algebra A is called a D-algebra if it is also a p-permutation module over G. The subgroup of $\widetilde{T_{<S}}(G)$ generated by D-algebras is denoted by $\widetilde{D}(G)$.*

Remark 4.0.14. *The map θ in the proof of Proposition 4.0.12 takes $D(G)$ onto $\widetilde{D}(G)$, so $\widetilde{D}(G) \cong D(G)/X(G)$.*

Remark 4.0.15. *If G is a p-group, then the notion of a D-algebra we consider here reduces to the notion of a D-algebra considered in [11].*

Finally, we'll give a description of the Jacobson radical of A^G for a $T_{\mathcal{S}}$-algebra. The proof is mimicked from the proof of Proposition 4.4 in [1]. Set

$$J_A = \sum_{P \in S} tr_P^G(A^P).$$

Proposition 4.0.16. *The Jacobson radical of A^G, denoted $J(A^G)$, is equal to J_A.*

Proof. Let $A[P]$ denote the sum of indecomposable summands of A with vertex P. Then, we have a decomposition

$$A = (\bigoplus_{P \in S} A[P]) \oplus A[S],$$

where $A[S]$ is the trivial module. Thus we have a decomposition of A^G

$$A^G = (\bigoplus_{P \in S} A[P]^G) \oplus A[S].$$

. Set $J_A[Q] = J_A \cap A[Q]^G$, so that

$$J_A = (\bigoplus_{P \in S} J_A[P]) \oplus J_A[S].$$

Now, we have

$$J_A[Q] = \sum_{P \in S} tr_P^G(A[Q]^P).$$

If $Q = S$, since for $P \in S$ the index $[G : P]$ is divisible by p, we get

$$J_A(S) = \sum_{P \in S} tr_P^G(A[S]) = 0.$$

If $Q \in \mathcal{S}$, there's a kQ-module M with

$$A[Q] \quad | \quad \mathrm{Ind}_Q^G(M)$$

so that

$$A[Q]^G \quad | \quad (\mathrm{Ind}_Q^G(M))^G$$

Since given $g_i, g_j \in G$ and $q \in Q$, there a solution to $xg_i = g_j q$ with $x \in G$, we get

$$(\mathrm{Ind}_Q^G(M))^G = (\bigoplus_{g \in G/Q} g \otimes M)^G = tr_Q^G(M^Q).$$

Thus $A[Q]^G \subset tr_Q^G(A[Q]^Q) \subset J_A[Q] \subset A[Q]^G$, which shows that

$$J_A = \bigoplus_{P \in S} A[P]^G.$$

Since J_A is an ideal with codimension one in A^G, which is a local algebra, we have $J_A = J(A^G)$. $\square$

5. CONCLUSION

We have developed the necessary mathematical tools which are needed to define the category of relatively stable modules for finite groups. Then, we have defined the notion of a relative endo-trivial module, giving a generalization of the group of endo-trivial modules which then makes a generalization of the Dade group for arbitrary finite groups possible. Our original contribution is the description of the group of relative endo-trivial modules in terms of certain algebras and the computation of the Jacobson radical of such algebras.

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