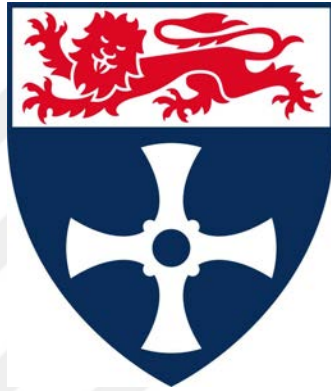


# A numerical investigation on laminar natural convection of non-Newtonian fluids in enclosed spaces



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This dissertation is submitted for the degree of

*Doctor of Philosophy*

September, 2018



To my mother and father...





## Declaration

I hereby declare that this thesis is my own work and that I have correctly acknowledge the work of others. This submission is in accordance with Newcastle University and School of Engineering guidance on good academic conduct. I certify that no part of the material offered has been previously submitted by me for a degree or other qualification in this or any other University. I confirm that the word length is within the prescribed range as advised by my school and faculty. I confirm that this thesis does not contain collaborative work. This thesis contains approximately 37000 words, 86 figures and 31 tables.

Sahin Yigit  
September, 2018



## Acknowledgements

Firstly, I would like to express my sincere thanks to my supervisor, Prof. Nilanjan Chakraborty, for his invaluable support, guidance and inspiration throughout this PhD. Also, I am grateful to Newcastle University for the Overseas Research Scholarship and the School of Engineering for the financial support.

I would like to convey my sincere appreciations to all my colleagues, Dr. Dipal Patel, Dr. Jiawei Lai, Dr. Bruno Machado, Sam Wood, Chris Stafford, Bill Papapostolou, and Gulcan Ozel for their pleasant company in the fluid group laboratory and numerous discussions.

I am also grateful to Dr. Osman Turan, Dr. Josef Hasslberger, Dr. Charles Turquand d'Auzay, Dr. Umair Ahmed for their encouragement and help during this PhD.

Finally, I would like to thank my family for great encouragement and support not only for this PhD but also throughout my life.



# Abstract

Natural convection in enclosed spaces has several applications such as electronic cooling, thermal energy storage systems, solar collectors and heating and preservation of food to name a few. Although natural convection of Newtonian fluids (i.e. fluids like water, air where viscous stress is directly proportional to strain rate) in enclosures has been analysed extensively in the existing literature, relatively limited effort has been directed to the natural convection of non-Newtonian fluids where the strain rate dependence of shear stresses is non-linear in nature. For example, yield stress fluid is a special type of non-Newtonian fluid, which acts as a solid and does not flow until a threshold stress is surpassed. Materials such as mud-slurries in oil drilling, molten chocolate and anti-drip paints are common examples of yield stress fluids. It is possible to modulate the yield stress based on electrical/magnetic field in electro-rheological/magneto-rheological fluids. Thus, it is possible to eliminate (or alter the strength of) convection by applying a magnetic/electric field, which can be useful for mitigating accidental damage in the case of nuclear meltdown and storage of cryogenic materials. Additionally, shear-thinning (shear-thickening) fluids are another special type of non-Newtonian fluid, which show a decrease (increase) in viscosity with increasing shear rate. Many common man-made and biological fluids exhibit shear-thinning (e.g. ketchup and blood) and shear-thickening (e.g. mixtures of corn starch and water; so-called “bulletproof” custard) behaviour. These types of fluids can also be very useful for designing new adaptive thermal management systems (e.g. cooling of electronics, solar and nuclear power systems, etc.).

Therefore, this thesis focuses on fundamental physical understanding and modelling of steady-state laminar natural convection of non-Newtonian fluids in enclosures using numerical simulations. A detailed parametric analysis has been

carried out to analyse the effects of yield stress and the shear-thinning/thickening nature of viscosity on heat and momentum transport within the rectangular and cylindrical annular enclosures, for both Rayleigh-Bénard and differentially heated side wall configurations. The bi-viscosity Bingham model and power-law model of viscosity have been used to mimic yield stress (i.e. Bingham) and shear-thinning/thickening fluids respectively. Accordingly, the steady-state laminar analyses have been performed to analyse the effects of nominal Rayleigh number  $Ra$ , Prandtl number  $Pr$ , Bingham number  $Bn$ , power-law index  $n$ , normalized internal radius  $r_i/L$  (where  $r_i$  is the internal cylinder radius and  $L$  is the difference between outer and inner radii) and aspect ratio  $AR$  ( $AR = H/L$  where  $H$  is the enclosure height) on the mean Nusselt number, as Buckingham's  $\pi$  theorem dilates  $Nu = f(Ra, Pr, Bn, n, r_i/L, AR)$  for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions.

Detailed scaling analysis has been utilised to explain the effects of all the aforementioned parameters considered here. This scaling analysis, in turn, has been utilised to propose correlations for the mean Nusselt number and these correlations have been validated with respect to numerical findings. These mean Nusselt number correlations have significant practical importance and are likely to be useful for designing and improving thermal systems in many important industrial applications involving both Bingham and power-law fluids.

# List of Publications

The results presented in this thesis (i.e. chapters 4-7) have been published in following journal articles.

## Journal articles

1. S. Yigit, R. J. Poole, N. Chakraborty, Effects of aspect ratio on laminar Rayleigh–Bénard convection of power-law fluids in rectangular enclosures: A numerical investigation, *International Journal of Heat and Mass Transfer* 91 (2015) 1292–1307.
2. S. Yigit, N. Chakraborty, Laminar natural convection of Bingham fluids in square cross-sectioned cylindrical annular cavity with differentially heated vertical walls subjected to constant heat fluxes, *Heat Transfer Engineering* 38 (13) (2017) 1171–1188.
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10. S. Yigit, J. Ford, R. J. Poole, N. Chakraborty, Numerical investigation of boundary condition effects on laminar natural convection of power law fluids in square cross-sectional cylindrical annular space with differentially heated vertical walls, *Computational Thermal Sciences: An International Journal* 7 (3) (2015) 261–282.
11. S. Yigit, N. Chakraborty, Influences of aspect ratio on natural convection of power-law fluids in cylindrical annular space with differentially heated vertical walls, *Thermal Science and Engineering Progress* 2 (2017) 151–164.
12. S. Yigit, N. Chakraborty, Rayleigh–Bénard power-law fluid convection in rectangular enclosures, *Journal of Thermophysics and Heat Transfer* 31 (4) (2017) 805–816.
13. S. Yigit, C. McRoberts, N. Chakraborty, Numerical investigation of laminar Rayleigh–Bénard convection of power-law fluids in square cross-sectional cylindrical annular enclosures, *International Communications in Heat and Mass Transfer* 78 (2016) 112–120.
14. O. Turan, S. Yigit, N. Chakraborty, Critical condition for Rayleigh–Bénard

convection of Bingham fluids in rectangular enclosures, *International Communications in Heat and Mass Transfer* 86 (2017) 117–125.

### **Conference proceedings**

A number of the results have been presented at the following international conferences.

1. S. Yigit, R. J. Poole, N. Chakraborty, “Aspect ratio effects on laminar Rayleigh-Bénard convection of power-law fluids in rectangular enclosures: A numerical investigation.” 6th International Symposium on Advances in Computational Heat Transfer, May 25-29, 2015, Piscataway, USA.
2. S. Yigit, N. Chakraborty, “Numerical investigation of laminar Rayleigh-Bénard convection of Bingham fluids in square cross-sectioned cylindrical enclosures.” 9th International Conference on Computational Heat, Mass and Momentum Transfer, May 23-26, 2016, Cracow, Poland.
3. S. Yigit, N. Chakraborty, “Numerical investigation of laminar Rayleigh-Bénard convection of power-law fluids in square cross-sectional cylindrical annular cavity.” 4th International Conference on Computational Methods for Thermal Problems, July 6-8, 2016, Georgia Tech, Atlanta, USA.
4. S. Yigit, O. Turan, N. Chakraborty, “Natural convection of power-law fluids in rectangular cross-sectional cylindrical annular enclosures with differentially heated vertical walls.” 7th International Symposium on Advances in Computational Heat Transfer, May 28 – June 2, 2017, Napoli, Italy.
5. S. Yigit, N. Chakraborty, “Natural convection of Bingham fluids in rectangular cross-sectional cylindrical annular enclosures with differentially heated vertical walls.” 10th International Conference on Computational Heat, Mass and Momentum Transfer, May 28 – June 1, 2017, Seoul, Korea.
6. S. Yigit, J. Hasslberger, N. Chakraborty and M. Klein, “Three-dimensional high-fidelity simulations of Rayleigh-Bénard convection of yield-stress fluids in cubic enclosures at high Rayleigh numbers.” 11th International Conference on Computational Heat, Mass and Momentum Transfer, May 21-24, 2018, Cracow, Poland.

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# Nomenclature

## Roman Symbols

|                 |                                                                                                                            |
|-----------------|----------------------------------------------------------------------------------------------------------------------------|
| $AR = H/L$      | Aspect ratio, dimensionless                                                                                                |
| $Bn$            | Bingham number, dimensionless                                                                                              |
| $Bn_c$          | Critical Bingham number, dimensionless                                                                                     |
| $Bn_{max}$      | Threshold value of Bingham number above which $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ settles to unity, dimensionless |
| $Bn_{max}^*$    | Scaled threshold value of Bingham number, dimensionless                                                                    |
| $c_p$           | Specific heat at constant pressure, $J/kgK$                                                                                |
| $e_{ij}$        | Rate of strain tensor                                                                                                      |
| $f_1, f_2, f_3$ | Functions, dimensionless                                                                                                   |
| $g$             | Gravitational acceleration, $m/s^2$                                                                                        |
| $Gr$            | Grashof number, dimensionless                                                                                              |
| $H$             | Height of the enclosure, $m$                                                                                               |
| $h$             | Heat transfer coefficient, $W/m^2K$                                                                                        |
| $K$             | Consistency, $kg/ms$                                                                                                       |
| $k$             | Thermal conductivity, $W/mK$                                                                                               |
| $L$             | Difference between inner and outer radius, $m$                                                                             |

|           |                                                                                                          |
|-----------|----------------------------------------------------------------------------------------------------------|
| $m$       | Stress growth exponent, $s$                                                                              |
| $n$       | Power-law index, dimensionless                                                                           |
| $Nu$      | Nusselt number, dimensionless                                                                            |
| $P$       | Pressure, $Pa$                                                                                           |
| $Pr$      | Prandtl number, dimensionless                                                                            |
| $q$       | Heat flux, $W/m^2$                                                                                       |
| $r$       | Radial coordinate, $m$                                                                                   |
| $R^2$     | Coefficient of determination, dimensionless                                                              |
| $r_i$     | Inner radius, $m$                                                                                        |
| $r_o$     | Outer radius, $m$                                                                                        |
| $Ra$      | Rayleigh number, dimensionless                                                                           |
| $T$       | Temperature, $K$                                                                                         |
| $T_C$     | Cold wall temperature, $K$                                                                               |
| $T_H$     | Hot wall temperature, $K$                                                                                |
| $U, W$    | Dimensionless radial ( $U = uL/\alpha$ ), vertical ( $U = wL/\alpha$ ) velocity component, dimensionless |
| $u, w$    | Radial, vertical velocity component, $m/s$                                                               |
| $U_{ref}$ | Reference velocity scale, $m/s$                                                                          |
| $x_{i,j}$ | i,j th velocity components, $m$                                                                          |
| $z$       | Coordinate in vertical direction, $m$                                                                    |

### Abbreviations

AURs      Apparently unyielded regions

|      |                                    |
|------|------------------------------------|
| CWHF | Constant wall heat flux            |
| CWT  | Constant wall temperature          |
| E.F. | Established flow initial condition |
| Q.F. | Quiescent flow initial condition   |

### Greek Symbols

|                                        |                                                                  |
|----------------------------------------|------------------------------------------------------------------|
| $\alpha$                               | Thermal diffusivity, $m^2/s$                                     |
| $\beta$                                | Coefficient of thermal expansion, $1/K$                          |
| $\Delta T$                             | Temperature difference, $K$                                      |
| $\delta$                               | Hydro-dynamic boundary layer thickness for vertical walls, $m$   |
| $\delta_1$                             | Hydro-dynamic boundary layer thickness for horizontal walls, $m$ |
| $\delta_{th}$                          | Thermal boundary-layer thickness, $m$                            |
| $\dot{\gamma}$                         | Second invariant of strain rate tensor, $s^{-1}$                 |
| $\mu$                                  | Dynamic viscosity, $Ns/m^2$                                      |
| $\mu_{yield}$                          | Yield viscosity, $Ns/m^2$                                        |
| $\phi$                                 | Azimuthal coordinate, dimensionless                              |
| $\psi$                                 | Stream function, $m^2/s$                                         |
| $\rho$                                 | Density, $kg/m^3$                                                |
| $\tau$                                 | Second invariant of viscous stress tensor, $Pa$                  |
| $\tau_{ij}$                            | Components of viscous stress tensor, $Pa$                        |
| $\tau_y$                               | Yield stress, $Pa$                                               |
| $\theta$                               | Dimensionless temperature, dimensionless                         |
| $\underline{\underline{\dot{\gamma}}}$ | Strain rate tensor, $s^{-1}$                                     |

|                                  |                                              |
|----------------------------------|----------------------------------------------|
| $\underline{\underline{\tau_y}}$ | Yield stress tensor, $Pa$                    |
| $\underline{\underline{\tau}}$   | Viscous stress tensor, $Pa$                  |
| $\Psi$                           | Dimensionless stream function, dimensionless |

### Subscripts

|        |                                  |
|--------|----------------------------------|
| $a$    | Apparent                         |
| $cen$  | Geometrical centre of the domain |
| $cond$ | Conduction                       |
| $CWHF$ | Constant wall heat flux          |
| $CWT$  | Constant wall temperature        |
| $eff$  | Effective value                  |
| $i$    | Inner periphery wall             |
| $max$  | Maximum value                    |
| $o$    | Outer periphery wall             |
| $Rec$  | Rectangular                      |
| $ref$  | Reference value                  |
| $sq$   | Square                           |

### Superscripts

|     |                                                       |
|-----|-------------------------------------------------------|
| $+$ | Non-dimensional value                                 |
| $H$ | Representative value in the horizontal boundary layer |
| $V$ | Representative value in the vertical boundary layer   |
| $-$ | Mean value                                            |

# Chapter 1

## Introduction

### 1.1 Natural Convection in Enclosures

Natural convection is a heat transport mechanism in which the fluid motion is generated by the buoyancy force arising from the change in the density of the fluid due to the rise in the temperature rather than by any external source (like a pump, fan, suction device, etc.). The laminar boundary layer development on a heated vertical plate (Fig. 1.1) can be taken as a classic example for buoyant flow induced by the temperature difference between a fluid and a surface immersed in it.

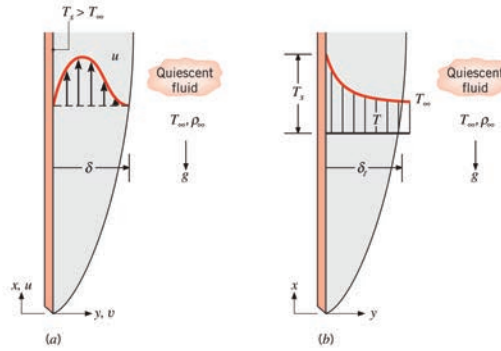


Figure 1.1: Laminar boundary layer development on a heated vertical plate: (a) Velocity boundary layer, (b) Thermal boundary layer [1]

As it can be seen from Fig. 1.1, the thickness of velocity boundary layer increases

in the vertical direction where the fluid velocity increases with increasing normal distance from the surface to reach a maximum value within the boundary layer, then gradually decreases to zero again. Accordingly, the temperature of the fluid in the vicinity of the surface increases due to heat transfer from the plate to the fluid. Consequently, the warmer fluid in the vicinity of the surface moves upwards along the vertical plate under the action of buoyancy forces arising from density variation of the fluid in response to temperature change.

The natural convection can be subdivided into two major categories. The first category deals with buoyant flows induced by temperature differences between a fluid and a surface immersed in it, which is called the external natural convection as shown in Fig. 1.1. The second category deals with buoyant flows which occur inside enclosures confined by surfaces at different temperatures and this category referred to as the internal natural convection. There are two generic configurations for natural convection in enclosures in the existing literature, which are discussed in detail in subsections 1.1.1 and 1.1.2.

### 1.1.1 Differentially Heated Side Walls Configuration

Natural convection in enclosures with differentially heated from side walls is one the most extensively analysed natural convection problem. Classical examples of two dimensional rectangular enclosures are shown schematically in Fig. 1.2 where the vertical walls of rectangular enclosures are kept at different temperature and the other boundaries are considered to be adiabatic.

The fluid flow generated by the buoyancy force inside the enclosure is usually a recirculating vortical motion in the core region, in addition to the boundary layers on top of the no-slip impenetrable walls (see Fig. 1.2). This type of natural convection in rectangular enclosures have two characteristic dimensions  $H$  is enclosure height,  $L$  is enclosure width (i.e. length between hot and cold walls). The enclosure is referred to as shallow when aspect ratio (i.e.  $AR = H/L$ ) is smaller than unity, while it is termed as tall enclosure for  $AR > 1$ .

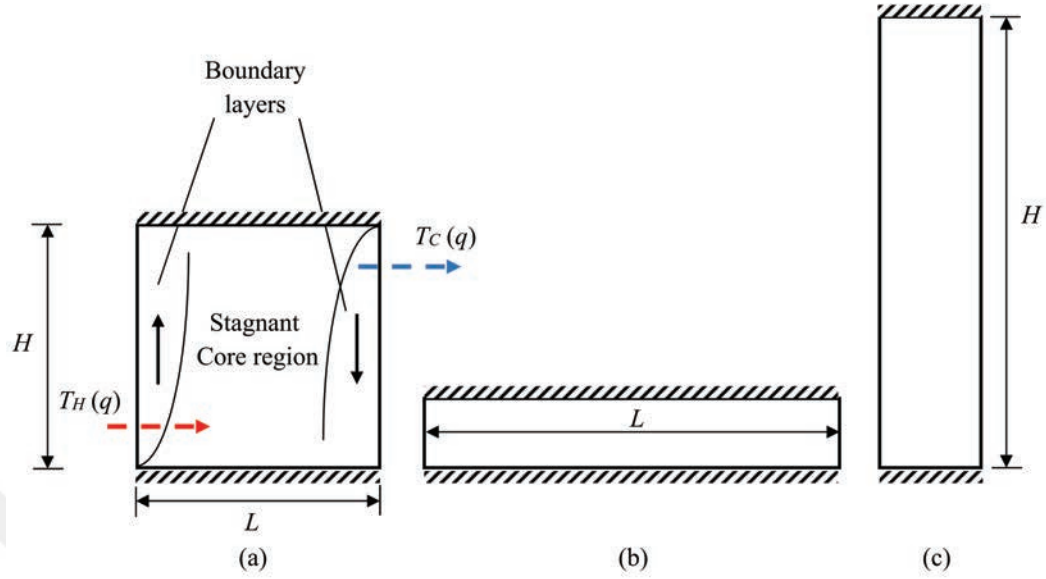


Figure 1.2: Classical examples of two dimensional rectangular enclosures with differentially heated from side walls a) Square enclosure, b) Shallow enclosures, c) Tall enclosures.

In this problem, the main purpose is to find the convective heat transfer coefficient (i.e.  $h$ ) which is a quantitative characteristic of convective heat transfer between a fluid and the active walls of enclosure. The convective heat transfer coefficient is dependent on both fluid and flow parameters such as density ( $\rho$ ), acceleration due to gravity ( $g$ ), volume expansion coefficient ( $\beta$ ), dynamic viscosity ( $\mu$ ), specific heat ( $c_p$ ), thermal conductivity, ( $k$ ), temperature difference between hot and cold walls ( $\Delta T$ ), characteristic length ( $L_c$  is the distance between hot and cold walls) respectively. (i.e.  $h = f(\rho, g, \beta, \mu, c_p, k, \Delta T, L_c)$ ). In heat transfer literature,  $h$  is presented in non-dimensional form as Nusselt number (i.e.  $Nu = hL_c/k$ ) which represents the ratio of convective to conductive heat transfer rate. it is also possible to express the Nusselt number as representative non-dimensional numbers that describe heat and fluid flow in natural convection problems (i.e.  $Nu = f_1(Ra, Pr, AR)$ ) using Buckingham's  $\pi$  theorem.

The  $Ra$  is Rayleigh number which represents the ratio of heat transfer due to

buoyancy forces and thermal diffusion,

$$Ra = \frac{\rho^2 c_p g \beta (\Delta T) L_c^3}{\mu k} = Gr Pr \quad (1.1)$$

where  $Gr$  is Grashof number which represents the ratio of buoyancy forces and viscous forces,

$$Gr = \frac{\rho^2 g \beta (\Delta T) L_c^3}{\mu^2} \quad (1.2)$$

and  $Pr$  is Prandtl number which represents the ratio of momentum diffusion to thermal diffusion.

$$Pr = \frac{\mu C_p}{k} \quad (1.3)$$

For temperature difference between hot and cold walls ( $\Delta T$ ) in Eq. 1.1 and 1.2, constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions are the two most common boundary conditions used in heat transfer literature. The temperature difference between hot and cold walls (i.e.  $\Delta T$ ) can be defined as  $T_H - T_C$  ( $qL_c/k$ ) for CWT (CWHF) boundary condition where  $T_H$ ,  $T_C$  are hot and cold wall temperatures and  $q$  is the constant wall heat flux as shown in Fig. 1.2. The CWT boundary condition can be implemented either by a thermostat or by condensing steam. By contrast, CWHF boundary condition can be obtained by electric coil heating in heat transfer experiments.

The regimes of natural convection in enclosures with differentially heated from the side wall configuration have been discussed in detail based on scaling analysis on thermal boundary layer thickness (i.e.  $\delta_{th} \sim H Ra_H^{-1/4}$ ) by Bejan [2]. It can be seen from Fig. 1.3 that flow characteristics within the enclosure depend strongly on the enclosure aspect ratio (i.e.  $AR = H/L$ ) and the Rayleigh number.

It is worth noting that the Rayleigh number  $Ra_L$  is generally defined based on the horizontal dimension  $L$  (i.e. length between hot and cold walls). Also from the definition of the Rayleigh number (see Eq. 1.1), the Rayleigh number  $Ra_H$  based on vertical dimension  $H$ , can be defined as  $Ra_H = Ra_L AR^3$ .

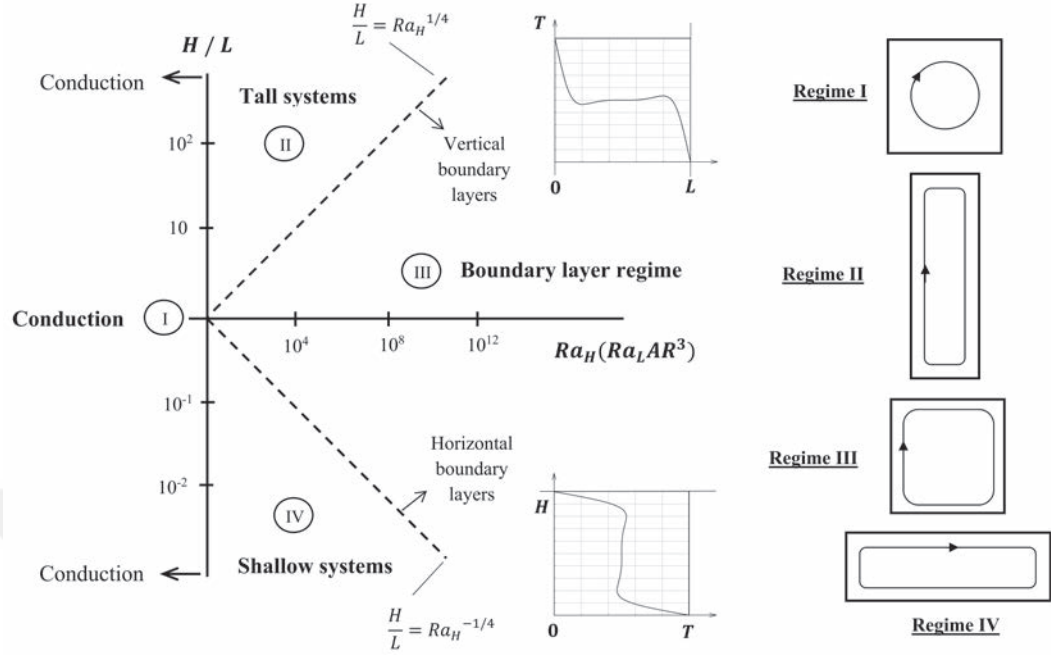


Figure 1.3: Four heat transfer regime with temperature profiles and circulation patterns for natural convection in an enclosure with differentially heated from the sidewalls [2].

The following criteria for distinct convective vertical and horizontal boundary layers are given by transport regimes defined in the Eq. (1.4) and Eq. (1.5) respectively

$$\frac{H}{L} < Ra_H^{1/4} = (Ra_L AR^3)^{1/4} \quad (1.4)$$

$$\frac{H}{L} > Ra_H^{-1/4} = (Ra_L AR^3)^{-1/4} \quad (1.5)$$

and the convective heat transfer requirements ( $Ra_H > 1$ ) divide the  $H/L$  and  $Ra_H$  phases into four sections [2] as shown in Fig. 1.3.

- **Regime I (Conduction limit:  $Ra_H < 1$ ):** The temperature varies linearly across the enclosure. The horizontal temperature gradient gives rise to a slow circulation; however, the heat transfer contribution of this flow is insignificant.

- **Regime II (Tall enclosure limit:  $H/L < Ra_H^{1/4}$ ):** For most of the enclosure height, the temperature varies linearly between the two sidewalls. The circulation pattern is characterized by horizontal distinct layers in the vicinity of the top and bottom walls.
- **Regime III (Boundary layer regime: large  $Ra_H$ ):** Vertical and horizontal thermal boundary layers form distinctly along the differentially heated sidewalls and adiabatic horizontal walls. The distribution of temperature between sidewalls remains non-linear.
- **Regime IV (Shallow enclosure limit:  $H/L > Ra_H^{-1/4}$ ):** The heat transfer mechanism is dominated by the presence of the vertical thermal boundary layers. An additional insulating effect is provided by the long horizontal core of the enclosure.

### 1.1.2 Differentially Heated Horizontal Walls Configuration

The second most extensively studied natural convection in enclosures problem is commonly as Rayleigh-Bénard convection configuration (i.e. natural convection in enclosure with differentially heated horizontal walls). Figure 1.4 is used here to explain differentially heated horizontal walls configuration in which the fluid is enclosed by two large, horizontal plates of different temperature ( $T_1 \neq T_2$ ). In case a, temperature of the lower plate exceeds that of the upper plate ( $T_1 < T_2$ ), the density decreases in the direction of gravitational force. If the temperature difference exceeds a critical value, the fluid density distribution becomes unstable and buoyancy force overcomes viscous forces to initiate the flow. As a result, the denser fluid descend under the action of gravity, while lighter fluid rises and gets colder as it moves. However, this condition does not hold for in case b, for which  $T_1 > T_2$  and the density no longer decreases in the direction of the gravitational force. Thus, lighter fluid sits on top of heavier fluid and represent a stable stratified condition and there is no bulk fluid motion. In case a, heat transfer occurs from the bottom to the top surface by natural convection whereas

for case b, heat transfer from the top to the bottom occurs purely due to thermal conduction.

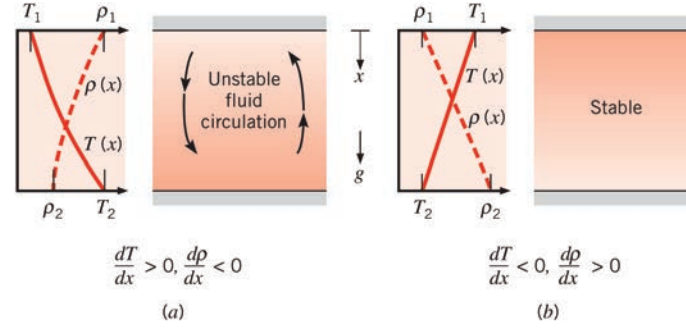


Figure 1.4: Conditions in a fluid between large horizontal plates at different temperatures: (a) Unstable temperature gradient, (b) Stable temperature gradient [1].

The configuration a, in Fig. 1.4 is called as Rayleigh-Bénard convection in the heat transfer literature. In Rayleigh-Bénard convection, fluid flow starts to influence convective heat transfer within the enclosures as temperature contours of Rayleigh-Bénard convection in horizontal fluid layer can be seen in Fig. 1.5.

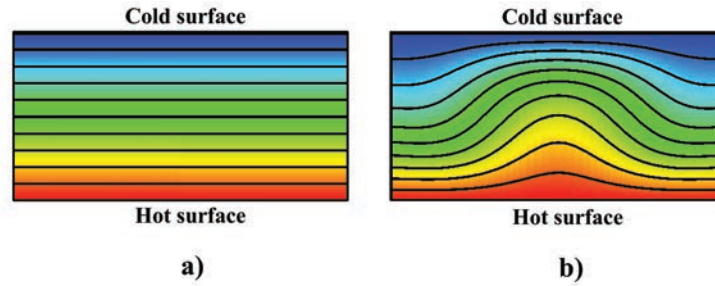


Figure 1.5: Temperature contours of natural convection in horizontal fluid layer differentially heated from below for a)  $Ra < 1708$ , b)  $Ra > 1708$ .

The configurations discussed in section 1.1.1 and 1.1.2 have been studied extensively in existing literature as the convective heat transfer in enclosures is one of the most encountered phenomena in industrial practice such as cooling of electronics [3], thermal energy storage systems [4], solar collectors [5], preservation of

canned foods [6]. However, the majority of the analyses have been restricted to Newtonian fluids (i.e. fluids where viscous stress is directly proportional to strain rate), whereas limited effort has been directed to the natural convection of non-Newtonian fluids where the strain rate dependence of shear stresses is non-linear in nature (see Fig. 1.6).

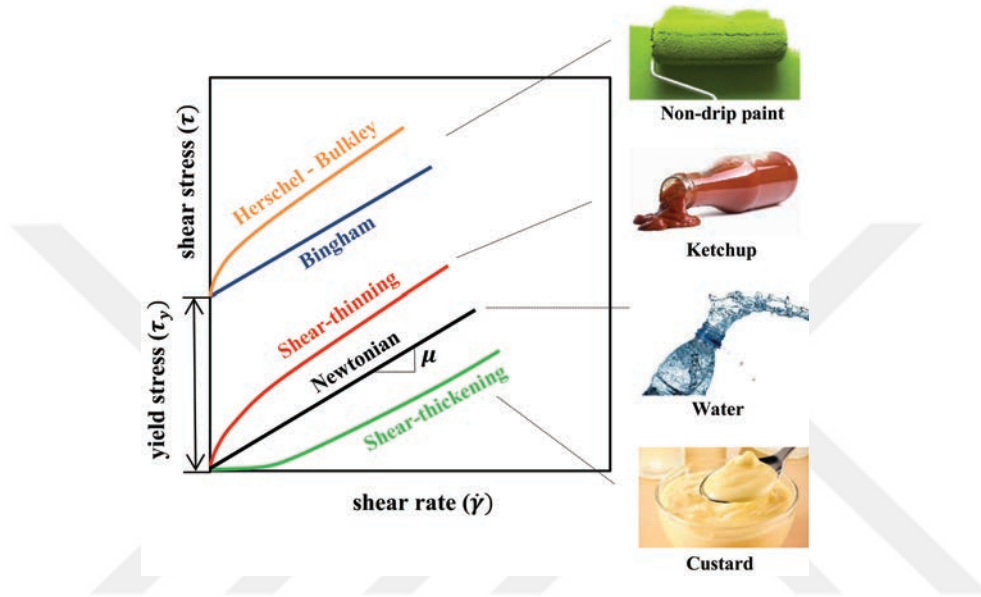


Figure 1.6: Types of fluid behaviour.

## 1.2 Motivation

It is logical that most previous work has concentrated on such Newtonian fluids due to most abundant fluids on the planet like water and air are Newtonian in character. However, virtually all synthetic, food-based, and biological fluids (see Table 1.1) are non-Newtonian in character and these fluids have been utilized for numerous purposes in daily life and industry for many applications. They can be used for designing new thermal management systems or improving existing ones for many applications. Thus, the knowledge of natural convection in more rheologically complex fluids than water or air is necessary from an engineering perspective.

The presence of a small amount of polymers in water-based solvents often ex-

Table 1.1: Examples of some substances exhibiting non-Newtonian Fluid behaviour [7].

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <ul style="list-style-type: none"> <li>• Adhesives (wall paper paste, carpet adhesive)</li> <li>• Animal waste slurries from cattle farms</li> <li>• Biological fluids (blood, synovial fluid, saliva)</li> <li>• Bitumen</li> <li>• Cement paste and slurries</li> <li>• Chalk slurries</li> <li>• Chocolates</li> <li>• Coal slurries</li> <li>• Cosmetics and personal care products (nail polish, lotions and creams, lipsticks, shampoos)</li> <li>• Dairy products and dairy waste streams (cheese, butter, yogurt, fresh cream, whey)</li> <li>• Drilling muds</li> <li>• Firefighting foams</li> <li>• Food stuffs (Fruit/vegetable purees and concentrates, sauces, mayonnaise etc.)</li> </ul> | <ul style="list-style-type: none"> <li>• Greases and lubricating oils</li> <li>• Mine tailings and mineral suspensions</li> <li>• Molten lava and magmas</li> <li>• Paints, polishes and varnishes</li> <li>• Paper pulp suspensions</li> <li>• Peat and lignite slurries</li> <li>• Polymer melts and solutions, reinforced plastics, rubber</li> <li>• Printing colours and inks</li> <li>• Pharmaceutical products (creams, foams, suspensions)</li> <li>• Sewage sludge</li> <li>• Wet beach sand</li> <li>• Waxy crude oils</li> </ul> |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

hibits shear thinning (e.g. fluids such as ketchup, blood etc.) or shear thickening (e.g. mixtures of corn starch and water, so-called “bullet-proof” custard etc.) behaviour which can be approximated as a power-law over a range of shear rates. Addition of polymers in Newtonian fluids often leads to drag reduction, which has been utilized to reduce pumping power requirements in cross-country pipelines (e.g. Alaska pipeline) but this technology is yet to be utilized widely for augmentation of heat transfer in spite of a growing demand of efficient electronic cooling. Recently, non-Newtonian liquid induced cooling has been investigated to improve electric performance of the solar cells [8]. It has been found that cooling and electrical performance increases with decreasing liquid layer thickness which covers solar cell due to augmentation of convection as a result of shear thinning

features of dielectric liquid. Accordingly, the direct liquid immersion cooling of solar cell was proposed as a solution for thermal management of concentrating photo-voltaic (CPV) systems (Fig. 1.7 a).

The passive cooling through natural convection has the advantages of simplicity and reliability for thermal management of electronic devices. However, the heat transfer coefficient resulting from traditional working fluids are insufficient to handle very high heat flux. Therefore, functionally thermal fluids, such as non-Newtonian phase change material slurry, produced by mixing together micro-encapsulated phase change material, are used to enhance the magnitude of the heat transfer coefficient [9, 10]. In addition, micro-encapsulated phase change material slurry is also used for thermal storage (e.g. wireless iron in Fig. 1.7 b).

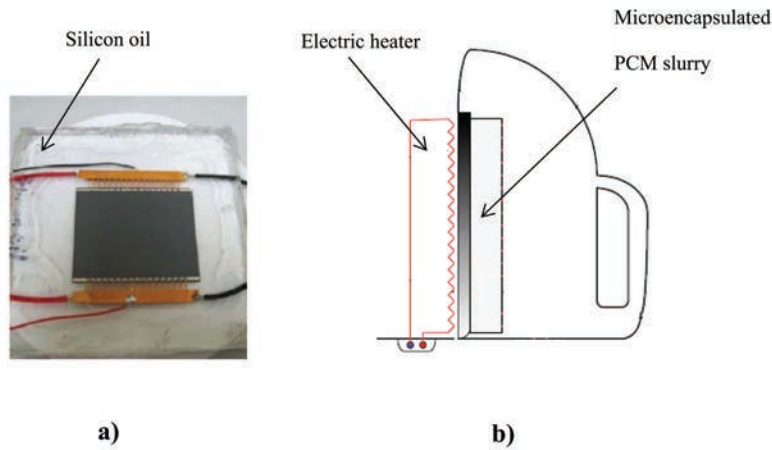


Figure 1.7: Practical examples for natural convection of non-Newtonian fluids in an enclosure: a) a photo-voltaic cell immersed in silicon oil [8], b) An application of micro-encapsulated PCM slurry in wireless iron.

Some electro-rheological/magneto-rheological fluids exhibit yield stress character and behave only as fluids once that yield stress is surpassed. It is possible to modulate the yield stress based on electrical/magnetic field in electro-rheological/magneto-rheological fluids [11]. Although magneto-rheological fluids have been used by automobile companies (e.g. BMW) for the active suspension, the use of magneto-rheological fluids for heat transfer applications is relatively rare. However, it is possible to eliminate (or alter the strength of) convection

(see Fig. 1.8) completely by changing the magnetic or electric field, which can be useful for reducing heat losses in storage of cryogenic materials and accidental damages of cryogenic materials. Moreover, it can be significantly useful in limiting accidental damages in the case of nuclear meltdown by stopping convection in the nuclear reactors. Olabi and Grunwald [12] analysed potential of magneto-rheological fluid (MRF) in various industries including the automotive industry. They reported that for every system where it is desirable to control motion using a fluid with changing viscosity, a solution based on MRF technology has a potential for improving functionality and costs [12].

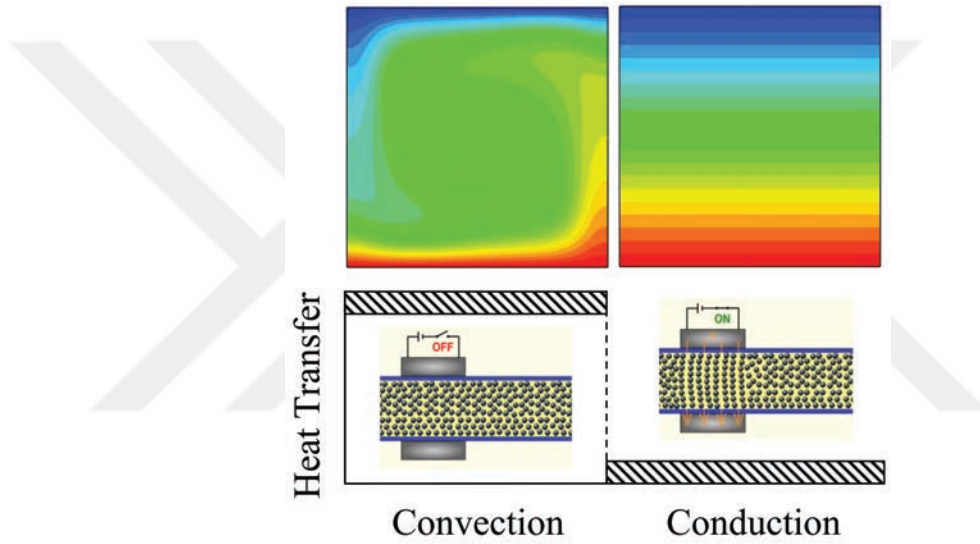


Figure 1.8: Schematic of the heat transfer control in enclosure with electro / magneto-rheological fluids.

### 1.3 Non-Newtonian Fluid Behaviour

Fluids can be classified in two different ways; either according to their response to externally applied pressure or according to their behaviour under the action of shear stress. The first classification divides the fluids as “compressible” and “incompressible” fluids, depending upon whether or not the volume of an element of fluid changes under external pressure. The second classification divides the fluids into “Newtonian” and “non-Newtonian” fluids, depending on the shear

rate dependence of the viscosity of the fluid. If shear stress ( $\tau$ ) is directly proportional to shear rate ( $\dot{\gamma}$ ), the fluid is called as the Newtonian fluid. On the other hand if the relation between  $\tau$  and  $\dot{\gamma}$  is non-linear, the fluid are called as the non-Newtonian fluid. In Fig. 1.9, the thin layer of a Newtonian fluid separated by two parallel planes by distance ( $d_y$ ) as shown. The fluid is subjected to a shear by the application of a force  $F$  on the upper surface with area  $A$  as shown in Fig. 1.9 a) and this force needs to be balanced by an equal and opposite internal frictional force in the fluid. For an incompressible Newtonian fluid in laminar flow, the resulting shear stress is equal to the product of the shear rate and viscosity. The slope of the linear relation between shear stress and shear rate in Fig. 1.9 b) provides the measurement of viscosity.

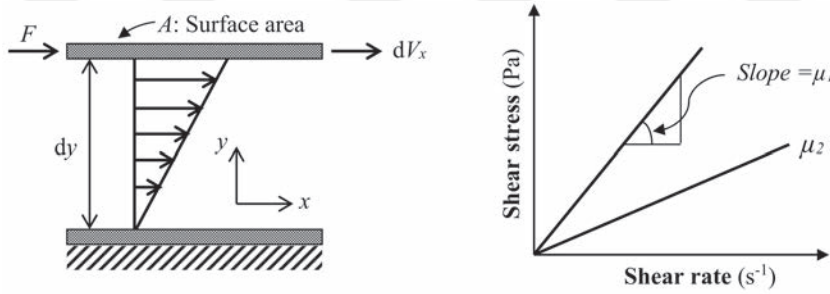


Figure 1.9: a) Schematic representation of unidirectional shearing flow, b) Typical shear stress shear rate behaviour for Newtonian fluids.

The shear rate for Newtonian fluids can be expressed as the velocity gradient in the direction perpendicular to that of the shear force:

$$\frac{F}{A} = \tau_{yx} = \mu \left( -\frac{dV_x}{d_y} \right) = \mu \dot{\gamma}_{yx} \quad (1.6)$$

For Newtonian fluids, the viscosity (i.e. ratio of shear stress to the shear rate) is constant and independent of shear stress or shear rate and depends on only the material and its temperature and pressure for a Newtonian fluids. However, the relation between shear stress and shear rate is non-linear for non-Newtonian fluids. This means that the viscosity of non-Newtonian fluids is not only constant at a given temperature and pressure but also dependent on flow conditions such

as shear rate. The non-Newtonian fluid behaviour can be classified into three general categories (see Fig. 1.10):

1. Fluids for which the rate of shear at any point is determined only by the value of the shear stress at that point at that instant; these fluids are variously known as ‘time independent’ or ‘generalized Newtonian fluids’ (GNF);
2. More complex fluids for which the relation between shear stress and shear rate depends, in addition, upon the duration of shearing and their kinematic history; they are called ‘time-dependent fluids’;
3. Substances exhibiting characteristics of both ideal fluids and elastic solids and showing partial elastic recovery, after deformation; these are categorized as ‘visco-elastic fluids’.

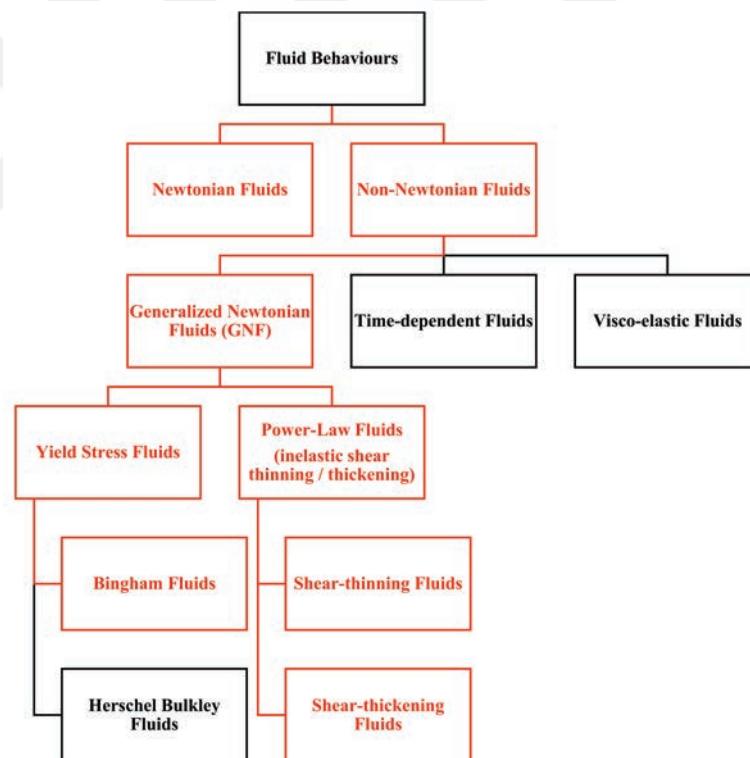


Figure 1.10: The classification of fluid behaviours. The red colour represents the scope of this study.

In this study, it will be concentrated on natural convection characteristic of the

incompressible Generalized Newtonian Fluids (GNF) as shown in the flow chart given by Fig. 1.10. The scope of this study is to investigate natural convection of yield stress (i.e. Bingham fluids) and power-law fluids (i.e. shear thinning and shear thickening fluids) in enclosed spaces.

## 1.4 Generalized Newtonian Fluids (GNF)

### 1.4.1 Yield Stress (Viscoplastic) Fluids

The yield stress (viscoplastic) fluid behaviour is characterized by the existence of a yield stress ( $\tau_0$ ) which must be exceeded for initiating fluid flow (see Fig. 1.11).

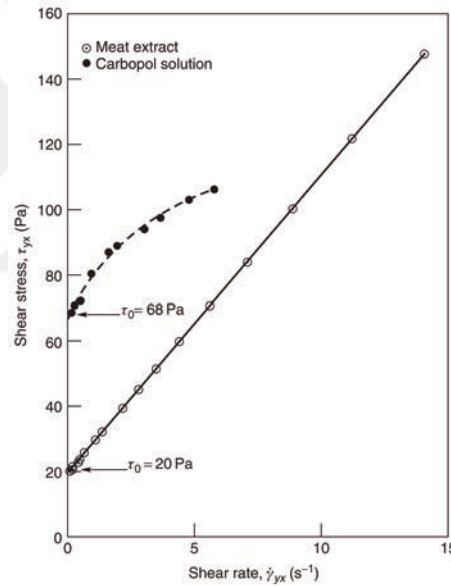


Figure 1.11: Representative shear stress–shear rate data showing viscoplastic behaviour in a meat extract (Bingham Plastic) and an aqueous carbopol polymer solution (yield-pseudoplastic) [7].

If the external stress ( $\tau_{yx}$ ) is greater than the yield stress ( $\tau_0$ ), fluid starts to flow, otherwise fluid act as a solid material. Once the magnitude of the external stress has exceeded the value of the yield stress, the relation between shear stress and shear rate may be linear or non-linear (Fig. 1.11). The yield stress definition

was pioneered by Bingham [13] with experimental work on several types of paint. Therefore, a fluid with a linear flow curve for  $\tau_{yx} > \tau_0$  is called as Bingham plastic fluid. Bingham fluids are characterized by a constant plastic viscosity (the slope of the shear stress versus shear rate) and a constant yield stress in the literature. On the other hand, a substance possessing a yield stress as well as a non-linear shear rate dependence of shear stress for  $\tau_{yx} > \tau_0$  is called a ‘yield-pseudo plastic’ material or Herschel Bulkley fluids (Fig. 1.11). The yield stress fluids are relevant to several practical applications as some materials (e.g. particulate suspensions, emulsions, foodstuffs, and drilling muds, etc. exhibit yield stress behaviour [14]).

### 1.4.2 Shear-Thinning Fluids

The shear-thinning fluid is one of the most common types of time-independent non-Newtonian fluid behaviour. It is characterized by an apparent viscosity which decreases with increasing shear rate hence it is called shear thinning. The most shear-thinning polymer solutions exhibit Newtonian behaviour at very low and at very high shear rates as shown schematically in Fig. 1.12.

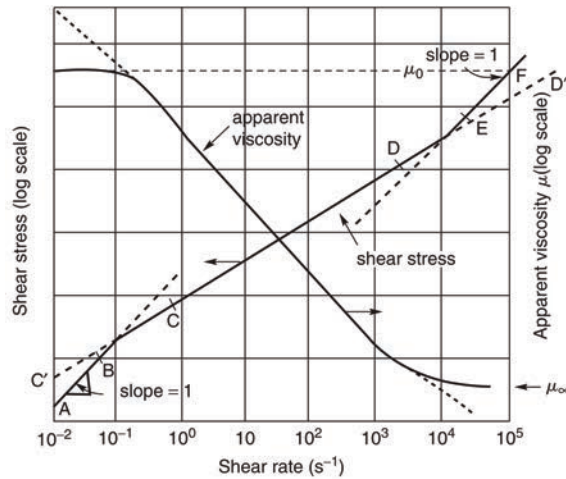


Figure 1.12: Schematic representation of shear-thinning behaviour [7].

The resulting values of the apparent viscosity at very low and high shear rates are known as the zero shear viscosity ( $\mu_0$ ) and the infinite shear viscosity ( $\mu_\infty$ ).

) respectively. Thus, the apparent viscosity of a shear-thinning fluid decreases from  $\mu_0$  to  $\mu_\infty$  with increasing shear rate [7]. Majority of synthetic and biological fluids exhibit shear-thinning behaviours (e.g. ketchup, blood, silicone oils and coatings).

### 1.4.3 Shear-Thickening Fluids

The shear-thickening fluid is also other common types of time-independent non-Newtonian fluid behaviour. It is characterized by an apparent viscosity which increases with the increasing shear rate, thus they are called shear-thickening fluids. Originally this type of behaviour was observed in concentrated suspensions; at low shearing levels, the liquid lubricates the motion of each particle past another to minimize solid-solid friction. Consequently, the resulting stresses are small. At high shear rates, however, the mixture expands (dilates) slightly (similar to that seen in sand dunes) so that the available liquid is no longer sufficient to fill the increased void space (Fig. 1.13) and to prevent direct solid-solid contacts (and friction). This leads to the development of much larger shear stresses than that seen in a pre-dilated sample at low shear rates. This mechanism causes the apparent viscosity to rise rapidly with the increasing rate of shear [7].

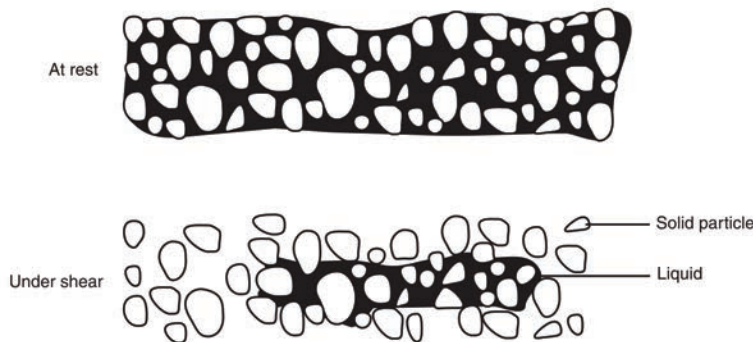


Figure 1.13: Schematic representation of shear-thickening behaviour [7].

The mixtures of corn starch and water, so-called “bullet-proof” custard are amongst the most well-known examples of shear thickening fluids.

## 1.5 Scope and Objectives of This Study

Natural convection of Newtonian fluids in enclosures have been extensively studied in the existing literature (see extensive reviews of Ostrach [15] and Boudenschatz [16]), whereas relatively limited attention has been given for non-Newtonian fluids despite their potential applications in cooling of electronics, food and chemical processing, and solar and nuclear power systems. Furthermore, the most of natural convection in enclosure studies were carried out for rectangular enclosures, in fact the annular cylindrical geometry (i.e. annular space around a cylindrical storage) is more common than rectangular enclosures in engineering applications such as chemical or biological fluid storage, anaerobic digestion tanks, and solar energy collectors. In this regard, an elaborate study is needed to provide the physical understanding of natural convection of non-Newtonian fluids in cylindrical annular enclosures.

Therefore here, a detailed parametric analysis has been carried out to analyse laminar natural convection of yield stress fluids (i.e. Bingham fluids) and power-law fluids (i.e. shear-thinning / thickening fluids) in cylindrical annular enclosures for both Rayleigh-Bénard and differentially heated from side wall configurations (see Fig. 1.14).

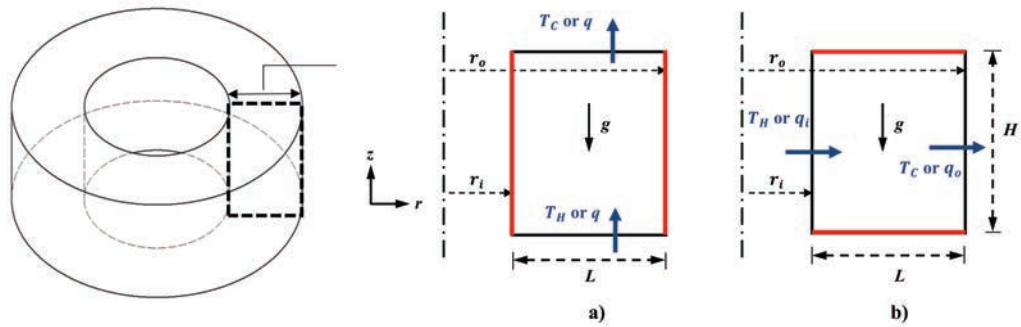


Figure 1.14: Schematic diagram of simulation domain: differentially heated a) horizontal, b) vertical walls for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions. The red colour represents insulated walls in the diagram.

Accordingly, the effects of

- nominal Rayleigh number ( $Ra$ ), Prandtl number ( $Pr$ ), Bingham number ( $Bn$ ) and power-law exponent ( $n$ ) ;
- different wall boundary conditions (e.g. constant wall temperature (CWT) and constant wall heat flux (CWHF))
- inner periphery normalised by length of cylinder ( $r_i/L$ ) ;
- aspect ratio ( $AR$  (radius to height ratio)).

on the mean Nusselt number has been analysed in this thesis as Buckingham's  $\pi$  theorem dilates  $\overline{Nu} = f(Ra, Pr, Bn, n, r_i/L, AR)$  in order to achieve the specific objectives of this study as follows:

- to provide physical insights into the fundamental understanding of natural convection of Bingham and power-law fluids in enclosures (for identifying conditions for high and low rates of heat transfer);
- to conduct detailed scaling analysis for natural convection of Bingham and power-law fluids in rectangular and cylindrical annular enclosures;
- to develop heat transfer correlations for engineering design calculations for natural convection of Bingham and power-law fluids in both rectangular and cylindrical annular enclosures.

## 1.6 Thesis Outline

This PhD thesis contains eight chapters (see Fig. 1.15), where this chapter provides the necessary introduction.

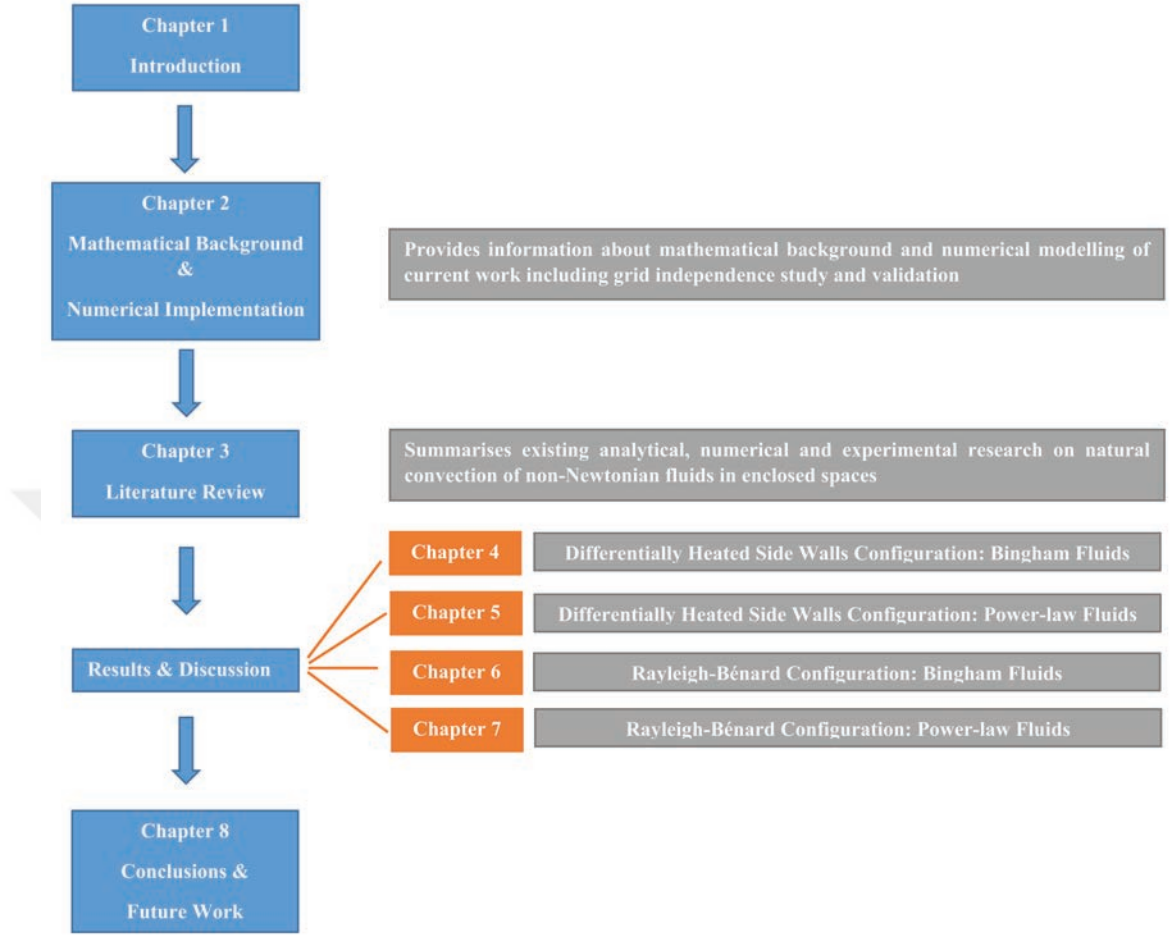


Figure 1.15: A flow chart of the structure of current thesis.

The rest of this PhD thesis organised as follows:

***Chapter 2:*** The necessary mathematical background and numerical implementations (i.e. validation, grid independence etc.) are provided in here.

***Chapter 3:*** A detailed literature review on laminar natural convection of Bingham and power-law fluids in enclosures is presented in here.

***Chapter 4:*** Numerical findings of the laminar natural convection of Bingham fluids in cylindrical enclosures with differentially heated side walls are presented here.

**Chapter 5:** Numerical findings of the laminar natural convection of power-law fluids in cylindrical enclosures with differentially heated side walls are presented here.

**Chapter 6:** Numerical findings of the laminar Rayleigh-Bénard convection of Bingham fluids in cylindrical enclosures are presented here.

**Chapter 7:** Numerical findings of the laminar Rayleigh-Bénard convection of power-law fluids in cylindrical enclosures are presented here.

**Chapter 8:** The main findings are summarised and conclusions are drawn along with the discussion on further research on this topic.



## Chapter 2

# Mathematical Background & Numerical Implementation

### 2.1 Yield stress Fluids

The Bingham fluid model is one of simplest representation of yield stress fluids, which assumes linear strain rate dependence of shear stress under the yielded state but acts as a solid under an unyielded state [13].

$$\dot{\underline{\underline{\gamma}}} = 0 \text{ for } \tau < \tau_y \quad (2.1)$$

$$\underline{\underline{\tau}} = (\mu + \tau_y/\dot{\gamma}) \dot{\underline{\underline{\gamma}}} \text{ for } \tau > \tau_y \quad (2.2)$$

where  $\dot{\gamma}_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)$  are the components of the rate of strain tensor  $\dot{\underline{\underline{\gamma}}}$ ,  $\underline{\underline{\tau}}$  is the stress tensor,  $\tau_y$  is the yield stress,  $\mu$  is the plastic viscosity,  $\tau$  and  $\dot{\gamma}$  are the second invariants of the stress and the rate of strain tensors respectively, which are expressed as:

$$\tau = \left[ \frac{1}{2} \underline{\underline{\tau}} : \underline{\underline{\tau}} \right]^{1/2} \quad (2.3)$$

$$\dot{\gamma} = \left[ \frac{1}{2} \dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}} \right]^{1/2} \quad (2.4)$$

However, the actual Bingham model is mathematically stiff and tends to predict an infinitely large viscosity for small values of shear rate [17]. Therefore, for the current analysis, the bi-viscosity regularization proposed by O'Donovan and Tanner [18] has been used to mimic the stress-shear rate characteristics for Bingham fluids:

$$\underline{\underline{\tau}} = \mu_{yield} \dot{\underline{\underline{\gamma}}} \text{ for } \dot{\gamma} \leq \tau_y / \mu_{yield} \quad (2.5)$$

$$\underline{\underline{\tau}} = \tau_y \left( \dot{\underline{\underline{\gamma}}} / \dot{\gamma} \right) + \mu \dot{\underline{\underline{\gamma}}} \text{ for } \dot{\gamma} > \tau_y / \mu_{yield} \quad (2.6)$$

where  $\mu_{yield}$  is the yield viscosity and  $\mu$  is the plastic viscosity. O'Donovan and Tanner [18] indicated that a value of  $\mu_{yield}$  equal to  $1000\mu$  mimics the true Bingham model in a satisfactory manner but here  $\mu_{yield}/\mu = 10^4$  is kept fixed to ensure higher fidelity of the simulations. Also, the Bingham model regularization proposed by Papanastasiou [19]. Papanastasiou's regularization [19] takes the following form:

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}_y (1 - \exp(-m\dot{\gamma})) + \mu \dot{\underline{\underline{\gamma}}} \quad (2.7)$$

where  $m$  is the stress growth exponent which has the dimension of time (i.e.  $m \gg 10L^2/\alpha$ ). The stress growth exponent has been chosen equal to  $m = 10^5 L^2/\alpha$  in this thesis to mimic true Bingham model in a satisfactory manner. Both Eqs. 2.5, 2.6 and 2.7 transform the “unyielded” region to a zone of high viscosity so that the numerical solutions predict vanishingly small magnitudes of velocity in the unyielded regions, and under this condition heat transfer takes place principally due to conduction, and fluid flow does not influence the thermal transport.

A limited number of simulations have also been carried out for laminar natural convection of Bingham fluids in square enclosures with differentially heated from side walls for  $Ra = 10^6$  and  $Pr = 10$  based on the regularizations proposed by

Table 2.1: Variations of  $\overline{Nu}_{sq}$  for different values of  $Bn$  for both bi-viscosity [18] and Papanastasiou [19] regularizations for  $Ra = 10^6$  and  $Pr = 10$ .

| $Bn$ | Bi-viscosity Reg.<br>( $\mu_{yield}/\mu = 10^4$ ) | Papanastasiou's<br>Reg. ( $m = 10^5 L^2/\alpha$ ) |
|------|---------------------------------------------------|---------------------------------------------------|
| 0    | 9.224                                             | 9.256                                             |
| 1    | 7.224                                             | 7.252                                             |
| 3    | 4.259                                             | 4.316                                             |
| 5    | 2.581                                             | 2.573                                             |
| 7    | 1.693                                             | 1.722                                             |
| 10   | 1.000                                             | 1.000                                             |

O'Donovan and Tanner [18] and Papanastasiou [19] in order to assess the sensitivity of the simulations on the choice of regularization. The maximum difference between the mean Nusselt numbers obtained from these two regularizations remains less than 2 % in the data provided in Table 2.1.

Additionally, several recent studies numerically investigated natural convection of Bingham fluids in square cavity with differentially heated side walls obeying a Bingham model without using any regularization [20, 21]. Comparisons have been made between the results obtained using the bi-viscosity regularization [18] with the ones without regularization [20, 21]. It has been reported by Refs. [20, 21] that the mean Nusselt number values obtained for low  $Ra$  (i.e.  $Ra = 10^4$ ) remain in good agreement, but small differences have been reported for high values of  $Ra$  (i.e.  $Ra = 10^5$ ) between the Bingham model without regularization and bi-viscosity regularization (i.e. 5 % difference in the mean Nusselt number has been reported for  $Bn = 3$ ,  $Ra = 10^5$  at  $Pr = 1$  between Refs. [20] and [22] in the Table 2 of Ref. [20]). However, this difference remains within typical computational uncertainty and may also arise because of the differences in the mesh and numerical schemes.

Although more computationally sophisticated but expensive techniques [20, 21] than the regularisation methods can offer higher degree of accuracy in capturing the onset of viscoplastic fluid motion, the analysis in this thesis focuses principally on convective heat transfer well beyond the critical condition for the onset of fluid movement. The fluid motion becomes too weak to impart any influence

on the convective heat transfer for Bingham numbers smaller than the threshold Bingham number value at which fluid flow entirely stops, so the regularisation based methods offer computational economy without compromising any physical accuracy for heat transfer problems.

### 2.1.1 Non-dimensional Numbers

The nominal Rayleigh, Prandtl and Bingham (i.e. ratio of yield stress to viscous stress [22]) numbers, which can be defined in the following manner:

$$Ra = \frac{\rho g \beta \Delta T_{ref} L_c^3}{\mu \alpha} = Gr Pr \quad \text{and} \quad Pr = \frac{\mu C_p}{k} \quad (2.8)$$

$$Bn = \frac{\tau_y L_c}{\mu \sqrt{g \beta \Delta T_{ref} L_c}} \quad (2.9)$$

In Eqs. 2.8 and 2.9,  $Gr = \rho^2 g \beta \Delta T_{ref} L_c^3 / \mu^2$  is the nominal Grashof number and  $\Delta T_{ref}$  can be taken to be  $\Delta T_{ref} = T_H - T_C$  for constant wall temperature (CWT) boundary condition while it can be taken to be  $\Delta T_{ref} = q L_c / k$  for constant wall heat flux (CWHF) boundary condition where  $L_c$  is the characteristic length between hot and cold walls.

In Eq. 2.9 the viscous straining (i.e.  $\mu \sqrt{g \beta \Delta T_{ref} L_c} / L_c$ ) is estimated based on velocity and length scales given by  $\sqrt{g \beta \Delta T_{ref} L_c}$  and  $L_c$  respectively. It is worth noting that in Bingham fluid flows, as the viscosity varies throughout the flow, an effective viscosity expressed as  $\mu_{eff} = \tau_y / \dot{\gamma} + \mu$  might be more representative of the viscous stress within the flow than the constant plastic viscosity  $\mu$ . Therefore the Rayleigh, Prandtl and Bingham numbers could have been defined more appropriately if  $\mu_{eff}$  was used instead of  $\mu$ . However,  $\dot{\gamma}$  is expected to show local variations in the flow domain so using a single characteristic value in the definitions of the non-dimensional numbers may not yield any additional benefit in comparison to the definitions given by Eqs. 2.8 and 2.9.

### 2.1.2 Governing Equations

The current analysis has been carried out in non-dimensional form for the sake of generalisation. The spatial co-ordinates, velocity components, pressure and temperature can be non-dimensionalised in the following manner:

$$r^+ = r/L_c; z^+ = z/L_c; u_i^+ = u_i/U_{ref}; P = P/\rho U_{ref}^2; \Theta = (T - T_{ref})/\Delta T_{ref} \quad (2.10)$$

where  $U_{ref}$  is taken to be  $\sqrt{g\beta\Delta T_{ref}L_c}$  based on the equilibrium of inertial and the buoyancy forces [22–25].

The incompressible steady-state non-dimensional governing equations for mass, momentum and energy for temperature-independent thermo-physical properties under the assumption of axisymmetry take the following form:

Non-dimensional mass conservation equation:

$$\frac{1}{r^+} \frac{\partial (r^+ u^+)}{\partial r^+} + \frac{\partial w^+}{\partial z^+} = 0 \quad (2.11)$$

Non-dimensional momentum conservation equations:

Radial direction:

$$u^+ \frac{\partial u^+}{\partial r^+} + w^+ \frac{\partial u^+}{\partial z^+} = -\frac{\partial P}{\partial r^+} + \frac{Pr^{1/2}}{Ra^{1/2}} \left[ \frac{1}{r^+} \frac{\partial (r^+ \tau_{rr}^+)}{\partial r^+} - \frac{\tau_{\phi\phi}^+}{r^+} + \frac{\partial (\tau_{rz}^+)}{\partial z^+} \right] \quad (2.12)$$

Vertical direction:

$$u^+ \frac{\partial w^+}{\partial r^+} + w^+ \frac{\partial w^+}{\partial z^+} = -\frac{\partial P}{\partial z^+} + \Theta + \frac{Pr^{1/2}}{Ra^{1/2}} \left[ \frac{1}{r^+} \frac{\partial (r^+ \tau_{rz}^+)}{\partial r^+} + \frac{\partial (\tau_{zz}^+)}{\partial z^+} \right] \quad (2.13)$$

Non-dimensional energy conservation equation:

$$u^+ \frac{\partial \Theta^+}{\partial r^+} + w^+ \frac{\partial \Theta^+}{\partial z^+} = \frac{1}{Pr^{1/2} Ra^{1/2}} \left[ \frac{1}{r^+} \frac{\partial}{\partial r^+} \left( r^+ \frac{\partial \Theta}{\partial r^+} \right) + \frac{\partial^2 \Theta}{\partial z^+ \partial z^+} \right] \quad (2.14)$$

In Eqs. 2.12 and 2.13,  $\tau_{ij}^+$  is the non-dimensional stress tensor which is given by:

$$\tau_{ij}^+ = \tau_{ij} L_c / \mu \sqrt{g \beta \Delta T_{ref} L_c} \quad (2.15)$$

where  $r$  is the radial coordinate,  $z$  axis is the vertical direction, and the axisymmetric flow is independent of the azimuthal direction  $\phi$ . However,  $\dot{\gamma}_{\phi\phi}$  takes the form  $\dot{\gamma}_{\phi\phi} = u/r$  in axisymmetric configuration and thus  $\tau_{\phi\phi}$  appears in Eq. 2.12. The components of viscous stress tensor (i.e.  $\tau_{rr}$ ,  $\tau_{\phi\phi}$ ,  $\tau_{rz}$  and  $\tau_{zz}$ ) are expressed according to Eqs. 2.5 and 2.6.

It is worth noting that the buoyancy term in Eq. 2.13 appears due to the Boussinesqs assumption which is strictly valid only for small temperature changes for which the density change remains with 10%. This assumption has been successfully used in several previous studies [26–30].

Furthermore, Peixinho et al. [31] reported based on their experimental findings that the yield stress is approximately independent of temperature and the plastic viscosity is only a weakly decreasing function of temperature for Carbopol aqueous solution in the temperature range 0 to 90 °C. Therefore, thermo-physical properties of Bingham fluids are taken to be independent of temperature for the sake of simplicity in the present study.

## 2.2 Power-law Fluids

For the Ostwald-De Waele (i.e. power law) model the viscous stress tensor  $\tau_{ij}$  is expressed as [32]:

$$\tau_{ij} = \mu_a e_{ij} = K (e_{kl} e_{kl} / 2)^{(n-1)/2} e_{ij} \quad (2.16)$$

where  $e_{ij} = (\partial u_i / \partial x_j + \partial u_j / \partial x_i)$  is the rate of strain tensor,  $K$  is the consistency,  $n$  is the power-law index and  $\mu_a$  is the apparent viscosity which is given by:

$$\mu_a = K (e_{kl} e_{kl} / 2)^{(n-1)/2} \quad (2.17)$$

According to Eq. 2.17, the apparent viscosity  $\mu_a$  decreases (increases) with increasing shear rate for  $n < 1$  ( $n > 1$ ) and thus fluids with  $n < 1$  ( $n > 1$ ) are referred to as shear-thinning (shear-thickening) fluids, whereas  $n = 1$  represents Newtonian fluids.

Many polymer melts and solutions exhibit shear-thinning character in the range of  $0.1 \leq n < 1$  depending upon the concentration and molecular weight of the polymer. However, fluid behaviour becomes extremely yield stress for very small values of  $n$  (e.g.  $n = 0.3$ ) [7]. Also, realistic shear-thickening fluids exhibit the value of  $n$  up to 1.8 (e.g. aerated poultry waste slurry) [7]. Therefore, the range of power-law index (i.e.  $0.6 \leq n \leq 1.8$ ) has been chosen in this thesis for analysing power-law fluids.

### 2.2.1 Non-dimensional Numbers

The nominal Rayleigh, Prandtl numbers defined here using a characteristic velocity scale (i.e.  $\alpha/L_c$ ) [33–35] and one can obtain following expression for nominal viscosity  $\mu$  of power-law fluids:

$$\mu \sim K (\dot{\gamma})^{n-1} \sim K \left( \frac{\alpha}{L_c^2} \right)^{n-1} \quad (2.18)$$

Equation 2.18 gives rise to following definitions of nominal Rayleigh  $Ra$  and Prandtl  $Pr$  numbers:

$$Ra = \frac{g\beta\Delta T_{ref}L_c^{2n+1}}{\alpha^n (K/\rho)} \quad (2.19)$$

$$Pr = \left( \frac{K}{\rho} \right) \alpha^{n-2} L_c^{2-2n} \quad (2.20)$$

In Eq. 2.19, the characteristic temperature difference  $\Delta T_{ref}$  can be taken to be  $\Delta T_{ref} = T_H - T_C$  for constant wall temperature (CWT) boundary condition while it can be taken to be  $\Delta T_{ref} = qL_c/k$  for constant wall heat flux (CWHF) boundary condition where  $L_c$  is the characteristic length between hot and cold walls.

However, a different definition of nominal viscosity  $\mu$  for power-law fluids was used earlier for analysing natural convection over a flat plate [36, 37] which is given as:

$$\mu_K = \rho \left( \frac{K}{\rho} \right)^{\frac{1}{2-n}} L_c^{\frac{2(1-n)}{2-n}} \quad (2.21)$$

Kim et al. [38] used definition given in Eq. 2.21 for analysing transient buoyant convection of a power-law non-Newtonian fluid in an enclosure. Accordingly, Equation 2.21 gives rise to following definitions of nominal Rayleigh  $Ra$  and Prandtl  $Pr$  numbers:

$$Ra_K = \frac{g\beta\Delta T L_c^3}{\alpha (K/\rho)^{1/(2-n)} L_c^{2(1-n)/(2-n)}} \quad (2.22)$$

$$Pr_K = \frac{(K/\rho)^{1/(2-n)} L_c^{2(1-n)/(2-n)}}{\alpha} \quad (2.23)$$

Eqs. 2.22 and 2.23 can be arranged along with nominal Rayleigh  $Ra$  and Prandtl  $Pr$  number definitions used in the present study (i.e. Eqs. 2.19 and 2.20) in following manner:

$$Ra = Ra_K Pr^{\frac{n-1}{2-n}} \quad (2.24)$$

$$Pr = Pr_K^{2-n} \quad (2.25)$$

It is worth noting that both  $Ra$  and  $Ra_K$  ( $Pr$  and  $Pr_K$ ) are valid definitions for the both nominal Rayleigh (Prandtl) numbers because the apparent viscosity  $\mu_a$  is a local property which varies throughout the flow field and cannot be adequately characterised by a single representative value.

Although the power-law model offers the simplest representation of shear-thinning behaviour, it does has some inadequacies. Generally, it applies over only a limited range of shear rates, however it does not predict the zero and infinite shear viscosities for given values of  $K$  and  $n$ . This is because  $K$  and  $n$  are dependent on the range of shear rates. Therefore, in the current analyses minimum and maximum

levels of  $\mu$  are taken to be  $\mu_{min} = 10^{-4}\mu_{n=1}$  and  $\mu_{max} = 10^4\mu_{n=1}$  respectively where  $\mu_{n=1}$  is the viscosity of the Newtonian fluid for the same nominal values of Rayleigh and Prandtl numbers. It has been checked that the results remain independent of the choices of  $\mu_{min}$  and  $\mu_{max}$  beyond these values.

### 2.2.2 Governing Equations

The current analysis has been carried out in non-dimensional form for the sake of generalisation. The spatial co-ordinates, velocity components, pressure and temperature can be non-dimensionalised in the following manner:

$$\begin{aligned} r^+ &= r/L_c; \quad z^+ = z/L_c; \quad u_i^+ = u_i/U_{ref}; \quad P = P/\rho U_{ref}^2; \\ \tau_{ij}^+ &= \tau_{ij}L_c/\rho\alpha U_{ref}; \quad \Theta = (T - T_{ref})/\Delta T_{ref} \end{aligned} \quad (2.26)$$

where  $U_{ref} = \alpha/L_c$  is the reference velocity scale. The incompressible steady-state non-dimensional governing equations for mass, momentum and energy for constant temperature-independent thermo-physical properties (e.g.  $k$ ,  $c_p$ ,  $n$  and  $K$ ) under the assumption of axisymmetry take the following form:

Non-dimensional mass conservation equation:

$$\frac{1}{r^+} \frac{\partial (r^+ u^+)}{\partial r^+} + \frac{\partial w^+}{\partial z^+} = 0 \quad (2.27)$$

Non-dimensional momentum conservation equations:

Radial direction:

$$u^+ \frac{\partial u^+}{\partial r^+} + w^+ \frac{\partial u^+}{\partial z^+} = -\frac{\partial P}{\partial r^+} + \frac{1}{r^+} \frac{\partial (r^+ \tau_{rr}^+)}{\partial r^+} - \frac{\tau_{\phi\phi}^+}{r^+} + \frac{\partial (\tau_{rz}^+)}{\partial z^+} \quad (2.28)$$

Vertical direction:

$$u^+ \frac{\partial w^+}{\partial r^+} + w^+ \frac{\partial w^+}{\partial z^+} = -\frac{\partial P}{\partial z^+} + RaPr\Theta + \frac{1}{r^+} \frac{\partial (r^+ \tau_{rz}^+)}{\partial r^+} + \frac{\partial (\tau_{zz}^+)}{\partial z^+} \quad (2.29)$$

Non-dimensional energy conservation equation:

$$u^+ \frac{\partial \Theta^+}{\partial r^+} + w^+ \frac{\partial \Theta^+}{\partial z^+} = \frac{1}{r^+} \frac{\partial}{\partial r^+} \left( r^+ \frac{\partial \Theta}{\partial r^+} \right) + \frac{\partial^2 \Theta}{\partial z^+ \partial z^+} \quad (2.30)$$

where  $r$  is the radial coordinate,  $z$  axis is the vertical direction, and the axisymmetric flow is independent of the azimuthal direction  $\phi$ . In Eq. 2.29, the buoyancy term is accounted for Boussinesq's approximation following several previous studies [34, 35, 39–42].

The analyses by Solomatov and Barr [43] and Kaddiri et al. [44] also indicate that the temperature dependence of consistency  $K$  do not significantly modify the qualitative nature of the temperature and velocity distributions in natural convection of power-law fluids in square enclosures. Therefore, the consistency  $K$  and apparent viscosity  $\mu_a$  are taken to be independent of temperature for the sake of simplicity in the present study. The results of the present analysis based on constant properties will still be valid at least in a qualitative sense because the underlying physics of heat and momentum transport is not affected by the thermo-physical property variation with temperature.

## 2.3 Boundary Conditions

### 2.3.1 Differentially Heated Side Walls Configuration

The simulation domain is schematically shown in Fig. 2.1 which indicates that the two horizontal walls of a rectangular cross-sectional cylindrical enclosure are insulated, whereas the vertical boundaries are taken to be differentially heated. Equations (2.11- 2.15) and (2.27 - 2.30) are subjected to the following boundary conditions in this configuration for both Bingham and power-law fluids respectively.

Velocity boundary conditions:

$u^+ = 0$  and  $w^+ = 0$  at  $z^+ = 0$  and  $z^+ = AR$  due to no-slip condition and impenetrability on horizontal walls and

$u^+ = 0$  and  $w^+ = 0$  at  $r^+ = r_i/L$  and  $r^+ = r_i/L + 1$  due to impenetrability and no-slip condition on vertical walls.

Temperature boundary conditions:

$\partial\Theta/\partial z^+ = 0$  at  $z^+ = 0$  and  $z^+ = AR$  due to adiabatic horizontal walls;

$\Theta = 1.0$  at  $r^+ = r_i/L$  and  $\Theta = 0.0$  at  $r^+ = r_i/L + 1$  for the CWT boundary condition;

$-\partial\Theta/\partial r^+ = 1.0$  at  $r^+ = r_i/L$  and  $-\partial\Theta/\partial r^+ = 1.0/(1 + L/r_i)$  at  $r^+ = r_i/L + 1$  for the CWHF boundary condition.

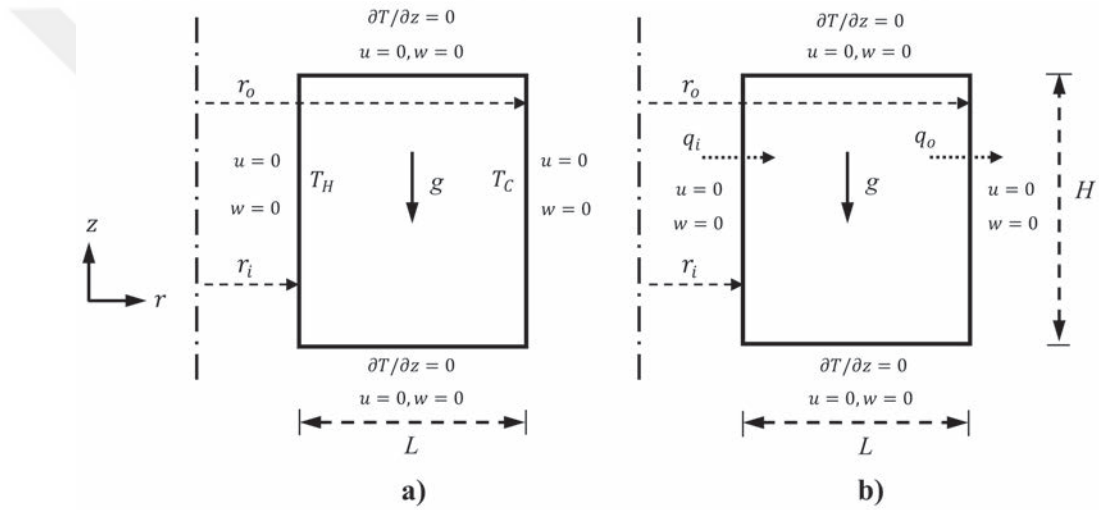


Figure 2.1: Schematic diagram of the simulation domain for differentially heated from side walls configuration: a) CWT, b) CWHF configurations.

Here,  $r_i/L$  is normalized internal radius (i.e. where  $r_i$  internal cylinder radius,  $L$  is the difference between outer and inner cylinder radius) and  $AR$ <sup>1</sup> is aspect ratio (i.e.  $AR = H/L$ , where  $H$  is the enclosure height). The characteristic length between hot and cold walls (i.e.  $L_c = L$ ) and  $\Delta T_{ref} = T_H - T_C$  ( $\Delta T_{ref} = q_i L/k$ ) for CWT (CWHF) boundary conditions yield following non-dimensional numbers for differentially heated side walls configuration:

<sup>1</sup>The aspect ratio has been defined as ratio of height to width of enclosures (i.e.  $AR = H/L$ ) by following previous studies [26, 40, 41, 45–47] on natural convection of Newtonian and non-Newtonian fluids.

**For Bingham fluids:**

$$Ra = \rho g \beta \Delta T_{ref} L^3 / \mu \alpha, \quad Pr = \mu C_p / k \quad \text{and} \quad Bn = \tau_y L / \mu \sqrt{g \beta \Delta T_{ref} L} ;$$

**For power-law fluids:**

$$Ra = \rho g \beta \Delta T_{ref} L^{2n+1} / \alpha^n (K/\rho), \quad Pr = (K/\rho) \alpha^{n-2} L^{2-2n}.$$

Accordingly, it is possible to express the Nusselt number for natural convection of Bingham and power-law fluids in rectangular cross-sectional cylindrical annular enclosures with differentially heated from side walls as:  $Nu = f_1(Ra, Pr, Bn, n, r_i/L, AR)$  using Buckingham's  $\pi$  theorem.

The local heat transfer coefficient  $h_i$  for the internal radius can be expressed as:

$$h_i = \left| -k \left( \frac{\partial T}{\partial r} \right)_{r=r_i} \times 1 / (T_{r=r_i} - T_{r=r_i+L}) \right| \quad (2.31)$$

The mean heat transfer coefficient  $\bar{h}_i$  and the mean Nusselt number  $\overline{Nu}_i$  based on the internal radius are evaluated as:

$$\bar{h}_i = \frac{1}{H} \int_0^H h_i dz \quad \overline{Nu}_i = \frac{\bar{h}_i L}{k} \quad (2.32)$$

According to steady state conditions, one obtains:

$$\frac{q_o}{q_i} = \frac{r_i}{r_o} \quad (2.33)$$

where  $q_i$  and  $q_o$  are the mean heat flux magnitudes at the internal and external radii respectively, which can be defined as:

$$q_i = \frac{1}{H} \int_0^H \left| k \left( \frac{\partial T}{\partial r} \right)_{r=r_i} \right| dz \quad q_o = \frac{1}{H} \int_0^H \left| k \left( \frac{\partial T}{\partial r} \right)_{r=r_o} \right| dz \quad (2.34)$$

This give rise to:

$$\bar{h}_o = \bar{h}_i \frac{r_i}{r_o}, \quad \overline{Nu} = \overline{Nu}_i \frac{r_i}{r_o} \quad (2.35)$$

According to the pure conduction solution, the heat flux on the internal surface can be expressed as:

$$(q_i)_{cond} = \frac{k |T_{r=r_i} - T_{r=r_o}|}{r_i \ln(1 + L/r_i)} \quad (2.36)$$

Using Eqs. 2.35 and 2.36, the Nusselt number due to pure conduction on the internal surface can be expressed as:

$$(\overline{Nu_i})_{cond} = \frac{(q_i)_{cond} L}{k (T_H - T_C)} = \frac{(L/r_i)}{\ln(1 + L/r_i)} \quad (2.37)$$

### 2.3.2 Rayleigh-Bénard Configuration

The simulation domain is schematically shown in Fig. 2.2 which indicates that the two vertical walls of a rectangular cross-sectional cylindrical enclosure are insulated, whereas the horizontal boundaries are taken to be differentially heated. Equations (2.11- 2.15) and (2.27 - 2.30) are subjected to the following boundary conditions in this configuration for Bingham and power-law fluids respectively.

Velocity boundary conditions:

$u^+ = 0$  and  $w^+ = 0$  at  $r^+ = r_i/H$  and  $r^+ = r_i/H + (1/AR)$  due to no-slip condition and impenetrability on vertical walls and

$u^+ = 0$  and  $w^+ = 0$  at  $z^+ = 0.0$  and  $z^+ = 1.0$  due to impenetrability and no-slip condition on horizontal walls.

Temperature boundary conditions:

$\partial\Theta/\partial r^+ = 0$  at  $r^+ = r_i/H$  and  $r^+ = r_i/H + (1/AR)$  due to adiabatic vertical walls;

$\Theta = 1.0$  at  $z^+ = 0.0$  and  $\Theta = 0.0$  at  $z^+ = 1.0$  for the CWT boundary condition;

$-\partial\Theta/\partial z^+ = 1.0$  at  $z^+ = 0.0$  and  $z^+ = 1.0$  for the CWHF boundary condition.

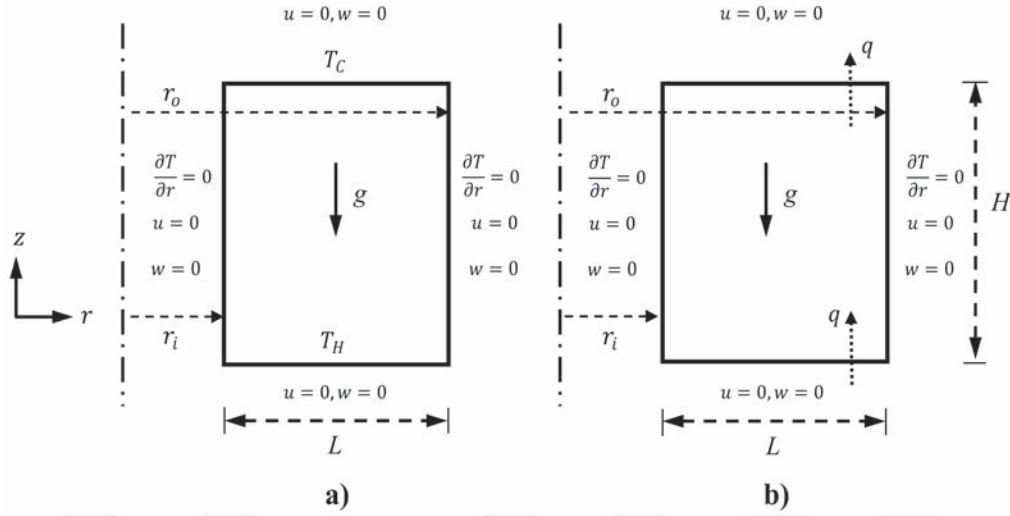


Figure 2.2: Schematic diagram of the simulation domain for Rayleigh-Bénard configuration : a) CWT, b) CWHF configurations.

Here, the characteristic length between hot and cold walls (i.e.  $L_c = H$ ) and  $\Delta T_{ref} = T_H - T_C$  ( $\Delta T_{ref} = qH/k$ ) for CWT (CWHF) boundary conditions lead to following non-dimensional numbers for Rayleigh-Bénard configuration:

**For Bingham fluids:**

$$Ra = \rho g \beta \Delta T_{ref} H^3 / \mu \alpha, \quad Pr = \mu C_p / k \quad \text{and} \quad Bn = \tau_y H / \mu \sqrt{g \beta \Delta T_{ref} H};$$

**For power-law fluids:**

$$Ra = \rho g \beta \Delta T_{ref} H^{2n+1} / \alpha^n (K/\rho), \quad Pr = (K/\rho) \alpha^{n-2} H^{2-2n}.$$

Accordingly, it is also possible to express the Nusselt number for Rayleigh-Bénard convection of both Bingham and power-law fluids in rectangular cross-sectional cylindrical annular enclosures as:  $Nu = f_1(Ra, Pr, Bn, n, r_i/L, AR)$  using Buckingham's  $\pi$  theorem.

The local heat transfer coefficient  $h_{cy}$  can be expressed as:

$$h_{cy} = \left| -k \left( \frac{\partial T}{\partial z} \right)_{z=0} \times 1 / (T_{z=0} - T_{z=H}) \right| \quad (2.38)$$

The mean heat transfer coefficient  $\bar{h}_{cy}$  and the mean Nusselt number  $\bar{Nu}_{cy}$  are

evaluated as:

$$\bar{h}_{cy} = \int_{r_i}^{r_i+L} 2\pi r h(r) dr / [\pi (r_i + L)^2 - \pi r_i^2] \quad \overline{Nu}_{cy} = \frac{\bar{h}_{cy} H}{k} \quad (2.39)$$

## 2.4 Numerical Methodology

The conservation equations (i.e. Eqs. (2.11- 2.15) and (2.27 - 2.30) for Bingham and power-law fluids, respectively) are solved in the framework of finite-volume methodology by using ANSYS-FLUENT and OpenFOAM solvers. These solvers was previously used successfully for simulating natural convection and non-Newtonian fluid flow problems [23, 48]. The convective terms are discretised using a second-order up-wind scheme, whereas the diffusive terms are discretised by a second-order central differencing scheme. The coupling of the pressure and velocity is accounted for by the well-known SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) algorithm [49].

In the framework of the finite-volume methodology the discretised equation of a general primitive variable  $\lambda$  at a grid point P takes the following form:  $a_p \lambda_p = \sum_{nb} a_{nb} \lambda_{nb} + b$  where the subscripts  $p$  and  $nb$  are used for the central node ‘P’ and the neighbourhood nodes respectively, and the coefficients associated with these nodes are referred to as  $a_p$  and  $a_{nb}$  respectively with  $b$  being the general source term. The discretised equations are solved iteratively until the relative (scaled) residual defined as  $R^\lambda = \sum_{cells} |\sum_{nb} a_{nb} \lambda_{nb} + b - a_p \lambda_p| / \sum_{cells} |a_p \lambda_p|$  attains a value smaller than a pre-determined small number  $\varepsilon$ , and thus  $R^\lambda \leq \varepsilon$  is considered to be the convergence criterion for the discretised set of equations. In this regard, the convergence criteria were set to  $10^{-6}$  for all the relative (scaled) residuals. In other words, the iterations continue until the all scaled residuals satisfy the criterion (i.e.  $R^\lambda \leq 10^{-6}$ ).

The contours of non-dimensional stream function  $\Psi$  with unyielded regions (i.e. grey regions) for  $AR = 4$  for differentially heated from side walls configuration, a)  $R^\lambda = 10^{-6}$ , b)  $R^\lambda = 10^{-7}$  at  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $Bn = 0.2$  and  $Pr = 500$  for CWHF boundary condition is shown in Fig. 2.3. It can be easily seen in Fig. 2.3

that the values of the maximum magnitude of non-dimensional stream function (i.e.  $\Psi_{max} = \psi_{max}/\alpha$ ) does not change for  $10^{-6} \leq R^\lambda \leq 10^{-7}$ . Also, the mean Nusselt number also changes in the 6<sup>th</sup> decimal place for  $10^{-6} \leq R^\lambda \leq 10^{-7}$ .



Figure 2.3: Contours of non-dimensional stream function  $\Psi$  with unyielded regions (i.e. grey regions) for  $AR = 4$  for differentially heated from side walls configuration, a)  $R^\lambda = 10^{-6}$ , b)  $R^\lambda = 10^{-7}$  at  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $Bn = 0.2$  and  $Pr = 500$  for CWHF boundary condition.

It is also worth indicating that the flow patterns presented here for Rayleigh-Bénard configuration are not artefacts of convergence criteria and reducing the convergence tolerance by an order of magnitude did not make any difference to the magnitudes of the stream functions (see Fig. 2.4) and also in the values of the mean Nusselt number (especially for the cases where the rolls are stacked upon one another). Lir and Lin [50] reported similar flow patterns for Rayleigh-Bénard convection of Newtonian fluid (i.e. air) in a rectangular shallow cavity based on their experimental flow visualisation. Mueller [51] also numerically reported similar flow patterns including rolls stacking over one another based on numerical simulations. Also, it has been shown by Yigit et al. [52] that the flow patterns in the Rayleigh-Bénard configuration depend significantly on the initial condition

and it is possible to have several steady state solutions based on different initial conditions for the CWT boundary condition. Thus, the flow patterns presented in this thesis (especially the ones where the rolls are stacked over one another and it should be noted that the circulation strength for both rolls are not the same and one is often much weaker than the other) need to be validated by experimental analyses.

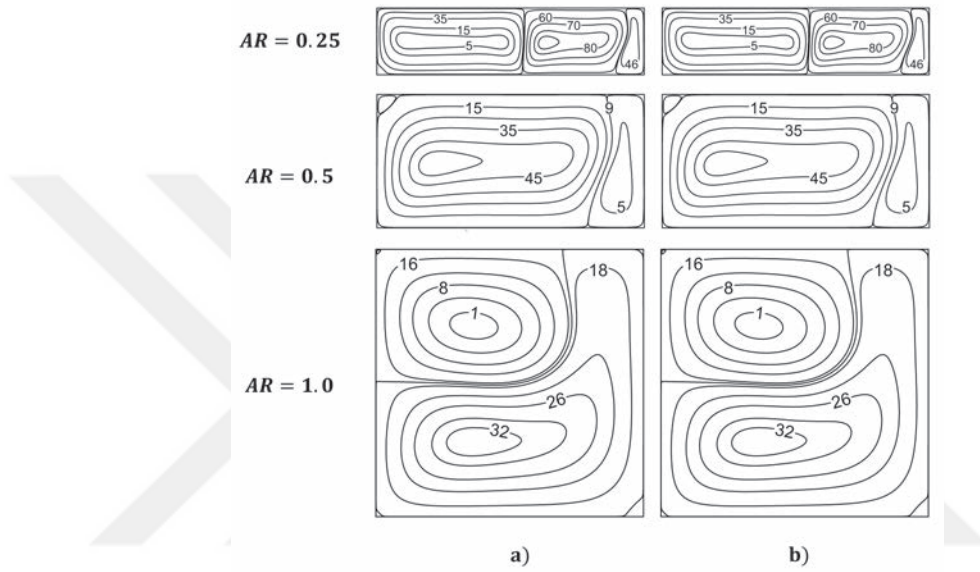


Figure 2.4: Contours of non-dimensional stream function  $\Psi$  for different values of  $AR$  for Rayleigh-Bénard configuration, a)  $R^\lambda = 10^{-6}$ , b)  $R^\lambda = 10^{-7}$  at  $r_i/L = 1.0$ ,  $Ra = 10^5$ ,  $Bn = 0$  and  $Pr = 500$  for CWT boundary condition.

Three-dimensional simulations of laminar natural convection of both Bingham and power-law fluids in cylindrical annular enclosures for both differentially heated from side walls and Rayleigh-Bénard configurations have been conducted for selected parameters to assess the validity of the assumption of axisymmetry. The distributions of non-dimensional temperature in the vertical plane, as obtained from three-dimensional simulations, are exemplarily shown in Fig. 2.5 for for Newtonian, Bingham (i.e.  $Bn = 0.02, Pr = 500$ ), shear-thinning (i.e.  $n = 0.8, Pr = 10^3$ ), shear-thickening (i.e.  $n = 1.2, Pr = 10^3$ ) at  $Ra = 10^4$ ,  $AR = 1.0$ ,  $r_i/L = 1.0$  in the CWT configuration. The isotherms in Fig. 2.5 indicate the assumption of axisymmetry remains valid.

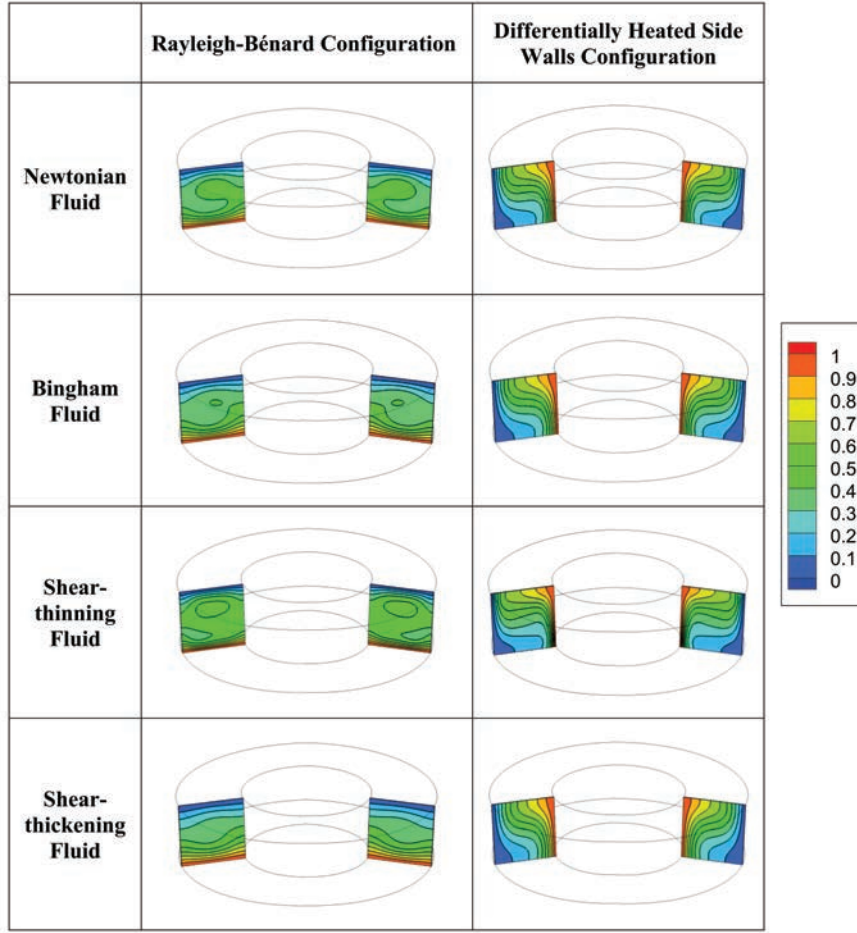


Figure 2.5: Variation of the non-dimensional temperature in the vertical mid-plane of three dimensional cylindrical annular enclosures for Newtonian, Bingham (i.e.  $Bn = 0.02, Pr = 500$ ), shear-thinning (i.e.  $n = 0.8, Pr = 10^3$ ), shear-thickening (i.e.  $n = 1.2, Pr = 10^3$ ) at  $Ra = 10^4$ ,  $AR = 1.0$ ,  $r_i/L = 1.0$  in the CWT configuration.

It is worth noting that three-dimensional simulations are considerably more expensive than two-dimensional calculations. For example, three-dimensional simulation is typically 10 times more expensive than the corresponding two-dimensional analysis for shear-thinning (i.e.  $n < 1$ ) fluids (e.g. CPU time is 36 minutes and 10 hours for two- and three-dimensional simulations respectively for  $n = 0.8$ ,  $Ra = 10^4$ ,  $AR = 1.0$ ,  $r_i/L = 1.0$  for CWT boundary condition in the Rayleigh-Bénard configuration. Here, it is also worth noting that the mesh is considerably

finer in two-dimensional analysis (e.g. number of nodes 57600 ( $240 \times 240$ )) than that in the corresponding three dimensional simulation (e.g. total number of nodes  $2.5 \times 10^5$  ( $50 \times 50$  in the vertical mid-plane)).

## 2.5 Grid Independency Study

The grid independence of the results has been established for both Bingham and power-law fluid analyses based on a careful analysis of different non-uniform meshes which are listed in Table 2.2.

The numerical uncertainties for the mean Nusselt number on the inner periphery  $\overline{Nu}_i$  is presented in Table 2.3 and Table 2.4 for both Newtonian, Bingham and power-law fluids respectively for the differentially heated from side walls configuration. The maximum numerical uncertainty associated with the mean Nusselt number  $\overline{Nu}_i$  based on the inner periphery for Newtonian (i.e.  $Bn = 0$ ) fluids and a representative Bingham (i.e.  $Bn = 0.2$ ) fluids; shear-thinning (i.e.  $n = 0.6$ ); shear-thickening (i.e.  $n = 1.8$ ) fluids has been found to be less than 1 % between M1, M2 and M3 mesh configurations for the ranges of parameters considered here. Accordingly, the M2 mesh configuration has been used for each  $AR$  for both Bingham and power-law fluids for the sake of accuracy of numerical results and computational economy in the differentially heated from side walls configuration.

Similarly, The numerical uncertainties for the mean Nusselt number  $\overline{Nu}_{cy}$  is presented in Table 2.5 and Table 2.6 for both Newtonian, Bingham and power-law fluids respectively for the Rayleigh-Bénard configuration. The maximum numerical uncertainty on the mean Nusselt number  $\overline{Nu}_{cy}$  for Newtonian (i.e.  $Bn = 0$ ) fluids and a representative Bingham (i.e.  $Bn = 0.05$ ) fluids; shear-thinning (i.e.  $n = 0.8$ ); shear-thickening (i.e.  $n = 1.4$ ) fluids has been found to be less than 1 % between M1, M2 and M3 mesh configurations for the ranges of parameters considered here.

Table 2.2: Summary of the non-uniform Cartesian meshes used in the current analysis with non-dimensional minimum cell distance ( $\Delta_{min,cell}/L$ ) and grid expansion ratio ( $r_e$ ) value.

|              |                       |                        |                        |                        |
|--------------|-----------------------|------------------------|------------------------|------------------------|
| $AR = 0.125$ | Grid                  | M1                     | M2                     | M3                     |
|              |                       | $(120 \times 180)$     | $(160 \times 200)$     | $(200 \times 220)$     |
|              | $\Delta_{min,cell}/L$ | $4.176 \times 10^{-4}$ | $3.135 \times 10^{-4}$ | $2.509 \times 10^{-4}$ |
| $AR = 0.25$  | $r_e$                 | 1.013                  | 1.01                   | 1.008                  |
|              | Grid                  | M1                     | M2                     | M3                     |
|              |                       | $(160 \times 180)$     | $(180 \times 220)$     | $(200 \times 240)$     |
| $AR = 0.5$   | $\Delta_{min,cell}/L$ | $6.27 \times 10^{-4}$  | $5.575 \times 10^{-4}$ | $5.018 \times 10^{-4}$ |
|              | $r_e$                 | 1.01                   | 1.009                  | 1.008                  |
| $AR = 1.0$   | Grid                  | M1                     | M2                     | M3                     |
|              |                       | $(180 \times 220)$     | $(200 \times 240)$     | $(220 \times 260)$     |
|              | $\Delta_{min,cell}/L$ | $1.115 \times 10^{-3}$ | $1.003 \times 10^{-3}$ | $9.127 \times 10^{-4}$ |
| $AR = 2.0$   | $r_e$                 | 1.009                  | 1.008                  | 1.007                  |
|              | Grid                  | M1                     | M2                     | M3                     |
|              |                       | $(180 \times 180)$     | $(240 \times 240)$     | $(260 \times 260)$     |
| $AR = 4.0$   | $\Delta_{min,cell}/L$ | $2.23 \times 10^{-3}$  | $1.673 \times 10^{-3}$ | $1.545 \times 10^{-4}$ |
|              | $r_e$                 | 1.009                  | 1.006                  | 1.006                  |
| $AR = 8.0$   | Grid                  | M1                     | M2                     | M3                     |
|              |                       | $(140 \times 300)$     | $(160 \times 320)$     | $(200 \times 400)$     |
|              | $\Delta_{min,cell}/L$ | $2.678 \times 10^{-3}$ | $2.508 \times 10^{-3}$ | $2.007 \times 10^{-3}$ |
| $AR = 4.0$   | $r_e$                 | 1.01                   | 1.01                   | 1.008                  |
|              | Grid                  | M1                     | M2                     | M3                     |
|              |                       | $(140 \times 400)$     | $(160 \times 480)$     | $(200 \times 600)$     |
| $AR = 8.0$   | $\Delta_{min,cell}/L$ | $2.865 \times 10^{-3}$ | $2.508 \times 10^{-3}$ | $2.007 \times 10^{-3}$ |
|              | $r_e$                 | 1.01                   | 1.01                   | 1.008                  |
| $AR = 8.0$   | Grid                  | M1                     | M2                     | M3                     |
|              |                       | $(140 \times 480)$     | $(160 \times 540)$     | $(180 \times 560)$     |
|              | $\Delta_{min,cell}/L$ | $2.865 \times 10^{-3}$ | $2.508 \times 10^{-3}$ | $2.23 \times 10^{-3}$  |
| $AR = 8.0$   | $r_e$                 | 1.01                   | 1.009                  | 1.009                  |

Table 2.3: Numerical uncertainty for the mean Nusselt number on the inner periphery  $\overline{Nu}_i$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.2$ ) fluids are shown for different  $AR$  and  $r_i/L$  at  $Ra = 10^6$  and  $Pr = 500$  in the differentially heated from side walls configuration for CWT boundary condition.

| $Bn$ | $AR$  | $r_i/L = 0.125$ |       |       | $r_i/L = 16.0$ |      |      |
|------|-------|-----------------|-------|-------|----------------|------|------|
|      |       | M1              | M2    | M3    | M1             | M2   | M3   |
| 0    | 0.125 | 4.21            | 4.21  | 4.21  | 1.16           | 1.16 | 1.16 |
| 0.2  |       | 1.00            | 1.00  | 1.00  | 1.00           | 1.00 | 1.00 |
| 0    | 0.25  | 13.94           | 13.93 | 13.93 | 5.82           | 5.82 | 5.82 |
| 0.2  |       | 1.00            | 1.00  | 1.00  | 1.00           | 1.00 | 1.00 |
| 0    | 0.5   | 19.34           | 19.34 | 19.34 | 9.51           | 9.55 | 9.55 |
| 0.2  |       | 9.17            | 9.17  | 9.17  | 4.52           | 4.54 | 4.54 |
| 0    | 1.0   | 18.92           | 18.92 | 18.92 | 9.5            | 9.51 | 9.52 |
| 0.2  |       | 13.48           | 13.49 | 13.49 | 6.70           | 6.71 | 6.71 |
| 0    | 2.0   | 17.27           | 17.36 | 17.36 | 8.53           | 8.6  | 8.6  |
| 0.2  |       | 13.88           | 13.86 | 13.86 | 6.77           | 6.75 | 6.75 |
| 0    | 4.0   | 15.32           | 15.32 | 15.32 | 7.4            | 7.46 | 7.46 |
| 0.2  |       | 13.00           | 13.01 | 13.01 | 6.20           | 6.23 | 6.23 |
| 0    | 8.0   | 13.43           | 13.44 | 13.44 | 6.36           | 6.39 | 6.40 |
| 0.2  |       | 11.75           | 11.78 | 11.78 | 5.10           | 5.12 | 5.12 |

Table 2.4: Numerical uncertainty for the mean Nusselt number on the inner periphery  $\overline{Nu}_i$  for shear-thinning (i.e.  $n = 0.6$ ) and shear-thickening (i.e.  $n = 1.8$ ) fluids are shown for different  $AR$  and  $r_i/L$  at  $Ra = 10^6$  and  $Pr = 10^3$  in the differentially heated from side walls configuration for CWT boundary condition.

| $n$ | $AR$  | $r_i/L = 0.125$ |       |       | $r_i/L = 16.0$ |       |       |
|-----|-------|-----------------|-------|-------|----------------|-------|-------|
|     |       | M1              | M2    | M3    | M1             | M2    | M3    |
| 0.6 | 0.125 | 53.78           | 53.69 | 53.62 | 25.10          | 25.07 | 25.04 |
| 1.8 |       | 3.64            | 3.64  | 3.64  | 1.03           | 1.029 | 1.03  |
| 0.6 | 0.25  | 74.95           | 74.74 | 74.66 | 39.55          | 39.44 | 39.4  |
| 1.8 |       | 3.72            | 3.72  | 3.72  | 1.055          | 1.057 | 1.056 |
| 0.6 | 0.5   | 73.60           | 73.53 | 73.48 | 39.69          | 39.66 | 39.63 |
| 1.8 |       | 5.24            | 5.24  | 5.24  | 1.75           | 1.75  | 1.75  |
| 0.6 | 1.0   | 67.51           | 67.32 | 67.28 | 36.47          | 36.37 | 36.35 |
| 1.8 |       | 6.73            | 6.73  | 6.73  | 2.63           | 2.63  | 2.63  |
| 0.6 | 2.0   | 59.96           | 60.89 | 60.77 | 32.39          | 32.24 | 32.18 |
| 1.8 |       | 6.63            | 6.645 | 6.645 | 2.59           | 2.59  | 2.59  |
| 0.6 | 4.0   | 52.32           | 52.25 | 52.17 | 28.09          | 28.05 | 28.01 |
| 1.8 |       | 6.06            | 6.06  | 6.06  | 2.30           | 2.30  | 2.3   |
| 0.6 | 8.0   | 46.03           | 45.98 | 45.19 | 24.28          | 24.26 | 24.24 |
| 1.8 |       | 5.32            | 5.32  | 5.32  | 1.89           | 1.89  | 1.89  |

Table 2.5: Numerical uncertainty for the mean Nusselt number  $\overline{Nu}_{cy}$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.05$ ) fluids are shown for different  $AR$  and  $r_i/L$  at  $Ra = 5 \times 10^4$  and  $Pr = 500$  in the Rayleigh-Bénard configuration for CWT boundary condition.

|      |      | $r_i/L = 0.0$ |       |       | $r_i/L = 16.0$ |      |      |
|------|------|---------------|-------|-------|----------------|------|------|
| $Bn$ | $AR$ | M1            | M2    | M3    | M1             | M2   | M3   |
| 0    | 0.25 | 3.08          | 3.074 | 3.074 | 3.65           | 3.68 | 3.68 |
| 0.05 |      | 2.86          | 2.86  | 2.86  | 3.38           | 3.39 | 3.39 |
| 0    | 0.5  | 3.19          | 3.19  | 3.19  | 3.55           | 3.56 | 3.56 |
| 0.05 |      | 2.83          | 2.84  | 2.84  | 3.19           | 3.20 | 3.20 |
| 0    | 1.0  | 1.96          | 1.96  | 1.96  | 3.30           | 3.30 | 3.31 |
| 0.05 |      | 2.84          | 2.84  | 2.84  | 2.93           | 2.93 | 2.93 |
| 0    | 2.0  | 3.11          | 3.11  | 3.11  | 2.97           | 2.97 | 2.97 |
| 0.05 |      | 1.00          | 1.00  | 1.00  | 1.00           | 1.00 | 1.00 |

Table 2.6: Numerical uncertainty for the mean Nusselt number  $\overline{Nu}_{cy}$  for shear-thinning (i.e.  $n = 0.8$ ) and shear-thickening (i.e.  $n = 1.4$ ) fluids are shown for different  $AR$  and  $r_i/L$  at  $Ra = 5 \times 10^4$  and  $Pr = 10^3$  in the Rayleigh-Bénard configuration for CWT boundary condition.

| $n$ | $AR$ | $r_i/L = 0.0$ |              |              | $r_i/L = 16.0$ |      |      |
|-----|------|---------------|--------------|--------------|----------------|------|------|
|     |      | M1            | M2           | M3           | M1             | M2   | M3   |
| 0.8 | 0.25 | no            | no           | no           | 4.22           | 4.25 | 4.26 |
|     |      | steady-state  | steady-state | steady-state |                |      |      |
| 1.4 |      | 2.04          | 2.04         |              | 2.24           | 2.24 | 2.24 |
| 0.8 | 0.5  | 2.64          | 2.64         | 2.64         | 4.36           | 4.36 | 4.36 |
| 1.4 |      | 2.00          | 2.00         | 2.00         | 2.15           | 2.15 | 2.15 |
| 0.8 | 1.0  | 3.09          | 3.09         | 3.09         | 4.56           | 4.56 | 4.56 |
| 1.4 |      | 2.02          | 2.02         | 2.02         | 1.98           | 1.98 | 1.98 |
| 0.8 | 2.0  | 3.01          | 3.01         | 3.01         | 4.19           | 4.19 | 4.19 |
| 1.4 |      | 1.08          | 1.08         | 1.08         | 1.05           | 1.05 | 1.05 |

Accordingly, the M2 mesh configuration has been used for each  $AR$  for both Bingham and power-law fluids for the sake of accuracy of numerical results and computational economy in the Rayleigh-Bénard configuration. The representative shear-thinning (i.e.  $n = 0.8$ ) fluid in Table 2.6 for Rayleigh-Bénard configuration has been chosen different than the representative shear-thinning (i.e.  $n = 0.6$ ) fluid in Table 2.4 for differentially heated from side walls configuration. This is because, the steady-state solution does not exist for  $n = 0.6$  for  $r_i/L = 0$  and  $AR < 1$  in the Rayleigh-Bénard configuration.

It is worth noting that in the course of the grid independency analyses, it has been also ensured that a change in the computational grid does not lead to a change in flow pattern within the enclosure.

## 2.6 Benchmarking & Validation

In addition to the grid-dependency study, the simulation results for Newtonian, Bingham and power-law fluids have also been compared with the existing data in the literature. The mean Nusselt numbers  $\overline{Nu}$  for laminar natural convection of Newtonian fluids in square enclosures with differentially heated from side walls for  $10^3 \leq Ra \leq 10^6$  and  $Pr = 0.71$  have been compared to the well known benchmark data of de Vahl Davis [30] in Table 2.7 . It is evident from Table 2.7 that an excellent agreement has been achieved between the present results and the benchmark data (maximum difference in mean Nusselt number is found to be less than 1 %).

Table 2.7: Comparison of the mean Nusselt number  $\overline{Nu}$  for laminar natural convection of Newtonian fluids in square enclosures with differentially heated from side walls for  $10^3 \leq Ra \leq 10^6$  and  $Pr = 0.71$  with the benchmark of de Vahl Davis [30], ANSYS Fluent and OpenFOAM solvers used in the present study.

| $Ra$   | de Vahl Davis<br>[30] | ANSYS Fluent<br>(present study) | OpenFOAM<br>(present study) |
|--------|-----------------------|---------------------------------|-----------------------------|
| $10^3$ | 1.118                 | 1.118                           | 1.117                       |
| $10^4$ | 2.243                 | 2.245                           | 2.330                       |
| $10^5$ | 4.519                 | 4.520                           | 4.518                       |
| $10^6$ | 8.800                 | 8.823                           | 8.865                       |

The Bingham fluid simulations also have been compared to the benchmark data reported by Vola et al. [29] for natural convection of Bingham fluids in square enclosures with vertical walls with different uniform temperatures. It is worth noting that Vola et al. [29] reported only the values of yield stress  $\tau_y$ . According to definition of  $Bn$  given in Eq. 2.9, the yield stress values reported in Table 5 of Vola et al. [29] give rise to  $Bn = 3.0, 0.95$  and  $0.3$  for  $Ra = 10^4, 10^5$  and  $10^6$  respectively. The mean Nusselt number values obtained for the current numerical methodology for the aforementioned values of  $Bn$  and  $Ra$  for  $Pr = 1.0$  (as used in Ref. [29]) are found to be in good agreement with the corresponding values

reported by Vola et al. [29] (maximum difference in  $\overline{Nu}$  is found to be less than 3 %).

Additionally, Bayazitoglu et al. [53] analytically investigated laminar natural convection of a Bingham material in a vertical parallel plane channel with a constant temperature differential across the walls. Their analyses have been directed for an illustrative example from the oil industry (i.e. distance between plates ( $H = 0.0254m$ ), density ( $\rho = 1121.292kg/m^3$ ), plastic viscosity ( $\mu = 0.002068kg/(ms)$ ), yield stress of Bingham material ( $\tau = 2.394Pa$ ) in the range of  $0 \leq \beta\Delta T \leq 0.22$ ). Finally, the velocity and stress distributions have been reported by Bayazitoglu et al. [53]. Here, the comparison of maximum dimensionless stress (i.e.  $\tau_{xy}/\tau_o$  at  $x/H = 0$ ) versus  $\beta\Delta T$  at  $\tau_o/(\rho g H) = 0.008571$  between Bayazitoglu et al. [53] and present numerical scheme can be seen in Fig. 2.6. The excellent agreement has been achieved between present numerical scheme and analytical solution of Bayazitoglu et al. [53] for Bingham fluid flow (see Fig. 2.6).

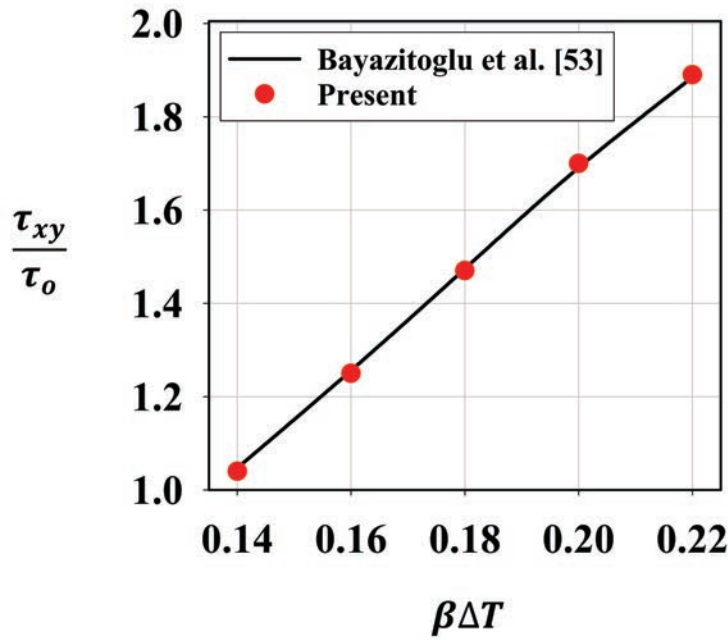


Figure 2.6: Comparison of maximum dimensionless stress (i.e.  $\tau_{xy}/\tau_o$  at  $x/H = 0$ ) versus  $\beta\Delta T$  between Bayazitoglu et al. [53] and present numerical scheme at  $\tau_o/(\rho g H) = 0.008571$ .

Moreover, the mean Nusselt number  $\overline{Nu}$  for laminar natural convection of Bingham fluids in square enclosures with differentially heated from side walls for  $Ra = 10^6$  and  $Pr = 100$  have been compared to data of Turan et al. [22] in Table 2.8 . It is evident from Table 2.8 that an excellent agreement has been achieved between the present results and the data of Turan et al. [22] (maximum difference in mean Nusselt number is found to be less than 2 %).

Table 2.8: Comparison of the mean Nusselt number  $\overline{Nu}$  for laminar natural convection of Bingham fluids in square enclosures with differentially heated from side walls for  $Ra = 10^6$  and  $Pr = 10^2$  with the data of Turan et al. [22], ANSYS Fluent and OpenFOAM solvers used in the present study.

| $Bn$ | Turan et al.<br>[22] | ANSYS Fluent<br>(present study) | OpenFOAM<br>(present study) |
|------|----------------------|---------------------------------|-----------------------------|
| 0    | 9.228                | 9.245                           | 9.284                       |
| 1    | 4.080                | 4.085                           | 4.064                       |
| 2    | 1.94                 | 1.95                            | 1.906                       |
| 3    | 1.044                | 1.05                            | 1.064                       |
| 3.5  | 1.000                | 1.00                            | 1.000                       |

Moreover, the variation of  $\overline{Nu}/\overline{Nu}_{n=1}$  with  $n$  for natural convection of power-law fluids in a square enclosure with differentially heated side walls subjected to CWT boundary condition presented by Turan et al. [54] has also been found to be excellent quantitative agreement with the numerical findings of Kim et al. [38] (see Fig. 2 of Ref. [54]). Accordingly, the mean Nusselt numbers  $\overline{Nu}$  for laminar natural convection of power-law fluids in square enclosures with differentially heated from side walls for  $Ra = 10^6$  and  $Pr = 10^3$  have been compared to data of Turan et al. [54] in Table 2.9. It is evident from Table 2.9 that an excellent agreement has been achieved between the present results and the data of Turan et al. [54] (maximum difference in mean Nusselt number is found to be less than 2 %). Here, it is also worth mentioning that the non-uniform Cartesian meshes used in the current analysis are more refined than Turan et al. [22] and Turan et al. [54] for Bingham and power-law fluids respectively.

Table 2.9: Comparison of the mean Nusselt number  $\overline{Nu}$  for laminar natural convection of power-law fluids in square enclosures with differentially heated from side walls for  $Ra = 10^6$  and  $Pr = 10^3$  with the data of Turan et al. [54], ANSYS Fluent and OpenFOAM solvers used in the present study.

| $n$ | Turan et al.<br>[54] | ANSYS Fluent<br>(present study) | OpenFOAM<br>(present study) |
|-----|----------------------|---------------------------------|-----------------------------|
| 0.6 | 35.629               | 35.625                          | 36.195                      |
| 0.8 | 16.264               | 16.260                          | 16.462                      |
| 1   | 9.246                | 9.250                           | 9.284                       |
| 1.4 | 4.276                | 4.275                           | 4.294                       |
| 1.8 | 2.560                | 2.560                           | 2.537                       |

## 2.7 Summary

This chapter presents all the necessary mathematical background (i.e. Bingham and power-law models) and numerical implementation (i.e. numerical scheme, grid independency study, validation) in detail for modelling natural convection of non-Newtonian fluids in enclosures. This chapter will be followed by the extensive literature review in chapter 3.

## Chapter 3

### Literature Review

Natural convection in enclosures filled with Newtonian fluids has been investigated progressively in the existing literature. Interested readers are referred to Ostrach [15], Bodenschatz et al. [16], Ganguli et al. [40], Catton [55], Gebhart et al. [56] and Khalifa [57] for an comprehensive review. In comparison, natural convection in enclosures filled with non-Newtonian fluids has received limited attention (see Fig. 3.1)

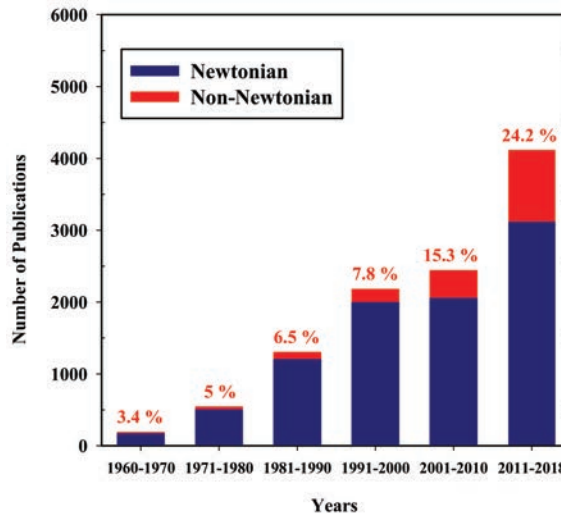


Figure 3.1: The published papers on natural convection in enclosure in the literature since 1960.

This can be easily seen in Fig. 3.1 which indicates the published papers on natural convection in enclosures for both Newtonian and non-Newtonian fluids in the literature since 1960. Fig. 3.1 is prepared by searching keywords “natural convection” and “enclosure” at the same time from 1960 to date in “Science Direct” academic search engine. It can also be easily seen from Fig. 3.1 that the number of the publications on natural convection in enclosures has increased dramatically for both Newtonian and non-Newtonian fluids but, the percentage of the studies have been directed on the case of non-Newtonian fluids is considerably lower than that for Newtonian fluids.

It is worth noting that this chapter, mainly concentrates the existing literature on natural convection of non-Newtonian fluids (i.e. yield stress and power-law fluids) in enclosures.

### 3.1 Natural Convection of Yield Stress Fluids in Enclosures

Yield stress fluid is a special type of non-Newtonian fluid, which acts as a solid and does not flow until a threshold yield stress is surpassed. Materials such as mud-slurries in oil drilling, toothpaste, molten chocolate and anti-drip paints are common examples of yield fluids [58]. Aqueous solutions of Carbopol is often used as a model yield stress fluid in laboratory experiments [24, 59, 60]. Some magneto- and electro-rheological fluids exhibit yield stress behaviour, and it is possible to modify the yield stress by applying electrical or magnetic fields [11, 61]. This may be useful in reducing heat losses or designing new adaptive thermal systems to control convective thermal transfer in enclosed spaces (i.e. preservation of canned food, chemical processing, storage of cryogenics, solar and nuclear power systems).

#### 3.1.1 Differentially Heated Side Walls Configuration

Vola et al. [29] pioneered analyses on natural convection of yield stress fluids obeying the Bingham model (the simplest form of yield stress fluids and shows

a linear strain rate dependence of viscous stress) in square enclosures with differentially heated vertical walls. However as the main interest of the work of Vola et al. [29] was in developing the numerical technique, the configuration was not investigated in detail and only limited results, for a single Prandtl number ( $Pr = 1$ ), were presented. It is also worth noting that  $Pr = 1$  is unrealistic for yield stress fluids. Accordingly, this analyses were extended by Turan et al. [22] and Turan et al. [62] to investigate the effects of nominal Rayleigh, Prandtl and Bingham numbers (i.e. non-dimensional yield stress) on the mean Nusselt number  $\overline{Nu}$  for the same configuration with both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions for a large range of Rayleigh ( $10^3 \leq Ra \leq 10^6$ ) and Prandtl numbers ( $0.1 \leq Pr \leq 100$ ). Their analyses indicated that the buoyancy-driven transport of yield stress fluids weakens with increasing Bingham number due to additional flow resistance arising from yield stress. This leads to a reduction in convective heat transfer rate, and thermal transport takes place purely due to conduction for large values of Bingham numbers since fluid flow practically stops under such a condition. Furthermore, new correlations were proposed by Turan et al. [22, 62] for the mean Nusselt number for natural convection of Bingham fluids in square enclosure with differentially heated vertical walls subjected to CWT [22] and CWHF [62] boundary conditions respectively.

It is well known that the aspect ratio  $AR$  (where  $H$  is the height and  $L$  is the enclosure length) plays a pivotal role in natural convection of Newtonian fluids in rectangular enclosures with differentially heated vertical side walls and interested readers are referred to references [40, 41] for further information. Here, it is worth noting that there is no compulsion for definition of the aspect ratio. Therefore in this thesis, the aspect ratio has been defined as ratio of height to width of enclosures (i.e.  $AR = H/L$ ) by following previous studies [26, 40, 41, 45–47].

The aspect ratio effects on natural convection of Bingham fluids in rectangular enclosures with differentially-heated side wall configuration has been numerically analysed for CWT [63] and CWHF [47] boundary conditions for nominal values of Rayleigh number in the range  $10^4 - 10^6$  and the aspect ratio range  $1/8$  to  $8$  for a single nominal Prandtl number ( $= 500$ ). It has been reported that for the CWT boundary condition for active walls, the aspect ratio at which the maximum

mean Nusselt number occurs is found to decrease with increasing Rayleigh number. In contrast, the value of mean Nusselt number increases monotonically with increasing aspect ratio in the case of CWHF boundary condition. Additionally, new correlations are proposed for the mean Nusselt number for natural convection of Bingham fluids in rectangular enclosures with differentially heated vertical walls subjected to CWT [63] and CWHF [47] boundary conditions respectively.

It is worth noting that the aforementioned analyses are restricted to rectangular enclosures but natural convection in cylindrical annular spaces is often more relevant in engineering applications, for example in insulating cylindrical tanks. Natural convection in vertical cylindrical annular enclosures has been studied both numerically [64] and experimentally [65, 66] for Newtonian fluids and the augmentation of the heat transfer rate has been reported with increasing internal radius. Thus, the curvature of cylindrical enclosures is also expected to play a key role in heat and momentum transport in natural convection of yield stress fluids.

Recently, laminar natural convection of yield stress fluids obeying Bingham model in square cross-sectional cylindrical annular enclosures with differentially heated vertical walls for both CWT [67] and CWHF [68] boundary conditions have been analysed numerically. It has been found that the heat transfer rate increases with increasing internal radius for Bingham fluids. The yield stress effects on natural convection in cylindrical annular enclosures are expected to be qualitatively similar to that in rectangular enclosures, but heat transfer rates are found to be quantitatively different to those in rectangular enclosures. Finally, new Nusselt number correlations have been proposed for laminar natural convection of both Newtonian and Bingham fluids in square cross-sectional cylindrical annular enclosures for both CWT and CWHF boundary conditions [67, 68]. These analyses have been extended for different aspect ratios of cylindrical annular enclosures  $AR$  (i.e.  $H/L$  where  $H$  is the enclosure height and  $L$  is the difference between outer and inner radius) by Yigit and Chakraborty [69] for both CWT and CWHF boundary conditions for active walls. Moreover, new Nusselt number correlations have been proposed for laminar natural convection of both Newtonian and Bingham fluids in rectangular cross-sectional cylindrical annular enclosures for both CWT and CWHF boundary conditions [69]. The main findings of studies di-

rected on natural convection of yield stress fluids in enclosures with differentially heated vertical walls in the existing literature are summarised in Table 3.1.

Table 3.1: Summary of the findings of existing analyses on natural convection of yield stress fluids in enclosures with differentially heated from side walls.

| Ref. | Enclosure                               | Boundary Cond. | $AR$    | $Ra$          | $Pr$         | $\overline{Nu}$ corr.      |
|------|-----------------------------------------|----------------|---------|---------------|--------------|----------------------------|
| [22] | Square                                  | CWT            | 1       | $10^3 - 10^6$ | $0.1 - 10^2$ | $f(Ra, Pr, Bn)$            |
| [62] | Square                                  | CWHF           | 1       | $10^3 - 10^6$ | $0.1 - 10^2$ | $f(Ra, Pr, Bn)$            |
| [63] | Rectangular                             | CWT            | 1/8 - 8 | $10^4 - 10^6$ | 7            | $f(Ra, Pr, Bn, AR)$        |
| [47] | Rectangular                             | CWT<br>CWHF    | 1/8 - 8 | $10^4 - 10^6$ | 500          | $f(Ra, Pr, Bn, AR)$        |
| [67] | Vertical Annulus<br>$r_i/L(0.125 - 16)$ | CWHF           | 1       | $10^3 - 10^6$ | $10 - 10^3$  | $f(Ra, Pr, Bn, r_i/L)$     |
| [68] | Vertical Annulus<br>$r_i/L(0.125 - 16)$ | CWT            | 1       | $10^3 - 10^6$ | $10 - 10^3$  | $f(Ra, Pr, Bn, r_i/L)$     |
| [69] | Vertical Annulus<br>$r_i/L(0.125 - 16)$ | CWT<br>CWHF    | 1/8 - 8 | $10^3 - 10^6$ | 500          | $f(Ra, Pr, Bn, AR, r_i/L)$ |

Refs. [67–69] are presented in this thesis.

### 3.1.2 Rayleigh-Bénard Configuration

Although superficially similar, natural convection in enclosures with differentially heated vertical walls is fundamentally different to that in the differentially heated horizontal wall configuration (e.g. Rayleigh-Bénard configuration). The convection sets in once a finite temperature difference is established between the active walls in differentially heated vertical wall configuration, whereas a critical Rayleigh number needs to be surpassed for the onset of fluid motion for the Rayleigh-Bénard configuration.

The analyses of Rayleigh-Bénard convection of yield stress fluids in the existing literature can be categorised into two broad categories. The first category deals with the critical temperature difference which determines the condition under which buoyancy force just overcomes the yield stress to induce flow within the enclosure, which has been extensively analysed using semi-analytical [70–72], numerical [73] and experimental [59, 60] methods. The second category focuses on the effects of Rayleigh, Prandtl and Bingham numbers on the heat transfer characteristics under conditions which are far beyond the critical condition for the onset of fluid motion. This aspect has principally been analysed based on semi-analytical [74] and numerical [23–25, 75–79] investigations.

The existing literature are summarised in Table 3.2 for both first and second categories about Rayleigh-Bénard convection of yield stress fluids in enclosures. In the first category, which deals with the critical conditions for onset of convection, Zhang et al. [70] reported that in the Rayleigh-Bénard configuration, Bingham fluid flow is unconditionally linearly stable under quiescent initial conditions. This suggests that only pure conduction results are obtained when a quiescent initial condition is used. However, an initial condition with an established flow leads to a convection-driven thermal transport for Bingham fluid flows. Accordingly, Balmforth and Rust [71] investigated the stability of yield stress fluids numerically and tested their numerical predictions experimentally on Carbopol solutions in a Rayleigh-Bénard configuration. Their findings suggest that yield strength will inhibit convection and heat transfer, and that the greater the yield strength, larger amplitude of the perturbation required to initiate convection within the enclosure [71]. Two experimental studies [59, 60] concentrated on the onset of

fluid motion of yield stress fluids in the Rayleigh-Bénard configuration , i.e. the subcritical or supercritical nature of the instability itself, rather than in the heat transfer characteristics well beyond critical conditions. As a result, they described the onset of the instability in terms of a critical yield number rather than in terms of a critical Rayleigh number.

Table 3.2: Summary of the findings of existing analyses on Rayleigh-Bénard convection of yield stress fluids in enclosures.

| Ref. | Enclosure                                         | Boundary Cond. | $AR$   | $Ra$              | $Pr$         | $\overline{Nu}$ corr.          |
|------|---------------------------------------------------|----------------|--------|-------------------|--------------|--------------------------------|
| [70] | Square                                            | CWT            | 1      | $Ra_{crit}$       | 1            | -                              |
| [71] | Square                                            | CWT            | 1      | $Ra_{crit}$       | Carbopol     | -                              |
| [72] | Rectangular                                       | CWT            | 0.5-5  | $Bn_{crit}$       | 1,7          | $Bn_{crit}$<br>$f(Bn, AR)$     |
| [73] | Rectangular                                       | CWT<br>CWHF    | 0.25-4 | $Ra_{crit}$       | 10-500       | $Ra_{crit}$<br>$f(Bn, Pr, AR)$ |
| [59] | Rectangular                                       | CWT            | 6-17.9 | $Bn_{crit}$       | Carbopol     | -                              |
| [60] | Rectangular                                       | CWT            | 19.3   | $Bn_{crit}$       | Carbopol     | -                              |
| [80] | Square                                            | CWT            | 1      | $1-5 \times 10^5$ | 10           | -                              |
| [25] | Square                                            | CWT            | 1      | $10^3 - 10^5$     | $0.1 - 10^2$ | $f(Ra, Pr, Bn)$                |
| [75] | Square                                            | CWHF           | 1      | $10^3 - 10^5$     | $0.1 - 10^2$ | $f(Ra, Pr, Bn)$                |
| [79] | Square<br>inclined<br>$\phi(0^\circ - 180^\circ)$ | CWT            | 1      | $10^3 - 10^5$     | 500          | $f(Ra, Pr, Bn, \phi)$          |
| [24] | Square                                            | CWHF           | 1      | $10^4 - 10^6$     | Carbopol     | $f(Ra, Y_o)$                   |
| [76] | Rectangular                                       | CWT            | 0.25-4 | $10^3 - 10^5$     | 500          | -                              |
| [77] | Rectangular                                       | CWHF           | 0.25-4 | $10^3 - 10^5$     | 500          | $f(Ra, Pr, Bn, AR)$            |
| [78] | Vertical Annulus<br>$r_i/L(0 - 24)$               | CWT<br>CWHF    | 1      | $10^3 - 10^5$     | 500          | $f$<br>$(Ra, Pr, Bn, r_i/L)$   |

Refs. [73, 76–78] are presented in this thesis.

The influence of viscoplastic rheological features on the Rayleigh-Bénard convection in square enclosure was investigated by Li et al. [81] numerically in order to compare with experimental results given by Darbouli et al. [59]. The yield stress fluid is modeled by a regularized Herschel-Bulkley model of viscosity and qualitatively good agreement obtained between numerical [81] and experimental results [59]. This analysis was extended for different aspect ratio (i.e. ratio of enclosure length to height) values by Aghighi et al. [82] using the regularized Herschel-Bulkley model of viscosity.

Davaille et al. [83] and Massmeyer et al. [84] investigated the development of a thermal plume in aqueous solution of Carbopol using experimental and numerical means respectively. Their results showed that thermal instabilities in a yield stress fluid behave very differently from those in Newtonian liquids. Also, the yield parameter, the ratio of stresses of thermal origin over the yield stress, is shown to be a key parameter which influences thermal transport [83, 84].

In a recent study, Turan et al. [73] carried out numerical simulations on laminar Rayleigh-Bénard convection of Bingham fluids in rectangular enclosures to analyse the critical Rayleigh number  $Ra_{crit}$  for which convection ceases to influence the thermal transport and thermal conduction becomes the principal heat transfer mechanism. It has been found that  $Ra_{crit}$  increases with increasing values of  $Bn$  and  $AR$  for both CWT and CWHF boundary conditions in case of both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids. Additionally,  $Ra_{crit}$  has been found to be insensitive to the change of  $Pr$  for Newtonian fluids (i.e.  $Bn = 0$ ), whereas  $Ra_{crit}$  increases with increasing  $Pr$  for Bingham fluids. Finally, based on a detailed scaling analysis, a correlation for  $Ra_{crit}$  has been proposed for Rayleigh-Bénard convection of Bingham fluids for both CWT and CWHF boundary conditions, which allows for a collapse of the critical Rayleigh number values for Bingham fluids in terms of a combination of  $Bn$ ,  $AR$  and  $Pr$ .

A number of investigations belonging to the second category (i.e. under conditions which are far beyond the critical condition for the onset of fluid motion), the effects of yield stress on the heat transfer characteristics specifically addressed where fluid flow is already initiated [23, 24, 62, 75–80]. Vikhansky [72] conducted numerical simulations to analyse effects of yield stress effects on Rayleigh-Bénard convection of Bingham fluids in square enclosures for high  $Ra$  and  $Bn$  values. It

has been observed that the presence of the yield stress significantly alters the dynamics of the system. In particular, the chaotic motion can stop suddenly sometimes, after a period of chaotic oscillations [72]. Subsequently, detailed parametric analyses have been directed on laminar Rayleigh-Bénard convection of Bingham fluids in square enclosures for CWT by Turan et al. [25] and for CWHF by Turan et al. [75] boundary conditions respectively for a large range of Rayleigh ( $10^3 \leq Ra \leq 10^5$ ) and Prandtl numbers ( $0.1 \leq Pr \leq 100$ ). As a result, it has been found that the mean Nusselt number  $\overline{Nu}$  decreases with increasing Bingham number in the case of yield stress fluids, and, for large values of Bingham number  $Bn$ , the value of  $\overline{Nu}$  rapidly drops to unity ( $\overline{Nu} = 1.0$ ) as thermal conduction dominates the thermal transport. The variation of  $\overline{Nu}$  in the Rayleigh-Bénard configuration was found to be markedly different from the corresponding variation of  $\overline{Nu}$  with  $Bn$  for the same nominal values of  $Ra$  and  $Pr$  in the differentially heated vertical sidewall configuration. Additionally, it has been demonstrated by Turan et al. [25, 75] that convection could be sustained up to a higher value of Bingham number due to stronger convection arising from higher temperature difference between horizontal walls in the case of CWT boundary condition than in the corresponding CWHF configuration. Furthermore, new correlations were proposed by Turan et al. [25, 75] for the mean Nusselt number for Rayleigh-Bénard convection of Bingham fluids in square enclosures for both CWT [25] and CWHF [75] boundary conditions.

Several analyses [23, 79] focused on the parameters such as inclination angle and partially heated bottom wall respectively on natural convection of Bingham fluids in square enclosures. Finally, they proposed new correlations which predict effects of inclination angle and partially heated bottom wall effects on the mean Nusselt number of Bingham fluids for natural convection in square enclosures [23, 79].

Yigit et al. [76] and Yigit and Chakraborty [77] investigated the effects of aspect ratio on Rayleigh-Bénard convection of Bingham fluids in rectangular enclosures for CWT and CWHF boundary conditions respectively. They indicated that the convective (diffusive) transport weakens (strengthens) with increasing aspect ratio (height: length) for Rayleigh-Bénard convection of Bingham fluids in rectangular enclosures. The relative contribution of convection to the overall thermal transport diminishes (strengthens) with increasing (decreasing)  $AR$  for a given

set of values of  $Ra$  and  $Pr$  for both Newtonian and Bingham fluids. Thus, the thermal transport is principally conduction-dominated for tall enclosures. Finally,  $\overline{Nu}$  correlations proposed in Refs. [25, 75] were extended by Yigit et al. [76] and Yigit and Chakraborty [77] for different  $AR$  values on Rayleigh-Bénard convection of Bingham fluids in rectangular enclosures for both CWT [76] and CWHF [77] boundary conditions.

Furthermore, all the analyses presented in Table 3.2 were carried out for rectangular enclosures except for Ref. [78]. Recent findings of Yigit et al. [78] revealed that the convective thermal transport strengthens with increasing internal cylinder radius for laminar Rayleigh-Bénard convection of Bingham fluids in square cross-sectional cylindrical annular enclosures. Also, it has been found that, the mean Nusselt number does not show a monotonic increase with increasing  $Ra$  for small values of normalised internal cylinder radius (i.e.  $r_i/L$ ) because of the change in flow pattern (i.e. number of convection rolls/cells). By contrast, the mean Nusselt number monotonically increases with increasing  $Ra$ , and only one cell flow pattern is obtained for large values of  $r_i/L$ . This analyses also extended in thesis (presented in chapter 6) for different aspect ratio for Rayleigh-Bénard convection of Bingham fluids in rectangular cross-sectional cylindrical annular enclosures for CWT and CWHF boundary conditions.

### 3.2 Natural Convection of Power-law Fluids in Enclosures

Shear-thinning (shear-thickening) fluids are special type of non-Newtonian fluids, which show a decrease (increase) in viscosity with increasing shear rate. Many common man made and biological fluids exhibit shear-thinning (e.g., ketchup and blood) and shear-thickening (e.g., mixtures of corn starch and water; so-called “bulletproof” custard) behaviours. The presence of a small amount of high molecular weight polymer in water-based solvents often exhibits shear thinning behaviour [85] that can be approximated as a power law over a range of shear rates. Thus, to a first approximation, a reasonable way to include non-Newtonian shear viscosity variations is through a simple generalized Newtonian fluid (GNF)

model of power law type, and these fluids has been termed as power-law fluids in the existing literature.

### 3.2.1 Differentially Heated Side Walls Configuration

Kim et al. [38] analysed transient natural convection of shear-thinning (i.e.  $n < 1$ ) power-law fluids in a square enclosure with differentially heated vertical side walls subjected to constant wall temperature (CWT) boundary condition. It has been found by Kim et al. [38] that the mean Nusselt number  $\overline{Nu}$  increases with decreasing power-law index  $n$  for a given set of values of nominal Rayleigh and Prandtl numbers (i.e.  $Ra$  and  $Pr$ ). Turan et al. [54, 86] proposed a Nusselt number correlation, guided by a scaling analysis, based on steady-state two-dimensional numerical simulations, and demonstrated that the strength of natural convection weakens and the mean Nusselt number  $\overline{Nu}$  decreases with increasing  $n$  for both CWT and constant wall heat flux (CWHF) boundary conditions for differentially heated vertical walls in square enclosures. The strengthening of heat transfer rate with shear-thinning behaviour is consistent with previous findings on natural convection in rectangular enclosures with differentially heated vertical walls [34, 35] and also in the findings for natural convection from a heated horizontal cylinder [87] and a heated tilted square [88] immersed in power-law fluids in the laminar flow regime.

Turan et al. [54, 86] also demonstrated that  $\overline{Nu}$  remains independent of  $Pr$  and depends only on  $Ra$  and  $n$ , which is consistent with previous analyses by Lamsaadi et al. [34, 35] who analysed the effects of the power-law index on natural convection in the high Prandtl number limit for both tall and shallow enclosures where the sidewalls were subjected to the CWHF boundary condition.

Turan et al. [89] also analysed the effects of aspect ratio (height: length) on natural convection of power-law fluids in rectangular enclosure with differentially heated vertical walls for a range of different values of  $Ra$  for  $1/8 \leq AR \leq 8$  (where  $AR = H/L$  is the aspect ratio with  $H$  and  $L$  being the enclosure height and length respectively) and proposed correlations for both CWT and CWHF boundary conditions. It was reported that  $\overline{Nu}$  follows a non-monotonic trend with the variation of aspect ratio  $AR$  for a given set of values of the  $Ra$  and

$Pr$  for shear-thinning ( $n < 1$ ), Newtonian ( $n = 1$ ) and shear-thickening ( $n > 1$ ) fluids for the CWT boundary condition whereas a monotonic increase in  $\overline{Nu}$  was obtained for increasing values of  $AR$  in the case of CWHF boundary condition irrespective of the value of  $n$ . This non-monotonic trend for the CWT boundary condition is caused by the competing effects of thermal convective and diffusive transports with  $AR$  variation. Finally, new correlations were proposed for  $\overline{Nu}$  for power-law fluids in both the CWT and CWHF boundary conditions by Turan et al. [89], which were shown to satisfactorily capture the correct qualitative and quantitative behaviour of  $\overline{Nu}$  in response to the changes in  $Ra$ ,  $AR$  and  $n$ .

Similarly, effects of aspect ratio on natural convection of power-law fluids in rectangular enclosure with differentially heated vertical walls analysed by Alloui and Vasseur [90] using Carreau–Yasuda and power-law non-Newtonian fluid models. A comparison is made between the predictions of the two rheological models. It has been expressed that for sufficiently high values of  $Ra$ , the boundary layer regime is reached for which the values of the mean Nusselt numbers are approximately the same according to the two models.

All these analyses have been carried out for rectangular enclosures but cylindrical enclosures are more relevant to practical applications (e.g. solar collectors, heat exchangers, etc.). Accordingly, the two-dimensional steady-state natural convection of power-law fluids is studied numerically between two concentric horizontal cylinders with different constant temperatures by Matin et al. [91]. The results indicate that with increasing the power-law index from 0.6 to 1.4, the mean Nusselt number decreases. In the best case among the range of parameters considered here the heat transfer rate for pseudo-plastic fluid ( $n = 0.6$ ) is 170 % higher than the Newtonian one and for dilatant fluid ( $n = 1.4$ ) the heat transfer rate is 43 % lower than the Newtonian fluid [91].

The effects of ratio of the internal radius to enclosure height (i.e.  $r_i/L$ ) on natural convection of power-law fluids in square cross-sectional cylindrical annular enclosures with differentially heated vertical walls have recently been investigated by Yigit et al. [92] and Yigit et al. [93] for both CWT and CWHF boundary conditions respectively. It has been found that convective transport strengthens with increasing  $r_i/L$  for natural convection of power-law fluids in square cross-sectional cylindrical annular spaces with differentially heated vertical walls

regardless of boundary conditions. Finally, new Nusselt number correlations have been proposed for laminar natural convection of both Newtonian and power-law fluids in square cross-sectional cylindrical annular enclosures for both CWT and CWHF boundary conditions [92, 93]. These analyses have been extended for different aspect ratio of cylindrical annular enclosures  $AR$  (i.e.  $H/L$  where  $H$  is the enclosure height and  $L$  is the difference between outer and inner radius) by Yigit and Chakraborty [94] for both CWT and CWHF boundary conditions. Subsequently, new Nusselt number correlations have been proposed for laminar natural convection of both Newtonian and power-law fluids in rectangular cross-sectional cylindrical annular enclosures for both CWT and CWHF boundary conditions [94]. The main findings of the existing studies directed to the analysis of natural convection of power-law fluids in enclosures with differentially heated vertical side walls in existing literature are summarised in Table 3.3.

Table 3.3: Summary of the findings of existing analyses on natural convection involving non-Newtonian fluids obeying power-law model of viscosity in enclosures with differentially heated from side walls.

| Ref. | Enclosure                                  | Boundary Cond. | $AR$  | $Ra$<br>$Pr$                   | $n$         | $\overline{Nu}$ corr.     |
|------|--------------------------------------------|----------------|-------|--------------------------------|-------------|---------------------------|
| [38] | Square                                     | CWT            | 1     | $10^5 - 10^7$<br>$10^2 - 10^4$ | $0.6 - 1$   | $f(Ra, Pr, n)$            |
| [54] | Square                                     | CWT            | 1     | $10^3 - 10^6$<br>$10 - 10^5$   | $0.6 - 1.8$ | $f(Ra, Pr, n)$            |
| [86] | Square                                     | CWHF           | 1     | $10^3 - 10^6$<br>$10 - 10^5$   | $0.6 - 1.8$ | $f(Ra, Pr, n)$            |
| [35] | Rectangular                                | CWHF           | 0.1-1 | $0 - 10^6$<br>$1 - 10^2$       | $0.6 - 1.4$ | $f(Ra, Pr, AR, n)$        |
| [34] | Rectangular                                | CWHF           | 1-12  | $0 - 10^6$<br>$1 - 10^2$       | $0.6 - 1.4$ | $f(Ra, Pr, AR, n)$        |
| [89] | Rectangular                                | CWT<br>CWHF    | 1/8-8 | $10^3 - 10^6$<br>$10^3$        | $0.6 - 1.8$ | $f(Ra, Pr, AR, n)$        |
| [92] | Vertical<br>Annulus<br>$r_i/L(0.125 - 16)$ | CWT            | 1     | $10^3 - 10^6$<br>$10^2 - 10^4$ | $0.6 - 1.8$ | $f(Ra, Pr, r_i/L, n)$     |
| [93] | Vertical<br>Annulus<br>$r_i/L(0.125 - 16)$ | CWHF           | 1     | $10^3 - 10^6$<br>$10^2 - 10^4$ | $0.6 - 1.8$ | $f(Ra, Pr, r_i/L, n)$     |
| [94] | Vertical<br>Annulus<br>$r_i/L(0.125 - 16)$ | CWT<br>CWHF    | 1/8-8 | $10^3 - 10^6$<br>$10^3$        | $0.6 - 1.8$ | $f(Ra, Pr, AR, r_i/L, n)$ |

Refs. [92–94] are presented in this thesis.

### 3.2.2 Rayleigh-Bénard Configuration

A number of studies also concentrated on Rayleigh-Bénard convection of power-law fluids [9, 10, 32, 39, 42–44, 52, 85, 95–103] over the last decade. The main

findings of these studies are summarised in Table 3.4. It is shown in Table 3.4 that several studies focused on the critical condition for the onset of the flow for power-law fluids [43, 44, 52, 96, 99, 100], while the others analysed the heat transfer characteristics of power-law fluids when the fluid flow is well established [9, 10, 32, 39, 42, 44, 52, 85, 95, 97, 98, 100–103].

Tien et al. [96] analysed onset of convection of shear-thinning (i.e.  $n < 1$ ) fluids in idealized situation of a horizontal layer of liquid of infinite extent confined between two rigid horizontal surfaces with heating from below both theoretically and experimentally (using aqueous solutions of carboxy-methyl-cellulose). It has been shown that critical Rayleigh number  $Ra_{crit}$ , which needs to be surpassed for initiating convection, decreases with decreasing value of  $n$ . Their theoretical results and experimental results agree substantially with the results based upon the approximate theoretical analysis.

Solomатов and Barr [43] and Kaddiri et al. [44] analysed the onset of convection in strongly temperature-dependent power-law fluids in rectangular enclosures with differentially heated horizontal walls with higher bottom wall temperature. The same configuration was investigated by Albaalbaki and Khayat [103] to analyse three-dimensional pattern formation and the onset of convection in Generalised Newtonian Fluids (GNF) using the Carreau-Bird model of viscosity (similar to the power-law model but with limiting zero and infinite shear rate viscosities). Additionally, Alloui et al. [99] conducted an analytical and numerical study of natural convection in a shallow enclosure filled with power-law fluids. It was demonstrated by Alloui et al. [99] that, for shear-thinning fluids, the onset of convection is subcritical. For shear thickening fluids, convection is found to occur at a supercritical Rayleigh number equal to zero [99]. However, Lamsaadi et al. [39] also reported that the presence of flow within the enclosure does not indicate appreciable heat transfer by advection for shear-thickening fluids. This essentially means that the heat transfer within the enclosure takes place predominantly by thermal conduction for nominal Rayleigh number (i.e.  $Ra$ ) is smaller than the threshold nominal Rayleigh number (i.e.  $Ra_{\overline{Nu}=1}$ ) for which mean Nusselt number values deviate from unity in the third decimal place (i.e.  $\overline{Nu} \approx 1.001$ ) for shear thickening fluids in spite of flow being present in the enclosure for  $Ra > 0$ . The findings of Alloui et al. [99] and Lamsaadi et al. [39] were also subsequently

confirmed by Yigit et al. [52] who also investigated the effects of aspect ratio  $AR$  (ratio of enclosure height to length) on  $Ra_{crit}$  of inelastic non-Newtonian fluids obeying the power-law model of viscosity within rectangular enclosures where the horizontal walls are subjected to constant wall temperatures with the bottom wall at higher temperature.

For analysing heat transfer characteristics of power-law fluids beyond the critical conditions, Ozoe and Churchill [32] pioneered the analysis of natural convection of power-law and Ellis model fluids in rectangular enclosures. Lamsaadi et al. [39] and Ohta et al. [95] analysed natural convection of shear-thinning fluids in a configuration where the horizontal walls are differentially heated with higher bottom wall temperature. Both Lamsaadi et al. [39] and Ohta et al. [95] reported the strengthening of natural convection in rectangular enclosures for shear-thinning fluids, which was in agreement with an experimental study undertaken by Inaba et al. [9] in the differentially heated horizontal wall configuration. Also, Barth and Carey [102] used limiting viscosities at low and high shear rates, to simulate three-dimensional natural convection in cubic enclosures with differentially heated horizontal walls and reported an augmented heat transfer rate for shear-thinning fluids in accordance with the experimental findings of Leung et al. [104].

Turan et al. [97] and Turan et al. [98] analysed the effects of nominal Rayleigh and Prandtl numbers (i.e.  $Ra$  and  $Pr$ ) and power-law exponent  $n$  for natural convection of both shear-thinning (i.e.  $n < 1$ ) and shear-thickening (i.e.  $n > 1$ ) fluids in square enclosures with differentially heated horizontal walls for both the CWT [97] and the CWHF [98] boundary conditions. Turan et al. [97] and Turan et al. [97, 98] also identified different regimes of laminar convection of power-law fluids in two-dimensional square enclosures and proposed correlations for the mean Nusselt number  $\overline{Nu}$ , which remain valid for a large range of values of  $Ra$ ,  $Pr$  and  $n$  for both CWT and CWHF boundary conditions. These analyses extended for different aspect ratio  $AR$  (i.e.  $H/L$  where  $H$  and  $L$  are the enclosure height and length respectively) on Rayleigh-Bénard convection of power-law fluids in rectangular enclosures for CWT [52] and CWHF [100] boundary conditions. Yigit et al. [52] showed that the flow pattern and the mean Nusselt number  $\overline{Nu}$  are dependent on initial conditions and it is possible to obtain different steady-state

solutions for different initial conditions. Furthermore, it is possible to obtain a steady solution for shear-thinning (i.e.  $n < 1$ ) fluids only for some initial conditions, whereas other initial conditions may yield unsteady flow patterns. This is consistent with previous analysis of Albaalbaki and Khayat [103] who demonstrated transitions between different flow patterns for shear-thinning (i.e.  $n < 1$ ) fluids in the supercritical regime of convection based on their analysis of non-Newtonian fluids with the Carreau-Bird model of viscosity. By contrast, convection takes place only by simple roll patterns in the case of shear-thickening (i.e.  $n > 1$ ) fluids according to Ref. [103], which is consistent with the observations of Yigit et al. [52]. The key simulation results have been explained based on scaling arguments and the scaling relations have been utilised to identify different regimes of natural convection of power-law fluids accounting for aspect ratio effects. Furthermore, the non-monotonic variation of the mean Nusselt number  $\overline{Nu}$  in response to the changes in  $Ra$ ,  $AR$ ,  $n$  and its dependence on the choice of initial condition severely limit the usefulness of a correlation for  $\overline{Nu}$  for these conditions. However, Yigit and Chakraborty [100] proposed correlations for the mean Nusselt number  $\overline{Nu}$ , which remain valid for a large range of values of  $Ra$ ,  $Pr$ ,  $AR$  and  $n$  for CWHF boundary condition for differentially heated horizontal walls.

The effects of inclination angle on natural convection of power-law fluids with different aspect ratios for different nominal values of Rayleigh and Prandtl number have been recently analysed by Khezzar et al. [42] for different values of  $AR$ . They proposed correlations for the mean Nusselt number  $\overline{Nu}$ , which remain valid for a large range of values of  $Ra$ ,  $Pr$ ,  $AR$ ,  $n$  and inclination angle for CWT boundary condition.

All the aforementioned analyses on Rayleigh-Bénard convection of power-law fluids in enclosures have been carried out for rectangular enclosures except for a few analyses [85, 101]. Recently, an experimental analysis by Darbouli et al. [85] reported that 0.1-0.2 % aqueous solutions of xanthan gum exhibit shear-thinning behaviour and significant enhancement of convective heat transfer rate in comparison to water has been observed for laminar Rayleigh-Bénard convection in cylindrical enclosures. Moreover, the effects of normalised cylinder radius (i.e.  $r_i/L$ ) have been analysed by Yigit et al. [101] on Rayleigh-Bénard convection

of power-law fluids in square cross-sectional cylindrical annular enclosures for both CWT and CWHF boundary conditions. It is found that the mean Nusselt number does not show a monotonic increase with increasing (decreasing)  $Ra$  ( $n$ ) especially for small values of  $r_i/L$  for a given value of  $Pr$  due to the changes in flow patterns (i.e. number of convection cells). However, the mean Nusselt number and flow patterns for large values of  $r_i/L$  approach those for square enclosures ( $r_i/L \rightarrow \infty$ ) for both CWT and CWHF boundary conditions. Additionally, the critical Rayleigh number  $Ra_{crit}$  for the onset of convection has been found to be mostly insensitive to the value of  $r_i/L$  for both CWT and CWHF boundary conditions [101]. This analyses also extended in this thesis (presented in chapter 7) for different aspect ratio for Rayleigh-Bénard convection of power-law fluids in rectangular cross-sectional cylindrical annular enclosures for CWT and CWHF boundary conditions.

Table 3.4: Summary of the findings of existing analyses on Rayleigh-Bénard convection involving non-Newtonian fluids obeying power-law model of viscosity in enclosures.

| Ref.  | Enclosure                                           | Boundary Cond. | $AR$  | $Ra$<br>$Pr$                       | $n$          | $\overline{Nu}$ corr.                            |
|-------|-----------------------------------------------------|----------------|-------|------------------------------------|--------------|--------------------------------------------------|
| [96]  | Horizontal Layers                                   | CWT            | -     | $Ra_{crit}$<br>CMC                 | 0.4-1        | -                                                |
| [43]  | Horizontal Layers                                   | CWT            | -     | $Ra_{crit}$<br>$\mu_\infty = f(T)$ | 1-4          | -                                                |
| [99]  | Rectangular                                         | CWHF           | 1/6   | $Ra_{crit}$                        | 0.8-1.2      | -                                                |
| [32]  | Square                                              | CWT            | 1     | $6 \times 10^3$<br>$0.1 - 10$      | 0.4-1        | -                                                |
| [97]  | Square                                              | CWHF           | 1     | $10^3 - 10^5$<br>$10 - 10^6$       | 0.6-1.8      | $f(Ra, Pr, n)$                                   |
| [98]  | Square                                              | CWT            | 1     | $10^3 - 10^5$<br>$10 - 10^6$       | 0.6-1.8      | $f(Ra, Pr, n)$                                   |
| [44]  | Square                                              | CWHF           | 1     | $0 - 10^5$<br>$\mu_\infty = f(T)$  | 0.6-1.4      | -                                                |
| [39]  | Rectangular                                         | CWHF           | 1/12  | $1 - 5 \times 10^3$<br>$1 - 10^2$  | 0.6-1.4      | -                                                |
| [52]  | Rectangular                                         | CWT            | 1/4-4 | $10^3 - 10^5$<br>$10^3$            | 0.6-1.8      | $Ra_{crit}$<br>$f(Ra, Pr, AR, n)$                |
| [100] | Rectangular                                         | CWHF           | 1/4-4 | $10^3 - 10^5$<br>$10^3$            | 0.6-1.8      | $Ra_{crit}, \overline{Nu}$<br>$f(Ra, Pr, AR, n)$ |
| [42]  | Rectangular inclined<br>$\phi(0^\circ - 180^\circ)$ | CWT            | 1/8-1 | $10^4 - 10^5$<br>$10^2 - 10^4$     | 0.6-1.4      | $f(Ra, Pr, AR, \phi, n)$                         |
| [101] | Vertical Annulus<br>$r_i/L(0 - 24)$                 | CWT<br>CWHF    | 1     | $10^3 - 10^5$<br>$10 - 10^4$       | 0.6 -<br>1.8 | $f(Ra, Pr, AR, r_i/L, n)$                        |

Refs. [52, 100, 101] are presented in this thesis.

### 3.3 Summary of Literature Review

This chapter presents an extensive review of studies about natural convection of both yield stress and power-law fluids in enclosures that have been studied over past decades. The findings of the effects of nominal Rayleigh, Prandtl numbers, thermal boundary conditions, and initial conditions on natural convection of both yield stress and power-law fluids in both rectangular and cylindrical annular enclosures were primary focus here with the heat transfer correlations. The key findings of this thesis have been presented in chapters 4 - 7 which are used to address gaps in the existing literature.



## Chapter 4

# Differentially Heated Side Wall Configuration: Bingham Fluids

Numerical findings on laminar natural convection of yield stress fluids obeying the Bingham model in axisymmetric rectangular cross-sectional cylindrical annular enclosures with differentially heated vertical side walls (see Fig. 2.1) have been presented in this chapter. Yield stress effects on heat and momentum transport have been investigated for a range of different values of normalized internal radius  $r_i/L$  (i.e.  $0.125 \leq r_i/L \leq 16$ , where  $r_i$  is the internal cylinder radius and  $L$  is the difference between outer and inner radii); aspect ratio  $AR$  (i.e.  $0.125 \leq AR = H/L \leq 8$  where  $H$  is the enclosure height); nominal Rayleigh number  $Ra$  (i.e.  $10^3 \leq Ra \leq 10^6$ ) and Prandtl number  $Pr$  range (i.e.  $10 \leq Pr \leq 10^3$ ) for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions.

### 4.1 Scaling Analysis

A detailed scaling analysis has been conducted here to elucidate the effects of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $AR$ ,  $r_i/L$  on the mean Nusselt number  $\overline{Nu}_i$ . The velocity component in the vertical direction (i.e.  $w$ ) can be scaled by equating the order of magnitudes of inertial and buoyancy terms as:  $w \sim \sqrt{g\beta(T_H - T_C)LAR}$  ( $w \sim \sqrt{g\beta q_i \delta_{th} LAR/k}$ )

for CWT (CWHF) boundary conditions where  $\delta_{th}$  is the thermal boundary layer thickness on the vertical walls. Based on the continuity equation one obtains:

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} \sim \left( \frac{u}{r} + \frac{u}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{H} \quad (4.1)$$

which leads to:

$$u \sim \frac{w}{AR(r+L)} \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{1}{\sqrt{AR}} \frac{\sqrt{Ra_{CWT}Pr}}{(1+L/r_i)} \quad (4.2)$$

$$u \sim \frac{w}{AR(r+L)} \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{1}{\sqrt{AR}} \frac{\sqrt{Ra_{CWHF}Pr}}{(1+L/r_i)} \sqrt{\frac{\delta_{th}}{L}} \quad (4.3)$$

Similarly, equating the order of magnitudes of inertial and viscous terms in the vertical direction yields:

$$\rho \frac{w^2}{H} \sim \frac{1}{\delta} \left( \tau_y + \mu \frac{w}{\delta} \right) \quad (4.4)$$

Using Eqs. 4.2 and 4.3 in Eq. 4.4 leads to:

$$\frac{\delta_{th}}{L} \sim \frac{1}{f_2} \sqrt{\frac{Pr}{Ra_{CWT}}} \left\{ \frac{Bn_{CWT}}{2} + \sqrt{\left( \frac{Bn_{CWT}}{2} \right)^2 + \frac{Ra_{CWT}AR}{Pr}} \right\} \quad (4.5)$$

$$\left( \frac{\delta_{th}}{L} \right)^{5/2} \sqrt{\frac{Ra_{CWHF}}{Pr}} \sim \frac{\sqrt{AR}}{f_3^2} + \frac{Bn_{CWHF}}{f_3} \left( \frac{\delta_{th}}{L} \right)^{1/2} \quad (4.6)$$

where  $\delta$  is the hydrodynamic boundary layer thickness on the vertical wall and  $f_2(Ra_{CWT}, Pr, Bn_{CWT}, AR, r_i/L)$ ,  $f_3(Ra_{CWHF}, Pr, Bn_{CWHF}, AR, r_i/L)$ , depicts the ratio of hydrodynamic and thermal boundary thickness (i.e.  $\delta/\delta_{th}$ ) for CWT and CWHF boundary conditions, respectively. Eq. 4.5 clearly shows that  $\delta_{th}/L$  increases with increasing  $AR(Bn_{CWT})$  for a given set of values of  $Ra$  and  $Pr$ . Moreover, Eq. 4.6 indicates that an exact analytical solution does not exist for the CWHF boundary condition. However, some behaviours can be obtained based on limiting assumptions. For instances, for small values of  $Bn_{CWHF}$  the contribution of  $Bn_{CWHF}(\delta_{th}/L)^{1/2}/f_3$  can be ignored in comparison to  $\sqrt{AR}/f_3^2$ ,

which leads to  $\delta_{th}/L \sim (AR)^{0.2}(Pr/Ra_{CWHF})^{0.2}f_3^{-0.8}$ . This limiting assumption also suggests that  $\delta_{th}/L$  increases with increasing  $AR$  ( $Bn_{CWHF}$ ) for a given set of values of  $Ra$  and  $Pr$ . Since the qualitative trend is expected to be the same for both CWT and CWHF boundary conditions, the scaling analysis is continued for cylindrical annular enclosures in the CWT boundary condition.

Based on the scaling estimates in Eqs. 4.2 and 4.5, it is possible to estimate the effective viscosity in vertical boundary layer (i.e.  $\mu_{eff}^V$ ) in the following manner ( $\mu_{eff}^V \sim \mu + \tau_y \delta/w$ ):

For CWT boundary condition:

$$\frac{\mu_{eff}^V}{\mu} \sim \left[ 1 + \frac{Bn_{CWT}}{\sqrt{AR}} \sqrt{\frac{Pr}{Ra_{CWT}}} \left\{ \frac{Bn_{CWT}}{2} + \sqrt{\left( \frac{Bn_{CWT}}{2} \right)^2 + \sqrt{\frac{Ra_{CWT} AR}{Pr}}} \right\} \right] \quad (4.7)$$

Similarly, equating order of magnitudes of inertial and viscous terms in the radial direction (i.e.  $\rho u^2/L \sim 1/\delta_1(\tau_y + \mu u/\delta_1)$ ) to estimate the effective viscosity in horizontal boundary layer (i.e.  $\mu_{eff}^H$ ) in the following manner ( $\mu_{eff}^H \sim \mu + \tau_y \delta_1/u$ ):

$$\frac{\mu_{eff}^H}{\mu} \sim \left\{ 1 + Bn_{CWT} \sqrt{AR} (1 + L/r_i) \sqrt{\frac{Pr}{Ra_{CWT}}} \right\} \times \left\{ \frac{Bn_{CWT} AR (1 + L/r_i)^2}{2} + \sqrt{\left( \frac{Bn_{CWT} AR (1 + L/r_i)^2}{2} \right)^2 + \sqrt{\frac{Ra_{CWT} AR}{Pr}} (1 + L/r_i)^2} \right\} \quad (4.8)$$

where  $\delta_1$  is the hydrodynamic boundary layer thickness on the horizontal wall. Using Eqs. 4.7 and 4.8, the effective Rayleigh numbers (i.e.  $Ra_{eff}^V$  and  $Ra_{eff}^H$ ) in both vertical and horizontal boundary layers can be estimated in the following

manner (i.e.  $Ra_{eff} = \rho g \beta (T_H - T_C) L^3 / \mu_{eff} \alpha$ ):

$$Ra_{eff}^V \sim Ra_{CWT} / \left[ 1 + \frac{Bn_{CWT}}{\sqrt{AR}} \sqrt{\frac{Pr}{Ra_{CWT}}} \left\{ \frac{Bn_{CWT}}{2} + \sqrt{\left( \frac{Bn_{CWT}}{2} \right)^2 + \sqrt{\frac{Ra_{CWT} AR}{Pr}}} \right\} \right] \quad (4.9)$$

$$Ra_{eff}^H \sim Ra_{CWT} / \left\{ 1 + Bn_{CWT} \sqrt{AR} (1 + L/r_i) \sqrt{\frac{Pr}{Ra_{CWT}}} \right\} \times \left\{ \frac{Bn_{CWT} AR (1 + L/r_i)^2}{2} + \sqrt{\left( \frac{Bn_{CWT} AR (1 + L/r_i)^2}{2} \right)^2 + \sqrt{\frac{Ra_{CWT} AR}{Pr}} (1 + L/r_i)^2} \right\} \quad (4.10)$$

Eqs. 4.9 and 4.10 indicate that the effective Rayleigh number  $Ra_{eff}$  remains smaller than the nominal Rayleigh number and  $Ra_{eff}$  decreases with increasing  $Bn$ . Furthermore, Eqs. 4.7 and 4.8 indicate that  $\mu_{eff}^H$  is expected to be greater than  $\mu_{eff}^V$  in cylindrical enclosures. This further implies that  $Ra_{eff}^H$  is expected to be smaller than  $Ra_{eff}^V$  in cylindrical enclosures as it is shown in Eqs. 4.9 and 4.10.

Finally, the wall heat flux for the internal radius  $q_i$  can be scaled as:  $q_i \sim h_i \Delta T \sim k \Delta T / \delta_{th}$  where  $\delta_{th}$  is the thermal boundary layer on the vertical wall. Accordingly, the scaling estimate of the mean Nusselt number can be expressed as  $\overline{Nu}_i \sim h_i L / k \sim L / \delta_{th}$ . It is not possible to obtain an analytical relation for  $\delta_{th}$  from Eq. 4.6 for CWHF boundary condition but it is possible to obtain a scaling estimate of  $\overline{Nu}_i$  using Eq. 4.5 for CWT boundary condition in the following manner:

$$\overline{Nu}_i \sim Max \left\{ \left( \overline{Nu}_i \right)_{cond}, \frac{(Ra_{CWT}/Pr)^{1/2}}{\left[ \frac{Bn_{CWT}}{2} + \frac{1}{2} \sqrt{Bn_{CWT}^2 + 4 \left( \frac{Ra_{CWT} AR}{Pr} \right)^{1/2}} \right]} \right\} \times f_2(Ra_{CWT}, Pr, Bn_{CWT}, AR, r_i/L) \quad (4.11)$$

It is worth noting that  $\overline{Nu}_i$  scaling estimation given in Eq. 4.11 is valid in the thermal boundary layer regime where the high values of temperature gradient are confined to two thin boundary layers adjacent to the vertical walls [105]. However, for shallow enclosures (i.e.  $AR \ll 1$ ) the limiting condition given by  $Ra_{CWT}AR^3 \rightarrow 0$  ( $Ra_{CWHF}AR^3 \rightarrow 0$ ) is referred to as the parallel-flow regime [105]. In the parallel flow regime, vertical velocity component at the core of the enclosure disappears and the fluid flow in the enclosure consist of two counter flowing horizontal streams. Interested readers are referred to Refs. [63, 105] for further detailed discussion on the parallel-flow regime.

## 4.2 Effects of Nominal Rayleigh Number $Ra$

The variations of non-dimensional temperature  $\theta = (T - T_{cen})/\Delta T_{ref}$  and non-dimensional axial (radial)  $W = wL/\alpha$  ( $U = uL/\alpha$ ) velocity components along the horizontal (vertical) mid-plane for different  $Ra$  values at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 500$  are shown in Fig. 4.1 for Newtonian (i.e.  $Bn = 0$ ) and a representative Bingham fluid case (i.e.  $Bn = 0.05$ ) for both CWT and CWHF boundary conditions. The distributions of  $\theta$  for the pure-conduction solution (where fluid flow does not affect thermal transport) are also shown in Fig. 4.1. Figure 4.1 shows that the non-dimensional temperature  $\theta$  distribution between the hot and cold walls of the enclosure deviates significantly from the pure conduction solution with increasing  $Ra$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluid case (i.e.  $Bn = 0.05$ )<sup>1</sup> which is indicative of the strengthening of convective thermal transport. This behaviour can also be seen from the increases in the magnitudes of  $W$  and  $U$  with increasing  $Ra$ , as shown in Fig. 4.1. It is also worth noting that the magnitudes of  $W$  and  $U$  for Bingham fluid case (i.e.  $Bn = 0.05$ ) remain smaller than the values obtained for Newtonian ( $Bn = 0$ ) fluid cases for the same set of values of  $Ra$ ,  $Pr$ ,  $AR$  and  $r_i/L$ .

<sup>1</sup>The lines for  $Ra = 10^3$  and  $Ra = 10^4$  collapse with each other and also with the pure conduction solution for  $Bn = 0.05$  in Fig. 4.1, since heat transfer takes place due to pure conduction for both  $Ra = 10^3$  and  $Ra = 10^4$  (see isotherms are parallel to the side walls in Fig. 4.2 for  $Bn = 0.05$  at  $Ra = 10^4$ )

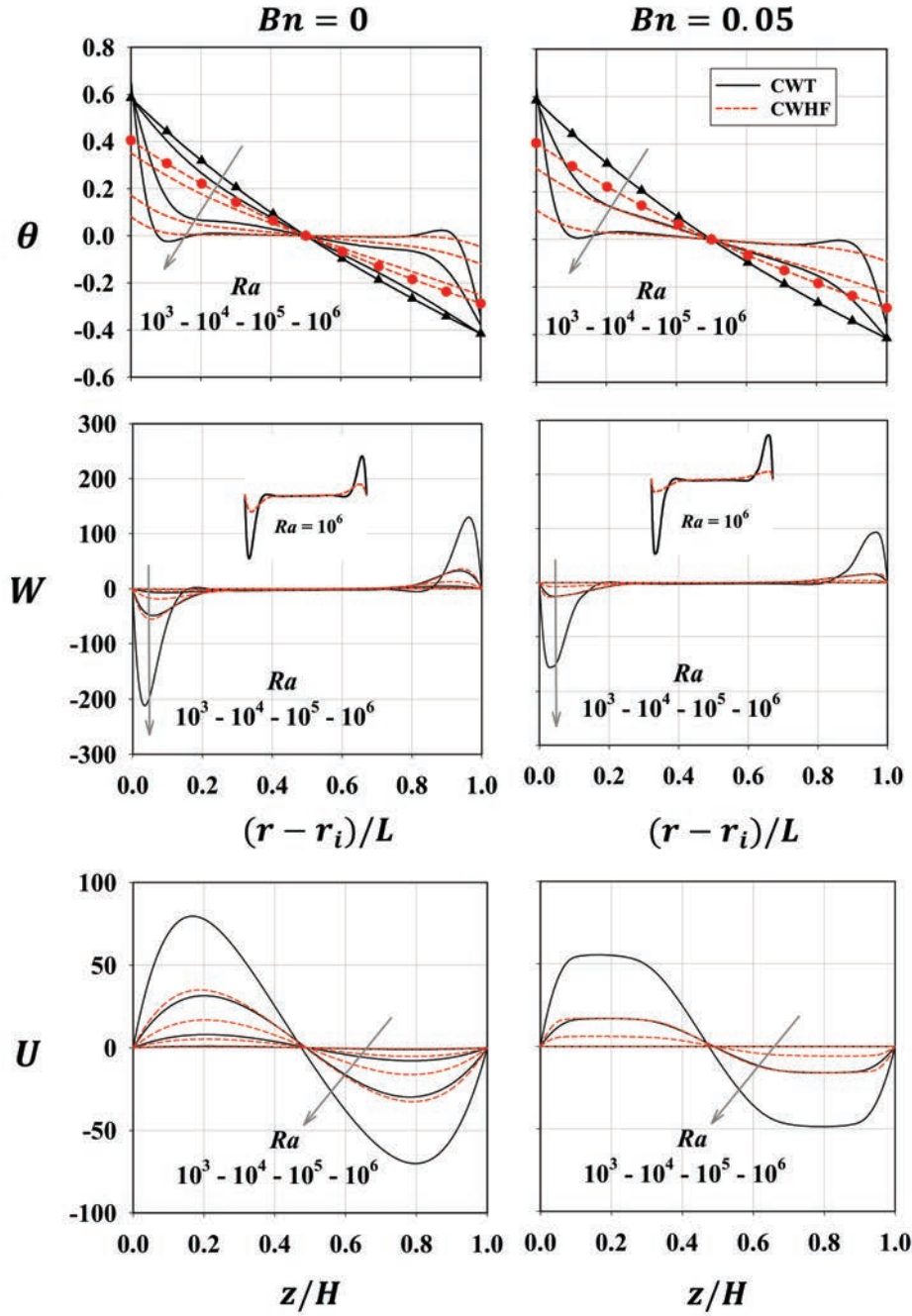


Figure 4.1: Variations of non-dimensional temperature  $\theta$  and non-dimensional axial (radial)  $W = wL/\alpha$  ( $U = uL/\alpha$ ) velocity components along the horizontal (vertical) mid-plane for different  $Ra$  values at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 500$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids. The pure conduction solution is shown by triangles (circles) for the CWT (CWHF) boundary conditions.

The viscous resistance strengthens with increasing  $Bn$ , which is reflected in the smaller magnitudes of  $W$  and  $U$  in the Bingham fluid cases than in the corresponding Newtonian fluid cases. Furthermore, the magnitudes of  $\theta$ ,  $W$  and  $U$  are found to be smaller in the CWHF boundary condition than the corresponding values in the CWT boundary condition, which is an indication of stronger convective thermal transport in the CWT boundary condition than in the CWHF boundary condition. This can also be confirmed by the scaling the wall heat flux on the inner periphery as  $q_i \sim k\Delta T/\delta_{th}$  where  $\Delta T$  and  $\delta_{th}$  are the characteristic temperature difference and the thickness of thermal boundary layer on vertical walls, respectively. This implies that  $\theta$  in the case of CWHF boundary condition scales as  $\theta \sim \Delta Tk/q_i L \sim q_i \delta_{th} k/q_i L k \sim O(\delta_{th}/L)$ , whereas  $\theta \sim O(1)$  for CWT boundary condition. Accordingly, the magnitude of  $\theta$  for the CWHF boundary condition is expected to decrease with increasing  $Ra$  due to the strengthening of buoyancy force. This smaller temperature difference between the vertical walls in the CWHF boundary condition induces a weaker convective transport than in the CWT boundary condition.

Additionally, the contours of non-dimensional temperature  $\theta$  and non-dimensional stream function  $\Psi = \psi/\alpha$  are shown in Fig. 4.2 for different values of  $Ra$  at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 500$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham fluid cases (i.e.  $Bn = 0.05$ ) for CWT boundary condition. Figure 4.2 shows that the magnitude of  $\Psi$  increases and isotherms become increasingly curved with increasing  $Ra$  due to the strengthening of convective thermal transport. Similar qualitative behaviour has been also observed for CWHF boundary condition but it is not shown here for the sake of conciseness.

The grey regions on the streamline plot in Fig. 4.2 shows the Apparently Unyielded Regions (AURs) (i.e. the regions where  $|\tau| \leq \tau_y$  (Mitsoulis and Zisis [106])). The distribution of AURs is dependent on the prediction of  $|\tau| \leq \tau_y$  which is a function of the choice of  $\mu_{yield}/\mu$ . Therefore the grey regions should not be treated as exact unyielded zones and for this very reason, Mitsoulis and Zisis [106] termed them as the Apparently Unyielded Regions (AURs). The value of  $\mu_{yield}$  affects only the size and distributions of AURs but qualitative and quantitative distributions of stream function and isotherms remain independent of the value of  $\mu_{yield}/\mu$  (at least for  $\mu_{yield}/\mu \geq 10^3$ ). Therefore, the precise shape and

size of AURs do not influence  $\overline{Nu}_i$  in the present study.

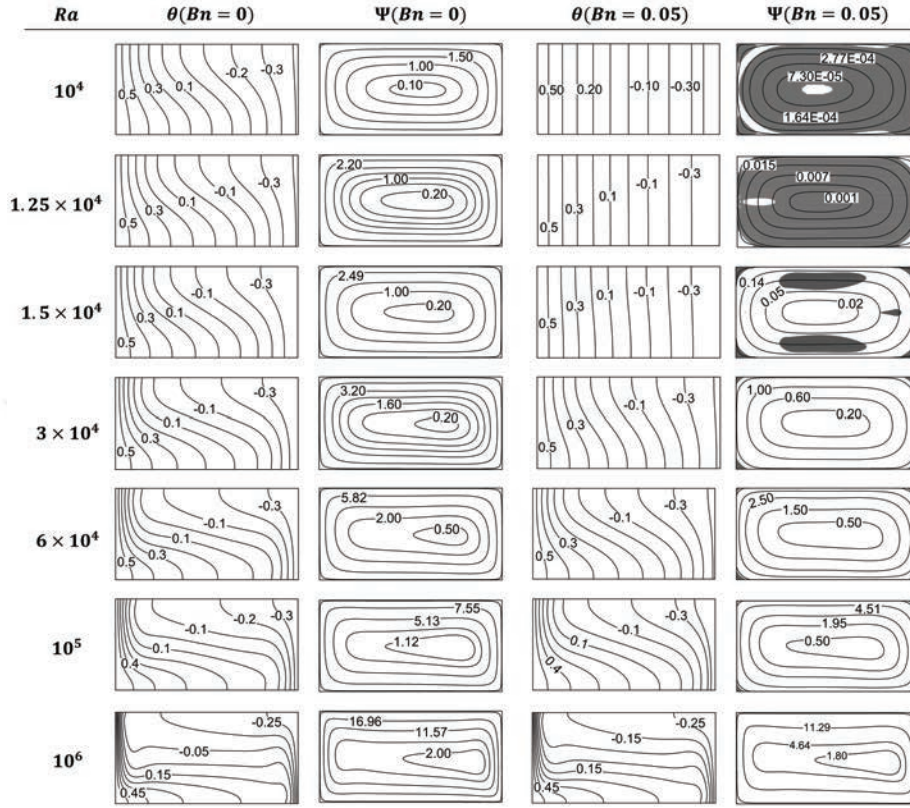


Figure 4.2: Contours of non-dimensional temperature  $\theta$  and non-dimensional stream function ( $\Psi = \psi/\alpha$ ) for different values of  $Ra$  at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 500$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham fluid case (i.e.  $Bn = 0.05$ ) for CWT boundary condition. The apparently unyielded regions (AURs) are shown by grey shading.

### 4.3 Effects of Bingham Number $Bn$

The variations of  $\theta$ ,  $W$  ( $U$ ) in the radial (axial) direction at  $z/L = 0.5$  ( $(r - r_i)/L = 0.5$ ) are shown in Fig. 4.3 for different values of  $Bn$  and  $AR$  at  $r_i/L = 1.0$ ,  $Ra = 10^6$  and  $Pr = 500$  for both CWT and CWHF boundary conditions. The distributions of  $\theta$  for pure conductive transport are shown in Fig. 4.3.

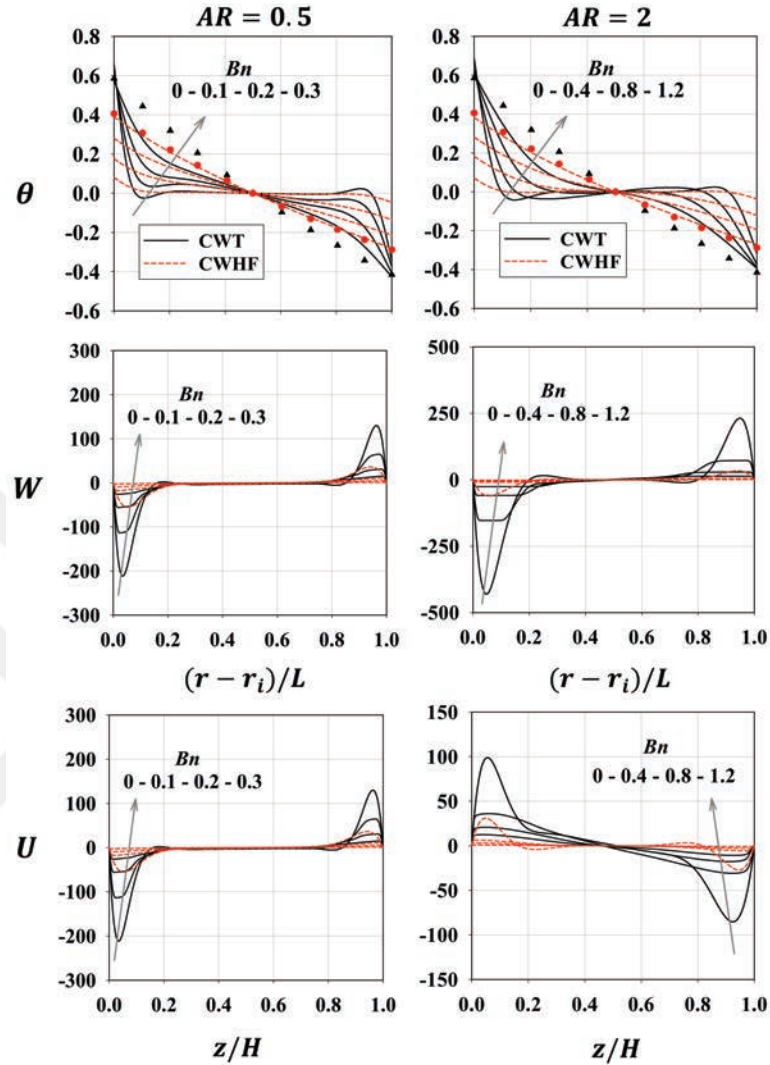


Figure 4.3: Variations of non-dimensional temperature  $\theta$  and non-dimensional axial (radial)  $W$  ( $U$ ) velocity components along the horizontal (vertical) mid-plane for different  $Bn$  values at  $r_i/L = 1.0$ ,  $Ra = 10^6$  and  $Pr = 500$  for  $AR = 0.5$  and 2. The pure conduction solution is shown by triangles (circles) for the CWT (CWHF) boundary conditions.

It can be seen from Fig. 4.3 that  $\theta$  approaches the pure conduction solution with increasing  $Bn$ , regardless of the boundary condition due to the weakening of buoyancy force relative to the viscous resistance. This is reflected in the decreasing magnitudes of  $W$  and  $U$  with increasing  $Bn$  as shown in Fig. 4.3 for

both CWT and CWHF boundary conditions.

The contours of non-dimensional temperature  $\theta$  and non-dimensional stream function  $\Psi$  are shown in Fig. 4.4 for different values of  $Bn$  at  $r_i/L = 1.0$ ,  $AR = 0.5$ ,  $Ra = 10^6$  and  $Pr = 500$  for the CWT boundary condition.

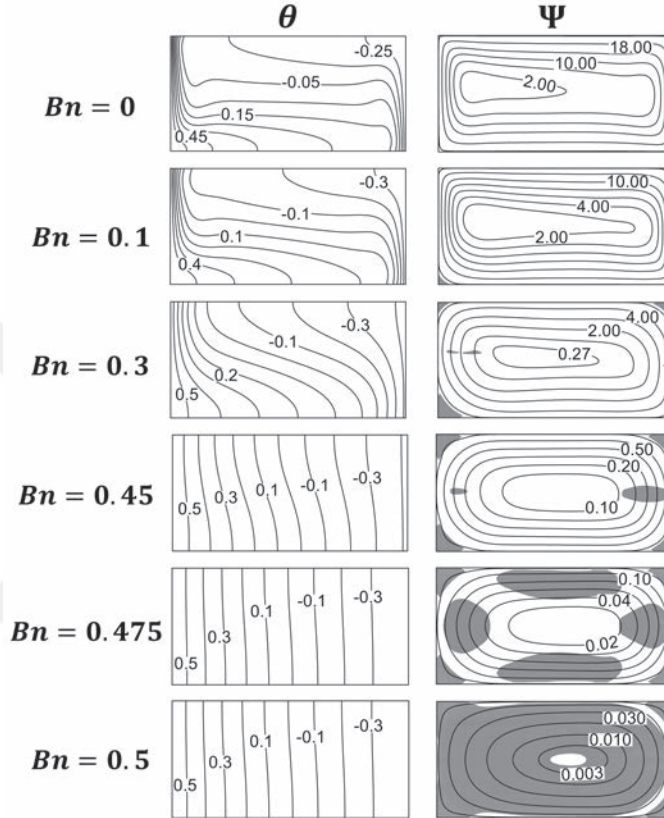


Figure 4.4: Contours of  $\theta$  and  $\Psi$  with AURs for different  $Bn$  values at  $r_i/L = 1.0$ ,  $AR = 0.5$ ,  $Ra = 10^6$  and  $Pr = 500$  for the CWT boundary condition.

Figure 4.4 shows that the magnitude of  $\Psi$  decreases and isotherms tend to be parallel to the active walls (i.e. hot and cold) due to the weakening of convective thermal transport with increasing  $Bn$ . Accordingly, the size of AURs increases with increasing  $Bn$  and heat transfer takes place purely due to conduction for large values of  $Bn$  because flow practically stops under such a condition.

Also, the findings based on Figs. 4.3 and 4.4 can be confirmed by the scaling estimation given in Eqs. 4.7 - 4.10. Equations 4.7 - 4.10 suggest that the effec-

tive viscosity  $\mu_{eff}$  (effective Rayleigh number  $Ra_{eff}$ ) increases (decreases), which indicates weakening of buoyancy effects in comparison to viscous forces, with increasing Bingham number.

## 4.4 Effects of Prandtl Number $Pr$

The variations of the mean Nusselt number based on the inner periphery  $\overline{Nu}_i$  normalised by the corresponding Nusselt number for pure conductive transport (i.e.  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ ) for different values of  $Ra$ ,  $Pr$  and  $r_i/L$  at  $AR = 1.0$  are shown in Fig. 4.5 for Newtonian fluids (i.e.  $Bn = 0$ ) in the case of both CWT and CWHF boundary conditions.

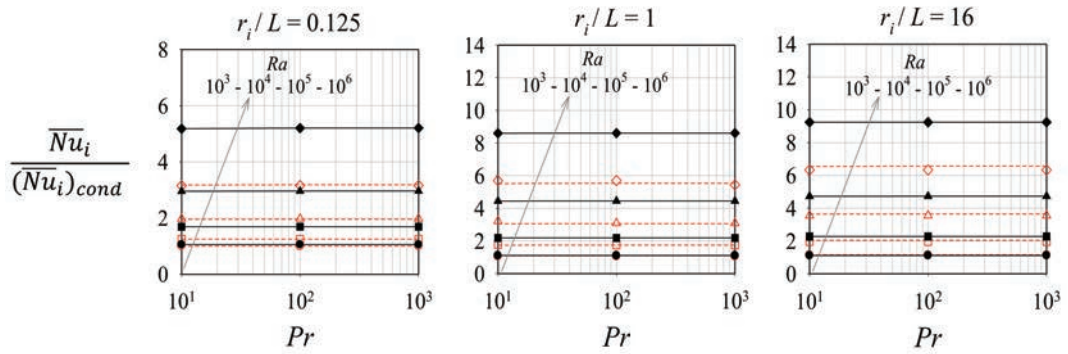


Figure 4.5: Variation of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $Pr$  for different values of  $Ra$  and  $r_i/L$  at  $AR = 1.0$  for Newtonian fluids (i.e.  $Bn = 0$ ). The values for CWT (CWHF) boundary conditions are shown by the solid (broken) line.

It is worth noting that Nusselt number for pure conductive transport  $(\overline{Nu}_i)_{cond}$  also depends on  $r_i/L$  as shown in Eq. 2.37 (i.e.  $(\overline{Nu}_i)_{cond} = (L/r_i) / (\ln(1 + L/r_i))$ ). It can be seen from Fig. 4.5 that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  does not change significantly with a change in  $Pr$  for Newtonian fluids (i.e.  $Bn = 0$ ) irrespective of the choice of boundary condition. For Newtonian fluids, a change in  $Pr$  for  $Pr \gg 1$  modifies the relative balance between the buoyancy and viscous forces because the thermal boundary layer remains much thinner than the hydro-dynamic boundary layer. Thus, thermal transport within thermal boundary layer does not get

affected by a change in  $Pr$  for  $Pr \gg 1$  in the case of Newtonian fluids (i.e.  $Bn = 0$ ) [2].

The variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $Bn$  for different values of  $Pr$  and  $r_i/L$  at  $Ra = 10^6$  and  $AR = 1.0$  are shown in Fig. 4.6 for both CWT and CWHF boundary conditions.

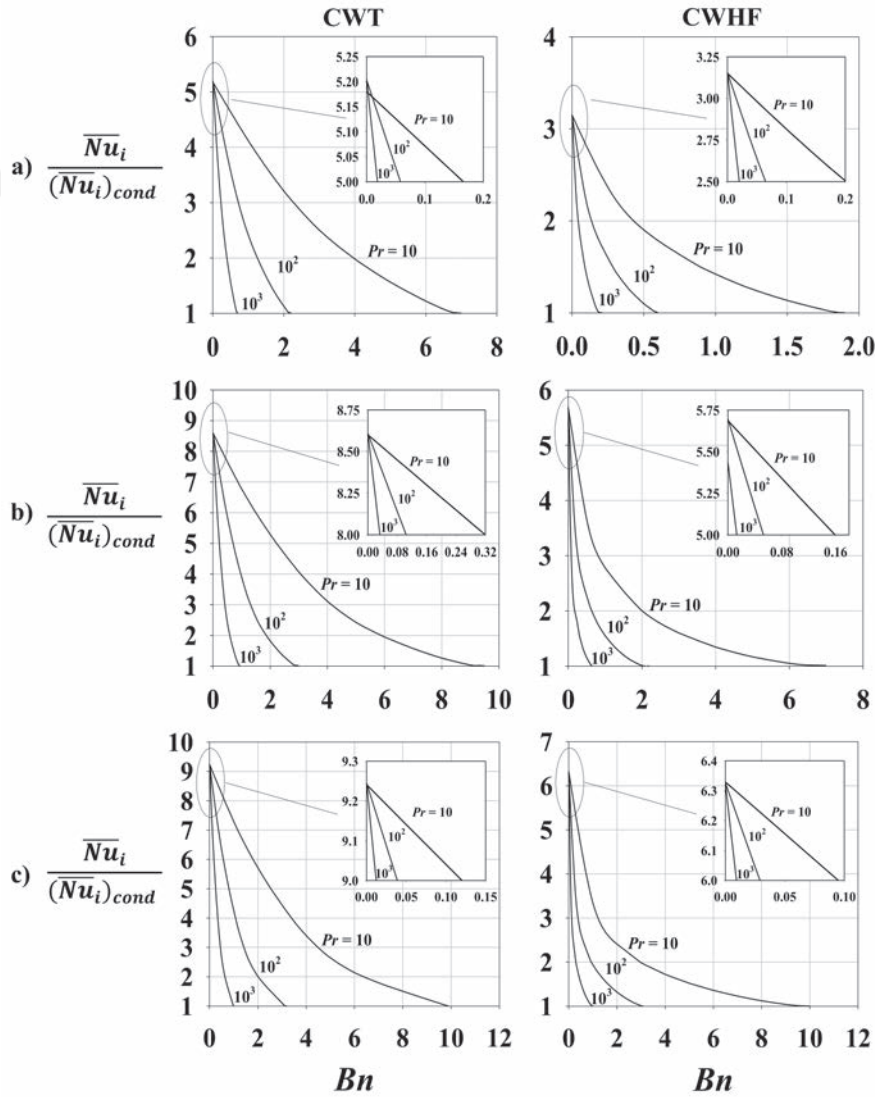


Figure 4.6: Variation of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $Bn$  for different values of  $Pr$  in the case of  $r_i/L$  a) 0.125, b) 1, c) 16 at  $Ra = 10^6$  and  $AR = 1.0$  for Bingham fluids both CWT and CWHF boundary conditions.

Figure 4.6 demonstrates that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  decreases with increasing  $Pr$  for large values of  $Bn$ , however an opposite trend has been observed for small values of  $Bn$  regardless of the boundary condition. This behaviour is qualitatively consistent with the previous findings for rectangular enclosures [22, 62] with differentially heated walls. It is evident from Eqs. 4.9 and 4.10 that both effective Rayleigh and Grashof numbers (i.e.  $Ra_{eff} = \rho g \beta \Delta T_{ref} L^3 / \mu_{eff} \alpha$  and  $Gr_{eff} = Ra_{eff} / Pr = \rho^2 g \beta \Delta T_{ref} L^3 / \mu_{eff}^2$ ) decrease with increasing  $Pr$  for large values of  $Bn$  since  $\mu_{eff}$  increases sharply for large values of  $Bn$ . However, an increase in  $Pr$  leads to a decrease in the thermal boundary thickness for small values of  $Bn$  for a given value of  $Ra$  and  $AR$ . This gives rise to a small increase in  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  for small values of  $Bn$  because the thinning of thermal boundary layer overcomes the strengthening of viscous resistance under this condition.

## 4.5 Effects of Aspect Ratio $AR$

The variations of non-dimensional temperature  $\theta$  and non-dimensional axial (radial)  $W$  ( $U$ ) velocity component along the horizontal (vertical) mid-plane for different  $AR$  values at  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $Bn = 0.2$  and  $Pr = 500$  are shown in Fig. 4.7 for both CWT and CWHF boundary conditions. The non-dimensional temperature  $\theta$  distributions obtained from a pure-conduction solution (where fluid flow does not affect thermal transport) are also shown in Fig. 4.7 for CWT (CWHF) boundary conditions. Figure 4.7 shows that the distribution of  $\theta$  deviates from the pure conduction solution with increasing  $AR$ , which indicates a strengthening of convection with increasing  $AR$  for both CWT and CWHF boundary conditions. Additionally, Fig. 4.7 shows that the magnitudes of  $\theta$ ,  $W$  and  $U$  in the CWHF boundary condition are smaller than the corresponding values obtained for the CWT boundary condition. This is also a confirmation of stronger convection for the CWT boundary condition than in the CWHF boundary condition for different  $AR$  values for a given set of values of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $r_i/L$ . Furthermore, it can be seen from Fig. 4.7 that the magnitudes of  $W$  and  $U$  increase with increasing  $AR$  for a given set of values of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $r_i/L$  for both CWT and CWHF boundary conditions.

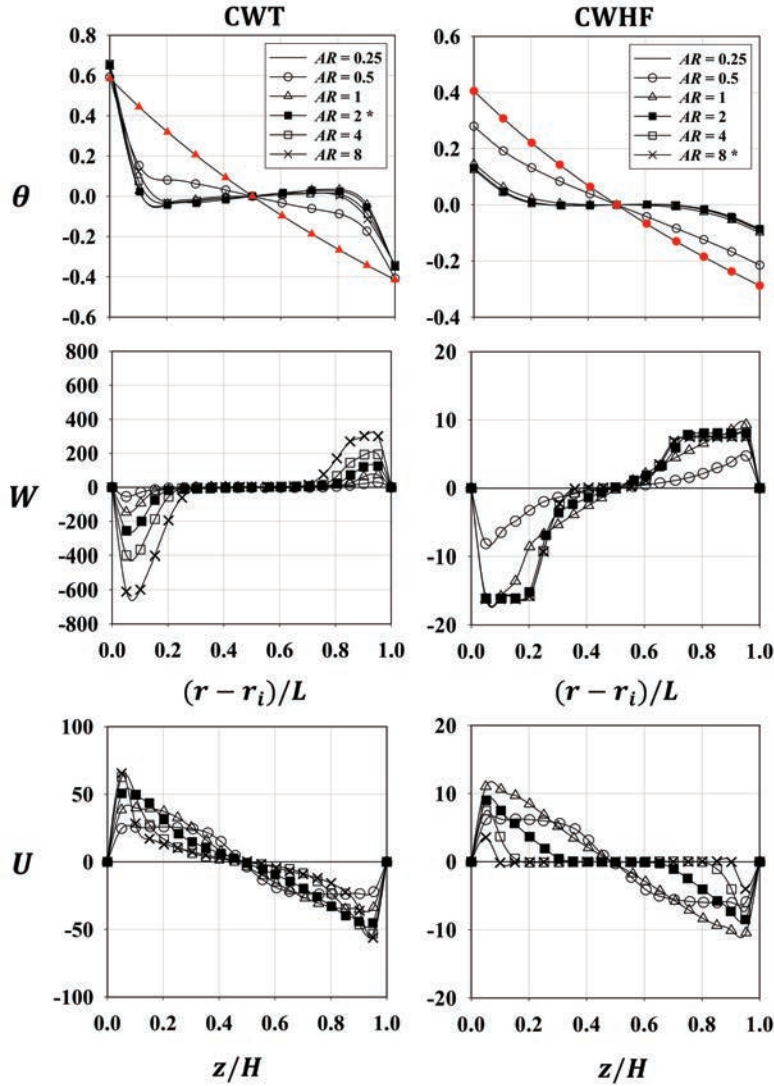


Figure 4.7: Variations of  $\theta$  and  $W$  ( $U$ ) velocity components along the horizontal (vertical) mid-plane for different  $AR$  values at  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $Bn = 0.2$  and  $Pr = 500$ . The pure conduction solution is shown by triangles (circles) for the CWT (CWHF) boundary conditions. The  $AR$  values indicated by an asterisk in the legend represent the aspect ratio at which the highest  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  is obtained as shown in Fig. 4.8.

The variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $AR$  for different values  $Ra$  at  $r_i/L = 1.0$  and  $Pr = 500$  are shown in Fig. 4.8 a) for Newtonian fluids (i.e.  $Bn = 0$ ) for both CWT and CWHF boundary conditions. Figure 4.8 a) shows that a

non-monotonic variation of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $AR$  for Newtonian fluids (i.e.  $Bn = 0$ ) in the case of CWT boundary condition, whereas a monotonic increase in  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  was obtained for increasing values of  $AR$  in the CWHF boundary condition.

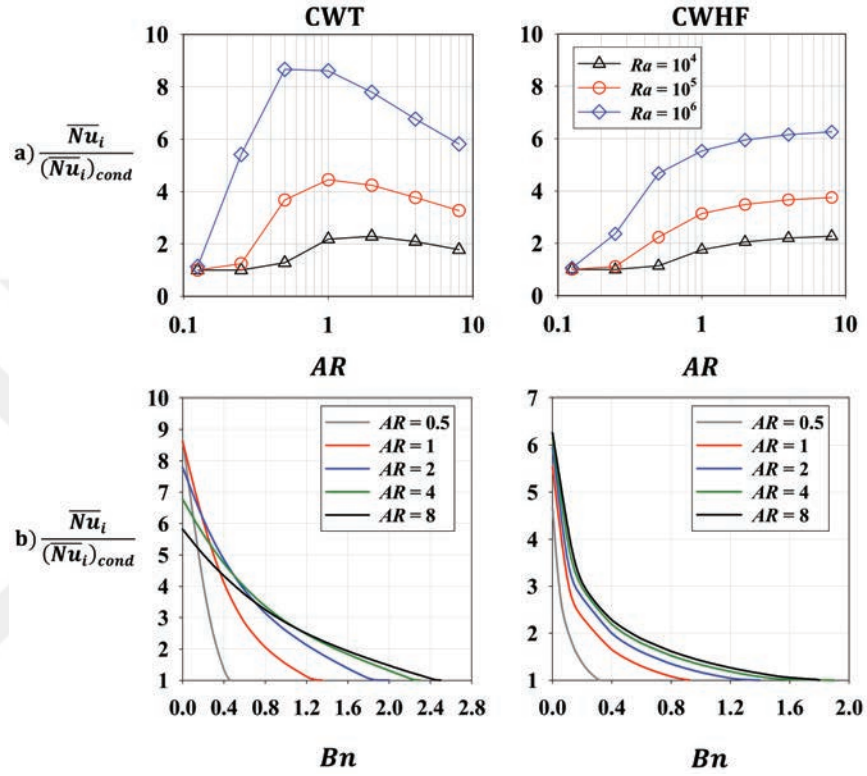


Figure 4.8: a) Variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $AR$  for different values of  $Ra$  at  $r_i/L = 1.0$  and  $Pr = 500$  for Newtonian fluids (i.e.  $Bn = 0$ ) for both CWT and CWHF boundary conditions, b) Variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $Bn$  for different values of  $AR$  at  $r_i/L = 1.0$ ,  $Ra = 10^6$  and  $Pr = 500$  for both CWT and CWHF boundary conditions.

Furthermore, the variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $Bn$  for different values  $AR$  at  $r_i/L = 1.0$ ,  $Ra = 10^6$  and  $Pr = 500$  are shown in Fig. 4.8 b) for both CWT and CWHF boundary conditions. Figure 4.8 b) demonstrates that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  decreases with increasing  $Bn$  and attains a value of unity (purely conductive thermal transport) for either equal to or greater than a threshold value of Bingham number  $Bn$  which is termed as  $Bn_{max}$  in the current analysis. Here,  $Bn_{max}$

is considered to be the value of Bingham number where  $\overline{Nu}_i / (\overline{Nu}_i)_{cond} = 1.01$  and  $\overline{Nu}_i / (\overline{Nu}_i)_{cond} < 1.01$  for  $Bn > Bn_{max}$ .

It is worth noting that  $Bn_{max}$  does not provide the estimation of the critical Bingham number  $Bn_c$  at which the flow ceases to take place in the context of a bi-viscosity regularisation. This is due to the fact that the flow always remains present for bi-viscosity regularisation but it no longer affects thermal advection for large values of  $Bn$  [63]. Thus,  $Bn_{max}$  is physically different from the critical Bingham number  $Bn_c$  at which the buoyancy force becomes just sufficient to overcome the yield stress. Under this situation the fluid is unyielded throughout the domain and  $\overline{Nu}_i$  becomes equal to  $(\overline{Nu}_i)_{cond}$ . For  $Bn = Bn_c$ , the equilibrium of buoyancy and yield stress contributions (i.e.  $\rho g \beta \Delta T \sim \tau_y / \delta \sim \tau_y / L f_2(Bn_c, Pr)$ ) gives rise to  $Bn_c \sim \sqrt{Ra / Pr} f_2(Bn_c, Pr)$ . This is consistent with the recent analytical results of Karimfazli et al. [21] and Vikhansky [72] for the onset of natural convection of Bingham fluids in rectangular enclosures with differentially heated vertical sidewalls. However, the quantity  $(Bn_{max})_i$  can be scaled for the CWT boundary condition by considering  $\overline{Nu}_i \sim L / \delta_{th} \sim L f_2 / \delta \sim (\overline{Nu}_i)_{cond}$ . This along with Eq. 4.11 leads to:  $(Bn_{max})_i \sim f_2(\overline{Nu}_i)_{cond}^{-1} \sqrt{Ra / Pr} - [\sqrt{AR} (\overline{Nu}_i)_{cond} / f_2]$  for the CWT boundary condition. It is worth noting that  $Bn > Bn_c$  ensures that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  remains equal to unity but it does not imply that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond} > 1$  when  $Bn$  is smaller than  $Bn_c$ . In the current analysis,  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  is only attains values greater than unity when  $Bn$  is smaller than  $(Bn_{max})_i$ , and fluid may remain in a yielded state under the condition  $(Bn_{max})_i < Bn < Bn_c$  even though the flow is weak enough to result in a unity value of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ .

Moreover, as shown in Figs. 4.7 and 4.8, the variation of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  shows a non-monotonic trend with  $AR$  for a given set of values of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $r_i / L$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids in the case of CWT boundary condition, whereas a monotonic increase in  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  was obtained for increasing values of  $AR$  for the CWHF boundary condition. For example, the highest value of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  is obtained at  $AR = 2$  for  $r_i / L = 1.0$ ,  $Ra = 10^6$ ,  $Bn = 0.2$  and  $Pr = 500$  in the CWT boundary condition. By contrast, the highest value of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  is obtained for the highest  $AR$  case in the CWHF

boundary condition, as indicated by an asterisk in Fig. 4.7. This is consistent with previous analyses on the influences of aspect ratio and boundary condition on natural convection in rectangular enclosures for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids [63], [47]). The non-monotonic (monotonic)  $AR$  dependence of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  for the CWT (CWHF) boundary condition can be explained in the following manner using the energy flux integral. The energy flux integral at the horizontal mid-plane can be given as:

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{cond} = \int_{r_i}^{r_o} \rho c_p T w 2\pi r dr - \int_{r_i}^{r_o} k \left( \frac{\partial T}{\partial z} \right) 2\pi r dr \quad (4.12)$$

where the first term on the right hand side represents the effects of convective transport, whereas the second term on the right hand side accounts for thermal conduction. Here,  $\dot{Q}_{conv}$  and  $\dot{Q}_{cond}$  can be scaled in the following manner:

$$\begin{aligned} \dot{Q}_{conv,CWT} &\sim \pi(2r_i + L) \rho c_p \Delta T w \delta \sim \\ &\quad k \Delta T \pi(2r_i + L) \sqrt{AR Pr} \\ &\quad \times \left[ \frac{Bn_{CWT}}{2} + \sqrt{\left( \frac{Bn_{CWT}}{2} \right)^2 + \sqrt{\frac{Ra_{CWT} AR}{Pr}}} \right] \end{aligned} \quad (4.13)$$

$$\begin{aligned} \dot{Q}_{conv,CWHF} &\sim \pi(2r_i + L) \frac{\rho c_p q_i \delta_{th} \delta}{k} \sim \\ &\quad q_i L \pi(2r_i + L) f_2 \sqrt{Ra_{CWHF} Pr AR} (\delta_{th}/L)^{5/2} \end{aligned} \quad (4.14)$$

$$\dot{Q}_{cond,CWT} \sim k \Delta T \pi(2r_i + L) / AR \quad (4.15)$$

$$\dot{Q}_{cond,CWHF} = \pi(2r_i + L) \frac{q_i \delta_{th}}{H} \sim q_i L \pi(2r_i + L) (\delta_{th}/L) (1/AR) \quad (4.16)$$

The above scaling estimates indicate that the convective transport strengthens with increasing  $AR$ , whereas thermal diffusion weakens with increasing  $AR$  in

the case of CWT boundary condition. The highest value of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  takes place for an optimum value of aspect ratio (i.e.  $AR_{max}$ ) due to the competition between an enhancement in advection and a weakening of thermal diffusion with an increase in  $AR$ . By contrast, the strengthening (weakening) of  $\dot{Q}_{conv,CWT}$  ( $\dot{Q}_{cond,CWHF}$ ) with increasing  $AR$  in the CWHF boundary condition is stronger (weaker) than  $\dot{Q}_{conv,CWT}$  ( $\dot{Q}_{cond,CWHF}$ ) in the CWT boundary condition. For example, in Newtonian fluids (i.e.  $Bn = 0$ ),  $\dot{Q}_{conv}$  scales as  $\dot{Q}_{conv,CWT} \sim k\Delta T\pi(2r_i + L)Ra_{CWT}^{0.25}Pr^{0.75}AR^{0.75}$  and  $\dot{Q}_{conv,CWHF} \sim q_iL\pi(2r_i + L)PrARf_2^{-1}$  for CWT and CWHF boundary conditions respectively. Similarly,  $\dot{Q}_{cond}$  scales as  $\dot{Q}_{cond,CWT} \sim k\Delta T\pi(2r_i + L)AR^{-1}$  and  $\dot{Q}_{cond,CWHF} \sim q_iL\pi(2r_i + L)Pr^{0.2}Ra^{-0.2}AR^{-0.8}f_2^{-0.8}$  for CWT and CWHF boundary conditions respectively. The quantities  $k\Delta T$  and  $q_iL$  are kept unaltered for CWT and CWHF boundary conditions, and thus, the strengthening (weakening) of the convective (conductive) transport with increasing  $AR$  in the CWHF boundary condition is stronger (weaker) than that in the CWT boundary condition for a given set of values of  $Ra$  and  $Pr$  for Newtonian fluids (i.e.  $Bn = 0$ ). A qualitatively similar behaviour has also been observed here for Bingham fluids. Thus,  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases monotonically with increasing  $AR$  in the CWHF boundary condition. This aspect is also shown in Fig. 4.9 with the contours of non-dimensional stream function  $\Psi$  with AURs for different  $AR$  values at  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $Bn = 0.2$  and  $Pr = 500$  for the CWHF boundary condition. Findings of Figs 4.7 - 4.9 are in qualitative agreement with previous analyses by Turan et al. [63], [47]) on rectangular enclosures for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids.

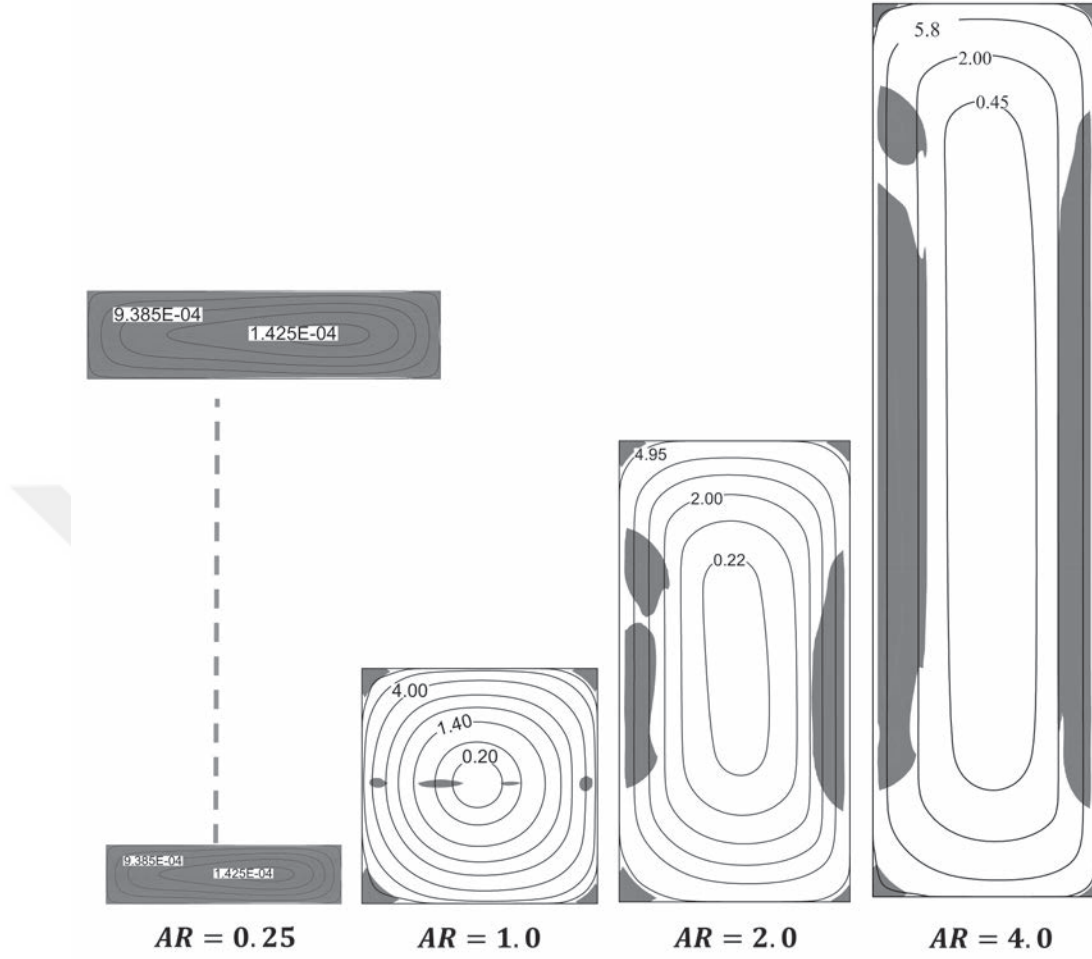


Figure 4.9: Contours of non-dimensional stream function  $\Psi$  with AURs for different  $AR$  values at  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $Bn = 0.2$  and  $Pr = 500$  for CWHF boundary condition.

## 4.6 Effects of Normalised Internal Cylinder Radius $r_i/L$

The variations of  $\theta$  and  $W$  ( $U$ ) in the radial (axial) direction at  $z/H = 0.5$  ( $(r - r_i)/L = 0.5$ ) are shown in Fig. 4.10 for different values  $r_i/L$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.5$ ) fluid cases at  $Ra = 10^6$ ,  $Pr = 10^2$  and  $AR = 1.0$  for both CWT and CWHF boundary conditions. Figure 4.10

indicates that the temperature difference between vertical walls increases with increasing  $r_i/L$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.5$ ) fluid cases for the CWHF boundary condition. It is important to understand that  $r_i/L$  dependence of the temperature difference between vertical walls  $\Delta T$  due to pure conductive transport in order to explain the wall curvature effects on the temperature difference between the vertical walls. It is worth noting that the temperature difference between the vertical walls  $\Delta T$  is dependent on the internal radius of the cylindrical annular enclosure when the thermal transport occurs purely by thermal conduction. The thermal resistance due to conduction in axisymmetric cylindrical geometry is given by  $\ln(r_o/r_i)/2\pi kL$ , which suggests that the steady state heat flow rate for pure conductive transport is given by:  $\dot{Q} = \Delta T_{cond} / \{\ln(r_o/r_i)/2\pi kL\}$ . Furthermore,  $\dot{Q}$  can be expressed in terms of heat flux on the inner periphery as:  $\dot{Q} = q_i 2\pi r_i L$ , which leads to:

$$\Delta T_{cond} = \frac{q_i L (r_i/L)}{k} \ln \left( 1 + \frac{L}{r_i} \right) \quad (4.17)$$

Eq. 4.17 indicates that  $\Delta T_{cond}$  increases with decreasing  $r_i/L$ . As the heat transfer coefficient  $h_i = q_i/\Delta T$  for the convective transport is expected to be greater than that for the pure conductive transport  $(h_i)_{cond} = q_i/\Delta T_{cond}$ , the temperature difference between the vertical walls remains smaller than that in the case of purely conductive transport in the boundary layer regime of convection.

Figure 4.10 further shows that  $U$  increases, whereas  $W$  decreases with increasing  $r_i/L$  for Newtonian (i.e.  $Bn = 0$ ) case. The isotherms and streamline patterns for different values  $r_i/L$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.5$ ) fluid cases at  $Ra = 10^6$ ,  $Pr = 10^2$  and  $AR = 1.0$  in Fig. 4.11 for the CWT boundary condition. It can be seen from Fig. 4.11 that the magnitude of  $\Psi$  increases and the size of the AURs decreases with increasing  $r_i/L$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.5$ ) fluid cases, which indicates that convective thermal transport strengthens with increasing  $r_i/L$ .

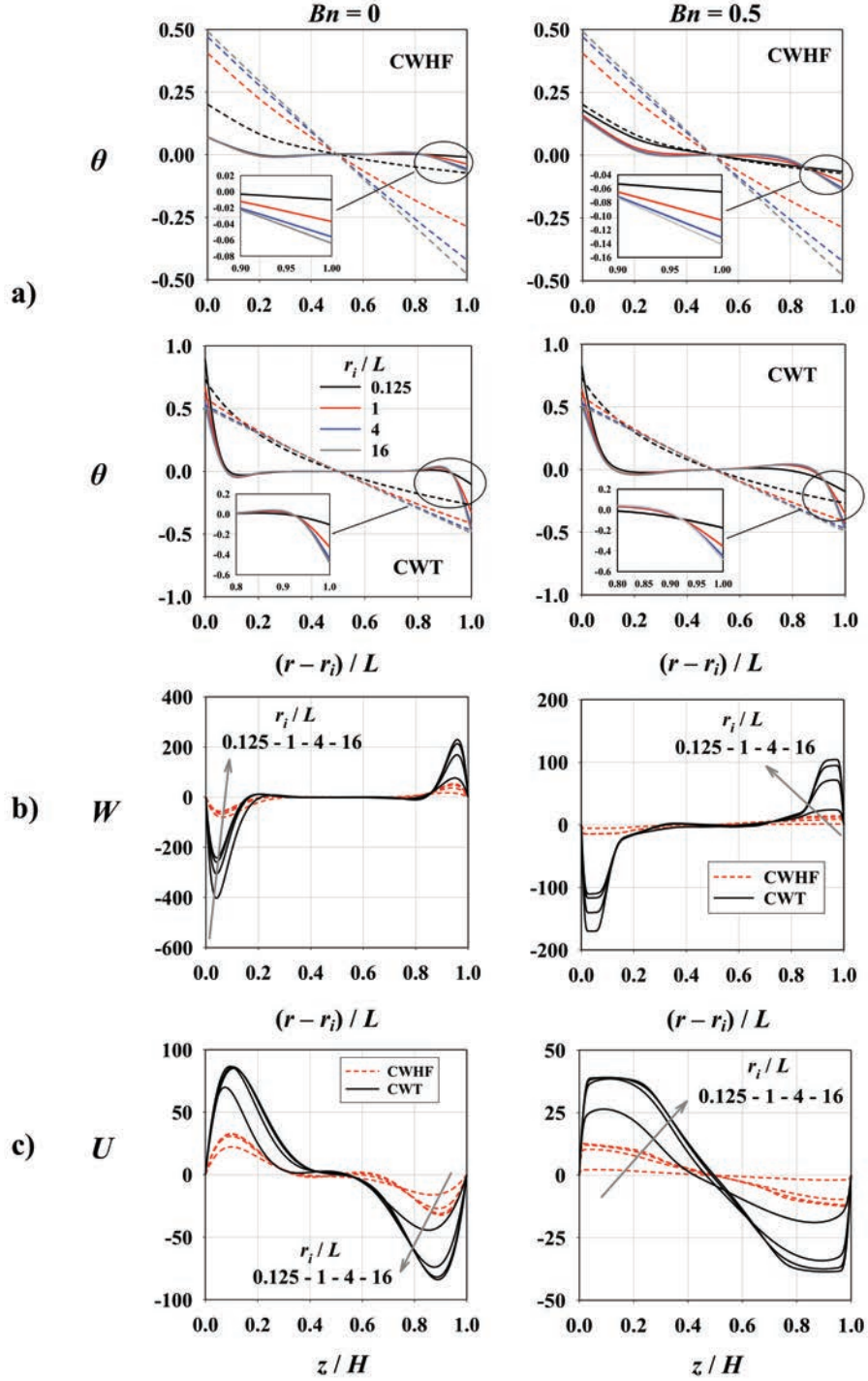


Figure 4.10: Variations of a)  $\theta$  and  $W$  with  $(r - r_i)/L$  at  $z/H = 0.5$ , c) Variations of  $U$  with  $z/H$  at  $0.5 (r_i + r_o)/L$  for different values of  $r_i/L$  at  $Ra = 10^6$ ,  $Pr = 10^2$  and  $AR = 1.0$  for Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.5$ ) fluid cases for both CWT and CWHF boundary conditions. The pure conduction solution for  $\theta$  is shown by the dashed lines for both CWT and CWHF boundary conditions.

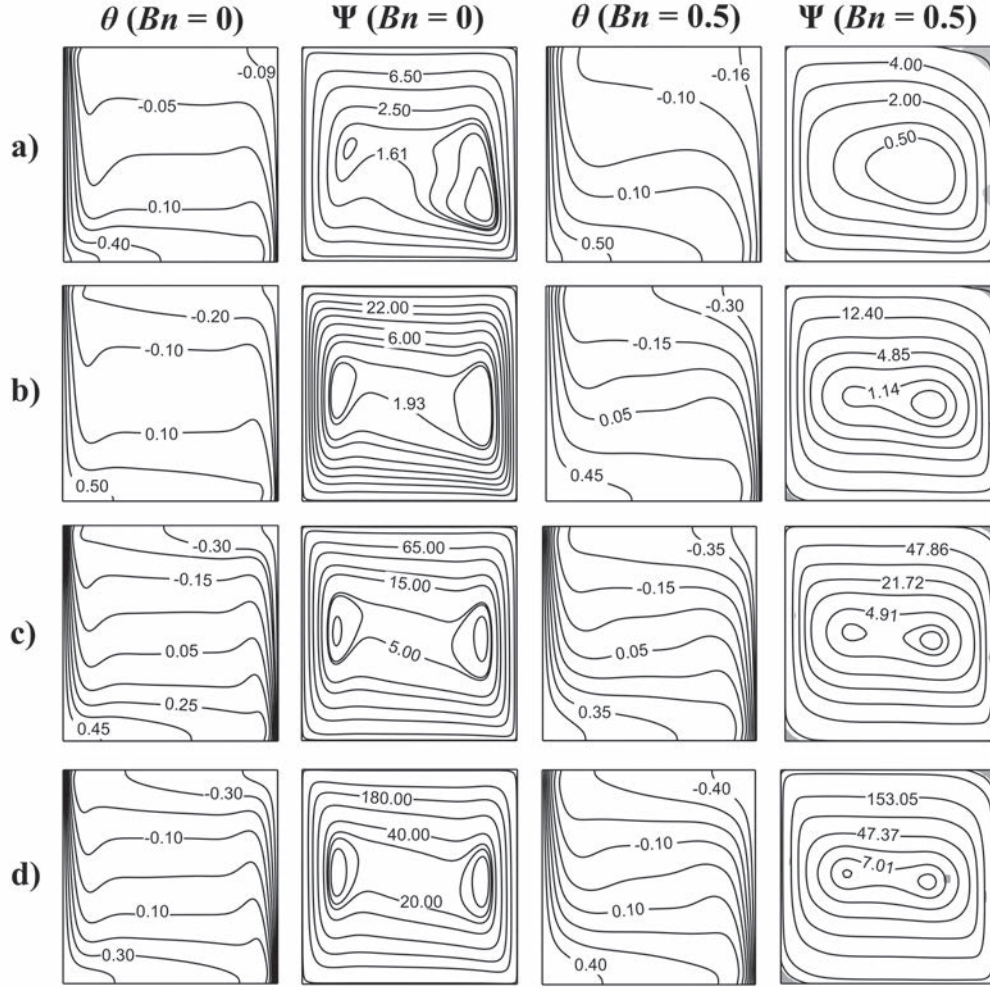


Figure 4.11: Contours of  $\theta$  and  $\Psi$  with AURs for Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.5$ ) fluid cases for  $r_i/L$  a) 0.125, b) 1, c) 4, d) 16 at  $Ra = 10^6$ ,  $Pr = 10^2$  and  $AR = 1.0$  for CWT boundary condition.

These numerical findings can be confirmed from the scaling estimation of the radial velocity component given by Eqs. 4.2 and 4.3 (i.e.  $u \sim w(1 + L/r_i)$ ) regardless of the boundary condition. As the variations  $W$  and  $U$  with  $r_i/L$  exhibit opposite behaviours, as shown in the Fig. 4.10, the relative contributions of thermal transport due to convection and conduction for different values of  $r_i/L$  can be explained from the variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $r_i/L$ . It is worth noting that  $(\overline{Nu}_i)_{cond}$  increases with decreasing  $r_i/L$  (see Eq. 2.37). Therefore, the

variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $r_i/L$  for different values of  $Bn$  at  $AR = 0.5$ ,  $Ra = 10^6$  and  $Pr = 500$  are shown in Fig. 4.12 for both CWT and CWHF boundary conditions. Figure 4.12 shows that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases with increasing  $r_i/L$  and approaches asymptotically to the value of the mean Nusselt number  $\overline{Nu}_{Rec}$  for the corresponding rectangular enclosures (i.e. in the limit of  $r_i/L \rightarrow \infty$ ). This suggests that the relative contribution of convection to the overall thermal transport increases with increasing  $r_i/L$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids. This is in a full agreement with previous results for Newtonian fluids [64–66, 107].

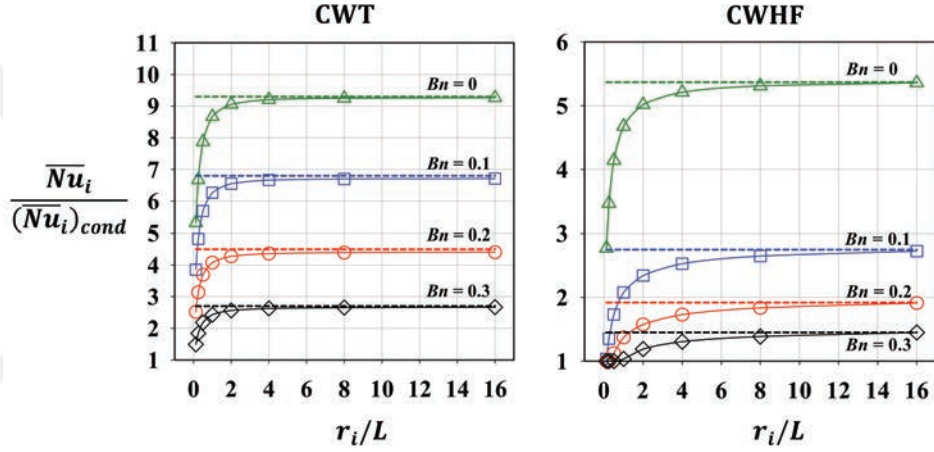


Figure 4.12: Variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $r_i/L$  for different values of  $Bn$  for  $AR = 0.5$ ,  $Ra = 10^6$  at  $Pr = 500$  for both CWT and CWHF boundary conditions. The dashed lines represent corresponding value of mean Nusselt number for rectangular enclosures  $\overline{Nu}_{Rec}$  for each  $Bn$  values.

## 4.7 Correlation of the Mean Nusselt Number $\overline{Nu}_i$

Bejan and Tien [105] proposed the following correlation for the mean Nusselt number  $\overline{Nu}_{Rec}$  on natural convection of Newtonian (i.e.  $Bn = 0$ ) fluids in rectangular enclosures with differentially heated vertical walls for the CWT boundary

condition:

$$\overline{Nu}_{Rec} = 1 + \left[ \left( \frac{Ra_{CWT}^2 AR^8}{362880} \right)^{-0.386} + \left( 0.623 Ra_{CWT}^{0.2} AR^{-0.4} \right)^{-0.386} \right]^{-1/0.386} \quad (4.18)$$

It is shown in elsewhere (Fig. 10 of [47]) that Eq. 4.18 captures the variation of  $\overline{Nu}_{Rec}$  with  $AR$  satisfactorily for  $AR \leq 1$ . It is worth noting that Eq. 4.18 underpredicts the value of  $\overline{Nu}_{Rec}$  for the aspect ratios of the order of unity (i.e.  $AR \sim 1$ ). Also, one of the most used  $\overline{Nu}_{Rec}$  correlations for tall (i.e.  $AR \gg 1$ ) enclosures was proposed by Elsherbiny et al. [108] in the following manner:

$$\overline{Nu}_{Rec} = \text{Max}(Nu_{1c}, Nu_{2c}, Nu_{3c}) \quad (4.19)$$

$$Nu_{1c} = 0.0605 Ra_{CWT}^{1/3}; \quad Nu_{2c} = \left[ 1 + \left( \frac{0.104 Ra_{CWT}^{0.293}}{1 + (6310/Ra_{CWT})^{1.36}} \right)^3 \right]^{1/3};$$

$$Nu_{3c} = 0.242 (Ra_{CWT}/AR)^{0.272} \quad (4.20)$$

The  $\overline{Nu}_{Rec}$  correlation given by Eqs. 4.19 and 4.20 satisfactorily captures the variation of  $\overline{Nu}_{Rec}$  with  $AR$  for  $AR \geq 1$  as it is shown in elsewhere (Fig. 10 of [47]). Similarly, Turan et al. [47] proposed the following correlation for  $\overline{Nu}_{Rec}$  for natural convection of Newtonian (i.e.  $Bn = 0$ ) fluids in rectangular enclosures with differentially heated vertical walls for the CWHF boundary condition:

For  $1/8 \leq AR \leq 1$ ;

$$\overline{Nu}_{Rec} = Nu_A \text{ for } Ra_{CWHF} AR^3 < 10^3; \quad \overline{Nu}_{Rec} = Nu_B \text{ for } Ra_{CWHF} AR^3 \geq 10^3 \quad (4.21)$$

$$Nu_A = 1 + Ra_{CWHF}^2 AR^8 / 362880;$$

$$Nu_B = 0.209 Ra_{CWHF}^{0.249} \left( \frac{Pr}{1 + Pr} \right)^{0.031} [c_B (1 - AR)^{c_c} + 1] \quad (4.22)$$

where  $c_B = -1.168$ ,  $c_c = 0.683 Ra_{CWHF}^{0.089}$ .

For  $1 < AR \leq 8$ ;

$$\overline{Nu}_{Rec} = 0.209 Ra_{CWHF}^{0.249} \left( \frac{Pr}{1 + Pr} \right)^{0.031} \left[ (0.737 Ra_{CWHF}^{-0.189}) \ln AR + 1 \right] \quad (4.23)$$

The correlation proposed in Eqs. 4.18 - 4.23 can be used for  $r_i/L \gg 1$  as  $\overline{Nu}_i$  is expected to approach  $\overline{Nu}_{Rec}$  and  $(\overline{Nu}_i)_{cond}$  approaches unity (i.e.  $\lim_{r_i/L \rightarrow \infty} (L/r_i) / (\ln(1 + L/r_i))$ ) in the limit of  $r_i/L \rightarrow \infty$ . In this study, the mean Nusselt number  $(\overline{Nu}_i)_{Bn=0}$  for natural convection of Newtonian fluids in rectangular cross-sectional cylindrical enclosures is parameterised in the following manner:

$$(\overline{Nu}_i)_{Bn=0} = (\overline{Nu}_{Rec})_{Bn=0} \frac{[m_o(L/r_i)]^{k_o}}{\ln[1 + (m_o(L/r_i))^{k_o}]} \quad (4.24)$$

For CWT boundary condition;

$$m_o = [0.712 + 0.016 \ln Ra]^{(\overline{Nu}_{Rec})_{Bn=0}^{-1}}; k_o = [0.656 + 0.022 \ln Ra]^{(\overline{Nu}_{Rec})_{Bn=0}^{-1}} \quad (4.25)$$

For CWHF boundary condition;

$$m_o = [0.151 (\ln Ra)^{0.66}]^{(\overline{Nu}_{Rec})_{Bn=0}^{-1}}; k_o = [0.694 + 0.016 \ln Ra]^{(\overline{Nu}_{Rec})_{Bn=0}^{-1}} \quad (4.26)$$

Equation 4.24 is not only valid for  $0.125 \leq r_i/L \leq 16$  but also remains applicable for rectangular enclosures because  $\overline{Nu}_i$  will be equal to the  $\overline{Nu}_{Rec}$  and  $(\overline{Nu}_i)_{cond}$

will be equal to unity (i.e.  $\lim_{r_i/L \rightarrow \infty} [(L/r_i)/\ln(1 + L/r_i)] = 1.0$ ) in the limit of  $r_i/L \rightarrow \infty$ . The predictions of Eq. 4.24 are shown in Fig. 4.13 for both CWT and CWHF boundary conditions, which indicate that this correlation satisfactorily predicts  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  (i.e.  $R^2 > 0.99$ ) for Newtonian fluids in rectangular crosssectional cylindrical annular enclosures for  $10^3 \leq Ra \leq 10^6$ ;  $10 \leq Pr \leq 10^3$ ;  $0.25 \leq AR \leq 8$ ;  $0.125 \leq r_i/L \leq 16$ . It is worth noting that Eq. 4.24 underpredicts for  $AR < 1$ , since the correlation given by Eq. 4.18 underpredicts the value of  $\overline{Nu}_{Rec}$  for  $AR < 1$  in the case of Newtonian fluids (i.e.  $Bn = 0$ ) (see Fig. 10 of [47]).

Using Eq. 4.11 one can obtain a scaling estimation of  $\overline{Nu}_i$  for Bingham fluids in the boundary layer regime for the CWT boundary condition in the following manner:

$$\overline{Nu}_i \sim \text{Max} \left\{ (\overline{Nu}_i)_{cond}, \frac{(\overline{Nu}_i)_{Bn=0}}{\left[ \frac{Bn^*}{2} + \frac{1}{2} \sqrt{Bn^{*2} + 4} \right]} \right\} f_2(Ra, Pr, Bn, AR, r_i/L) \quad (4.27)$$

where  $Bn^* = Bn \left[ (RaAR/Pr)^{-1/4} \right]$ .

Eq. 4.27 is extended to correlate  $\overline{Nu}_i$  for natural convection of Bingham fluids in rectangular crosssectional cylindrical annular enclosures for  $0 \leq Bn \leq Bn_{max}$ :

$$\frac{[\overline{Nu}_i / (\overline{Nu}_i)_{cond}] - 1}{[(\overline{Nu}_i)_{Bn=0} / (\overline{Nu}_i)_{cond}]} = \frac{2 \left[ 1 - (Bn^*/Bn_{max}^*)^{b_i} \right]^{c_i}}{Bn^* + \sqrt{Bn^{*2} + 4}} \quad \text{when} \quad (\overline{Nu}_i)_{Bn=0} / (\overline{Nu}_i)_{cond} > 1 \quad (4.28)$$

$$\overline{Nu}_i / (\overline{Nu}_i)_{cond} = 1 \quad \text{when} \quad (\overline{Nu}_i)_{Bn=0} / (\overline{Nu}_i)_{cond} = 1 \quad (4.29)$$

where  $Bn_{max}^* = Bn_{max} \left[ (RaAR/Pr)^{-1/4} \right]$ .

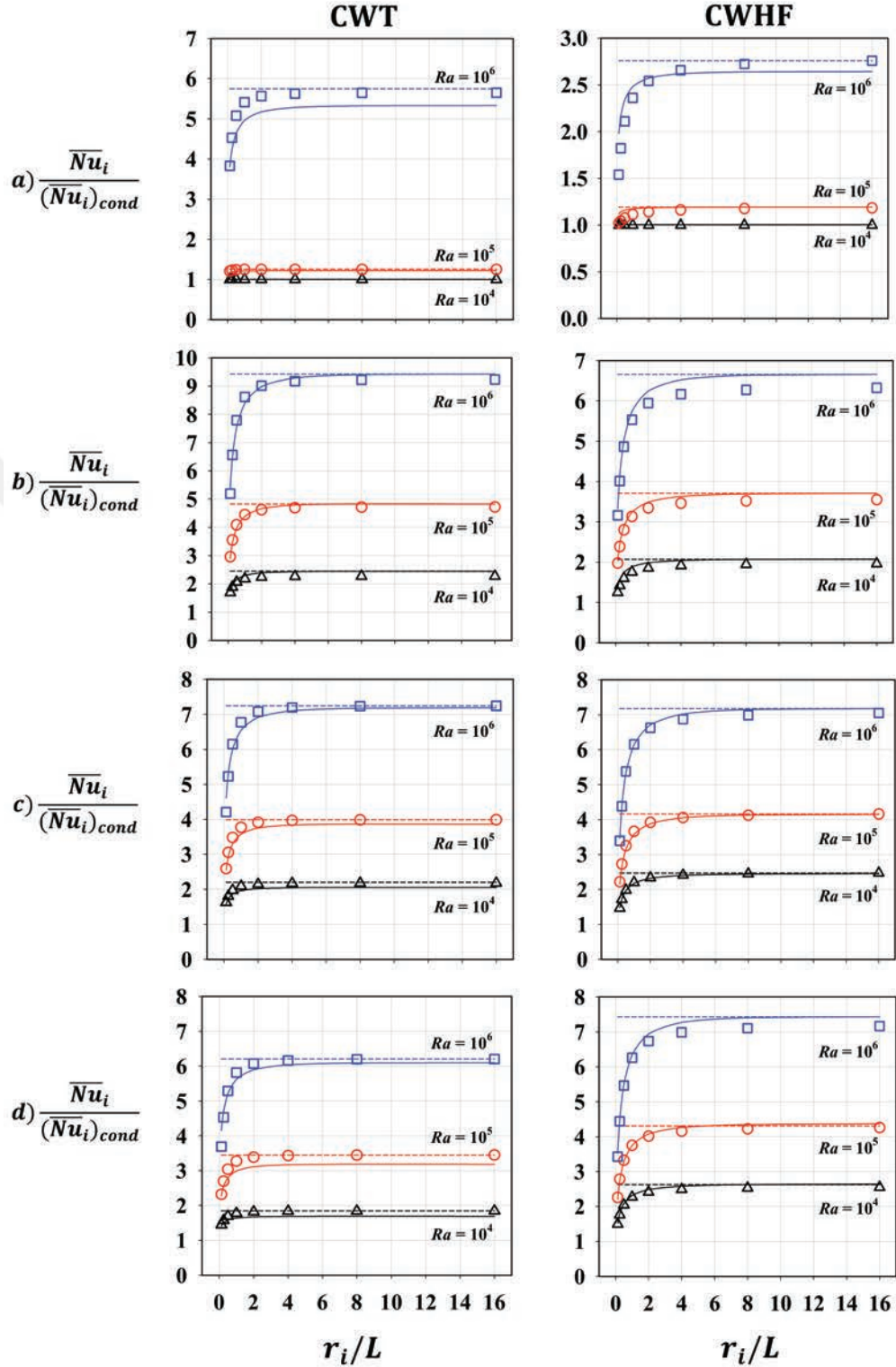


Figure 4.13: The variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $r_i/L$  for Newtonian fluids (i.e.  $Bn = 0$ ) for different values of  $Ra$  and  $AR$ : a) 0.25, b) 1, c) 4, d) 8 at  $Pr = 500$  along with predictions of Eq. 4.24 (solid line). The corresponding values for the  $\overline{Nu}_{Rec}$  are shown by the dashed line.

Turan et al. [22, 62] also proposed an empirical correlation for  $(Bn_{max})_{sq}$ , which is relevant to natural convection of Bingham fluids in square enclosures for both CWT [22] and CWHF [62] boundary conditions for  $10^3 \leq Ra \leq 10^6$  and  $0.1 \leq Pr \leq 10^2$ . Here,  $(Bn_{max})_{i,sq}$  in the square cross-sectional cylindrical annular enclosures is parameterised using correlation of  $(Bn_{max})_{sq} = 0.019Ra^{0.56}Pr^{-0.46}$  for both CWT and CWHF boundary conditions.

For CWT boundary condition:

$$(Bn_{max})_{i,sq} = (Bn_{max})_{sq} - [m_1(L/r_i)^{k_1}] \quad (4.30)$$

For CWHF boundary condition:

$$(Bn_{max})_{i,sq} = (Bn_{max})_{sq} / [1 + m_1(L/r_i)^{k_1}] \quad \text{for } r_i/L \leq 1 \quad (4.31)$$

$$(Bn_{max})_{i,sq} = (Bn_{max})_{sq} - [m_1(L/r_i)^{k_1}] \quad \text{for } r_i/L > 1 \quad (4.32)$$

where  $m_1$  and  $k_1$  are the correlation parameters, which are listed in Table 4.1. It can be seen Fig. 4.14 that correlations given in Table 4.1 satisfactorily capture (i.e.  $R^2 > 0.99$ ) the variation of  $(Bn_{max})_{i,sq}$  for different values of  $10^3 \leq Ra \leq 10^6$ ,  $10 \leq Pr \leq 10^3$ ,  $0.125 \leq r_i/L \leq 16$  at  $AR = 1.0$ .

Similarly, Turan et al. [47] proposed the following correlation for threshold value of Bingham number  $(Bn_{max}^*)_{Rec}$  for rectangular enclosures, which is valid for both CWT and CWHF boundary conditions:

$$(Bn_{max}^*)_{Rec} = [0.019 + 0.01\text{erf}(2AR - 2)] Ra^{0.31} Pr^{-0.21} AR^{-0.25} \quad (4.33)$$

Table 4.1: Summary of  $(Bn_{max})_{i,sq}$  correlations in the square cross-sectional (i.e.  $AR = 1$ ) cylindrical annular enclosures for the ranges of  $10^3 \leq Ra \leq 10^6$ ,  $10 \leq Pr \leq 10^3$  and  $0.125 \leq r_i/L \leq 16$  given by Eqs. 4.30 - 4.32 for CWT and CWHF boundary conditions respectively.

| CWT            |                                                                                                                                                                                                                                                                  |
|----------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $r_i/L \leq 1$ | $(Bn_{max})_{i,sq} = (Bn_{max})_{sq} - [m_1(L/r_i)^{k_1}]$ $m_1 = \left[ \frac{1+0.0028Pr}{656.8+20.4Pr} \right] Ra^{0.615}$ $k_1 = \left[ \frac{8.82+0.35Pr}{21.8+Pr} \right] - 0.017 \ln Ra$                                                                   |
| $r_i/L > 1$    | $(Bn_{max})_{i,sq} = (Bn_{max})_{sq} - [m_1(L/r_i)^{k_1}]$ $m_1 = \left[ \frac{1+0.003Pr}{565.2+18.15Pr} \right] Ra^{0.6}$ $k_1 = \left[ \frac{1+0.006Pr}{1.397+0.038Pr} \right] - \left[ \frac{1+0.006Pr}{19.85+0.547Pr} \right] \ln Ra$                        |
| CWHF           |                                                                                                                                                                                                                                                                  |
| $r_i/L \leq 1$ | $(Bn_{max})_{i,sq} = (Bn_{max})_{sq} / [1 + m_1(L/r_i)^{k_1}]$ $m_1 = \left[ \frac{50.8+0.97Pr}{77.6+Pr} \right] Ra^{0.048}$ $k_1 = \left[ \frac{1.145+0.117Pr}{1+0.117Pr} \right] - 0.02 \ln Ra$                                                                |
| $r_i/L > 1$    | $(Bn_{max})_{i,sq} = (Bn_{max})_{sq} - [m_1(L/r_i)^{k_1}]$ $m_1 = (0.077 - 0.01 \ln Pr) + \left[ \frac{1+0.0028Pr}{311.4+11Pr} \right] Ra^{0.585}$ $k_1 = \left[ \frac{1+0.153Pr}{0.064+0.372Pr} \right] - \left[ \frac{1-0.226Pr}{63.76+9.63Pr} \right] \ln Ra$ |

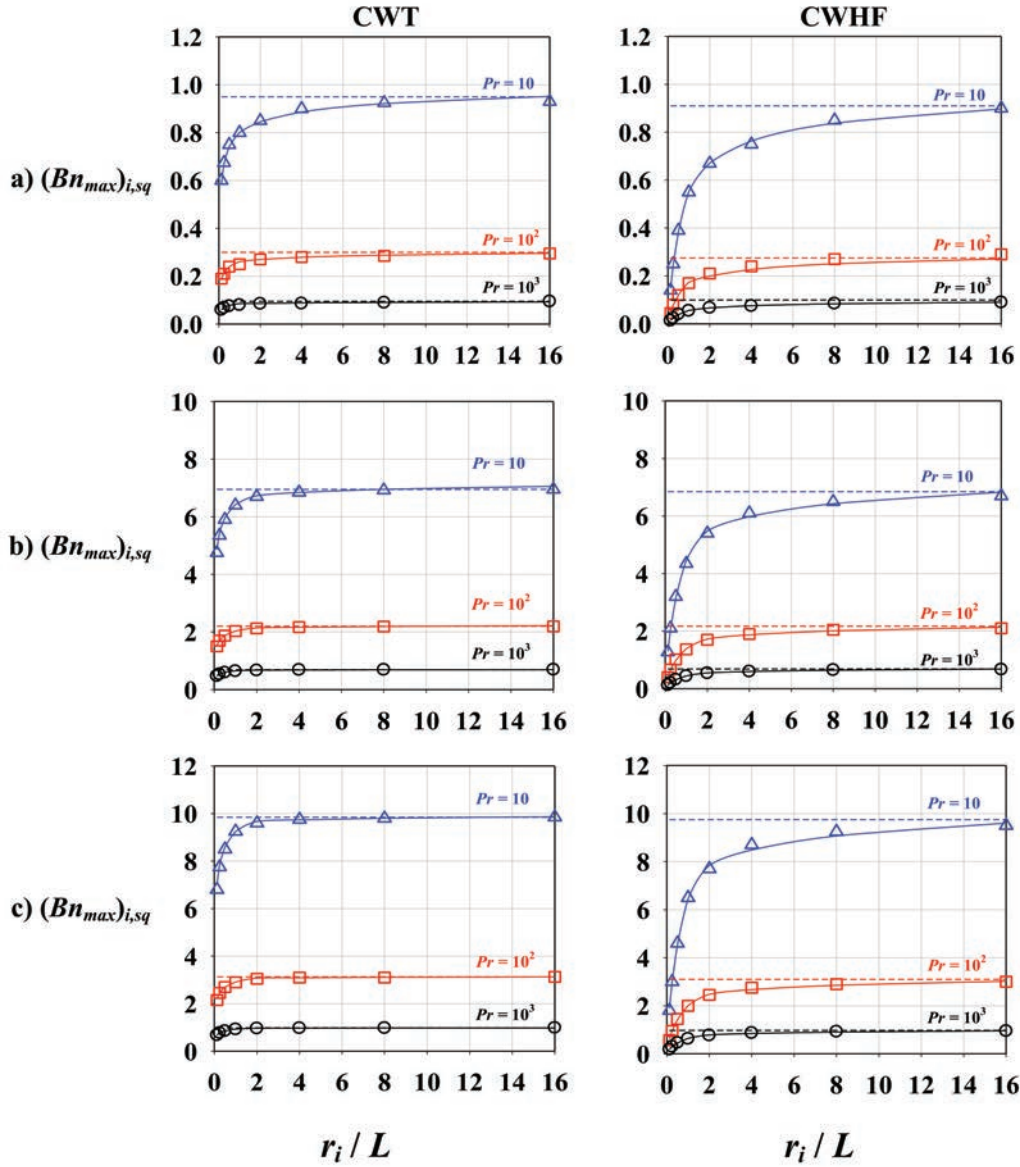


Figure 4.14: Variation of  $(Bn_{max})_{i,sq}$  with  $r_i/L$  for different values of  $Pr$  and  $Ra$ : a)  $10^4$ , b)  $5 \times 10^5$ , c)  $10^6$  at  $Ar = 1.0$  along with the predictions of the correlations (solid line) in Table 4.1 for both CWT and CWHF boundary conditions. The corresponding  $(Bn_{max})_{sq}$  values are shown by dashed lines.

It is worth noting that Turan et al. [47] used the  $\overline{Nu}_{Rec} = 1.0$  as the criterion for  $(Bn_{max}^*)_{Rec}$  in Eq. 4.33 but this is different from the criterion followed here (i.e. where  $\overline{Nu}_i / (\overline{Nu}_i)_{cond} = 1.01$ ). This correlation has been modified here in the

following manner based on the revised  $\overline{Nu}_{Rec}$  criterion (i.e.  $\overline{Nu}_{Rec} = 1.01$ ).

$$(Bn_{max}^*)_{Rec} = c_1 Ra^{0.31} Pr^{-0.21} AR^{-0.25} \text{ for } CWT \quad (4.34)$$

$$(Bn_{max}^*)_{Rec} = c_2 Ra^{0.31} Pr^{-0.21} AR^{-0.25} \text{ for } CWHF \quad (4.35)$$

where  $c_1$  and  $c_2$  are the correlation parameters, which are listed in Table 4.2.

The correlation proposed in Eqs. 4.34 and 4.35 can be used for  $r_i/L \gg 1$  in the limit of  $r_i/L \rightarrow \infty$ . In this study,  $(Bn_{max})_{Rec} = (Bn_{max}^*)_{Rec} [(RaAR/Pr)^{1/4}]$  is expressed using Eqs. 4.36 and 4.37 in the following manner:

$$(Bn_{max})_{i,Rec} = (Bn_{max})_{Rec} - m_2(L/r_i)^{k_2} \text{ for } CWT \quad (4.36)$$

$$(Bn_{max})_{i,Rec} = (Bn_{max})_{Rec} / [1 + m_3(L/r_i)^{k_3}] \text{ for } CWHF \quad (4.37)$$

where  $m_2$ ,  $k_2$ ,  $m_3$  and  $k_3$  are the correlation parameters, which are listed in Table 4.3. It can be seen Fig. 4.15 that correlations given in Table 4.3 satisfactorily capture (i.e.  $R^2 > 0.99$ ) the variation of  $(Bn_{max})_{i,Rec}$  for different values of  $10^3 \leq Ra \leq 10^6$ ,  $0.125 \leq AR \leq 8$ ,  $0.125 \leq r_i/L \leq 16$  at  $Pr = 10^3$ .

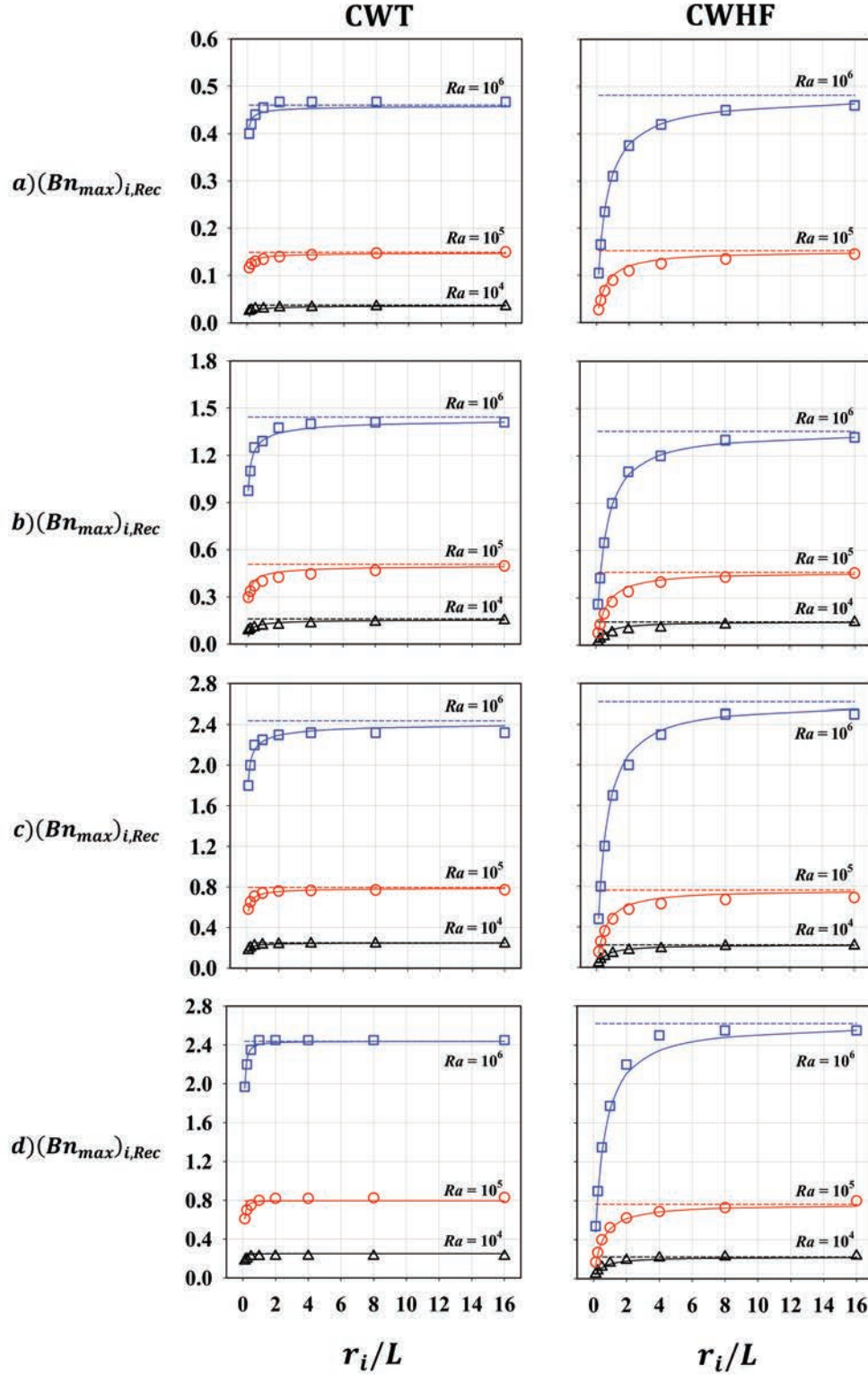


Figure 4.15: Variation of  $(Bn_{max})_{i,Rec}$  with  $r_i/L$  for different values of  $Ra$  and  $AR$ : a) 0.5, b) 1, c) 4, d) 8 at  $Pr = 500$  along with the predictions of the correlations (solid line) in Table 4.3 for both CWT and CWHF boundary conditions. The corresponding  $(Bn_{max})_{Rec}$  values are shown by dashed lines.

Table 4.2: Summary of  $(Bn_{max}^*)_{Rec}$  correlation given by Eqs. 4.34 and 4.35.

| $(Bn_{max}^*)_{Rec} = c_1 Ra^{0.31} Pr^{-0.21} AR^{-0.25} (CWT)$  |                                                                                                                            |
|-------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------|
| $AR \leq 1$                                                       | $c_1 = \left[ \frac{1-0.062 \ln Ra}{90.5-5.117 \ln Ra} \right] + [0.021 - 0.001 \ln Ra] \operatorname{erf} [2AR - 1.5]$    |
| $AR > 1$                                                          | $c_1 = \left[ \frac{1-0.052 \ln Ra}{85.34-3.742 \ln Ra} \right] + [0.024 - 0.001 \ln Ra] \operatorname{erf} [2AR - 3.352]$ |
| $(Bn_{max}^*)_{Rec} = c_2 Ra^{0.31} Pr^{-0.21} AR^{-0.25} (CWHF)$ |                                                                                                                            |
| $AR \leq 1$                                                       | $c_2 = \left[ \frac{1-0.054 \ln Ra}{94.696-4 \ln Ra} \right] + [0.0175 - 0.0008 \ln Ra] \operatorname{erf} [2AR - 1.402]$  |
| $AR > 1$                                                          | $c_2 = \left[ \frac{1-0.085 \ln Ra}{166} \right] + 0.021 \operatorname{erf} [2AR - 3.097]$                                 |

Table 4.3: Summary of  $(Bn_{max})_{i,Rec}$  correlation given by Eqs. 4.36 and 4.37.

| CWT                                                                                                                   |                                                                                                                                                                                       |
|-----------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $(Bn_{max})_{i,Rec} = (Bn_{max})_{Rec} - m_2(L/r_i)^{k_2}$ ; $m_2 = d_o Ra^{d_1}$ ; $k_2 = d_2 - d_3 \ln Ra$          |                                                                                                                                                                                       |
| $AR \leq 1$                                                                                                           | $d_o = 0.002 / [1 + \exp((AR - 7.812) / 0.51)]$ $d_1 = 0.009 + 0.303 AR^{1.16}$ $d_2 = 1.096 / [1 + \exp(-AR / 1504.99)], d_3 = 0$                                                    |
| $AR > 1$                                                                                                              | $d_o = 0.0002 / [1 + \exp((AR - 6.633) / 0.501)]$ $d_1 = 0.069 + 0.321 AR^{0.215}$ $d_2 = 15.598 / [1 + \exp((16.819 - AR) / 5.11)]$ $d_3 = 0.077 / [1 + \exp((3.986 - AR) / 0.028)]$ |
| CWHF                                                                                                                  |                                                                                                                                                                                       |
| $(Bn_{max})_{i,Rec} = (Bn_{max})_{Rec} / [1 + m_3(L/r_i)^{k_3}]$ ; $m_3 = d_4 Ra^{d_5}$ ;<br>$k_2 = d_6 - d_7 \ln Ra$ |                                                                                                                                                                                       |
| $AR \leq 1$                                                                                                           | $d_4 = 0.525 / [1 + \exp((-AR - 1.748) / 0.1)], d_5 = 0$ $d_6 = (23.403 AR - 1) / (17.154 AR + 0.912)$ $d_7 = 0.028 / [1 + \exp((0.545 - AR) / 14.617)]$                              |
| $AR > 1$                                                                                                              | $d_4 = 0.44 / [1 + \exp((-AR - 1.748) / 0.1)]$ $d_5 = 0.0189 / [1 + \exp((-AR - 600.2) / 4878.034)]$ $d_6 = (39.647 AR - 1) / (37.748 AR), d_7 = 0$                                   |

Turan et al. [22] and Turan et al. [62] proposed  $b_{sq} = 1$  and  $c_{sq} = 0.42Ra^{0.13}Pr^{0.12}$  using Eqs. 4.28 and 4.29 for CWT boundary condition [22] and  $b_{sq} = 0.643$  and  $c_{sq} = 0.143Ra^{0.226}Pr^{0.062}$  for CWHF boundary condition [62] in the case of square enclosures. Here,  $b_{i,sq}$  and  $c_{i,sq}$  have been parameterised here for square cross-sectional cylindrical annular enclosures as follows:

For CWT boundary condition:

$$b_{i,sq} = b_{sq} \text{ and } c_{i,sq} = c_{sq} - [m_4(L/r_i)^{k_4}] \quad (4.38)$$

where  $m_4 = \frac{1.047Pr^{0.2}}{1+(Ra/16880)^{-0.53}}$  and  $k_4 = \frac{4.807+0.031Pr}{51+Pr}$ .

For CWHF boundary condition:

$$b_{i,sq} = b_{sq} \left[ \left( \frac{\ln Ra - 15.48}{0.512 \ln Ra - 9.76} \right) (L/r_i)^{0.01} \right] \text{ and } c_{i,sq} = c_{sq} - [m_4(L/r_i)^{k_4}] \quad (4.39)$$

where  $m_4 = \left[ \frac{0.015}{1-0.05 \ln Pr} \right] Ra^{0.37} - \left[ \frac{0.733}{1+0.267 \ln Pr} \right];$

$k_4 = \left[ \frac{1-0.072 \ln Ra}{\ln Ra - 7.643} \right] + \left[ \frac{1-0.06 \ln Ra}{0.22 \ln Ra - 1.98} \right] \exp(-n_o \ln Pr)$  and  $n_o = \frac{1-0.064 \ln Ra}{0.148 \ln Ra - 0.994}.$

The predictions of Eqs. 4.38 and 4.39 are shown in Fig. 4.16, which shows that the correlation satisfactorily predicts ( $R^2 > 0.99$  according to non-linear regression reported by Brown [109])  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  for Bingham fluids in square cross-sectional cylindrical annular enclosures for  $10^3 \leq Ra \leq 10^6$ ,  $10 \leq Pr \leq 10^3$  and  $0.125 \leq r_i/L \leq 16$  at  $AR = 1.0$  for both CWT and CWHF boundary conditions.

Similarly, Turan et al. [47] proposed the mean Nusselt number  $\overline{Nu}_{Rec}$  correlation for natural convection of Bingham fluids in rectangular enclosures in the following manner by utilising Eq. 4.28 and 4.29:

For CWT boundary condition;

$$\frac{\overline{Nu}_{Rec} - 1}{(\overline{Nu}_{Rec})_{Bn=0} - 1} = \left\{ \left[ 1 - \left( \frac{1}{4.55} \right)^{-2} (Bn^{**})_{CWT} \right]^{2/n_1} + \left[ \frac{2 \left[ 1 - (Bn^*/Bn_{max}^*)^{b_o} \right]^{b_1}}{Bn^* + \sqrt{Bn^{*2} + 4}} \right]^{1/n_1} \right\}^{n_1} \quad (4.40)$$

In Eq. 4.40,  $n_1 = -0.02$  is when  $\overline{Nu}_{Rec} > 1$  and  $\overline{Nu}_{Rec} = 1$  is when  $(\overline{Nu}_{Rec})_{Bn=0} = 1$ ;  $b_o = 1.37$  and  $b_1 = 0.676 Ra_{CWT}^{0.131} Pr^{0.11}$ ;  $(Bn^{**})_{CWT} = Bn_{CWT} / [(Ra_{CWT}/Pr)^{1/2} AR^2]$ .

For CWHF boundary condition;

$$\frac{\overline{Nu}_{Rec} - 1}{(\overline{Nu}_{Rec})_{Bn=0} - 1} = \left\{ \left[ 1 - \left( \frac{1}{5.5} \right)^{-2} (Bn^{**})_{CWHF} \right]^{2/n_2} + \left[ \frac{2 \left[ 1 - (Bn^*/Bn_{max}^*)^{b_2} \right]^{b_3}}{Bn^* + \sqrt{Bn^{*2} + 4}} \right]^{1/n_2} \right\}^{n_2} \quad (4.41)$$

In Eq. 4.41,  $n_2 = -0.02$  is when  $\overline{Nu}_{Rec} > 1$  and  $\overline{Nu}_{Rec} = 1$  is when  $(\overline{Nu}_{Rec})_{Bn=0} = 1$ ;  $b_2 = 0.867$  and  $b_3 = 0.25 Ra_{CWHF}^{0.207} Pr^{0.062}$ ;

$$(Bn^{**})_{CWHF} = Bn_{CWT} / [(Ra_{CWHF}/Pr)^{1/2} AR^2].$$

It is worth noting that first and second terms on the right hand side of Eqs. 4.40 and 4.41 account for the contributions of the parallel flow and boundary layer regimes of convection respectively.

In current study,  $\overline{Nu}_{Rec}$  is replaced by  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  and  $b_o$ ,  $b_1$ ,  $b_2$  and  $b_3$  (see Eqs. 4.40 and 4.41) are modified in the following form for evaluating for natural convection of Bingham fluids in rectangular cross-sectional cylindrical enclosures:

$$(b_o)_i = 1.37 / [1 + k_5 (L/r_i)^{0.01}] \text{ and } (b_1)_i = (0.676 Ra_{CWT}^{0.131} Pr^{0.11}) / [1 + k_6 (L/r_i)^{0.01}] \quad (4.42)$$

$$(b_2)_i = 0.867 / \left[ 1 + k_7 (L/r_i)^{0.01} \right] \text{ and}$$

$$(b_3)_i = \left( 0.25 Ra_{CWHF}^{0.207} Pr^{0.062} \right) / \left[ 1 + k_8 (L/r_i)^{0.01} \right] \quad (4.43)$$

where  $k_5$ ,  $k_6$ ,  $k_7$  and  $k_8$  are the correlation parameters. The resultant correlation for  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  in the case of rectangular cross-sectional cylindrical enclosures is summarised in Table 4.4.

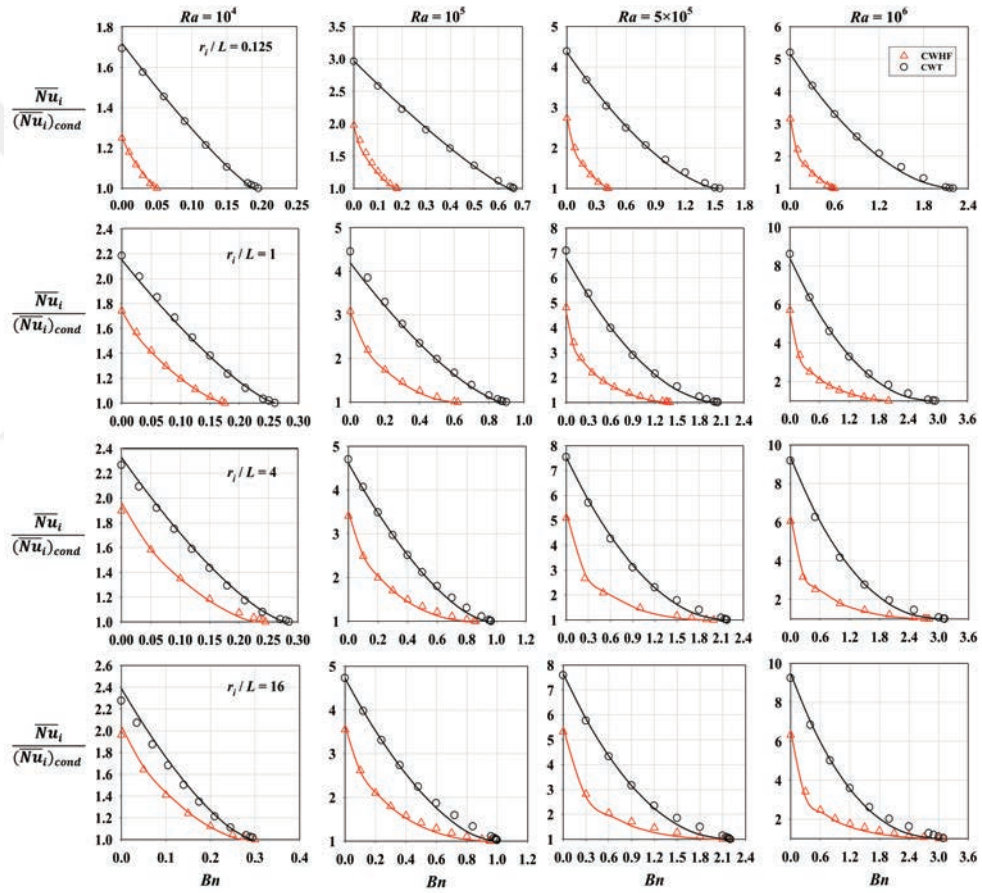


Figure 4.16: Variation of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $Bn$  for different values of  $r_i/L$  and  $Ra$  at  $Pr = 10^2$  and  $AR = 1.0$  for both CWT and CWHF boundary conditions along with the prediction of Eqs. 4.38 and 4.39.

The correlation in Table 4.4 is not only valid for  $0.125 \leq r_i/L \leq 16$  but also

remains applicable for rectangular enclosures because  $(b_o)_i$ ,  $(b_1)_i$ ,  $(b_2)_i$ ,  $(b_3)_i$  become  $b_o$ ,  $b_1$ ,  $b_2$ ,  $b_3$  respectively (i.e.  $\lim_{r_i/L \rightarrow \infty} [(L/r_i) / \ln(1 + L/r_i)] = 1.0$ ) in the limit of  $r_i/L \rightarrow \infty$ . Accordingly,  $\overline{Nu}_i$  will be equal to the  $\overline{Nu}_{Rec}$  and  $(\overline{Nu}_i)_{cond}$  assumes a value of unity (i.e.  $\lim_{r_i/L \rightarrow \infty} [(L/r_i) / \ln(1 + L/r_i)] = 1.0$ ) in the limit of  $r_i/L \rightarrow \infty$ . The predictions of the  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  correlation listed in Table 4.4 are shown in Fig. 4.17 for both CWT and CWHF boundary conditions, which indicates that this correlation satisfactorily predicts  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  (i.e.  $R^2 > 0.98$ ) for Bingham fluids in rectangular cross-sectional cylindrical annular enclosures for  $10^3 \leq Ra \leq 10^6$ ,  $0.5 \leq AR \leq 8$  and  $0.125 \leq r_i/L \leq 16$  at  $Pr = 500$  for both CWT and CWHF boundary conditions.

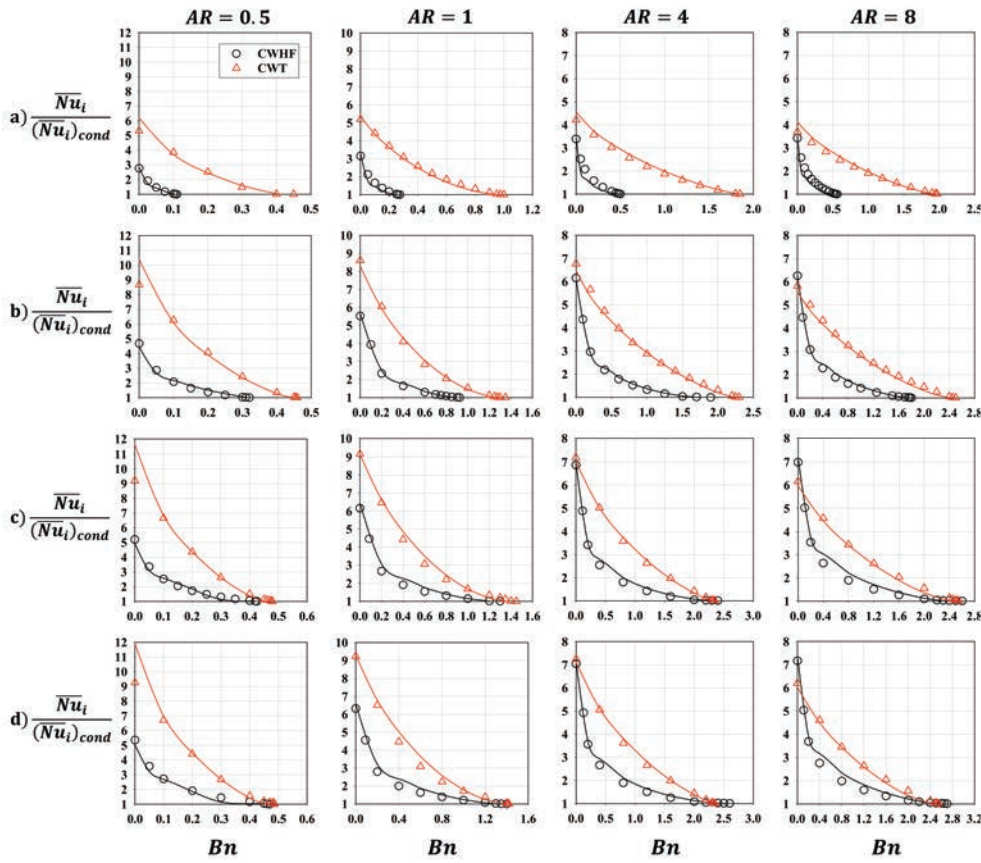


Figure 4.17: Variation of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $Bn$  for different values of  $AR$  and  $r_i/L$ : a) 0.125, b) 1, c) 4, d) 16 at  $Ra = 10^6$  and  $Pr = 500$  for both CWT and CWHF boundary conditions along with the prediction of correlations given in Table 4.4.

Table 4.4: Summary of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  correlation for rectangular cross-sectional cylindrical annular enclosures for CWT and CWHF boundary conditions.

| CWT                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |                                                                                                                                      |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------|
| $\frac{[\overline{Nu}_i / (\overline{Nu}_i)_{cond}]^{-1}}{[(\overline{Nu}_i)_{Bn=0} / (\overline{Nu}_i)_{cond}]^{-1}} = \left\{ \left[ 1 - \left( \frac{1}{4.55} \right)^{-2} (Bn^{**})_{CWT} \right]^{2/n_1} + \left[ \frac{2[1 - (Bn^{**} / Bn_{max}^{*})^{(b_o)_i}]^{(b_1)_i}}{Bn^{*} + \sqrt{Bn^{*2} + 4}} \right]^{1/n_1} \right\}^{n_1}$ <p>for <math>\overline{Nu}_i / (\overline{Nu}_i)_{cond} &gt; 1</math> and <math>\overline{Nu}_i / (\overline{Nu}_i)_{cond} = 1</math> is when <math>(\overline{Nu}_i)_{Bn=0} / (\overline{Nu}_i)_{cond} = 1</math><br/> <math>(b_o)_i = 1.37 / [1 + k_5(L/r_i)^{0.01}]</math> and <math>(b_1)_i = (0.676 Ra_{CWT}^{0.131} Pr^{0.11}) / [1 + k_6(L/r_i)^{0.01}]</math></p>   |                                                                                                                                      |
| $0.5 \leq AR \leq 1$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $k_5 = [1.1 - 1.018 AR] Ra^{0.208 AR - 0.059}$ $k_6 = [0.607 - 0.147 AR] \ln Ra - [1.79 + 0.6 AR]$                                   |
| $1 < AR \leq 8$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | $k_5 = [0.137 / (1 + \exp((AR - 2.326) / 0.33))] Ra^{0.067 AR^{1.45}}$ $k_6 = (0.513 AR^{0.26}) \ln Ra - (44.95 AR^{0.064} - 44.52)$ |
| CWHF                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |                                                                                                                                      |
| $\frac{[\overline{Nu}_i / (\overline{Nu}_i)_{cond}]^{-1}}{[(\overline{Nu}_i)_{Bn=0} / (\overline{Nu}_i)_{cond}]^{-1}} = \left\{ \left[ 1 - \left( \frac{1}{5.5} \right)^{-2} (Bn^{**})_{CWHF} \right]^{2/n_2} + \left[ \frac{2[1 - (Bn^{**} / Bn_{max}^{*})^{(b_2)_i}]^{(b_3)_i}}{Bn^{*} + \sqrt{Bn^{*2} + 4}} \right]^{1/n_2} \right\}^{n_2}$ <p>for <math>\overline{Nu}_i / (\overline{Nu}_i)_{cond} &gt; 1</math> and <math>\overline{Nu}_i / (\overline{Nu}_i)_{cond} = 1</math> is when <math>(\overline{Nu}_i)_{Bn=0} / (\overline{Nu}_i)_{cond} = 1</math><br/> <math>(b_2)_i = 0.867 / [1 + k_7(L/r_i)^{0.01}]</math> and <math>(b_3)_i = (0.25 Ra_{CWHF}^{0.207} Pr^{0.062}) / [1 + k_8(L/r_i)^{0.01}]</math></p> |                                                                                                                                      |
| $0.5 \leq AR \leq 8$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $k_7 =$ $[0.168 / (1 + \exp((14.1 - AR) / 2.282))] Ra^{0.6 - 0.028 AR}$                                                              |
| $0.5 \leq AR \leq 1$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       | $k_8 = [0.009 AR^{0.585}] Ra^{0.542 - 0.1 AR}$                                                                                       |
| $1 < AR \leq 8$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | $k_8 = \left[ \frac{1 - 0.479 AR}{318.7 - 157.9 AR} \right] (\ln Ra)^{k_9}$ $k_9 = \frac{2.764}{1 + \exp((1.49 - AR) / 0.671)}$      |

## 4.8 Summary of Key Findings

The effects of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $AR$ ,  $r_i/L$  and boundary condition (i.e. CWT and CWHF) on steady-state laminar natural convection of Bingham fluids in rectangular cross-sectional cylindrical annular enclosures with differentially heated vertical side walls have been numerically investigated in this chapter. It is found that the mean Nusselt number based on the inner periphery  $\overline{Nu}_i$  increases (decreases) with increasing  $Ra$  ( $Bn$ ) due to the strengthening of buoyancy force (viscous resistance), regardless of the boundary condition for the vertical walls. Accordingly, the mean Nusselt number  $\overline{Nu}_i$  normalised by the corresponding Nusselt number due to pure conductive transport (i.e.  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ ) decreases with increasing  $Bn$ , and attains a value of unity (purely conductive thermal transport) for large values of  $Bn$  irrespective of the boundary condition. This behaviour is obtained because the fluid flow practically stops and thermal transport becomes conduction-dominated for large values of  $Bn$  regardless of boundary conditions. Also,  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  does not change significantly with increasing  $Pr$  for Newtonian fluids, but it decreases with increasing  $Pr$  for large values of Bingham number for both CWT and CWHF boundary conditions. Additionally,  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases with increasing  $r_i/L$  before approaching an asymptotic value equal to the mean Nusselt number for rectangular enclosures in the limit of  $r_i/L \rightarrow \infty$ . This indicates that the ratio of convective to diffusive thermal transport increases with increasing  $r_i/L$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids for both CWT and CWHF boundary conditions. The effects of aspect ratio  $AR$  have also been investigated in detail and it is found that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  shows a non-monotonic trend (i.e. an increase with increasing  $AR$  before reaching a maximum and subsequently shows a decreasing trend) with increasing  $AR$  for a given set of values of  $Ra$ ,  $Pr$ ,  $r_i/L$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids for the CWT boundary condition, whereas  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases monotonically with increasing  $AR$  in the CWHF boundary condition. The influences of convective thermal transport strengthen whereas thermal diffusive transport weakens with increasing  $AR$  in this boundary condition, and these competing effects are responsible for the non-monotonic  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  variation with  $AR$  in the CWT boundary condition. However, the strengthening of con-

vective transport with increasing  $AR$  dominates over the weakening of thermal diffusion in the CWHF boundary condition for the aspect ratio range considered here. Furthermore, conductive thermal transport plays a dominant role for small values of  $AR$  in the case of both CWT and CWHF boundary conditions. Detailed scaling analysis is utilised to explain the observed influences of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $AR$  and  $r_i/L$  on the mean Nusselt number  $\overline{Nu}_i$  and new correlations have been proposed for the mean Nusselt number  $\overline{Nu}_i$ , which have been shown to predict numerical findings accurately for the range of parameters considered here.





## Chapter 5

# Differentially Heated Side Wall Configuration: Power-law Fluids

Numerical findings on laminar natural convection of shear thinning / thickening fluids obeying a power-law model of viscosity in axisymmetric rectangular cross-sectional cylindrical annular enclosures with differentially heated vertical side walls (see Fig. 2.1) have been presented in this chapter. Effects of shear-thinning / thickening character of viscosity on heat and momentum transport have been investigated for a range of different values of normalized internal radius  $r_i/L$  (i.e.  $0.125 \leq r_i/L \leq 16$ , where  $r_i$  is the internal cylinder radius and  $L$  is the difference between outer and inner radii); aspect ratio  $AR$  (i.e.  $0.125 \leq AR = H/L \leq 8$  where  $H$  is the enclosure height); nominal Rayleigh number  $Ra$  (i.e.  $10^3 \leq Ra \leq 10^6$ ), Prandtl number  $Pr$  (i.e.  $10^2 \leq Pr \leq 10^4$ ) and power-law index  $n$  (i.e.  $0.6 \leq n \leq 1.8$ ) for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions.

### 5.1 Scaling Analysis

A detailed scaling analysis has been performed here to explain the influences of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$ ,  $r_i/L$  on the mean Nusselt number  $\overline{Nu_i}$ . Equating the order of magnitudes of inertial and buoyancy terms yields  $w \sim \sqrt{g\beta(T_H - T_C)LAR}$

( $w \sim \sqrt{g\beta q_i \delta_{th} LAR/k}$ ) for CWT (CWHF) boundary conditions. Using continuity equation for axisymmetric geometry, one gets:

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} \sim \left( \frac{u}{r} + \frac{u}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{H} \quad (5.1)$$

which leads to scaling estimate of  $u$ :

$$u \sim \frac{w}{AR(r+L)} \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{1}{\sqrt{AR}} \frac{\sqrt{Ra_{CWT} Pr}}{(1+L/r_i)} \quad (5.2)$$

$$u \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{Ra_{CWHF}^{(n+1)/(n+4)} Pr^{(n+2)/(n+4)} AR^{-2/(n+4)}}{(1+L/r_i) f_3^{3/(n+4)}} \quad (5.3)$$

Similarly, equating the order of magnitudes of inertial and viscous terms in the vertical (radial) direction (i.e.  $\rho w^2/H \sim [(K/\delta)(w/\delta)^n]$  ( $\rho u^2/L \sim [(K/\delta_1)(u/\delta_1)^n]$ ), it is possible to estimate hydrodynamic boundary layer thickness where  $\delta$  ( $\delta_1$ ) is the hydrodynamic boundary layer thickness on the vertical (horizontal) walls respectively:

$$\delta \sim L Ra_{CWT}^{(n-2)/(2n+2)} Pr^{n/(2n+2)} AR^{n/(2n+2)} \quad (5.4)$$

$$\delta \sim L Ra_{CWHF}^{(n-2)/(n+4)} Pr^{n/(n+4)} AR^{n/(n+4)} f_3^{(n-2)/(n+4)} \quad (5.5)$$

$$\delta_1 \sim \frac{L Ra_{CWT}^{(n-2)/(2n+2)} Pr^{n/(2n+2)} AR^{(2-n)/(2n+2)}}{(1+L/r_i)} \quad (5.6)$$

$$\delta_1 \sim \frac{L Ra_{CWHF}^{(n-2)/(n+4)} Pr^{n/(n+4)} AR^{(4-2n)/[(n+4)(n+1)]} f_3^{(6-3n)/[(n+4)(n+1)]}}{(1+L/r_i)^{(n-2)/(n+1)}} \quad (5.7)$$

where the function  $f_2(Ra_{CWT}, Pr, n, AR, r_i/L)$ ,  $f_3(Ra_{CWHF}, Pr, n, AR, r_i/L)$ , represents the ratio of hydrodynamic and thermal boundary layer thickness (i.e.  $\delta/\delta_{th}$ ) on the vertical wall for CWT and CWHF boundary conditions, respectively. Using Eqs. 5.4 - 5.7, it is possible to estimate the effective viscosity in the vertical

(horizontal) boundary layers (i.e.  $\mu_{eff}^V \sim K (w/\delta)^{n-1}$  ( $\mu_{eff}^H \sim K (u/\delta_1)^{n-1}$ )) in order to predict the effective Rayleigh numbers (i.e.  $Ra_{eff} \sim \frac{\rho^2 c_p g \beta \Delta T_{ref} L^3}{u_{eff}^k}$ ) in the vertical and horizontal boundary layers in the following manner:

$$Ra_{CWT,eff}^V \sim Ra_{CWT}^{(5-n)/(2n+2)} Pr^{(1-n)/(2n+2)} AR^{(1-n)/(2n+2)} \quad (5.8)$$

$$Ra_{CWHF,eff}^V \sim \frac{Ra_{CWHF}^{(7-2n)/(n+4)} Pr^{(2-2n)/(n+4)} AR^{(2-2n)/(n+4)}}{f_3^{[(n+1)(n-1)]/(n+4)}} \quad (5.9)$$

$$Ra_{CWT,eff}^H \sim \frac{Ra_{CWT}^{(5-n)/(2n+2)} Pr^{(1-n)/(2n+2)} AR^{(3-3n)/(2n+2)}}{(1 + L/r_i)^{(3-3n)/(n+1)}} \quad (5.10)$$

$$Ra_{CWHF,eff}^H \sim \frac{Ra_{CWHF}^{(7-2n)/(n+4)} Pr^{(2-2n)/(n+4)} AR^{(6-6n)/(n+4)} f_3^{(9-9n)/[(n+4)(n+1)]}}{(1 + L/r_i)^{(3-3n)/(n+1)}} \quad (5.11)$$

Eqs. 5.8 - 5.11 shows that the effective value of Rayleigh number attains significantly larger than its nominal value for a decrease in values of  $n$  for shear thinning (i.e.  $n < 1$ ) fluids. Furthermore,  $Ra_{CWT,eff}^H$  and  $Ra_{CWHF,eff}^H$  decrease with decreasing  $r_i/L$  for shear thinning (i.e.  $n < 1$ ) fluids. However,  $Ra_{CWT,eff}^H$  and  $Ra_{CWHF,eff}^H$  increase with decreasing  $r_i/L$  for shear thickening (i.e.  $n > 1$ ) fluids.

The mean Nusselt number  $\overline{Nu}_i$  can be estimated by using the wall heat flux scaling  $q_w \sim k\Delta T/\delta_{th}$  as:

$$\overline{Nu}_{i,CWT} \sim Ra_{CWT}^{\frac{2-n}{2n+2}} Pr^{\frac{-n}{2n+2}} AR^{\frac{-n}{2n+2}} f_2(Ra_{CWT}, Pr, AR, n, r_i/L) \quad (5.12)$$

$$\overline{Nu}_{i,CWHF} \sim Ra_{CWHF}^{\frac{2-n}{n+4}} Pr^{\frac{-n}{n+4}} AR^{\frac{-n}{n+4}} f_3(Ra_{CWHF}, Pr, AR, n, r_i/L)^{\frac{2n+2}{n+4}} \quad (5.13)$$

Equations 5.12 and 5.13 imply that the exponent of Rayleigh number increases with decreasing  $n$  but the exponent remains positive for the range of the values

of  $n$  considered here. Thus, the mean Nusselt number  $\overline{Nu}_i$  is expected to increase with an increase in (a decrease in) nominal Rayleigh number (power-law exponent).

## 5.2 Effects of nominal Rayleigh number $Ra$ and power-law index $n$

The distributions of dimensionless temperature  $\theta = (T - T_{cen})/\Delta T_{ref}$  and dimensionless axial (radial)  $W = wL/\alpha$  ( $U = uL/\alpha$ ) velocity components along the horizontal (vertical) mid-plane for different  $Ra$  and values at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 10^3$  are shown in Fig. 5.1 for both CWT and CWHF boundary conditions. The distributions of  $\theta$  for the pure-conduction solution (where fluid flow does not affect thermal transport) are also shown in Fig. 5.1 by triangles (circles) for CWT (CWHF) boundary conditions. Figure 5.1 demonstrates that the  $\theta$  distribution between the hot and cold walls of the enclosure deviates significantly from the pure conduction solution with increasing (decreasing)  $Ra$  ( $n$ ), which is indicative of the strengthening of convective thermal transport for both CWT and CWHF boundary conditions. This behaviour can also be seen from the increases in the magnitudes of  $W$  and  $U$  with increasing (decreasing)  $Ra$  ( $n$ ), as shown in Fig. 5.1, regardless of the boundary condition. Furthermore, the magnitudes of  $\theta$ ,  $W$  and  $U$  are found to be smaller in the CWHF boundary condition than the corresponding values in the CWT boundary condition, which is an indication of stronger convective thermal transport in the CWT boundary condition in comparison to that in the CWHF boundary condition. This can also be confirmed by scaling of wall heat flux on the inner periphery as  $q_i \sim k\Delta T/\delta_{th}$  where  $\Delta T$  and  $\delta_{th}$  are the characteristic temperature difference and the thickness of thermal boundary layer on vertical walls, respectively. This implies that  $\theta$  in the case of CWHF boundary condition scales as  $\theta \sim \Delta Tk/q_i L \sim q_i \delta_{th} k/q_i L k \sim O(\delta_{th}/L)$ , whereas  $\theta \sim O(1)$  for CWT boundary condition. Accordingly, the magnitude of  $\theta$  for the CWHF boundary condition is supposed to decrease with an increase in (a decrease in)  $Ra$  ( $n$ ) because of the thinning of thermal boundary layer thickness.

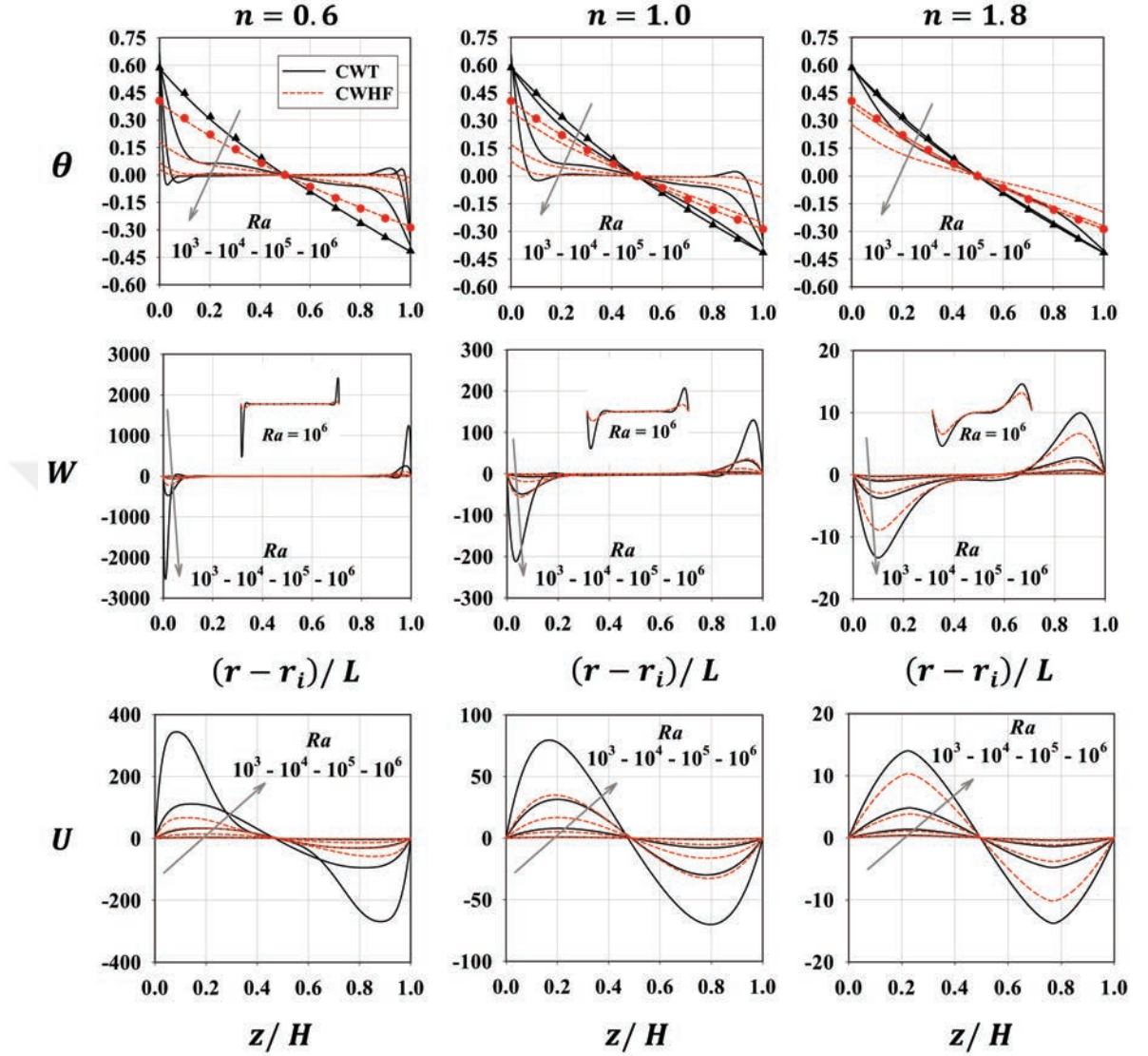


Figure 5.1: Variations of dimensionless temperature  $\theta$  and dimensionless axial (radial)  $W = wL/\alpha$  ( $U = uL/\alpha$ ) velocity components along the horizontal (vertical) mid-plane for different  $Ra$  and  $n$  values at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 10^3$ . The pure conduction solution is shown by the triangle (circle) for the CWT (CWHF) boundary conditions.

This smaller temperature difference between the vertical walls in the CWHF boundary condition induces to weaker convection than in the CWT boundary

condition. This is consistent with scaling analysis presented in Eqs. 5.8 - 5.11, which indicates that the effective Rayleigh number in the CWHF boundary condition remains smaller than the corresponding value in the CWT boundary condition for same set of numerical values of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$  and  $r_i/L$ . The contours of dimensionless temperature  $\theta$  and dimensionless stream function  $\Psi = \psi/\alpha$  are demonstrated in Fig. 5.2 for different values of  $Ra$  and  $n$  for  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 10^3$  in the case of boundary condition. Figure 5.2 shows that the magnitude of  $\Psi$  increases and dimensionless temperature contours become increasingly curved with an increase in (a decrease in)  $Ra$  ( $n$ ) due to the strengthening of convective thermal transport. Similar qualitative behaviour has been also observed for CWHF boundary condition but it is not shown here for the sake of conciseness. It is worth noting that these findings are consistent with previous findings [92, 93] which dealt with natural convection of power-law fluids in annular spaces with square cross-section (i.e. same boundary condition for  $AR = 1.0$ ). The weakening viscous effects can be seen from Fig. 5.3, which shows that the possibility of obtaining fluid viscosities higher than viscosity of Newtonian fluids (i.e.  $\mu_{n=1}$ ) decreases with decreasing  $n$  because of the increasing extent of shear thinning. This is consistent with the scaling arguments presented in here, which demonstrate that the effective viscosity decreases with decreasing  $n$ . Furthermore, Eqs. 5.8 - 5.11 show that the effective Rayleigh number increases significantly in comparison to the nominal Rayleigh number, especially for shear-thinning (i.e.  $n < 1$ ) fluids, which suggests that the strength of buoyancy-driven transport strengthens in comparison to viscous resistance with decreasing  $n$ , as confirmed in Figs. 5.1 - 5.3.

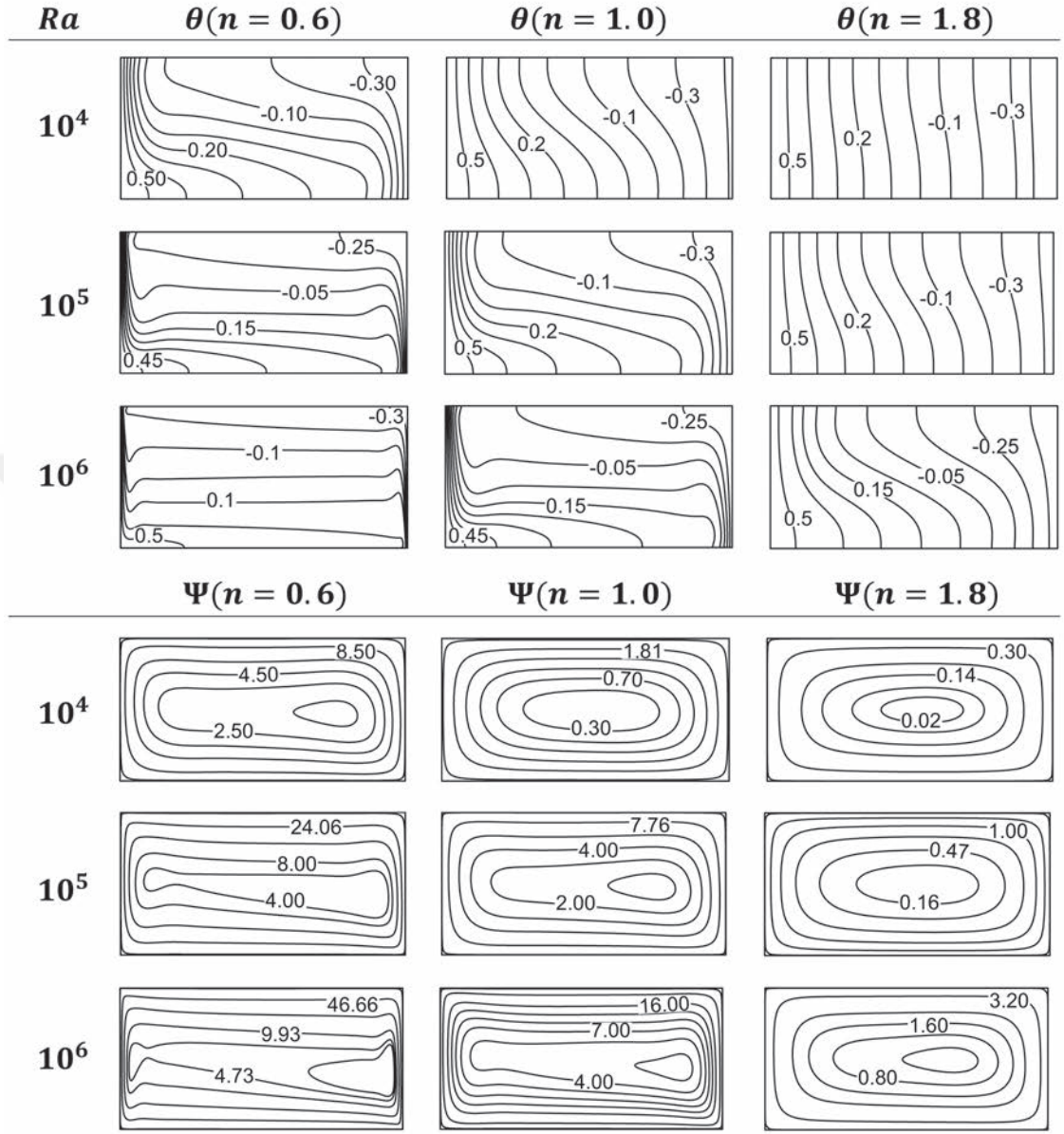


Figure 5.2: Contours of the dimensionless isotherms ( $\theta$ ) and dimensionless stream functions ( $\Psi$ ) for different values of  $Ra$  and  $n$  at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 10^3$  for the CWT boundary condition.

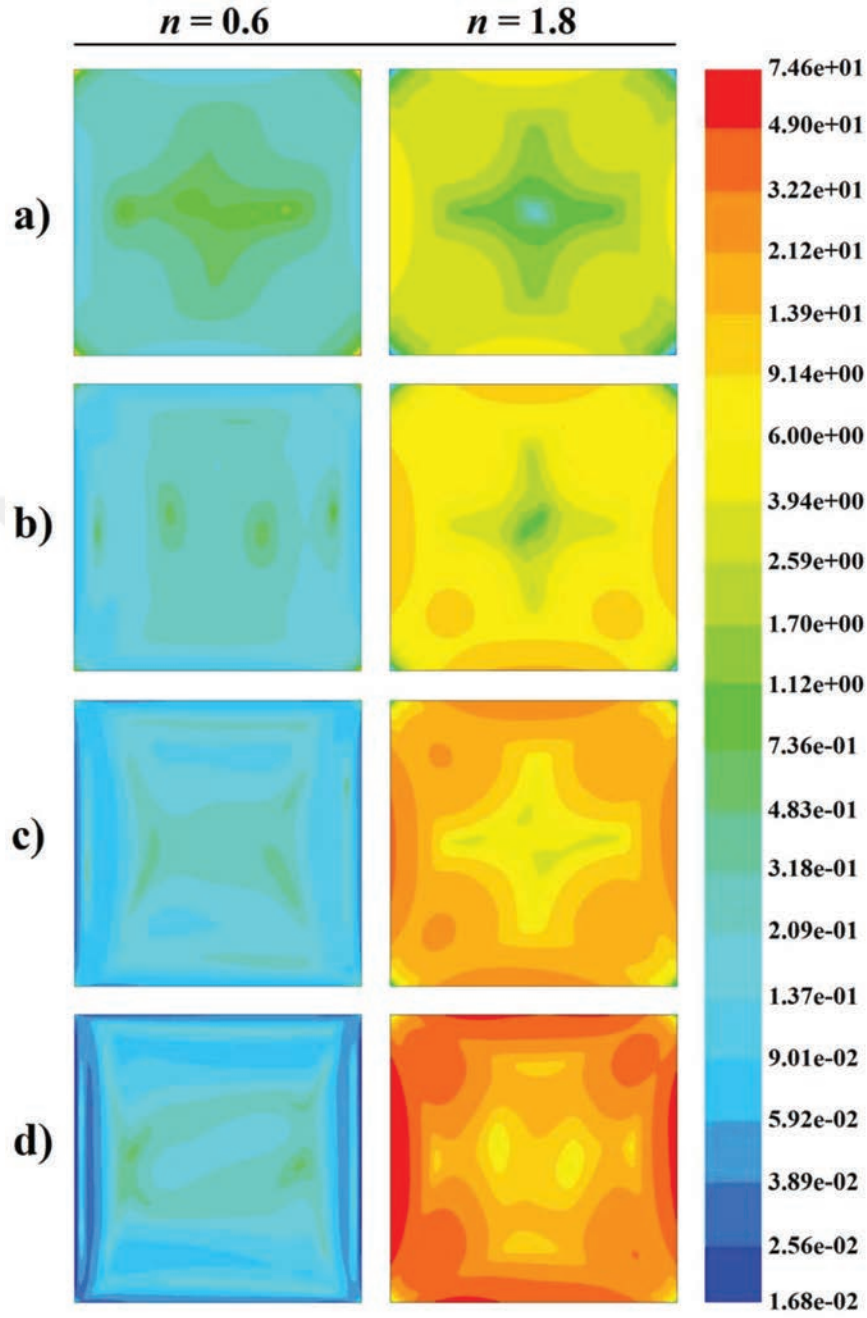


Figure 5.3: Contours of non-dimensional viscosity  $\mu/\mu_{n=1}$  for  $n = 0.6, 1.8$  at  $Ra$ : a)  $10^3$ , b)  $10^4$ , c)  $10^5$ , d)  $10^6$  at  $Pr = 10^3$  and  $r_i/L = 1.0$ .

### 5.3 Effects of Prandtl Number $Pr$

Variations of  $\overline{Nu}_i$  with  $Pr$  for different values of  $Ra$  and  $n$  at  $AR = 1.0$  for a)  $r_i/L = 0.125$ , b)  $r_i/L = 16$  are shown in Fig. 5.4 for both CWT and CWHF boundary conditions.

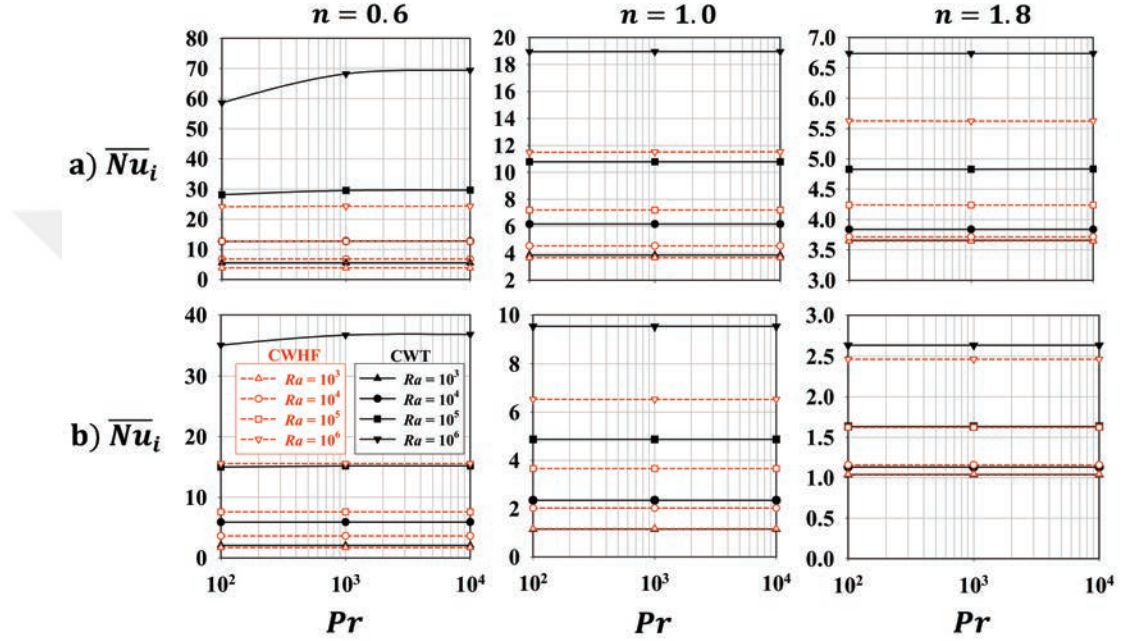


Figure 5.4: Variation of  $\overline{Nu}_i$  with  $Pr$  for different values of  $Ra$  and  $n$  at  $AR = 1.0$  for a)  $r_i/L = 0.125$ , b)  $r_i/L = 16$ .

It can be seen from Fig. 5.4 that a change in  $Pr$  does not have any significant influence on the variation of  $\overline{Nu}_i$ , which is consistent with several previous findings [54, 86, 97, 98]. For  $Pr \gg 1$ , the thermal boundary layer is thinner than the hydrodynamic boundary layer thickness, so a change in Prandtl number modifies the relative balance between the buoyancy and viscous forces and does not affect the thermal transport within the thermal boundary layer. Thus, the Prandtl number does not have any significant influence on the variation of  $\overline{Nu}_i$  for all values of  $Ra$ ,  $n$ , and  $r_i/L$  when  $Pr > 10^3$ . The scaling estimates of thermal boundary layer  $\delta_{th} = \delta/f_2$  ( $\delta_{th} = \delta/f_3$ ) for CWT (CWHF) boundary condition in Eqs. 5.4 and 5.5 suggests that an increase in  $Pr$  gives rise to a significant

decrease in  $\delta_{th}$  for small values of  $n$  and this leads to an increase in  $\overline{Nu}_i$  with increasing  $Pr$ , which can be observed only for  $10^2 \leq Pr < 10^3$  at  $n = 0.6$  in the parameter range studied here.

Figure 5.4 demonstrates that  $\overline{Nu}_i$  assumes higher values in the case of the CWT boundary condition than in the case of the CWHF boundary condition. It has already been shown in Fig. 5.1 that the temperature difference between the vertical walls and velocity magnitude in the CWHF boundary condition remain smaller than in the case of the CWT boundary condition (for the same nominal values of Rayleigh number), which indicates that the convection strength is weaker in the CWHF boundary condition than in the CWT boundary condition for a given set of values of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$  and  $r_i/L$ . Moreover, it can be seen from Eqs. 5.12 and 5.13 that the exponent of Rayleigh number in the scaling estimate of  $\overline{Nu}_i$  remains greater in the CWT boundary condition than in the CWHF boundary condition ( $[(2-n)/(2n+2)] > [(2-n)/(n+4)]$  for the range of  $n$  considered here. However, the difference between  $(2-n)/(2n+2)$  and  $(2-n)/(n+4)$  decreases with increasing  $n$ . Thus, the difference between  $\overline{Nu}_i$  values between CWT and CWHF boundary conditions is small for small (large) values of nominal Rayleigh number (power law exponent) where the effects of convection are weak.

## 5.4 Effects of Aspect Ratio $AR$

The distributions of  $\theta$  and  $W$  ( $U$ ) along the horizontal (vertical) mid-plane for different  $AR$  values for  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $n = 0.6$  and  $Pr = 10^3$  are shown in Fig. 5.5 for both CWT and CWHF boundary conditions. The  $\theta$  distributions obtained from the pure-conduction solutions (where fluid flow does not affect thermal transport) are also shown in Fig. 5.5 by the red dashed lines for CWT and CWHF boundary conditions. Figure 5.5 exhibits that the distribution of  $\theta$  deviates from the pure conduction solution with increasing  $AR$ , which indicates a strengthening of convection with an increase in  $AR$  for both CWT and CWHF boundary conditions. Additionally, Fig. 5.5 indicates that the magnitudes of  $\theta$ ,  $W$  and  $U$  in the CWHF boundary condition are smaller than the corresponding values observed for the CWT boundary condition.

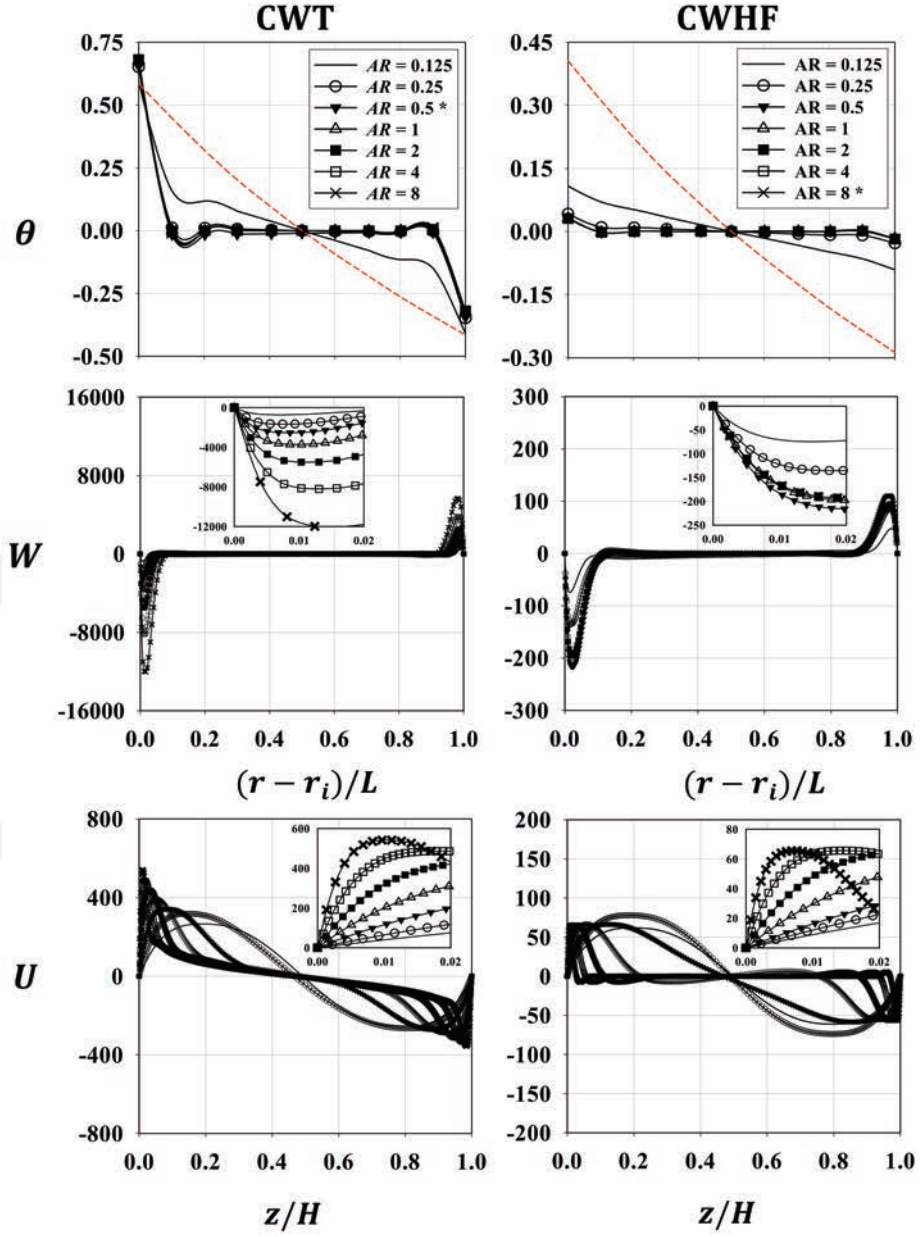


Figure 5.5: Variations of  $\theta$  and  $W$  ( $U$ ) along the horizontal (vertical) mid-plane for different  $AR$  values at  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $n = 0.6$  and  $Pr = 10^3$  for both CWT and CWHF boundary conditions. The pure conduction solution is shown by the dashed line. The  $AR$  values indicated by an asterisk in the legend represent highest  $\overline{Nu}_i$  is obtained as shown in Fig. 5.6 .

This is also a confirmation of stronger convection in CWT boundary condition in comparison to that in the CWHF boundary condition for different  $AR$  values with the same set of values of  $Ra$ ,  $Pr$ ,  $n$ ,  $r_i/L$ . Furthermore, it can be noticed from Fig. 5.5 that the magnitudes of  $W$  and  $U$  increase with increasing  $AR$  for a given set of values of  $Ra$ ,  $Pr$ ,  $n$ ,  $r_i/L$  for both CWT and CWHF boundary conditions. This is consistent with the qualitative trends predicted by scaling relations given by Eqs. 5.2 and 5.3. Furthermore, the aspect ratio  $AR$  (i.e.  $AR_{max}$ ) for which the highest  $\overline{Nu}_i$  is obtained is indicated by an asterisk in the legend of Fig. 5.5. This reveals that a non-monotonic  $AR$  dependence of the  $\overline{Nu}_i$  is obtained in the CWT boundary condition (e.g. the highest  $\overline{Nu}_i$  is obtained at  $AR = 0.5$  for  $r_i/L = 1.0$ ,  $Ra = 10^6$ ,  $n = 0.6$  and  $Pr = 10^3$ ). By contrast, the highest value of  $\overline{Nu}_i$  is obtained for the highest  $AR$  case in the CWHF boundary condition. The variations of  $\overline{Nu}_i$  with  $AR$  for different values of  $Ra$  and  $n$  at  $r_i/L = 1.0$  and  $Pr = 10^3$  are shown in Fig. 5.6 for both CWT and CWHF boundary conditions. It can clearly be seen from Fig. 5.6 that the  $\overline{Nu}_i$  shows a non-monotonic  $AR$  dependence for a given set values of  $Ra$ ,  $Pr$ ,  $r_i/L$  for shear thinning (i.e.  $n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear thickening (i.e.  $n > 1$ ) fluids for CWT boundary condition, while a monotonic increase in  $\overline{Nu}_i$  was obtained for an increase in values of  $AR$  in the CWHF boundary condition, regardless of the value of  $n$ . It can also be noticed from Fig. 5.6 that  $AR_{max}$  is dependent on values of  $Ra$  and  $n$  for same set of values of  $r_i/L$  and  $Pr$  in CWT boundary condition. This is in agreement with previous analyses on the influences of aspect ratio and boundary condition on natural convection in rectangular enclosures for both Newtonian (i.e.  $n = 1.0$ ) [41] and power-law fluids [89].

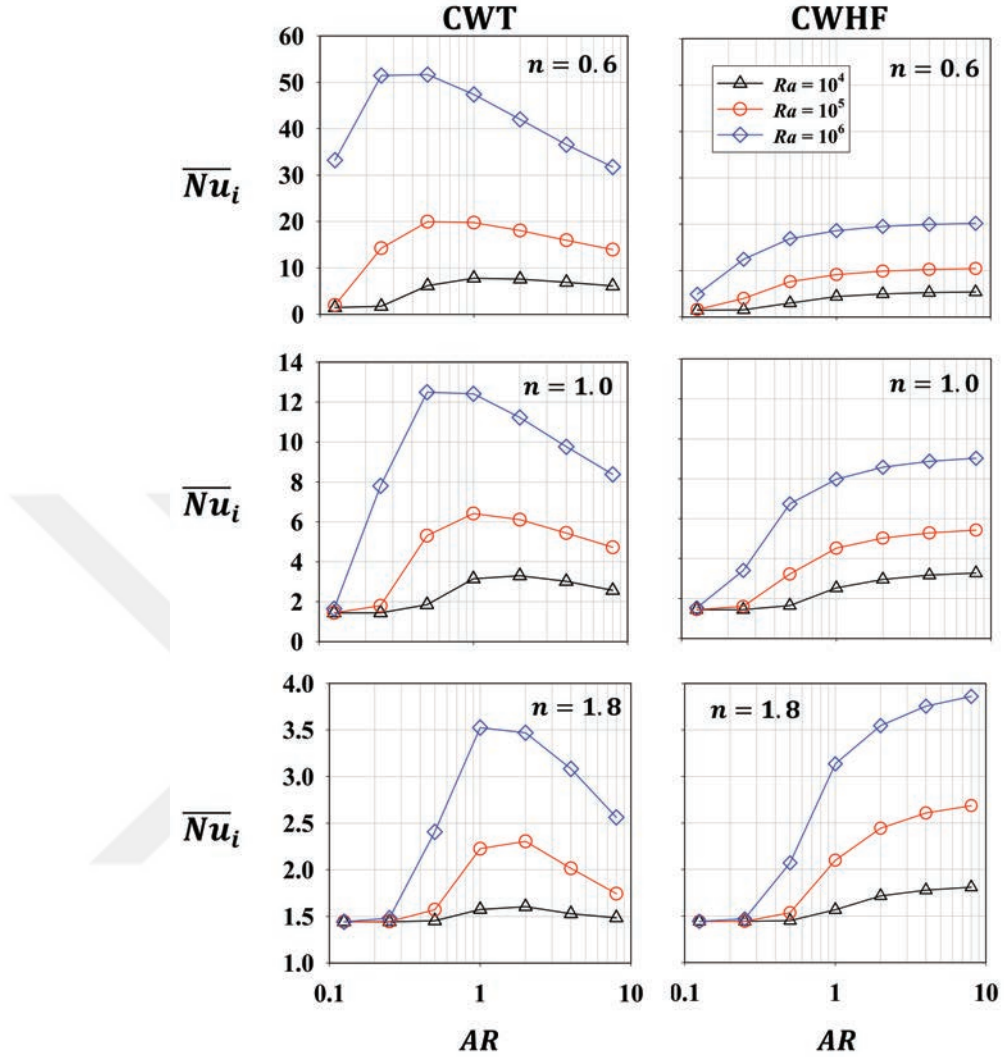


Figure 5.6: Variations of  $\overline{Nu}_i$  with  $AR$  for different values of  $Ra$  and  $n$  at  $r_i/L = 1.0$  and  $Pr = 10^3$ .

The non-monotonic (monotonic)  $AR$  dependence of  $\overline{Nu}_i$  for CWT (CWHF) boundary condition can be explained in the following manner using the energy flux integral between active walls. The energy flux integral can be shown as follows:

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{cond} = \int_{r_i}^{r_o} \rho c_p T w 2\pi r dr - \int_{r_i}^{r_o} k \left( \frac{\partial T}{\partial z} \right) 2\pi r dr \quad (5.14)$$

where the first term on the right-hand side represents the effects of convective transport, whereas the second term on the right-hand side accounts for thermal conduction. Here,  $\dot{Q}_{conv}$  and  $\dot{Q}_{cond}$  can be scaled in the following manner by using Eqs. 5.1 and 5.7:

$$\dot{Q}_{conv,CWT} \sim \pi(2r_i + L)\rho c_p \Delta T w \delta \sim k \Delta T \pi(2r_i + L) Ra_{CWT}^{(2n-1)/(2n+2)} Pr^{(2n+1)/(2n+2)} AR^{(2n+1)/(2n+2)} \quad (5.15)$$

$$\dot{Q}_{conv,CWHF} \sim \pi(2r_i + L) \frac{\rho c_p w q_i \delta_{th} \delta}{k} \sim \frac{q_i L \pi(2r_i + L) Ra_{CWHF}^{(3n-3)/(n+4)} Pr^{(3n+2)/(n+4)} AR^{(3n+2)/(n+4)}}{f_3^{(3n-3)/(n+4)}} \quad (5.16)$$

$$\dot{Q}_{cond,CWT} \sim k \Delta T \pi(2r_i + L) / AR \quad (5.17)$$

$$\dot{Q}_{cond,CWHF} \sim \pi(2r_i + L) \frac{q_i \delta_{th}}{H} \sim q_i L \pi(2r_i + L) \frac{Ra_{CWHF}^{(n-2)/(n+4)} Pr^{n/(n+4)} AR^{-4/(n+4)}}{f_3^{(2n+2)/(n+4)}} \quad (5.18)$$

The scaling estimation in Eqs. 5.14 - 5.18 suggest that the influences of convection strengthen with an increase in  $AR$ , but the diffusive thermal transport effects weaken with an increase in  $AR$  for CWT boundary condition, which is in agreement with previous findings of for power-law and Newtonian fluids [26, 40, 89, 105]. The highest value of  $\overline{Nu_i}$  is obtained for an optimum value of aspect ratio (i.e.  $AR_{max}$ ) due to the competition between an enhancement in advection and a weakening of thermal diffusion with an increase in  $AR$ . It can be noted from Eq. 5.15 and 5.17 that the value of  $AR_{max}$  for CWT boundary condition is dependent on  $Ra$  for a given set of values of  $n$  and  $Pr$ , which is consistent with the numerical results presented in Fig. 5.6. By contrast, the

strengthening (weakening) of  $\dot{Q}_{conv,CWHF}$  ( $\dot{Q}_{cond,CWHF}$ ) with increasing  $AR$  in the CWHF boundary condition is stronger (weaker) than  $\dot{Q}_{conv,CWT}$  ( $\dot{Q}_{cond,CWT}$ ) in the CWT boundary condition. Thus,  $\overline{Nu}_i$  increases monotonically with increasing  $AR$  in the CWHF boundary condition, which is in agreement with previous analyses [41, 89] on rectangular enclosures with both Newtonian (i.e.  $n = 1.0$ ) and power-law fluids.

## 5.5 Effects of Normalised Internal Cylinder Radius $r_i/L$

The distributions of dimensionless temperature  $\theta$  along the horizontal mid-plane for different  $r_i/L$  and  $n$  values are shown in Fig. 5.7 at  $AR = 0.5$ ,  $Ra = 10^6$  and  $Pr = 10^3$  in the case of both CWT and CWHF boundary conditions. Figure 5.7 indicates that the temperature difference increases with an increase in  $r_i/L$  for shear thinning (i.e.  $n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear thickening (i.e.  $n > 1$ ) fluids in the CWHF boundary condition. It is important to understand the  $r_i/L$  dependence of the temperature difference between vertical walls  $\Delta T$  due to pure conductive transport in order to explain the wall curvature effects on the temperature difference between vertical walls. It is worth indicating that the temperature difference between vertical walls  $\Delta T$  is dependent on the internal radius of the cylindrical annular enclosure when the thermal transport occurs purely by conduction. The thermal resistance due to conduction in axisymmetric cylindrical geometry is given by  $\ln(r_o/r_i)/2\pi kL$ , which suggests that the steady state heat flow rate for pure conductive transport is given by:  $\dot{Q} = \Delta T_{cond} / \{\ln(r_o/r_i)/2\pi kL\}$ . Furthermore,  $\dot{Q}$  can be expressed in terms of heat flux on the inner periphery as:  $\dot{Q} = q_i 2\pi r_i L$ , which leads to:

$$\Delta T_{cond} = \frac{q_i L (r_i/L)}{k} \ln \left( 1 + \frac{L}{r_i} \right) \quad (5.19)$$

Equation 5.19 indicates that  $\Delta T_{cond}$  increases with a decrease in values of  $r_i/L$ . As the heat transfer coefficient  $h_i = q_i/\Delta T$  for convective transport is expected to be greater than that for pure conductive transport  $(h_i)_{cond} = q_i/\Delta T_{cond}$ , the

temperature difference between the vertical walls remains smaller than that in the case of purely conductive transport for the transport regime, where convection plays a significant role in the overall thermal transport.

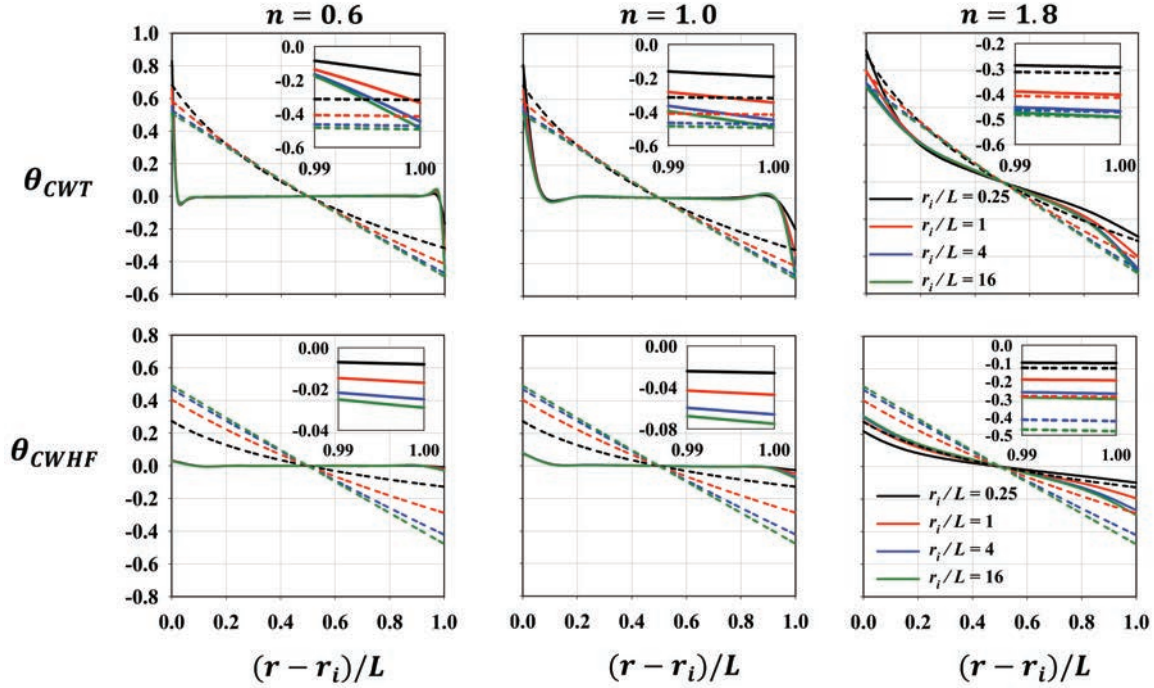


Figure 5.7: Variations of  $\theta$  along the horizontal mid-plane for different  $r_i/L$  and  $n$  values at  $AR = 0.5$ ,  $Ra = 10^6$ ,  $Pr = 10^3$  for both CWT and CWHF boundary conditions. The pure conduction solution is shown by the dashed line.

Furthermore, the distributions of dimensionless axial (radial)  $W$  ( $U$ ) velocity components along the horizontal (vertical) mid-plane for different  $r_i/L$  and  $n$  values are shown in Fig. 5.8 at  $AR = 0.5$ ,  $Ra = 10^6$  and  $Pr = 10^3$  in the case of both CWT and CWHF boundary conditions. It can be seen from Fig. 5.8 that  $U$  increases while  $W$  decreases with an increase in  $r_i/L$  for shear thinning (i.e.  $n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear thickening (i.e.  $n > 1$ ) fluids in both CWT and CWHF boundary conditions. This numerical findings can be confirmed by scaling estimation of the radial velocity component given in Eq. 5.2 and 5.3 (i.e.  $u \sim w/AR(1 + L/r_i)$ ) for both Newtonian and power-law fluids

regardless of the boundary condition. As the variations  $W$  and  $U$  with  $r_i/L$  exhibit opposite behaviours, as shown in the Fig. 5.8, the relative contributions of thermal transport due to convection and conduction for different values of  $r_i/L$  can be explained from the variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $r_i/L$ .

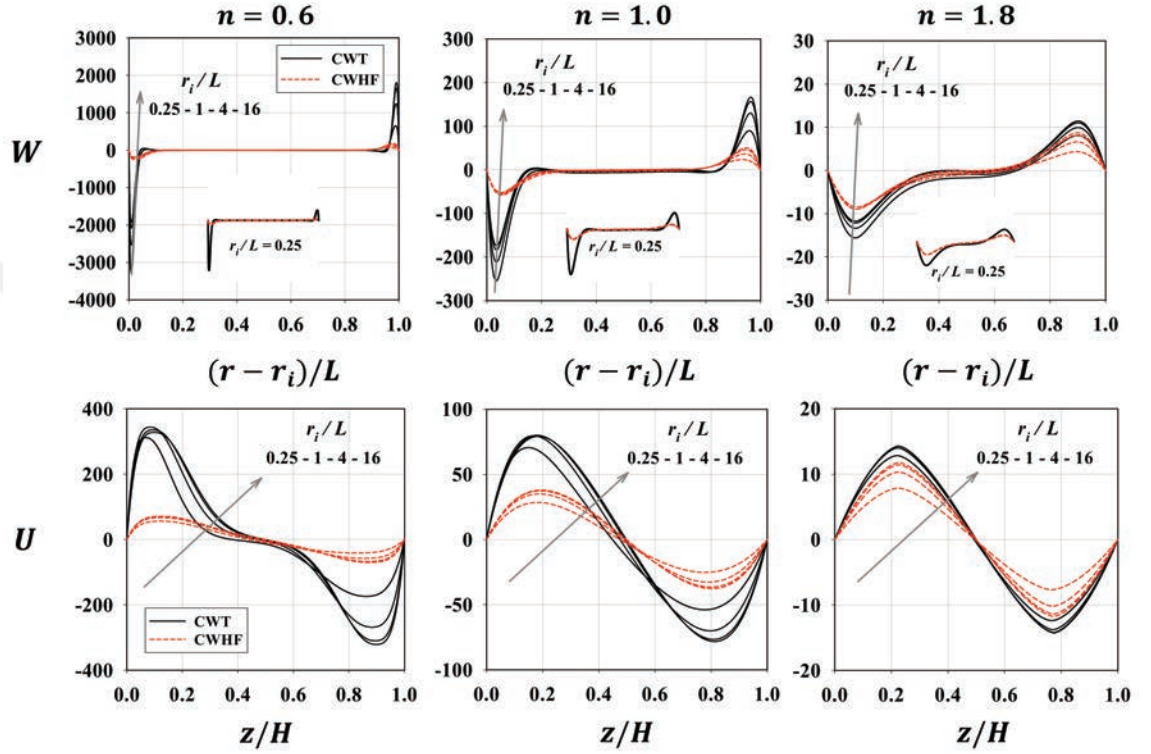


Figure 5.8: Variations of  $W$  ( $U$ ) along the horizontal (vertical) mid-plane for different  $r_i/L$  and  $n$  values at  $AR = 0.5$ ,  $Ra = 10^6$ ,  $Pr = 10^3$  for both CWT and CWHF boundary conditions.

It is worth noting that  $(\overline{Nu}_i)_{cond}$  increases with a decrease in  $r_i/L$  (see Eq. 2.37). Therefore, the variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $r_i/L$  for different values of  $n$  and  $AR$  at  $Ra = 10^6$  and  $Pr = 10^3$  are shown in Fig. 5.9 for both CWT and CWHF boundary conditions. Figure 5.9 exhibits that  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases with an increase in  $r_i/L$  and approaches asymptotically to the value of the mean Nusselt number  $\overline{Nu}_{Rec}$  for the corresponding rectangular enclosures (i.e. in the limit of  $r_i \rightarrow \infty$ ).

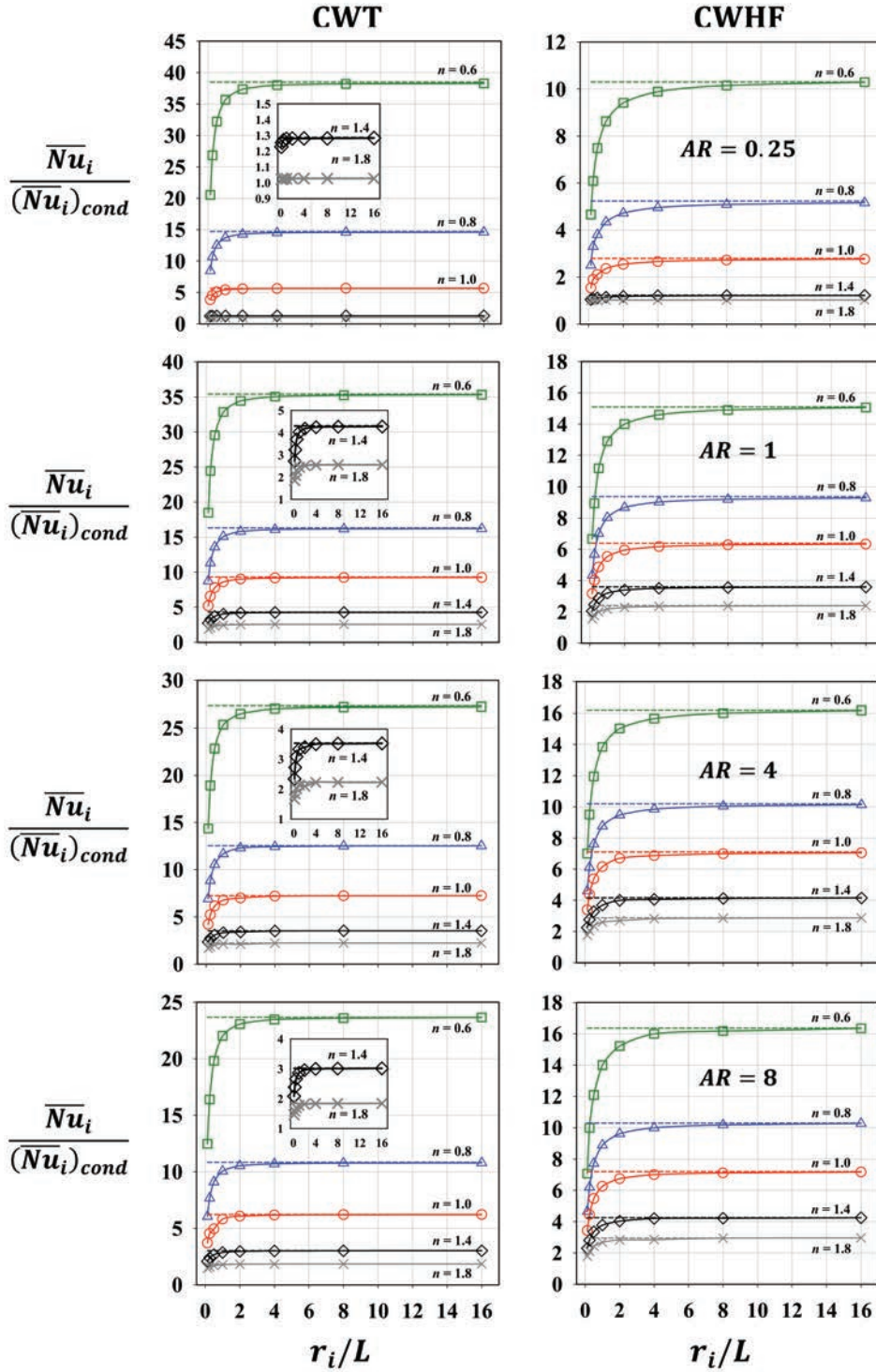


Figure 5.9: Variations of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with  $r_i/L$  for different values of  $n$  and  $AR$  at  $Ra = 10^6$  and  $Pr = 10^3$  for both CWT and CWHF boundary conditions. The dashed lines represent corresponding value of mean Nusselt number for rectangular enclosures for each  $AR$  values.

This suggests that the relative contribution of convection to overall thermal transport increases with an increase in  $r_i/L$ . This is in a full agreement with previous results for Newtonian fluids [64–66, 107] and for power-law fluids [92, 93]. This behaviour of  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with increasing  $r_i/L$  agrees with scaling relations presented in Eqs. 5.8 - 5.11, which indicates  $Ra_{eff}^H$  remains smaller than  $Ra_{eff}^V$  in cylindrical annular enclosures, regardless of the boundary condition. The above findings indicate that relative influence of convection on the overall thermal transport is weaker in cylindrical annular enclosures than in rectangular enclosures and the relative strength of convection augments with an increase in internal cylinder radius.

## 5.6 Correlations for Mean Nusselt Number $\overline{Nu}_i$

Turan et al. [89] proposed a correlation for the mean Nusselt number for natural convection of power-law fluids in rectangular enclosures (i.e.  $\overline{Nu}_{Rec}$ ) with differentially heated from side walls for the range of nominal Rayleigh number  $10^4 \leq Ra \leq 10^6$ , range of Prandtl number  $10 \leq Pr \leq 10^5$ , range of aspect ratio  $0.125 \leq AR \leq 8$ , and range of power-law index  $0.6 \leq n \leq 1.8$  which are listed in the Table 5.1 and 5.2 for both CWT and CWHF boundary conditions respectively.

It is worth noting the correlation given in Table 5.1 and 5.2 can be utilised for  $r_i/L \gg 1$  where  $(\overline{Nu}_i)_{cond}$  approaches unity in the limit of  $r_i/L \rightarrow \infty$ . Based on numerical findings, the mean Nusselt number  $\overline{Nu}_i$  is parametrised here as follows:

$$\overline{Nu}_i = \overline{Nu}_{Rec} \frac{[m(L/r_i)]^d}{\ln \{1 + [m(L/r_i)]^d\}} \quad (5.20)$$

In Eq. 5.20,  $m$  and  $d$  are correlation parameters which are listed in Table 5.3 and 5.4 for both CWT and CWHF boundary conditions respectively. According to Eq. 5.20,  $\overline{Nu}_i$  becomes equal to  $\overline{Nu}_{Rec}$  in the limit of  $r_i/L \rightarrow \infty$ . Furthermore,  $m$  and  $d$  assume the values of unity when  $\overline{Nu}_{Rec} = 1.0$  which indicates that heat transfer occurs purely due to conduction. Accordingly, the unity value of  $\overline{Nu}_{Rec}$  also ensures  $\overline{Nu}_i = (\overline{Nu}_i)_{cond} = \frac{(L/r_i)}{1 + \ln(L/r_i)}$ , which is the value of the

mean Nusselt number for pure conduction in cylindrical annular enclosures with differentially heated from side walls. The correlation of given by Eq. 5.20 has been obtained using a non-linear regression method [109]. The predictions of Eq. 5.20 are shown in Fig. 5.10, which demonstrates that the correlation satisfactorily ( $R^2 > 0.98$  and 3% maximum error percentage apart from overprediction in the case of  $AR = 0.25$  for shear thinning fluids) predicts  $\overline{Nu}_i$  for  $0.6 \leq n \leq 1.8$ ,  $10^4 \leq Ra \leq 10^6$ ,  $10^2 \leq Pr \leq 10^4$ ,  $0.25 \leq AR \leq 8$ ,  $0.125 \leq r_i/L \leq 16$  for both CWT and CWHF boundary conditions.



Table 5.1: Summary of  $\overline{Nu}_{Rec}$  correlation [89] for rectangular enclosures for CWT boundary condition.

| CWT                                                                                                                                                                                                                                                    |                                                                                                                                                                                                                                  |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $RaAR^7 \geq 10^4$                                                                                                                                                                                                                                     | $\overline{Nu}_{Rec} = \text{Max} \left\{ 1.0, \frac{0.162 \frac{Ra^{0.043} Pr^{0.341}}{(1+Pr)^{0.091}} \left[ \frac{Ra^{2-n}}{(Pr^n AR^n)} \right]^{\frac{1}{2(n+1)}}}{e^{b_{CWT}(1-n)} [e^{AR^{-0.212}} - 1]} \right\}$        |
| $RaAR^7 < 10^4$<br>$RaAR^5/n^9 \geq 40$                                                                                                                                                                                                                | $\overline{Nu}_{Rec} = \text{Max} \left\{ 1.0, \frac{0.162 \frac{Ra^{0.043} Pr^{0.341}}{(1+Pr)^{0.091}} \left[ \frac{Ra^{2-n}}{(Pr^n AR^n)} \right]^{\frac{1}{2(n+1)}}}{e^{b_{CWT}(1-n)} [1 - 1.5(1 - AR)^{b_2}]^{-1}} \right\}$ |
| $RaAR^7 < 10^4$<br>$RaAR^5/n^9 < 40$                                                                                                                                                                                                                   | $\overline{Nu}_{Rec} = 1 + \left( \frac{1}{4.95} \right)^{4+4/n} Ra^{2/n} AR^{4+4/n}$                                                                                                                                            |
| $b_{CWT} = 1.343 Ra^{0.065} Pr^{0.036} AR^{0.099}$ for $n \leq 1$<br><br>$b_{CWT} = 0.858 Ra^{0.071} Pr^{0.034} AR^{-0.006}$ for $n > 1$<br><br>$b_2 = 0.156 Ra^{0.222} n^{-0.887}$ ( $b_2 = 0.725 Ra^{0.104} n^{-1.030}$ ) for $n \leq 1$ ( $n > 1$ ) |                                                                                                                                                                                                                                  |

Table 5.2: Summary of  $\overline{Nu}_{Rec}$  correlation [89] for rectangular enclosures for CWHF boundary condition.

| CWHF                                                                                                                                                                                                                                                                                                                                                                  |                                                                                                                                                                                                                                          |
|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $AR \geq 1$                                                                                                                                                                                                                                                                                                                                                           | $\overline{Nu}_{Rec} = \text{Max} \left\{ 1.0, \frac{0.209 \frac{Ra^{0.049} Pr^{0.231}}{(1+Pr)^{0.031}} \left[ \frac{Ra^{1-n/2}}{(Pr^{n/2} AR^{n/2})} \right]^{\frac{1}{n/2+2}}}{e^{b_{CWT}(1-n)} [c_3 \ln(AR) + 1]^{-1}} \right\}$      |
| $AR < 1$<br>$\frac{Ra AR^5}{n^9} \geq 150$                                                                                                                                                                                                                                                                                                                            | $\overline{Nu}_{Rec} = \text{Max} \left\{ 1.0, \frac{0.209 \frac{Ra^{0.049} Pr^{0.231}}{(1+Pr)^{0.031}} \left[ \frac{Ra^{1-n/2}}{(Pr^{n/2} AR^{n/2})} \right]^{\frac{1}{n/2+2}}}{e^{b_{CWT}(1-n)} [1 - 1.168(1 - AR)c_2]^{-1}} \right\}$ |
| $AR < 1$<br>$\frac{Ra AR^5}{n^9} < 150$                                                                                                                                                                                                                                                                                                                               | $\overline{Nu}_{Rec} = 1 + \left( \frac{1}{4.95} \right)^{4+4/n} Ra^{2/n} AR^{4+4/n}$                                                                                                                                                    |
| $b_{CWHF} = 0.965 Ra^{0.038} Pr^{0.072} AR^{0.034}$ for $n \leq 1$<br><br>$b_{CWHF} = 0.815 Ra^{0.052} Pr^{0.063} AR^{0.046}$ for $n > 1$<br><br>$c_2 = 0.382 Ra^{0.121} n^{-0.586}$ ( $c_2 = 0.683 Ra^{0.069} n^{-0.186}$ ) for $n \leq 1$ ( $n > 1$ )<br><br>$c_3 = 0.737 Ra^{-0.061} n^{0.347}$ ( $c_3 = 0.737 Ra^{-0.061} n^{0.595}$ ) for $n \leq 1$ ( $n > 1$ ) |                                                                                                                                                                                                                                          |

Table 5.3: Summary of  $\overline{Nu}_i$  correlation for rectangular cross-sectional cylindrical annular enclosures for CWT boundary condition.

| CWT                                                  |                                                                                                                                                                                                                                                        |
|------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $AR < 1, n \leq 1$<br>$\frac{RaAR^5}{n^9} \leq 3125$ | $m = 0.975^{(\overline{Nu}_{Rec}-1)}$<br>$d = \left\{ 1.471 - 0.075 \ln \left( \frac{RaAR^5}{n^9} \right) \right\}^{(\overline{Nu}_{Rec}-1)}$                                                                                                          |
| $AR < 1, n \leq 1$<br>$\frac{RaAR^5}{n^9} > 3125$    | $m = 0.975^{(\overline{Nu}_{Rec}-1)}$<br>$d = \left\{ 0.955 - 0.002 \ln \left( \frac{RaAR^5}{n^9} \right) \right\}^{(\overline{Nu}_{Rec}-1)}$                                                                                                          |
| $AR \geq 1, n \leq 1$                                | $m = \left\{ (1.218 - 0.506n) + 0.016n^{2.52} \ln Ra \right\}^{(\overline{Nu}_{Rec}-1)}$<br>$d = \left\{ \left( \frac{0.592 - 1.385n}{1 - 2.208n} \right) + \left( \frac{1 - 2.274n}{37.5 - 95.43n} \right) \ln Ra \right\}^{(\overline{Nu}_{Rec}-1)}$ |
| $AR \geq 1, n > 1$                                   | $m = d = (0.9)^{(\overline{Nu}_{Rec}-1)}$                                                                                                                                                                                                              |

Table 5.4: Summary of  $\overline{Nu}_i$  correlation for rectangular cross-sectional cylindrical annular enclosures for CWHF boundary condition.

| CWHF                                                |                                                                                                                                                                                                                                                                                         |
|-----------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $AR < 1, n \leq 1$<br>$\frac{RaAR^5}{n^9} \leq 970$ | $m = 0.85^{(\overline{Nu}_{Rec}-1)}$<br>$d = \left\{ 0.746 - 0.039 \ln \left( \frac{RaAR^5}{n^9} \right) \right\}^{(\overline{Nu}_{Rec}-1)}$                                                                                                                                            |
| $AR < 1, n \leq 1$<br>$\frac{RaAR^5}{n^9} > 970$    | $m = 0.85^{(\overline{Nu}_{Rec}-1)}$<br>$d = \left\{ 0.309 - 0.042 \ln \left( \frac{RaAR^5}{n^9} \right) \right\}^{(\overline{Nu}_{Rec}-1)}$                                                                                                                                            |
| $AR \geq 1, n \leq 1$                               | $m = \left\{ \left( 0.208 - 0.057n^{4.19} \right) \ln Ra^{\left( \frac{1-0.932n}{17.63-1.66n} \right)} \right\}^{(\overline{Nu}_{Rec}-1)}$<br>$d = \left\{ \left( 0.794 - 0.1n^{3.84} \right) + \left( \frac{1-0.605n}{85.83-61.63n} \right) \ln Ra \right\}^{(\overline{Nu}_{Rec}-1)}$ |
| $AR \geq 1, n > 1$                                  | $m = d = (0.7 + 0.01 \ln Ra)^{(\overline{Nu}_{Rec}-1)}$                                                                                                                                                                                                                                 |

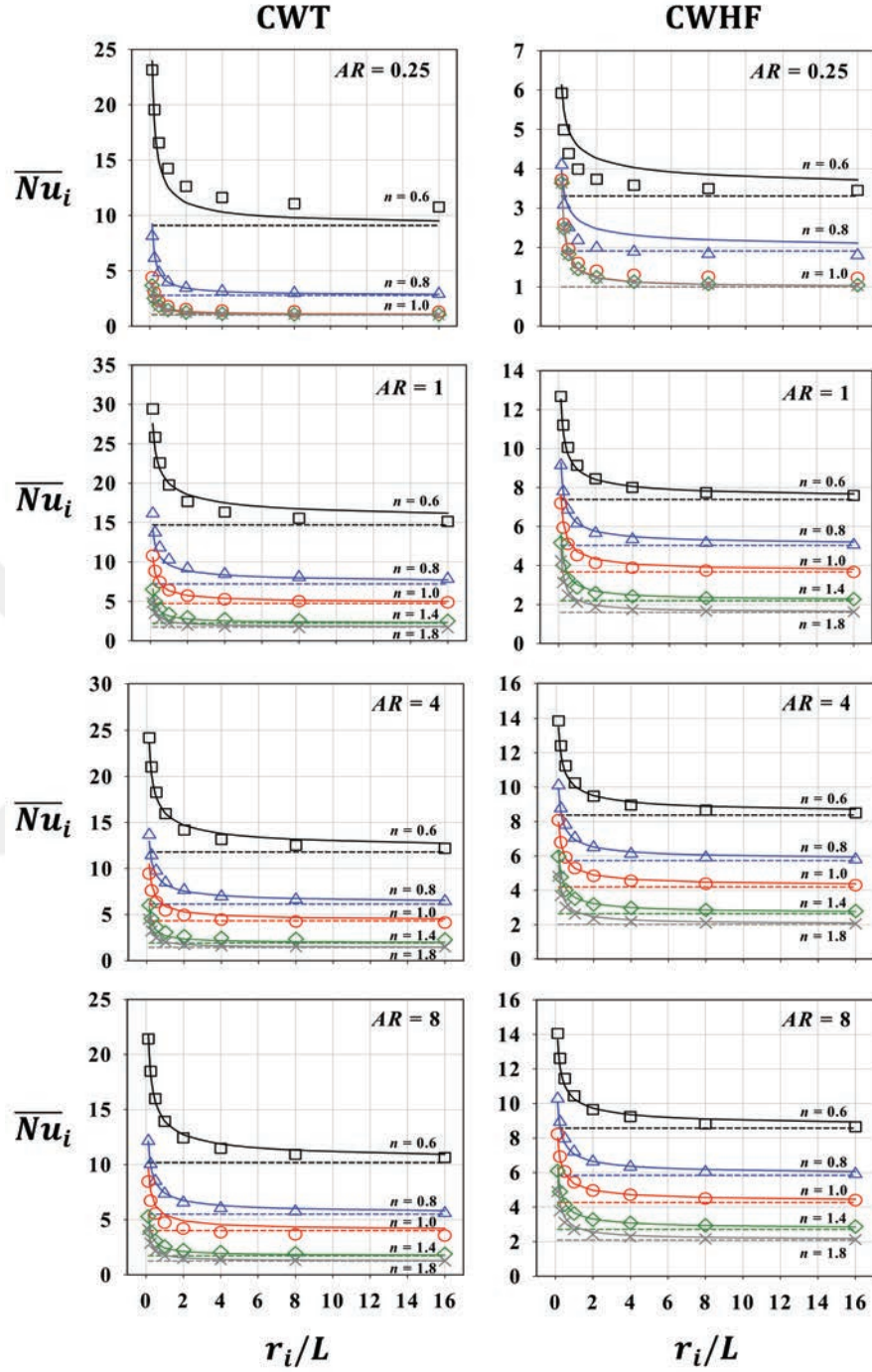


Figure 5.10: Variations of  $\overline{Nu}_i$  with  $r_i/L$  for different values of  $n$  and  $AR$  at  $Ra = 10^5$  and  $Pr = 10^3$  along with predictions of Eq. 20 (solid lines) for both CWT and CWHF boundary conditions. The dashed lines represent correlations given in Table 5.1 and 5.2 which are corresponding value of the mean Nusselt number for rectangular enclosures for CWT and CWHF boundary conditions respectively.

## 5.7 Summary of Key Findings

The effects of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$ ,  $r_i/L$  and boundary condition (i.e. CWT and CWHF) on steady-state laminar natural convection of power-law fluids in rectangular cross-sectional cylindrical annular enclosures with differentially heated vertical side walls have been numerically investigated in this chapter. The influence of thermal convection has been found to be stronger (weaker) in shear thinning (i.e.  $n < 1$ ) (shear thickening (i.e.  $n > 1$ )) fluids than Newtonian fluids (i.e.  $n = 1.0$ ) for the same set of values of  $Ra$ ,  $Pr$ ,  $AR$ ,  $r_i/L$  regardless of the boundary condition. This is reflected in an increase in the mean Nusselt number  $\overline{Nu}_i$  with increase (decrease) in  $Ra$  ( $n$ ). However,  $\overline{Nu}_i$  has been found to be essentially insensitive to the change in  $Pr$  except for some minor  $Pr$  dependence for small values of  $n$  regardless of boundary condition. The  $\overline{Nu}_i$  in the CWT boundary condition has been shown to follow a non-monotonic  $AR$  dependence for shear thinning (i.e.  $n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear thickening (i.e.  $n > 1$ ) fluids, whereas  $\overline{Nu}_i$  increases monotonically with increasing  $AR$  in the case of CWHF boundary condition irrespective of the value of  $n$ . This non-monotonic  $AR$  dependence of  $\overline{Nu}_i$  originates as a result the competition between the strengthening of convective thermal transport and the weakening of thermal diffusion with increasing  $AR$ . The aspect ratio  $AR_{max}$ , at which the mean Nusselt number  $\overline{Nu}_i$  attains its peak value in the CWT boundary condition, depends on the nominal value of Rayleigh number for shear thinning (i.e.  $n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear thickening (i.e.  $n > 1$ ) fluids. The strengthening of thermal convection with increasing  $AR$  is much stronger than the weakening of thermal diffusion in the case of CWHF boundary condition which gives rise to a monotonic increase of  $\overline{Nu}_i$  with an increase in  $AR$ . Moreover, the mean Nusselt number  $\overline{Nu}_i$  decreases with increasing  $r_i/L$  before approaching the corresponding mean Nusselt number value for a rectangular enclosure in the limit of  $r_i/L \rightarrow \infty$ . By contrast  $\overline{Nu}_i$  normalised by the corresponding Nusselt number for pure conduction (i.e.  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$ ) increases with increasing  $r_i/L$  because  $(\overline{Nu}_i)_{cond}$  decreases with increasing  $r_i/L$  for cylindrical annular enclosures (i.e.  $(\overline{Nu}_i)_{cond} = (L/r_i) / \ln(1 + L/r_i)$ ). An increase in  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  with increasing  $r_i/L$  is a reflection of the augmentation of the ratio of convective to conductive

transport. Detailed scaling analysis is utilised to explain the observed influences of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$  and  $r_i/L$  on the mean Nusselt number  $\overline{Nu_i}$  and new correlations have been proposed for the mean Nusselt number  $\overline{Nu_i}$ , which have been shown to predict numerical findings accurately for the range of parameters considered here.





## Chapter 6

# Rayleigh-Bénard Configuration: Bingham Fluids

Numerical findings on laminar Rayleigh-Bénard convection of yield stress fluids obeying Bingham model in axisymmetric rectangular cross-sectional cylindrical annular enclosures (see Fig. 2.2) have been presented in this chapter. Yield stress effects on heat and momentum transport have been investigated for a range of different values of normalized internal radius  $r_i/L$  (i.e.  $0 \leq r_i/L \leq 16$ , where  $r_i$  is the internal cylinder radius and  $L$  is the difference between outer and inner radii); aspect ratio  $AR$  (i.e.  $0.25 \leq AR = H/L \leq 4$  where  $H$  is the enclosure height); nominal Rayleigh number  $Ra$  (i.e.  $10^3 \leq Ra \leq 10^5$ ) and Prandtl number  $Pr$  range (i.e.  $10 \leq Pr \leq 10^3$ ) for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions.

### 6.1 Scaling Analysis

A detailed scaling analysis has been conducted here to elucidate the effects of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $AR$ ,  $r_i/L$  on the mean Nusselt number  $\overline{Nu}_{cy}$ . The velocity component in the vertical direction (i.e.  $w$ ) can be scaled by equating the order of magnitudes of inertial and buoyancy terms as:  $w \sim \sqrt{g\beta(T_H - T_C)H}$  ( $w \sim \sqrt{g\beta q\delta_{th}H/k}$ ) for CWT (CWHF) boundary conditions where  $\delta_{th}$  is the thermal boundary layer

thickness on the vertical walls. Based on the continuity equation one obtains:

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} \sim \left( \frac{u}{r} + \frac{u}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{H} \quad (6.1)$$

which leads to:

$$u \sim \frac{w}{AR(r+L)} \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{1}{AR^2} \frac{\sqrt{Ra_{CWT}Pr}}{(1+L/r_i)} \quad (6.2)$$

$$u \sim \frac{w}{AR(r+L)} \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{1}{AR^2} \frac{\sqrt{Ra_{CWHF}Pr}}{(1+L/r_i)} \sqrt{\frac{\delta_{th}}{H}} \quad (6.3)$$

Similarly, equating the order of magnitudes of inertial and viscous terms in the radial direction yields:

$$\rho \frac{u^2}{L} \sim \frac{1}{\delta_1} \left( \tau_y + \mu \frac{u}{\delta_1} \right) \quad (6.4)$$

Using Eqs. 6.2 and 6.3 in Eq. 6.4 leads to:

$$\begin{aligned} \frac{\delta_1}{H} &\sim \sqrt{\frac{Pr}{Ra_{CWT}}} \\ &\times \left\{ \frac{Bn_{CWT}}{2} AR(1+L/r_i)^2 + \sqrt{\left( \frac{Bn_{CWT}}{2} AR(1+L/r_i)^2 \right)^2 + \sqrt{\frac{Ra_{CWT}}{Pr}} (1+L/r_i)} \right\} \end{aligned} \quad (6.5)$$

$$\frac{1}{f_3^{1/2}} \left( \frac{\delta_1}{H} \right)^{5/2} \sqrt{\frac{Ra_{CWHF}}{Pr}} \sim (Bn_{CWHF} AR) f_3^{1/2} \left( \frac{\delta_1}{H} \right)^{1/2} (1+L/r_i)^2 + AR(1+L/r_i) \quad (6.6)$$

where  $\delta_1$  is the hydrodynamic boundary layer thickness on the horizontal walls and  $f_2(Ra_{CWT}, Pr, Bn_{CWT}, AR, r_i/L)$ ,  $f_3(Ra_{CWHF}, Pr, Bn_{CWHF}, AR, r_i/L)$  depicts the ratio of hydrodynamic and thermal boundary thickness (i.e.  $\delta_1/\delta_{th}$ ) for CWT and CWHF boundary conditions, respectively. Eq. 6.5 clearly shows

that  $\delta_1/H$  increases with increasing  $AR$  ( $Bn_{CWT}$ ) for a given set values  $Ra$  and  $r_i/L$ . Moreover, Eq. 6.6 indicates that an exact analytical solution does not exist for the CWHF boundary condition. However, some behaviours can be obtained based on limiting assumptions. For instances,  $Bn = 0$  (i.e. Newtonian fluid) Eq. 6.6 yields:  $\delta_1/H \sim (Pr/Ra_{CWHF})^{0.2} AR^{0.4} (1 + L/r_i)^{0.4} f_3^{0.2}$ , whereas for large values of  $(BnAR)$ , one gets:  $\delta_1/H \sim (Bn_{CWHF} AR)^{0.5} (Pr/Ra_{CWHF})^{0.25} f_3^{0.5}$ . This limiting assumptions also suggest that  $\delta_1/H$  increases with increasing  $AR$  ( $Bn_{CWHF}$ ) for a given set values  $Pr$  and  $r_i/L$ . Here, since qualitative trend is expected to be the same for both CWT and CWHF boundary conditions, this scaling analysis is continued for cylindrical annular enclosures in the case of CWT boundary condition.

Based on the scaling estimates in Eqs. 6.2 and 6.5, it is possible to estimate the effective viscosity in horizontal boundary layer (i.e.  $\mu_{eff}^H$ ) in the following manner ( $\mu_{eff}^H \sim \mu + \tau_y \delta_1/u$ ) for CWT boundary condition:

$$\frac{\mu_{eff}^H}{\mu} \sim \left\{ 1 + Bn_{CWT} AR (1 + L/r_i) \sqrt{\frac{Pr}{Ra_{CWT}}} \right\} \times \left\{ Bn_{CWT} AR (1 + L/r_i)^2 + \sqrt{\left( Bn_{CWT} AR (1 + L/r_i)^2 \right)^2 + \sqrt{\frac{Ra_{CWT}}{Pr}} (1 + L/r_i)} \right\} \quad (6.7)$$

Similarly, equating order of magnitudes of inertial and viscous terms in the vertical direction (i.e.  $\rho w^2/H \sim 1/\delta(\tau_y + \mu w/\delta)$ ) to estimate the effective viscosity in horizontal boundary layer (i.e.  $\mu_{eff}^V$ ) in the following manner ( $\mu_{eff}^V \sim \mu + \tau_y \delta/w$ ):

$$\frac{\mu_{eff}^V}{\mu} \sim \left[ 1 + Bn_{CWT} \sqrt{\frac{Pr}{Ra_{CWT}}} \left\{ \frac{Bn_{CWT}}{2} + \sqrt{\left( \frac{Bn_{CWT}}{2} \right)^2 + \sqrt{\frac{Ra_{CWT}}{Pr}}} \right\} \right] \quad (6.8)$$

where  $\delta$  is the hydrodynamic boundary layer thickness on the vertical wall. Using Eqs. 6.7 and 6.8, the effective Rayleigh numbers (i.e.  $Ra_{eff}^V$  and  $Ra_{eff}^H$ ) in both vertical and horizontal boundary layers can be estimated in the following manner

(i.e.  $Ra_{eff} = \rho g \beta (T_H - T_C) H^3 / \mu_{eff} \alpha$ ):

$$Ra_{eff}^V \sim Ra_{CWT} / \left[ 1 + Bn_{CWT} \sqrt{\frac{Pr}{Ra_{CWT}}} \left\{ \frac{Bn_{CWT}}{2} + \sqrt{\left( \frac{Bn_{CWT}}{2} \right)^2 + \sqrt{\frac{Ra_{CWT}}{Pr}}} \right\} \right] \quad (6.9)$$

$$Ra_{eff}^H \sim Ra_{CWT} / \left\{ 1 + Bn_{CWT} AR (1 + L/r_i) \sqrt{\frac{Pr}{Ra_{CWT}}} \right\} \times \left\{ Bn_{CWT} AR (1 + L/r_i)^2 + \sqrt{\left( \frac{Bn_{CWT} AR (1 + L/r_i)^2}{2} \right)^2 + \sqrt{\frac{Ra_{CWT}}{Pr}} (1 + L/r_i)} \right\} \quad (6.10)$$

Equations 6.9 and 6.10 indicates that the effective Rayleigh number  $Ra_{eff}$  remains smaller than the nominal Rayleigh number for Bingham fluids and  $Ra_{eff}$  decreases with increasing  $Bn$ . Furthermore, Eqs. 6.7 and 6.8 indicate that  $\mu_{eff}^H$  is expected to be greater than  $\mu_{eff}^V$  in cylindrical enclosures. This further implies that  $Ra_{eff}^H$  is expected to be smaller than  $Ra_{eff}^V$  in cylindrical enclosures as it is shown in Eqs. 6.9 and 6.10.

Finally, the wall heat flux  $q$  can be scaled as:  $q \sim h\Delta T \sim k\Delta T/\delta_{th}$  where  $\delta_{th}$  is the thermal boundary layer on the horizontal wall. Accordingly, the scaling estimate of the mean Nusselt number can be expressed as  $\overline{Nu}_{cy} \sim hH/k \sim H/\delta_{th} \sim (H/f_2/\delta_1)$ . It is not possible to obtain an analytical relation for  $\delta_{th}$  from Eq. 6.6 for CWHF boundary condition but it is possible to obtain a scaling estimate of  $\overline{Nu}_{cy}$  using Eq. 6.5 in the case of CWT boundary condition in the following manner:

$$\overline{Nu}_{cy} \sim \text{Max} \left\{ 1, \frac{(Ra_{CWT}/Pr)^{1/2}}{\left[ \frac{Bn_{CWT}}{2} \frac{AR}{(1+L/r_i)^{-2}} + \sqrt{\left[ \frac{Bn_{CWT}}{2} \frac{AR}{(1+L/r_i)^{-2}} \right]^2 + \sqrt{\frac{Ra_{CWT}}{Pr}} (1 + L/r_i)} \right]} \right\} \times f_2(Ra_{CWT}, Pr, Bn_{CWT}, AR, r_i/L) \quad (6.11)$$

Equation 6.11 shows that  $\overline{Nu}_{cy}$  is expected to increase with an increase in  $Ra$  and  $r_i/L$ , whereas an opposite behaviour is expected with an increase in  $Bn$  and  $AR$ .

## 6.2 Effects of Bingham Number $Bn$

The contours of dimensionless temperature  $\theta_{CWHF} = (T - T_{cen})k/qL$  and stream function  $\Psi = \psi/\alpha$  are shown in Fig. 6.1 for different values of  $Bn$  for  $r_i/L = 1.0$ ,  $AR = 0.5$ ,  $Ra = 10^5$  and  $Pr = 500$  in the case of CWHF boundary condition. Figure 6.1 indicates that the magnitude of  $\Psi$  decreases and isotherms become progressively parallel to the horizontal walls with increasing  $Bn$ . This is an indicator of weakening of convection with an increase in  $Bn$  and heat transfer can only occur purely due to conduction for large values of  $Bn$ . The grey regions on the stream function distribution in Fig. 6.1 indicate the Apparently Unyielded Regions (AURs) (regions where  $|\tau| \leq \tau_y$  [106]). The AURs can be considered as the regions of slow moving fluids because of the high viscosity ( $\mu_{yield} \geq 10^3\mu$ ) in the context of bi-viscosity regularisation. The shapes and sizes of AURs are dependent on the choice of  $\mu_{yield}$  [25, 75–78] but the qualitative and quantitative distributions of stream function and isotherms remain independent of the value of  $\mu_{yield}$  for  $\mu_{yield} \geq 10^3\mu$ , and thus a fixed value  $\mu_{yield} \geq 10^4\mu$  is used here for all the simulations for the sake of high-fidelity. The precise shapes and sizes of AURs do not affect the mean Nusselt number in the current analysis. As it is shown in Fig. 6.1, the size of the AURs increases, and thermal transport becomes increasingly conduction-dominated with an increase in  $Bn$ . The fluid flow becomes so weak that fluid flow does not contribute to thermal transport for large values of  $Bn$ .

It is worth noting that the distributions of  $\theta_{CWT} = (T - T_{cen}) / (T_H - T_C)$  and  $\Psi$  for the CWT boundary condition remain qualitatively similar to those obtained for the CWHF boundary condition, and thus are not explicitly shown here for the sake of conciseness.

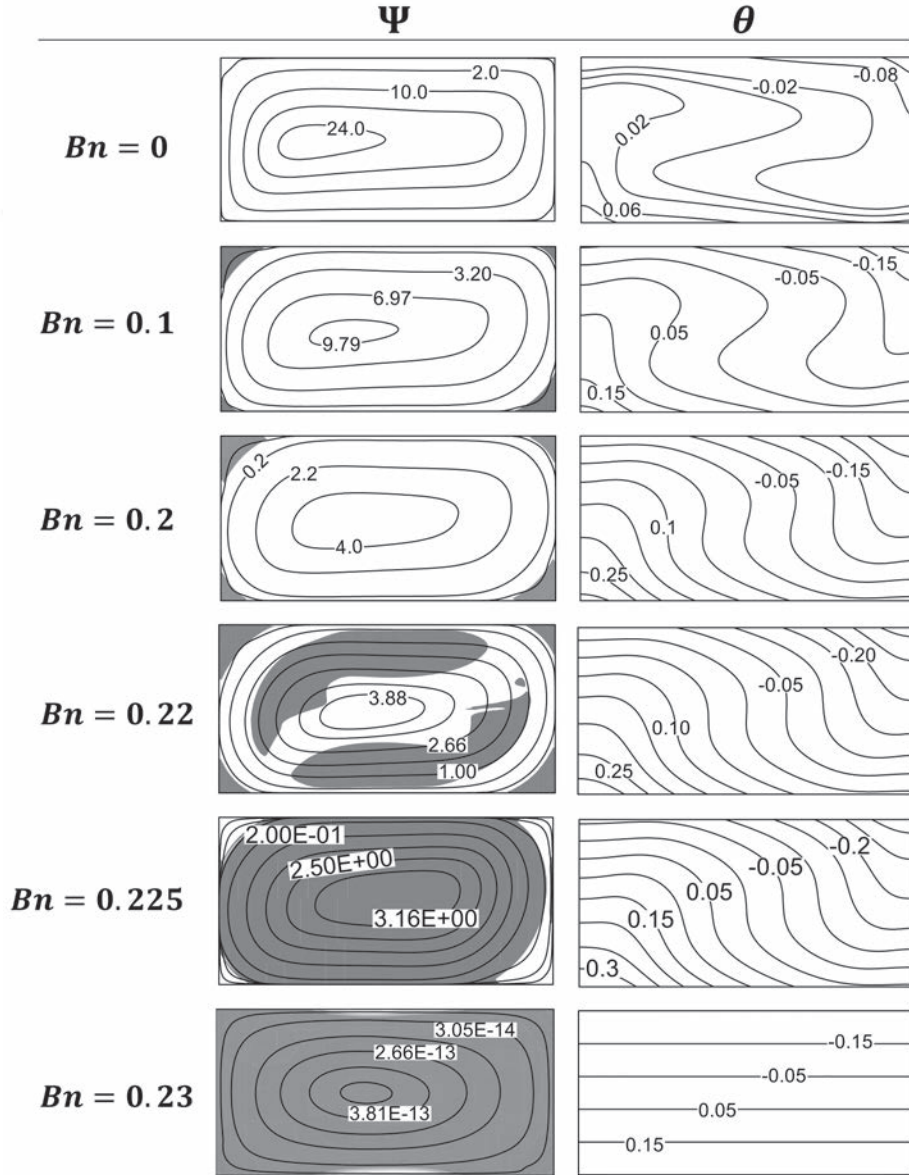


Figure 6.1: Contours of  $\theta$  and  $\Psi$  with AURs (shown in grey) for  $r_i/L = 1.0$ ,  $AR = 0.5$ ,  $Ra = 10^5$  and  $Pr = 500$  in the CWHF boundary condition.

### 6.3 Effects of Nominal Rayleigh Number $Ra$ and Normalised Cylinder Radius $r_i/L$

The variations of dimensionless temperature  $\theta = (T - T_{cen})/\Delta T_{ref}$  and dimensionless vertical ( $W = wL/\alpha$ ) (radial ( $U = uL/\alpha$ )) velocity components along with the horizontal (vertical) mid-plane for different  $Ra$  and  $r_i/L$  at  $AR = 1.0$  and  $Pr = 500$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.03$ ) fluid cases in the both CWT and CWHF boundary conditions. It can be seen from Fig. 6.2 that the temperature difference between hot and cold walls is smaller in the CWHF boundary condition than that for the CWT boundary condition for a given set of values of  $Ra$ ,  $Pr$  and  $Bn$  and  $AR$ . For the CWHF boundary condition, the wall heat flux can be estimated as  $q \sim k\Delta T/\delta_{th}$  where  $\Delta T$  and  $\delta_{th}$  are the characteristic temperature difference and the thickness of thermal boundary layer on horizontal walls, respectively. This suggests that the non-dimensional temperature  $\theta$  in the case of CWHF boundary condition scales as  $\theta \sim \Delta Tk/qL \sim q\delta_{th}k/qLk \sim O(\delta_{th}/L)$ , whereas  $\theta \sim O(1)$  for the CWT boundary condition. The above discussion suggests that  $\theta \sim O(\delta_{th}/L)$  for the CWHF boundary condition is expected to increase with increasing  $\delta_{th}$ , as the thermal boundary layer thickness increases with an increase in Bingham number due to the weakening of convection strength (see Fig. 6.2). The convection strength also weakens with decreasing  $Ra$ , which is reflected in the thickening of thermal boundary layer. Thus, the temperature difference between horizontal walls also increases with decreasing  $Ra$  for the CWHF boundary condition. Higher temperature difference between the horizontal walls leads to stronger convection in the CWT boundary condition than in the corresponding CWHF boundary condition for the same set of values of  $r_i/L$ ,  $Ra$ ,  $Pr$ , and  $Bn$ . This can be verified from Fig. 6.2 which shows that the magnitudes of  $W$  and  $U$  are greater in the CWT boundary condition than in the CWHF boundary condition for a given set of values of  $r_i/L$ ,  $Ra$ ,  $Pr$  and  $Bn$ .

Figure 6.2 further shows that  $r_i/L$  does not significantly affect the temperature difference between horizontal walls for both  $Bn = 0$  and  $Bn = 0.03$  regardless of the boundary condition at low nominal  $Ra$  values (i.e.  $Ra = 10^4$ ). However, the temperature difference between horizontal walls is significantly influenced by  $r_i/L$

for high values of  $Ra$  (i.e.  $Ra = 10^5$ ) for  $r_i/L < 1$  for both CWT and CWHF boundary conditions. This behaviour arises due to a change in the streamline pattern for  $r_i/L < 1$  and  $Ra \geq 5 \times 10^4$  (see in Fig. 6.3), which leads to a modification of temperature distribution between hot and cold walls regardless of the boundary condition.

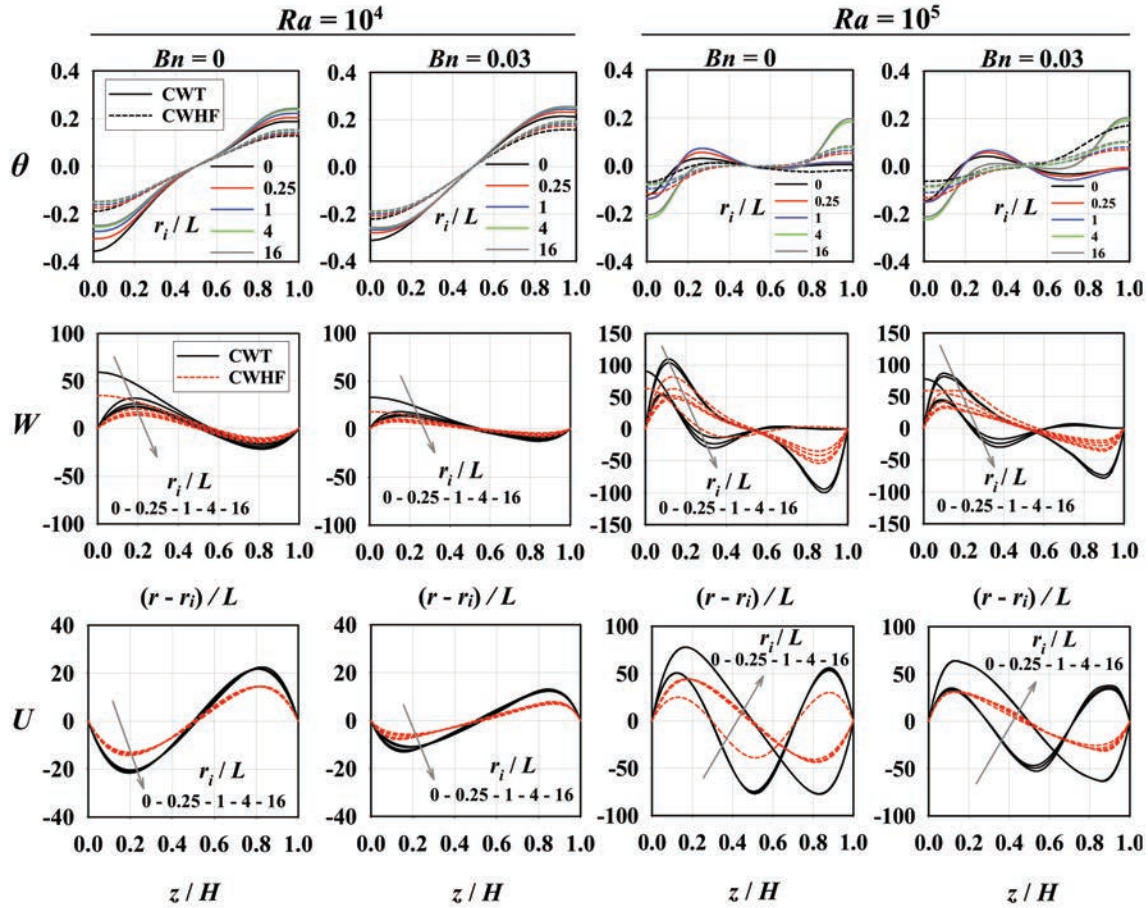


Figure 6.2: Variations of  $\theta$  and  $W$  ( $U$ ) along with the horizontal (vertical) mid-plane for different  $Ra$  and  $r_i/L$  at  $AR = 1.0$  and  $Pr = 500$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.03$ ) fluid cases in the both CWT and CWHF boundary conditions.

It is worth noting that the magnitudes of  $W$  and  $U$  for  $Ra = 10^5$  are greater than the corresponding values for  $Ra = 10^4$  due to strengthening of buoyancy forces

with increasing  $Ra$  for both CWT and CWHF boundary condition. It can be also seen from Fig. 6.2 that non-dimensional axial velocity component  $W$  decreases while the non-dimensional radial velocity component  $U$  increases with increasing  $r_i/L$  for both  $Bn = 0$  and  $Bn = 0.03$  cases. This is consistent with the scaling relation given by Eqs. 6.2 and 6.3 (i.e.  $u \sim w/AR(1 + L/r_i)$ ). Additionally, the magnitudes of  $W$  and  $U$  are smaller in the Bingham fluid cases than in the corresponding Newtonian fluid cases for a given set of values of nominal Rayleigh and Prandtl numbers.

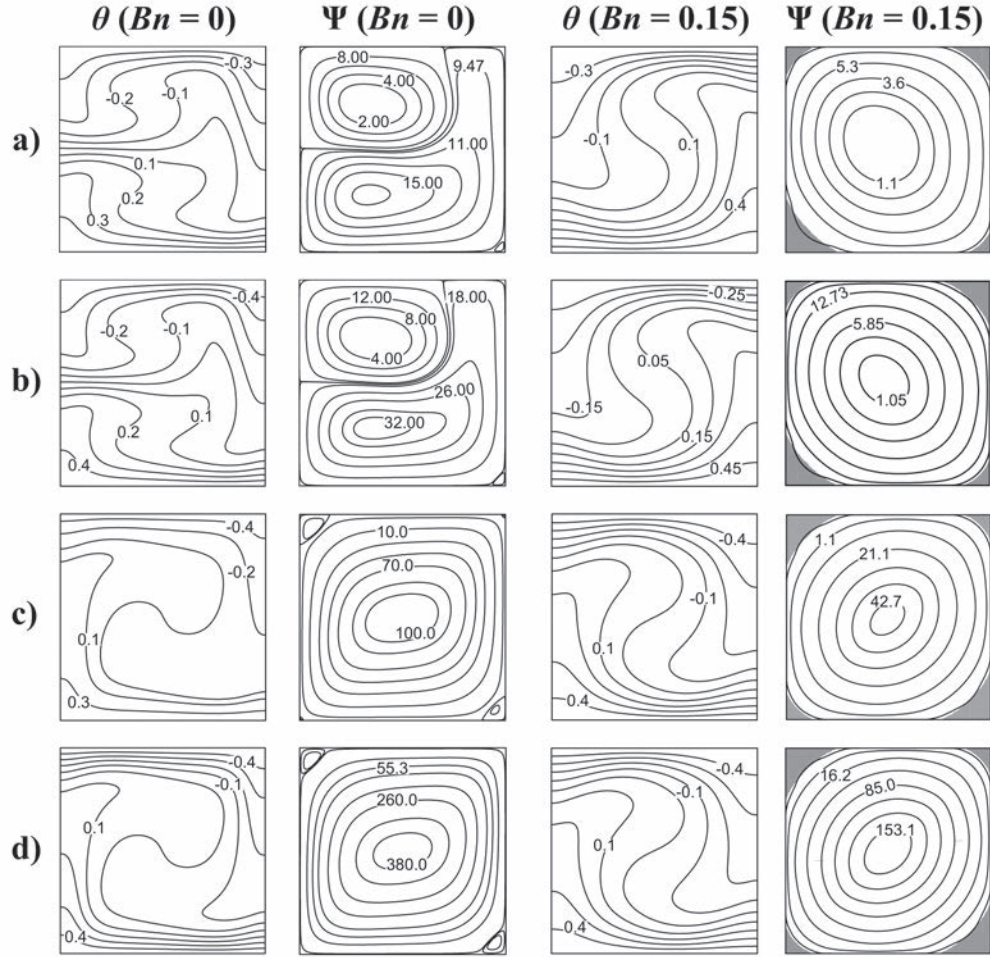


Figure 6.3: Contours of  $\theta$  and  $\Psi$  with AURs (shown in grey) for  $Bn = 0$  and  $Bn = 0.15$  in case of  $r_i/L = 1.0$  a) 0.25 , b) 1 , c) 4 , d) 16 at  $Ra = 10^5$ ,  $AR = 1.0$  and  $Pr = 500$  for CWT boundary condition.

Moreover, Fig. 6.3 indicates that the contours  $\theta$  and  $\Psi$  are shown for different values of  $r_i/L$  at  $Ra = 10^5$ ,  $AR = 1.0$  and  $Pr = 500$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.15$ ) fluid cases in the CWT boundary condition. It can be seen from Fig. 6.3 that the magnitude of  $\Psi$  increases with an increase in  $r_i/L$  in the CWT boundary condition. This behaviour remains qualitatively similar to those obtained for the CWHF boundary condition, and thus are not explicitly shown here for the sake of conciseness.

## 6.4 Effects of Prandtl Number $Pr$

The variations of the mean Nusselt number  $\overline{Nu}_{cy}$  with  $Bn$  for different values of  $Pr$  at  $r_i/L = 1.0$ ,  $Ra = 10^4$  and  $AR = 1.0$  for both CWT and CWHF boundary conditions are shown in Fig. 6.4.

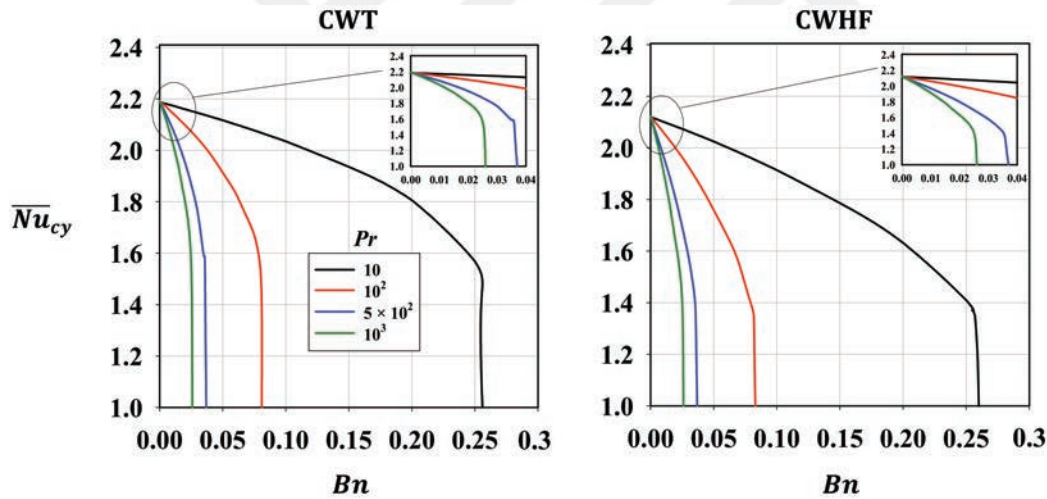


Figure 6.4: Variation of  $\overline{Nu}_{cy}$  with  $Bn$  for different values of  $Pr$  at  $r_i/L = 1.0$ ,  $Ra = 10^4$  and  $AR = 1.0$  for both CWT and CWHF boundary conditions.

It can be seen from Fig 6.4 that  $\overline{Nu}_{cy}$  decreases with increasing  $Bn$  which is indicative of the weakening of convective transport as a result of additional flow resistance due to the yield stress. Furthermore, Fig. 6.4 shows that  $\overline{Nu}_{cy}$  settles to unity once a threshold value of Bingham number  $(Bn_{max})_{cy}$  is surpassed. Here  $(Bn_{max})_{cy}$  is taken to be the nominal Bingham number where  $\overline{Nu}_{cy} = 1.01$ ,

so that  $\overline{Nu}_{cy} < 1.01$  for  $Bn > (Bn_{max})_{cy}$ . The fluid flow becomes so weak for  $Bn > (Bn_{max})_{cy}$  that it does not impart any influence on the thermal transport, which takes place purely due to conduction under this situation. Fig. 6.4 further shows that  $\overline{Nu}_{cy}$  does not change significantly with a change in  $Pr$  for Newtonian fluids (i.e.  $Bn = 0$ ) irrespective of the choice of boundary condition. The heat transfer characteristics in the present boundary condition depend on the relative strengths of inertial, viscous, and buoyancy forces. For  $Pr > 1$ , change in the  $Pr$  modifies the relative balance between the buoyancy and viscous forces because the thermal boundary layer remains much thinner than the hydro-dynamic boundary layer. Thus, thermal transport within thermal boundary layer does not get affected by a change in  $Pr$  for  $Pr > 1$  in the case of Newtonian fluids (i.e.  $Bn = 0$ ) [2]. By contrast,  $\overline{Nu}_{cy}$  decreases with increasing  $Pr$  for large values of Bingham number for both CWT and CWHF boundary conditions as it is shown in Fig. 6.4. Moreover, the value of Bingham number  $(Bn_{max})_{cy}$  for which  $\overline{Nu}_{cy}$  approaches unity decreases with increasing  $Pr$ . This behavior is qualitatively consistent with the previous findings of Rayleigh-Bénard convection of Bingham fluids in rectangular enclosures [25, 75]. It is evident from Eqs. 6.9 and 6.10 that both effective Rayleigh and Grashof numbers (i.e.  $Ra_{eff} = \rho g \beta \Delta T_{ref} L^3 / \mu_{eff} \alpha$ ;  $Gr_{eff} = Ra_{eff} / Pr = \rho^2 g \beta \Delta T_{ref} L^3 / \mu_{eff}^2$ ) decrease with increasing  $Pr$  for large values of  $Bn$  since  $\mu_{eff}$  increases sharply for large values of  $Bn$ . However, an increase in  $Pr$  leads to a decrease in the thermal boundary thickness for small values of  $Bn$  for a given value of  $Ra$ . This gives rise to a small increase in  $\overline{Nu}_{cy}$  for small values of  $Bn$  because the thinning of thermal boundary layer overcomes the strengthening of viscous resistance under this condition.

## 6.5 Effects of Aspect Ratio $AR$

The variations of  $\theta$  and  $W$  along the vertical (horizontal) mid-plane for different  $AR$  and  $Ra$  values at  $Bn = 0.03$ ,  $r_i/L = 1.0$  and  $Pr = 500$  are shown in Fig. 6.5 for both CWT and CWHF boundary conditions. The extent of non-linearity in  $\theta$  variation in Fig. 6.5 provides a measure of convection strength (because a linear variation is indicative of pure-conductive transport) for the both CWT and CWHF boundary conditions. Figure 6.5 shows that the variation of  $\theta$  with  $z/H$

becomes increasingly linear and the magnitude of  $W$  decreases with increasing  $AR$  due to the weakening of convective transport. This behaviour can be confirmed from Fig. 6.6 where the contours of  $\theta$  and  $\Psi$  in the CWT boundary condition are shown for different  $AR$  values at  $Bn = 0.03$ ,  $r_i/L = 1.0$  and  $Pr = 500$ . It is evident from Fig. 6.6 that the magnitude of  $\Psi$  decreases and the isotherms become increasingly parallel to the horizontal walls with increasing  $AR$ , indicating conduction-driven thermal transport. Figure 6.6 also shows the number of rolls within the enclosure changes with  $AR$  and thus the qualitative nature of the distribution of  $W$  with  $(r - r_i)/L$  also changes with the change in  $AR$  in Fig. 6.5.

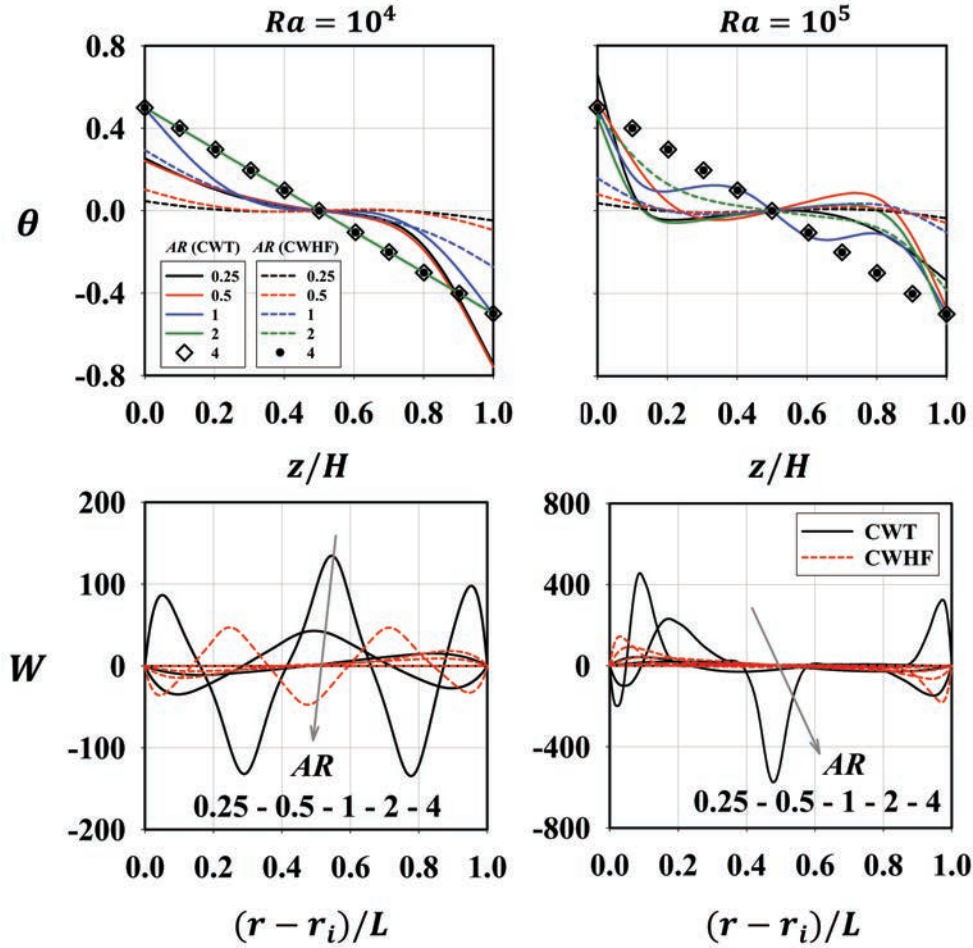


Figure 6.5: Variations of  $\theta$  ( $W$ ) along the vertical (horizontal) mid-plane for different  $AR$  and  $Ra$  at  $Bn = 0.03$ ,  $r_i/L = 1.0$  and  $Pr = 500$  for both CWT and CWHF boundary conditions.

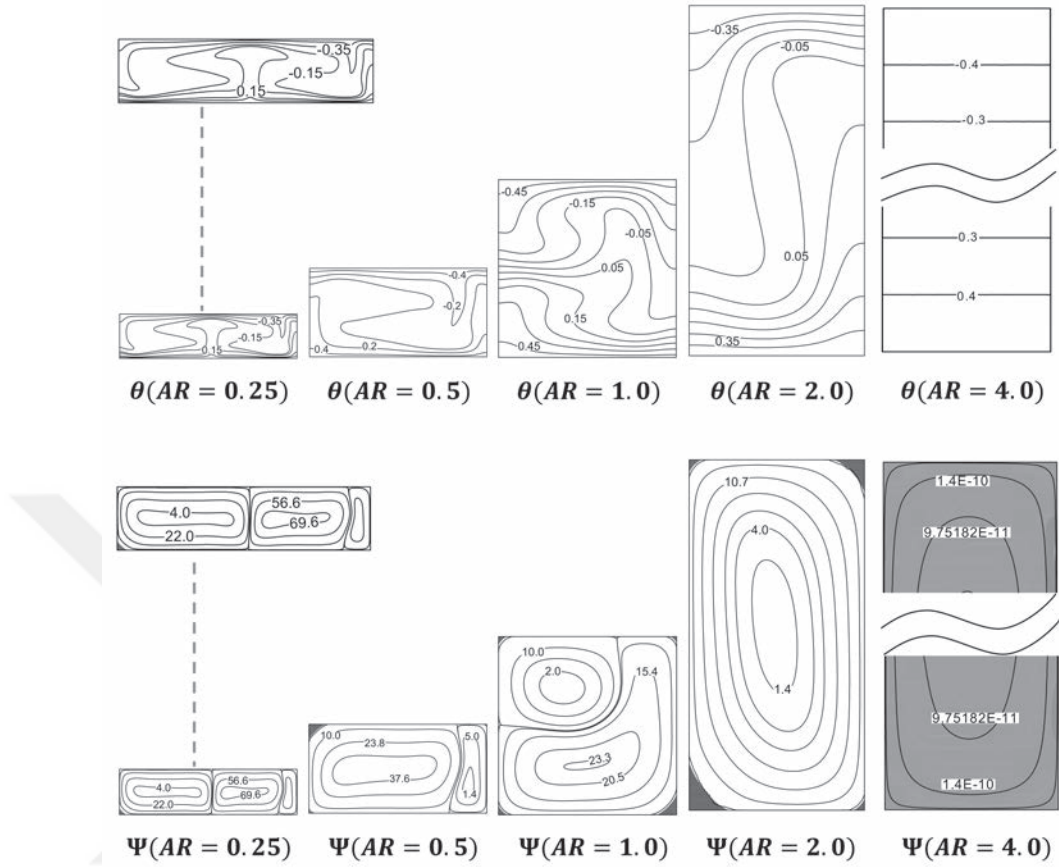


Figure 6.6: Contours of  $\theta$  and  $\Psi$  with AURs (shown in grey) for different  $AR$  at  $Bn = 0.03$ ,  $r_i/L = 1.0$ ,  $Ra = 10^5$  and  $Pr = 500$  in the CWT boundary condition.

The weakening (strengthening) of advective (conductive) transport with increasing  $AR$  can be explained with the help of the energy flux integral at the vertical mid-plane [77]:

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{cond} = \int_0^H \rho c_p T u dz - \int_0^H k \left( \frac{\partial T}{\partial r} \right) dz \quad (6.12)$$

where the first term on the right hand side represents the effects of convective transport, whereas the second term on the right hand side accounts for the contribution of thermal conduction. The quantities  $\dot{Q}_{conv}$  and  $\dot{Q}_{cond}$  can be scaled in the following manner:

$$\begin{aligned} \dot{Q}_{conv,CWT} &\sim \rho c_p \Delta T u \delta_1 \sim Pr k \Delta T \\ &\times \left[ \frac{Bn_{CWT}}{2} (1 + L/r_i) + \sqrt{\left( \frac{Bn_{CWT}}{2} \right)^2 (1 + L/r_i)^2 + \sqrt{\frac{Ra_{CWT}}{Pr}} \frac{1}{AR^2} \frac{1}{(1 + L/r_i)}} \right] \end{aligned} \quad (6.13)$$

$$\dot{Q}_{conv,CWHF} \sim \frac{\rho c_p u q \delta_{th} \delta_1}{k} \sim q L f_3 \frac{\sqrt{Ra_{CWHF} Pr}}{AR^{5/2}} \left( \frac{\delta_{th}}{L} \right)^{5/2} \quad (6.14)$$

$$\dot{Q}_{cond,CWT} \sim k \frac{\Delta T}{L} H \sim k \Delta T AR \quad (6.15)$$

$$\dot{Q}_{cond,CWHF} \sim k q \frac{\delta_{th}}{k} \frac{H}{L} \sim L q AR \left( \frac{\delta_{th}}{L} \right) \quad (6.16)$$

Equations 6.13 - 6.16 indicates that  $\dot{Q}_{conv}$  ( $\dot{Q}_{cond}$ ) weakens (strengthens) with increasing  $AR$  when  $k$ ,  $\Delta T$ ,  $Ra$ ,  $q$  and  $Pr$  are kept constant for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids, which is consistent with the observations made from Figs. 6.5 and 6.6.

## 6.6 Behaviour on Mean Nusselt Number of Cylindrical Enclosures $\overline{Nu}_{cy}$

The variation of  $\overline{Nu}_{cy}$  with  $r_i/L$  for different  $Ra$  and  $AR$  is shown in Fig. 6.7 for Newtonian fluids (i.e.  $Bn = 0$ ) at  $Pr = 500$  for both CWT and CWHF boundary conditions. It can be seen from Fig. 6.7 that  $\overline{Nu}_{cy}$  increases with increasing  $Ra$  and asymptotically approaches the corresponding mean Nusselt number  $\overline{Nu}_{Rec}$  for rectangular enclosures in the CWT boundary condition. However,  $\overline{Nu}_{cy}$  increases with increasing  $Ra$  but remains insensitive to the changes in  $r_i/L$  for  $r_i/L > 1$  for the CWHF boundary condition.

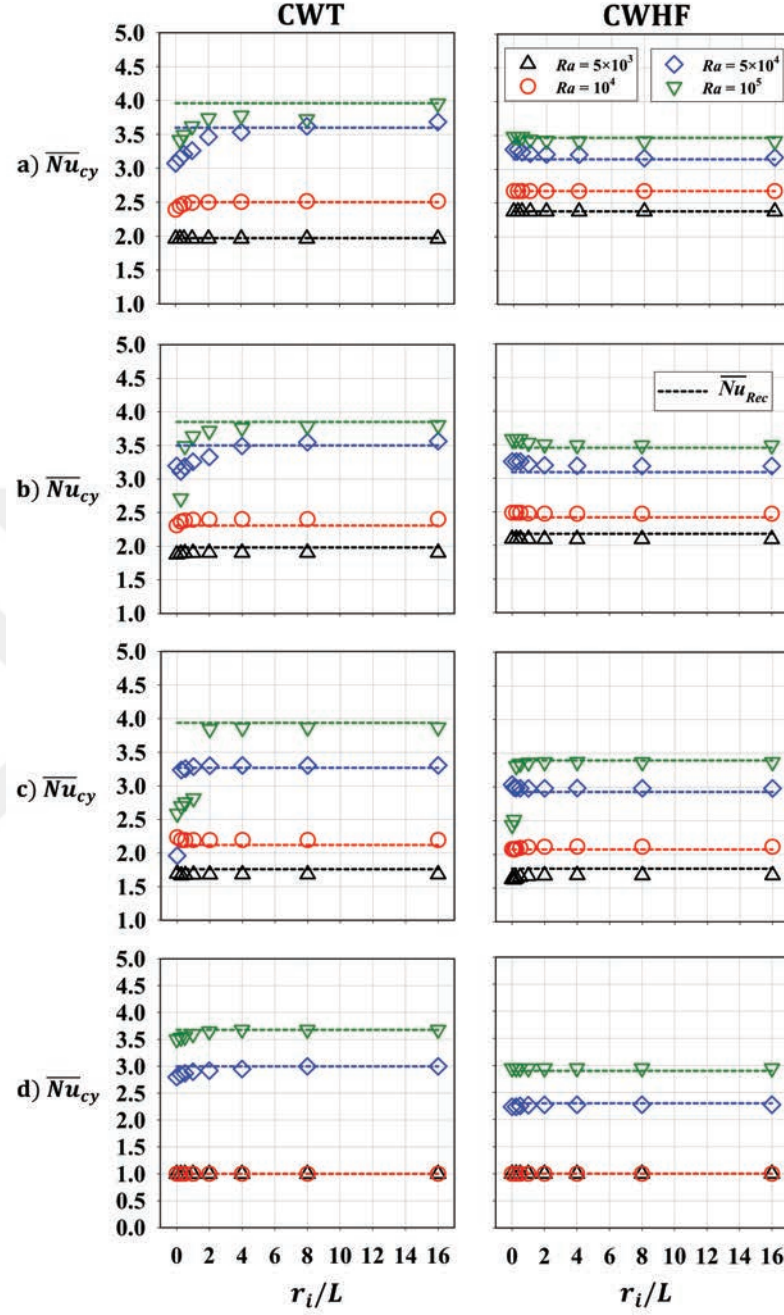


Figure 6.7: Variation of  $\overline{Nu}_{cy}$  with  $r_i/L$  for Newtonian fluids (i.e.  $Bn = 0$ ) for different  $Ra$  and  $AR$  a) 0.25, b) 0.5, c) 1, d) 2 at  $Pr = 500$  for both CWT and CWHF boundary conditions. The corresponding values of the mean Nusselt number  $\overline{Nu}_{Rec}$  for rectangular enclosures is given by dashed lines (which is correlated in Eq. 6.17 by Ref. [77]).

In order to explain this, the contours of  $\Psi$  for different values of  $Ra$ ,  $r_i/L$  and  $AR$  at  $Pr = 500$  are shown in Fig. 6.8 in the both CWT and CWHF boundary conditions in the case of Newtonian fluids (i.e.  $Bn = 0$ ). It can be noted from Fig. 6.8 that the number of rolls within the enclosure changes with  $Ra$ ,  $AR$  and  $r_i/L$  in the CWT boundary condition, whereas changes in  $AR$  and  $r_i/L$  values do not affect the flow pattern in the CWHF boundary condition. The changes in the flow patterns modify the distributions of isotherms between hot and cold walls, which is reflected in the variation of  $\overline{Nu}_{cy}$  (see Fig. 6.7). Using Eqs. 6.5 and 6.6 one obtains the following estimate of mean Nusselt number  $\overline{Nu}_{cy} \sim (H/\delta_1) f_{2,3}$  for natural convection of Newtonian fluids by putting  $Bn = 0$  as:  $\overline{Nu}_{cy} \sim (Ra_{CWT}/Pr)^{0.25} f_2/(1 + L/r_i)^{0.5}$ ;  $\overline{Nu}_{cy} \sim (Ra_{CWHF}/Pr)^{0.2} f_3^{0.8}/AR^{0.4}(1 + L/r_i)^{0.4}$  for CWT and CWHF boundary conditions, respectively. The above scaling estimates suggest that  $\overline{Nu}_{cy}$  is expected to increase with increasing  $Ra$  and  $r_i/L$ , which are consistent with the observations made from Fig. 6.7.

The flow patterns indicated in Fig 6.8 are consistent with previous analyses [110], which dealt with laminar Rayleigh-Bénard convection of Newtonian fluids in a cylinder (i.e.  $r_i/L = 0$ ). Additionally, Lir and Lin [50] reported similar flow patterns for Rayleigh-Bénard convection of Newtonian fluid (i.e. air) in a rectangular shallow cavity based on their experimental flow visualisation. Mueller [51] also numerically reported similar flow patterns including rolls stacking over one another based on numerical simulations. Moreover it is worth indicating that the flow patterns presented here are not artefacts of convergence criteria and reducing the convergence tolerance by an order of magnitude did not make any difference to the magnitudes of the stream functions and also in the values of the mean Nusselt number (especially for the cases where the rolls are stacked upon one another). It is also worth noting that the flow patterns in the Rayleigh-Bénard configuration depend significantly on the initial condition [52] and it is possible to have several steady state solutions based on different initial conditions for the CWT boundary condition. Thus, the flow patterns presented in this study (especially the ones where the rolls are stacked over one another and it should be noted that the circulation strength for both rolls are not the same and one is often much weaker than the other) need to be validated by experimental analyses.

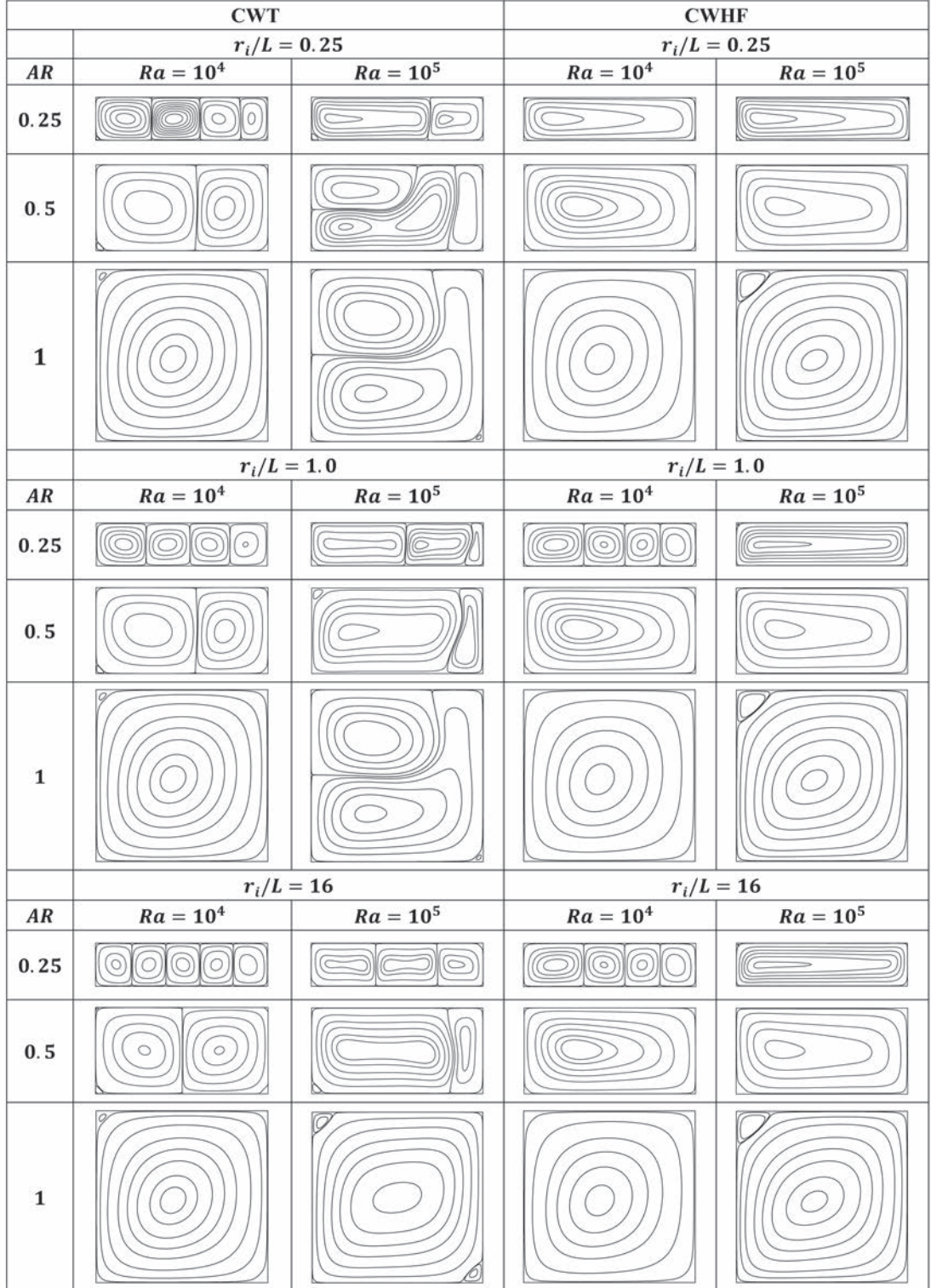


Figure 6.8: Contours of  $\Psi$  for different values of  $Ra$ ,  $r_i/L$  and  $AR$  at  $Pr = 500$  in the case of both CWT and CWHF boundary conditions for Newtonian fluids (i.e.  $Bn = 0$ ).

The following correlation for the mean Nusselt number was proposed in Refs. [76, 77] for natural convection of Newtonian fluids in rectangular enclosures  $\overline{Nu}_{Rec}$ :

$$\overline{Nu}_{Rec} = [\{m_0 / [1 + \exp \{(AR - x_0) / n_0\}]\} - y_0] |AR - 1|^{0.01} + \overline{Nu}(AR = 1) \quad (6.17)$$

where  $\overline{Nu}(AR = 1)$  is the mean Nusselt number for square enclosures (i.e.  $AR = 1$ ) [75] and  $m_0, x_0, n_0, y_0$  are the correlation parameters, which are listed in Table 6.1 for both CWT and CWHF boundary conditions. The predictions of Eq. 6.17 are shown in Fig. 6.7 (dashed lines) and Fig 6.9, which indicates that  $\overline{Nu}_{cy}$  for the CWHF boundary condition can be satisfactorily predicted with the help of Eq. 6.17 apart from  $AR = 1.0, r_i/L \leq 0.25$ . Moreover, Eq. 6.17 is only valid for  $r_i/L \geq 4$  in the CWT boundary condition for  $AR \leq 1$  and  $Ra \geq 5 \times 10^4$  (see Fig. 6.7).

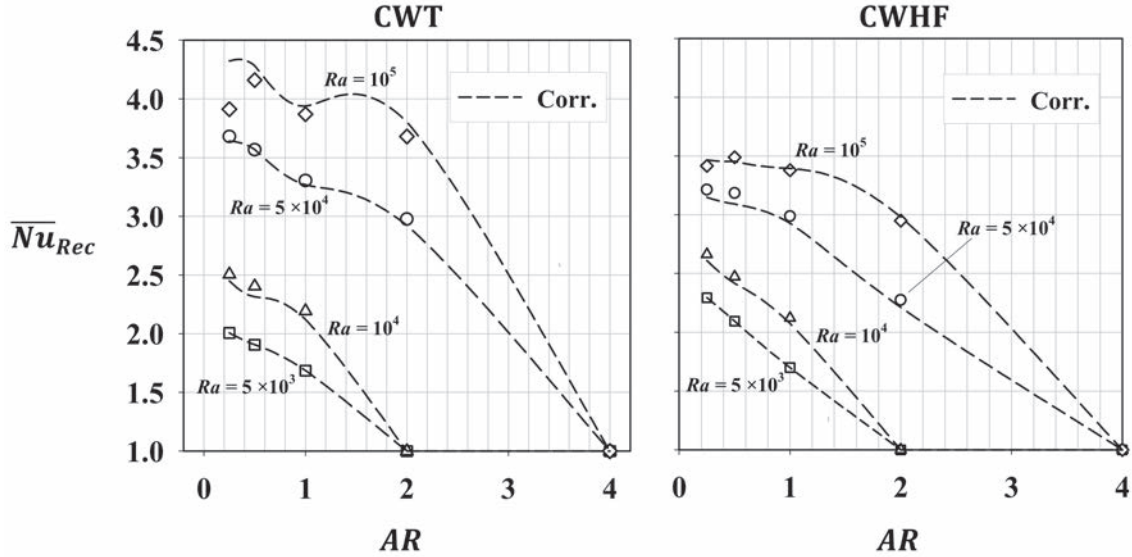


Figure 6.9: Variation of  $\overline{Nu}_{Rec}$  with  $AR$  for Newtonian fluids (i.e.  $Bn = 0$ ) for different values of  $Ra$  at  $Pr = 500$  along with the prediction of Eq. 6.17 for both CWT and CWHF boundary conditions.

Table 6.1: Summary of  $\overline{Nu}_{Rec}$  correlation given by Eq. 6.17 for rectangular enclosures for both CWT and CWHF boundary conditions.

|                                                                                                                   |                                                                                                                                                                                                                                                                                                                                                |
|-------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\overline{Nu}_{Rec} = [\{m_0 / [1 + \exp \{(AR - x_0) / n_0\}]\} - y_0]  AR - 1 ^{0.01} + \overline{Nu}(AR = 1)$ |                                                                                                                                                                                                                                                                                                                                                |
| CWT                                                                                                               | $\overline{Nu}(AR = 1) = 0.178 Ra^{0.269} [Pr / (1 + Pr)]^{0.02}$ $x_0 = 0.55 + 29.25 \exp \left[ -0.5 \{(\ln Ra - 11.12) / 0.306\}^2 \right]$ $y_0 = 0.455 + 4097 / [1 + \exp \{(46754 - Ra) / 4258\}]$ $m_0 = 0.765 + 4097 / [1 + \exp \{(46743 - Ra) / 4270\}]$ $n_0 = 0.022 + 3.31 \exp \left[ -0.5 \{(\ln Ra - 11.1) / 0.301\}^2 \right]$ |
| CWHF                                                                                                              | $\overline{Nu}(AR = 1) = 0.289 Ra^{0.214} [Pr / (1 + Pr)]^{0.017}$ $x_0 = 0.613 + 2.21 / [1 + \exp \{(10.43 - \ln Ra) / 0.42\}]$ $y_0 = -33.6 + 40.95 / [1 + \exp \{(-\ln Ra - 4.658) / 8.03\}]$ $m_0 = 0.767 + 1.917 / [1 + \exp \{(9.194 - \ln Ra) / 0.715\}]$ $n_0 = 0.04 + 0.69 / \left\{ 1 + [(\ln Ra - 11) / 0.644]^2 \right\}$          |

The variations of the mean Nusselt number  $\overline{Nu}_{cy}$  with  $Bn$  for different values

of  $r_i/L$ ,  $AR$  and  $Ra$  at  $Pr = 500$  for both CWT and CWHF boundary conditions are shown in Figs. 6.10 and 6.11 for Bingham fluids. It can be seen from Figs. 6.10 and 6.11 that the  $\overline{Nu}_{cy}$  decreases with increasing  $Bn$  which is indicative of the weakening of convective transport as a result of additional flow resistance due to the yield stress. Moreover, the simulations have been conducted by increasing (decreasing)  $Bn$  until  $(Bn_{max})_{cy}$  ( $Bn = 0$ ) is reached from  $Bn = 0$  ( $Bn = (Bn_{max})_{cy}$ ) in order to assess if the variation of  $\overline{Nu}_{cy}$  with  $Bn$  shows any hysteresis. It is found that hysteresis occurs (i.e.  $\overline{Nu}_{cy}$  is indeed multi-valued for a given value of  $Bn$  and different flow patterns are obtained for a given value of  $Bn$ ) especially for  $Ra \geq 5 \times 10^4$  and  $AR \leq 1$ , depending on the initial condition (see Figs. 6.10 and 6.11) for both CWT and CWHF boundary conditions.

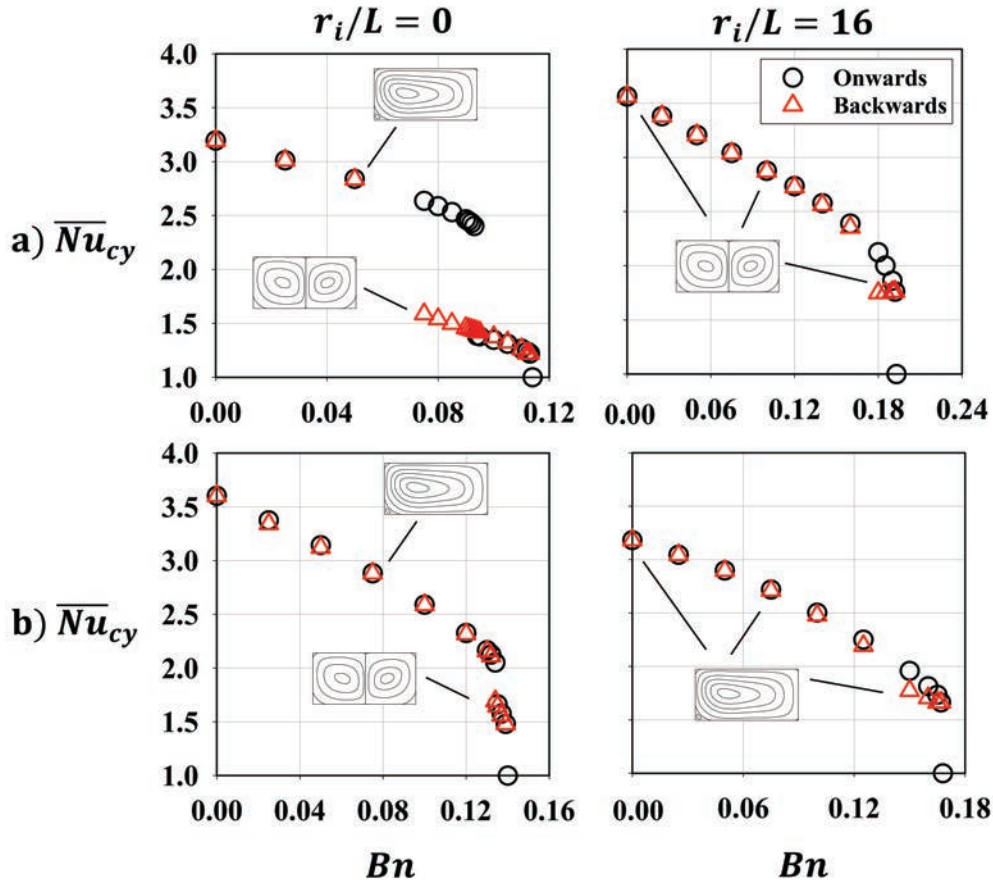


Figure 6.10: Variation of  $\overline{Nu}_{cy}$  with  $Bn$  for Bingham fluids at  $Ra = 5 \times 10^4$ ,  $AR = 0.5$  and  $Pr = 500$  for a) CWT, b) CWHF boundary conditions.

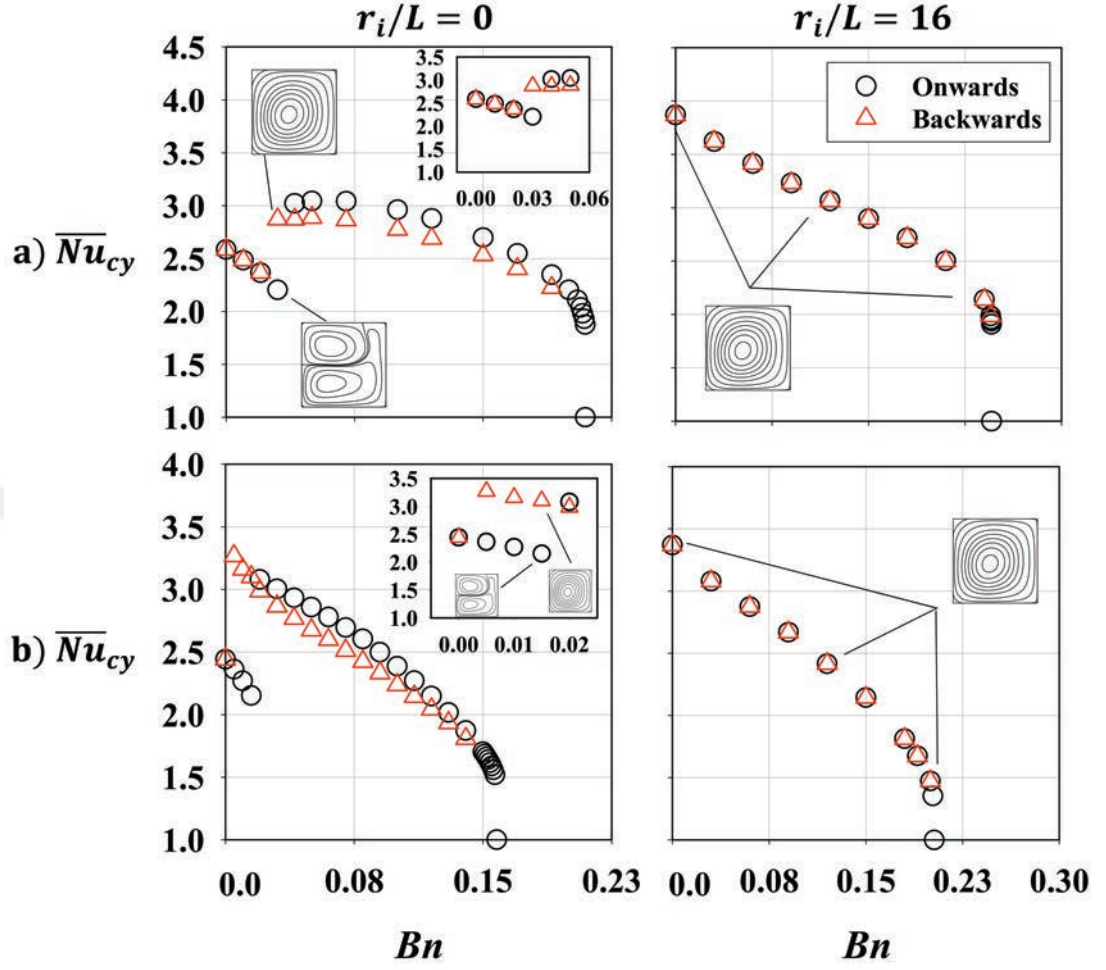


Figure 6.11: Variation of  $\overline{Nu}_{cy}$  with  $Bn$  for Bingham fluids at  $Ra = 10^5$ ,  $AR = 1.0$  and  $Pr = 500$  for a) CWT, b) CWHF boundary conditions.

The quantity  $(Bn_{max})_{cy}$  can be estimated for the CWT boundary condition by considering  $\overline{Nu}_{cy} \sim H/\delta_{th} \sim Hf_2/\delta_1 \sim O(1)$ . This along with Eq. 6.5 yields a scaling estimate of  $(Bn_{max})_{cy}$  for the CWT boundary condition:

$$(Bn_{max})_{cy} \sim AR^{-1} \left[ \frac{f_2 \sqrt{Ra/Pr}}{(1 + L/r_i)^2} - \frac{1}{(1 + L/r_i)f_2} \right] \quad (6.18)$$

Using  $r_i/L \rightarrow \infty$  in Eq. 6.18 yields the estimate of the corresponding threshold value of Bingham number for rectangular enclosures for the CWT boundary

condition:

$$(Bn_{max})_{Rec} \sim AR^{-1} \left[ f_2 \sqrt{Ra/Pr} - 1/f_2 \right] \quad (6.19)$$

The variation of  $(Bn_{max})_{cy}$  with  $r_i/L$  for different values of  $Ra$  and  $AR$  at  $Pr = 500$  for both CWT and CWHF boundary conditions is shown in Fig. 6.12. It is evident from Fig. 6.12 that  $(Bn_{max})_{cy}$  increases (decreases) with increasing  $Ra$  ( $AR$ ) and thus convection could be sustained up to high values of Bingham number for large (small) values of  $Ra$  ( $AR$ ). Figure 6.12 also shows that  $(Bn_{max})_{cy}$  assumes comparable values in both CWT and CWHF boundary conditions for  $Ra \leq 10^4$ . However,  $(Bn_{max})_{cy}$  with  $r_i/L$  in the CWT boundary condition is greater than in the case of CWHF boundary condition for the same set of values of  $AR$ ,  $Pr$  and  $r_i/L$  due to stronger convection induced by higher temperature difference between active walls in the CWT boundary condition (see Fig. 6.2). Furthermore, it can be seen from Fig. 6.12 that  $(Bn_{max})_{cy}$  increases with increasing  $r_i/L$  before approaching the corresponding value for the rectangular enclosure  $(Bn_{max})_{Rec}$  in the limit of  $r_i/L \rightarrow \infty$ . This indicates that convection strengthens with increasing  $r_i/L$  for Bingham fluids for both CWT and CWHF boundary conditions. It can be seen from Eq. 6.18 that  $(Bn_{max})_{cy}$  is expected to increase (decrease) with increasing  $Ra$  ( $AR$ ) for a given set of values of  $Pr$  and  $r_i/L$ , as can be observed from Fig. 6.12. Moreover,  $(Bn_{max})_{cy}$  is expected to increase with increasing  $r_i/L$  and asymptotically approaches  $(Bn_{max})_{Rec}$  (see Eq. 6.19) in the limit of  $r_i/L \rightarrow \infty$ , which are consistent with the findings of Fig. 6.12. The expression for  $(Bn_{max})_{cy}$  in the case of CWHF boundary condition cannot be analytically obtained from Eq. 6.6 but the same qualitative behaviour as that of the CWT boundary condition can be expected for the CWHF boundary condition.

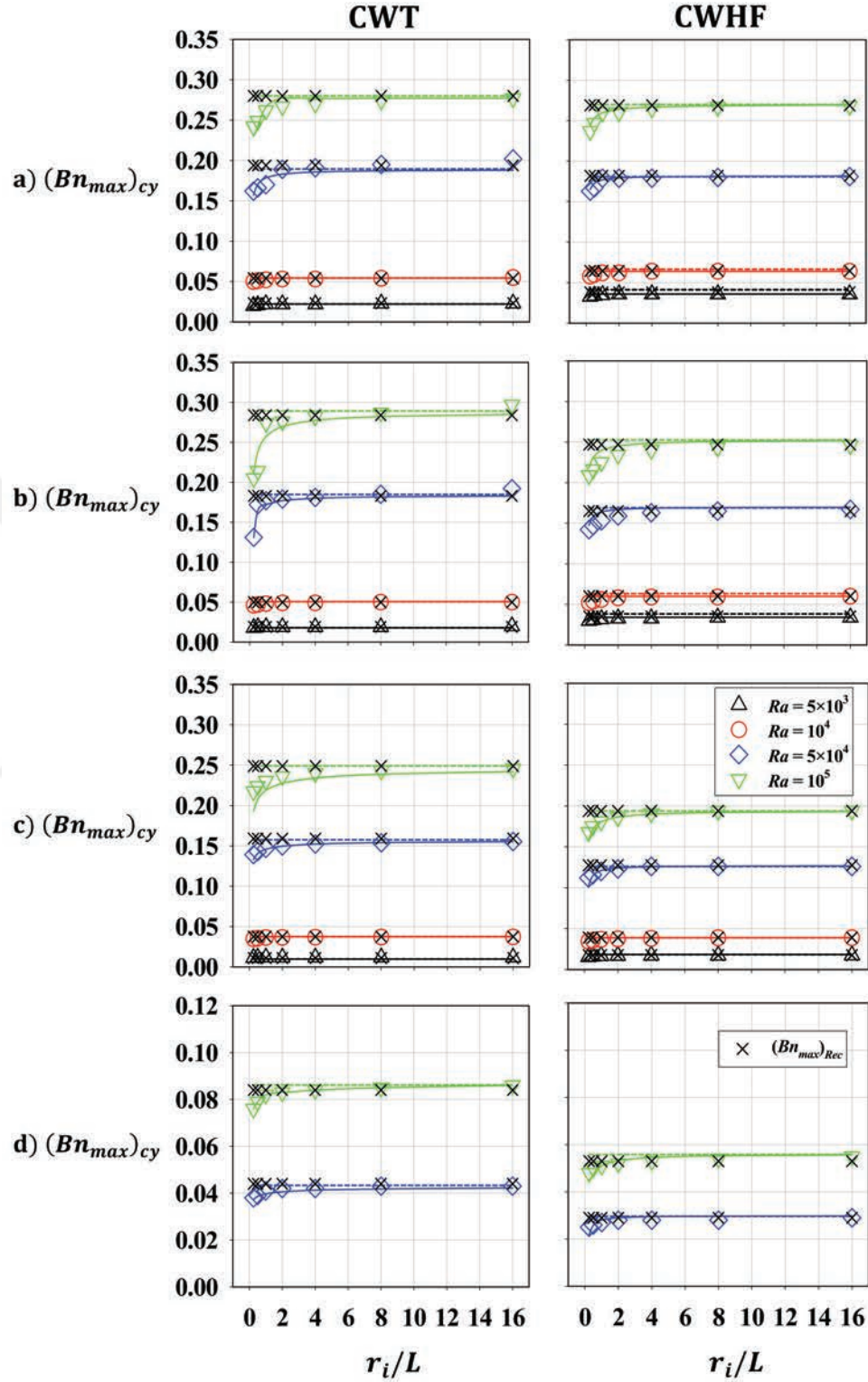


Figure 6.12: Variation of  $(Bn_{max})_{cy}$  with  $r_i/L$  for different values of  $Ra$  and  $AR$  (a) 0.25, (b) 0.5, (c) 1.0, (d) 2.0 at  $Pr = 500$  along with predictions of the Eq. 6.21 (solid lines) in the case of both CWT and CWHF boundary conditions. The corresponding values of the mean Nusselt number  $(Bn_{max})_{Rec}$  for rectangular enclosures is given by dashed lines (which is correlated in Eq. 6.20 by Ref. [77]).

The quantity  $(Bn_{max})_{Rec}$  is parameterised by Yigit and Chakraborty [77] in the following manner for  $10 \leq Pr \leq 500$ :

$$(Bn_{max})_{Rec} = [\{m_1/[1 + \exp\{(AR - x_1)/n_1\}]\} - y_1] |AR - 1|^{0.01} + Bn_{max}(AR = 1) \quad (6.20)$$

where  $Bn_{max}(AR = 1)$  is  $Bn_{max}$  for square enclosures (i.e.  $AR = 1$ ), and  $m_1$ ,  $x_1$ ,  $n_1$ ,  $y_1$  are the correlation parameters, which are listed in Table 6.2 for both CWT and CWHF boundary conditions. The predictions of the Eq. 6.20 can be easily seen in Fig. 6.13.

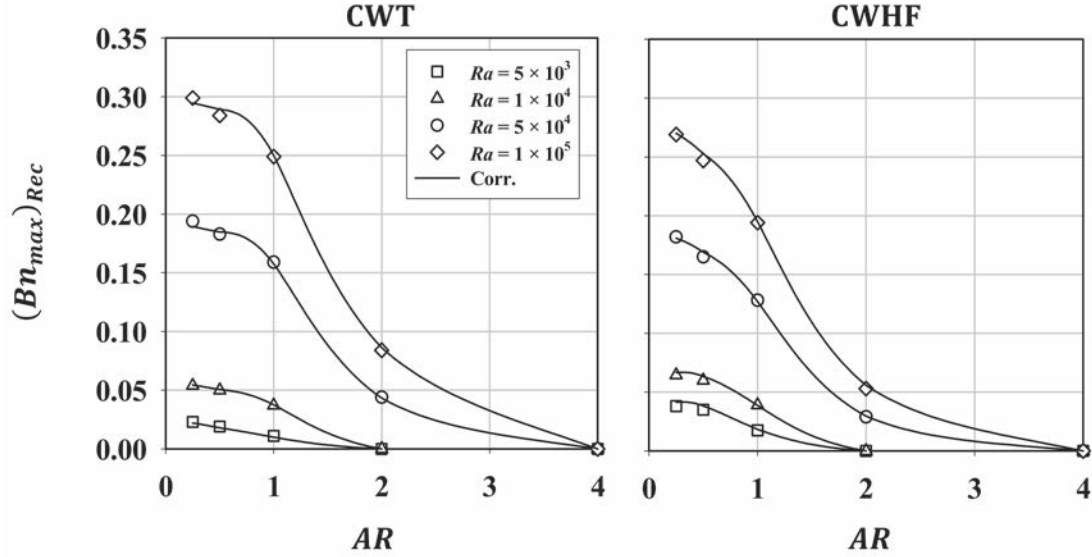


Figure 6.13: Variation of  $(Bn_{max})_{Rec}$  with  $AR$  for different values of  $Ra$  at  $Pr = 500$  along with the prediction of Eq. 6.20 in the case of both CWT and CWHF boundary conditions.

Table 6.2: Summary of  $(Bn_{max})_{Rec}$  correlation given by Eq. 6.20 for rectangular enclosures for both CWT and CWHF boundary conditions.

|                                                                                                      |                                                                                                                                                                                                                                                                                                                                                                |
|------------------------------------------------------------------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $(Bn_{max})_{Rec} = [\{m_1/[1 + \exp\{(AR - x_1)/n_1\}]\} - y_1]  AR - 1 ^{0.01} + Bn_{max}(AR = 1)$ |                                                                                                                                                                                                                                                                                                                                                                |
| CWT                                                                                                  | $Bn_{max}(AR = 1) =$ $[\{(0.00048Pr + 0.132)/(Pr + 18.15)\} Ra^{0.525}] -$ $[(0.0389Pr + 7.687)/(Pr + 13.56)]$ $x_1 = 1.686 - 1.154 \exp[-0.5 \{(\ln Ra - 9.04)/0.841\}^2]$ $y_1 = -0.033 + 1.105/[1 + \exp\{(13.04 - \ln Ra)/1.417\}]$ $n_1 = 0.354 - 0.43 \exp[-0.5 \{(\ln Ra - 9.01)/0.282\}^2]$ $m_1 = -0.01 + 0.804/[1 + \exp\{(12.07 - \ln Ra)/1.158\}]$ |
| CWHF                                                                                                 | $Bn_{max}(AR = 1) =$ $[\{(Pr + 236)/(2707Pr + 40280)\} Ra^{0.525}] -$ $[(Pr + 128)/(44.6Pr + 271)]$ $x_1 = -34.35 + 35.72/[1 + \exp\{(0.819 - \ln Ra)/1.733\}]$ $y_1 = -0.0337 + 7.967/[1 + \exp\{(18.52 - \ln Ra)/1.986\}]$ $n_1 = 0.214 + 0.266/[1 + \exp\{(10.2 - \ln Ra)/0.398\}]$ $m_1 = 0.0014 + 1.034/[1 + \exp\{(12.7 - \ln Ra)/1.315\}]$              |

Equation 6.20 is consistent with the scaling estimate presented in Eq. 23ii. Using Eq. 6.18,  $(Bn_{max})_{cy}$  is correlated here in the following manner:

$$(Bn_{max})_{cy} = (Bn_{max})_{Rec} / \left[ 1 + a_o(L/r_i)^{b_o} \right] \quad (6.21)$$

where  $a_o = [0.088 \ln Ra - 0.872] AR^{0.23}$  ( $a_o = 0.042 \ln Ra - 0.406$ )  $AR^{0.39}$ ) and

$$b_o = [(0.745 \ln Ra - 6.98) / (1 + (1.72 \ln Ra - 17.8) AR)]$$

( $b_o = (6.269 - 0.474 \ln Ra) AR^{0.15}$ ) for CWT (CWHF) boundary conditions.

The predictions of Eq. 6.21 are shown in Fig. 6.12 which shows that this correlation satisfactorily predicts ( $R^2 = 0.99$ )  $(Bn_{max})_{cy}$  for Bingham fluids in rectangular cross-sectional cylindrical enclosures for  $10^3 \leq Ra \leq 10^5$ ,  $0.25 \leq AR \leq 4$ ,  $0 \leq r_i/L \leq 16$  at  $Pr = 500$ . It is worth noting that  $(Bn_{max})_{cy}$  becomes equal to  $(Bn_{max})_{Rec}$  in the limit of  $r_i/L \rightarrow \infty$  which ensures that asymptotic condition  $(Bn_{max})_{cy} = (Bn_{max})_{Rec}$  for  $r_i/L \rightarrow \infty$  is satisfied.

A correlation for  $\overline{Nu}_{Rec}$  was proposed in [76, 77] based on scaling relation of  $\overline{Nu}_{Rec}$  (Eq. 6.11 in the limit of  $r_i/L \rightarrow \infty$ ) for natural convection of Bingham fluids in rectangular enclosures in the range  $0 \leq Bn \leq Bn_{max}$ . The same methodology of [76, 77] has been utilised here to propose a correlation of  $\overline{Nu}_{cy}$  for  $0 \leq Bn \leq (Bn_{max})_{cy}$  in the following manner for the CWT boundary condition:

$$\frac{\overline{Nu}_{cy} - 1}{(\overline{Nu}_{cy})_{Bn=0} - 1} = \frac{2 \left[ 1 - (Bn^*/Bn_{max}^*)^b \right]^c}{Bn^* + \sqrt{Bn^{*2} + 4}} \quad \overline{Nu}_{cy} = 1 \quad \text{when} \quad (\overline{Nu}_{cy})_{Bn=0} = 1 \quad (6.22)$$

where  $Bn_{max}^* = Bn_{max} AR (Ra/Pr)^{-1/4} (1 + L/r_i)^{1.5}$  and  $b$  and  $c$  are the correlation parameters. Following Refs. [76, 77], the functional forms given by Eq. 6.22 are taken to be valid also for the CWHF boundary condition. Yigit and Chakraborty [77] proposed  $b_{Rec}$  and  $c_{Rec}$  in the following manner for both CWT and CWHF

boundary conditions:

$$b_{Rec} = b(AR = 1) \quad c_{Rec} = c(AR = 1) - m_2 \exp \left[ -0.5 \{ (AR - 0.366) / n_2 \}^2 \right] |AR - 1| \quad (6.23)$$

where  $m_2, n_2$  are the correlation parameters which are listed in Table 6.3 for both CWT and CWHF boundary conditions. The predictions of 6.22 and 6.23 can be seen in Fig. 6.14.

Table 6.3: Summary of  $b$  and  $c$  correlation given by Eq. 6.23 for rectangular enclosures for both CWT and CWHF boundary conditions.

|                                                                                                                    |                                                                                                                                                                                                  |
|--------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $b_{Rec} = b(AR = 1) \quad c_{Rec} = c(AR = 1) - m_2 \exp \left[ -0.5 \{ (AR - 0.366) / n_2 \}^2 \right]  AR - 1 $ |                                                                                                                                                                                                  |
| CWT                                                                                                                | $b(AR = 1) = 0.6; c(AR = 1) = 0.025 Ra^{0.171} Pr^{0.095}$<br>$m_2 = 0.03 + 0.0334 / [1 + \exp \{ (9.48 - \ln Ra) / 0.06 \}]$<br>$n_2 = 0.09 + 0.032 / [1 + \exp \{ (\ln Ra - 9.13) / 0.083 \}]$ |
| CWHF                                                                                                               | $b(AR = 1) = 0.75; c(AR = 1) = 0.0818 Ra^{0.1019} Pr^{0.054}$<br>$m_2 = -2091 + 28070 / [1 + \exp \{ (9.419 - \ln Ra) / 0.583 \}]$<br>$n_2 = 0.024$                                              |

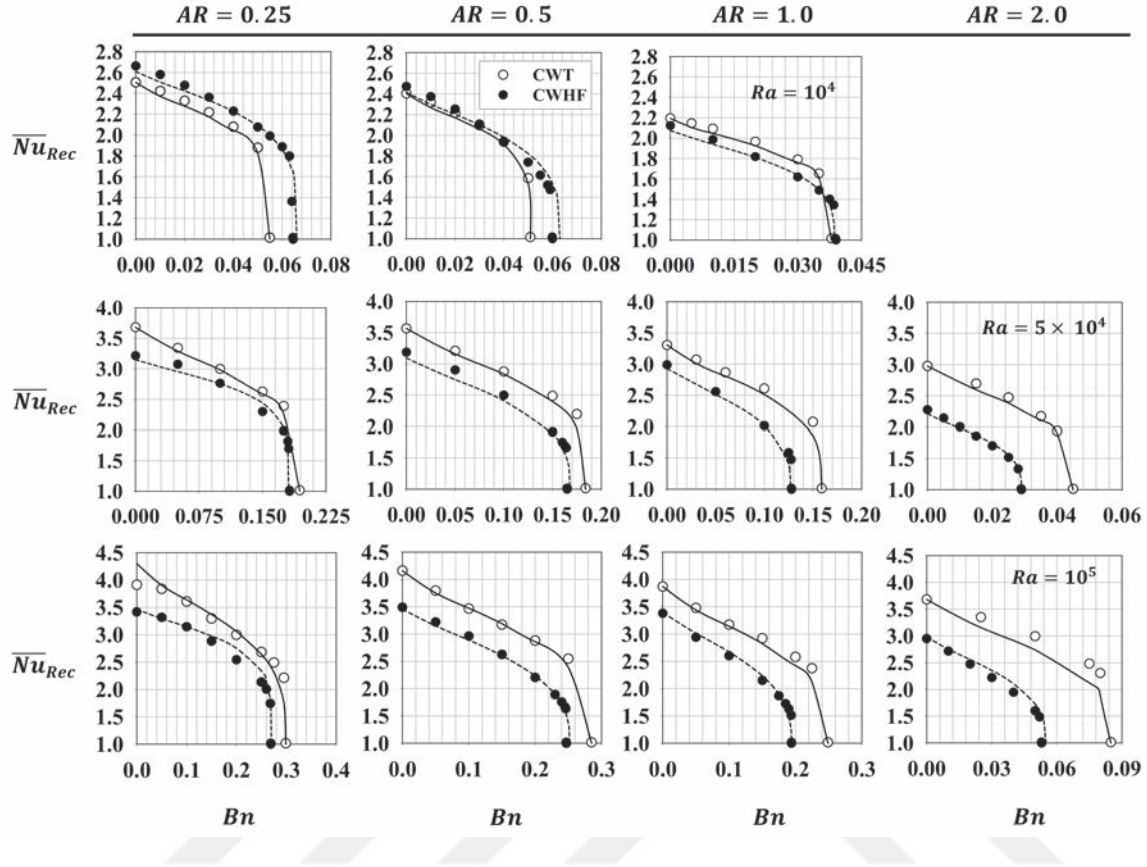


Figure 6.14: Variation of  $\overline{Nu}_{Rec}$  with  $Bn$  for different values of  $Ra$  and  $AR$  at  $Pr = 500$  along with the prediction of Eq. 6.22 and 6.23 in the case of both CWT and CWHF boundary condition.

Accordingly, in the current study,  $b$  and  $c$  have been parameterized in the range of  $0.25 \leq r_i/L \leq 16$ ,  $0.25 \leq AR \leq 2$  and  $5 \times 10^4 \leq Ra \leq 10^5$  for the CWHF boundary condition in the following manner:

$$b_{cy} = b_{Rec} \quad c_{cy} = c_{Rec} / [1 + m_3(L/r_i)^{n_3}] \quad (6.24)$$

$m_3 = 0.4 / [1 + \exp\{(AR - 0.75) / 0.136\}]$  and  $n_3 = 0.41 \exp(1.4AR)$  for CWHF boundary condition. This range is determined by the parameter space where the variation of  $\overline{Nu}_{cy}$  with  $Bn$  is monotonic and a single cell convection pattern is observed. It is worth noting that the changes in the flow pattern causes non-

monotonic variation of  $\overline{Nu}_{cy}$  with  $Bn$  in the CWT boundary condition for  $AR \leq 1$ ,  $Ra \geq 5 \times 10^4$ . As a result, a correlation for  $\overline{Nu}_{cy}$  would be valid for very limited parameter range. Therefore, a correlation for  $\overline{Nu}_{cy}$  has not been proposed for the CWT boundary condition in this analysis.

The predictions of Eq. 6.22 are shown in Fig. 6.15 which shows that this correlation satisfactorily predicts ( $R^2 = 0.99$  and 3% maximum percentage error)  $\overline{Nu}_{cy}$  for natural convection of Bingham fluids in rectangular-cross sectional cylindrical enclosures for CWHF boundary condition. It is also worth noting that  $c_{cy}$  becomes equal to  $c_{Rec}$  in the limit of  $r_i/L \rightarrow \infty$ , which ensures that asymptotic condition  $\overline{Nu}_{cy} = \overline{Nu}_{Rec}$  for  $r_i/L \rightarrow \infty$ , is satisfied by Eq. 6.24.

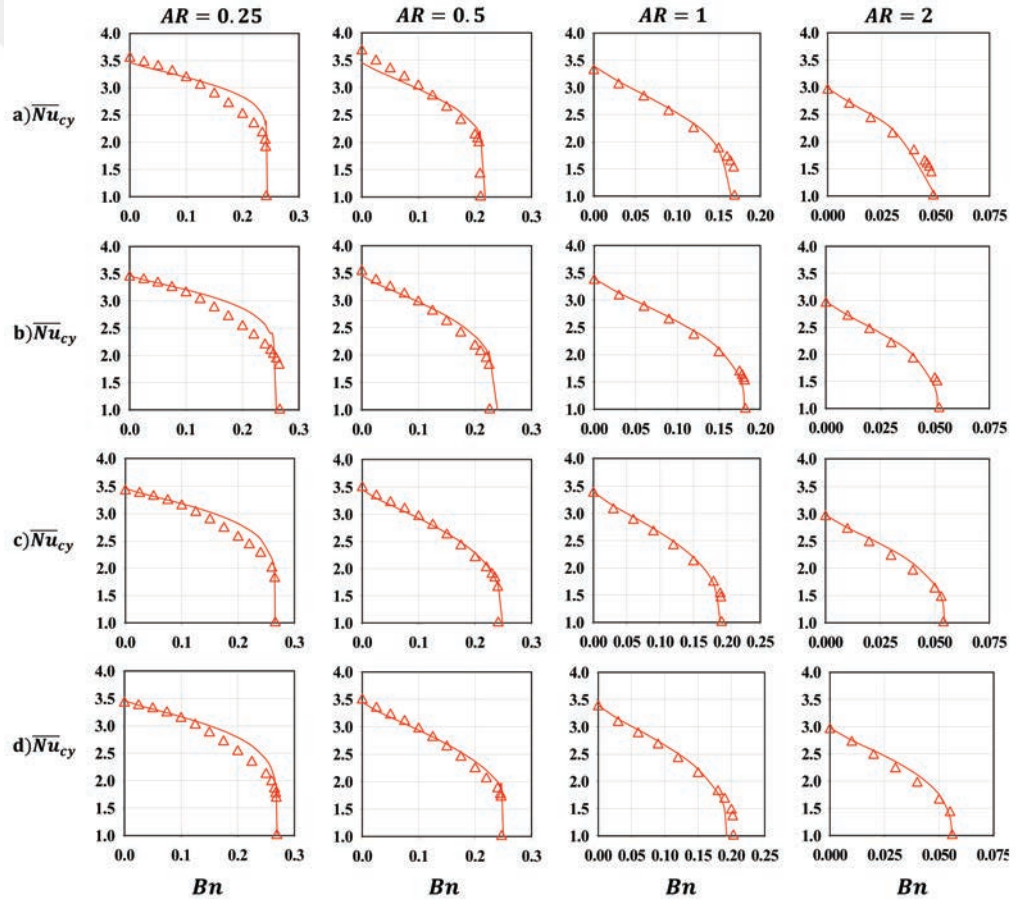


Figure 6.15: Variation of  $\overline{Nu}_{cy}$  with  $Bn$  for different values of  $AR$  and  $r_i/L$  (a) 0.25, (b) 1.0, (c) 4.0, (d) 16.0 at  $Ra = 10^5$  and  $Pr = 500$  along with the prediction of Eq. 6.22 and 6.24 (solid line) in the case of CWHF boundary condition.

## 6.7 Onset of Convection of Bingham Fluids

It is well known that a critical Rayleigh number  $Ra_{crit}$  needs to be exceeded to initiate the fluid motion in the case of Rayleigh-Bénard convection. The effects of yield stress on the critical Rayleigh number for laminar Rayleigh-Bénard convection of Bingham fluids in rectangular cross-sectional cylindrical annular enclosures have been analysed in this section. It is worth noting that Bingham fluid flow is unconditionally stable under the quiescent flow initial condition [70], and thus the steady-state solutions for Newtonian (i.e.  $Bn = 0$ ) fluids are used as the initial conditions for Bingham fluid simulations in the Rayleigh-Bénard convection boundary condition. Therefore, the values of  $Ra_{crit}$  are evaluated by reducing  $Ra$  in steps of 50 from an established flow condition until  $\overline{Nu}_{cy}$  reduces to 1.00 (i.e. indicating thermal conduction-driven transport) for a given set of values of  $Bn$ ,  $Pr$ ,  $AR$  and  $r_i/L$ . Table 6.4 lists the nominal values of Rayleigh number  $Ra$ , above which  $\overline{Nu}_{cy}$  values deviate from unity, which has been taken for the critical Rayleigh number  $Ra_{crit}$  for the onset of convection in this analysis for Bingham fluid case (i.e.  $Bn = 0.02$ ) for different values of  $AR$  and  $r_i/L$  at  $Pr = 500$ . Table 6.4 shows that  $Ra_{crit}$  decreases with decreasing  $AR$ , however,  $Ra_{crit}$  values are found to independent of  $r_i/L$ . Thus, effects of yield stress (i.e. Bingham number) has been analysed for different values of  $AR$ ,  $Pr$  in rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) for the current analysis.

Table 6.4: Values of critical Rayleigh number  $Ra_{crit}$  for different values of  $AR$  and  $r_i/L$  for both CWT and CWHF boundary conditions at  $Bn = 0.02$  and  $Pr = 500$ .

|         | CWT        |          |          | CWHF       |          |          |
|---------|------------|----------|----------|------------|----------|----------|
| $r_i/L$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ |
| 0       | 5050       | 6500     | 29500    | 3050       | 5500     | 38000    |
| 1       | 5050       | 6500     | 29500    | 3050       | 5500     | 38000    |
| 16      | 5050       | 6500     | 29500    | 3050       | 5500     | 38000    |
| Rec.    | 5050       | 6500     | 29500    | 3050       | 5500     | 38000    |

Accordingly, using Eqs. 6.2, 6.3 and 6.5, 6.6 in Eq. 6.4 for rectangular enclosures

(i.e.  $r_i/L \rightarrow \infty$ ),  $\mu_{eff}$  can be estimated, which provides an estimation of effective Rayleigh number (i.e.  $Ra_{eff} = \rho g \beta (T_H - T_C) H^3 / \mu_{eff} \alpha$ ) as:

$$Ra_{CWT,eff} = \frac{Ra}{1 + 0.5 Bn AR (Pr/Ra)^{1/2} \left[ Bn AR \sqrt{(Bn AR)^2 + 4 (Ra/Pr)^{1/2}} \right]} \quad (6.25)$$

$$Ra_{CWHF,eff} = \frac{Ra}{1 + (Bn AR) f_3^{1/2} (\delta_1/H)^{1/2}} \quad (6.26)$$

The variation of  $\overline{Nu}_{Rec}$  with  $Ra$  is shown in Fig. 6.16 for both CWT and CWHF boundary conditions at  $Pr = 10^2$  and  $AR = 0.5$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.02$ ) fluids. In the case of Newtonian fluids (i.e.  $Bn = 0$ ), the instability is known to be super-critical since  $\overline{Nu}_{Rec}$  decreases gradually with decreasing  $Ra$  and eventually settles to 1.00 indicating a purely conduction-driven transport. For  $Ra \sim Ra_{crit}$ , it is possible to equate the order of magnitudes of the advective and diffusive terms of the energy transport equation to obtain:  $q/H \sim \rho c_p w \Delta T / H$ , which leads to  $h = q / \Delta T \sim kw / \alpha$  or  $Nu = hH/k \sim \sqrt{g\beta\Delta TH^3/\alpha}$ . Thus, the mean for Newtonian fluids is expected to scale as:  $Nu \sim \sqrt{g\beta\Delta TH^3/\alpha} \sim \sqrt{RaPr}$  ( $Nu \sim \sqrt{g\beta\Delta TH^3/\alpha} \sim \sqrt{RaPr} \times \sqrt{\delta_{th}/H}$ ) in the case of CWT (CWHF) boundary condition for  $Ra \sim Ra_{crit}$ .

It can be seen from Fig. 6.16 that  $\overline{Nu}$  varies as  $Ra^{0.5}$  ( $Ra^{0.35}$ ) for  $Ra \sim Ra_{crit}$  (represented by green lines in Fig. 6.16) in the case of CWT (CWHF) boundary conditions. In the case of  $Ra > Ra_{crit}$  but for  $Ra \approx Ra_{crit}$  one can further simplify  $\overline{Nu} \sim \sqrt{Ra/Pr}$  using  $Ra = Ra_{crit} + x$  (where  $x = (Ra - Ra_{crit}) \ll Ra_{crit}$ ), which upon series expansion yields:  $\overline{Nu} \sim 0.5 [Ra_{crit}^{0.5} + Ra/Ra_{crit}] \sqrt{Pr}$ . It can be seen from Fig. 6.16 that  $\overline{Nu}_{Rec}$  in Newtonian fluids indeed shows a linear variation (shown by black solid lines in Fig. 6.16) with  $Ra$  close to the critical condition for CWT boundary condition. A similar linear variation is observed for CWHF boundary condition, which can be explained by a series expansion as done for CWT boundary condition. For an established boundary layer transport regime, substituting  $Bn = 0$  in Eq. 6.5 for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ), and using  $\overline{Nu} \sim qH/\Delta Tk \sim H/\delta_{th}$  lead to:  $\overline{Nu} \sim (Ra/Pr)^{0.25} f_3$  for CWT

boundary condition, which is consistent with the behaviour shown by the dashed line, which is given by the mean Nusselt number correlation (see Table 6.1 for its mathematical expression) proposed by Yigit et al. [76, 77].

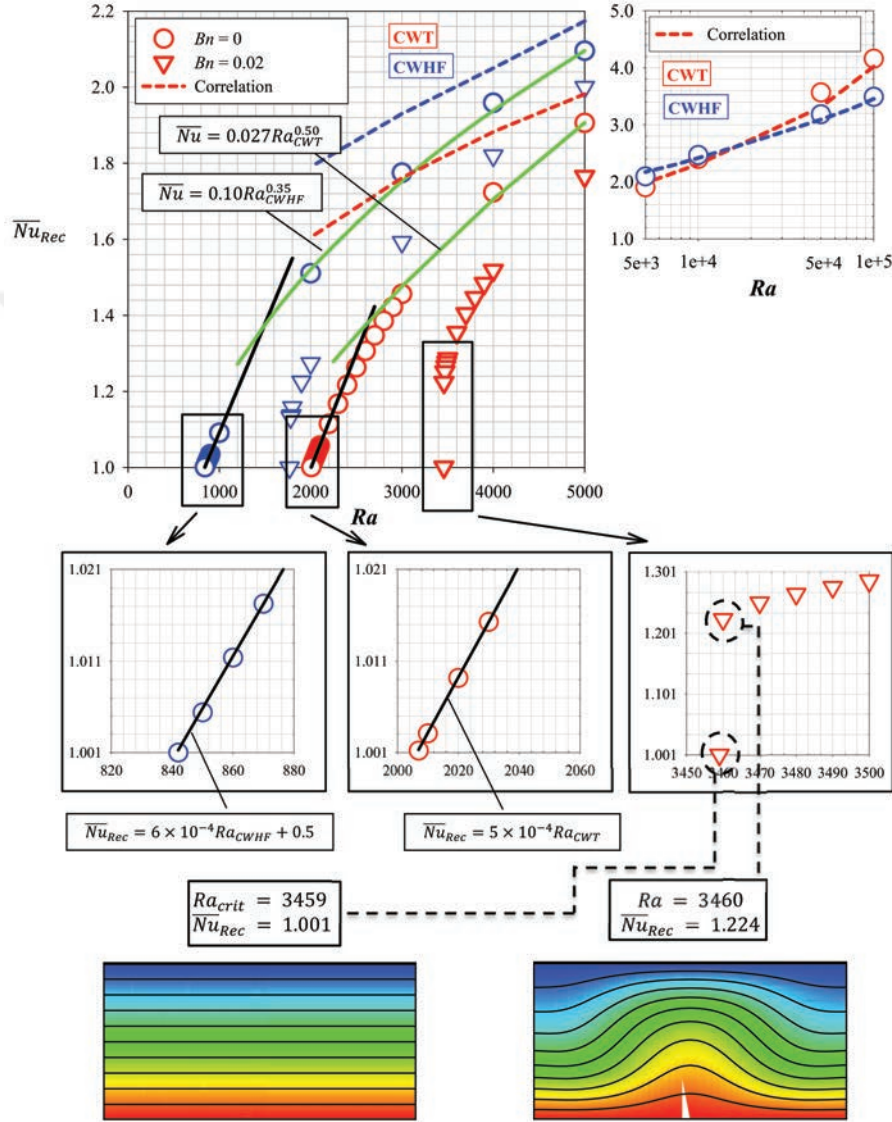


Figure 6.16: The variation of  $\overline{Nu}_{Rec}$  with  $Ra$  for both CWT and CWHF boundary conditions at  $Pr = 10^2$  and  $AR = 0.5$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham (i.e.  $Bn = 0.02$ ) fluids and the temperature isotherms for  $Ra_{crit} = 3459$  and for  $Ra_{crit} = 3460$  Bingham (i.e.  $Bn = 0.02$ ) fluid case for CWT boundary condition.

For CWHF boundary condition,  $\delta_{th}/H$  is expected to be of the order of unity (i.e.  $\delta_{th}/H \sim 1$ ) for  $Ra \approx Ra_{crit}$ , and thus the mean Nusselt number for Newtonian fluids, shows a linear variation with  $Ra$  before showing a power-law in terms of  $Ra$  with a positive exponent smaller than unity (i.e.  $\overline{Nu} \sim Ra^{0.35}$ ), as it can be seen in Fig. 6.16. It is worth noting that the mean Nusselt number scales as  $\overline{Nu} \sim \sqrt{RaPr} \times \sqrt{\delta_{th}/H} \sim Ra^{0.35}$  instead of  $\overline{Nu} \sim Ra^{0.5}$  for  $Ra \sim Ra_{crit}$  in the CWHF case since  $\delta_{th}/H$  is also dependent on  $Ra$ . Putting  $Bn = 0$  in Eq. 6.6 for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) reveals that  $H/\delta_1 \sim (Ra/Pr)^{0.2} f_3^{0.2}$  for the boundary layer transport for well-established Rayleigh- Bénard convection (i.e.  $Ra \gg Ra_{crit}$ ), which leads to  $\overline{Nu} \sim H/\delta_{th} \sim (Ra/Pr)^{0.2} f_3^{0.8}$ , which is consistent with the correlation proposed by Yigit and Chakraborty [77], as summarised in Table 6.1.

In contrast to the behaviour of Newtonian fluids, the variation of  $\overline{Nu}$  with  $Ra$  for a representative Bingham fluid case exhibits a rapid drop of  $\overline{Nu}$  to 1.0 at  $Ra = Ra_{crit}$  in Fig. 6.16 after a gradual decrease with decreasing  $Ra$ , regardless of the boundary condition. At  $\overline{Nu} = 1.0$ , the fluid flow ceases to influence the thermal transport and the isotherms become parallel to the horizontal walls indicating conduction-dominated thermal transport. Here, the instability is found to be sub-critical for Bingham fluids, since thermal convection ceases abruptly once Rayleigh number becomes smaller than a threshold critical value (i.e.  $Ra \approx Ra_{crit}$ ).

### 6.7.1 Effects of Bingham Number ( $Bn$ ) on $Ra_{crit}$

The variations of  $Ra_{crit}$  with  $Bn$  are shown in Fig. 6.17 for different values of  $Pr$  and  $AR$  for both CWT and CWHF boundary conditions. It can be seen from Fig. 6.17 that  $Ra_{crit}$  increases with increasing  $Bn$ , regardless of the boundary condition. The viscous flow resistance in comparison to buoyancy force strengthens with increasing  $Bn$  and as a result higher values of  $Ra_{crit}$  are needed to overcome the viscous forces to initiate flow within the enclosure. This statement can be supported by scaling estimations given by Eqs. 6.25 and 6.26, which indicate that  $Ra_{CWT,eff}$  and  $Ra_{CWHF,eff}$  decrease with increasing  $Bn$  as a result of the weakening of advective transport. Thus, higher values of  $Ra$  in the Bingham fluid

cases are needed than the corresponding Newtonian fluid cases in order to have values of  $Ra_{eff}$  which allow for convection within the enclosure. Furthermore, the values of  $Ra_{crit}$  for Bingham fluids in the case of CWT boundary condition has been found to be greater than those in the CWHF boundary condition for  $AR \leq 1$ , whereas an opposite trend has been obtained for  $AR > 1$ .

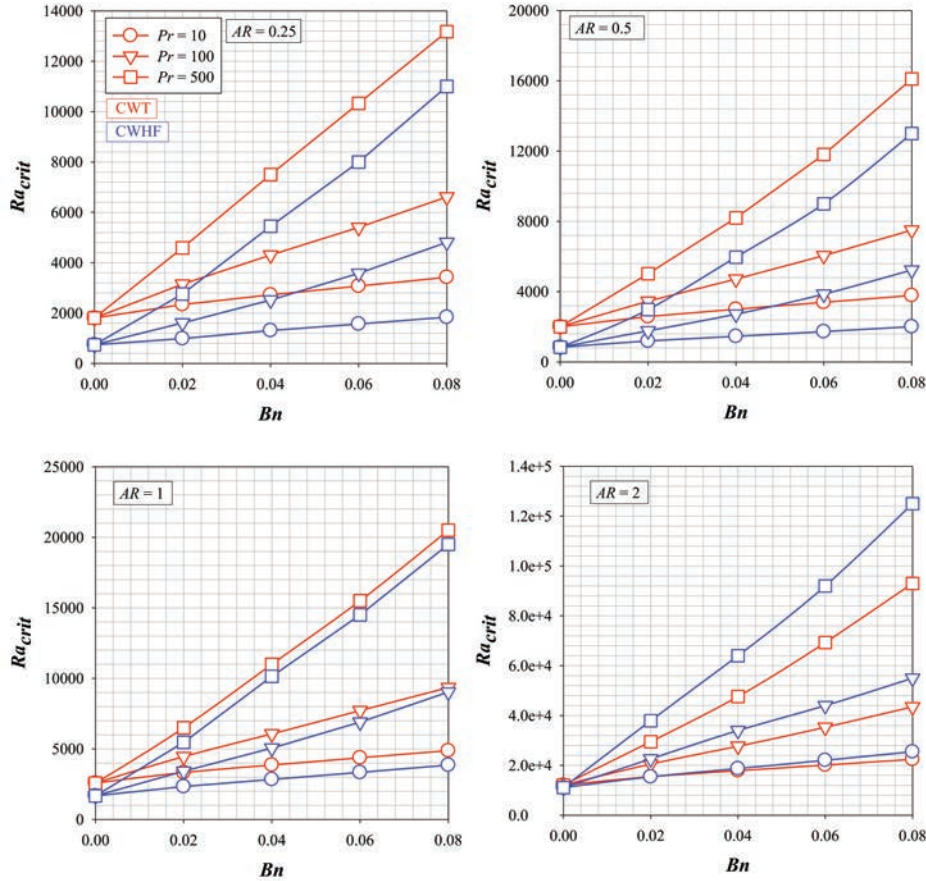


Figure 6.17: The variations of  $Ra_{crit}$  with  $Bn$  for different values of  $Pr$  and  $AR$  for both CWT and CWHF boundary conditions..

It is also worth noting that above scaling estimations are only valid when convection is well-established but useful insights can still be extracted regarding the relative influences of advection (i.e.  $\dot{Q}_{conv}$ ) and diffusion (i.e.  $\dot{Q}_{cond}$ ) on overall thermal transport. It is not possible to express Eq. 6.6 analytically, but useful limiting conditions can be inferred from the scaling relations (i.e. Eqs. 6.12-

6.16). For large values of  $(BnAR)$  for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ), Eq. 6.6 becomes:  $\delta_1/H \sim (BnAR)^{1/2}(Pr/Ra)^{1/4}f_3^{1/2}$ . This provides an estimation of the maximum magnitude of  $\dot{Q}_{cond}$  in the case of CWHF boundary condition as  $\dot{Q}_{cond} \sim qH(Pr/Ra)^{1/4}Bn^{1/2}AR^{3/2}f_3^{-1}$ . This scaling estimation shows that  $\dot{Q}_{cond}$  strengthens with increasing  $AR$  and  $Bn$  when  $q$ ,  $Ra$  and  $Pr$  are held constant for the CWHF boundary condition. Similarly,  $\dot{Q}_{cond}$  strengthens with increasing  $AR$  (see Eq. 6.14 for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ )) in the case of CWT boundary condition. By contrast, the contribution of  $\dot{Q}_{conv}$  strengthens with decreasing  $AR$  for both boundary conditions. A comparison between  $\dot{Q}_{cond} \sim qH(Pr/Ra)^{1/4}Bn^{1/2}AR^{3/2}f_3^{-1}$  (for CWHF) and  $\dot{Q}_{cond} \sim (k\Delta T)AR$  (for CWT) reveals that the augmentation of diffusive transport with increasing  $Bn$  is stronger in the CWHF boundary condition than in the CWT boundary condition and this trend strengthens (weakens) with increasing  $AR$  for  $AR > 1$  ( $AR < 1$ ), as can be substantiated from Fig. 6.17. The above scaling relations also suggest that a greater (smaller) value of  $Ra_{crit}$  is needed to overcome the diffusive transport for the CWT boundary condition than in the case of CWHF boundary condition for  $AR > 1$  ( $AR < 1$ ).

### 6.7.2 Effects of Prandtl Number ( $Pr$ ) on $Ra_{crit}$

The effects of  $Pr$  on  $Ra_{crit}$  for both CWT and CWHF boundary conditions are shown in Fig. 6.18 at different values of  $Bn$  and  $AR$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluid cases. It is evident from Fig. 6.18 that  $Ra_{crit}$  has been found to be independent to the variation of  $Pr$  for Newtonian fluids (i.e.  $Bn = 0$ ), whereas  $Ra_{crit}$  increases with increasing  $Pr$  for Bingham fluids for both CWT and CWHF boundary conditions. Thermal boundary layer is much thinner than the hydrodynamic boundary layer thickness for  $Pr \gg 1$  for Newtonian fluids (i.e.  $Bn = 0$ ) and thus a change in  $Pr$  modifies the relative balance between the buoyancy and viscous forces, and does not affect the thermal transport within the thermal boundary layer [2]. Therefore,  $Ra_{crit}$  remains unaffected by Prandtl number  $Pr$  for Newtonian fluids. On the other hand, for Bingham fluids, the scaling predictions for  $Ra_{eff}$  given by Eqs. 6.25 and 6.26 (note that  $f_3$  is expected to increase with increasing  $Pr$ ) indicate that both  $Ra_{CWT,eff}$  and  $Ra_{CWHF,eff}$

decrease with increasing  $Pr$  for a given set of values of  $Bn$  and  $AR$ .

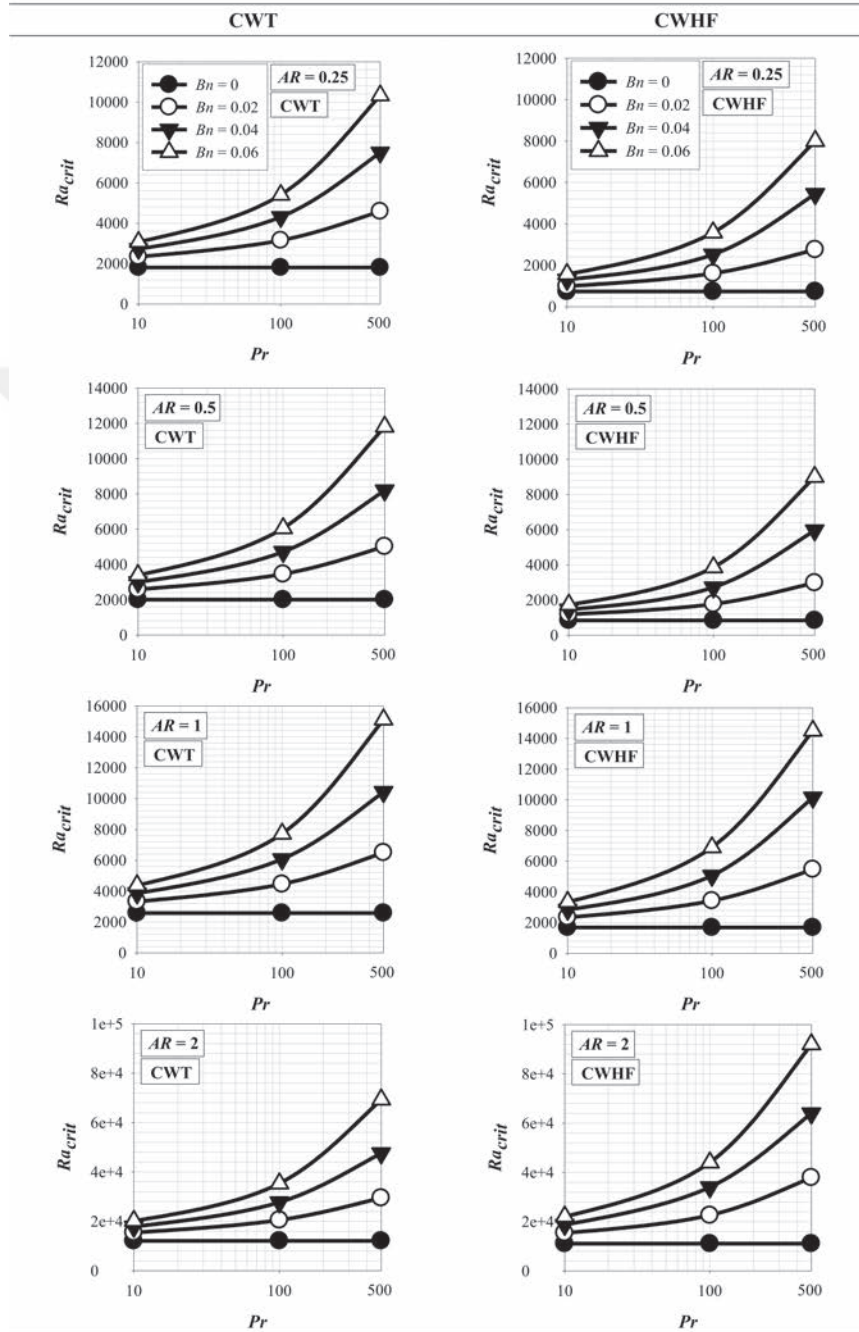


Figure 6.18: The variations of  $Ra_{crit}$  with  $Pr$  for different values of  $Bn$  and  $AR$  for both CWT and CWHF boundary conditions..

This is an evidence of that the relative strength of buoyancy force in comparison to viscous force weakens with increasing  $Pr$ . Accordingly, this weakening of buoyancy force with respect to viscous force leads to an increase in  $Ra_{crit}$  with increasing  $Pr$  for Bingham fluids for both CWT and CWHF boundary conditions.

### 6.7.3 Effects of Aspect Ratio ( $AR$ ) on $Ra_{crit}$

The influence of  $AR$  on  $Ra_{crit}$  has been shown in Fig. 6.19 for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluid cases for both CWT and CWHF boundary conditions. It can be seen from Fig. 6.19 that  $Ra_{crit}$  increases with increasing  $AR$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids, regardless of the boundary condition. This can be seen as an indicator of weakening of advective transport with increasing  $AR$ . This can be explained by the scaling estimations of  $\dot{Q}_{conv}$  and  $\dot{Q}_{cond}$  given by Eqs. 6.13 - 6.15 for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) as it is shown in Eqs. 6.27 - 6.30.

$$\dot{Q}_{conv,CWT} \sim 0.5 (k\Delta T) Pr \left[ Bn + \sqrt{Bn^2 + 4 (Ra/Pr)^{1/2} / AR^2} \right] \quad (6.27)$$

$$\dot{Q}_{conv,CWHF} \sim qH \sqrt{RaPr} (\delta_1/H)^{5/2} AR^{-1} f_3^{-3/2} \quad (6.28)$$

$$\dot{Q}_{cond,CWT} \sim (k\Delta T) AR \quad (6.29)$$

$$\dot{Q}_{cond,CWHF} \sim qH (\delta_1/H) AR f_3^{-1} \quad (6.30)$$

Equations. 6.27 - 6.30 indicate that  $\dot{Q}_{conv}$  ( $\dot{Q}_{cond}$ ) weakens (strengthens) with increasing  $AR$  for both Newtonian and Bingham fluids in the case of both CWT and CWHF boundary conditions. For this reason, the value of  $Ra_{crit}$  increases with increasing  $AR$  for both Newtonian and Bingham fluids (see Fig. 6.19), regardless of the boundary condition. The observation for Bingham fluids in Fig. 6.19 can further be explained by the scaling estimation of  $Ra_{eff}$  (i.e. Eqs. 6.25 and 6.26). It is expected from Eqs. 6.25 and 6.26 that  $Ra_{CWT,eff}$  and  $Ra_{CWHF,eff}$  decrease

with increasing  $AR$ , which indicates that the advective transport weakens with increasing  $AR$ .

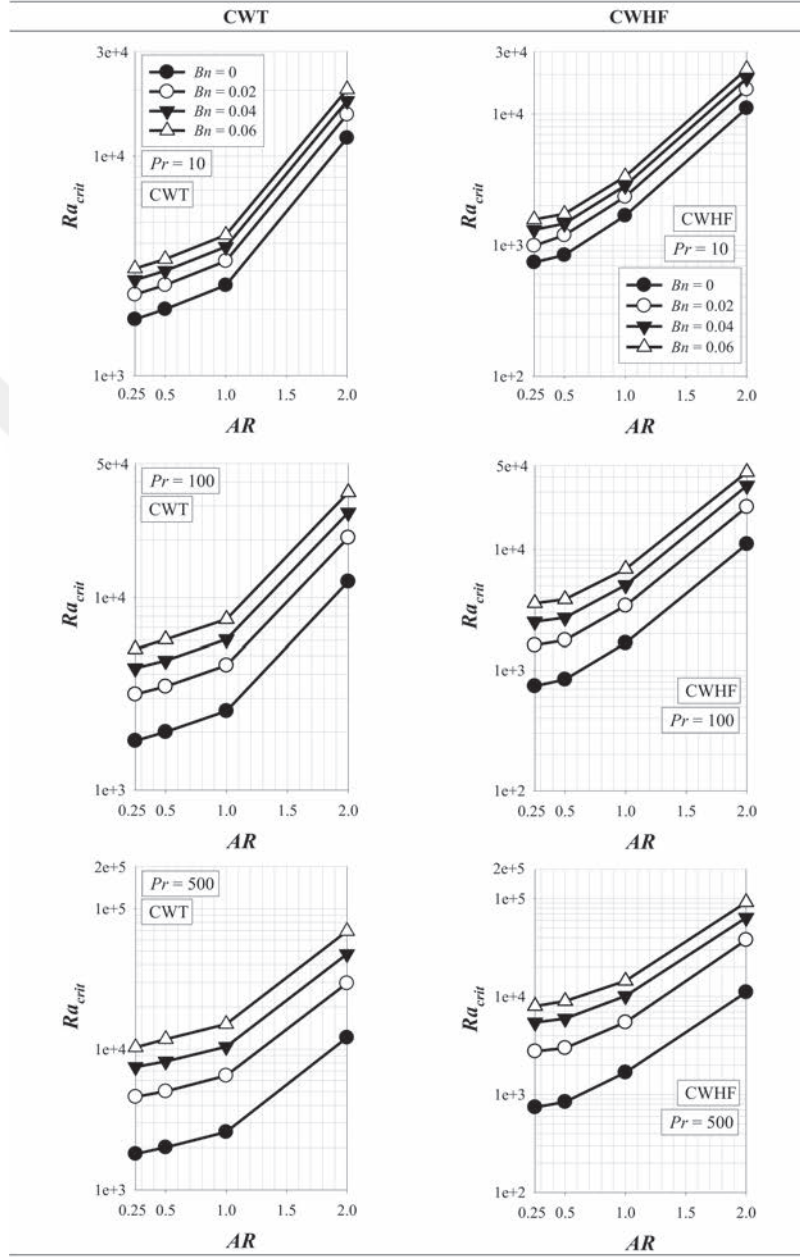


Figure 6.19: The variations of  $Ra_{crit}$  with  $AR$  for different values of  $Bn$  and  $Pr$  for both CWT and CWHF boundary conditions..

Therefore, the convection starts to play a significant role in the overall thermal

transport at a greater value of  $Ra_{crit}$  for higher  $AR$  for both CWT and CWHF boundary conditions.

#### 6.7.4 Parameterisation of Critical Rayleigh Number $Ra_{crit}$

The value of effective Rayleigh number for the critical Rayleigh number  $Ra_{crit}$  can be scaled using Eq. 6.25 as:

$$Ra_{eff} = \frac{Ra_{crit}}{1 + 0.5BnAR (Pr/Ra_{crit})^{1/2} \left[ BnAR \sqrt{(BnAR)^2 + 4 (Ra_{crit}/Pr)^{1/2}} \right]} \quad (6.31)$$

It is worth noting that Eq. 6.26 indicates that an exact analytical solution does not exist for CWHF boundary condition. However, the qualitative trend is expected to be the same for both CWT and CWHF boundary conditions, thus, Eq. 6.31 can be used  $Ra_{crit}$  scaling for both CWT and CWHF boundary conditions. Eq. 6.31 can be rewritten in the following manner:

$$\begin{aligned} [(Ra_{crit}/Ra_{eff}) - 1] &\sim 1 + 0.5BnAR (Pr/Ra_{crit})^{1/2} \\ &\times \left[ BnAR \sqrt{(BnAR)^2 + 4 (Ra_{crit}/Pr)^{1/2}} \right] \end{aligned} \quad (6.32)$$

Obtaining an expression for  $[(Ra_{crit}/Ra_{eff}) - 1] (Ra_{crit}/Pr)^{0.5} (2/BnAR)$  from Eq. 6.32 and squaring it leads to:

$$\left[ \frac{(Ra_{crit}/Ra_{eff})}{(Ra_{eff})^{1/2} (Ra_{crit})^{1/4}} \right] \sim BnAR Pr^{1/4} \quad (6.33)$$

Eq. 6.33 can be recast in the following manner:

$$\left[ \frac{(Ra_{crit}/(Ra_{crit})_{Bn=0})}{((Ra_{crit})_{Bn=0})^{1/2} (Ra_{crit})^{1/4}} \right] \sim BnARPr^{1/4} \frac{Ra_{eff}^{1/2}}{((Ra_{crit})_{Bn=0})^{1/2}} + \frac{Ra_{eff} - (Ra_{crit})_{Bn=0}}{((Ra_{crit})_{Bn=0})^{1/2} (Ra_{crit})^{1/4}} \quad (6.34)$$

In Eq. 6.34,  $(Ra_{eff} - (Ra_{crit})_{Bn=0}) / ((Ra_{crit})_{Bn=0})^{1/2} (Ra_{crit})^{1/4}$  can be neglected because, the numerator (i.e.  $Ra_{eff} - (Ra_{crit})_{Bn=0}$ ) is expected to be much smaller in comparison to the denominator (i.e.  $((Ra_{crit})_{Bn=0})^{1/2} (Ra_{crit})^{1/4}$ ) of this term. Thus, Eq. 6.34 can be simplified in the following manner:

$$\left[ \frac{(Ra_{crit}/(Ra_{crit})_{Bn=0})}{((Ra_{crit})_{Bn=0})^{1/2} (Ra_{crit})^{1/4}} \right] \sim BnARPr^{1/4} \frac{Ra_{eff}^{1/2}}{((Ra_{crit})_{Bn=0})^{1/2}} \sim Bn^a AR^b Pr^c \quad (6.35)$$

The quantity  $Ra_{eff}^{1/2} / ((Ra_{crit})_{Bn=0})^{1/2}$  is expected to be dependent on  $Bn$ ,  $AR$  and  $Pr$  and thus the right hand side of Eq. 6.35 can be taken to scale with  $Bn^a AR^b Pr^c$ . Finally, correlations have been proposed for the coefficients of  $a$ ,  $b$  and  $c$  in Eq. 6.35.

Fig. 6.20 shows that the variation of  $R = \frac{Ra_{crit}/(Ra_{crit})_{Bn=0}}{((Ra_{crit})_{Bn=0})^{0.5} (Ra_{crit})^{0.25}}$  with  $B = Bn^a AR^b Pr^c$  can be approximated by  $R = 20.35B^{0.85}$  for the range of  $Bn$ ,  $Pr$  and  $AR$  considered in this analysis.

Here,  $(Ra_{crit})_{Bn=0}$  also correlation proposed for both CWT and CWHF boundary conditions as  $(Ra_{crit})_{Bn=0} = k_1 AR^{k_2} + k_3$  where  $k_1 = 718$ ,  $k_2 = 3.83$ ,  $k_3 = 1874$  for CWT boundary condition, and  $k_1 = 940$ ,  $k_2 = 3.6$ ,  $k_3 = 740$  for CWHF boundary condition. It can be seen from Fig. 6.20 that a satisfactory level of collapse has been obtained for all  $Bn$ ,  $Pr$  and  $AR$  values considered here for both CWT and CWHF boundary conditions. Moreover, the collapse of data in Fig. 6.20 suggests that numerical / experimental data for  $Ra_{crit}$  in the case of Bingham fluids for a combination of  $Bn$ ,  $AR$  and  $Pr$  in either of the boundary conditions will be sufficient to predict  $Ra_{crit}$  (i.e. approximately 7% average uncertainty) for the range  $Bn$ ,  $AR$  and  $Pr$  values and boundary conditions considered in this analysis.

It is also worth noting that  $Ra_{crit}$  parameterisation given in Eq. 6.35 is valid for laminar Rayleigh- Bénard convection of Bingham fluids in rectangular cross-sectional cylindrical annular enclosures as  $Ra_{crit}$  values are found to independent of  $r_i/L$ .

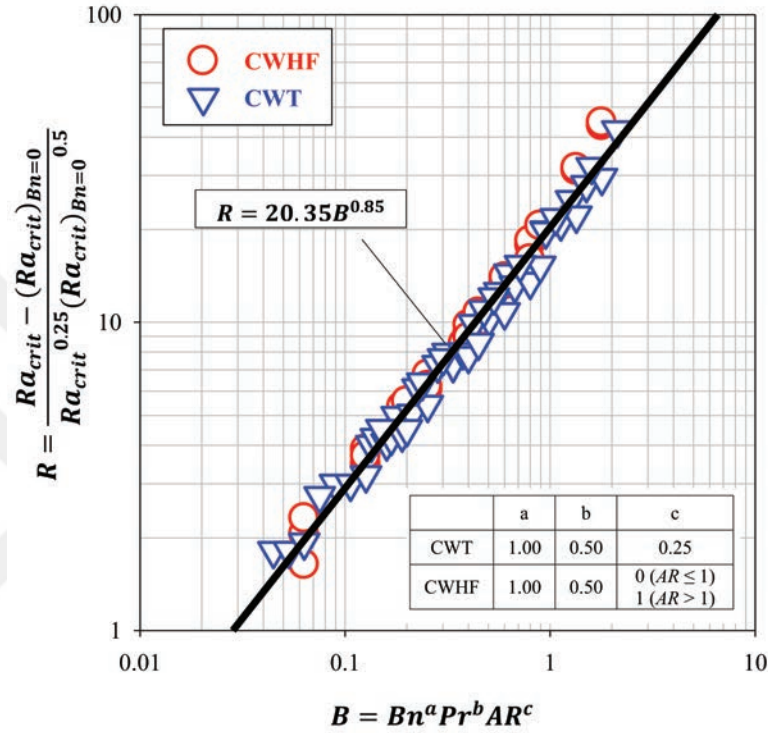


Figure 6.20: The variation of  $R = (Ra_{crit} - (Ra_{crit})_{Bn=0}) / ((Ra_{crit}^{0.25} (Ra_{crit})_{Bn=0})^{0.5})$  with  $B = Bn^a Pr^b AR^c$  for different values of  $Bn$ ,  $Pr$  and  $AR$  on log-log plot for both CWT and CWHF boundary conditions.

## 6.8 Summary of Key Findings

The effects of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $AR$ ,  $r_i/L$  and boundary condition (i.e. CWT and CWHF) on steady-state laminar Rayleigh- Bénard convection of Bingham fluids in rectangular cross-sectional cylindrical annular enclosures have been numerically investigated in this chapter. It is found that the buoyancy-driven transport strengthens with increasing (decreasing) with  $Ra$  ( $AR$ ) for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids, regardless of the boundary condition. The mean

Nusselt number  $\overline{Nu}_{cy}$  for Bingham fluids has been found to be smaller than the corresponding values obtained for Newtonian fluids (i.e.  $Bn = 0$ ) for both CWT and CWHF boundary conditions due to additional flow resistance arising from the yield stress in the case of Bingham fluids. Moreover,  $\overline{Nu}_{cy}$  decreases with increasing  $Bn$  and it assumes a value equal to unity (i.e.  $\overline{Nu}_{cy} = 1.0$ ) for large values of  $Bn$ , which indicates that thermal transport takes place purely due to thermal conduction since fluid flow becomes extremely weak to influence thermal transport under such conditions. Bingham fluids are shown to exhibit a non-monotonic  $Pr$  dependence on  $\overline{Nu}_{cy}$  and a detailed physical explanation has been provided for this behavior. Although variation of  $\overline{Nu}_{cy}$  in response to changes in  $Ra$ ,  $Pr$ ,  $Bn$  remains qualitatively similar for both CWT and CWHF boundary conditions for given  $AR$  and  $r_i/L$ ,  $\overline{Nu}_{cy}$  for the CWHF boundary condition assumes smaller values than in the CWT boundary condition for large values of  $Ra$  for a given set of values of  $Pr$ ,  $Bn$ ,  $AR$  and  $r_i/L$ . Furthermore,  $\overline{Nu}_{cy}$  increases with increasing  $r_i/L$  and approaches the corresponding value obtained for rectangular enclosures ( $r_i/L \rightarrow \infty$ ) for both CWT and CWHF boundary conditions. It has also been found that the flow pattern (i.e. number of rolls within the enclosure) changes with the variations of  $Ra$ ,  $AR$  and  $r_i/L$  in the CWT boundary condition, whereas changes in  $AR$  and  $r_i/L$  do not significantly affect the flow patterns in the CWHF boundary condition. The changes in the flow pattern gives rise to non-monotonic variation of  $\overline{Nu}_{cy}$  with  $Bn$  for the CWT boundary condition prevalently. It has been found that a correlation of the mean Nusselt number for CWT boundary condition is of limited value because of its non-monotonic variation with  $Bn$ . Thus, numerical findings have been used here to propose a correlation for  $\overline{Nu}_{cy}$  in the range of  $0.25 \leq r_i/L \leq 16$ ,  $0.25 \leq AR \leq 2$  and  $5 \times 10^4 \leq Ra \leq 10^5$  only for the CWHF boundary condition.

Furthermore, the effects of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $AR$ ,  $r_i/L$  and boundary condition (i.e. CWT and CWHF) on critical Rayleigh number  $Ra_{crit}$  for which convection ceases to influence the thermal transport and thermal conduction becomes the principal heat transfer mechanism have been also numerically investigated in this chapter. It has been found that  $Ra_{crit}$  increases with increasing values of  $Bn$  and  $AR$ , however,  $Ra_{crit}$  values are found to independent of  $r_i/L$  regardless of the boundary condition. The values of  $Ra_{crit}$  have been found to be greater in the case of the

CWT boundary condition than in the CWHF boundary condition for  $AR \leq 1$ , whereas an opposite trend is obtained for  $AR > 1$  for Bingham fluids. Additionally,  $Ra_{crit}$  has been found to be insensitive to the change of  $Pr$  for Newtonian fluids (i.e.  $Bn = 0$ ), whereas  $Ra_{crit}$  increases with increasing  $Pr$  for Bingham fluids irrespective of the boundary condition. A detailed scaling analysis has also been performed to elucidate the effects of  $Bn, Pr, AR$  and  $r_i/L$  on  $Ra_{crit}$  for Bingham fluids. The results of scaling analysis and numerical findings have been utilised to propose a new correlation for  $Ra_{crit}$  for both Newtonian and Bingham fluids in the case of both CWT and CWHF boundary conditions.





# Chapter 7

## Rayleigh-Bénard Configuration: Power-law Fluids

Numerical findings on laminar Rayleigh-Bénard convection of shear thinning / thickening fluids obeying power-law model in axisymmetric rectangular cross-sectional cylindrical annular enclosures (see Fig. 2.2) have been presented in this chapter. Effects of shear-thinning / thickening character of viscosity on heat and momentum transport have been investigated for a range of different values of normalized internal radius  $r_i/L$  (i.e.  $0 \leq r_i/L \leq 16$ , where  $r_i$  is the internal cylinder radius and  $L$  is the difference between outer and inner radii); aspect ratio  $AR$  (i.e.  $0.25 \leq AR = H/L \leq 4$  where  $H$  is the enclosure height); nominal Rayleigh number  $Ra$  (i.e.  $10^3 \leq Ra \leq 10^5$ ), Prandtl number  $Pr$  range (i.e.  $10 \leq Pr \leq 10^4$ ) and power-law index  $n$  (i.e.  $0.6 \leq n \leq 1.8$ ) for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions.

### 7.1 Scaling Analysis

A detailed scaling analyses conducted to explain the influences of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$ ,  $r_i/L$  on the mean Nusselt number  $\overline{Nu}_{cy}$ . Equating the order of magnitudes of inertial and buoyancy terms yields  $w \sim \sqrt{g\beta(T_H - T_C)H}$  ( $w \sim \sqrt{g\beta q_i \delta_{th} H/k}$ ) for CWT (CWHF) boundary conditions. Using continuity equation for axisymmetric

geometry, one gets:

$$\frac{1}{r} \frac{\partial(ru)}{\partial r} \sim \left( \frac{u}{r} + \frac{u}{L} \right) \sim \frac{\partial w}{\partial z} \sim \frac{w}{H} \quad (7.1)$$

which leads to scaling estimate of  $u$ :

$$u \sim \frac{w}{AR(r+L)} \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{1}{AR^2} \frac{\sqrt{Ra_{CWT}Pr}}{(1+L/r_i)} \quad (7.2)$$

$$u \sim \frac{w}{AR(1+L/r_i)} \sim \frac{\alpha}{L} \frac{1}{AR^2} \frac{Ra_{CWHF}^{(n+1)/(n+4)} Pr^{(n+2)/(n+4)}}{(1+L/r_i) f_3^{-(n+1)/(n+4)}} \quad (7.3)$$

Similarly, equating the order of magnitudes of inertial and viscous terms in the vertical (radial) direction (i.e.  $\rho w^2/H \sim [(K/\delta)(w/\delta)^n]$  ( $\rho u^2/L \sim [(K/\delta_1)(u/\delta_1)^n]$ ), it is possible to estimate hydrodynamic boundary layer thickness for  $r_i \neq 0$  where  $\delta$  ( $\delta_1$ ) is the hydrodynamic boundary layer thickness on the vertical (horizontal) walls respectively:

$$\delta \sim H Ra_{CWT}^{(n-2)/(2n+2)} Pr^{n/(2n+2)} \quad (7.4)$$

$$\delta \sim H Ra_{CWHF}^{(n-2)/(n+4)} Pr^{n/(n+4)} f_3^{(2-n)/(n+4)} \quad (7.5)$$

$$\delta_1 \sim \frac{H Ra_{CWT}^{(n-2)/(2n+2)} Pr^{n/(2n+2)} AR^{(1-n)/(n+1)}}{(1+L/r_i)^{(n-2)/(n+1)}} \quad (7.6)$$

$$\delta_1 \sim \frac{H Ra_{CWHF}^{(n-2)/(n+4)} Pr^{n/(n+4)} AR^{(1-n)/(n+1)} f_3^{(2-n)/(n+4)}}{(1+L/r_i)^{(n-2)/(n+1)}} \quad (7.7)$$

where the function  $f_2(Ra_{CWT}, Pr, n, AR, r_i/L)$ ,  $f_3(Ra_{CWHF}, Pr, n, AR, r_i/L)$ , represents the ratio of hydrodynamic and thermal boundary layer thickness (i.e.  $\delta/\delta_{th}$ ) on the vertical wall for CWT and CWHF boundary conditions, respectively. Using Eqs. 7.4 - 7.7, it is possible to estimate the effective viscosity in the vertical (horizontal) boundary layers (i.e.  $\mu_{eff}^V \sim K (w/\delta)^{n-1}$  ( $\mu_{eff}^H \sim K (u/\delta_1)^{n-1}$ )) in

order to predict the effective Rayleigh numbers (i.e.  $Ra_{eff} \sim \frac{\rho^2 c_p g \beta \Delta T_{ref} H^3}{u_{eff} k}$ ) in the vertical and horizontal boundary layers in the following manner:

$$Ra_{CWT,eff}^V \sim Ra_{CWT}^{(5-n)/(2n+2)} Pr^{(1-n)/(2n+2)} \quad (7.8)$$

$$Ra_{CWHF,eff}^V \sim \frac{Ra_{CWHF}^{(7-2n)/(n+4)} Pr^{(2-2n)/(n+4)}}{f_3^{(3-3n)/(n+4)}} \quad (7.9)$$

$$Ra_{CWT,eff}^H \sim \frac{Ra_{CWT}^{(5-n)/(2n+2)} Pr^{(1-n)/(2n+2)} AR^{(2n-2)/(n+1)}}{(1 + L/r_i)^{(3-3n)/(n+1)}} \quad (7.10)$$

$$Ra_{CWHF,eff}^H \sim \frac{Ra_{CWHF}^{(7-2n)/(n+4)} Pr^{(2-2n)/(n+4)} AR^{(2n-2)/(n+1)} f_3^{(3n-3)/(n+4)}}{(1 + L/r_i)^{(3-3n)/(n+1)}} \quad (7.11)$$

Eqs. 7.8 - 7.11 shows that the effective value of Rayleigh number attains significantly larger than its nominal value for a decrease in values of  $n$  for shear thinning (i.e.  $n < 1$ ) fluids. Furthermore,  $Ra_{CWT,eff}^H$  and  $Ra_{CWHF,eff}^H$  decrease with decreasing  $r_i/L$  for shear thinning (i.e.  $n < 1$ ) fluids. However,  $Ra_{CWT,eff}^H$  and  $Ra_{CWHF,eff}^H$  increase with decreasing  $r_i/L$  for shear thickening (i.e.  $n > 1$ ) fluids.

The mean Nusselt number  $\overline{Nu}_{cy}$  can be estimated by using the wall heat flux scaling  $q_w \sim k\Delta T/\delta_{th}$  as:

$$\overline{Nu}_{cy,CWT} \sim \frac{Ra_{CWT}^{(2-n)/(2n+2)} Pr^{n/(2n+2)} AR^{(n-1)/(n+1)}}{f_2 (1 + L/r_i)^{(2-n)/(n+1)}} \quad (7.12)$$

$$\overline{Nu}_{cy,CWHF} \sim \frac{Ra_{CWHF}^{(2-n)/(n+4)} Pr^{n/(n+4)} AR^{(n-1)/(n+1)}}{f_3^{-6/(n+4)} (1 + L/r_i)^{(2-n)/(n+1)}} \quad (7.13)$$

Equations 7.12 and 7.13 imply that the exponent of Rayleigh number increases with decreasing  $n$  but the exponent remains positive for the range of the values of  $n$  considered here. Thus, the mean Nusselt number  $\overline{Nu}_{cy}$  is expected to in-

crease with an increase in (a decrease in) nominal Rayleigh number (power-law exponent).

## 7.2 Effects of Nominal Rayleigh Number $Ra$ and Power-law Index $n$

Venturi et al. [111] identified multiple steady two-dimensional Newtonian (i.e.  $n = 1.0$ ) flow patterns in the super-critical regime (when the critical Rayleigh number for linear instability is surpassed) of convection in the differentially heated horizontal wall boundary condition with higher bottom wall temperature for square (i.e.  $AR = 1.0$ ) enclosures. Furthermore, the analyses by Albaalbaki and Khayat [103] indicated that it may be possible to obtain convection along a sub-critical branch (i.e. a non-linear bifurcation from infinity) for Carreau-Bird shear-thinning (i. e.  $n < 1$ ) fluids when the power-law exponent  $n$  is smaller than a threshold value. This is also confirmed by Yigit et al. [52] for shear-thinning (i. e.  $n < 1$ ) fluids obeying the power-law model of viscosity. From the foregoing it is evident that the simulations for both Newtonian and power-law fluids in the current boundary condition can be sensitive to the initial conditions. Thus, the simulations in the current analysis have been carried out for two types of initial conditions. For the Newtonian fluid (i.e.  $n = 1.0$ ) the simulations have been initialised using two methodologies. In one set of simulations, the quiescent flow condition is used for the initial condition, whereas the steady-state simulation results obtained for a smaller value of  $Ra$  is used for the initial condition for the cases with  $Ra \geq 2.5 \times 10^3$  in the second method. These two methodologies will henceforth be referred to as the “quiescent flow” and “established flow” initial conditions respectively, in this study. For example, one obtains a pure conduction steady-state solution for  $Ra = 10^3$  in the case of Newtonian fluids (i.e.  $n = 1.0$ ) as it is well below the critical  $Ra$  to initiate flow. This solution is used to initialise the Newtonian simulations for  $Ra = 2.5 \times 10^3$ . For the “established flow” initial condition the steady-state solutions for  $Ra = 2.5 \times 10^3$  are used for the initialisation of  $Ra = 5 \times 10^3$  simulations involving Newtonian (i.e.  $n = 1.0$ ) fluids. Similarly the steady-state solutions for  $Ra = 5 \times 10^3$  are used for the initialisation

of  $Ra = 7.5 \times 10^3$  (and this continues with using previous steady-state solutions as initial conditions for  $Ra = 10^4, 1.75 \times 10^4, 2.5 \times 10^4, 3.75 \times 10^4, 5 \times 10^4, 6 \times 10^4, 7 \times 10^4, 8 \times 10^4, 9 \times 10^4$  until  $Ra = 10^5$  is reached) simulations in the context of established flow initial condition for Newtonian (i.e.  $n = 1.0$ ) fluids. The steady-state solution obtained for Newtonian (i.e.  $n = 1.0$ ) fluids based on established flow initial condition for the given set of values of  $Ra$  and  $Pr$  is used as an initial condition for one set of simulations involving non-Newtonian power-law fluids, whereas the quiescent condition is used as an initial condition for the other set of power-law fluid simulations.

The variation of mean Nusselt number  $\overline{Nu}_{cy}$  with nominal  $Ra$  for  $0.6 \leq n \leq 1.8$  at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 10^3$  is shown in Fig. 7.1 with representative streamline patterns and isotherms for both established (E.F.) and quiescent (Q.F.) flow initial conditions in both CWT and CWHF boundary conditions. Figure 7.1 shows that  $\overline{Nu}_{cy}$  monotonically increases with an increase in  $Ra$  for both shear-thickening (i.e.  $n < 1$ ) and Newtonian (i.e.  $n = 1.0$ ) fluids regardless of the initial and boundary conditions. By contrast, the variation of  $\overline{Nu}_{cy}$  with  $Ra$  is sensitive to the both initial and boundary conditions for shear-thinning (i.e.  $n < 1$ ) fluids. A non-monotonic trend for the variation of  $\overline{Nu}_{cy}$  with  $Ra$  is observed for shear-thinning (i.e.  $n < 1$ ) fluids for both choices of initial and boundary conditions. This behaviour can be explained by the change in the streamline patterns with the alteration of  $Ra$ , as it shown in Fig. 7.1. This change in flow pattern modifies the distance between the isotherms in the vicinity of the active walls (i.e. hot and cold walls), and is responsible for the change in the values of  $\overline{Nu}_{cy}$ . Furthermore, it can be seen in Fig. 7.1 that  $\overline{Nu}_{cy}$  decreases with increasing  $n$  due to augmented viscous resistance before assuming a value equal to unity (indicating conduction-driven transport) for high values of  $n$ . Moreover, the streamline patterns change with the alteration of  $n$  for both choices of initial and boundary conditions, especially for shear-thinning (i.e.  $n < 1$ ) fluids, which is consistent with previous analyses [52, 100, 103], which demonstrated the transition between different preferred flow patterns for shear-thinning (i.e.  $n < 1$ ) fluids. By contrast, a single roll pattern is observed in the case of shear-thickening (i.e.  $n > 1$ ) fluids irrespective of the choice of initial and boundary conditions, which is consistent with the observations from Fig 7.1.

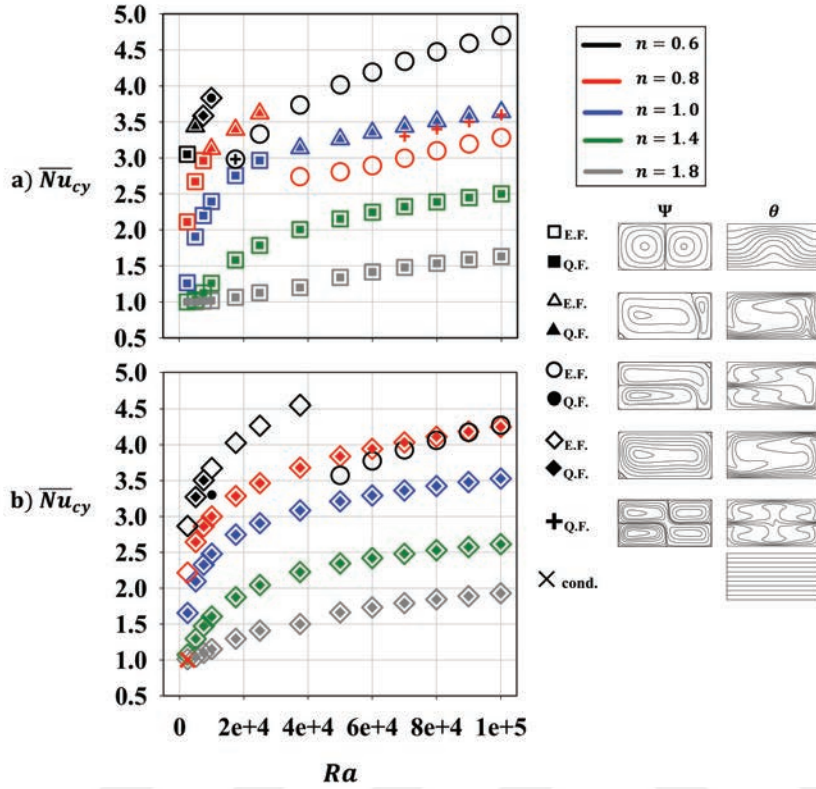


Figure 7.1: Variation of  $\overline{Nu}_{cy}$  with nominal  $Ra$  for different values of  $n$  at  $r_i/L = 1.0$ ,  $AR = 0.5$  and  $Pr = 10^3$  for with representative streamline and isotherm patterns for both established (E.F.) and quiescent (Q.F.) flow initial conditions in both a) CWT and b) CWHF boundary conditions.

It is also worth noting that the findings of Fig. 7.1 can be confirmed by the scaling estimation given in Eqs. 7.12 and 7.13 (i.e.  $\overline{Nu}_{cy}$  increases with an increase in (a decrease in) nominal Rayleigh number (power-law exponent)). Moreover, Eqs. 7.8 - 7.11 indicate that the effective Rayleigh number  $Ra_{eff}$  becomes increasingly larger than its nominal value for decreasing values of  $n$  for shear thinning fluids (i.e.  $n < 1$ ) for given values of  $Pr$ ,  $AR$ ,  $r_i/L$ . Thus, steady state solution corresponding to a particular type of flow pattern such as one cell structure, which is obtained for small values of  $Ra_{eff}$ , may not be possible to obtain for large values of  $Ra_{eff}$ . Therefore, steady state solution may exhibit multiple cell patterns for large values of  $Ra_{eff}$  but multiple cell structure acts to reduce the mean Nusselt number.

### 7.3 Effects of Prandtl Number $Pr$

The variation of  $\overline{Nu}_{cy}$  with  $Pr$  for different values of  $Ra$ ,  $n$ ,  $r_i/L$  and different initial conditions (i.e. E.F. and Q.F.) at  $AR = 1.0$  is shown in Figs. 7.2 and 7.3 with representative flow patterns in the case of CWT and CWHF boundary conditions, respectively.

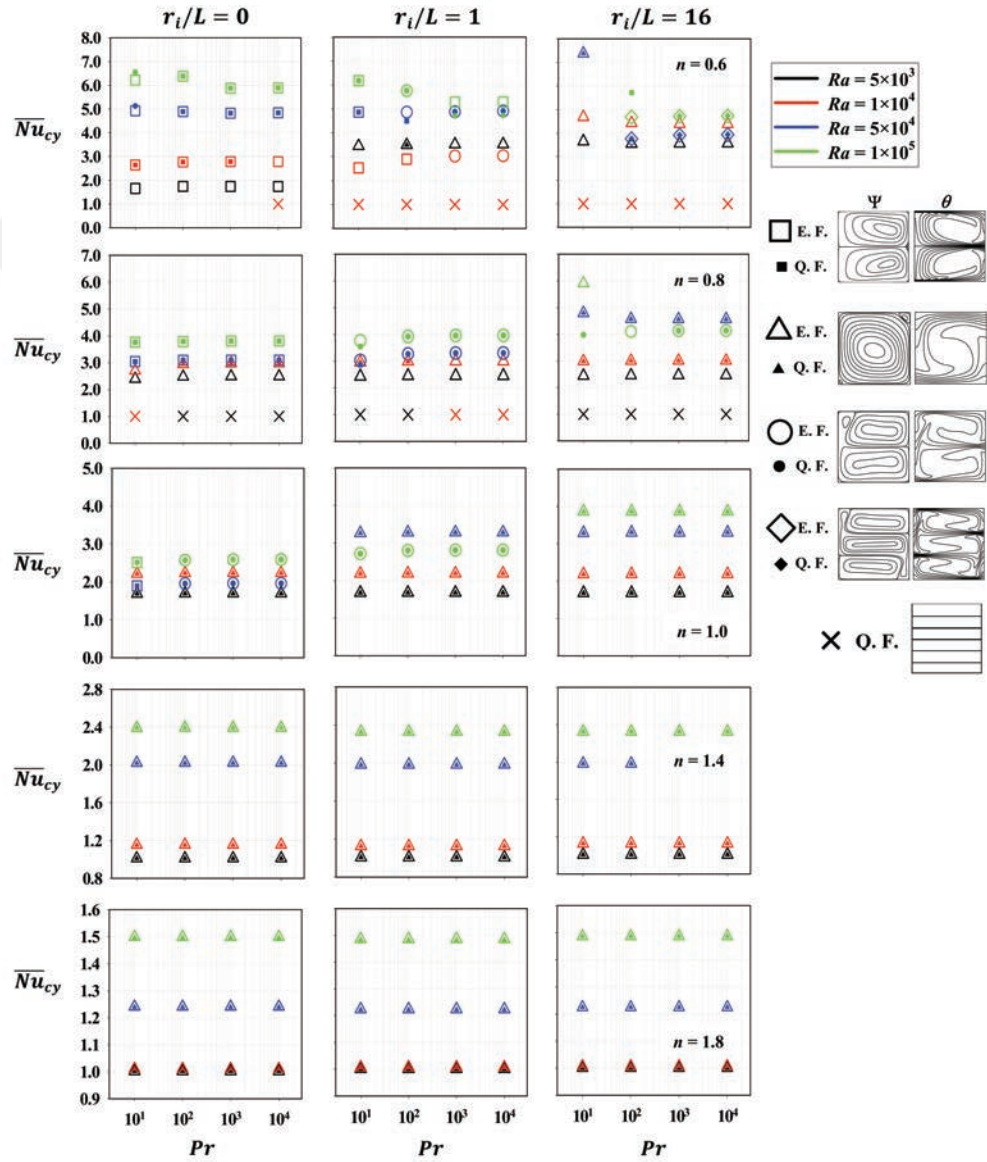


Figure 7.2: Variation of  $\overline{Nu}_{cy}$  with  $Pr$  for different values of  $Ra$ ,  $n$  and  $r_i/L$  at  $AR = 1.0$  for CWT boundary condition for both E.F and Q.F initial conditions.

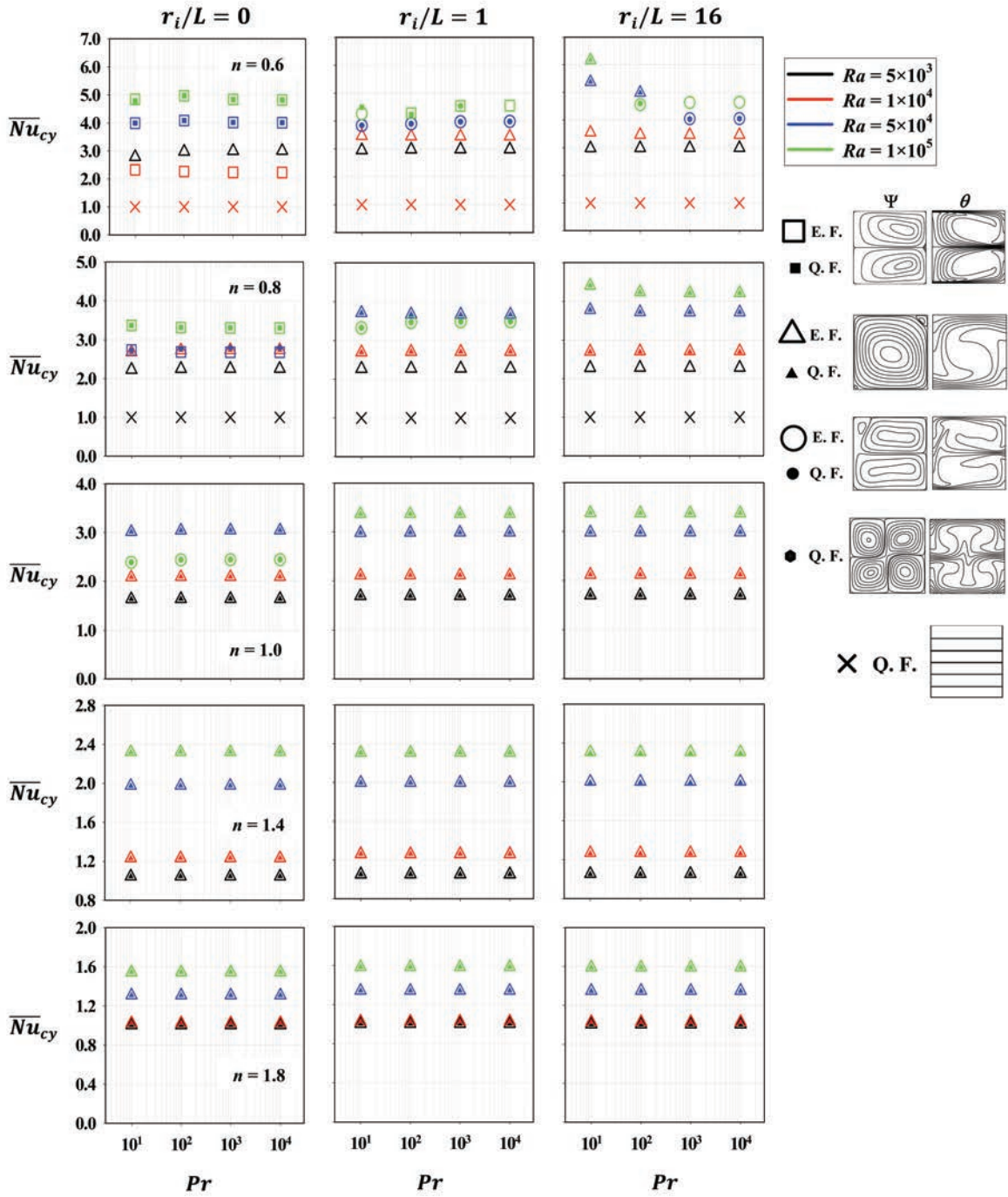


Figure 7.3: Variation of  $\overline{Nu}_{cy}$  with  $Pr$  for different values of  $Ra$ ,  $n$  and  $r_i/L$  at  $AR = 1.0$  for CWHF boundary condition for both E.F and Q.F initial conditions.

The variations of the  $\overline{Nu}_{cy}$  and the streamline pattern are found to be insensitive to the choices of  $Pr$  and initial condition for both Newtonian (i.e.  $n = 1.0$ ) and shear thickening fluids (i.e.  $n > 1$ ), whereas the influences of  $Pr$  and initial condition are found to be significant in the case of shear thinning fluids (i.e.  $n < 1$ ) for a given set of values of  $Ra$ ,  $AR$  and  $r_i/L$  for both CWT and CWHF boundary conditions. The thermal boundary layer remains much thinner than the hydro-dynamic boundary layer for  $Pr \gg 1$  so the relative balance between the buoyancy and viscous forces is principally affected by nominal Prandtl number [2, 92, 93, 98], which does not alter the thermal transport within the thermal boundary layer for Newtonian fluids and shear-thickening fluids. This gives rise to a weak  $Pr$  dependence of  $\overline{Nu}_{cy}$  for Newtonian fluids and shear-thickening fluids. Scaling estimates given by Eqs. 7.4 - 7.7 suggest that the thickness of hydro-dynamic boundary layer decreases with decreasing (increasing)  $n$  ( $Ra$ ). Thus, both hydrodynamic and thermal boundary layers remain thin for a combination of high nominal Rayleigh number and small power-law exponent, and therefore a change in  $Pr$  alters the thermal boundary layer thickness and in turn  $\overline{Nu}_{cy}$  significantly as shown in Figs. 7.2 and 7.3.

## 7.4 Effects of Aspect Ratio $AR$

The variation of  $\overline{Nu}_{cy}$  with  $AR$  for different values of  $Ra$  and  $n$  is shown in Fig. 7.4 at  $r_i/L = 1.0$  and  $Pr = 10^3$  for both E.F. and Q.F. initial conditions in both CWT (left column) and CWHF (right column) boundary conditions. It can be seen from Fig. 7.4 that  $\overline{Nu}_{cy}$  shows a mostly decreasing trend with increasing  $AR$  for different values of  $Ra$  and  $n$  even though some non-monotonic trends can be observed mainly for shear-thinning (i.e.  $n = 0.8$ ) and Newtonian (i.e.  $n = 1.0$ ) fluids in the cases with  $AR \leq 1$  due to the change in flow structure within the enclosure. This can be seen in Fig. 7.5, which shows streamlines patterns (i.e.  $\Psi$ ) for different values of  $Ra$  and  $AR$  for shear-thinning fluids (i.e.  $n = 0.8$ ), Newtonian (i.e.  $n = 1.0$ ) and shear-thickening fluids (i.e.  $n = 1.4$ ) at  $r_i/L = 1.0$  and  $Pr = 10^3$  for the E.F. initial condition in the case of both CWT and CWHF boundary conditions. It is worth noting that the streamline patterns in Fig. 7.5 are in good agreement with the findings of Leong [110] on Rayleigh-Bénard

convection of Newtonian (i.e.  $n = 1.0$ ) fluids in a vertical cylinder (i.e.  $r_i/L = 0$ ).

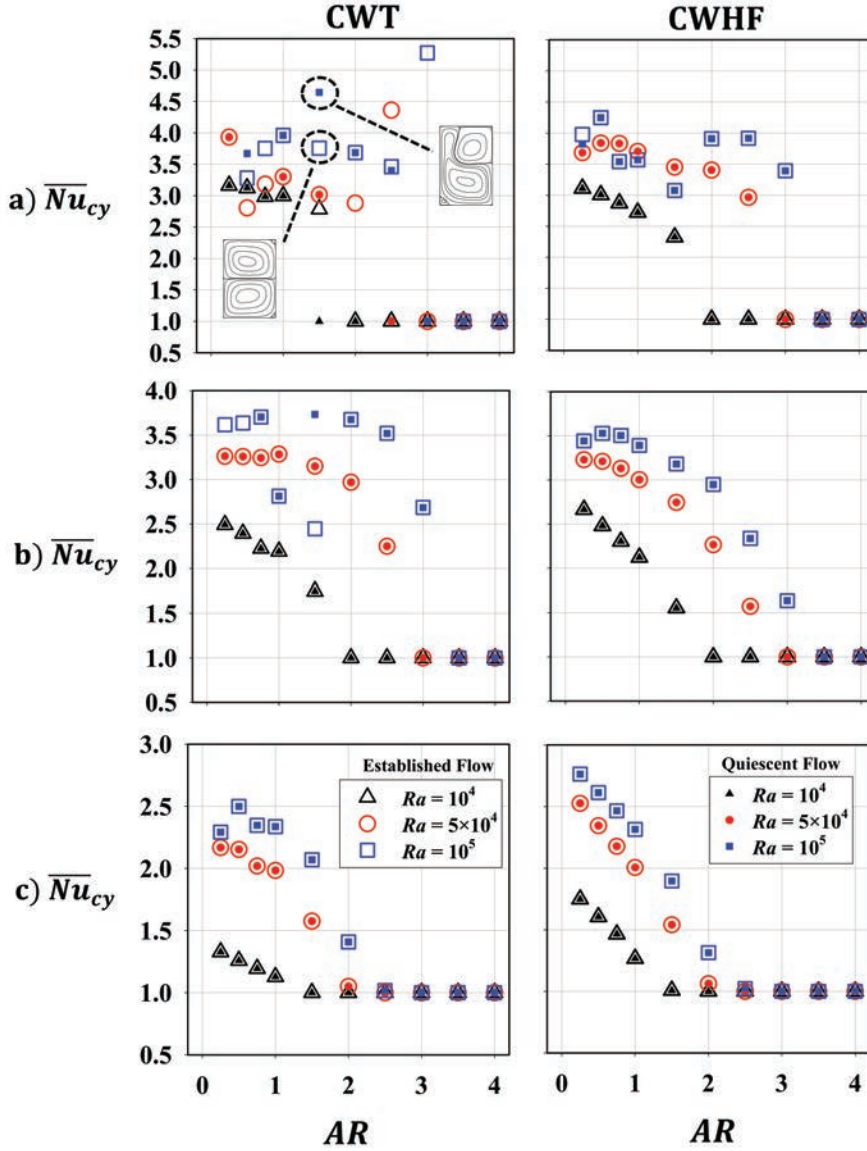


Figure 7.4: Variation of  $\overline{Nu}_{cy}$  with  $AR$  for a) shear-thinning fluids (i.e.  $n = 0.8$ ), b) Newtonian fluids (i.e.  $n = 1.0$ ) and c) shear-thickening fluids (i.e.  $n = 1.4$ ) at  $r_i/L = 1.0$  and  $Pr = 10^3$  for both established (E.F.) and quiescent (Q.F.) flow initial conditions in the case of both CWT (left column) and CWHF (right column) boundary conditions.

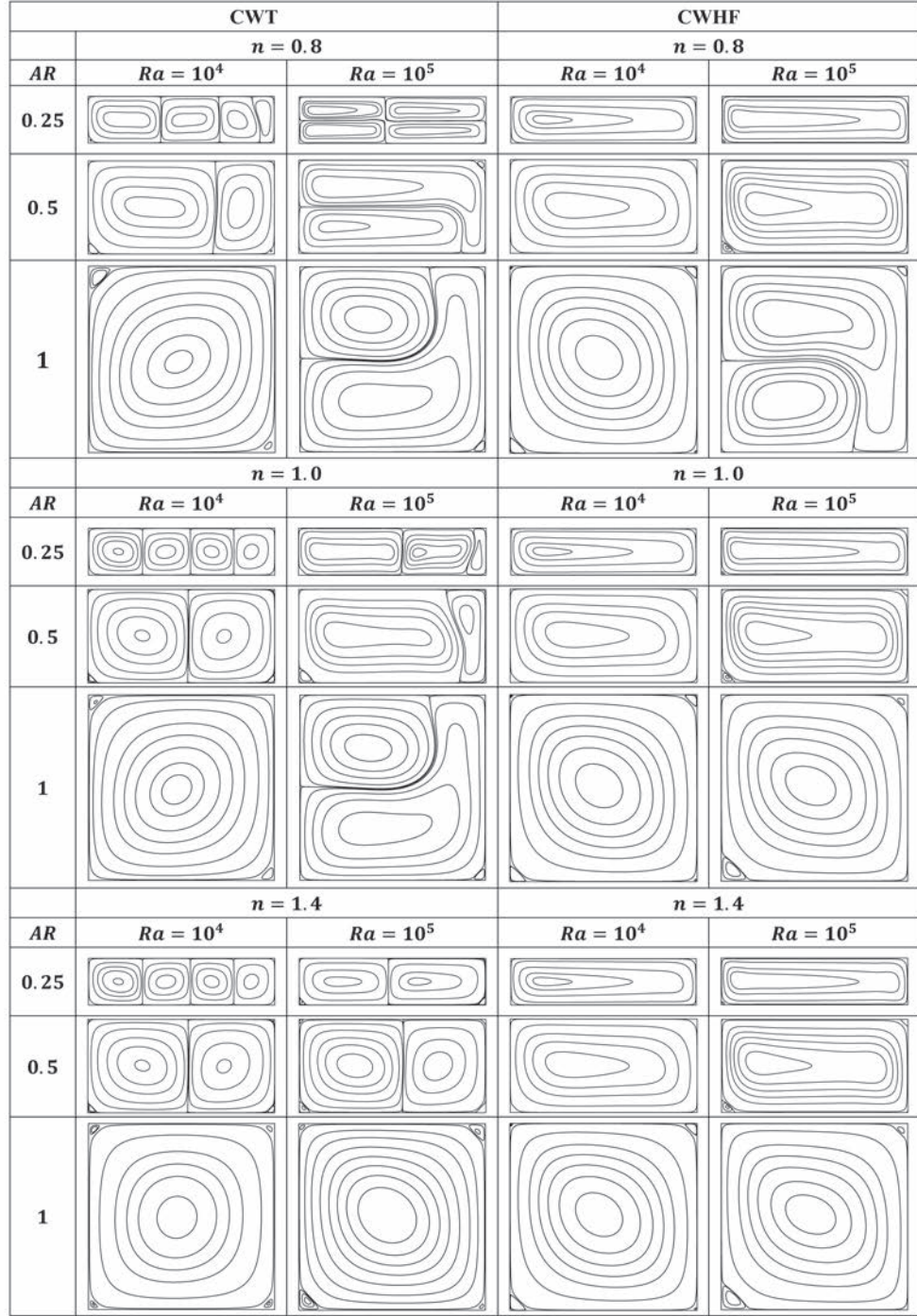


Figure 7.5: Contours of non-dimensional stream function  $\Psi$  for different values of  $Ra$  and  $AR$  for shear-thinning fluids (i.e.  $n = 0.8$ ), Newtonian fluids (i.e.  $n = 1.0$ ) and shear-thickening fluids (i.e.  $n = 1.4$ ) at  $r_i/L = 1.0$  and  $Pr = 10^3$  for established flow (E.F.) initial condition in the case of both CWT (left column) and CWHF (right column) boundary conditions.

Figure 7.4 further shows that the variation of  $\overline{Nu}_{cy}$  with  $AR$  is sensitive to the initial conditions and the steady-state solution is not obtained for the Q.F. initial condition especially for  $AR < 1$  for shear-thinning (i.e.  $n < 1$ ) fluids in the CWT boundary condition. Moreover, Fig. 7.4 shows that  $\overline{Nu}_{cy}$  eventually attains a value equal to unity (indicating conduction-driven transport) for tall enclosures (i.e.  $AR > 1$ ). This also can be confirmed from Fig. 7.6, which shows that the isotherms for the  $AR = 4.0$  case remain parallel to the horizontal walls indicating a conduction-dominated transport. Moreover, a conduction dominated transport is obtained for  $AR > 3$  irrespective of the values of  $Ra$  and  $n$  for all the initial and boundary conditions considered here.

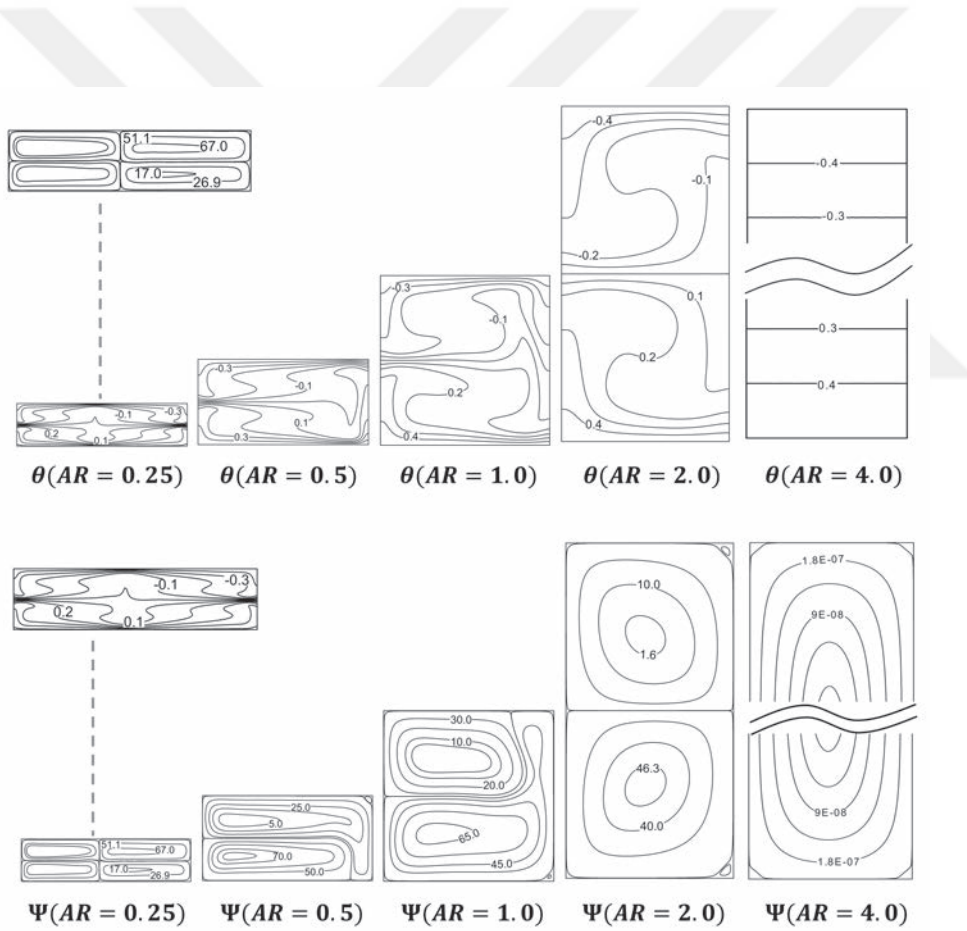


Figure 7.6: Contours of  $\theta$  and  $\Psi$  for different  $AR$  at  $r_i/L = 1.0$ ,  $Ra = 10^5$  and  $Pr = 10^3$  for shear-thinning (i.e.  $n = 0.8$ ) fluid case for established flow initial condition in the case of CWT boundary condition.

The  $AR$  dependence of  $\overline{Nu}_{cy}$  for CWT (CWHF) boundary condition can be explained in the following manner using the energy flux integral between hot and cold walls. The energy flux integral can be shown as follows:

$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{cond} = \int_0^H \rho c_p T u dz - \int_0^H k \left( \frac{\partial T}{\partial r} \right) dz \quad (7.14)$$

where the first term on the right-hand side represents the effects of convective transport, whereas the second term on the right-hand side accounts for thermal conduction. Here,  $\dot{Q}_{conv}$  and  $\dot{Q}_{cond}$  can be scaled in the following manner by using Eqs. 7.1 and 7.7:

$$\dot{Q}_{conv,CWT} \sim \rho c_p \Delta T u \delta_{th} \sim \frac{k \Delta T Ra_{CWT}^{(2n-1)/(2n+2)} Pr^{(2n+1)/(2n+2)} AR^{-2n/(n+1)}}{(1 + L/r_i)^{(2n-1)/(n+1)}} \quad (7.15)$$

$$\dot{Q}_{conv,CWHF} \sim \frac{\rho c_p u q \delta_{th} \delta_1}{k} \sim \frac{q L Ra_{CWHF}^{(3n-3)/(n+4)} Pr^{(3n+2)/(n+4)} AR^{(2-2n)/(n+1)}}{f_3^{4n+1/(n+4)} (1 + L/r_i)^{(3n-3)/(n+1)}} \quad (7.16)$$

$$\dot{Q}_{cond,CWT} \sim (k \Delta T) AR \quad (7.17)$$

$$\dot{Q}_{cond,CWHF} \sim q AR \delta_{th} \sim q AR H \frac{Ra_{CWHF}^{(n-2)/(n+4)} Pr^{n/(n+4)} AR^{(1-n)/(n+1)}}{f_3^{(2n+2)/(n+4)} (1 + L/r_i)^{(n-2)/(n+1)}} \quad (7.18)$$

Equations 7.15 - 7.18 suggest relative weakening (strengthening) of the contribution of  $\dot{Q}_{conv}$  ( $\dot{Q}_{cond}$ ) to the overall thermal transport with increasing  $AR$  for both shear-thinning (i.e.  $n < 1$ ) and shear-thickening (i.e.  $n > 1$ ) fluids for both CWT and CWHF boundary conditions.

For shear-thinning (i.e.  $n < 1$ ) fluids, the exponent of  $AR$  remains negative in Eqs. 7.12 and 7.13 (i.e.  $\overline{Nu}_{cy} \sim AR^{(n-1)/(n+1)}$ ) and this exponent (i.e.  $(n -$

$1)/(n+1))$  becomes increasingly negative with decreasing  $n$ , which suggests that  $\overline{Nu}_{cy}$  is expected to attain higher values than that obtained for Newtonian (i.e.  $n = 1.0$ ) fluids for the combination of small values of  $n$  and  $AR$  for a given set of values of  $Ra$ ,  $Pr$  and  $r_i/L$ . By contrast, the exponent of  $AR$  assumes increasingly positive values with increasing  $n$  value for shear-thickening (i.e.  $n > 1$ ) fluids independent of boundary conditions. Therefore,  $\overline{Nu}_{cy}$  for shear-thickening (i.e.  $n > 1$ ) fluids is expected to attain smaller values than that obtained for Newtonian (i.e.  $n = 1.0$ ) fluids for a given set of values of  $Ra$ ,  $Pr$ ,  $AR$  and  $r_i/L$  for both CWT and CWHF boundary conditions. It is worth noting that the aforementioned inferences drawn in Eqs. 7.15 - 7.18, are entirely consistent with Fig. 7.4, where a single roll flow pattern (i.e. streamline patterns  $\Psi$  are given in Fig. 7.5 for Newtonian fluids (i.e.  $n = 1.0$ ) at  $Ra = 10^4$  for different  $AR$  in the CWT and CWHF boundary conditions) is obtained for Rayleigh-Bénard convection for both CWT and CWHF boundary conditions.

## 7.5 Effects of Normalised Internal Cylinder Radius $r_i/L$

The variations of  $\overline{Nu}_{cy}$  with  $r_i/L$  for different  $Ra$ ,  $n$ , and initial conditions (i.e. E.F. and Q.F.) at  $AR = 1.0$  and  $Pr = 10^3$  are shown in Figs. 7.7 and 7.8 with representative flow patterns in the CWT and CWHF boundary conditions for  $0 \leq r_i/L \leq 1$  and  $1 \leq r_i/L \leq 24$  respectively. It can be seen from Figs. 7.7 and 7.8 that the values of  $\overline{Nu}_{cy}$  and flow patterns are not affected by the modification of  $r_i/L$  for shear thickening fluids (i.e.  $n > 1$ ) irrespective of the choices of initial and boundary conditions. However,  $\overline{Nu}_{cy}$  shows a non-monotonic variation with  $r_i/L$  due to changes in flow pattern for both Newtonian (i.e.  $n = 1.0$ ) and shear thinning fluids (i.e.  $n < 1$ ) for a given set of values of  $Ra$ ,  $Pr$  and  $AR$  irrespective of the choices of initial and boundary conditions. This tendency is prevalent for small values of  $r_i/L$  for cases with a combination of high values of  $Ra$  and small values of  $n$ . Equations 7.8 - 7.11 indicate that the effective Rayleigh number  $Ra_{eff}$  becomes increasingly larger than its nominal value for increasing values of  $r_i/L$  for shear thinning fluids (i.e.  $n < 1$ ).

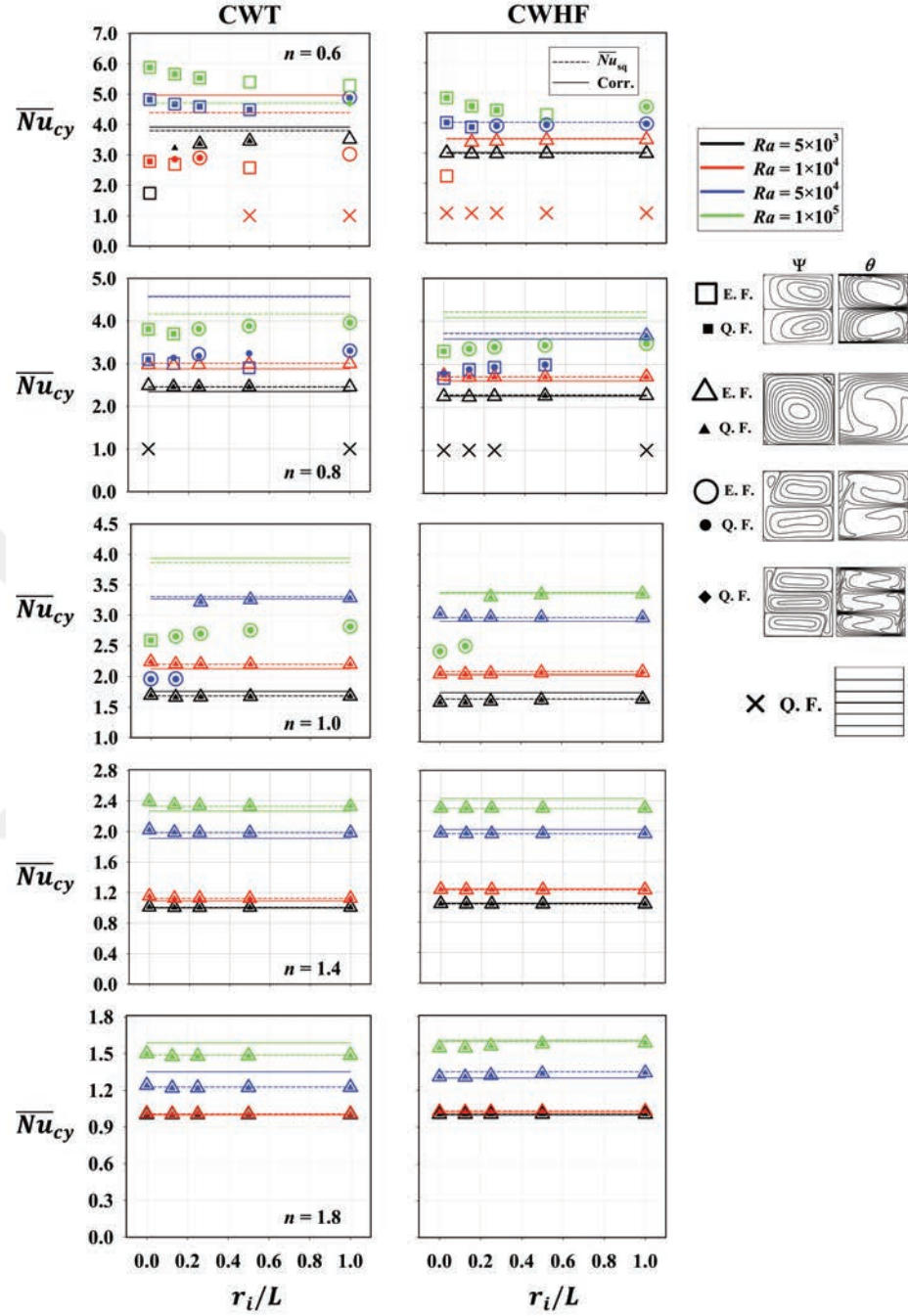


Figure 7.7: Variation of  $\overline{Nu}_{cy}$  with  $r_i/L$  (for  $r_i/L \leq 1$ ) for different values of  $Ra$  and  $n$  at  $AR = 1$  and  $Pr = 10^3$  for both CWT and CWHF boundary conditions in the cases of E.F and Q.F initial conditions. The corresponding values of mean Nusselt number for square enclosures (i.e.  $\overline{Nu}_{sq}$ ) are shown with broken lines. The correlation results for  $\overline{Nu}_{sq}$  in Table 1 are shown by solid lines.

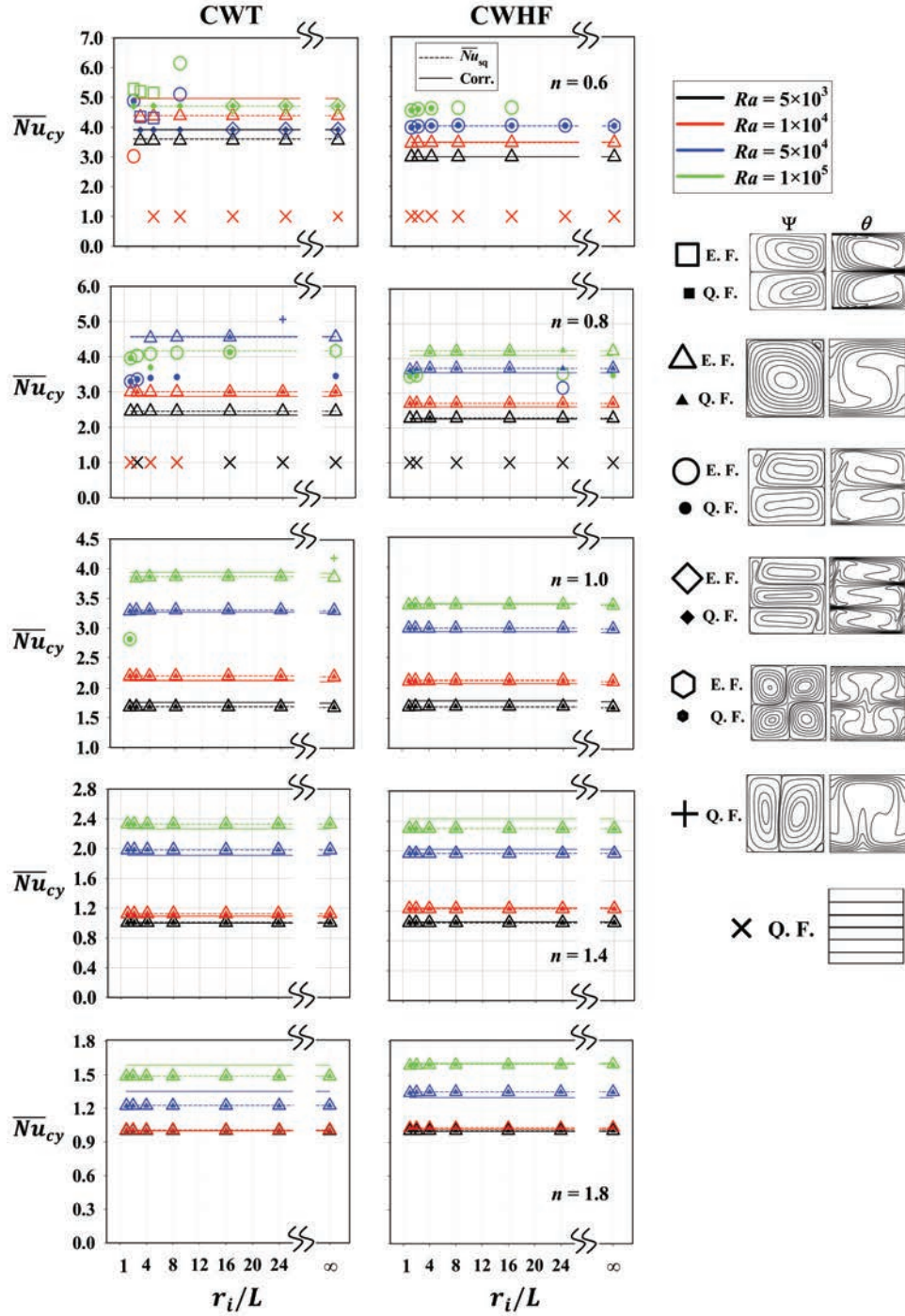


Figure 7.8: Variation of  $\overline{Nu}_{cy}$  with  $r_i/L$  (for  $r_i/L > 1$ ) for different values of  $Ra$  and  $n$  at  $AR = 1$  and  $Pr = 10^3$  for both CWT and CWHF boundary conditions in the cases of E.F and Q.F initial conditions. The corresponding values of mean Nusselt number for square enclosures (i.e.  $\overline{Nu}_{sq}$ ) are shown with broken lines. The correlation results for  $\overline{Nu}_{sq}$  in Table 1 are shown by solid lines.

The variations of  $\overline{Nu}_{cy}$  with  $r_i/L$  for different values of  $n$  and  $AR$  at  $Ra = 5 \times 10^4$  and  $Pr = 10^3$  are shown along with representative flow patterns in Figs. 7.9- 7.11 for both E.F. and Q.F. initial conditions in the case of both CWT and CWHF boundary conditions.

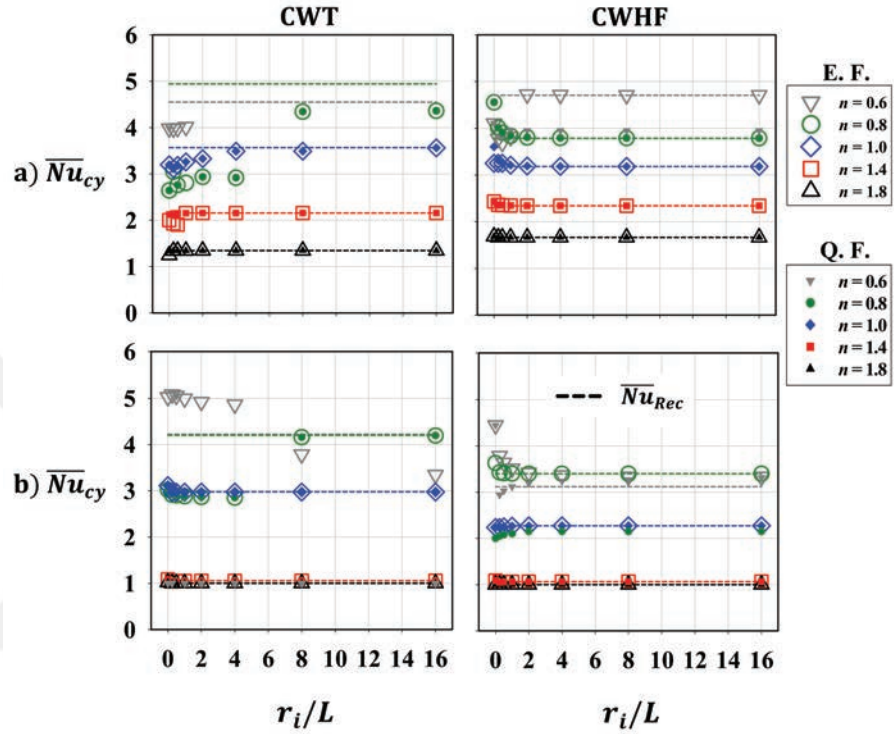


Figure 7.9: Variation of  $\overline{Nu}_{cy}$  with  $r_i/L$  for different values of  $n$  and a)  $AR = 0.5$ , b)  $AR = 2.0$  at  $Ra = 5 \times 10^4$  and  $Pr = 10^3$  for both established (E.F.) and quiescent (Q.F.) flow initial conditions in the case of both CWT (left column) and CWHF (right column) boundary conditions.

It can be seen from Figs. 7.7 - 7.9 that the values of  $\overline{Nu}_{cy}$  and flow patterns are not significantly affected by the modification of  $r_i/L$  for shear-thickening fluids (i.e.  $n > 1$ ) for both E.F. and Q.F. initial conditions in the case of CWHF boundary condition. However, the values of  $\overline{Nu}_{cy}$  and flow patterns are sensitive to the choices of  $r_i/L$  and initial conditions especially for  $r_i/L < 1$  and  $AR < 1$  in the CWHF boundary condition for shear-thinning fluids (i.e.  $n < 1$ ) and Newtonian (i.e.  $n = 1.0$ ) fluids. Furthermore,  $\overline{Nu}_{cy}$  shows a non-monotonic variation with  $r_i/L$  due to the changes in flow patterns for shear-thinning (i.e.

$n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear-thickening fluids (i.e.  $n > 1$ ) for a given set of values of  $Ra$  and  $Pr$  for both E.F. and Q.F. initial conditions in the case of CWT boundary condition. This tendency is prevalent for small values of  $r_i/L$  and  $AR$  for cases with a small values of  $n$ .

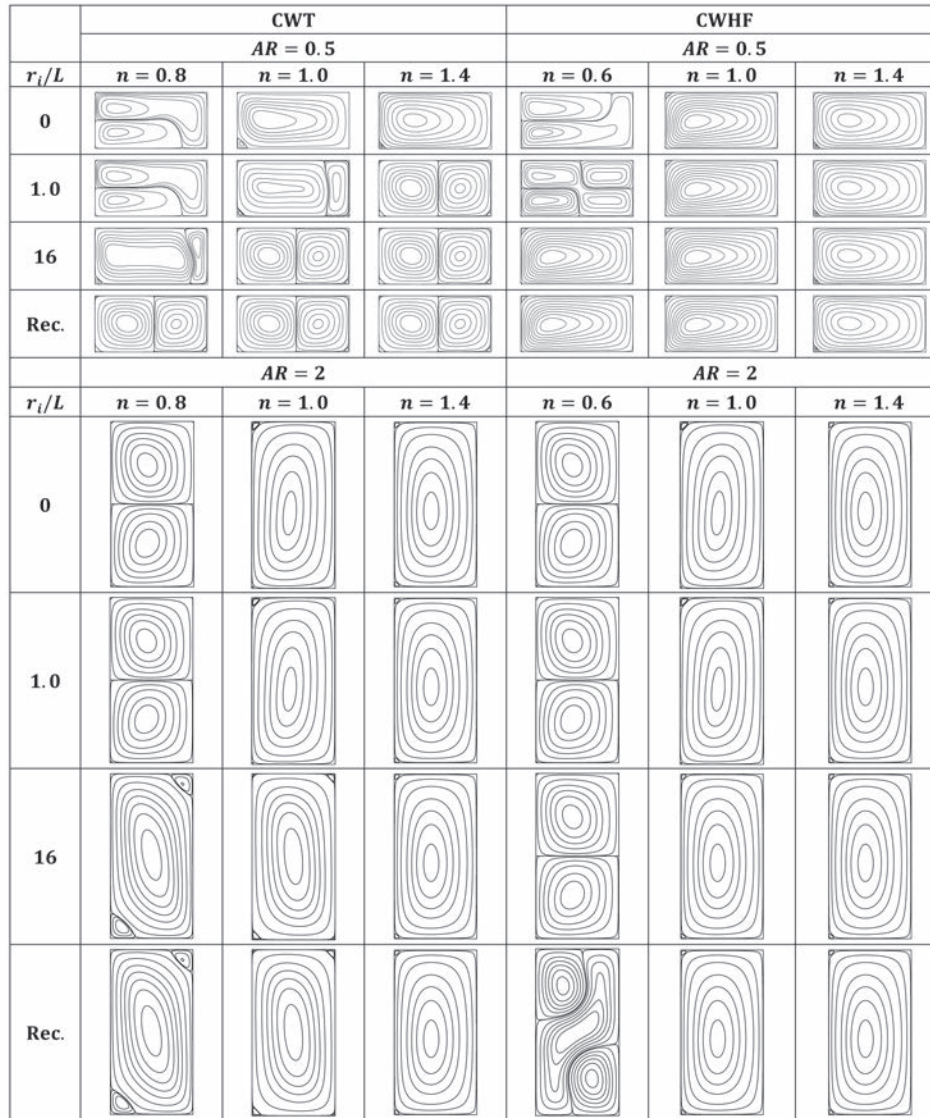


Figure 7.10: Contours of  $\Psi$  for different values of  $n$ ,  $r_i/L$  and  $AR$  at  $Ra = 5 \times 10^4$  and  $Pr = 10^3$  for established flow (E.F.) initial condition in the case of both CWT (left column) and CWHF (right column) boundary conditions.

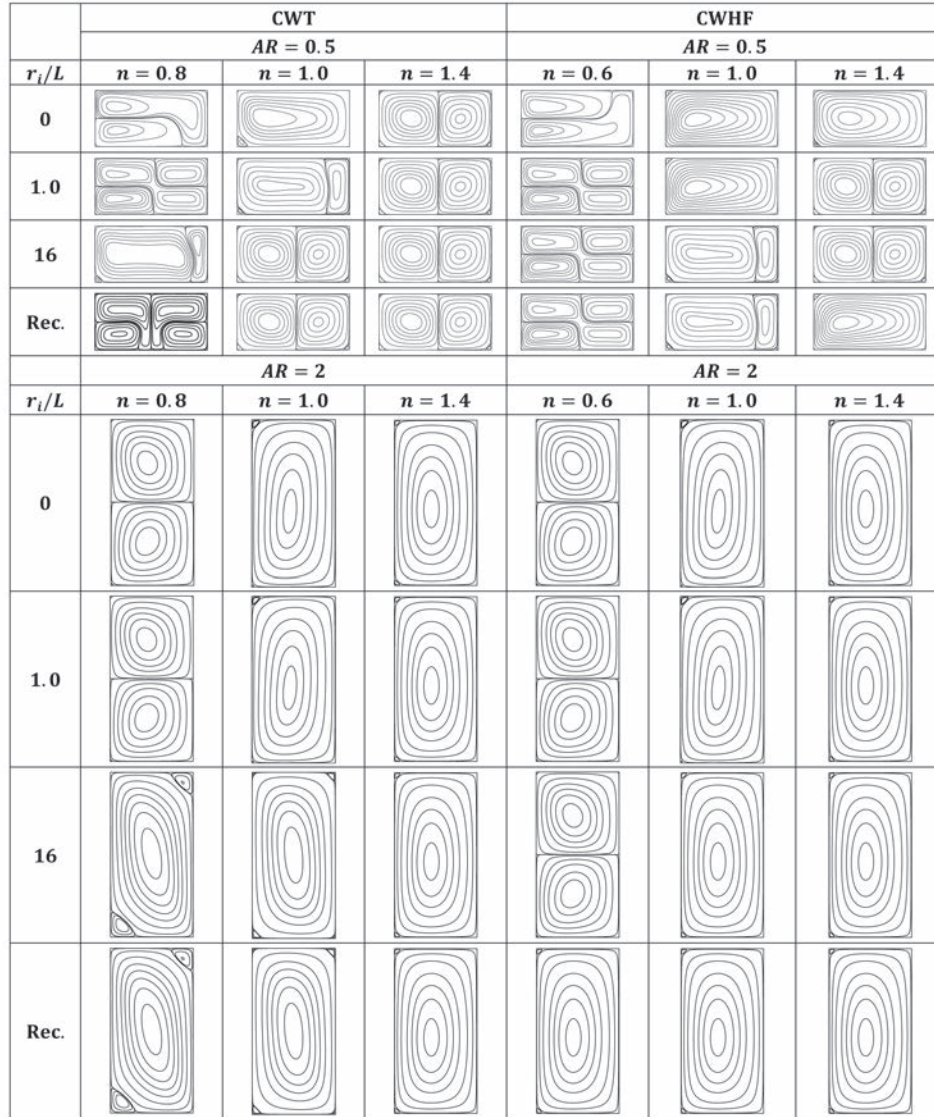


Figure 7.11: Contours of  $\Psi$  for different values of  $n$ ,  $r_i/L$  and  $AR$  at  $Ra = 5 \times 10^4$  and  $Pr = 10^3$  for quiescent flow (Q.F.) initial condition in the case of both CWT (left column) and CWHF (right column) boundary conditions.

It is worth noting that only the cases for which steady two-dimensional solution exists are shown in Figs. 7.7 - 7.11. For example, the steady state solution does not exist for  $n = 0.6$ ,  $AR = 0.5$  and  $r_i/L > 1$  in the case of CWT boundary condition for both initial conditions considered here. The effective Rayleigh number  $Ra_{eff}$  (see. Eqs. 7.8 - 7.11) becomes increasingly larger than its nominal value for

decreasing (increasing) values of  $AR$  ( $r_i/L$ ) for shear-thinning fluids (i.e.  $n < 1$ ) regardless of boundary condition. Thus, the steady state solution corresponding to a particular type of flow pattern associated with small values of  $Ra_{eff}$ , may not be possible to obtain for large values of  $Ra_{eff}$ , where the steady state solution yields a different flow pattern. The flow patterns indicated in Figs. 7.2 - 7.11 are consistent with previous analyses [50, 51, 110] for Newtonian fluids. These flow patterns are not artefacts of convergence criteria and reducing the convergence tolerance by an order of magnitude did not make any difference to the magnitudes of the stream functions and also in the values of the mean Nusselt number. Also, it is possible to have several steady state solutions based on different initial conditions in the Rayleigh-Bénard configuration [52]. Thus, the flow patterns presented in this study need to be validated by experimental analyses for power-law fluids

Moreover, the mean Nusselt number  $\overline{Nu}_{cy}$  assumes higher values for the CWT boundary condition than for the CWHF boundary condition for large (small) values of  $Ra$  ( $n$ ) for a given set of values  $AR$ ,  $Pr$ , and  $r_i/L$  but an opposite behaviour is observed for small (large) values of  $Ra$  ( $n$ ) provided the flow pattern remains unchanged with the variations of  $Ra$  and  $n$ . This can be explained by the scaling estimates of  $\overline{Nu}_{cy}$  as given by Eqs. 7.12 and 7.13. The exponent of  $Ra_{CWT}$  (i.e.  $(2-n)/(2n+2)$ ) assumes greater values than that of  $Ra_{CWHF}$  (i.e.  $(2-n)/(n+4)$ ) and this difference widens with decreasing  $n$  for the same set values of  $Pr$ ,  $AR$  and  $r_i/L$ . By contrast, the exponent of  $Pr$  assumes greater negative value for CWT boundary condition (i.e.  $-n/(2n+2)$ ) than the corresponding value in the case of CWHF boundary condition (i.e.  $-n/(n+4)$ ). The effects of higher exponent of nominal Rayleigh number dominate over the influences of higher magnitude of negative exponent of Prandtl number for high (small) values of  $Ra$  ( $n$ ) for the same set values of  $AR$  and  $r_i/L$ , which leads to higher values of mean Nusselt number  $\overline{Nu}_{cy}$  in the CWT boundary condition than in the case of CWHF boundary condition as long as flow pattern remains unchanged. However, the effects of higher magnitude of negative exponent of Prandtl number overcome the influences of higher exponent of nominal Rayleigh number for small (high) values of  $Ra$  ( $n$ ) for the same set values of  $AR$  and  $r_i/L$ , which leads to higher values of mean Nusselt number  $\overline{Nu}_{cy}$  in the CWHF boundary condition

than in the case of same flow pattern under CWT boundary condition. Although  $\overline{Nu}_{cy}$  exhibits a non-monotonic variation with  $r_i/L$  for small values of  $r_i/L$  and  $AR$  (see Figs. 7.7 - 7.9), the mean Nusselt number  $\overline{Nu}_{cy}$  and the flow pattern for large values of  $r_i/L$  approach those for rectangular enclosures ( $r_i/L \rightarrow \infty$ ) irrespective of the choices of initial and boundary conditions provided the steady-state solution remains valid.

Turan et al. [98] proposed correlation for mean Nusselt number  $\overline{Nu}_{sq,CWHF}$  for laminar Rayleigh- Bénard convection of power-law fluids in square enclosures for CWHF boundary condition (i.e.  $AR = 1.0$ ) for  $10 \leq Pr \leq 10^4$  based on scaling relations given by Eqs. 7.12 and 7.13 as follows:

$$\overline{Nu}_{sq,CWHF} = 0.289 Ra_{CWHF}^{0.014} \frac{Pr^{0.217}}{(1 + Pr)^{0.017}} \left( \frac{Ra_{CWHF}^{1-n/2}}{Pr^{n/2}} \right)^{1/(n/2+2)} A > 1 \quad (7.19)$$

$$A = \exp[(n - 1) c_1 Ra^{c_2} Pr^{c_3}] \quad (7.20)$$

$$c_1 = 0.345; c_2 = 0.129 c_3 = 0.103 \text{ for } n \leq 1 \text{ and } 5 \times 10^3 \leq Ra \leq 10^5 \quad (7.21)$$

$$c_1 = 0.014; c_2 = 0.458 c_3 = 0.097 \text{ for } n > 1 \text{ and } 5 \times 10^3 \leq Ra \leq 10^4 \quad (7.22)$$

$$c_1 = 0.259; c_2 = 0.153 c_3 = 0.073 \text{ for } n > 1 \text{ and } 10^4 \leq Ra \leq 10^5 \quad (7.23)$$

$$\overline{Nu}_{sq,CWHF} = 1 \text{ when } \overline{Nu}_{sq,CWHF} \leq 1 \quad (7.24)$$

This correlation was improved by Yigit et al. [101] for shear-thickening (i.e.

$n > 1$ ) fluids, which is given as:

$$A = n^d \text{ for } n > 1 \quad (7.25)$$

$$d = \left[0.136Pr^{0.04}\ln Ra\right]^{1.5} \text{ for } 5 \times 10^3 \leq Ra \leq 10^4 \quad (7.26)$$

$$d = \left[0.145Pr^{0.04}\ln Ra\right]^{1.5} \text{ for } 10^4 \leq Ra \leq 10^5 \quad (7.27)$$

$$\overline{Nu}_{sq,CWHF} = 1 \text{ when } \overline{Nu}_{sq,CWHF} \leq 1 \quad (7.28)$$

It is worth noting that Eqs. 7.19 - 7.28 are valid for where convection takes place as one cell flow pattern. Equations 7.19 - 7.28 have been modified by Yigit and Chakraborty [100] to include  $AR$  dependence of  $\overline{Nu}_{Rec,CWHF}$  in the following manner for single cell flow pattern for  $0.25 \leq AR \leq 4$ ,  $5 \times 10^3 \leq Ra \leq 10^5$  and  $10 \leq Pr \leq 10^4$ :

$$\overline{Nu}_{Rec,CWHF} = \overline{Nu}_{sq,CWHF} + [(1.987 - 0.157\ln Ra)(1 - AR)] \text{ for } n \leq 1 \quad (7.29)$$

$$\overline{Nu}_{Rec,CWHF} = \overline{Nu}_{sq,CWHF} + \left[ \left( \frac{1 - 0.124\ln Ra}{1.452 - 0.183\ln Ra} \right) (1 - AR) \right] \text{ for } n > 1 \quad (7.30)$$

Equations 7.29 and 7.30 are also valid for where single cell flow pattern is realised and can be utilised for cylindrical annular enclosures with  $r_i/L > 1$  in the CWHF boundary condition. It can be seen from Figs. 7.7 - 7.9 that Eqs. 7.29 and 7.30 satisfactorily predicts  $\overline{Nu}_{cy,CWHF}$  in the CWHF boundary condition. The predictions of Eqs. 7.29 and 7.30 are compared with  $\overline{Nu}_{Rec,CWHF}$  in 7.12, which shows a good agreement in the parameter range considered here.

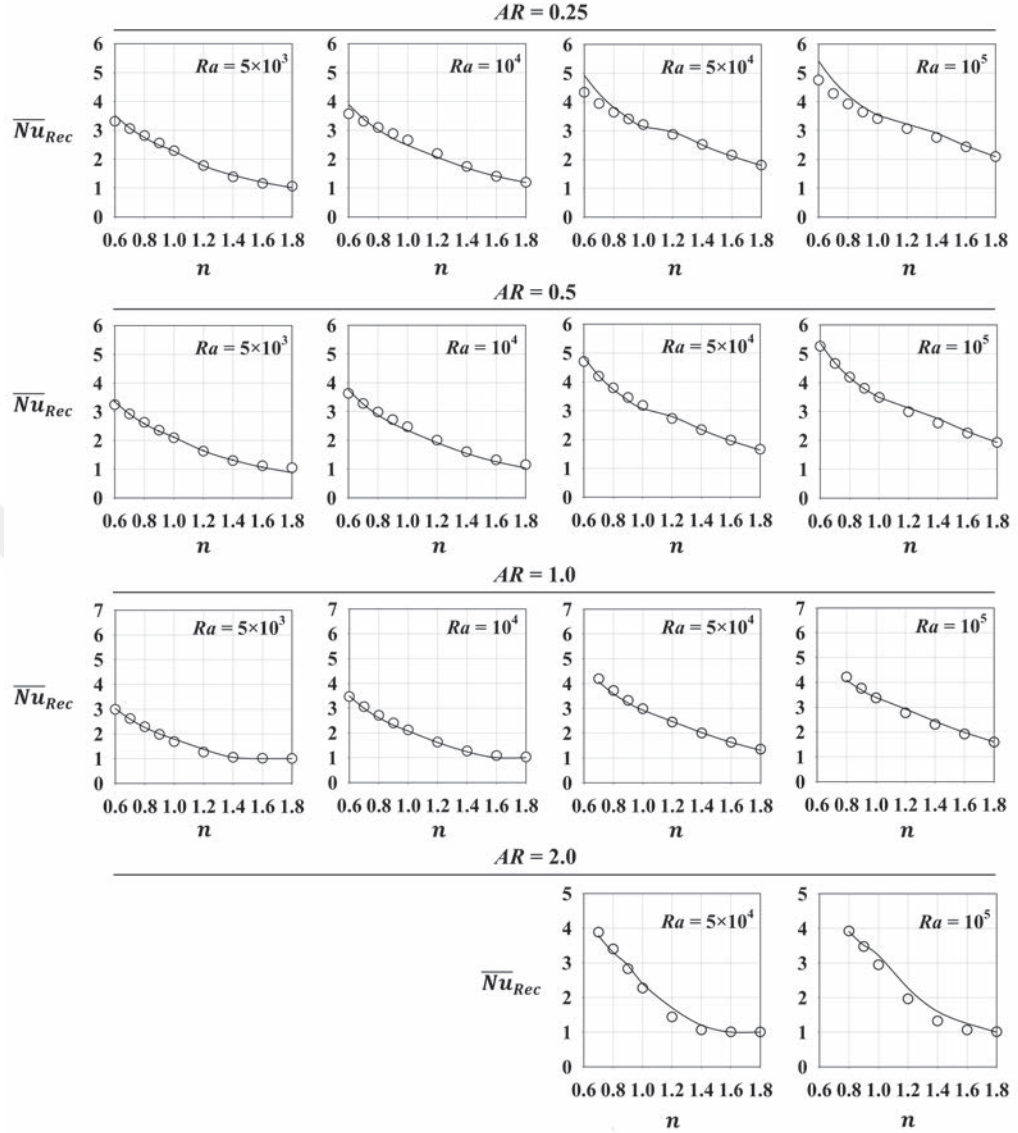


Figure 7.12: Variation of  $\overline{Nu}_{Rec,CWHF}$  with  $n$  for different values of  $Ra$  and  $AR$  at  $Pr = 10^3$ , along with the prediction of Eqs. 7.29 and 7.30.

By contrast, the non-monotonic variation of the mean Nusselt number in response to the changes in  $Ra$ ,  $Pr$ ,  $AR$ ,  $r_i/L$ ,  $n$  and initial condition severely limit the usefulness of a correlation for  $\overline{Nu}_{cy}$  for CWT boundary condition for both rectangular [52] and cylindrical annular enclosures, therefore is not attempted here for CWT boundary condition.

## 7.6 Onset of Convection of Power-law Fluids

It is well-known that a threshold value of nominal Rayleigh number needs to be exceeded for the onset of fluid motion in Rayleigh-Bénard boundary condition. The value of nominal Rayleigh number  $Ra$ , above which  $\overline{Nu}_{Rec}$  values deviate from unity has been taken as the critical Rayleigh number for the onset of convection for shear-thinning (i.e.  $n < 1$ ) and Newtonian (i.e.  $n = 1.0$ ) fluids in this analysis. Table 7.1 lists the nominal values of critical Rayleigh number  $Ra_{crit}$  for the onset of convection for shear-thinning (i.e.  $n < 1$ ) and Newtonian (i.e.  $n = 1.0$ ) fluids for both CWT and CWHF boundary condition. It is evident from Table 7.1 that the onset of convection needs higher values of  $Ra_{crit}$  for increasing values of  $AR$  and  $n$  for both CWT and CWHF boundary conditions. The decrease in  $Ra_{crit}$  with decreasing  $n$  is consistent with the analysis by Tien et al. [96]. The values of  $Ra_{crit}$  for Newtonian (i.e.  $n = 1.0$ ) fluids in square (i.e.  $AR = 1.0$ ) enclosures is found for CWT boundary condition ( $Ra_{crit} = 2568$ ) with linear regression where the instability is super-critical (see later Fig. 7.13 for more discussion). This is consistent with previous findings that  $Ra_{crit}$  was found 2585 by Venturi et al. [111] and 2586 by Ternik et al. [112]. For shear-thinning fluids, where the instability is sub-critical, the  $Ra_{crit}$  is estimated by reducing  $Ra$  in steps of 50, 250, 1000 for  $AR \leq 1$ ,  $AR = 2$  and  $AR = 4$  respectively from an established flow condition where  $\overline{Nu}_{Rec} \gg 1$  as it is listed in Table 7.1 for both CWT and CWHF boundary conditions.

A number of simulations have been also carried out by Yigit et al. [52] to identify the condition for which mean Nusselt number  $\overline{Nu}_{Rec}$  deviates from unity for Rayleigh-Bénard convection of power-law fluids in rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) for CWT boundary condition as shown in Fig. 7.13. The variations of  $(\overline{Nu}_{Rec} - 1)$  with  $Ra$  for  $n = 0.6$  and  $n = 1.0$  are shown in Fig. 7.13 a) for  $AR = 0.25$ , 1.0 and 4.0 for CWT boundary condition. For Newtonian (i.e.  $n = 1.0$ ) fluids, where the instability is super-critical, the critical Rayleigh number is estimated by linear regression of  $(\overline{Nu}_{Rec} - 1)$  versus  $Ra$ . In contrast, for shear-thinning fluids, where the instability is sub-critical, the critical Rayleigh number is estimated by reducing  $Ra$  from an established flow condition where  $\overline{Nu}_{Rec} \gg 1.0$ .

Table 7.1: Values of critical Rayleigh number  $Ra_{crit}$ . Here  $Ra_{crit}$  is estimated by linear regression for Newtonian fluids and reducing  $Ra$  in steps of 50, 250, 1000 for  $AR \leq 1$ ,  $AR = 2$  and  $AR = 4$  respectively for shear-thinning (i.e.  $n < 1$ ) fluids from an established flow condition where  $\overline{Nu}_{Rec} \gg 1$  for both CWT and CWHF boundary conditions. The critical Rayleigh number is equal to zero for the shear thickening (i.e.  $n > 1$ ) fluids as it is shown in Fig. 7.13, which is consistent with previous findings by Alloui et al. [99].

| CWT  |             |            |            |            |            |
|------|-------------|------------|------------|------------|------------|
| $n$  | $AR = 0.25$ | $AR = 0.5$ | $AR = 1.0$ | $AR = 2.0$ | $AR = 4.0$ |
| 0.6  | 600         | 650        | 800        | 2750       | 16500      |
| 0.7  | 850         | 900        | 1100       | 4250       | 30000      |
| 0.8  | 1100        | 1250       | 1600       | 6250       | 55000      |
| 0.9  | 1500        | 1650       | 2200       | 9000       | 100500     |
| 1.0  | 1808        | 2006       | 2568       | 12,022     | 141670     |
| CWHF |             |            |            |            |            |
| $n$  | $AR = 0.25$ | $AR = 0.5$ | $AR = 1.0$ | $AR = 2.0$ | $AR = 4.0$ |
| 0.6  | 350         | 400        | 650        | 3000       | 22500      |
| 0.7  | 450         | 500        | 900        | 4250       | 37500      |
| 0.8  | 550         | 650        | 1150       | 6250       | 60000      |
| 0.9  | 700         | 750        | 1450       | 8750       | 95000      |
| 1.0  | 743         | 841        | 1676       | 11107      | 138850     |

According to power law model, viscosity assumes large values at low shear rate values for shear-thinning fluids ( $n < 1$ ). Thus, it is not surprising to obtain sub-critical instability as shown by abrupt jumps of  $(\overline{Nu}_{Rec} - 1)$  and the maximum value of non-dimensional stream function  $\Psi_{max}$  from zero values with an increase in  $Ra$ . However,  $(\overline{Nu}_{Rec} - 1)$  cannot be used for a sufficiently accurate proof of the onset of fluid motion for shear-thickening (i.e.  $n > 1$ ) fluids since in these cases  $\overline{Nu}_{Rec}$  remains almost equal to unity for large range of  $Ra$ . The values of nominal Rayleigh number  $Ra$  (i.e.  $Ra_{\overline{Nu}_{Rec}=1}$ ) for which  $\overline{Nu}_{Rec}$  values deviate from unity in the third decimal place (i. e.  $\overline{Nu}_{Rec} \approx 1.001$ ) for shear thickening

fluids (i.e.  $n > 1$ ) are listed in Table 7.2 for both CWT and CWHF boundary conditions.

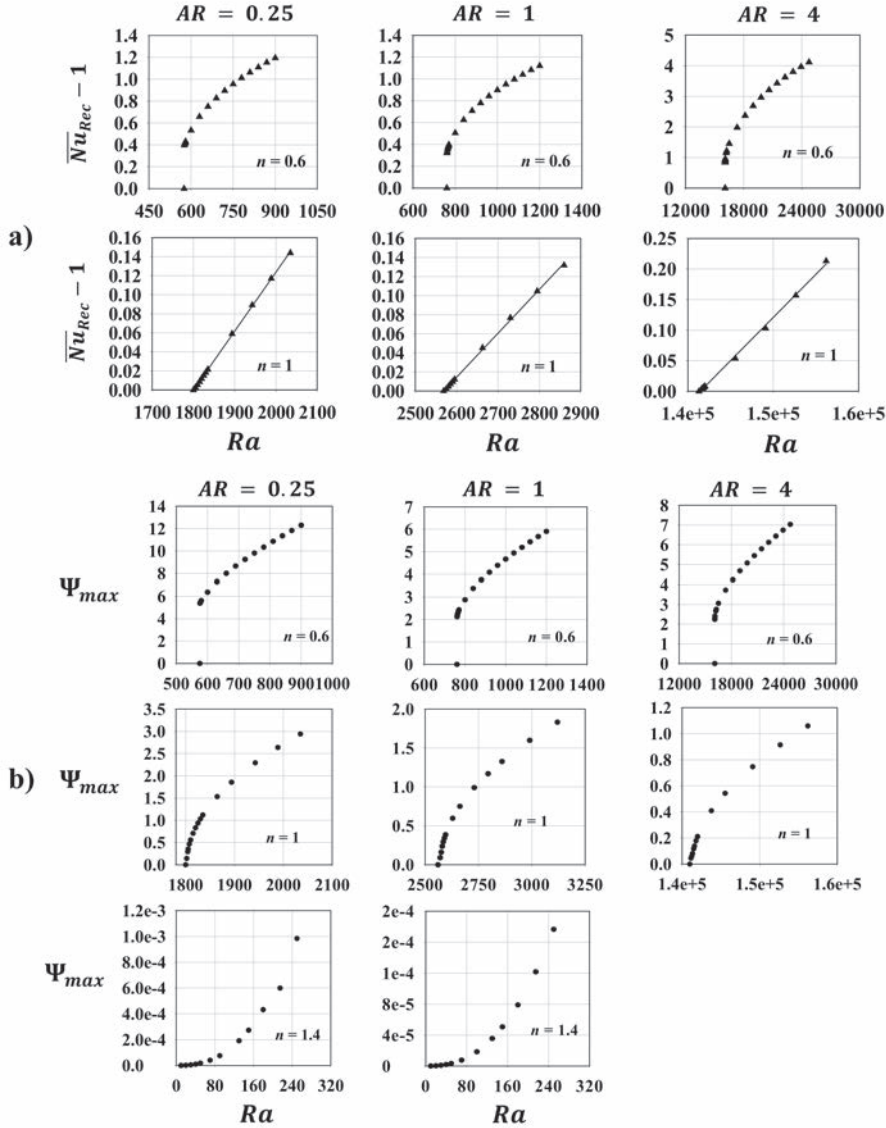


Figure 7.13: (a) Variation of  $(\overline{Nu}_{Rec} - 1)$  with  $Ra$  for different values of  $AR$  for  $n = 0.6$  and  $n = 1.0$  for CWT boundary condition, (b) variation of the maximum value of the non-dimensional stream function  $\Psi_{max}$  with  $Ra$  for different values of  $AR$  for  $n = 0.6$ ,  $n = 1.0$  and  $n = 1.4$  for CWT boundary condition. The variation of  $\Psi_{max}$  with  $Ra$  is not shown for  $AR = 4.0$  in the case of  $n = 1.4$  because  $\Psi_{max}$  values remain smaller than  $10^{-8}$  for small values of  $Ra$  (i.e.  $Ra \rightarrow 0$ ).

Table 7.2: Values of Rayleigh number  $Ra_{\overline{Nu}_{Rec}=1}$  for which  $\overline{Nu}_{Rec}$  values deviate from unity in the third decimal place (i.e.  $\overline{Nu}_{Rec} \approx 1.001$ ) for shear thickening fluids (i.e.  $n > 1$ ).

| CWT  |             |            |            |            |            |
|------|-------------|------------|------------|------------|------------|
| $n$  | $AR = 0.25$ | $AR = 0.5$ | $AR = 1.0$ | $AR = 2.0$ | $AR = 4.0$ |
| 1.2  | 2000        | 2250       | 3000       | 17000      | 250000     |
| 1.4  | 2500        | 2750       | 3500       | 30000      | 450000     |
| 1.6  | 3000        | 3000       | 4500       | 40000      | 750000     |
| 1.8  | 3500        | 4000       | 5000       | 65000      | 1500000    |
| CWHF |             |            |            |            |            |
| $n$  | $AR = 0.25$ | $AR = 0.5$ | $AR = 1.0$ | $AR = 2.0$ | $AR = 4.0$ |
| 1.2  | 775         | 875        | 1850       | 15000      | 250000     |
| 1.4  | 800         | 900        | 2000       | 20000      | 425000     |
| 1.6  | 825         | 925        | 2200       | 30000      | 1050000    |
| 1.8  | 850         | 950        | 2500       | 40000      | 1400000    |

Moreover, the value of  $(\overline{Nu}_{Rec} - 1)$  is likely to be affected by finite mesh resolution and convergence criteria. For this reason, the critical Rayleigh number for shear-thickening (i.e.  $n > 1$ ) fluids is estimated by the deviation of the maximum value of non-dimensional stream function  $\Psi_{max}$  from a zero value. The variation of  $\Psi_{max}$  with  $Ra$  for  $n = 1.4$  in Fig. 7.13 b) reveals that fluid flow initiates for any non-zero value of  $Ra$ . Thus, the critical Rayleigh number for shear-thickening (i.e.  $n > 1$ ) fluids can be considered as  $Ra_{crit} = 0$ . The viscosity assumes vanishingly small values for small shear rate values in the case of shear thickening fluids ( $n > 1$ ) and thus it does not offer any resistance to fluid motion once  $Ra$  approaches zero value. This is found to be consistent with previous findings of Alloui et al. [99]. However, it is worth noting that the presence of flow within the enclosure does not indicate appreciable heat transfer by advection for shear-thickening fluids. It is clear from Table 7.2 that the heat transfer within the enclosure takes place predominantly by thermal conduction for  $Ra < Ra_{\overline{Nu}_{Rec}=1}$  in spite of flow being present in the enclosure for  $Ra > 0$ .

The values of  $Ra_{crit}$  in Table 7.1 for the onset of convection for shear-thinning and Newtonian fluids ( $n \leq 1$ ) and the values of  $Ra_{\overline{Nu}_{Rec}=1}$  in Table 7.2 for shear-thickening fluids (i.e.  $n > 1$ ) are parameterised which are listed in Table 7.3. It is worth noting that correlation for  $Ra_{crit}$  given in Table 7.3 provides comparable predictions as that of the correlation proposed by Turan et al. [98]. Figures 7.14 and 7.15 shows the variations of  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{Rec}=1}$  with  $AR$  along with the predictions of correlations given in Table 7.3 respectively for both CWT and CWHF boundary conditions.

Table 7.3: Summary of  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{Rec}=1}$  correlation for both CWT and CWHF boundary conditions [52, 100].

| CWT        |                                                                                                                                                                                                                                                                                          |
|------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $n \leq 1$ | $Ra_{crit} = 1620 \exp(0.48AR) n^{2.25}$ for $0.25 \leq AR \leq 1$<br>$Ra_{crit} = [1964 \exp(1.08AR) - 4168] n^a$ for $1 < AR \leq 4$<br>$a = [534 / (1 + \exp((-6.23 - AR) / 1.28))] - 530$                                                                                            |
| $n > 1$    | $Ra_{\overline{Nu}_{Rec}=1} = 660 \exp(0.5AR + 0.85n)$ for $0.25 \leq AR \leq 1$<br>$Ra_{\overline{Nu}_{Rec}=1} = a_1 \exp(b_0 n)$ for $1 < AR \leq 4$<br>$a_1 = 1400$ ; $b_0 = 0.64AR^{1.73}$ for $1 < AR \leq 2$<br>$a_1 = 5600$ ; $b_0 = 1.15 + 0.08 \exp(0.8AR)$ for $2 < AR \leq 4$ |
| CWHF       |                                                                                                                                                                                                                                                                                          |
| $n \leq 1$ | $Ra_{crit} = 1700n^2 + [(AR - 1)(a_0 n^b)]$ for $0.25 \leq AR \leq 4$<br>$a_0 = 65860 / [1 + \exp((3.387 - AR) / 0.789)]$<br>$b = 2.38 + 0.27AR$                                                                                                                                         |
| $n > 1$    | $Ra_{\overline{Nu}_{Rec}=1} = 720 + \exp(3.85AR) [13.79 / (1 - 0.352n)]$ for $0.25 \leq AR \leq 1$<br>$Ra_{\overline{Nu}_{Rec}=1} = [1370 \exp(a_2 AR)] - 4000$ for $1 < AR \leq 4$<br>$a_2 = 1.281 + 0.462 / (1 + \exp((1.462 - n) / 0.09))$                                            |

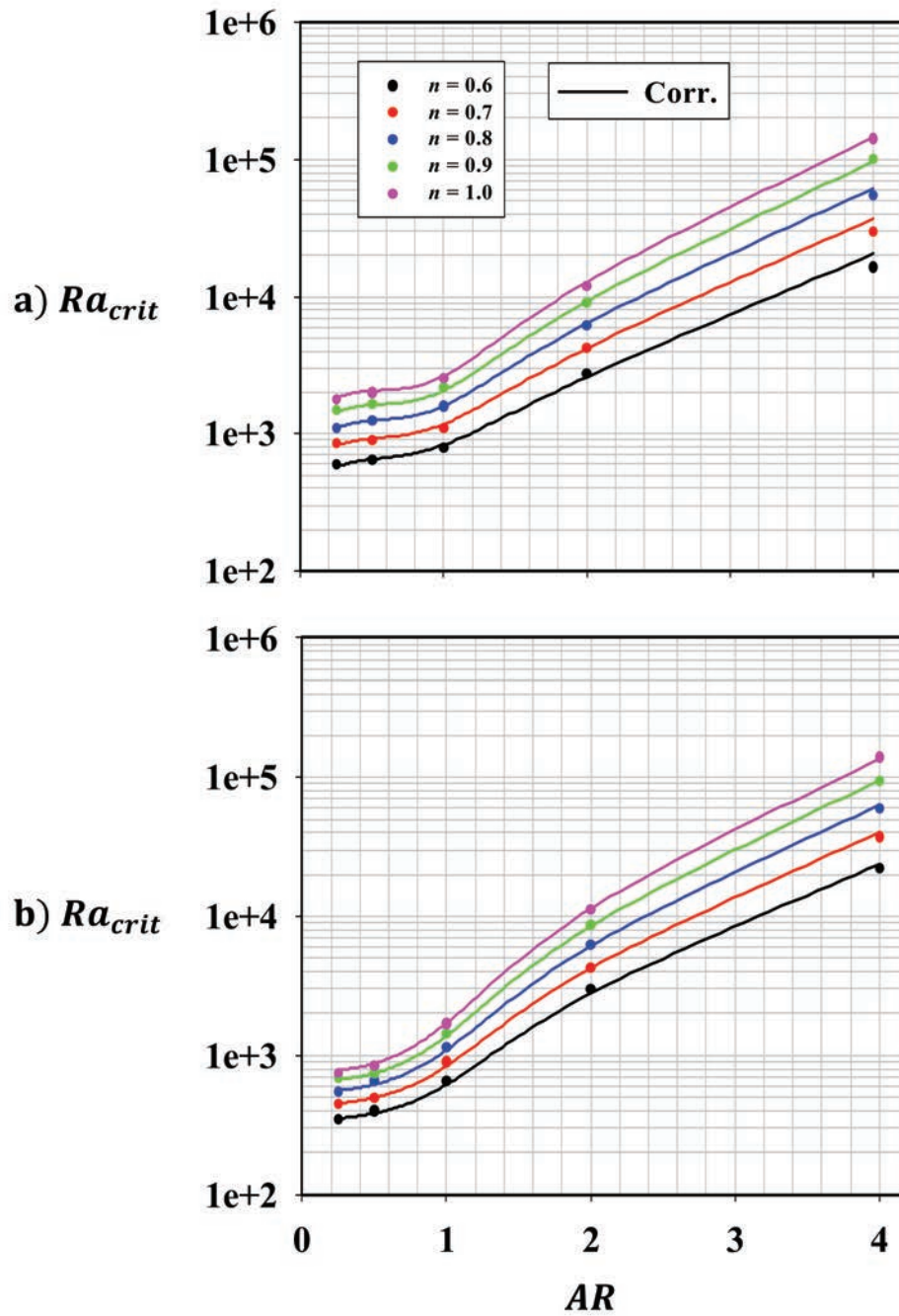


Figure 7.14: a) Variations of  $Ra_{crit}$  with  $AR$  with for different values of  $n$  for a) CWT, b) CWHF boundary conditions along with the correlation given in Table 7.3.

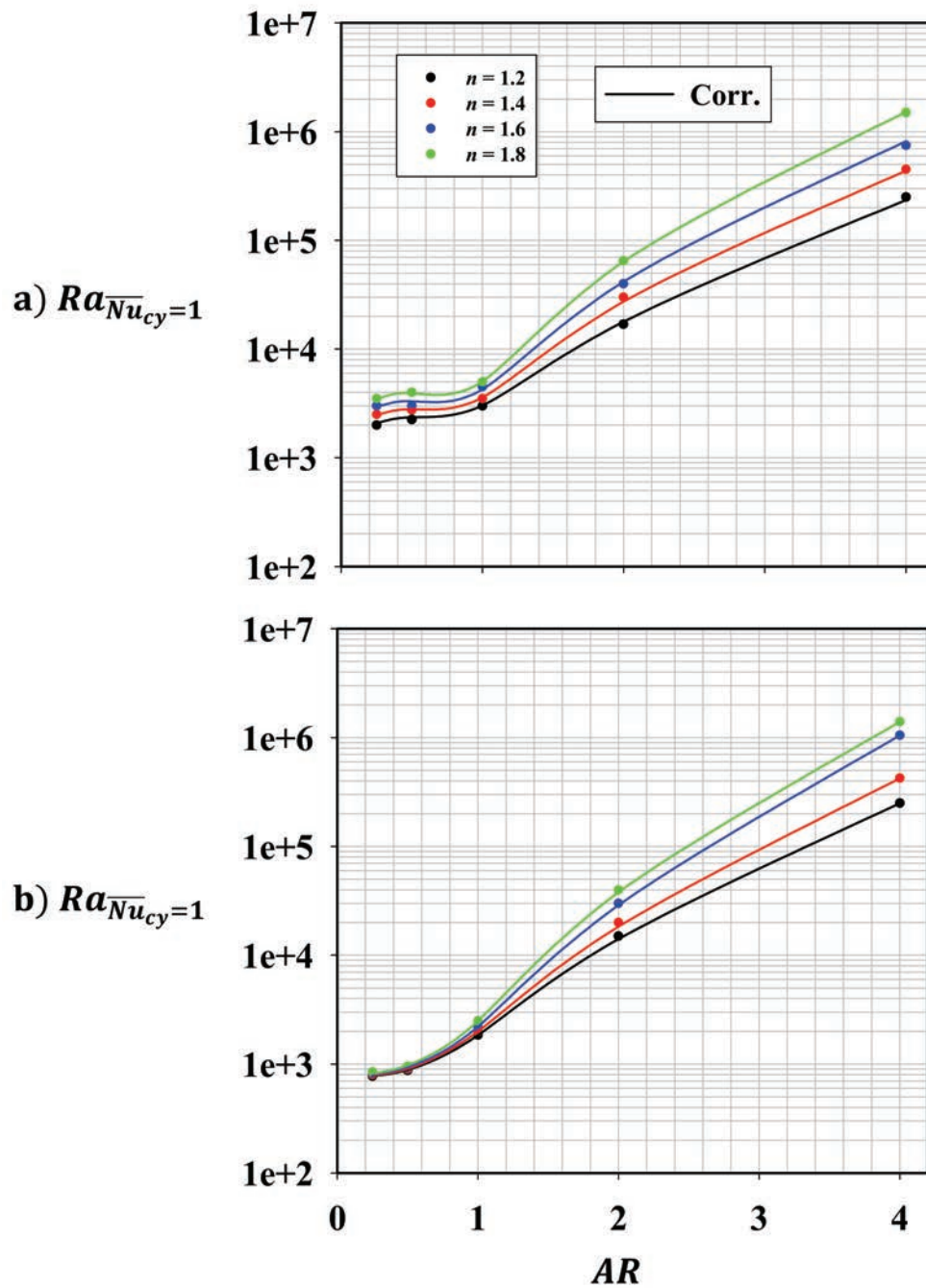


Figure 7.15: a) Variations of  $Ra_{\overline{Nu}_{cy}=1}$  with  $AR$  for different values of  $n$  for a) CWT, b) CWHF boundary conditions along with the correlation given in Table 7.3.

It can be seen from Fig. 7.14 and 7.15 that correlations given in Table 7.3 satisfactorily predict the variation of  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{Rec}=1}$  with  $AR$  for the range of  $0.25 \leq AR \leq 4$  and  $0.6 \leq n \leq 1.8$  for both CWT and CWHF boundary conditions. It is worth noting that Turan et al. [98] indicated that change in the  $Pr$  does not alter the values of  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{Rec}=1}$  for power-law fluids in the range of  $10 \leq Pr \leq 10^4$  for  $AR = 1.0$  regardless of boundary conditions. Therefore, the correlations are given in Table 7.3 for  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{Rec}=1}$  are valid for the range of  $0.6 \leq n \leq 1.8$ ,  $0.25 \leq AR \leq 4$  and  $10 \leq Pr \leq 10^4$  for both CWT and CWHF boundary conditions.

Accordingly, Table 7.4 lists the critical Rayleigh number  $Ra_{crit}$  for the onset of convection for shear-thinning and Newtonian fluids and the threshold Rayleigh number  $Ra_{\overline{Nu}_{cy}=1}$  for which  $\overline{Nu}_{cy}$  values deviate from unity in the third decimal place (i.e.  $\overline{Nu}_{cy} \approx 1.001$ ) for shear-thickening fluids which are consistent with previous findings of Refs. [52, 100] for the CWT and CWHF boundary conditions, respectively. Table 7.4 indicates that  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{cy}=1}$  remains largely independent of  $r_i/L$  for different  $AR$  in the both CWT and CWHF boundary conditions but  $Ra_{crit}$  for  $r_i/L = 0$  has been found to be slightly smaller in the case of CWT boundary condition. Therefore, the correlations proposed by Refs. [52, 100] (Table 7.3) for  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{cy}=1}$  in the case of laminar Rayleigh-Bénard convection of power-law fluids in rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) are listed in Table 7.4, which shows that these correlations can also be used to predict  $Ra_{crit}$  and  $Ra_{\overline{Nu}_{cy}=1}$  in the case of rectangular cross-sectional cylindrical annular enclosures for both CWT and CWHF boundary conditions.

Additionally, for shear-thinning (i.e.  $n < 1$ ) fluids the effective Rayleigh number may assume much larger values than  $Ra$  so that laminar steady-state two dimensional solutions cease to exist for  $n < 1$  at, what appear to be, quite low nominal values of  $Ra$  [98]. This is consistent with the observations made from Figs. 7.1 - 7.11 which show that a steady two-dimensional solution could not be found under some conditions involving small values of  $r_i/L$ ,  $AR$  and/or  $n$ . Thus a threshold value  $Ra_{ss}$  can be expected for the effective Rayleigh number such that a steady two-dimensional solution does not exist when  $Max(Ra_{eff}^H, Ra_{eff}^V) > Ra_{ss}$ .

Table 7.4: Values of critical Rayleigh number  $Ra_{crit}$  for different values of  $n$  and  $r_i/L$  for both CWT and CWHF boundary conditions at  $Pr = 10^3$ . Values of Rayleigh number  $Ra_{\overline{Nu}_{cy}=1}$  for which  $\overline{Nu}_{cy}$  values deviate from unity in the third decimal place (i.e.  $\overline{Nu}_{cy} \approx 1.001$ ) for shear-thickening fluids (i.e.  $n = 1.8$ ) for both CWT and CWHF boundary conditions at  $Pr = 10^3$ .

|         | CWT        |          |          | CWHF       |          |          |
|---------|------------|----------|----------|------------|----------|----------|
|         | $n = 0.6$  |          |          | $n = 0.6$  |          |          |
| $r_i/L$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ |
| 0       | 700        | 750      | 2500     | 400        | 650      | 3000     |
| 1       | 650        | 800      | 2750     | 400        | 650      | 3000     |
| 16      | 650        | 800      | 2750     | 400        | 650      | 3000     |
| Rec.    | 650        | 800      | 2750     | 400        | 650      | 3000     |
|         | $n = 1.0$  |          |          | $n = 1.0$  |          |          |
| $r_i/L$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ |
| 0       | 1850       | 2250     | 11000    | 850        | 1700     | 11250    |
| 1       | 2000       | 2600     | 12250    | 850        | 1700     | 11250    |
| 16      | 2000       | 2600     | 12250    | 850        | 1700     | 11250    |
| Rec.    | 2000       | 2600     | 12250    | 850        | 1700     | 11250    |
|         | $n = 1.8$  |          |          | $n = 1.8$  |          |          |
| $r_i/L$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ | $AR = 0.5$ | $AR = 1$ | $AR = 2$ |
| 0       | 4000       | 5000     | 65000    | 950        | 2500     | 40000    |
| 1       | 4000       | 5000     | 65000    | 950        | 2500     | 40000    |
| 16      | 4000       | 5000     | 65000    | 950        | 2500     | 40000    |
| Rec.    | 4000       | 5000     | 65000    | 950        | 2500     | 40000    |

A number of simulations have been carried out for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) in CWT boundary condition for different values of  $Ra$  and  $n$  within the aspect ratio range  $0.25 \leq AR \leq 4$ , and it has been found that a converged steady two-dimensional solution cannot be obtained when  $Max(Ra_{eff}^H, Ra_{eff}^V) \leq Ra_{ss} = 2.1 \times 10^8 AR^{1.75}$  (unsteady numerical solutions can be obtained however),

which essentially leads to the following condition for steady two-dimensional solutions to exist in this boundary condition. The aforementioned conditions given are shown on a regime diagram in Fig. 7.16 for  $n = 0.6, 1.0$  and  $1.4$  for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) in CWT boundary condition. For  $Ra \leq Ra_{crit}$  ( $Ra \leq Ra_{\overline{Nu}_{Rec}=1}$ ) the heat transfer takes place principally due to thermal conduction for  $n \leq 1$  ( $n > 1$ ) and thus this regime has been termed as the ‘conduction-dominated regime’ in Fig. 7.16. The region given by  $Max(Ra_{eff}^H, Ra_{eff}^V) \leq Ra_{ss} = 2.1 \times 10^8 AR^{1.75}$  and  $Ra > Ra_{crit}$  ( $Ra > Ra_{\overline{Nu}_{Rec}=1}$ ) for  $n \leq 1$  ( $n > 1$ ) in Fig. 7.16 is termed as the ‘steady two-dimensional laminar convection regime’. As steady two-dimensional laminar solutions do not exist for  $Max(Ra_{eff}^H, Ra_{eff}^V) > Ra_{ss} = 2.1 \times 10^8 AR^{1.75}$  the corresponding regime is referred to as the ‘unsteady convection regime’. The validity of the above regime diagram for  $n = 0.6$  has been substantiated from the conditions A, B, and C, shown on the regime diagram. The quiescent condition initial condition is used for condition A, whereas the steady-state solution obtained for  $Ra = 5 \times 10^3$ ,  $n = 0.6$  and  $AR = 0.5$  based on established flow initial condition is used as the initial condition for conditions B and C. For the conditions A and B the mean Nusselt number  $\overline{Nu}_{Rec}$  assumes constant values (i.e. condition A:  $\overline{Nu}_{Rec} = 1.0$ ) as predicted by the regime diagram. For the condition C, the temporal evolution of  $\overline{Nu}_{Rec}$  shows a complex temporal oscillation. Thus, the numerical results for conditions A, B and C substantiate the validity of the regime diagram and the regime boundaries shown in Fig. 7.16. It is worth noting that the current analysis primarily deals with the steady-state of laminar natural convection in rectangular enclosures for  $0.25 \leq AR \leq 4$ ,  $10^3 \leq Ra \leq 10^5$  at  $Pr = 10^3$  for CWT boundary condition and the unsteady simulations discussed in Fig. 7.16 for conditions A, B and C have been carried out only to validate the regime diagram. The unsteady regime solution for power-law fluids in rectangular enclosures could be an interesting subject for future investigation but it is beyond the scope of the current analysis. Moreover, the non-monotonic variation of the mean Nusselt number  $\overline{Nu}_{Rec}$  in response to the changes in  $Ra$ ,  $AR$ ,  $n$  and its dependence on the choice of initial condition severely limit the usefulness of a correlation for  $\overline{Nu}_{Rec}$  and therefore one is not attempted here. However, the regime diagram given by Fig. 7.16 provides a useful guide for the effects of aspect ratio on Rayleigh–Bénard convection of

power-law fluids in rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) for CWT boundary condition.

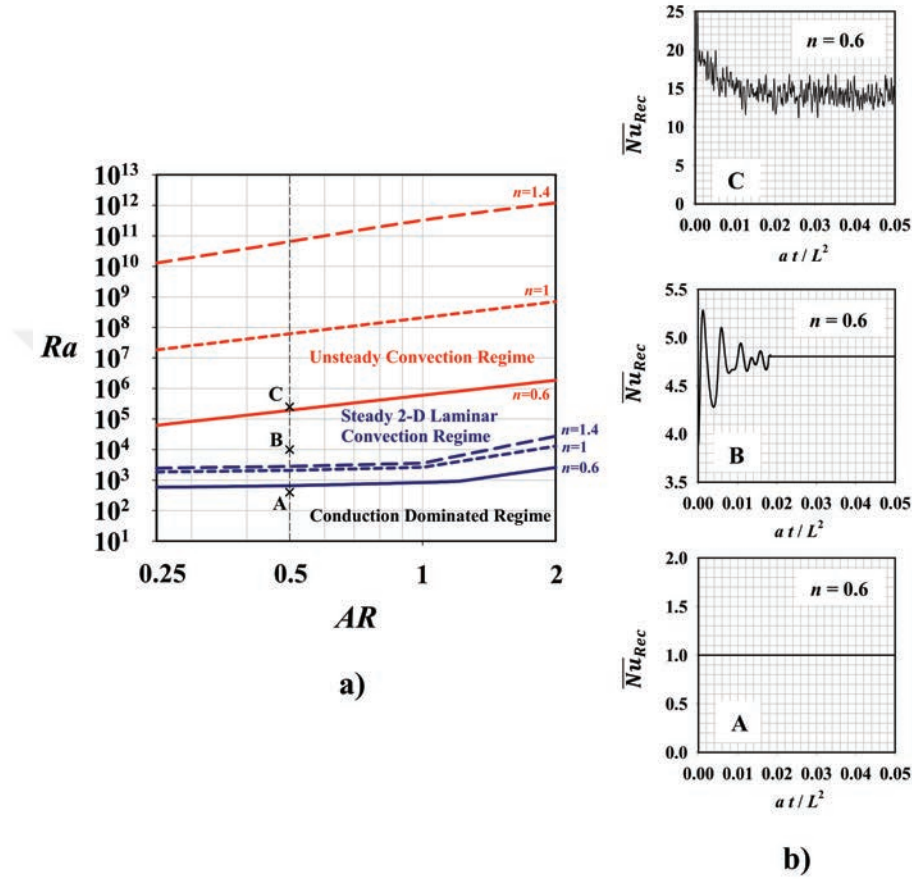


Figure 7.16: (a) Different regimes of convection for  $n = 0.6, 1.0$  and  $1.4$ . The boundary between the steady 2-D laminar convection and unsteady convection regimes are shown by the red lines for CWT boundary condition. The boundary between the conduction dominated and steady 2-D laminar convection regimes are distinguished by the blue lines, (b) Temporal evolution of  $\overline{Nu}_{Rec}$  with dimensionless time  $at/L^2$  for  $n = 0.6$  and  $Pr = 10^3$ . (A) conduction dominated regime  $Ra = 400$ ; (B) steady 2-D laminar convection Regime  $Ra = 10^4$ ; (C) unsteady convection regime  $Ra = 2.5 \times 10^5$ . The quiescent condition initial condition is used for condition A, whereas the steady-state solution obtained for  $Ra = 5 \times 10^3$ ,  $n = 0.6$  and  $AR = 0.5$  based on established flow initial condition is used as the initial condition for conditions B and C.

It is important to note that the boundaries, which distinguish one regime to the other on Fig. 7.16, are based on scaling arguments so these boundaries should not be treated rigidly but need to be considered only in an order of magnitude sense.

## 7.7 Summary of Key Findings

The effects of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$ ,  $r_i/L$ , boundary condition (i.e. CWT and CWHF) and initial condition (i.e. established (E.F.) and quiescent (Q.F.) flow) on steady-state laminar Rayleigh- Bénard convection of power-law fluids in rectangular cross-sectional cylindrical annular enclosures have been numerically investigated in this chapter. It has been found that strength of thermal convection augments with increasing (decreasing)  $Ra$  ( $n$ ) regardless of the boundary condition. However, the mean Nusselt number  $\overline{Nu}_{cy}$  has been found to be essentially insensitive to the change in  $Pr$  for both CWT and CWHF boundary conditions. Moreover,  $\overline{Nu}_{cy}$  decreases with increasing  $AR$  and attains unity value (indicating pure conduction) for  $AR > 3$  for shear-thinning fluids (i.e.  $n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear-thickening (i.e.  $n > 1$ ) fluids irrespective of the choices of  $Ra$ ,  $Pr$ ,  $r_i/L$ , initial and boundary conditions. Furthermore,  $\overline{Nu}_{cy}$  assumes higher values for CWT boundary condition than that in the case for CWHF boundary condition for large (small) values of  $Ra$  ( $n$ ), whereas an opposite trend is observed for small (large) values of  $Ra$  ( $n$ ) provided the qualitative nature of the flow pattern remains unchanged. Additionally,  $\overline{Nu}_{cy}$  and flow patterns for large values of  $r_i/L$  approach those for rectangular enclosures ( $r_i/L \rightarrow \infty$ ) for both CWT and CWHF boundary conditions. Moreover,  $\overline{Nu}_{cy}$  dependence of  $Ra$ ,  $Pr$ ,  $AR$ ,  $r_i/L$ ,  $n$  and initial condition shows complex non-monotonic trends due to the changes in flow pattern (i.e. number of cells) for shear-thinning fluids (i.e.  $n < 1$ ). However, this behaviour is much weaker for shear-thickening fluids (i.e.  $n > 1$ ) for both CWT and CWHF boundary conditions. Detailed scaling analysis is utilised to explain the observed influences of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$  and  $r_i/L$  on the mean Nusselt number  $\overline{Nu}_{cy}$  and the non-monotonic variation of the mean Nusselt number in response to the changes in  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$  and  $r_i/L$  and initial condition severely limit the usefulness of a correlation for  $\overline{Nu}_{cy}$  for CWT boundary condition for

cylindrical annular enclosures, therefore is not attempted here.

The effects of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$ ,  $r_i/L$  and boundary condition (i.e. CWT and CWHF) on critical Rayleigh number  $Ra_{crit}$  for which convection ceases to influence the thermal transport and thermal conduction becomes the principal heat transfer mechanism have been also numerically investigated in this chapter. It has been found that  $Ra_{crit}$  increases with increasing values of  $n$  and  $AR$ , however,  $Ra_{crit}$  values are found to independent of  $Pr$  and  $r_i/L$ . regardless of the boundary condition. The results of scaling analysis and numerical findings have been utilised to propose a new correlation for  $Ra_{crit}$  for both Newtonian and power-law fluids in the case of both CWT and CWHF boundary conditions.

# Chapter 8

## Main Findings & Future Work

### 8.1 Main Findings

In this thesis, the steady-state laminar natural convection of Bingham and power-law fluids in both rectangular and cylindrical annular enclosures have been investigated numerically. The detailed numerical simulations have been conducted to analyse the effects of nominal Rayleigh number  $Ra$ , Prandtl number  $Pr$ , Bingham number  $Bn$ , power-law index  $n$ , normalized internal radius  $r_i/L$  and aspect ratio  $AR$  ( $AR = H/L$  where  $H$  is the enclosure height) on mean Nusselt number using Buckingham's  $\pi$  theorem (i.e.  $Nu = f_1(Ra, Pr, Bn, n, r_i/L, AR)$ ). Two main configurations (i.e. differentially heated side walls and Rayleigh-Bénard configurations) have been considered for both constant wall temperature (CWT) and constant wall heat flux (CWHF) boundary conditions in the chapters 4 - 7 for the parameters considered here. The following important results have been found in consequence of present analyses.

#### 8.1.1 Differentially Heated Side Walls Configuration

Effects of yield stress (i.e. Bingham number) and shear-thinning / thickening character of viscosity (i.e. power-law index) on heat and momentum transport have been investigated for a range of different values of  $r_i/L$  (i.e.  $0.125 \leq r_i/L \leq$

16);  $AR$  (i.e.  $0.125 \leq AR \leq 8$ );  $Ra$  (i.e.  $10^3 \leq Ra \leq 10^6$ ) and  $Pr$  (i.e.  $10 \leq Pr \leq 10^3$ ) for both CWT and CWHF boundary conditions in chapters 4 and 5.

### 8.1.1.1 Bingham Fluids

- The thermal convection strengthens (i.e. which reflects as increase in the mean Nusselt number  $\overline{Nu}_i$ ) with increasing  $Ra$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids regardless of boundary conditions.
- The mean Nusselt number  $\overline{Nu}_i$  decreases with increasing  $Bn$ , which is the indicator of weakening of the thermal convection. Therefore, the  $\overline{Nu}_i$  obtained for Bingham fluids are smaller than those obtained in the case of Newtonian fluids (i.e.  $Bn = 0$ ) for the same values of  $Ra$ ,  $Pr$ ,  $AR$ ,  $r_i/L$  for both CWT and CWHF boundary conditions.
- The mean Nusselt number normalized by the corresponding value obtained for pure conductive transport  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases with increasing  $r_i/L$  before asymptotically approaching the mean Nusselt number  $\overline{Nu}_{Rec}$  for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ). This suggest that the ratio of the convective to the conductive transport strengthens with increasing cylinder radius in the cylindrical annular enclosures for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids irrespective of the boundary conditions.
- The mean Nusselt number normalised by the corresponding value for pure conduction  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  decreases with increasing  $Pr$  for large values of  $Bn$ , however an opposite trend has been observed for small values of  $Bn$  cases for a given value of the  $Ra$ ,  $AR$ ,  $r_i/L$  for both CWT and CWHF boundary conditions.
- The mean Nusselt number normalised by the corresponding value for pure conduction  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  decreases with increasing  $Bn$  and attains a value of unity (purely conductive thermal transport,  $\overline{Nu}_i / (\overline{Nu}_i)_{cond} = 1.0$ ) for greater than a threshold value of  $Bn$  (i.e.  $(Bn_{max})_i$  for this configuration where  $(\overline{Nu}_i / (\overline{Nu}_i)_{cond} = 1.01)$ ) as the heat transfer takes place due to purely conduction.

- The value of  $(Bn_{max})_i$  increases with increasing  $r_i/L$  before asymptotically approaching the corresponding threshold value of  $(Bn_{max})_{Rec}$  for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ). This suggests that the conduction-dominated regime occurs at higher values of  $(Bn_{max})_i$  for increasing (decreasing) values of  $Ra$  ( $Pr$ ) for given value of  $AR$  and  $r_i/L$  regardless of boundary conditions.
- The mean Nusselt number normalised by the corresponding value for pure conduction  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  shows a non-monotonic trend with increasing  $AR$  in the CWT boundary condition, whereas  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases monotonically with increasing  $AR$  in the CWHF boundary condition for a given set of values of  $Ra$ ,  $Pr$ ,  $r_i/L$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids.
- The influences of convective thermal transport strengthen while thermal diffusive transport weakens with increasing  $AR$ , and these competing effects are responsible for the non-monotonic  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  variation with  $AR$  in the CWT boundary condition.
- Numerical findings based on detailed scaling analysis have been utilised to propose a new correlation for the  $\overline{Nu}_i$  for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluids, which have been shown to predict numerical findings accurately for the range of parameters considered here.

#### 8.1.1.2 Power-law Fluids

- The mean Nusselt number  $\overline{Nu}_i$  increases with decreasing (increasing)  $n$  ( $Ra$ ) due to strengthening of thermal advection irrespective of boundary condition. However,  $\overline{Nu}_i$  has been found to be insensitive to the change in  $Pr$  regardless of the value of  $n$  considered here for both CWT and CWHF boundary conditions.
- The mean Nusselt number  $\overline{Nu}_i$  assumes smaller values for the CWHF boundary condition than in the CWT boundary condition for the same set of values of  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$  and  $r_i/L$ .

- The mean Nusselt number normalized by the corresponding value obtained for pure conductive transport  $\overline{Nu}_i / (\overline{Nu}_i)_{cond}$  increases with increasing  $r_i/L$  before asymptotically approaching the mean Nusselt number  $\overline{Nu}_{Rec}$  for rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ). This indicates that the contribution of convection to overall thermal transport strength increases with an increase in  $r_i/L$  for regardless of the value of  $n$ , since the Nusselt number for pure conductive transport  $(\overline{Nu}_i)_{cond}$  decreases with an increase in  $r_i/L$  for cylindrical annular spaces.
- The mean Nusselt number  $\overline{Nu}_i$  exhibits a non-monotonic variation with an increase in  $AR$  for same set of values of  $Ra$ ,  $Pr$ ,  $r_i/L$  for shear thinning ( $n < 1$ ), Newtonian ( $n = 1.0$ ) and shear thickening ( $n > 1$ ) fluids in the CWT configuration. However,  $\overline{Nu}_i$  increases monotonically with an increase in  $AR$  in the CWHF configuration irrespective of the value of  $n$ .
- The competition between the strengthening of thermal convection and the weakening of conductive thermal transport with an increase in  $AR$  is responsible for the non-monotonic  $AR$  dependence of  $\overline{Nu}_i$  in the CWT boundary condition.
- Numerical findings based on detailed scaling analysis have been utilised to propose a new correlation for the  $\overline{Nu}_i$  for power-law fluids, which have been shown to predict numerical findings accurately for the range of parameters considered here.

### 8.1.2 Rayleigh- Bénard Configuration

Effects of yield stress (i.e. Bingham number) and shear-thinning / thickening character of viscosity (i.e. power-law index) on heat and momentum transport have been investigated for a range of different values of  $r_i/L$  (i.e.  $0 \leq r_i/L \leq 16$ );  $AR$  (i.e.  $0.25 \leq AR \leq 4$ );  $Ra$  (i.e.  $10^3 \leq Ra \leq 10^5$ ) and  $Pr$  (i.e.  $10 \leq Pr \leq 10^3$ ) for both CWT and CWHF boundary conditions in chapters 6 and 7.

### 8.1.2.1 Bingham Fluids

- The mean Nusselt number  $\overline{Nu}_{cy}$  for Bingham fluids has been found to be smaller than the corresponding values obtained for Newtonian fluids (i.e.  $Bn = 0$ ) for both CWT and CWHF boundary conditions due to additional flow resistance arising from the yield stress in the case of Bingham fluids.
- The mean Nusselt number  $\overline{Nu}_{cy}$  decreases with increasing  $Bn$  and it assumes a value equal to unity (i.e.  $\overline{Nu}_{cy} = 1.0$ ) for large values of  $Bn$  for greater than a threshold value of  $Bn$  (i.e.  $(Bn_{max})_{cy}$ ) for this configuration where  $\overline{Nu}_{cy} = 1.01$ , which indicates that thermal transport takes place purely since fluid flow becomes extremely weak to influence thermal transport under such conditions.
- The corresponding variation of  $\overline{Nu}_{cy}$  with  $Bn$  in this configuration is found to be significantly different from the one in the differentially-heated side walls configuration for the same values of  $Ra$ ,  $Pr$ ,  $Bn$ ,  $AR$ ,  $r_i/L$ .
- The mean Nusselt number  $\overline{Nu}_{cy}$  decreases with increasing  $Pr$  for large values of  $Bn$ , however, the  $\overline{Nu}_{cy}$  increases with increasing  $Pr$  for very small values of  $Bn$  for both CWT and CWHF boundary conditions.
- The value of  $(Bn_{max})_{cy}$  decreases with increasing  $Pr$  for same set of values of  $Ra$ ,  $AR$ ,  $r_i/L$  for both CWT and CWHF boundary conditions.
- The mean Nusselt number  $\overline{Nu}_{cy}$  increases with increasing  $r_i/L$  and approaches the corresponding value obtained for rectangular enclosures ( $r_i/L \rightarrow \infty$ ) for both CWT and CWHF boundary conditions.
- The flow pattern (i.e. number of rolls within the enclosure) changes with the variations of  $Ra$ ,  $AR$  and  $r_i/L$  in the CWT configuration, whereas changes in  $AR$  and  $r_i/L$  do not significantly affect the flow patterns in the CWHF boundary condition.
- The changes in the flow pattern gives rise to non-monotonic variation of  $\overline{Nu}_{cy}$  with  $Bn$  for the CWT configuration.

- Numerical findings based on detailed scaling analysis have been utilised to propose a new correlation for the  $\overline{Nu}_{cy}$  for Bingham fluids in cylindrical annular enclosures for CWHF boundary condition. The correlation of the  $\overline{Nu}_{cy}$  in cylindrical annular enclosures for CWT boundary condition is not attempted because of its non-monotonic variation with  $Bn$ .
- The correlation of the mean Nusselt number  $\overline{Nu}_{Rec}$  for Bingham fluids in rectangular enclosures (i.e.  $r_i/L \rightarrow \infty$ ) have been also proposed for both CWT and CWHF boundary conditions for values of  $Ra$ ,  $Pr$ ,  $AR$  considered here.
- The critical Rayleigh number  $Ra_{crit}$  for the onset of convection increases with increasing values of  $Bn$  and  $AR$ , however,  $Ra_{crit}$  values are found to independent of  $r_i/L$  regardless of the boundary condition.
- The values of the critical Rayleigh number  $Ra_{crit}$  for the onset of convection have been found to be greater in the case of CWT boundary condition than in the CWHF boundary condition for  $AR \leq 1$ , whereas an opposite trend is obtained for  $AR > 1$  for Bingham fluids.
- The critical Rayleigh number  $Ra_{crit}$  for the onset of convection has been found be insensitive to the change of  $Pr$  for Newtonian fluids (i.e.  $Bn = 0$ ), whereas  $Ra_{crit}$  increases with increasing  $Pr$  for Bingham fluids irrespective of the boundary condition.
- Based on a detailed scaling analysis, new correlation for  $Ra_{crit}$  has been purposed, which allows for a collapse of the  $Ra_{crit}$  values for Bingham fluids in terms of a combination of  $Bn$ ,  $AR$  and  $Pr$  for both CWT and CWHF boundary conditions.

#### 8.1.2.2 Power-law Fluids

- The strength of thermal convection augments (i.e. which reflects as increase (decrease) in the  $\overline{Nu}_{cy}$ ) with increasing (decreasing)  $Ra$  ( $n$ ) regardless of the boundary condition. However,  $\overline{Nu}_{cy}$  has been found to be essentially

insensitive to the change in  $Pr$  for both CWT and CWHF boundary conditions.

- The mean Nusselt number  $\overline{Nu}_{cy}$  dependence of  $Ra$ ,  $Pr$ ,  $AR$ ,  $r_i/L$ ,  $n$  and initial condition shows complex non-monotonic trends due to the changes in flow pattern (i.e. number of cells) for shear-thinning fluids (i.e.  $n < 1$ ). However, this behaviour is much weaker for shear-thickening fluids (i.e.  $n > 1$ ) for both CWT and CWHF boundary conditions.
- The mean Nusselt number  $\overline{Nu}_{cy}$  decreases with increasing  $AR$  and attains unity value (indicating pure conduction) for  $AR > 3$  for shear-thinning fluids (i.e.  $n < 1$ ), Newtonian (i.e.  $n = 1.0$ ) and shear-thickening (i.e.  $n > 1$ ) fluids irrespective of the choices of  $Ra$ ,  $Pr$ ,  $r_i/L$ , initial and boundary conditions.
- The mean Nusselt number  $\overline{Nu}_{cy}$  assumes higher values for CWT boundary condition than that in the case for CWHF boundary condition for large (small) values of  $Ra$  ( $n$ ), whereas an opposite trend is observed for small (large) values of  $Ra$  ( $n$ ) for  $Pr$ ,  $AR$ ,  $r_i/L$  for the case of the qualitative nature of the flow pattern remains unchanged.
- The mean Nusselt number  $\overline{Nu}_{cy}$  and flow patterns for large values of  $r_i/L$  approach those for rectangular enclosures ( $r_i/L \rightarrow \infty$ ) for both CWT and CWHF boundary conditions.
- Based on a detailed scaling analysis, new correlations have been proposed for the  $\overline{Nu}_{cy}$  for power-law fluids for CWHF boundary condition. However, the non-monotonic variation of the  $\overline{Nu}_{cy}$  in response to the changes in  $Ra$ ,  $Pr$ ,  $n$ ,  $AR$  and  $r_i/L$  and initial condition severely limit the usefulness of a correlation for  $\overline{Nu}_{cy}$  for CWT boundary condition for cylindrical annular enclosures, therefore is not attempted here.
- The critical Rayleigh number  $Ra_{crit}$  for the onset of convection  $Ra_{crit}$  increases with increasing values of  $n$  and  $AR$  for both CWT and CWHF boundary conditions. However,  $Ra_{crit}$  values are found to independent of  $Pr$  and  $r_i/L$ . regardless of the boundary condition.

- Numerical findings based on detailed scaling analysis have been utilised to propose a new correlation for  $Ra_{crit}$  for both Newtonian and power-law fluids in the case of both CWT and CWHF boundary conditions.

## 8.2 Summary of Main Findings

In this thesis, wide range of parametric analysis has been directed on laminar natural convection of non-Newtonian fluids in enclosed spaces. Accordingly, it has been provided physical insights into fundamental understanding of topic by developing comprehensive scaling analysis and mean Nusselt number correlations. Consequently, the outcomes of present thesis (i.e. scaling relations, the mean Nusselt correlations) will be not only be useful for future experiments on this topic but also for designing and improving thermal systems in many important industrial applications (i.e. food and chemical processing, solar collectors, cooling of electronic equipment, cryogenic storage, nuclear safety systems).

## 8.3 Future Work

The present study addressed some major issues on laminar natural convection of non-Newtonian fluids in enclosures, however, more effort needs to be directed for further improvement of fundamental understanding on this topic. The further discussion on research on this topic can be listed below:

- Although temperature-dependent thermo-physical properties are not expected to change the qualitative behaviour observed here [44], the inclusion of temperature dependence of plastic viscosity  $\mu$ , yield stress  $\tau_y$ , consistency  $K$ , power-law index  $n$  and thermal conductivity  $k$  might be necessary for more accurate quantitative predictions.
- The present topic has been investigated based on numerical analyses predominantly due to experimental difficulties. However, comprehensive experimental studies need to be carried out in this topic.

- The current analysis focuses on steady laminar flow, however, there is no information about turbulent natural convection of non-Newtonian fluids in enclosures in the existing literature. Therefore, Direct Numerical Simulation (DNS) needs to be directed to investigate turbulent natural convection of non-Newtonian fluids in enclosures.

An, Open Foam solver (i.e. “buoyantBoussinesqPimpleFoam”) has been developed for high-fidelity simulation of turbulent natural convection of Newtonian fluids in three dimensional enclosures. This numerical scheme has been validated for turbulent Rayleigh–Bénard convection of Newtonian fluids in cubic domains with respect to the results reported by Kaczorowski et al. [113]. Also, Vasco et al. [114] presented three-dimensional simulation results for laminar natural convection of Bingham fluids in a cube with differentially heated side walls, while the other boundaries kept adiabatic. The comparison of the mean Nusselt number have been listed in Table 8.1 with existing data in the literature for both Newtonian [113] and Bingham [114] fluid cases. Table 8.1 shows that an excellent agreement (less than 2% uncertainty on mean Nusselt number) has been obtained here for both Newtonian (i.e.  $Bn = 0$ ) and Bingham fluid cases.

Table 8.1: Comparison of the mean Nusselt number with the existing data in the literature for both Newtonian [113] and Bingham [114] fluid cases.

| $\overline{Nu}$ | Kaczorowski<br>et al. [113] | Present<br>study |
|-----------------|-----------------------------|------------------|
| $Ra = 10^7$     |                             |                  |
| $Pr = 4.38$     | 16.1                        | 16.3             |
| $Bn = 0$        |                             |                  |
| $\overline{Nu}$ | Vasco et al.<br>[114]       | Present<br>study |
| $Ra = 10^6$     |                             |                  |
| $Pr = 1.0$      | 8.946                       | 8.958            |
| $Bn = 0.1$      |                             |                  |

Direct Numerical Simulations (DNS) have been conducted to analyse unsteady three dimensional Rayleigh–Bénard convection of yield stress fluids obeying a

Bingham model in cubic domains (i.e. differentially heated horizontal walls heated from below where other walls are adiabatic) at high values of nominal Rayleigh number. The simulations have been carried out for a nominal Rayleigh number  $Ra = 10^8$  for a representative value of nominal Prandtl number (i.e.  $Pr = 320$  which corresponds to a 0.05 % Carbopol-water solution [60]). The distribution of non-dimensional iso-surfaces  $\theta = (T - T_C)/(T_H - T_C)$  is shown in Fig. 8.1 for different  $Bn$  at  $Ra = 10^8$  and  $Pr = 320$ . It can be seen from Fig. 8.1 that the non-dimensional temperature iso-surfaces become less deformed and thermal plumes from the bottom wall become less apparent as  $Bn$  increases. These are the indications of the relative weakening of buoyancy force due to the augmented yield stress.

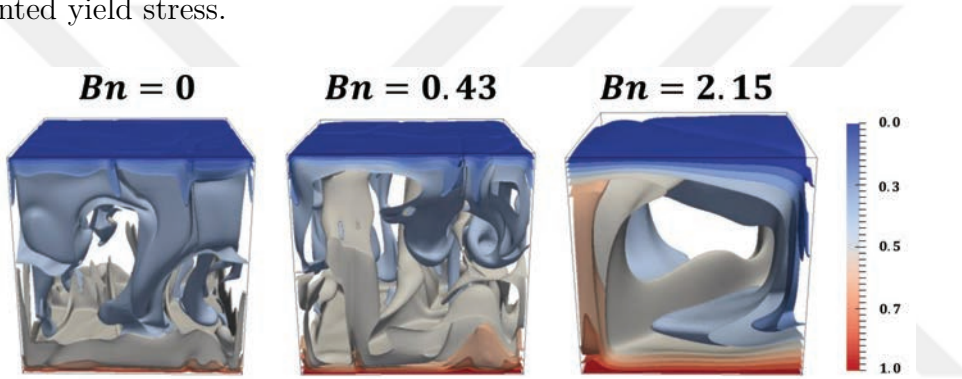


Figure 8.1: Distributions of non-dimensional temperature iso-surfaces for different  $Bn$ : a) 0, b) 0.43, c) 2.15 at  $Ra = 10^8$  and  $Pr = 320$ .

This also leads to the weakening of convection with increasing  $Bn$ , which can be substantiated from Fig. 8.2 where the pathlines coloured by non-dimensional velocity magnitude (i.e.  $\sqrt{u_i u_i} H / \alpha$ ) for the corresponding cases are shown. Figure 8.2 shows that the non-dimensional velocity magnitude decreases with increasing  $Bn$  due to the weakening of the convective thermal transport.

The unyielded regions in Bingham fluid flows can be indicated by the locations of the Apparently Unyielded Regions (AURs) (i.e. the regions where  $|\tau| \leq \tau_y$  (Mitsoulis and Zisis [106])). The distributions of AURs are shown in Fig. 9 for different  $Bn$  at  $Ra = 10^8$  and  $Pr = 320$ . It can be seen from Fig. 9 that the size and the probability of finding AURs increase with increasing  $Bn$  as the convective transport weakens.

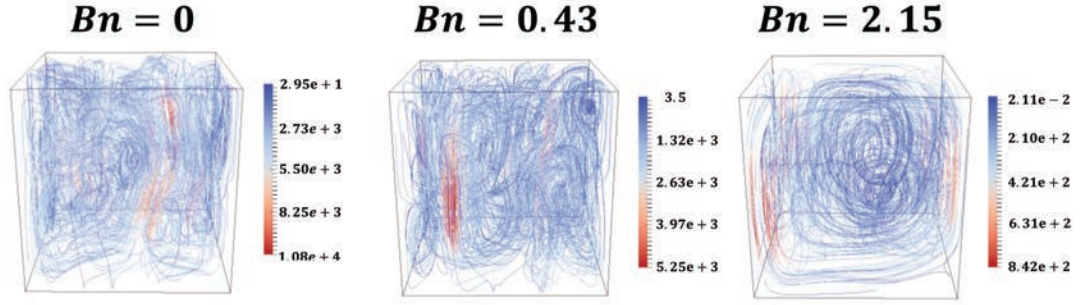


Figure 8.2: The pathlines coloured by non-dimensional velocity magnitude for different  $Bn$ : a) 0, b) 0.43, c) 2.15 at  $Ra = 10^8$  and  $Pr = 320$ .

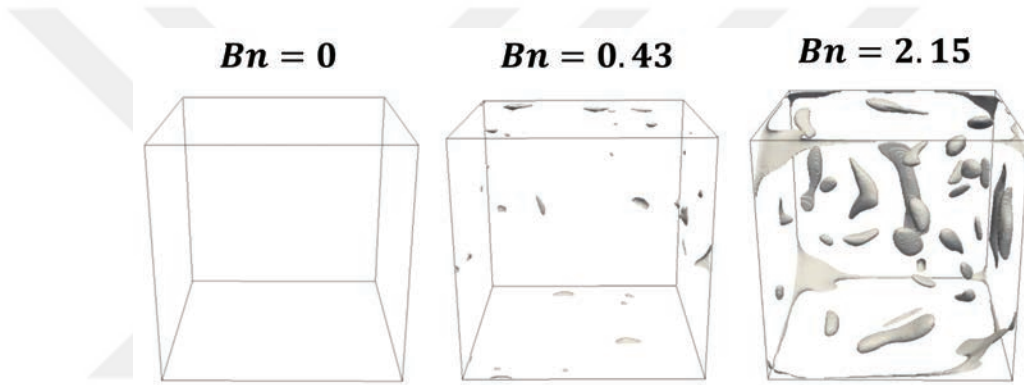


Figure 8.3: Distributions of AURs for different  $Bn$ : a) 0, b) 0.43, c) 2.15 at  $Ra = 10^8$  and  $Pr = 320$ .

As a conclusion, it can be easily seen that the flow domain increasingly becomes unyielded and the effects of Rayleigh-Bénard instability become less pronounced as the Bingham number increases.



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