Some Problems Associated With Linear Difference Operators

by

Ahmet Delil

University of Wales Cardiff

This thesis is submitted to the University of Wales
in fulfilment of the requirements of
candidature for the degree of

Doctor of Philosophy

September 1997

DECLARATION

I hereby declare that this thesis is the result of my own independent investigation except where I have indicated my indebtedness to other sources.

I hereby certify that this thesis has not already been accepted in substance for any other degree, nor is it being submitted concurrently for any other degree.

I hereby give consent for my thesis, if accepted, to be available for photocopying and for inter-library loan, and for the title and summary to be made available to outside organisations.

....
(candidate)

....
(supervisor)

17th October 1997
(date)

ACKNOWLEDGMENTS

I would especially like to thank my supervisor of studies Prof. W.D. Evans for his invaluable guidance and encouragement over past few years. I am also grateful to him for the time he spent reviewing this manuscript and for the fact that he always made himself available whenever I had problems.

I would also like to thank Dr. M.J. Beynon for providing me his numerical algorithm. Considerable thanks to Dr. B.M. Brown and Dr. D.K.R. McCormack for their help on computing; to Prof. V.I. Burenkov and Dr. A.M. Cohen for their both pleasent and informative lectures on Differential Equations and Numerical Analysis. Many thanks to Dr. G.R.H. Graves for helping me on my TEX problems; to my colleagues Dr. Alex Balinsky, Yelena Khazan, Dr. Jan Lang, Dr. Amiran Gogatishvili, Phil Bartley, Matthew Nassau, Richard Salmon for their help and encouragements.

For each member of my family and my friends what should I say? Thank you all for standing by me.

Finally, I am grateful to Celal Bayar and Balikesir Universities (Turkey) for their financial support, and to the people who were directly or indirectly involved in any stage of this research.

Dedicated:

To my mother and sisters
(who were not given opportunity)

SUMMARY

In this thesis we develop a theory for Kolmogorov-type inequalities of the form

$$\sum_{n=j}^{\infty} |Mx_n|^2 w_n \leq K \Big(\sum_{n=-1}^{\infty} |x_n|^2 w_n \Big)^{1/2} \Big(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n \Big)^{1/2}, \; j = -1, 0,$$

and apply both analytical and numerical methods for obtaining the value of the best constant in these inequalities.

The existence of the above inequalities is shown to be determined by the behaviour of the null set of the least eigenvalue of a real, symmetric matrix of 4×4 size whose elements are functions of the Hellinger-Nevanlinna $m-$function. The criterion for a valid inequality is given in terms of the behaviour of the Hellinger-Nevanlinna $m-$function associated with the second-order real, formally symmetric, linear difference expression M of limit-point type.

CONTENTS

LIST OF FIGURES AND TABLES

Chapter 1

INTRODUCTION

§1.1 Historical background for the problem

The investigation of inequalities involving derivatives have a long history. Here we give a brief account of those that are directly or indirectly relevant to us.

Landau (1913) and Hadamard (1914) established the inequality

$$\|f'\|^2 \leq 4\|f\|\,\|f''\| \tag{1.1}$$

in the $L^\infty[0,\infty)$ setting, and showed that 4 is the best possible constant. The analogue of (1.1) in $L^2[0,\infty)$ and $L^2(-\infty,\infty)$ norms were given by Hardy and Littlewood (1932). Kolmogorov (1939) extended (1.1) to the inequalities of the form

$$\|f^{(m)}\| \leq K_{n,m}\|f\|^{\frac{(n-m)}{n}}\|f^{(n)}\|^{\frac{m}{n}} \tag{1.2}$$

where $\|f\| = \sup\limits_{-\infty<x<\infty} |f(x)|$ and the best possible constants

$$K_{n,m} = K_{n-m}K_n^{\frac{m-n}{n}}, \qquad K_j = \frac{4}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p(j+1)}}{(2p+1)^{j+1}}$$

are given explicitly.

Everitt (1971/1972) extended (1.1) to a general class of inequalities in $L^2_w(a,b)$, which involve the second order formally symmetric differential expression

$$Nf := \frac{1}{w}\big[-(pf')' + qf\big], \tag{1.3}$$

namely

$$\left(\int_a^b (p|f'|^2 + q|f|^2)dx\right)^2 \leq K\int_a^b |f|^2 w dx \int_a^b |Nf|^2 w dx \tag{1.4}$$

which are generally called the HELP inequalities after Hardy, Everitt, Littlewood and Polya.

p, q and w in (1.4) are real-valued functions on $[a,b)$ $(-\infty < a < b \leq \infty)$ and assumed to satisfy minimal smoothness conditions in order for the equation

$$Nf = \lambda f, \qquad \lambda \in \mathbb{C} \tag{1.5}$$

to be regular at a and also to satisfy the so-called strong limit-point (SLP) condition at the singular end point b. The inequality (1.4) is required to hold on a domain that makes the right hand side defined and finite. Everitt proved that the existence of the inequality and the value of the best constant depend explicitly on the behavior of the Titchmarsh-Weyl $m-$function for the equation (1.5). Details of the $m-$function can be found in survey papers by Bennewitz&Everitt [7], Evans&Everitt [25] and the references contained therein.

Evans&Zettl (1978) gave an alternative proof of Everitt's result by using the theory of linear operators in Hilbert space.

Phóng [54] gave a necessary and sufficient condition for the validity of the Kolmogorov type inequality

$$\|N^m f\| \leq C_{n,m}\|f\|^{\frac{n-m}{n}}\|N^n f\|^{\frac{m}{n}},\ n > m,\ n \text{ and } m \text{ are integers} \tag{1.6}$$

in which $||.||$ is the L^2_w norm over the interval $[0, \infty)$ and Nf is given by (1.3).

Beynon [8], in his Ph.D. thesis, works on the inequality (1.6) in the special case when n is even and $m = n/2$, i.e.

$$||N^{\frac{n}{2}} f||^2 \leq K_n ||f|| ||N^n f||, \; n = 2, 4, 6, ..., \tag{1.7}$$

on the maximal domain that makes the right hand side of (1.7) finite in the $L^2_w(a, b)$ setting. He gives a criterion for a valid inequality in (1.7) in terms of the Titchmarsh-Weyl $m-$function, which is suitable for computational techniques, unlike Phóng's theoretical results. By using the von Neumann formula, i.e. the orthogonal decomposition of the maximal operator related to the differential expression Nf, he reduces the criterion to the positive definiteness of a certain real and symmetric $2n \times 2n$ matrix which involves the $m(\lambda)$ function and the spectral parameter λ in (1.5) with $\lambda \in \mathbb{C} \backslash \mathbb{R}$. He also gives a numerical method for computing the value of the best constant of the inequality, and applies it to certain different cases of the coefficients p, q and w for $n = 2, 4, 6$. The case $n = 2$

$$||Nf||^2 \leq K ||f|| ||N^2 f||, \tag{1.8}$$

with the $L^2_w[0, \infty)$ norm is the subject of Beynon, Brown, Evans [9].

It is the discrete analogue of (1.8) that is going to be our main concern in this thesis.

The discrete analogue of the inequality (1.1) in $\ell^2(0, \infty)$ was proved by Copson (1979): if $\{x_n\}_0^\infty$ is a sequence of real numbers such that $\sum_{n=0}^\infty x_n^2$ and $\sum_{n=0}^\infty (\Delta^2 x_n)^2$ are convergent, where $\Delta x_n = x_{n+1} - x_n$ and $\Delta^2 x_n = \Delta(\Delta x_n)$, then $\sum_{n=0}^{\infty}(\Delta x_n)^2$ is convergent and the inequality

$$\left(\sum_{n=0}^{\infty}(\Delta x_n)^2\right)^2 \leq 4\sum_{n=0}^{\infty} x_n^2 \sum_{n=0}^{\infty}(\Delta^2 x_n)^2 \tag{1.9}$$

holds with the constant 4 being best possible.

Brown&Evans (1992) extended Copson's inequality to the discrete HELP inequalities:

$$\left(\sum_{n=0}^{\infty}\left(p_n|\Delta x_n|^2 + q_n|x_n|^2\right) + p_{-1}|\Delta x_{-1}|^2\right)^2 \leq K \sum_{n=-1}^{\infty} |x_n|^2 w_n \sum_{n=0}^{\infty} |Mx_n|^2 w_n \tag{1.10}$$

where

$$Mx_n := \begin{cases} \frac{1}{w_n}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n], & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}\Delta x_n, & n = -1 \end{cases} \tag{1.11}$$

where $\{p_n\}_{-1}^{\infty}$, $\{q_n\}_0^{\infty}$ and $\{w_n\}_{-1}^{\infty}$ are real valued with

$$p_n \neq 0, \qquad \text{and } w_n > 0 \qquad \forall n = -1, 0, 1, \ldots. \tag{1.12}$$

The inequality is required to hold on the domain

$$D := \left\{x = \{x_n\}_{-1}^{\infty} : x \in \ell_w^2 \text{ and } \sum_{n=0}^{\infty} |Mx_n|^2 w_n < \infty\right\} \tag{1.13}$$

and ℓ_w^2 is the Hilbert space of complex valued sequences $x = \{x_n\}_{-1}^{\infty}$ such that $\sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty$. The criterion they obtain for the validity of (1.10) has a similar form to that of Everitt for (1.4), with the Titchmarsh-Weyl $m-$function now replaced by the Hellinger-Nevanlinna $m-$function for the difference equation

$$Mx_n = \lambda x_n, \; n \geq 0, \; \lambda \in \mathbb{C} \tag{1.14}$$

introduced simultaneously by Hellinger (1922) and Nevanlinna (1922).

Brown, Evans, Littlejohn [15] examine the inequality (1.10) when (1.14) is the recurrence relation for the classical orthogonal polynomials.

Since the Hellinger-Nevanlinna $m-$function will play an important role in our problem, we give a brief account of the construction and the properties of it in §1.3 below. For a full discussion Akhiezer [1,Chapter 1] and Atkinson [3, Chapter 5] may be consulted.

§1.2 Objectives and the method

Our main concern in this thesis is the investigation of an inequality of Kolmogorov type associated with the second order formally symmetric difference expression M defined in (1.11), namely

$$\sum_{n=0}^{\infty} |Mx_n|^2 w_n \leq K\left(\sum_{n=-1}^{\infty} |x_n|^2 w_n\right)^{1/2} \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n\right)^{1/2} \tag{1.15}$$

where $M^2 x_n = M(Mx_n)$. The inequality is required to hold on the maximal domain that makes the right hand side of (1.15) finite. The difference expressions M and M^2 are required to satisfy the so-called *limit-point* (LP) conditions, which are that the equation (1.14) must have precisely one (up to constant multiples) ℓ_w^2 solution for $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and the equation

$$M^2 x_n = \lambda^2 x_n, \; n \geq 1, \; \lambda^2 \in \mathbb{C} \tag{1.16}$$

must have precisely two linearly independent ℓ_w^2 solutions for $\lambda^2 \in \mathbb{C}\backslash\mathbb{R}$. We will also require M^2 to satisfy the so-called *partial separation* (PS) condition: if the right-hand side of (1.15) converges then the left-hand side converges. We give a full description and discussion of these terms in Chapter 3.

The inequality (1.15) is the discrete case analogue of the problem that was considered by Beynon, Brown, Evans [9]. But, the operator theoretical proof that they give is no longer valid in our case. As we shall see, the main reason for this is because the minimal operator related to the expression M does not have a dense domain in ℓ^2_w, and hence a unique adjoint can not be defined for it. For this reason, we will make use of the theory of symmetric relations and their adjoints on ℓ^2_w developed by Bennewitz [5-6] and Pleijel [55-56]. This is one of the main ingredients that Brown&Evans [12] used in their investigation of the inequality (1.10).

We will also examine the inequality

$$\sum_{n=-1}^{\infty} |Mx_n|^2 w_n \leq K \left(\sum_{n=-1}^{\infty} |x_n|^2 w_n \right)^{1/2} \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n \right)^{1/2} \tag{1.17}$$

under the same conditions as those for the inequality (1.15), and give examples for specific cases of the coefficients p, q and w for both inequalities (1.15) and (1.17).

In Chapter 2, we examine some relationships between the so-called *Dirichlet* (D), *conditional Dirichlet* (CD), *weak Dirichlet* (WD), *strong limit-point* (SLP) and *limit-point* (LP) properties of the difference expression M with complex coefficients. In studying inequalities involving the expression M, such as HELP and Kolmogorov-type inequalities, these properties and the relationships between them are crucial. The work we present in Chapter 2 is the direct analogue of work by Race [58] for differential expressions.

As we pointed out earlier, for our inequalities to be valid, M^2 is required to satisfy the so-called *partial separation* (PS) condition. In Chapter 3, we give necessary and sufficient conditions for the expression M^2 to be partially separated. This is the discrete case of important results by Kauffmann, Read, Zettl [43, Theorem 2.6, p.60].

In our main result (Theorem 4.2.5; but see also Theorem 4.3.2) a criterion is established for the validity of (1.15), i.e. the inequality holds for some finite constant K, the precise value of the best constant is determined, and the sequences which give equality, if they exist, are given. This criterion is given in terms of the Hellinger-Nevanlinna $m-$function of M, and even in the relatively easy examples we consider (cf. Chapter 5), the analysis is formidable. We have therefore followed the lead of Beynon in [8], and used computational methods. Whenever possible we have used asymptotic analysis to confirm the main features of the result.

§1.3 Hellinger-Nevanlinna circles

In this section we are going to give a summary of the derivation and properties of the $m-$function related to the equation (1.14). We shall assume throughout that, for all relevant values of n, (1.12) is satisfied. We refer to Akhiezer [1, Chapter 1] and Atkinson [3, Chapter 5] for further information.

Let $\{\theta_n(\lambda)\}_{-1}^{\infty}$ and $\{\phi_n(\lambda)\}_{-1}^{\infty}$ be the two solutions of (1.14), satisfying

$$\left.\begin{aligned} &\theta_{-1}(\lambda) = 1,\ \phi_{-1}(\lambda) = 0, \\ &\theta_0(\lambda) = 0,\ p_{-1}\phi_0(\lambda) = 1. \end{aligned}\right\} \tag{1.18}$$

Define the Möbius transformations

$$f_k(\lambda, z) := -\frac{p_{k-1}\theta_k(\lambda) + z\theta_{k-1}(\lambda)}{p_{k-1}\phi_k(\lambda) + z\phi_{k-1}(\lambda)}, \; z \in \mathbb{C}, \; \lambda \in \mathbb{C}_+ \tag{1.19}$$

where $\mathbb{C}_+(\mathbb{C}_-)$ is the open upper (lower) half plane of $\mathbb{C}$. Then, the image of $\mathbb{R}$ under these transformations are circles $C(k+1, \lambda)$ with radius

$$\begin{aligned} r_{k+1} &= \frac{1}{|[\phi\phi]_{k+1}|} \\ &= \frac{1}{|2\Im(\lambda)| \sum_{n=0}^{k} |\phi_n(\lambda)|^2 w_n}, \; \Im(\lambda) > 0 \end{aligned} \tag{1.20}$$

and centres

$$a_{k+1} = -\frac{[\theta\phi]_{k+1}}{[\phi\phi]_{k+1}} \tag{1.21}$$

where $[\theta\phi]_k := p_{k-1}\big(\theta_k(\lambda)\overline{\phi_{k-1}(\lambda)} - \theta_{k-1}(\lambda)\overline{\phi_k(\lambda)}\big)$, $k = 0, 1, 2, \ldots$. Also, the image of $\mathbb{C}_-$ is the disc $D(k+1, \lambda)$ within the circles $C(k+1, \lambda)$. The circles $C(k+1, \lambda)$ are the Hellinger-Nevanlinna circles, and are nested as $k \to \infty$, since the disc $D(k+2, \lambda)$ is inside the disc $D(k+1, \lambda)$. So, they converge either to a limit point $m_+(\lambda)$ or a limit circle $C_+(\infty, \lambda)$ (similarly for $m_-(\lambda)$ and $C_-(\infty, \lambda)$ when $\lambda \in \mathbb{C}_-$, the disk $D(k+1, \lambda)$ now being the image of $\mathbb{C}_+$ under the transformation (1.19)). These two cases are known accordingly as M being in the *limit − point* (LP) and *limit − circle* (LC) cases respectively. We also have the following:

(i) m_+ (m_-) is analytic in $\mathbb{C}_+$ ($\mathbb{C}_-$);

(ii) $m_+ : \mathbb{C}_+ \to \mathbb{C}_+$, $m_- : \mathbb{C}_- \to \mathbb{C}_-$;

(iii) $\overline{m_+(\lambda)} = m_-(\bar{\lambda})$; $(\lambda \in \mathbb{C}_+)$;

(iv) if $\theta(\lambda)$, $\phi(\lambda)$ are the solutions of (1.14) satisfying the conditions (1.18)

then

$$\Psi_\pm(\lambda) := \theta(\lambda) + m_\pm(\lambda)\phi(\lambda) \in \ell^2_w, \; (\lambda \in \mathbb{C}_\pm). \tag{1.22}$$

We shall use the notation $m(\lambda) = m_+(\lambda)$ $(\lambda \in \mathbb{C}_+)$ and $m(\lambda) = m_-(\lambda)$ $(\lambda \in \mathbb{C}_-)$ throughout.

Two consequences of (iv) together with (1.20) are: in the limit-point (LP) case, $\phi(\lambda) = \{\phi_n(\lambda)\}_{-1}^{\infty} \notin \ell^2_w$ for $\lambda \notin \mathbb{R}$ and $\Psi(\lambda) = \{\Psi_n(\lambda)\}_{-1}^{\infty}$ is the unique, up to constant multiples, ℓ^2_w solution of (1.14) for $\lambda \notin \mathbb{R}$. In the limit-circle (LC) case all solutions of (1.14) are in ℓ^2_w. In fact, the following theorem by Atkinson [3, Theorem 5.6.1, p.131] shows that this classification is independent of the choice of λ.

1.3.1 Theorem (Atkinson)

If all solutions of (1.14) are in ℓ^2_w for some $\lambda = \lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ then all solutions of (1.14) are in ℓ^2_w for all $\lambda \in \mathbb{C}$.

An equivalent characterisation for M to be LP is:

$$M \text{ is LP} \iff \lim_{m\to\infty} p_m(\overline{x_{m+1}}y_m - y_{m+1}\overline{x_m}) = 0, \qquad \forall x, y \in D(M)$$

where

$$D(M) := \left\{x \in \ell^2_w : \sum_{n=0}^{\infty} |Mx_n|^2 w_n < \infty\right\}$$

and this was given by Hinton&Lewis [39, p.425], Atkinson [3]. We give a detailed proof of this statement in Chapter 3.

For some LP-LC criteria for M, Hinton&Lewis [39, p.435], Atkinson [3] may be consulted. Here we quote one without proof:

1.3.2 Theorem (Hinton&Lewis [39, Theorem 10, p.435])

The expression M is LP if (1.12) holds and

$$\sum_{n=0}^{\infty} \frac{(w_n w_{n+1})^{1/2}}{|p_n|} = \infty.$$

1.3.3 Definition If the difference equation (1.16) has precisely two linearly independent ℓ^2_w solutions for $\Im(\lambda^2) \neq 0$, then the expression M^2 is said to be LP. If M^2 is not LP, then there exists either 3 or 4 linearly independent ℓ^2_w solutions of (1.16) for $\Im(\lambda^2) \neq 0$. For convenience, in case of 4 linearly independent ℓ^2_w solutions of (1.16) we will say that M^2 is in the LC case.

For $\Im(\lambda^2) \neq 0$ it is known that (1.16) has at least 2 linearly independent solutions in ℓ^2_w (see, for example, Billigheimer [10, p.70]). Also the classification of Definition 1.3.3 is independent of λ^2 for $\Im(\lambda^2) \neq 0$, i.e. if (1.16) has k linearly independent solutions in ℓ^2_w for $\lambda^2 = \lambda_0^2 \notin \mathbb{R}$, then the same is true for all $\lambda^2 \notin \mathbb{R}$. Furthermore, in the LC case all solutions of (1.16) are in ℓ^2_w for all $\lambda^2 \in \mathbb{C}$.

In Chapter 3, we will also prove the following: M^2 is LP if and only if,

$$\lim_{m\to\infty} p_m\left(\overline{x_{m+1}}My_m - My_{m+1}\overline{x_m} + \overline{Mx_{m+1}}y_m - y_{m+1}\overline{Mx_m}\right) = 0 \;\forall x, y \in D(M^2)$$

where

$$D(M^2) := \left\{x \in \ell^2_w : \sum_{n=1}^{\infty} |M^2 x_n|^2 w_n < \infty\right\}.$$

§1.4 Some basic results from the theory of difference equations

In this section we give some useful preliminaries for difference equations, most of which can be found in Benammar [4]. We start with the existence and uniqueness theorem.

1.4.1 Theorem Let $f = \{f_n\}_{-1}^{\infty}$ be a complex valued sequence. Then given any complex numbers t_0, t_1, there exists a unique solution $x = \{x_n\}_{-1}^{\infty}$ of

$$Mx_n = \lambda x_n + f_n, \; n \geq 0, \; \lambda \in \mathbb{C} \tag{1.23}$$

satisfying $x_{-1} = t_0$, $x_0 = t_1$.

Proof: For $n = 0$ we have

$$\begin{aligned} Mx_0 &= \frac{1}{w_0}\big[-\Delta(p_{-1}\Delta x_{-1}) + q_0 x_0\big] \\ &= \frac{1}{w_0}\big[-p_0x_1 + (p_{-1}+p_0+q_0)x_0 - p_{-1}x_{-1}\big] \\ &= \lambda x_0 + f_0 \end{aligned}$$

and, given x_{-1}, x_0 and f_0, we can find

$$x_1 = \frac{1}{p_0}\big[(p_{-1}+p_0+q_0)x_0 - p_{-1}x_{-1} - \lambda x_0 w_0 - f_0 w_0\big].$$

For $n = 1$, we have

$$\begin{aligned} Mx_1 &= \frac{1}{w_1}\big[-p_1x_2 + (p_0+p_1+q_1)x_1 - p_0x_0\big] \\ &= \lambda x_1 + f_1 \end{aligned}$$

and x_2 can be evaluated. Continuing this way recursively, we find that the solution of (1.23) is uniquely determined by the initial values x_{-1}, x_0. Note that, if $x_{-1} = x_0 = 0$ then the unique solution of $Mx_n = 0$, $(n \geq 0)$, is the null sequence.

1.4.2 Lemma (The summation by parts formula)

For arbitrary sequences $x = \{x_n\}_k^m$ and $y = \{y_n\}_k^m$ $(a \leq b,\ a, b \in \mathbb{N}_0)$, we have that

$$\sum_{n=k}^{m} x_n \Delta y_n = x_{m+1}y_{m+1} - x_k y_k - \sum_{n=k}^{m} y_{n+1}\Delta x_n. \tag{1.24}$$

The simple proof of this lemma is to be found in Chapter 2, Lemma 2.1.1.

1.4.3 Lemma Let $\phi = \{\phi_n\}_{-1}^{\infty}$, $\psi = \{\psi_n\}_{-1}^{\infty}$ be solutions of

$$Mx_n = \lambda x_n,\ (n \in \mathbb{N}_0,\ \lambda \in \mathbb{C}).$$

Then

$$[\phi,\psi]_n := p_{n-1}(\phi_n\Delta\psi_{n-1} - \psi_n\Delta\phi_{n-1})$$

$$= p_{n-1}(\phi_{n-1}\psi_n - \phi_n\psi_{n-1}),$$

is constant for all $n \in \mathbb{N}_0$. This constant is non-zero if and only if ϕ and ψ are linearly independent.

For the proof of this lemma we refer to Benammar [4, p.14], but we give another version of this lemma in Chapter 3, Lemma 3.2.5. The proof of the following lemma can also be found in the same reference.

1.4.4 Lemma (The variation of parameters formula)

Let $\phi = \{\phi_n\}_{-1}^{\infty}$ and $\psi = \{\psi_n\}_{-1}^{\infty}$ be linearly independent solutions of $Mx_n = \lambda x_n$ $(n \in \mathbb{N}_0,\ \lambda \in \mathbb{C})$ and suppose that $[\phi,\psi]_n = 1$ for all n. Then $\Phi = \{\Phi_n\}_{-1}^{\infty}$ defined by

$$\left.\begin{aligned} &\Phi_n = \sum_{m=0}^{n} \big(-\psi_m\phi_n + \phi_m\psi_n\big)w_m f_m;\ (n \in \mathbb{N}_0), \\ &\Phi_{-1} = 0 \end{aligned}\right\} \tag{1.25}$$

satisfies

$$\left.\begin{aligned} &M\Phi_n = \lambda\Phi_n + f_n, \qquad n \in \mathbb{N}_0; \lambda \in \mathbb{C}, \\ &\Phi_{-1} = \Phi_0 = 0. \end{aligned}\right\} \tag{1.26$a-b$}$$

Any solution of (1.26)(a) is of the form

$$\Psi = \Phi + A\phi + B\psi, \tag{1.27}$$

for some constants $A, B \in \mathbb{C}$.

We now give the Green's formulas for the expressions M and M^2, which will be of crucial importance in the sequel.

Let M be defined by (1.11). Then, by using the summation by parts formula, for $k \in \mathbb{N}_0,\ m \geq k$, we have

$$\sum_{n=k}^{m} \overline{x_n} M y_n w_n = \sum_{n=k}^{m} q_n y_n \overline{x_n} - \sum_{n=k}^{m} \Delta(p_{n-1}\Delta y_{n-1})\overline{x_n}$$

$$= \textstyle\sum_{n=k}^{m} q_n y_n \overline{x_n} + \sum_{n=k}^{m}(p_n \Delta y_n)\overline{\Delta x_n} - p_m \Delta y_m \overline{x_{m+1}} + p_{k-1}\Delta y_{k-1}\overline{x_k} \tag{1.28}$$

and similarly

$$\begin{aligned}\sum_{n=k}^{m} y_n \overline{M x_n} w_n = \sum_{n=k}^{m} q_n \overline{x_n} y_n + \sum_{n=k}^{m}(p_n \overline{\Delta x_n})\Delta y_n \\ - p_m \overline{\Delta x_m} y_{m+1} + p_{k-1}\overline{\Delta x_{k-1}} y_k.\end{aligned} \tag{1.29}$$

Subtracting (1.29) from (1.28) we obtain

$$\begin{aligned}\sum_{n=k}^{m} (\overline{x_n} M y_n - y_n \overline{M x_n}) w_n = p_m(\overline{\Delta x_m} y_{m+1} - \Delta y_m \overline{x_{m+1}}) \\ - p_{k-1}(\overline{\Delta x_{k-1}} y_k - \Delta y_{k-1}\overline{x_k})\end{aligned}$$

and, letting $m \to \infty$ now yields

$$\begin{aligned}\sum_{n=k}^{\infty} (\overline{x_n} M y_n - y_n \overline{M x_n}) w_n = \lim_{m\to\infty} p_m(\overline{x_{m+1}} y_m - y_{m+1}\overline{x_m}) \\ - p_{k-1}(\overline{x_k} y_{k-1} - y_k \overline{x_{k-1}}).\end{aligned} \tag{1.30}$$

The formula (1.30) is called the Green's formula related with the expression M.

Now, let $M^2 x_n := M(M x_n)$, more precisely

$$M^2 x_n := \begin{cases} \frac{1}{w_n}\big[-\Delta(p_{n-1}\Delta M x_{n-1}) + q_n M x_n\big], & n \geq 1 \\ \frac{1}{w_0}\big[-\Delta(p_{-1}\Delta M x_{-1}) + q_0 M x_0\big], & n = 0 \\ -\frac{p_{-1}}{w_{-1}}\Delta M x_{-1}, & n = -1. \end{cases} \tag{1.31}$$

Following (1.30), for $k \in \mathbb{N}_0$ and $m \geq k$, we have

$$\begin{aligned}\sum_{n=k}^{\infty} (\overline{x_n} M^2 y_n - M y_n \overline{M x_n}) w_n = \lim_{m\to\infty} p_m(\overline{x_{m+1}} M y_m - M y_{m+1}\overline{x_m}) \\ - p_{k-1}(\overline{x_k} M y_{k-1} - M y_k \overline{x_{k-1}})\end{aligned}$$

and

$$\sum_{n=k}^{\infty}\left(\overline{Mx_n}My_n - y_n\overline{M^2x_n}\right)w_n = \lim_{m\to\infty} p_m(\overline{Mx_{m+1}}y_m - y_{m+1}\overline{Mx_m})$$
$$- p_{k-1}(\overline{Mx_k}y_{k-1} - y_k\overline{Mx_{k-1}}).$$

Hence,

$$\begin{aligned}&\sum_{n=k}^{\infty}\left(\overline{x_n}M^2y_n - y_n\overline{M^2x_n}\right)w_n \\ &= \lim_{m\to\infty} p_m\left(\overline{x_{m+1}}My_m - My_{m+1}\overline{x_m} + \overline{Mx_{m+1}}y_m - y_{m+1}\overline{Mx_m}\right) \\ &\quad - p_{k-1}\left(\overline{x_k}My_{k-1} - My_k\overline{x_{k-1}} + \overline{Mx_k}y_{k-1} - y_k\overline{Mx_{k-1}}\right)\end{aligned} \tag{1.32}$$

which is called the Green's formula for M^2.

§1.5 Notation and plan of the thesis

We denote by $\mathbb{N}_0 = \{0, 1, 2, ...\}$, the set of non-negative integers and $\mathbb{N} = \mathbb{N}_0\backslash\{0\}$. $\mathbb{R}$ and $\mathbb{C}$ represent the real and complex number fields respectively. By $\mathbb{C}_+(\mathbb{C}_-)$ we mean the open upper (lower) half planes of $\mathbb{C}$ respectively defined as follows:

$$C_+ = \{\lambda \in \mathbb{C} : \Im(\lambda) > 0\} \text{ and } C_- = \{\lambda \in \mathbb{C} : \Im(\lambda) < 0\},$$

where $\Im(.)$ denotes the imaginary part of a complex number. By ℓ^1 we mean the set of absolutely summable complex sequences $x = \{x_n\}_{-1}^{\infty}$, i.e.

$$\ell^1 := \left\{x = \{x_n\}_{-1}^{\infty} : \sum_{n=-1}^{\infty} |x_n| < \infty\right\}.$$

ℓ_w^2 represents the weighted Hilbert space of complex sequences $x = \{x_n\}_{-1}^{\infty}$, i.e.

$$\ell_w^2 := \left\{x = \{x_n\}_{-1}^{\infty} : \sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty\right\},$$

and it is denoted by ℓ^2 for the unit weight $w = 1$.

In this thesis, chapters are divided into sections. For example §3.2 is section 2 in Chapter 3. Theorems, lemmas, corollaries and remarks are numbered consecutively within each chapter: Theorem 3.2.1 means the first theorem in the second section of Chapter 3. Also equation numbers are given consecutively in each chapter, for example, (4.11) means the eleventh formulae in the fourth chapter. Figures are numbered consecutively in the same chapter, and also tables are numbered consecutively in the same chapter.

The plan of the thesis is as follows.

In Chapter 2, we establish some relationships between the so-called *strong limit-point* (SLP), *weak Dirichlet* (WD), *conditional Dirichlet* (CD), *Dirichlet* (D) and *limit-point* (LP) properties of a second order difference expression M with complex coefficients p and q. In Chapter 3 we establish orthogonal decomposition of the maximal graph related to the expression M^2 defined in (1.31). Also, we give necessary and sufficient conditions for *partial separation* (PS) of M^2. A connection between the PS condition and the deficiency indices of the minimal operators related to M and M^2 is also given in this chapter. Chapter 4 is where we establish our criterion for the validity of the inequalities (1.15) and (1.17), the evaluation of the best constants and the determination of sequences which give equality, whenever possible. In Chapter 5 we give specific examples of the general inequality (1.15). The values of the best constants are estimated by computational techniques, but confirmed, whenever possible, by asymptotic analysis. Examples of the inequality (1.17) are also investigated. Concluding remarks and future directions of research are given in Chapter 6.

There is a list of references at the end of this thesis, which have been consulted or which are of relevance to the problems considered.

Chapter 2

SOME RESULTS RELATED WITH STRONG LIMIT-POINT AND DIRICHLET PROPERTIES

In this chapter, we give some results which are the direct analogue of work by Race [**58**]. We investigate some relationships between *strong limit-point* (SLP), *limit-point* (LP), *weak Dirichlet* (WD), *conditional Dirichlet* (CD) and *Dirichlet* (D) properties of the second order formally-symmetric difference expression M.

§2.1. Preliminaries

We have the difference equation

$$Mx_n = \lambda x_n \qquad n = 0, 1, 2, ..., \lambda \in \mathbb{C} \tag{2.1}$$

where the difference expression M is defined by

$$Mx_n := \begin{cases} \frac{1}{w_n}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n], & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}\Delta x_n, & n = -1 \end{cases} \tag{2.2}$$

or, more specifically,

$$Mx_n := \begin{cases} \frac{1}{w_n}[-p_n x_{n+1} + (p_{n-1} + p_n + q_n)x_n - p_{n-1}x_{n-1}], & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}(x_0 - x_{-1}), & n = -1. \end{cases} \tag{2.3}$$

The sequences $p = \{p_n\}_{-1}^{\infty}$, $q = \{q_n\}_{-1}^{\infty}$ are complex valued with $p_n \neq 0$, $q_{-1} = 0$; $w = \{w_n\}_{-1}^{\infty}$ is the positive weight and Δ is the forward difference operator, i.e. $\Delta x_n = x_{n+1} - x_n$.

Note that, defining the expression M by (2.3) makes the difference equation (2.1) a 3-term recurrence relation. We now associate a maximal operator, $T(M)$, with M on $D_{T(M)} \subset \ell^2_w$ into ℓ^2_w as follows:

$$[Tx]_n = Tx_n := Mx_n, \qquad n = -1, 0, 1, 2, ... \tag{2.4}$$

$$D_{T(M)} := \{x = \{x_n\}_{-1}^{\infty} : x \in \ell^2_w, \quad \sum_{n=0}^{\infty} |Tx_n|^2 w_n < \infty\}. \tag{2.5}$$

Here, ℓ^2_w is the complex Hilbert space of all complex sequences $x = \{x_n\}_{-1}^{\infty}$ which satisfy

$$\sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty \tag{2.6}$$

with the inner product and the norm $(x, y) = \sum_{n=-1}^{\infty} x_n \bar{y}_n w_n$, $\|x_n\| = (x, x)^{\frac{1}{2}}$.

ℓ^1 is the set of all absolutely summable complex sequences, i.e.

$$\ell^1 = \{x = \{x_n\}_{-1}^{\infty} : \sum_{n=-1}^{\infty} |x_n| < \infty\}.$$

We now have the following summation by parts formula in order to be able to get some important formulas.

2.1.1 Lemma: (Summation by parts formula)

For arbitrary sequences $x = \{x_n\}_{-1}^{\infty}$ and $y = \{y_n\}_{-1}^{\infty}$ we have that

$$\sum_{n=a}^{b} x_n \Delta y_n = x_{b+1} y_{b+1} - x_a y_a - \sum_{n=a}^{b} y_{n+1} \Delta x_n, \qquad -1 \le a \le b, \ a, b \in \mathbb{Z}. \tag{2.7}$$

Proof: The difference of a product is

$$\Delta(x_n y_n) = x_{n+1} y_{n+1} - x_n y_n$$

$$= x_{n+1} y_{n+1} - x_n y_n + x_n y_{n+1} - x_n y_{n+1}$$

$\Rightarrow$

$$\Delta(x_n y_n) = y_{n+1}\Delta x_n + x_n \Delta y_n. \tag{2.8}$$

Now, by using

$$\sum_{n=a}^{b} \Delta(x_n y_n) = x_{b+1}y_{b+1} - x_a y_a$$

and summing (2.8) over n we have

$$\sum_{n=a}^{b} \Delta(x_n y_n) = \sum_{n=a}^{b} y_{n+1}\Delta x_n + \sum_{n=a}^{b} x_n \Delta y_n$$

$\Rightarrow$

$$x_{b+1}y_{b+1} - x_a y_a = \sum_{n=a}^{b} y_{n+1}\Delta x_n + \sum_{n=a}^{b} x_n \Delta y_n$$

and hence (2.7).

Now, using (2.2) and (2.7), for $m \in \mathbb{N}_0$, we have

$$\begin{aligned}\sum_{n=0}^{m} \bar{x}_n M y_n w_n &= \sum_{n=0}^{m} q_n y_n \bar{x}_n - \sum_{n=0}^{m} \Delta(p_{n-1}\Delta y_{n-1})\bar{x}_n \\ &= \sum_{n=0}^{m} q_n y_n \bar{x}_n + \sum_{n=0}^{m} (p_n \Delta y_n)\Delta \bar{x}_n - p_m \Delta y_m \bar{x}_{m+1} + p_{-1}\Delta y_{-1}\bar{x}_0\end{aligned}$$

and finally, letting $m \to \infty$, for $x, y \in D_{T(M)}$,

$$\sum_{n=0}^{\infty} (p_n \Delta y_n \Delta \bar{x}_n + q_n y_n \bar{x}_n) = \sum_{n=0}^{\infty} (\bar{x}_n T y_n) w_n + \lim_{m\to\infty} p_m \Delta y_m \bar{x}_{m+1} - p_{-1}\Delta y_{-1}\bar{x}_0. \tag{2.9}$$

The left hand side of (2.9) is called the Dirichlet sum, and (2.9) is called the Dirichlet formula. Using (2.9), we have for $x, y \in D_{T(M)}$,

$$\begin{aligned}\sum_{n=0}^{m} (x_n T y_n - y_n T x_n) w_n &= p_m(\Delta x_m y_{m+1} - \Delta y_m x_{m+1}) \\ &\quad - p_{-1}(\Delta x_{-1} y_0 - \Delta y_{-1} x_0) \\ &= \sum_{n=0}^{m} \Delta[p_{n-1}(y_n \Delta x_{n-1} - x_n \Delta y_{n-1})].\end{aligned} \tag{2.10}$$

We may take the sum out from (2.10) to obtain

$$(x_nTy_n - y_nTx_n)w_n = \Delta[p_{n-1}(y_n\Delta x_{n-1} - x_n\Delta y_{n-1})], \qquad x, y \in D_{T(M)}. \quad (2.11)$$

Following the Dirichlet formula (2.9), we have for $x, y \in D_{T(M)}$,

$$\sum_{n=0}^{m}(p_n|\Delta x_n|^2 + q_n|x_n|^2) = p_m\Delta x_m\bar{x}_{m+1} - p_{-1}\Delta x_{-1}\bar{x}_0 + \sum_{n=0}^{m}(\bar{x}_nTx_n)w_n. \quad (2.12)$$

It is a consequence of (2.10) and (2.5) that, letting $m \to \infty$ in (2.10) we get the result

$$\lim_{m\to\infty} p_m(y_{m+1}\Delta x_m - x_{m+1}\Delta y_m) \text{ exists and is finite } \forall x, y \in D_{T(M)}. \qquad (2.13)$$

Morover, the limit in (2.13) is a constant when x, y are solutions of (2.1) by Lemma 1.4.3 in Chapter 1.

2.1.2 Definition If there is precisely one solution (up to constant multiples) of (2.1) for $\Im(\lambda) \neq 0$, which satisfies (2.6) then M is said to be in the Limit-Point (LP) case; otherwise all solutions of (2.1) satisfy (2.6) for all $\lambda \in \mathbb{C}$ and M is said to be in the Limit-Circle (LC) case (Atkinson [3], Hinton-Lewis [39]).

An alternative but equivalent characterization of M being LP is that the limit in (2.13) be zero for all $x, y \in D_{T(M)}$ (Hinton-Lewis [39,p.435]). We give the full proof of this statement in Chapter 3, Theorem 3.4.6.

2.1.3 Definition We say that M is

(i) Strong-Limit-Point (SLP) on $D_{T(M)}$ if

$$\lim_{m\to\infty} p_m\Delta y_m\bar{x}_{m+1} = 0 \ \forall x, y \in D_{T(M)}; \qquad (2.14)$$

(ii) Dirichlet (D) on $D_{T(M)}$ if

$$\{|p_n|^{1/2}\Delta x_n\}_{-1}^{\infty}, \quad \{|q_n|^{1/2}x_n\}_{-1}^{\infty} \in \ell^2 \ \forall x \in D_{T(M)}; \tag{2.15}$$

(iii) Conditional Dirichlet (CD) on $D_{T(M)}$ if

$$\{|p_n|^{1/2}\Delta x_n\}_{-1}^{\infty} \in \ell^2 \text{ and } \sum_{n=0}^{\infty} q_n|x_n|^2 \text{ is convergent } \forall x \in D_{T(M)}; \tag{2.16}$$

(iv) Weak Dirichlet (WD) on $D_{T(M)}$ if

$$\sum_{n=0}^{\infty}(p_n\Delta x_n\Delta y_n + q_n x_n y_n) \text{ is convergent } \forall x, y \in D_{T(M)}. \tag{2.17}$$

We observe that (2.14) is equivalent to

$$\lim_{m\to\infty} p_m\Delta x_m\bar{x}_{m+1} = 0, \qquad \forall x \in D_{T(M)}. \tag{2.18}$$

Because, it is clear that (2.14) implies (2.18), and (2.18) implies, for $\alpha \in \mathbb{C}$,

$$\lim_{m\to\infty} p_m[\Delta(x_m + \alpha y_m)(\bar{x}_{m+1} + \bar{\alpha}\bar{y}_{m+1})] = 0$$

$\Rightarrow$

$$\lim_{m\to\infty} p_m[\bar{\alpha}\Delta x_m\bar{y}_{m+1} + \alpha\Delta y_m\bar{x}_{m+1}] = 0$$

and taking $\alpha = 1$,i we get (2.14). It is also clear that, (2.18) is equivalent to

$$\lim_{m\to\infty} p_m\Delta x_m x_{m+1} = 0, \qquad \forall x \in D_{T(M)} \tag{2.19}$$

since,

$$\begin{aligned}
\lim_{m\to\infty} p_m\Delta x_m\bar{x}_{m+1} = 0 &\Longleftrightarrow \lim_{m\to\infty} |p_m\Delta x_m\bar{x}_{m+1}| = 0 \\
&\Longleftrightarrow \lim_{m\to\infty} |p_m\Delta x_m||\bar{x}_{m+1}| = 0 \\
&\Longleftrightarrow \lim_{m\to\infty} |p_m\Delta x_m||x_{m+1}| = 0 \\
&\Longleftrightarrow \lim_{m\to\infty} |p_m\Delta x_m x_{m+1}| = 0 \\
&\Longleftrightarrow \lim_{m\to\infty} p_m\Delta x_m x_{m+1} = 0.
\end{aligned}$$

For another proof of the equivalence of (2.18) and (2.19), see Benammar [4, p.84]. By the Dirichlet formula we also observe that the WD property is equivalent to

$$\lim_{m\to\infty} p_m \Delta y_m \bar{x}_{m+1} \qquad \text{exists and is finite} \qquad \forall x, y \in D_{T(M)}. \tag{2.20}$$

Now, using (2.13) this is equivalent to

$$\lim_{m\to\infty} p_m \Delta x_m x_{m+1}, \text{ exists and is finite } \forall x \in D_{T(M)}.$$

Also note that in (2.16)

$$\left\{|p_n|^{1/2}\Delta x_n\right\}_{-1}^{\infty} \in \ell^2 \iff \left\{p_n(\Delta x_n)^2\right\}_{-1}^{\infty} \in \ell^1 \iff \left\{p_n \Delta x_n \Delta y_n\right\}_{-1}^{\infty} \in \ell^1, \tag{2.21}$$

$\forall x, y \in D_{T(M)}$.

2.1.4 Remark Following the above definitions and subsequent comments, we have the following implications:

$$\text{D} \Rightarrow \text{CD} \Rightarrow \text{WD}, \ \text{SLP} \Rightarrow \text{WD}, \ \text{SLP} \Rightarrow \text{LP}.$$

§2.2 The main results

2.2.1 Theorem

(a) If either $w = \{w_n\}_{-1}^{\infty} \notin l^1$ or $\frac{1}{p} = \{\frac{1}{p_n}\}_{-1}^{\infty} \notin l^1$ or $\sum_{n=0}^{\infty} q_n$ is not convergent then CD $\Rightarrow$ SLP.

(b) If w, $\frac{1}{p} \in l^1$ but $q \notin l^1$ then D$\Rightarrow$SLP.

(c) If w, $\frac{1}{p}$, $q \in l^1$ then M is both D and LC.

Proof:

(a)$_{(i)}$ We assume that $w \notin \ell^1$ and M is CD on $D_{T(M)}$. Let $x, y \in D_{T(M)}$ then, by the Dirichlet formula (2.9), or by (2.20) since CD$\Rightarrow$WD,

$$\alpha = \lim_{m\to\infty} p_m \Delta y_m \bar{x}_{m+1} < \infty.$$

We need to prove that $\alpha = 0$ under the conditions in the hypothesis. Suppose the contrary that $\alpha \neq 0$, then for some $m_0 \in \mathbb{N}$

$$|p_m \Delta y_m x_{m+1}| \geq \frac{|\alpha|}{2}, \ \forall m \geq m_0,$$

which implies

$$|p_m \Delta y_m \Delta x_m| \geq \frac{|\alpha|}{2} \left|\frac{\Delta x_m}{x_{m+1}}\right|, \ \forall m \geq m_0, \ \forall x, y \in D_{T(M)}. \tag{2.22}$$

However M is CD and this implies that, summing over m, the left hand side of (2.22) belongs to ℓ^1. Thus,

$$\sum_{n=-1}^{\infty} \left|\frac{\Delta x_n}{x_{n+1}}\right| < \infty$$

and hence in particular $|\frac{\Delta x_n}{x_{n+1}}| \to 0$ as $n \to \infty$. So, as $n \to \infty$

$$\left|\log \frac{x_{n+1}}{x_n}\right| = \left|-\log(1 - \frac{\Delta x_n}{x_{n+1}})\right| \sim \left|\frac{\Delta x_n}{x_{n+1}}\right|$$

since

$$\lim_{t\to 0} \frac{\log(1-t)}{t} = -1.$$

Hence, since absolute convergence implies convergence,

$$\sum_{n=-1}^{\infty} \left|\log \frac{x_{n+1}}{x_n}\right| < \infty \Rightarrow \sum_{n=-1}^{\infty} \log \frac{x_{n+1}}{x_n} \quad \text{is convergent.}$$

$\Rightarrow$

$$\lim_{N\to\infty}\sum_{n=m_0}^{N}\log\frac{x_{n+1}}{x_n} \text{ exists for } m_0\in\mathbb{N}_0.$$

This implies

$$\lim_{N\to\infty}\sum_{n=m_0}^{N}\Delta(\log x_n)=\lim_{N\to\infty}(\log x_{N+1}-\log x_{m_0}) \text{ exists.}$$

So,

$$\lim_{N\to\infty}x_N\neq 0 \tag{2.23}$$

which is a contradiction, since $x\in\ell_w^2$ and $w\notin\ell^1$. So, $\alpha=0$ and M is SLP.

(a)$_{(ii)}$ Assume $\frac{1}{p}\notin\ell^1$ and M is CD. As above we suppose that, for $x,y\in D_{T(M)}$,

$$\alpha=\lim_{m\to\infty}p_m\Delta y_m x_{m+1}\neq 0.$$

Then, we obtain from (2.23) that

$$\beta=\lim_{m\to\infty}x_m\neq 0.$$

Thus,

$$\lim_{m\to\infty}p_m\Delta y_m=\alpha\beta^{-1},$$

and for some $m_0\in\mathbb{N}$,

$$|p_m(\Delta y_m)^2|\geq\frac{1}{4}|\alpha\beta^{-1}|^2|p_m^{-1}|\qquad\forall m\geq m_0. \tag{2.24}$$

However, summing over m, the left hand side of (2.24) belongs to ℓ^1 by the hypothesis that M is CD. Hence, so does the right hand side of (2.24) which is a contradiction to saying that $\frac{1}{p}\notin\ell^1$. Hence $\alpha=0$, proving M is SLP.

$(\mathbf{a})_{(iii)}$ Assume $p^{-1} \in \ell^1$ but $\sum_{n=0}^{\infty} q_n$ is not convergent and M is CD. Let $x \in D_{T(M)}$ and as in $(\mathbf{a})_{(i)}$ above, suppose that

$$\alpha = \lim_{m\to\infty} p_m x_{m+1} \Delta x_m \neq 0.$$

Then, $\lim_{m\to\infty} x_m = \beta \neq 0$ exists and, it follows that,

$$\lim_{m\to\infty} p_m \Delta x_m = \alpha\beta^{-1} \neq 0 \Rightarrow \lim_{m\to\infty} \Delta x_m = \lim_{m\to\infty} \alpha\beta^{-1} p_m^{-1}.$$

So, since $p^{-1} \in \ell^1$, we have

$$\sum_{m=-1}^{\infty} |\Delta x_m| < \infty, \qquad \text{i.e.} \qquad \Delta x \in \ell^1 \quad (x \in D_{T(M)}). \tag{2.25}$$

By the definition of $D_{T(M)}$,

$$\sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty \qquad \text{and} \qquad \sum_{n=-1}^{\infty} |Tx_n|^2 w_n < \infty,$$

i.e.

$$\sum_{-1}^{\infty} |x_n w_n^{1/2}|^2 < \infty \; and \sum_{-1}^{\infty} |[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n] w_n^{-1/2}| < \infty.$$

Now, by Cauchy-Schwarz inequality in ℓ^2, we have

$$\sum_{n=-1}^{\infty} |x_n w_n^{1/2}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n] w_n^{-1/2}|$$
$$\leq \left(\sum_{n=-1}^{\infty} |x_n w_n^{1/2}|^2\right)^{1/2} \left(\sum_{n=-1}^{\infty} |[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n] w_n^{-1/2}|^2\right)^{1/2}$$

which gives

$$\sum_{n=-1}^{\infty} |x_n[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n]| < \infty$$

and, since $\lim_{m\to\infty} x_m = \beta \neq 0$, we have that

$$\sum_{n=-1}^{\infty} |[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n]| < \infty. \tag{2.26}$$

Now,

$$\sum_{n=0}^{\infty}[-\Delta(p_{n-1}\Delta x_{n-1})+q_nx_n] = -\lim_{m\to\infty} p_m\Delta x_m + p_{-1}\Delta x_{-1} + \sum_{n=0}^{\infty} q_nx_n$$

$\Rightarrow$

$$\sum_{n=0}^{\infty} q_nx_n = \lim_{m\to\infty} p_m\Delta x_m - p_{-1}\Delta x_{-1} + \sum_{n=0}^{\infty}[-\Delta(p_{n-1}\Delta x_{n-1})+q_nx_n] \quad (2.27)$$

which proves the convergence of the sum $\sum_{n=0}^{\infty} q_nx_n$.

Since $\beta = \lim_{m\to\infty} x_m \neq 0$, $x_m \neq 0$ for all large $m \in \mathbb{N}_0$. On the other hand, using the summation by parts formula

$$\sum_{n=k}^{m} u_n\Delta v_n = u_{m+1}v_{m+1} - u_kv_k - \sum_{n=k}^{m} v_{n+1}\Delta u_n,$$

and supposing $k \in \mathbb{N}_0$ is such that $x_n \neq 0$ for all $n \geq k$, we have

$$\begin{aligned}
\sum_{n=k}^{m} q_n &= \sum_{n=k}^{m} \frac{1}{x_n}(q_nx_n) \\
&= \frac{1}{x_{m+1}}\sum_{s=k-1}^{m} q_sx_s - \frac{1}{x_k}\sum_{s=k-1}^{k-1} q_sx_s \\
&\quad -\sum_{n=k}^{m}\Big(\sum_{s=k-1}^{n} q_sx_s\Big)\Delta\Big(\frac{1}{x_n}\Big) \qquad\qquad (2.28)\\
&= \frac{\sum_{n=k-1}^{m} q_nx_n}{x_{m+1}} - \frac{q_{k-1}x_{k-1}}{x_k} \\
&\quad +\sum_{n=k}^{m}\Big(\sum_{s=k-1}^{n} q_sx_s\Big)\Big(\frac{\Delta x_n}{x_{n+1}x_n}\Big).
\end{aligned}$$

In (2.28) we've made use of the following:

$$u_n = \frac{1}{x_n}, \qquad v_n = \sum_{s=k-1}^{n-1} q_sx_s \Rightarrow q_nx_n = \Delta v_n \qquad (n \geq k).$$

Now, letting $m \to \infty$, we get that the right hand-side of (2.28) tends to a finite limit since $\sum_{n=0}^{\infty} q_nx_n$ is convergent and $\lim_{n\to\infty} x_n = \beta \neq 0$, which contradicts the hypothesis that $\sum_{n=0}^{\infty} q_n$ is divergent. This proves $\alpha = 0$ which guarantees that M is SLP.

(b) If $p^{-1} \in \ell^1$, $w \in \ell^1$ but $q \notin \ell^1$ then D$\Rightarrow$SLP.

For, assume that M is D and $q \notin \ell^1$. As in the proof of $(\mathbf{a})_{(i)}$ above, if for some $x, y \in D_{T(M)}$

$$\lim_{m\to\infty} p_m \Delta y_m x_{m+1} \neq 0$$

then we obtain the existence of $\lim_{m\to\infty} x_m \neq 0$. But, as M is D, $|q|^{1/2}x \in \ell^2$ and these two facts contradict the assumption that $q \notin \ell^1$. So, indeed

$$\lim_{m\to\infty} p_m \Delta y_m x_{m+1} = 0, \qquad \forall x, y \in D_{T(M)}$$

proving M is SLP.

(c) If $\frac{1}{p}$, w, $q \in \ell^1$ then M is LC and D. For the proof, we need the matrix representation of (2.1), i.e. for $n \geq 0$,

$$-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n = \lambda w_n x_n$$

$$\Downarrow$$

$$\Delta(p_{n-1}\Delta x_{n-1}) = (-\lambda w_n + q_n)x_n$$

or, equivalently, we obtain the recurrence relation

$$p_n(x_{n+1} - x_n) = (-\lambda w_n + q_n)x_n + p_{n-1}(x_n - x_{n-1}). \tag{2.29}$$

In order to obtain the matrix form of (2.29), we put

$$\begin{aligned} y_n &= p_n(x_{n+1} - x_n) \\ &= p_n \Delta x_n. \end{aligned}$$

So,

$$x_n = x_{n-1} + y_{n-1}/p_{n-1}$$
$$y_n = (x_{n-1} + y_{n-1}/p_{n-1})(-\lambda w_n + q_n) + y_{n-1} \tag{2.30}$$

or, equivalently, having

$$X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad A_n = \begin{pmatrix} 0 & 1/p_{n-1} \\ (-\lambda w_n + q_n) & (-\lambda w_n + q_n)/p_{n-1} \end{pmatrix}$$

we get

$$X_n = (I + A_n)X_{n-1}, \; n = 0, 1, 2, ... \tag{2.31}$$

where I is the identity matrix.

We are going to give the proof for the LC and D cases separately.

(i) The LC case:

For the LC case, it is sufficient to prove that, for some λ, say $\lambda = 0$, for all solutions of (2.29) $\sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty$ holds. Moreover, since $\sum_{n=-1}^{\infty} w_n < \infty$, it is sufficient to prove that all solutions of (2.29), with $\lambda = 0$, are bounded. For this purpose, we have the following theorem due to Atkinson [3, p.447].

2.2.2 Theorem (Atkinson)

Let the sequence of k-by-k matrices, in our case k=2,

$$A_n, \quad n = 0, 1, 2, 3, ...; \quad A_n = (a_{n\,rs}), \quad r, s = 1, 2, 3, ..., k$$

satisfy

$$\sum_{n=0}^{\infty} |A_n| < \infty \tag{2.32}$$

where

$$|A_n| := \sum_{r=1}^{k} \sum_{s=1}^{k} |a_{n\,rs}|. \tag{2.33}$$

Then the solutions of the recurrence relation

$$X_n - X_{n-1} = A_{n-1}X_{n-1},\ n = 0, 1, 2, ..., \tag{2.34}$$

where X_n is a $k-$vector, converge as $n \to \infty$ (which is stronger than the boundedness that we require). If in addition, the matrices $I + A_n$ are all non-singular, then $\lim_{n\to\infty} X_n \neq 0$, unless all the X_n are zero vectors.

Proof:

We write X_{nr} (r=1,2,3,...,k) for the entries of X_n, i.e.

$$X_n = \begin{pmatrix} X_{n1} \\ X_{n2} \\ \vdots \\ X_{nk} \end{pmatrix},$$

and define its norm by

$$|X_n| := \sum_{r=1}^{k} |X_{nr}|.$$

Using (2.34), we have

$$\begin{aligned} |X_{nr} - X_{n-1\,r}| &= |\sum_{s=1}^{k} a_{n\,rs}X_{n-1\,s}| \\ &\leq \sum_{s=1}^{k} |a_{n\,rs}| \sum_{s=1}^{k} |X_{n-1\,s}| \\ &= \sum_{s=1}^{k} |a_{n\,rs}||X_{n-1}| \end{aligned}$$

and summing over r, we have

$$|X_n - X_{n-1}| \leq |A_n||X_{n-1}|. \tag{2.35}$$

Now, by using the property $|a| - |b| \leq |a - b|$ in (2.35), we get

$$|X_n| \leq |X_{n-1}| + |A_n||X_{n-1}|.$$

It follows that

$$|X_n| \leq |X_{n-1}| + |A_n||X_{n-1}|$$

$$= (1 + |A_n|)|X_{n-1}|,$$

and so,

$$\begin{aligned}
&|X_0| \leq (1 + |A_0|)|X_{-1}| \\
&|X_1| \leq (1 + |A_1|)|X_0| \leq (1 + |A_1|)(1 + |A_0|)|X_{-1}| \\
&\quad \vdots \\
&|X_n| \leq |X_{-1}| \prod_{m=0}^{n} (1 + |A_m|).
\end{aligned} \tag{2.36}$$

Now, using (2.32)

$$|X_n| \leq K < \infty, \qquad \forall n \in \mathbb{N}_0 \tag{2.37}$$

and $\{|X_n|\}_0^\infty$ has a fixed upper bound, as asserted.

We now deduce that the sequence $\{X_n\}_{n=0}^\infty$ converges in the sense that the sequences $\{X_{nr}\}_{n=0}^\infty$ $(r = 1, 2, 3, ..., k)$ converge. For this, by (2.35)

$$\sum_{n=0}^{m} |X_n - X_{n-1}| \leq \sum_{n=0}^{m} |A_n||X_{n-1}|$$

and, by (2.32) and (2.37), we have that

$$\sum_{n=0}^{\infty} |X_n - X_{n-1}| < \infty.$$

In particular

$$\sum_{n=1}^{m} |X_{nr} - X_{n-1\,r}| < \infty, \qquad r = 1, 2, 3, ..., k. \tag{2.38}$$

So, the sequences $\{X_{nr}\}_0^\infty$, the entries of $\{X_n\}_0^\infty$, $(r = 1, 2, 3, ..., k)$ are all convergent as was to be proved.

For the last statement of the theorem, that $\{X_n\}_0^\infty$ do not tend to zero, we observe first that, from (2.34), if the $I + A_n$ are non-singular, then $\{X_n\}_0^\infty$ will be all zero or all non-zero. From (2.35) we now deduce that

$$|X_n| \geq |X_{n-1}| - |X_n - X_{n-1}|$$

$$\geq |X_{n-1}|(1 - |A_n|)$$

and so, if n is so large that $|A_m| < 1$ for all $m \geq n$,

$$|X_{n+p}| \geq |X_{n-1}| \prod_{m=n}^{n+p} (1 - |A_m|).$$

Making $p \to \infty$, the product on the right converges to a positive limit by (2.32), and so $\{|X_{n+p}|\}_0^\infty$ has a positive (non-zero) lower bound. Note that, in (2.38) we have a stronger conclusion than the convergence of the sequences $\{X_{nr}\}_{n=0}^\infty$ (r=1,2,3,...,k), namely the fact that they are bounded. This proves theorem 2.2.2.

In our case, for $\lambda = 0$,

$$|A_n| = \sum_{r=1}^{2}\sum_{s=1}^{2} |a_{nrs}|$$
$$= 1/|p_{n-1}| + |q_n| + |q_n|/|p_{n-1}|.$$

Hence, $\sum_{n=0}^\infty |A_n| < \infty$ by the hypothesis of **(c)**, and the theorem of Atkinson is applicable. So, applying this theorem, $\{X_n\}_0^\infty$ is convergent, i.e. the entries of X_n

$$\{X_{n1}\}_0^\infty = \{x_n\}_0^\infty \quad and \quad \{X_{n2}\}_0^\infty = \{y_n\}_0^\infty = \{p_n \Delta x_n\}_0^\infty$$

are convergent, so they are bounded and hence **(c)-(i)**.

(ii) The *Dirichlet* (D) case:

We now prove that $x \in D_{T(M)} \Rightarrow \sum_{n=-1}^\infty |p_n||\Delta x_n|^2 < \infty$, $\sum_{n=0}^\infty |q_n||x_n|^2 < \infty$ under the assumptions.

We know that all solutions of (2.29) are bounded by the theorem of Atkinson, but this is not true for all $x \in D_{T(M)}$ by the choice of w. We will state this for $\lambda = 0$ only but the proof also applies to all $\lambda \in \mathbb{C}$.

Let $x \in D_{T(M)}$ and define $f = \{f_n\}_{-1}^{\infty}$ by

$$f_n = Mx_n.$$

Then $\sum_{n=-1}^{\infty} |f_n|^2 w_n < \infty$. Also by the variation of parameters formula (cf. Chapter 1, Lemma 1.4.4), if $\varphi = \{\varphi_n\}_{-1}^{\infty}$ and $\psi = \{\psi_n\}_{-1}^{\infty}$ are linearly independent solutions of

$$Mx_n = \lambda x_n, \qquad n \in \mathbb{N}_0, \lambda \in \mathbb{C})$$

with

$$[\varphi, \psi]_n := p_{n-1}(\varphi_n \Delta \psi_{n-1} - \psi_n \Delta \varphi_{n-1}) = 1 \qquad \forall n \in \mathbb{N}_0,$$

then any solution of

$$Mx_n = \lambda x_n + f_n$$

is of the form

$$x_n = \Phi_n + A\varphi_n + B\psi_n \tag{2.39}$$

in which A and B are constants, and

$$\Phi_n = \sum_{m=0}^{n} (\psi_m \varphi_n - \varphi_m \psi_n) w_m f_m, \qquad n \in \mathbb{N}_0, \Phi_{-1} = 0.$$

Since $\{\varphi\}_{-1}^{\infty}$ and $\{\psi\}_{-1}^{\infty}$ are bounded by **(c)-(i)** case, using also Cauchy-Schwarz inequality in ℓ^2, it follows that for constants C_1, C_2, and K,

$$\begin{aligned}
|\Phi_n| &\leq C_1 \sum_{m=0}^{n} w_m |f_m| \\
&\leq C_2 (\sum_{m=0}^{n} w_m)^{1/2} (\sum_{m=0}^{n} w_m |f_m|^2)^{1/2} \\
&\leq K
\end{aligned}$$

$\Rightarrow \Phi$ is bounded.

This implies that $\{x_n\}_{-1}^{\infty}$ is bounded from the fact that $\{A\varphi_n + B\psi_n\}_{-1}^{\infty}$ and $\{\Phi_n\}_{-1}^{\infty}$ are bounded in (2.39). So, since $q \in \ell^1$ and following the above result,

$$\sum_{n=0}^{\infty} |q_n||x_n|^2 < \infty.$$

For the last part, we need to prove that $\sum_{n=0}^{\infty} |p_n||\Delta x_n|^2 < \infty$. From (2.39),

$$p_n \Delta x_n = p_n \Delta \Phi_n + p_n \Delta (A\varphi_n + B\psi_n)$$

and

$$\begin{aligned}
p_n \Delta \Phi_n &= p_n \Delta \sum_{m=0}^{n} (\psi_m \varphi_n - \varphi_m \psi_n) w_m f_m \\
&= p_n \sum_{m=0}^{n+1} (\psi_m \varphi_{n+1} - \varphi_n \psi_{n+1}) w_m f_m - p_n \sum_{m=0}^{n} (\psi_m \varphi_n - \varphi_m \psi_n) w_m f_m \\
&= p_n(\psi_0 \varphi_{n+1} - \varphi_0 \psi_{n+1}) w_0 f_0 - p_n(\psi_0 \varphi_n - \varphi_0 \psi_n) w_0 f_0 \\
&\quad + p_n(\psi_1 \varphi_{n+1} - \varphi_1 \psi_{n+1}) w_1 f_1 - p_n(\psi_1 \varphi_n - \varphi_1 \psi_n) w_1 f_1 \\
&\quad \vdots \\
&\quad + p_n(\psi_n \varphi_{n+1} - \varphi_n \psi_{n+1}) w_n f_n - p_n(\psi_n \varphi_n - \varphi_n \psi_n) w_n f_n \\
&\quad + p_n(\psi_{n+1} \varphi_{n+1} - \varphi_{n+1} \psi_{n+1}) w_{n+1} f_{n+1} \\
&= p_n(\psi_0 \Delta \varphi_n - \varphi_0 \Delta \psi_n) w_0 f_0 \\
&\quad + p_n(\psi_1 \Delta \varphi_n - \varphi_1 \Delta \psi_n) w_1 f_1 \\
&\quad \vdots \\
&\quad + p_n(\psi_n \Delta \varphi_n - \varphi_n \Delta \psi_n) w_n f_n
\end{aligned}$$

$\Rightarrow$

$$p_n \Delta \Phi_n = \sum_{m=0}^{n} [\psi_m (p_n \Delta \varphi_n) - \varphi_m (p_n \Delta \psi_n)] w_m f_m$$

and since, $\{p_n \Delta \varphi_n\}_{-1}^{\infty}$, $\{p_n \Delta \psi_n\}_{-1}^{\infty}$, $\{\varphi_n\}_{-1}^{\infty}$ and $\{\psi_n\}_{-1}^{\infty}$ are bounded by the theorem of Atkinson, $\{p_n \Delta \Phi_n\}_{-1}^{\infty}$ is also bounded, and so is $\{p_n \Delta x_n\}_{-1}^{\infty}$.

Now, since

$$\sum_{n=0}^{\infty} |p_n||\Delta x_n|^2 = \sum_{n=0}^{\infty} (|p_n||\Delta x_n|)^2/|p_n|$$

and by the hypothesis that $p^{-1} \in \ell^1$,

$$\begin{aligned}\sum_{n=0}^{\infty} |p_n||\Delta x_n|^2 &= \sum_{n=0}^{\infty} (|p_n||\Delta x_n|)^2/|p_n| \\ &< \infty.\end{aligned}$$

This completes the proof.

2.2.3 Corollary

From the Dirichlet formula (2.9), (2.18) and Theorem 2.2.1(a) we may deduce that if either $p^{-1} \notin \ell^1$ or $w \notin \ell^1$ or $\sum_{n=0}^{\infty} q_n$ is not convergent, then M is CD implies that

$$\sum_{n=0}^{\infty} (p_n|\Delta x_n|^2 + q_n|x_n|^2) \quad \text{is convergent for} \quad x \in D_{T(M)}. \tag{2.40}$$

Because, for $x \in D_{T(M)}$,

$$\sum_{n=0}^{\infty} (p_n|\Delta x_n|^2 + q_n|x_n|^2) = \lim_{N\to\infty} p_N \Delta x_N \bar{x}_{N+1} - p_{-1}\Delta x_{-1}\bar{x}_0 + \sum_{n=0}^{\infty} \bar{x}_n M x_n$$

and, since CD implies SLP by Theorem 2.2.1(a), $\lim_{m\to\infty} p_m \Delta x_m \bar{x}_{m+1} = 0$ by (2.18) and hence the result (2.40). From (2.40) and CD property we also deduce that $\sum_{n=0}^{\infty} q_n|x_n|^2$ is convergent for all $x \in D_{T(M)}$.

2.2.4 Remarks

(1) Under the conditions of **(a)** we have

$$\text{D} \Rightarrow \text{CD} \Rightarrow SLP \Rightarrow LP.$$

(2) For any $w \in \ell^1$ we have the following example in which $p^{-1} \in \ell^1$, $q \notin \ell^1$ but $\sum_{n=0}^{\infty} q_n$ is convergent, i.e. $\{q_n\}_{-1}^{\infty}$ is conditionally convergent, and for which M

is CD and LC, and hence not SLP. This proves that $q \notin \ell^1$ is not sufficient to ensure CD $\Rightarrow$ SLP.

2.2.5 Example

We want to establish an expression M of the form (2.2) such that $\{q_n\}_0^\infty$ is conditionally convergent, $\{w_n\}_{-1}^\infty$, $\{p_n^{-1}\}_{-1}^\infty \in \ell^1$ while M is CD and LC, hence not SLP, at the same time.

Let $\{r_n\}_0^\infty$ be a conditionally convergent real sequence. Choose a constant C_1 so that the sequence

$$\{R_n\}_0^\infty = \{\sum_{k=0}^{n} r_k\}_0^\infty + C_1$$

be positive, i.e. $R_n > 0$ $\forall n = 0, 1, 2, \ldots$. Then $\{R_n\}_0^\infty$ is bounded and, for $p_n > 0$ for all n=0,1,2,..., the sequence

$$\{x_n\}_0^\infty = \{\sum_{k=0}^{n} \frac{R_{k-1}}{p_{k-1}}\}_0^\infty + C_2,\ R_{-1} = 0, p_{n-1} > 0, \forall n \in \mathbb{N}_0,\ \text{and } x_{-1} \geq x_0$$

is also positive, i.e. $x_n > 0$ for any $C_2 > 0$. Note that $\{x_n\}_{-1}^\infty$ is monotonic increasing, i.e. $x_{n+1} \geq x_n$ $\forall n = -1, 0, 1, 2, \ldots$, from the fact that x_n are the sum of positive numbers. Now,

$$X = \lim_{n\to\infty} x_n \quad \text{exists} \tag{2.41}$$

since $\{R_n\}_{-1}^\infty$ is bounded and $p^{-1} = \{p_n^{-1}\}_{-1}^\infty \in \ell^1$. Morover, $x \in \ell_w^2$ since, $w \in \ell^1$ and $\{x_n\}_{-1}^\infty$ is bounded.

Now, we show that $\{x_n\}_{-1}^\infty$ is a solution of (2.1) with $\lambda = 0$, in which $\{q_n\}_{-1}^\infty$ is given by

$$q_n = \frac{r_n}{x_n},\ n \geq 0, q_{-1} = 0. \tag{2.42}$$

Really, for $n \geq 0$,

$$
\begin{aligned}
-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n &= -\Delta(p_{n-1}(x_n - x_{n-1})) + r_n \\
&= -\Delta(p_{n-1}\frac{R_{n-1}}{p_{n-1}}) + r_n \\
&= -(R_n - R_{n-1}) + r_n \\
&= 0.
\end{aligned}
$$

Note that, in

$$|q_n| = \frac{|r_n|}{x_n} \geq \frac{|r_n|}{X}, \ \forall n$$

summing over n we have $\{q_n\}_{-1}^{\infty} \notin \ell^1$ from the fact that $\{r_n\}_0^{\infty}$ is a conditionally convergent sequence, i.e. $r \notin \ell^1$ but $\sum_{n=-1}^{\infty} r_n$ is convergent.

Now, summation by parts formula gives, $\forall N \in \mathbb{N}_0$,

$$
\begin{aligned}
\sum_{n=0}^{N} q_n &= \sum_{n=0}^{N} \frac{r_n}{x_n} \\
&= \sum_{n=0}^{N} \frac{\Delta R_{n-1}}{x_n} \\
&= \sum_{n=-1}^{N-1} \frac{\Delta R_n}{x_{n+1}} \\
&= \frac{R_N}{x_N} - \frac{R_{-1}}{x_{-1}} - \sum_{n=-1}^{N-1} R_n \Delta(\frac{1}{x_n}) \\
&= \frac{R_N}{x_N} - \sum_{n=-1}^{N-1} \frac{R_n}{x_{n+1}} + \sum_{n=-1}^{N-1} \frac{R_n}{x_n}.
\end{aligned}
$$

For the first expression on the right-hand side, $\lim_{n\to\infty} R_n$ and $\lim_{n\to\infty} x_n$ exist and $X = \lim_{n\to\infty} x_n > 0$. For the sums on the right, since $\sum_{n=0}^{\infty} R_n$ is convergent and $\{\frac{1}{x_n}\}_{-1}^{\infty}$ is positive and decreasing, the Abel's convergence test that we quote down without proof is applicable (see Hyslop [40], Theorem 30, p.61)

2.2.6 Theorem (Abel's Test)

If $\{b_n\}_k^\infty$ $(k \in \mathbb{Z})$ is positive, monotonic decreasing sequence and if $\sum_k^\infty a_n$ is convergent, then $\sum_k^\infty a_n b_n$ is also convergent.

So, both

$$\sum_{n=-1}^{N} \frac{R_n}{x_{n+1}} \text{ and } \sum_{n=-1}^{N} \frac{R_n}{x_n}$$

are convergent and therefore $\{q_n\}_0^\infty$ is convergent.

Now, let $\{y_n\}_{-1}^\infty$ be another solution of (2.29) together with (2.42) complementary to $\{x_n\}_{-1}^\infty$, i.e. such that

$$[x, y]_n := p_{n-1}(y_n x_{n-1} - y_{n-1} x_n)$$

be constant, or equivalently, $[x, y]_n = 1$. Then,

$$\begin{aligned}\Delta(\frac{y_{n-1}}{x_{n-1}}) &= \frac{y_n x_{n-1} - x_n y_{n-1}}{x_n x_{n-1}} \\ &= \frac{1}{p_{n-1} x_n x_{n-1}}\end{aligned}$$

$\Rightarrow$

$$y_{n-1} = x_{n-1} \sum_{k=0}^{n-1} \frac{1}{p_{k-1} x_k x_{k-1}}$$

$\Rightarrow$

$$y_n = x_n \sum_{k=0}^{n} \frac{1}{p_{k-1} x_k x_{k-1}}.$$

So, since $\{y_n\}_{-1}^\infty$ is bounded and increasing,

$$\lim_{n \to \infty} y_n \text{ exists.} \tag{2.43}$$

We note that $\sum_{k=0}^\infty \frac{1}{p_{k-1} x_k x_{k-1}}$ is absolutely convergent, since for all n

$$\left|\frac{1}{p_{k-1} x_k x_{k-1}}\right| \le \frac{1}{\delta_2^2} \frac{1}{p_{k-1}},$$

and $p^{-1} \in \ell^1$. So, $y \in \ell^2_w$ since $w \in \ell^1$.

We see that, $My_n = 0$ too. Really,

$$
\begin{aligned}
My_n &= -\Delta(p_{n-1}\Delta y_{n-1}) + \frac{r_n}{x_n} x_n \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} \\
&= -\Delta\left(p_{n-1}\left[x_n \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} - x_{n-1} \sum_{k=0}^{n-1} \frac{1}{p_{k-1}x_k x_{k-1}}\right]\right) \\
&\quad + r_n \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} \\
&= -\Delta\left(p_{n-1}\Delta x_{n-1} \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} + \frac{1}{x_n}\right) + r_n \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} \\
&= -\Delta\left(R_{n-1} \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} + \frac{1}{x_n}\right) + r_n \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} \\
&= \frac{-R_n}{p_n x_n x_{n+1}} - \Delta R_{n-1} \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} - \Delta(\frac{1}{x_n}) + r_n \sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}} \\
&= \frac{-p_n \Delta x_n}{p_n x_n x_{n+1}} - \Delta(\frac{1}{x_n}) \\
&= 0.
\end{aligned}
$$

Hence we have shown that M is LC and hence not SLP, since $x, y \in \ell^2_w$ and x, y are linearly independent solutions of $Mx_n = \lambda x_n$, $\lambda \in \mathbb{C}$.

We now show that M is CD. Let $z \in D_{T(M)}$. Then,

$$\{Tz_n\}_{-1}^{\infty} = \{Mz_n\}_{-1}^{\infty} = \{f_n\}_{-1}^{\infty} \in \ell^2_w,\ w \in \ell^1.$$

The method of variation of parameters gives

$$z_n = Ax_n + By_n + \sum_{m=0}^{n} (x_n y_m - y_n x_m) f_m w_m \ (z_{-1} = 0, n \in \mathbb{N}_0)$$

where A and B are constants. Note that, from (2.42), (2.43) and the fact that

$$\lim_{n\to\infty} \sum_{m=0}^{n} (x_n y_m - y_n x_m) f_m w_m < \infty,$$

we have

$$\lim_{n\to\infty} z_n \text{ exists.} \tag{2.44}$$

Now,

$$p_n^{1/2}\Delta z_n = A(p_n^{1/2}\Delta x_n)+B(p_n^{1/2}\Delta y_n)+\sum_{m=0}^{n}[y_m(p_n^{1/2}\Delta x_n)-x_m(p_n^{1/2}\Delta y_n)]w_m f_m$$

and we have the following three facts:

(i)

$$\left|p_n^{1/2}\Delta x_n\right| = \frac{R_n}{p_n^{1/2}} \leq \frac{C}{p_n^{1/2}}, \qquad \text{C is a constant}$$

implies $\{p_n^{1/2}\Delta x_n\}_{-1}^{\infty} \in \ell^2$ since $\{R_n\}_0^{\infty}$ is bounded and $\{p_n^{-1}\}_{-1}^{\infty} \in \ell^1$.

(ii)

$$\begin{aligned} \left|p_n^{1/2}\Delta y_n\right| &= \left|\frac{1}{p_n^{1/2}x_n} + (p_n^{1/2}\Delta x_n)\sum_{k=0}^{n}\frac{1}{p_{k-1}x_k x_{k-1}}\right| \\ &\leq \frac{C_1}{p_n^{1/2}} + \frac{C_2}{p_n^{1/2}} \qquad (C_1, C_2 \text{ are constants}) \\ &= \frac{K}{p_n^{1/2}}, \qquad (K \text{ is a consant}), \end{aligned} \tag{2.45}$$

since $x_n \geq \delta > 0$ and

$$\sum_{k=0}^{n}\frac{1}{p_{k-1}x_k x_{k-1}} \leq \frac{1}{\delta^2}\sum_{k=0}^{n}\frac{1}{p_{k-1}} < C_2$$

in (2.45). This implies that $p_n^{1/2}\Delta y_n \in \ell^2$.

(iii)

$$\begin{aligned} &\sum_{m=0}^{n}[y_m(p_n^{1/2}\Delta x_n) - x_m(p_n^{1/2}\Delta y_n)]f_m w_m \\ &\leq \frac{C}{p_n^{1/2}}\sum_{m=0}^{n}|f_m w_m| \\ &\leq \frac{C}{p_n^{1/2}}\left[(\sum_{m=0}^{n} w_m)^{1/2}(\sum_{m=0}^{n} w_m|f_m|^2)^{1/2}\right] \\ &\leq \frac{K}{p_n^{1/2}} \end{aligned}$$

in which we have used (i), (ii) and the Cauchy-Schwarz inequality with C, K are constants. From (i), (ii) and (iii) it follows that

$$\{p_n^{1/2}\Delta z_n\}_{-1}^{\infty} \in \ell^2. \tag{2.46}$$

Finally, we have the following:

(i)

$$\lim_{n\to\infty} p_n \Delta x_n = \lim_{n\to\infty} R_n < \infty.$$

(ii)

$$\lim_{n\to\infty} p_n \Delta y_n = \lim_{n\to\infty} \left(\frac{1}{x_n} + (p_n\Delta x_n)\sum_{k=0}^{n} \frac{1}{p_{k-1}x_k x_{k-1}}\right) < \infty,$$

since the limits $\lim_{n\to\infty} \frac{1}{x_n}$ and $\lim_{n\to\infty} p_n\Delta x_n$ exist and $\sum_{k=0}^{\infty} \frac{1}{p_{k-1}x_k x_{k-1}}$ is absolutely convergent.

(iii)

$$\begin{aligned}\lim_{n\to\infty} |p_n\Delta x_n \sum_{m=0}^{n} y_m(w_m f_m)| &\le C_1 \lim_{n\to\infty} |p_n\Delta x_n \sum_{m=0}^{n}(w_m f_m)| \\ &\le C_2 \lim_{n\to\infty}\left(\sum_{m=0}^{n} w_m\right)^{1/2}\left(\sum_{m=0}^{n} w_m|f_m|^2\right)^{1/2} \\ &< \infty\end{aligned}$$

and similarly

(iv)

$$\lim_{n\to\infty} |p_n\Delta y_n \sum_{m=0}^{n} x_m(w_m f_m)| \le C \lim_{n\to\infty} |p_n\Delta y_n \sum_{m=0}^{n} w_m f_m| < \infty.$$

A consequence of (i), (ii), (iii) and (iv) is that

$$\lim_{n\to\infty} p_n\Delta z_n \text{ exists.}$$

We know also that, $\lim_{n\to\infty} z_n$ exists from (2.44). Therefore,

$$\lim_{n\to\infty} p_n \Delta z_n \bar{z}_{n+1} \qquad \text{exists.} \tag{2.47}$$

It is a consequence of (2.46) and (2.47) that M is CD. Because, from the identity

$$p_n \Delta z_n \bar{z}_{n+1} = p_{-1}\Delta z_{-1}\bar{z}_0 - \sum_{m=0}^{n} w_m \bar{z}_n T z_n + \sum_{m=0}^{n} p_n |\Delta z_n|^2 + \sum_{m=0}^{n} q_n |z_n|^2,$$

the CD property is equivalent to

(a) $\quad \{p_n |\Delta z_n|^2\}_{-1}^{\infty} \in \ell^1$

(b) $\quad \lim_{n\to\infty} p_n \Delta z_n \bar{z}_{n+1}$ exists $\forall z \in D_{T(M)}$.

This completes the desired example.

2.2.7 Theorem

Suppose $\{p_n\}_{-1}^{\infty}$ is real and $p_n > 0$ for all n, although $\{q_n\}_{-1}^{\infty}$ may still be complex.

(a) Suppose either $\{w_m \sum_{n=-1}^{m} p_n^{-1}\}_{m=-1}^{\infty} \notin \ell^1$ or $\{q_n\}_{-1}^{\infty} \notin \ell^1$, then

$$M \text{ is D} \iff \{|q_n|^{1/2} x_n\}_{-1}^{\infty} \in \ell^2 \qquad x \in D_{T(M)}.$$

(b) Suppose $\left\{w_m \sum_{n=-1}^{m} p_n^{-1}\right\}_{m=-1}^{\infty} \notin \ell^1$, then

$$M \text{ is CD} \iff \sum_{n=-1}^{\infty} q_n |x_n|^2 \text{ is convergent} \qquad x \in D_{T(M)}.$$

Proof:

(a) Since M is D $\Rightarrow \{|q_n|^{1/2} x_n\}_{-1}^{\infty} \in \ell^2$ for all $x \in D_{T(M)}$, we need only to prove the other implication.

Suppose $\{|q_n|^{1/2} x_n\}_{-1}^{\infty} \in \ell^2$ for all $x \in D_{T(M)}$. From the formula

$$\sum_{n=0}^{m} p_n |\Delta x_n|^2 = p_m \Delta x_m \bar{x}_{m+1} - p_{-1}\Delta x_{-1}\bar{x}_0 + \sum_{n=0}^{m} \bar{x}_n M x_n - \sum_{n=0}^{m} q_n |x_n|^2 \tag{2.48}$$

if we let $m \to \infty$, then the sums on the right converge. So, since $\{p_n\}_{-1}^{\infty}$ is positive, $\{p_n^{1/2}|\Delta x_n|\}_{-1}^{\infty} \notin \ell^2$ only if

$$\lim_{m\to\infty} p_m \Delta x_m \bar{x}_{m+1} = \infty$$

$$\Downarrow$$

$$\lim_{m\to\infty} p_m |\Delta x_m \bar{x}_{m+1}| = \infty.$$

But,

$$\begin{aligned} p_m|\Delta x_m \bar{x}_{m+1}| &\leq p_m|\Delta x_m|(|x_{m+1}| + |x_m|) \\ &\leq p_m \Delta(|x_m|^2) \end{aligned}$$

and hence

$$\lim_{m\to\infty} p_m \Delta(|x_m|^2) = \infty. \tag{2.49}$$

This implies, since $p_m > 0$ for all $m \in \mathbb{N}_0$, that $\{|x_n|^2\}_{-1}^{\infty}$ is monotonic increasing, i.e. $\Delta|x_n|^2 \geq 0$ for all large n, and so is $\{|x_n|\}_{-1}^{\infty}$.

We now have two cases: Either $\{q_n\}_{-1}^{\infty} \notin \ell^1$ or $\{q_n\}_{-1}^{\infty} \in \ell^1$.

If $\{q_n\}_{-1}^{\infty} \notin \ell^1$ we then get a contradiction to the assumption since this would imply $\{|q_n|^{1/2} x_n\}_{-1}^{\infty} \notin \ell^1$. So, $\{q\}_{-1}^{\infty}$ must be in ℓ^1. Then from (2.42),

$$\Delta(|x_n|^2) > p_n^{-1} \text{ since, from (2.49), } p_n\Delta(|x_n|^2) > 1 \text{ for large enough } n \in \mathbb{N}_0.$$

This implies, for some $m_0 \in \mathbb{N}_0$,

$$\begin{aligned} |x_m|^2 &\geq |x_m|^2 - |x_{m_0-1}|^2 \\ &> \sum_{n=m_0}^{m} p_{n-1}^{-1} \qquad m \in \mathbb{N},\ m > m_0. \end{aligned}$$

So,

$$\infty > \sum_{n=m_0}^{\infty} w_n |x_n|^2 > \sum_{n=m_0}^{\infty} w_n \left(\sum_{k=m_0}^{n} p_{k-1}^{-1} \right) \tag{2.50}$$

which is a contradiction to the assumption that $\{w_m \sum_{n=-1}^{m} p_n^{-1}\}_{m=-1}^{\infty} \notin \ell^1$ and hence $\{p_n^{1/2}|\Delta x_n|\}_{-1}^{\infty}$ is in ℓ^2, and M is D.

(b) The assumption concerning w and p, i.e. $\left\{w_m \sum_{n=-1}^{m} p_n^{-1}\right\}_{m=-1}^{\infty} \notin \ell^1$, implies that at least one of the sequences $w = \{w_n\}_{-1}^{\infty}$, $p^{-1} = \{p_n^{-1}\}_{-1}^{\infty}$ does not belong to ℓ^1. So, Corollary 2.2.3 gives the implication in one direction. Conversely, if $\sum_{n=m_0}^{\infty} q_n |x_n|^2$ is convergent for $x \in D_{T(M)}$ then (2.48), (2.49) and (2.50) again hold, so $\{p_n^{1/2}|\Delta x_n|\}_{-1}^{\infty} \in \ell^2$ by (a) above. It then follows from (2.48) that

$$\lim_{N\to\infty} p_N \Delta x_N \bar{x}_{N+1} \quad \forall x \in D_{T(M)} \quad \text{exists.}$$

The argument in the proof of Theorem 2.2.1(a)(i-ii) shows that this limit must be zero, i.e.

$$\lim_{N\to\infty} p_N \Delta x_N \bar{x}_{N+1} = 0 \quad \forall x \in D_{T(M)}$$

or equivalently,

$$\lim_{N\to\infty} p_N \Delta x_N y_{N+1} = 0 \quad \forall x, y \in D_{T(M)},$$

since at least one of $\{w_n\}_{-1}^{\infty}$, $\{p_n^{-1}\}_{-1}^{\infty}$ is not in ℓ^1. This leads us to the conclusion that M is SLP by Theorem 2.2.1(a)(i-ii)).

In (2.17), since $\{p_n^{1/2}\Delta x_n\}_{-1}^{\infty}$, $\{p_n^{1/2}\Delta y_n\}_{-1}^{\infty} \in \ell^2$ we deduce the convergence of

$$\sum_{n=m_0}^{\infty} q_n x_n \bar{y}_n \quad \forall x, y \in D_{T(M)},$$

since $\{p_n^{1/2}\Delta x_n\}_{-1}^{\infty}$, $\{p_n^{1/2}\Delta y_n\}_{-1}^{\infty} \in \ell^2$ and Cauchy-Schwarz inequality yields $\sum_{n=0}^{\infty} p_n \Delta x_n \Delta \bar{y}_n < \infty$. So, using (2.21), and the fact that absolute convergence implies convergence, M is CD and the theorem is therefore proved.

2.2.8 Remarks

1. $w \notin \ell^1$ is a sufficient condition for **(a)** and **(b)** to hold. But, if $w \in \ell^1$ then the condition on p and w, i.e.,

$$\{w_m \sum_{n=-1}^{m} p_n^{-1}\}_{m=-1}^{\infty} \notin \ell^1$$

is in general stronger than the requirement that $p^{-1} \notin \ell^1$.

2. If $w \in \ell^1$ then summation by parts shows that, for any $m \in \mathbb{N}_0 \cup \{-1\}$,

$$\sum_{n=-1}^{m} w_n(\sum_{k=-1}^{n} p_k^{-1}) = \sum_{n=-1}^{m} p_n^{-1}(\sum_{k=n}^{m} w_k) \qquad n < m. \tag{2.51}$$

This follows by using the summation by parts formula

$$\sum_{n=-1}^{m} u_{n+1}\Delta v_n = u_{m+1}v_{m+1} - u_{-1}v_{-1} - \sum_{n=-1}^{m} v_n \Delta u_n$$

where

$$v_n = \sum_{k=-1}^{n-1} w_k \Rightarrow \Delta v_n = w_n \text{ and } u_n = \sum_{k=-1}^{n-1} p_k^{-1} \Rightarrow \Delta u_n = p_n^{-1}.$$

For,

$$\begin{aligned}\sum_{n=-1}^{m} w_n(\sum_{k=-1}^{n} p_k^{-1}) &= (\sum_{k=-1}^{m} w_k)(\sum_{k=-1}^{m} p_k^{-1}) - 0 - \sum_{n=-1}^{m} p_n^{-1}(\sum_{k=-1}^{n-1} w_k) \\ &= \sum_{n=-1}^{m} p_n^{-1}(\sum_{k=-1}^{m} w_k - \sum_{k=-1}^{n-1} w_k) \\ &= \sum_{n=-1}^{m} p_n^{-1}(\sum_{k=n}^{m} w_k).\end{aligned}$$

If we let $m \to \infty$, we see that the condition in **(a)** and **(b)** is equivalent to the condition that

$$\{p_n^{-1} \sum_{k=n}^{\infty} w_k\}_{n=-1}^{\infty} \notin \ell^1 \text{ when } w \in \ell^1.$$

For example, if $m < \infty$ and $w = 1$, this condition becomes

$$\sum_{n=-1}^{\infty} (m-n)p_n^{-1} = \infty. \tag{2.52}$$

2.2.9 Theorem Suppose that $p_n > 0\ \forall n$ and w satisfy either

(a) $w = 1$ and $\sum_{m=-1}^{\infty} p_m^{-1} = \infty$

or

(b) $\left\{\frac{w_m}{w_{m+1}}\right\}_{-1}^{\infty}$ is bounded above and $\sum_{m=-1}^{\infty} p_m^{-1} w_m = \infty$,

then M is SLP $\iff$ M is WD.

Proof:

Since SLP always implies WD, we need only to prove that WD $\Rightarrow$ SLP under the conditions in the hypothesis. So, suppose that M satisfies the WD property, i.e. $\beta = \lim_{m\to\infty} p_n \Delta x_n x_{n+1}$ exists and is finite $\forall x \in D_{T(M)}$, but M is not SLP, i.e. $\beta \neq 0$. We show that $\beta \neq 0$ leads to a contradiction under (a) or (b) and hence M is SLP.

(a)

Suppose that

$$\beta = \lim_{m\to\infty} p_m \Delta x_m x_{m+1} \neq 0 \qquad \forall x \in D_{T(M)}.$$

Now, multiplying both sides of the following by $\bar{\beta}$, and summing over m

$$x_{m+1}\Delta x_m = x_{m+1}^2 - x_m x_{m+1} \tag{2.53}$$

we get

$$\sum_{m=-1}^{\infty} \bar{\beta}(p_m \Delta x_m x_{m+1}) p_m^{-1} = \bar{\beta} \sum_{m=-1}^{\infty} x_{m+1}^2 - \bar{\beta} \sum_{m=-1}^{\infty} x_m x_{m+1}.$$

Here, the right hand-side is finite since $x \in D_{T(M)}$, but the left hand-side is infinite if $\sum_{m=-1}^{\infty} p_m^{-1} = \infty$, which is a contradiction. So, $\beta = 0$ and M is SLP.

(b)

From (2.53), we have

$$\sum_{m=0}^{\infty}(\bar{\beta} p_m \Delta x_m x_{m+1}) w_m p_m^{-1} = \bar{\beta}\{\sum_{m=0}^{\infty} w_{m+1} x_{m+1}^2 (\frac{w_m}{w_{m+1}}) - \sum_{m=0}^{\infty}(w_m w_{m+1})^{1/2} x_m x_{m+1} (\frac{w_m}{w_{m+1}})^{1/2}\}.$$

If $\{\frac{w_m}{w_{m+1}}\}_{-1}^{\infty}$ is bounded above and $\sum_{m=0}^{\infty} w_m p_m^{-1} = \infty$ then we again get a contradiction, hence $\beta = 0$ and M is SLP.

Chapter 3

OPERATORS AND THEIR GRAPHS, PARTIAL SEPARATION

In this chapter, we arm ourselves with the tools that we need to use in the next chapter where we give our inequality.

§3.1. Operators and their graphs

In the differential operators case, by using von Neumann theory, the maximal operator is expressible as an orthogonal sum of the minimal operator and the deficiency sub-spaces with respect to an appropriate inner-product as

$$D(T_0^*) = D(T) = D(T_0) \oplus N(\lambda^2) \oplus N(\bar{\lambda}^2), \qquad \Im(\lambda^2) \neq 0 \tag{3.1}$$

where T is the maximal and T_0 is the minimal operator related to the square of the second order formally symmetric differential expression and $N(\lambda^2)$, $N(\bar{\lambda}^2)$ are the related deficiency subspaces (see Beynon [8], p.23).

We now define the maximal and the minimal operators related to the recurrence expressions M and M^2 and ask if we can apply the same theory. We remind ourselves that the recurrence expressions, i.e. the difference expressions are defined as

$$Mx_n := \begin{cases} \frac{1}{w_n}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n], & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}\Delta x_n, & n = -1 \end{cases} \tag{3.2}$$

and

$$M^2x_n = [M(Mx)]_n := \begin{cases} M(Mx_n), & n \geq 1 \\ \frac{1}{w_0}[-\Delta(p_{n-1}\Delta Mx_{n-1}) + q_nMx_n], & n = 0 \\ -\frac{p_{-1}}{w_{-1}}\Delta Mx_n, & n = -1. \end{cases} \tag{3.3}$$

We use the notation $T(M^k)$ for the maximal operators, $T_0'(M^k)$ for the pre-minimal operators and $T_0(M^k)$ for the minimal operators related to M^k, $k = 1, 2$, where $M^2 := M(M)$. We now define the operators related to M and M^2 as follows. The maximal operators are

$$T(M) : D_{T(M)} \subset \ell^2_w \to \ell^2_w, \qquad [T(M)x]_n = T(M)x_n := Mx_n \tag{3.4}$$

$$D_{T(M)} := \{x \in \ell^2_w : \sum_{n=0}^{\infty} |Mx_n|^2 w_n < \infty\}$$

$$T(M^2) : D_{T(M^2)} \subset \ell^2_w \to \ell^2_w, \qquad [T(M^2)x]_n = T(M^2)x_n := M^2x_n \tag{3.5}$$

$$D_{T(M^2)} := \{x \in \ell^2_w : \sum_{n=1}^{\infty} |M^2x_n|^2 w_n < \infty\},$$

the pre-minimal operators are

$$T_0'(M) := T(M) \restriction_{D_{T_0'(M)}} \text{ and } T_0'(M^2) := T(M^2) \restriction_{D_{T_0'(M^2)}} \tag{3.6}$$

on the domains

$$D_{T_0'(M)} := \{x \in D_{T(M)} : x_{-1} = 0 \text{ and } x_n = 0 \text{ for all but a finite number of values of } n\} \tag{3.7}$$

$$D_{T_0'(M^2)} := \{x \in D_{T(M^2)} : x_{-1} = x_0 = 0, x_n = 0 \text{ for all but a finite number of values of } n\} \tag{3.8}$$

and the minimal operators $T_0(M)$ and $T_0(M^2)$ are defined as the closures of the pre-minimal operators $T_0'(M)$ and $T_0'(M^2)$ respectively in ℓ^2_w.

Now, when we attempt to prove that $T_0^*(M^2) = T(M^2)$, we see that this is not possible since the domain of $T_0(M^2)$ is not dense in ℓ^2_w and hence a unique adjoint for $T_0(M^2)$ can not be defined. This means that we can not apply the theory that Beynon *et al.* **[9]** used for their inequality. So, we are forced to use the theory of linear relations and their adjoints developed by Bennewitz **[5-6]** and Pleijel **[55-56]**. Bennewitz ([**6**], p.35) defines the adjoint of a linear relation on an arbitrary Hilbert space H as follows.

3.1.1 Definition (Adjoint for a linear relation) Let the set E be a linear relation on H, i.e. $E \subset H \times H$. The adjoint relation E^* is defined as

$$< U, V > \in E^* \iff (U, v)_H = (V, u)_H \quad \forall < u, v > \in E. \tag{3.9}$$

E is called symmetric relation if $E \subseteq E^*$.

Note that, we use the notations $< .,. >$ for an element in a relation, i.e. it is a pair in $H \times H$, and $(.,.)_H$, or simply $(.,.)$ when there is no ambiguity, for the inner product defined on a Hilbert space H.

3.1.2 Definition (Graph of a linear operator)

The graph of a linear operator T is defined as

$$G(T) := \{< x, x^* >: x \in D(T) \text{ and } x^* = Tx\}. \tag{3.10}$$

Now, since the graphs $G\big(T_0'(M^2)\big)$ and $G\big(T_0(M^2)\big)$ are linear subsets of $\ell^2_w \times \ell^2_w$ and since they are linear relations in the terminology of Bennewitz, the adjoint relations $G^*\big(T_0'(M^2)\big)$ and $G^*\big(T_0(M^2)\big)$ can be defined as closed linear relations on $\ell^2_w \times \ell^2_w$. This is the technique that Brown and Evans **[12]** used in order to give a proof for the HELP inequality for second order formally symmetric difference expressions, and we are now going to make use of the same ideas. For simplicity, G_2, G'_{02} and G_{02} will represent the graphs $G\big(T(M^2)\big)$, $G\big(T'_0(M^2)\big)$ and $G\big(T_0(M^2)\big)$ respectively.

If G_{02} is the closure of G'_{02}, then

$$(G_{02})^* = (G'_{02})^*. \tag{3.11}$$

Now, define the maximal graph as

$$G := \big\{ < x, x^* >: x^*_n = M^2 x_n,\ (n \geq 1),\ x \in D_{T(M^2)} \big\}. \tag{3.12}$$

Note that, x^*_{-1} and x^*_0 can be anything and $G_2 \subset G$. We then have the following lemma.

3.1.3 Lemma $G^*_{02} = G$.

Proof:

(a) To prove that $G \subset G^*_{02}$, let $< x, x^* > \in G$ and $< y, y^* > = < y, T'_0(M^2)y > \in G'_{02}$. Following (3.9), we need to show that

$$(T'_0(M^2)y, x) - (y, x^*) = 0, \qquad \forall < y, y^* > \in G'_{02}.$$

We have, since $y_{-1} = y_0 = My_{-1} = 0$,

$$(T'_0(M^2)y, x) - (y, x^*) = \sum_{n=-1}^{\infty} M^2 y_n \bar{x}_n w_n - \sum_{n=-1}^{\infty} y_n \bar{x}^*_n w_n$$

$$= \sum_{n=1}^{\infty} (\bar{x}_n M^2 y_n - y_n \overline{M^2 x_n}) w_n + M^2 y_0 \bar{x}_0 w_0 + M^2 y_{-1} \bar{x}_{-1} w_{-1} - y_{-1} \bar{x}^*_{-1} w_{-1}$$

$$- y_0 \bar{x}^*_0 w_0$$

$$= p_0(\bar{x}_0 M y_1 - \bar{x}_1 M y_0 + \overline{Mx_0} y_1 - \overline{Mx_1} y_0)$$

$$+ [-\Delta(p_{-1}\Delta My_{-1}) + q_0 M y_0]\bar{x}_0 - p_{-1}\Delta My_{-1}\bar{x}_{-1}$$

$$= p_0(\bar{x}_0 M y_1 - \bar{x}_1 M y_0 + \overline{Mx_0} y_1)$$

$$+ (-p_0 \Delta M y_0 + p_{-1}\Delta M y_{-1} + q_0 M y_0)\bar{x}_0 - p_{-1} M y_0 \bar{x}_{-1} + p_{-1} M y_{-1} \bar{x}_{-1}$$

$$= p_0(\bar{x}_0 My_1 - \bar{x}_1 My_0 + \overline{Mx}_0 y_1) - p_0 My_1\bar{x}_0 + p_0 My_0\bar{x}_0$$

$$+ p_{-1}My_0\bar{x}_0 - p_{-1}My_{-1}\bar{x}_0 + q_0 My_0\bar{x}_0 - p_{-1}My_0\bar{x}_{-1}$$

$$= -p_0\bar{x}_1 My_0 + p_0 y_1\overline{Mx}_0 + p_0 My_0\bar{x}_0 + p_{-1}My_0\bar{x}_0 + q_0 My_0\bar{x}_0 - p_{-1}My_0\bar{x}_{-1}$$

$$= My_0(-p_0\bar{x}_1 + p_0\bar{x}_0 + p_{-1}\bar{x}_0 + q_0\bar{x}_0 - p_{-1}\bar{x}_{-1}) + p_0 y_1\overline{Mx}_0$$

$$= My_0\overline{Mx}_0 w_0 + p_0 y_1\overline{Mx}_0$$

$$= (w_0 My_0 + p_0 y_1)\overline{Mx}_0$$

$$= 0.$$

(b) To prove $G^*_{02} \subset G$ let $< x, x^* > \in G^*_{02} = (G'_{02})^*$ and $< y, T'_0(M^2)y > \in G'_{02}$.

Then, since

$$(T'_0(M^2)y, x) = (y, x^*),$$

$$
\begin{aligned}
0 &= (T'_0(M^2)y, x) - (y, x^*) \\
&= \sum_{n=1}^{\infty} \bar{x}_n M^2 y_n w_n + M^2 y_0\bar{x}_0 w_0 + M^2 y_{-1}\bar{x}_{-1}w_{-1} - \sum_{n=1}^{\infty} y_n\bar{x}^*_n w_n \\
&\quad - y_0\bar{x}^*_0 w_0 - y_{-1}\bar{x}^*_{-1}w_{-1} \\
&= \sum_{n=1}^{\infty}(\bar{x}_n M^2 y_n - \overline{M^2x}_n y_n)w_n + \sum_{n=1}^{\infty} y_n\overline{M^2x}_n w_n - \sum_{n=1}^{\infty} y_n\bar{x}^*_n w_n \\
&\quad - p_{-1}My_0\bar{x}_{-1} - p_0 My_1\bar{x}_0 + (p_{-1} + p_0 + q_0)My_0\bar{x}_0 \\
&= p_0(\bar{x}_0 My_1 - \bar{x}_1 My_0 + y_1\overline{Mx}_0 - y_0\overline{Mx}_1) + \sum_{n=1}^{\infty} y_n(\overline{M^2x}_n - \bar{x}^*_n)w_n \\
&\quad - p_{-1}My_0\bar{x}_{-1} - p_0 My_1\bar{x}_0 + (p_{-1} + p_0 + q_0)My_0\bar{x}_0 \\
&= \sum_{n=1}^{\infty} y_n(\overline{M^2x}_n - \bar{x}^*_n)w_n + p_0 y_1\overline{Mx}_0 + My_0[-p_0\bar{x}_1 + (p_{-1} + p_0 + q_0)\bar{x}_0 \\
&\quad - p_{-1}\bar{x}_{-1}] \\
&= \sum_{n=1}^{\infty} y_n(\overline{M^2x}_n - \bar{x}^*_n)w_n + \overline{Mx}_0(My_0 w_0 + p_0 y_1) \\
&= \sum_{n=1}^{\infty} y_n(\overline{M^2x}_n - \bar{x}^*_n)w_n.
\end{aligned}
$$

Now, take $y_n = \delta_{n,k}$, where $k, n = 1, 2, 3, ...$, to obtain

$$\bar{x}_k^* = \overline{M^2 x_k} \text{ for } k = 1, 2, 3, ...$$

and $\bar{x}_{-1}^*$, $\bar{x}_0^*$ can be anything since $y_{-1} = y_0 = 0$. So,

$$x_n^* = M^2 x_n, \text{ for } n \geq 1$$

which means that $< x, x^* > \in G$. The lemma is therefore proved.

3.1.4 Lemma Let M^2 be limit-point. Then G_{02} is a closed, symmetric relation on ℓ_w^2 and

$$< y, y^* > \in G_{02} \iff \begin{cases} y_n^* = M^2 y_n, n \geq 1 \\ y_{-1} = y_0 = 0 \\ y_{-1}^* = M^2 y_{-1} = \frac{p_0 p_{-1}}{w_0 w_{-1}} y_1 \\ y_0^* = M^2 y_0 = \frac{p_0 p_1}{w_0 w_1} y_2 - \frac{p_0}{w_0} \left(\frac{p_0 + p_1 + q_1}{w_1} + \frac{p_{-1} + p_0 + q_0}{w_0} \right) y_1 \end{cases} \tag{3.13}$$

Proof: G_{02} is closed by its definition, and it is symmetric since

$$G_{02} = \overline{G'_{02}} \subset G = G_{02}^*. \tag{3.14}$$

Note that, this also shows that G'_{02} is symmetric, since $G_{02}^* = (G'_{02})^*$. Now, the rest of the proof:

Necessity of (3.13) ($\Rightarrow$): Let $< y, y^* > \in G_{02} \subset G = G_{02}^*$. We then have $y_n^* = M^2 y_n$ $(n = 1, 2, 3, ...)$. Also let $< x, x^* > \in G = G_{02}^*$ where $x = \{x_n\}_{-1}^{\infty}$ has only a finite number of non-zero x_n's but arbitrary, otherwise. Then, since $G_{02}^* = G$,

$$\begin{aligned} 0 &= (y^*, x) - (y, x^*) \\ &= \sum_{n=1}^{\infty} (M^2 y_n \bar{x}_n - y_n \overline{M^2 x_n}) w_n + (y_{-1}^* \bar{x}_{-1} - y_{-1} \bar{x}_{-1}^*) w_{-1} + (y_0^* \bar{x}_0 - y_0 \bar{x}_0^*) w_0 \\ &= p_0 (\bar{x}_0 M y_1 - \bar{x}_1 M y_0 + \overline{M x_0} y_1 - \overline{M x_1} y_0) \\ &\quad + y_{-1}^* \bar{x}_{-1} w_{-1} - y_{-1} \bar{x}_{-1}^* w_{-1} + y_0^* \bar{x}_0 w_0 - y_0 \bar{x}_0^* w_0 \end{aligned}$$

$$= p_0\bar{x}_0\frac{1}{w_1}[-p_1y_2+(p_0+p_1+q_1)y_1-p_0y_0]-p_0\bar{x}_1\frac{1}{w_0}[-p_0y_1$$

$$+(p_{-1}+p_0+q_0)y_0-p_{-1}y_{-1}]+p_0y_1\frac{1}{w_0}[-p_0\bar{x}_1+(p_{-1}+p_0+q_0)\bar{x}_0$$

$$-p_{-1}\bar{x}_{-1}]-p_0y_0\overline{Mx_1}+y^*_{-1}\bar{x}_{-1}w_{-1}-y_{-1}\bar{x}^*_{-1}w_{-1}+y^*_0\bar{x}_0w_0-y_0\bar{x}^*_0w_0$$

$$=\bar{x}_0[y^*_0w_0-\frac{p_0p_1}{w_1}y_2+\frac{p_0(p_0+p_1+q_1)}{w_1}y_1+\frac{p_0(p_{-1}+p_0+q_0)}{w_0}y_1]$$

$$+y_{-1}(\frac{p_{-1}p_0}{w_0}x_1-x^*_{-1}w_{-1})+\bar{x}_{-1}(y^*_{-1}w_{-1}-\frac{p_{-1}p_0}{w_0}y_1)$$

$$+y_0[-\frac{p_0^2}{w_1}\bar{x}_0-\frac{p_0(p_{-1}+p_0+q_0)}{w_0}\bar{x}_1-p_0\overline{Mx_1}-\bar{x}^*_0w_0].$$

Now, since the x_n and x^*_n are arbitrary, it follows that

$$\begin{cases} y_{-1}=y_0=0 \\ y^*_{-1}=\frac{p_0p_{-1}}{w_0w_{-1}}y_1=M^2y_{-1} \\ y^*_0=\frac{p_0p_1}{w_0w_1}y_2-\frac{p_0}{w_0}(\frac{p_0+p_1+q_1}{w_1}+\frac{p_{-1}+p_0+q_0}{w_0})y_1=M^2y_0. \end{cases} \tag{3.15}$$

Sufficiency of (3.13) ($\Leftarrow$): For this implication, we need to show that the elements $<y,y^*>\in G$ satisfying (3.15) are in G_{02}, i.e. in G^*_2. So, take an arbitrary $<x,x^*>\in G$ and $<y,\ y^*>\in G$ satisfying (3.15). Then, we show that $(y^*,x)-(y,x^*)=0$, i.e. $<y,y^*>\in G^*_2=G_{02}$. Really, since M^2 is limit-point (cf. section 3.2) by the hypothesis,

$$(y^*,x)-(y,x^*)$$

$$=\sum_{n=1}^{\infty}(M^2y_n\bar{x}_n-y_n\overline{M^2x_n})w_n+(y^*_{-1}\bar{x}_{-1}-y_{-1}\bar{x}^*_{-1})w_{-1}+(y^*_0\bar{x}_0-y_0\bar{x}^*_0)w_0$$

$$=\frac{p_0p_{-1}}{w_0w_{-1}}y_1\bar{x}_{-1}w_{-1}+\frac{p_0}{w_0}[\frac{p_1}{w_1}y_2-(\frac{p_0+p_1+q_1}{w_1}+\frac{p_{-1}+p_0+q_0}{w_0})y_1]\bar{x}_0w_0$$

$$+p_0(\bar{x}_0My_1-\bar{x}_1My_0+\overline{Mx_0}y_1-\overline{Mx_1}y_0)$$

$$=\frac{p_0p_{-1}}{w_0}y_1\bar{x}_{-1}+p_0\bar{x}_0[\frac{p_1}{w_1}y_2-(\frac{p_0+p_1+q_1}{w_1}+\frac{p_{-1}+p_0+q_0}{w_0})y_1]+$$

$$p_0\bar{x}_0\frac{1}{w_1}[-p_1y_2+(p_0+p_1+q_1)y_1-p_0y_0]-p_0\bar{x}_1\frac{1}{w_0}[-p_0y_1+(p_{-1}+p_0+q_0)y_0$$

$$-p_{-1}y_{-1}]+y_1p_0\frac{1}{w_0}[-p_0\bar{x}_1+(p_{-1}+p_0+q_0)\bar{x}_0-p_{-1}\bar{x}_{-1}]$$

$$=0$$

since $y_{-1}=y_0=0$, and hence the result.

We are now able to give the orthogonal direct sum of $G_{02}^* = G$ in terms of G_{02} and the deficiency subspaces $N(\bar{\lambda}^2)$ and $N(\lambda^2)$ with respect to the graph inner product

$$(< a, a^* >, < b, b^* >)_{\lambda^2} := (a^* - \epsilon a, b^* - \epsilon b) + \eta^2(a, b), \ \ \lambda^2 = \epsilon + \mathrm{i}\eta \ (\eta \neq 0) \quad (3.16)$$

defined on G. In (3.16), $(.,.)$ is the ℓ^2_w inner product and the norm on G is the norm produced by the graph inner product. The deficiency subspaces are defined as

$$\begin{aligned} N(\lambda^2) &:= \{< x, \lambda^2 x > \in G_{02}^* = G\} \\ &= \{< x, \lambda^2 x > \in \ell^2_w \times \ell^2_w : M^2 x_n = \lambda^2 x_n, \ n = 1, 2, 3, ...\} \qquad (3.17) \\ N(\bar{\lambda}^2) &:= \{< x, \bar{\lambda}^2 x > \in \ell^2_w \times \ell^2_w : M^2 x_n = \bar{\lambda}^2 x_n, \ n = 1, 2, 3, ...\}. \end{aligned}$$

We have the following lemma, which establishes the orthogonal decomposition of the maximal graph with respect to the graph inner-product defined by (3.16).

3.1.5 Lemma Let $\lambda^2 = \epsilon + \mathrm{i}\eta$, $\Im(\lambda^2) \neq 0$. Then,

$$G = G_{02} \oplus N(\lambda^2) \oplus N(\bar{\lambda}^2) \quad (3.18)$$

where $\oplus$ denotes the orthogonal sum with respect to the graph inner-product (3.16) defined on G.

Proof:

(a) $G_{02} \perp N(\lambda^2)$: Let $< y, y^* > \in G_{02}$ and $< u, \lambda^2 u > \in N(\lambda^2)$, then

$$\begin{aligned} (< y, y^* >, < u, \lambda^2 u >)_{\lambda^2} &= (y^* - \epsilon y, \lambda^2 u - \epsilon u) + \eta^2(y, u) \\ &= (y^* - \epsilon y, \mathrm{i}\eta u) + \eta^2(y, u) \\ &= -\mathrm{i}\eta(y^*, u) + \mathrm{i}\epsilon\eta(y, u) + \eta^2(y, u) \\ &= -\mathrm{i}\eta(y, \lambda^2 u) + \mathrm{i}\epsilon\eta(y, u) + \eta^2(y, u) \end{aligned}$$

$$= -\mathrm{i}\eta\epsilon(y,u) - \eta^2(y,u) + \mathrm{i}\epsilon\eta(y,u) + \eta^2(y,u)$$

$$= 0$$

since $< u, \lambda^2 u > \in G_{02}^* = G$ and by Lemma 3.1.3.

(b) $G_{02} \perp N(\bar{\lambda}^2)$**:** It follows similar to (a).

(c) $N(\lambda^2) \perp N(\bar{\lambda}^2)$**:** Let, $< u, \lambda^2 u > \in N(\lambda^2)$ and $< v, \bar{\lambda}^2 v > \in N(\bar{\lambda}^2)$, then

$$(< u, \lambda^2 u >, < v, \bar{\lambda}^2 v >)_{\lambda^2} = (\lambda^2 u - \epsilon u, \bar{\lambda}^2 v - \epsilon v) + \eta^2(u,v)$$

$$= (\mathrm{i}\eta u, -\mathrm{i}\eta v) + \eta^2(u,v)$$

$$= 0.$$

§3.2. Limit-point (LP) classification for M^2

In this section, we aim to give a classification result that establishes a connection between the number of ℓ_w^2 solutions of

$$M^2 x_n = \lambda^2 x_n, \qquad n \geq 1, \Im(\lambda^2) \neq 0 \tag{3.19}$$

and self-adjointness of the operators related to (3.3). We start with the following definition.

3.2.1 Definition If there are precisely 2 ℓ_w^2 solutions of

$$M^2 x_n = \lambda^2 x_n, \qquad n \geq 1 \tag{3.20}$$

for $\Im(\lambda^2) \neq 0$ then M^2 is said to be *limit-point* (LP).

3.2.2 Definition If M^2 is not LP then (3.20) has either 3 or 4 linearly independent solutions in ℓ_w^2. For convenience, we shall say that M^2 is in the *limit-circle* case (LC) when (3.20) has 4 linearly independent solutions in ℓ^2_w, i.e. all solutions are in ℓ^2_w for all $\lambda \in \mathbb{C}$.

We now state and prove the following result which is analogous to that for M (cf. Theorem 3.4.6).

M^2 is LP $\iff$

$$\lim_{m\to\infty} p_m(\overline{x_{m+1}}My_m - \bar{x}_m My_m + \overline{Mx_{m+1}}y_m - \overline{Mx_m}y_{m+1}) = 0 \tag{3.21}$$

$\forall x, y \in D(T_2)$. To start with, we define the space

$$\ell^2_{w,1} := \left\{x = \{x_n\}_1^\infty : \sum_{n=1}^{\infty} |x_n|^2 w_n < \infty\right\} \tag{3.22}$$

with the inner product $(x, y) := \sum_{n=1}^{\infty} x_n \bar{y}_n w_n$. We identify this Hilbert space of sequences with $\left\{x = \{x_n\}_{-1}^\infty \in \ell^2_w : x_{-1} = x_0 = 0\right\}$. Now define the operators A_0' and A

$$A_0' : \ell^2_{w,1} \to \ell^2_{w,1}, \qquad A : \ell^2_{w,1} \to \ell^2_{w,1}$$

on the domains

$$D(A_0') := \left\{x \in \ell^2_{w,1} : x_n = 0 \text{ for all but a finite number of values of n}\right\},$$

$$D(A) := \left\{x : x, M^2 x \in \ell^2_{w,1}\right\}$$

as

$$A_0' x := M^2 x, \text{ and } Ax = M^2 x, \text{ i.e., } A, A_0' : \{x_n\}_1^\infty \to \{M^2 x_n\}_1^\infty.$$

Our first step is the following lemma.

3.2.3 Lemma $D(A_0')$ is dense, and furthermore A_0' is a symmetric operator in $\ell^2_{w,1}$.

Proof: $D(A'_0)$ is dense in $\ell^2_{w,1}$ since there is no non-zero element in $\ell^2_{w,1}$ orthogonal to $D(A'_0)$. Now, by using the Green's formula related to the expression M^2 (cf. Chapter 1, (1.32)) we have

$$\begin{aligned}(A'_0 y, x) - (y, A'_0 x) &= \sum_{n=1}^{\infty}\{\bar{x}_n M^2 y_n - y_n \overline{M^2 x_n}\}w_n \\ &= \lim_{m\to\infty} p_m(\overline{x_{m+1}} M y_m - \bar{x}_m M y_{m+1} \\ &\qquad + \overline{Mx}_{m+1} y_m - \overline{Mx}_m y_{m+1}) \\ &= 0\end{aligned}$$

for all $x, y \in D(A'_0)$ since $y_{-1} = y_0 = x_{-1} = x_0 = 0$ and $\bar{x}_m, y_m = 0$ for large $m \in \mathbb{N}$. This shows that $A'_0 \subset (A'_0)^*$, and hence that A'_0 is symmetric. The proof of the lemma is therefore complete.

Now, if we define the closure of A'_0 by A_0, then also A_0 is symmetric. The following lemma is also very important in the sequel.

3.2.4 Lemma

(i) $A_0^* = A$

(ii) $A^* = A_0$

Proof: For **(i)**, we need to show both $A \subset A_0^*$ and $A_0^* \subset A$. For the first inclusion, let $x \in D(A'_0)$ and $y \in D(A)$. By using the Green's formula related to M^2, we have

$$\begin{aligned}(A'_0 x, y) - (x, Ay) &= \sum_{n=1}^{\infty}\{M^2 x_n \bar{y}_n - x_n \overline{M^2 y_n}\}w_n \\ &= p_0(\bar{y}_1 M x_0 - \bar{y}_0 M x_1 + x_0 \overline{My}_1 - x_1 \overline{My}_0) \\ &\quad - \lim_{m\to\infty} p_m(\bar{y}_{m+1} M x_m - \bar{y}_m M x_{m+1} \\ &\quad + x_m \overline{My}_{m+1} - x_{m+1} \overline{My}_m)\end{aligned}$$

$$= p_0(\bar{y}_1 M x_0 - x_1 \overline{My}_0)$$

$$= 0$$

since $x_{-1} = x_0 = y_{-1} = y_0 = 0$ and $x_m = 0$ for large m. So,

$$(A_0' x, y) = (x, Ay)$$

~~and hence the first inclusion. For the second inclusion, we need to show that any~~ $y \in D(A_0^*)$ is an element of $D(A)$ and $A_0^* y = Ay$.

Let $x \in D(A_0')$ and $y^* = A_0^* y$. Then, since $A_0' \subset A_0 = A_0^{**}$, we have

$$\begin{aligned}
0 &= (A_0' x, y) - (x, A_0^* y) \\
&= \sum_{n=1}^{\infty} (\bar{y}_n M^2 x_n - \overline{M^2 y}_n x_n) w_n + \sum_{n=1}^{\infty} (\overline{M^2 y}_n x_n - \bar{y}_n^* x_n) w_n \\
&= -p_0(M x_0 \bar{y}_1 - M x_1 \bar{y}_0 + x_0 \overline{My}_1 - x_1 \overline{My}_0) \\
&\quad + \lim_{m \to \infty} p_m (M x_m \bar{y}_{m+1} - M x_{m+1} \bar{y}_m \\
&\quad + x_m \overline{My}_{m+1} - x_{m+1} \overline{My}_m) \\
&\quad + \sum_{n=1}^{\infty} (\overline{M^2 y}_n - \bar{y}_n^*) x_n w_n \\
&= p_0(\bar{y}_1 M x_0 - x_1 \overline{My}_0) + \sum_{n=1}^{\infty} (\overline{M^2 y}_n - \bar{y}_n^*) x_n w_n \\
&= \sum_{n=1}^{\infty} (\overline{M^2 y}_n - \bar{y}_n^*) x_n w_n
\end{aligned}$$

since $x_{-1} = x_0 = y_{-1} = y_0 = 0$ and $x_m = 0$ for large m. Now, taking $x_n = \delta_{n,k}$ where $k, n = 1, 2, 3, \ldots$, we get

$$\bar{y}_k^* = \overline{M^2 y}_k \qquad k = 1, 2, 3, \ldots.$$

This means that, $y_n^* = M^2 y_n$ for $n = 1, 2, 3, \ldots$. So, $y \in D(A)$ and $y^* = Ay$. Hence $A_0^* \subset A$ and (i) is proved. Finally, since A_0 is closed, $A_0^{**} = A_0$ and (ii) follows from this. Hence the proof of the lemma follows.

The following lemma gives a criterion for the linear-independence of any two solutions of

$$Mx_n = \lambda x_n, \qquad n \geq 0,\ \lambda \in \mathbb{C}. \tag{3.23}$$

which is similar to that of Benammar [4]. Before we give the lemma, we define

$$[\phi, \theta]_k := p_{k-1}\big(\phi_{k-1}\theta_k - \phi_k\theta_{k-1}\big), \qquad k = 0, 1, 2, ...$$

and we remind ourselves that $p_n \neq 0, n \geq -1$.

3.2.5 Lemma Let $\phi = \{\phi_n\}_{-1}^{\infty}$ and $\theta = \{\theta_n\}_{-1}^{\infty}$ be any two solutions of (3.23). Then these solutions are linearly independent if and only if

$$[\phi, \theta]_0 := p_{-1}\big(\phi_{-1}\theta_0 - \phi_0\theta_{-1}\big) \neq 0. \tag{3.24}$$

Proof: Suppose that (3.24) holds and ϕ, θ are not linearly-independent. Then there exist α, β both not zero such that,

$$\alpha\phi_n + \beta\theta_n = 0 \qquad n = -1, 0, 1, ...$$

In particular,

$$\alpha\phi_{-1} + \beta\theta_{-1} = 0,\ \alpha\phi_0 + \beta\theta_0 = 0.$$

This implies, since α and β are not both zero, $\phi_{-1}\theta_0 - \phi_0\theta_{-1} = 0$ contradicting (3.24). Suppose now that, ϕ and θ are linearly independent and (3.24) does not hold. Then, there exist not both zero $\alpha, \beta \in \mathbb{C}$ such that $\alpha\phi_{-1} + \beta\theta_{-1} = 0$ and $\alpha\phi_0 + \beta\theta_0 = 0$. Now, if we let

$$\Psi_n = \alpha\phi_n + \beta\theta_n$$

then

$$M\Psi_n = \lambda\Psi_n, \qquad n \geq 0, \qquad \lambda \in \mathbb{C}. \tag{3.25}$$

Then by inserting the initial values $\Psi_{-1} = \Psi_0 = 0$ in (3.25) we get that $\Psi_n = \alpha\phi_n + \beta\theta_n = 0$ for all n, proving the linear independence of the solutions ϕ and θ.

We now have the following important theorem, which we have been aiming for. It gives the connection between the limit (3.21) and the LP condition on M^2. The equivalence of (i) and (v) below was obtained for M previously by Hinton-Lewis **[39]**.

3.2.6 Theorem The following are equivalent.

(i) M^2 is LP.

(ii) A_0 is self-adjoint.

(iii) A is self-adjoint.

(iv) A is symmetric.

(v) (3.21) holds for all $x, y \in D(A)$.

Proof:

(ii)$\Rightarrow$(iii): Using Lemma 3.2.4, $A_0 = A_0^* \Rightarrow A^* = A$.

(iii)$\Rightarrow$(iv): $A = A^* \Rightarrow A \subset A^*$ follows clearly.

(iv)$\Rightarrow$(ii): Starting from (iv), the argument $A \subset A^* = A_0 \subset A_0^* = A$ gives the self-adjointness of A_0.

(iv) $\Longleftrightarrow$ (v): Since, for all $x, y \in D(A)$,

$$(Ay, x) - (y, Ax) = \sum_{n=1}^{\infty} \{\bar{x}_n M^2 y_n - y_n \overline{M^2 x_n}\} w_n$$

$$= \lim_{m\to\infty} p_m(\bar{x}_{m+1}My_m - \bar{x}_m My_{m+1}$$

$$+ \overline{Mx}_{m+1}y_m - \overline{Mx}_m y_{m+1}),$$

clearly $(Ay, x) = (y, Ax) \iff$ (3.22) holds. Finally, we prove that (i) and (ii) are equivalent which is the most important part of the theorem.

(i)$\Rightarrow$(ii): Suppose that (i) holds, i.e. $\exists$ precisely 2 linearly independent $\psi^{(1)}$ and $\psi^{(2)}$ ℓ_w^2 solutions of

$$(M^2 - \mathrm{i})x_n = 0, \qquad n \geq -1 \tag{3.26}$$

and let $\{\psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)}\}$ be a basis for the solution space of (3.26). Suppose that

$$(A - \mathrm{i}I)y = 0. \tag{3.27}$$

On setting $x_n = y_n$, $n \geq 1$, and $x_{-1} = x_0 = 0$, we have that $x = \{x_n\}_{-1}^{\infty} \in \ell_w^2$ and

$$(M^2 - \mathrm{i})x_n = 0, \qquad n \geq 1.$$

Hence

$$x_n = C_1\psi_n^{(1)} + C_2\psi_n^{(2)} + C_3\psi_n^{(3)} + C_4\psi_n^{(4)}, \qquad n \geq -1$$

for constants C_1, C_2, C_3, C_4. It then follows that $\Phi = C_3\psi^{(3)} + C_4\psi^{(4)} \in \ell_w^2$. But since M^2 is assumed to be limit-point, $\psi_n^{(1)}, \psi_n^{(2)}, \Phi$ can not be linearly independent. Hence we must have

$$x_n = C_1'\psi_n^{(1)} + C_2'\psi_n^{(2)}, \qquad n \geq -1$$

for some $C_1', C_2' \in \mathbb{C}$. Then, from $x_{-1} = x_0 = 0$ we have

$$C_1'\psi_{-1}^{(1)} + C_2'\psi_{-1}^{(2)} = C_1'\psi_0^{(1)} + C_2'\psi_0^{(2)} = 0$$

which, in view of Lemma 3.2.5, implies $C_1' = C_2' = 0$ and hence $x_n = 0$, $n \geq 1$. Consequently $y = 0$ and $nul(A - \mathrm{i}I) = 0$. Since A_0 is real and $A = A_0^*$, it follows that A_0 is self-adjoint.

(ii)⇒(i): Let $\psi^{(k)}, k = 1, 2, 3, 4$ be a basis of solutions of

$$M^2 x_n = \lambda^2 x_n, \qquad n \geq 1, \qquad \lambda^2 \notin \mathbb{R} \tag{3.28}$$

where $\psi^{(1)}$ and $\psi^{(2)}$ are such that

$$[\psi^{(1)}, \psi^{(2)}]_0 = p_{-1}(\psi_{-1}^{(1)}\psi_0^{(2)} - \psi_0^{(1)}\psi_{-1}^{(2)}) \neq 0.$$

This condition ensures that $\psi^{(1)}$ and $\psi^{(2)}$ are linearly independent in view of Lemma 3.2.5. We take $\psi^{(1)}$ and $\psi^{(2)}$ as

$$M\psi^{(1)} = \lambda\psi^{(1)}, \qquad M\psi^{(2)} = -\lambda\psi^{(2)} \qquad \lambda^2 \notin \mathbb{R}$$

since we know that 2 solutions of (3.28) are the solutions of

$$Mx_n = \lambda x_n, \qquad n \geq 0$$

and 2 are the solutions of

$$Mx_n = -\lambda x_n, \qquad n \geq 0.$$

Then any solution of (3.28) is of the form

$$x = \{x_n\}_{-1}^{\infty}, x_n = \sum_{k=1}^{4} a_k \psi_n^{(k)}.$$

Suppose $x \in \ell_w^2$. If we set $x_{-1} = x_0 = 0$, then $x \in D(A)$ and $(A - \mathrm{i}I)x = 0$. The condition $x_{-1} = x_0 = 0$ gives a_1, a_2 in terms of a_3, a_4:

$$a_1 = \frac{a_3[\psi^{(2)}, \psi^{(3)}]_0 + a_4[\psi^{(2)}, \psi^{(4)}]_0}{[\psi^{(1)}, \psi^{(2)}]_0}$$

$$a_2 = \frac{a_3[\psi^{(3)}, \psi^{(1)}]_0 + a_4[\psi^{(4)}, \psi^{(1)}]_0}{[\psi^{(1)}, \psi^{(2)}]_0}.$$

However, since A_0 is assumed to be self-adjoint, we must have $nul(A - \mathrm{i}I) = 0$ and hence $x_n = 0$ for all n. This implies $a_3 = a_4 = 0$ and hence

$$x = a_1\psi^{(1)} + a_2\psi^{(2)}$$

in other words M^2 is limit-point and this completes the proof.

§3.3 Partial separation (PS)

In proving our inequality, it may happen that the left hand side of the inequality may not be finite while the right hand side is. The partial separation condition is precisely the condition that avoids this situation.

3.3.1 Definition: (Partial Separation Condition)

We say that M^2 is PS (partially separated) if

$$D_{T(M^2)} \subseteq D_{T(M)} \subset \ell^2_w,$$

in other words, if

$$x, M^2x \in \ell^2_w \Rightarrow Mx \in \ell^2_w.$$

This definition leads us to the following lemma.

3.3.2 Lemma

(a) If M^2 is PS and M is LP then M^2 is LP,

(b) If M is LC then M^2 is LC.

Proof: Let us start with the following implication:

(a) If M^2 is PS then, M is LP $\Rightarrow M^2$ is LP. (3.29)

To prove this, following Theorem 3.2.6, we need to prove that

$$\lim_{m\to\infty} p_m(\bar{x}_{m+1}My_m - \bar{x}_m My_{m+1} + \overline{Mx}_{m+1}y_m - \overline{Mx}_m y_{m+1}) = 0 \qquad (3.30)$$

$\forall x, y \in D_{T(M^2)}$. This does indeed hold, since by the PS condition $x, y, Mx, My \in D_{T(M)}$, and since M is LP,

$$\lim_{m\to\infty} p_m(\bar{x}_{m+1}My_m - \bar{x}_m My_{m+1}) = 0 \qquad \forall x, My \in D_{T(M)}$$

$$\lim_{m\to\infty} p_m(\overline{Mx}_{m+1}y_m - \overline{Mx}_m y_{m+1}) = 0 \qquad \forall Mx, y \in D_{T(M)}.$$

Hence we get (3.30), and (3.29) is proved. A consequence of (3.29) is the following implication.

If M^2 is PS then M^2 is not LP $\Rightarrow M$ is LC.

If M^2 is PS then M^2 is LC $\Rightarrow M$ is LC. (3.31)

For, assume that M^2 is PS and LC, and M is not LC. Then M must be LP which implies, by (3.29), that M^2 is LP which is a contradiction. Hence (3.31) is proved. Now, we shall prove the following implication.

(b) M is LC $\Rightarrow M^2$ is LC. (3.32)

To prove (3.32) consider the following equality,

$$(M^2 - I\lambda^2)\varphi = (M - I\lambda)(M + I\lambda)\varphi = 0 \qquad \Im(\lambda^2) \neq 0. \qquad (3.33)$$

Let $\phi^{(1)}, \phi^{(2)}$ be linearly independent solutions of $(M - \lambda I)x = 0$ and $\phi^{(3)}, \phi^{(4)}$ be linearly independent solutions of $(M + \lambda I)x = 0$. Since M is assumed to be LC, all these sequences are in ℓ^2_w, and they all satisfy (3.33). We will have proved that M^2 is LC if we prove that the $\phi^{(k)}, k = 1, 2, 3, 4$, are linearly independent. Suppose that

$$\alpha_1\phi^{(1)} + \alpha_2\phi^{(2)} + \alpha_3\phi^{(3)} + \alpha_4\phi^{(4)} = 0. \tag{3.34}$$

Then, applying $M \pm \lambda I$ on (3.34), we get

$$(M - \lambda I)(\alpha_3\phi^{(3)} + \alpha_4\phi^{(4)}) = 0, \quad (M + \lambda I)(\alpha_1\phi^{(1)} + \alpha_2\phi^{(2)}) = 0.$$

Thus,

$$2\lambda(\alpha_3\phi^{(3)} + \alpha_4\phi^{(4)}) = 0, \quad -2\lambda(\alpha_1\phi^{(1)} + \alpha_2\phi^{(2)}) = 0$$

which imply $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ since $\{\phi^{(1)}\ ,\ \phi^{(2)}\}$ and $\{\phi^{(3)}\ ,\ \phi^{(4)}\}$ are linearly independent. Hence, $\{\phi^{(k)}\}, k = 1, 2, 3, 4$, is a linearly independent set of solutions of (3.33), which belong to ℓ^2_w and hence M^2 is LC.

A consequence of (3.32) is the following implication,

$$M^2 \text{ is not LC } \Rightarrow M \text{ is LP.}$$

In particular,

$$M^2 \text{ is LP} \Rightarrow M \text{ is LP.} \tag{3.35}$$

To show (3.35), assume that M^2 is LP. Now, if M is not LP then M has to be LC, which implies, by (3.32), that M^2 is LC and which is against our assumption. So, M is LP and hence (3.35). The proof of Lemma 3.3.2 is now completed.

§3.4 The conditions LP, PS and self-adjoint operators

In this section, we examine the PS condition in terms of operators generated by M on subspaces of ℓ^2_w. This brings out more clearly the true significance of the condition. We shall see that the situation is more complicated in our case of difference expression M than when M is a differential operator, mainly because of the fact noted in §3.1, that the minimal operator T_0 is not densely defined.

In [**43**],(p.63), Kauffman, Read and Zettl give the necessary and sufficient conditions for a formally symmetric differential expression M^k, $k \geq 2$, to be partially separated. The question they originally investigate is to find under which circumstances

$$T_0(M^k) = \left[T_0(M)\right]^k$$

where M is a formally symmetric differential expression. They found out that $T_0(M^k) \subseteq \left[T_0(M)\right]^k$ follows from the theorem of Glazman [**34**], below and $T_0(M^k) \supseteq \left[T_0(M)\right]^k$ is precisely the partial separation condition. Furthermore, they give the result

$$"def\big(T_0(M^k)\big) = k\ def\big(T_0(M)\big) \iff M^k \text{ is partially separated}"$$

which enables us to get sufficient conditions on the coefficients for all powers M^k, $k \geq 2$, to be limit-point, i.e. minimal deficiency indices.

In applying the same ideas to the difference expression M, we face the density problem of the operators related to M in the first place. To overcome this problem we introduce the following notation.

3.4.1. Notation

As usual, we define our difference expression as

$$Mx_n := \begin{cases} \frac{1}{w_n}[-p_n x_{n+1} + (p_n + p_{n-1} + q_n)x_n - p_{n-1}x_{n-1}], & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}(x_0 - x_{-1}), & n = -1. \end{cases} \tag{3.36}$$

Now, we introduce the expressions M_k, $k = -1, 0, 1, 2, ...,$

$$M_k x_n := \begin{cases} \frac{1}{w_n}[-p_n x_{n+1} + (p_n + p_{n-1} + q_n)x_n - p_{n-1}x_{n-1}], & n \geq k+1 \\ \frac{1}{w_k}[-p_k x_{k+1} + (p_k + p_{k-1} + q_k)x_k], & n = k \end{cases} \tag{3.37}$$

where $\{p_n\}_{-1}^{\infty}$, $\{q_n\}_0^{\infty}$, $\{w_n\}_{-1}^{\infty}$ are real with $p_n \neq 0$, $w_n > 0$. Note that M_k is defined to be M for $n \geq k$ with $x_{-1} = x_0 = ... = x_{k-1} = 0$ inserted. Note also that $M_{-1} := M$. The maximal, pre-minimal and minimal operators related to M_k will be represented by $T(M_k)$, $T_0'(M_k)$ and $T_0(M_k)$ respectively.

We define the Hilbert spaces

$$\ell^2_{w,k} := \{x = \{x_n\}_{n=k}^{\infty} : \sum_{n=k}^{\infty} |x_n|^2 w_n < \infty\}, \ k = -1, 0, 1, ... \tag{3.38}$$

with the inner-product and norm

$$(x, y)_k := \sum_{n=k}^{\infty} x_n \bar{y}_n w_n, \ \|x\|_k := \sum_{n=k}^{\infty} |x_n|^2 w_n. \tag{3.39}$$

Note that, $k = -1$ is the special case where we get $\ell^2_{w,-1} := \ell^2_w$. Now, for k=0,1,2,..., we define our operators as follows.

The maximal operators are defined as

$$D_{T(M_k)} := \{x \in \ell^2_{w,k} : \sum_{n=k}^{\infty} |M_k x_n|^2 w_n < \infty\}$$

$$T(M_k)x_n := M_k x_n.$$

The pre-minimal operators are

$$D_{T_0'(M_k)} := \{x \in D_{T(M_k)} : \ x_m = 0 \text{ for all but a finite number of values of } m\}$$

$$T_0'(M_k) := T(M_k) \restriction_{D_{T_0'(M_k)}},$$

and, the minimal operators are defined to be the closures of the pre-minimal operators, i.e.

$$T_0(M_k) := \overline{T_0'(M_k)}.$$

3.4.2 Lemma $T_0'(M_k)$ is a symmetric operator in $\ell^2_{w,k}$.

Proof: We need to show that $T_0'(M_k) \subseteq T_0'^*(M_k)$. First of all, a unique adjoint $T_0'^*(M_k)$ can be defined in $\ell^2_{w,k}$ since there is no non-zero element in $\ell^2_{w,k}$ which is orthogonal to $D_{T_0'(M_k)}$ and hence $D_{T_0'(M_k)}$ is dense in $\ell^2_{w,k}$. For the inclusion, let $x, y \in D_{T_0'(M_k)}$. Then,

$$\begin{aligned}
&\left(T_0'(M_k)x, y\right)_k - \left(x, T_0'(M_k)y\right)_k \\
&= \sum_{n=k}^{\infty} (\bar{y}_n M_k x_n - x_n \overline{M_k y_n}) w_n \\
&= -p_{k-1}(\bar{y}_k x_{k-1} - \bar{y}_{k-1} x_k) + \lim_{m\to\infty} p_m(\bar{y}_{m+1} x_m - \bar{y}_m x_{m+1}) \\
&= 0
\end{aligned}$$

since $x_m = y_m = 0$ for large m, and $p_{k-1}(\bar{y}_k x_{k-1} - \bar{y}_{k-1} x_k) = 0$. Hence the desired inclusion.

3.4.3 Theorem $T(M_k) = T_0^*(M_k)$.

Proof: We need to show both implications $T(M_k) \subseteq T_0^*(M_k)$ and $T_0^*(M_k) \subseteq T(M_k)$. For the first, let $x \in D_{T_0'(M_k)}$ and $y \in D_{T(M_k)}$. Then,

$$\begin{aligned}
\left(T_0'(M_k)x, y\right)_k - \left(x, T(M_k)y\right)_k &= \sum_{n=k}^{\infty} (M_k x_n \bar{y}_n - x_n \overline{M_k y_n}) w_n \\
&= -\lim_{m\to\infty} p_m(\bar{y}_m x_{m+1} - \bar{y}_{m+1} x_m) \\
&= 0
\end{aligned}$$

$\Rightarrow T(M_k) \subseteq T_0'^*(M_k) = T_0^*(M_k)$, and hence the first inclusion.

Now, for the second inclusion, let $y \in D_{T_0^*(M_k)}$ and $y^* = T_0^*(M_k)y$. Then, since $T_0'(M_k) \subseteq T_0(M_k) = T_0^{**}(M_k)$, for all $x \in D_{T_0'(M_k)}$,

$$\begin{aligned}
0 &= \left(T_0^*(M_k)y, x\right)_k - \left(y, T_0'(M_k)x\right)_k \\
&= \sum_{n=k}^{\infty} \left(\bar{x}_n y_n^* - y_n \overline{M_k x_n}\right) w_n \\
&= \sum_{n=k}^{\infty} \left(\bar{x}_n y_n^* - \bar{x}_n M_k y_n\right) w_n + \sum_{n=k}^{\infty} \left(\bar{x}_n M_k y_n - y_n \overline{M_k x_n}\right) w_n \\
&= \sum_{n=k}^{\infty} \left(y_n^* - M_k y_n\right) \bar{x}_n w_n - \lim_{m \to \infty} p_m (\bar{x}_m y_{m+1} - \bar{x}_{m+1} y_m) \\
&= \sum_{n=k}^{\infty} \left(y_n^* - M_k y_n\right) \bar{x}_n w_n.
\end{aligned}$$

Now, taking $x_n := \delta_{n,p}$; $n, p = k, k+1, k+2, \ldots$, we get

$$y_n^* = M_k y_n \text{ for } n \geq k$$

since m is arbitrary. So, $y \in D_{T(M_k)}$ and $y^* = T(M_k)y$. Hence, we have the result that

$$y \in D_{T_0^*(M_k)} \Rightarrow y \in D_{T(M_k)}$$

which gives the desired inclusion and, hence, the proof of the theorem follows.

3.4.4 Corollary

(1) Following Theorem 3.4.3,

$$T^*(M_k) = T_0^{**}(M_k) = T_0(M_k)$$

since $T_0(M_k)$ is a closed operator by definition.

(2) $T_0(M_k)$ is also a symmetric operator:

$$T_0(M_k) \subseteq T(M_k) = T_0^*(M_k)$$

which follows from Theorem 3.4.3.

3.4.5 Lemma $0 \leq def\big[T_0(M_k) - \lambda\big] \leq 1$, $\lambda \notin \mathbb{R}$.

Proof: We need to prove that $nul\big[T_0^*(M_k) - \lambda\big] \leq 1$, i.e. we need to show that $nul\big[T(M_k) - \lambda\big] \leq 1$, since $T(M_k) = T_0^*(M_k)$ by Theorem 3.4.3. So, suppose that

$$M_k x_n = \lambda x_n;\ n \geq k,\ x \in D_{T(M_k)}. \tag{3.40}$$

It is then obvious that the recurrence relation (3.40) has a unique solution for given the value of x_k (cf. (3.37)). Hence,

$$T(M_k)x_n = \lambda x_n;\ n \geq k$$

has at most one solution, and hence $def\big[T_0(M_k) - \lambda\big] \leq 1$.

The following theorem shows more clearly what the LP condition is for a difference expression M. We see that, no matter where and how the initial terms of the expression are defined, the LP condition is related with what happens at ∞.

3.4.6 Theorem The following are equivalent for $k \geq 0$.

(1) M is limit-point,

(2) $T_0(M_k)$ is self-adjoint,

(3) $T(M_k)$ is self-adjoint,

(4) $T(M_k)$ is symmetric,

(5) $\lim_{m\to\infty} p_m(\bar{x}_{m+1}y_m - \bar{x}_m y_{m+1}) = 0 \quad \forall x, y \in D_{T(M_k)}$.

3.4.7 Remark This theorem shows that the LP condition for the expression M is all about what happens at infinity and nothing to do with the initial values.

Also note that, $T(M_k)$ and $T_0(M_k)$ are the same operators under the conditions given in the theorem.

Proof of Theorem 3.4.6: The equivalence of (2) and (3) follows from Theorem 3.4.3. For (3) and (4), let $T(M_k)$ be symmetric. Then,

$$T(M_k) \subseteq T^*(M_k) = T_0(M_k) \subseteq T(M_k) \iff T(M_k) = T^*(M_k)$$

and hence (3), as (4) obviously follows from (3). The equivalence of (4) and (5) follows immediately from Green's formula: For all $x, y \in D_{T(M_k)}$,

$$\big(T(M_k)x, y\big)_k - \big(x, T(M_k)y\big)_k = -\lim_{m\to\infty} p_m(\bar{y}_{m+1}x_m - \bar{y}_m x_{m+1}).$$

Here is the most important part of the theorem, (1) $\iff$ (2):

(1)$\Rightarrow$(2): Let M be LP and $\{\psi_m\}_{-1}^{\infty}$ be the unique ℓ_w^2 solution of

$$Mx_n = \lambda x_n,\ n \geq 0,\ \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{3.41}$$

Let θ and ϕ be the usual basis of all solutions: $\theta_{-1} = 0$, $\theta_0 = 1$, $\phi_{-1} = 1$, $\phi_0 = 0$ say. Then, we know that

$$\psi_{-1} = \theta_{-1} + m(\lambda)\phi_{-1} = m(\lambda) \neq 0$$

where $m(.)$ is the Hellinger-Nevanlinna $m(\lambda)$ function for M. To prove that $T(M_k)$ is self-adjoint, it is sufficient to prove that

$$\big[T(M_k) - \mathrm{i}\big]y_n = 0 \Rightarrow y_n = 0,\ n \geq k$$

since $T(M_k)$ is real and $T(M_k) = T_0^*(M_k)$. So, suppose that $y \in D_{T(M_k)}$ and $\big[T(M_k) - \mathrm{i}\big]y_n = 0$ $n \geq k$. Then, with

$$x_n = \begin{cases} y_n, & n \geq k \\ 0, & n \leq k-1 \end{cases}$$

we have

$$Mx_n = \mathrm{i}x_n, \ n \geq 0, \ \{x_n\}_{-1}^{\infty} \in \ell_w^2.$$

Hence, since M is LP, ψ and x must be linearly dependent and there exists $c \in \mathbb{C}$ such that $x_n = c\psi_n, \ n \geq -1$. Thus, $x_{-1} = 0$ implies that $c = 0$ since $\psi_{-1} \neq 0$. So, $x_n = 0 \ n \geq -1$ and hence $y_n = 0 \ \forall n$ which proves that $T(M_k)$ is self-adjoint.

(2)⇒(1): For this implication, we start from the special case $k = 0$, i.e., $T(M_0)$ is self-adjoint ⇒ M is LP. Suppose M is not LP in which case there are two linearly independent solutions of (3.41). Let θ and ϕ be the usual basis of solutions: $\theta_{-1} = 0$, $\theta_0 = 1$, $\phi_{-1} = 1$, $\phi_0 = 0$ again. Then, $\{\theta_n\}_{-1}^{\infty}$ satisfies

$$M\theta_n = \lambda\theta_n, \ n \geq 0, \ \theta_{-1} = 0,$$

i.e.

$$M_0\theta_n = \lambda\theta_n, \ n \geq 0, \ \{\theta_n\}_{-1}^{\infty} \in \ell_{w,0}^2,$$

i.e.

$$T(M_0)\theta_n = \lambda\theta_n, \ n \geq 0.$$

Now, since $\{\theta_n\}_0^{\infty} \not\equiv 0$ we have that $T(M_0)$ is not self-adjoint which is a contradiction.

Now, returning to our main implication, it is sufficient to prove that $T(M_k)$ is self-adjoint ⇒ $T(M_0)$ is self-adjoint, and here is the proof: Suppose

$$T(M_0)x_n = \lambda x_n, \ n \geq 0,$$

i.e.

$$M_0x_n = \lambda x_n, \ n \geq 0, \ \{x_n\}_0^{\infty} \in \ell_{w,0}^2.$$

We set

$$x_n = \begin{cases} y_n, & n \geq k \\ 0, & n < k. \end{cases}$$

Then,

$$M_k y_n = \lambda y_n, \ n \geq k, \ \{y_n\}_k^\infty \in \ell^2_{w,k},$$

i.e. $T(M_k)y = \lambda y$. Now, since $T(M_k)$ is self-adjoint, $y_n = 0$ for $n \geq k$, and hence $x_n = 0$ for $n \geq 0$ which implies that $T(M_0)$ is self-adjoint.

We have proved that

$$T(M_k) \text{ is self-adjoint } \Rightarrow T(M_0) \text{ is self-adjoint } \Rightarrow M \text{ is LP.}$$

Hence, the proof of the theorem is complete.

The problem we investigate now is to find under which circumstances the equality $T_0(M_1^2) = \left[T_0(M_1)\right]^2$ holds. Our starting point is to show that $T_0'(M_1^2) = \left[T_0'(M_1)\right]^2$. Following (3.37), we have

$$M_1 x_n = \begin{cases} \frac{1}{w_n}[-p_n x_{n+1} + (p_n + p_{n-1} + q_n)x_n - p_{n-1}x_{n-1}], & n \geq 2 \\ \frac{1}{w_1}[-p_1 x_2 + (p_1 + p_0 + q_1)x_1], & n = 1 \end{cases}$$

$$M_1^2 x_n := \begin{cases} \frac{1}{w_n}[-p_n M_1 x_{n+1} + (p_n + p_{n-1} + q_n)M_1 x_n - p_{n-1}M_1 x_{n-1}], & n \geq 2 \\ \frac{1}{w_1}[-p_1 M_1 x_2 + (p_1 + p_0 + q_1)M_1 x_1], & n = 1. \end{cases}$$

Then the maximal operators related to these expressions are denoted by $T(M_1)$ and $T(M_1^2)$ with the following domains, respectively.

$$D_{T(M_1)} = \left\{x \in \ell^2_{w,1} : \sum_{n=1}^{\infty} |M_1 x_n|^2 w_n < \infty\right\}$$

$$D_{T(M_1^2)} = \left\{x \in \ell^2_{w,1} : \sum_{n=1}^{\infty} |M_1^2 x_n|^2 w_n < \infty\right\}.$$

The pre-minimal operators are

$$T_0'(M_1) = T(M_1) \restriction_{D_{T_0'(M_1)}}$$

$$T_0'(M_1^2) = T(M_1^2) \restriction_{D_{T_0'(M_1^2)}}$$

where

$$D_{T_0'(M_1)} = \{x \in D_{T(M_1)} : x_m = 0 \text{ for large m}\}$$

$$D_{T_0'(M_1^2)} = \{x \in D_{T(M_1^2)} : x_m = 0 \text{ for large m}\}.$$

It is now obvious that

$$T_0'(M_1^2) = [T_0'(M_1)]^2. \tag{3.42}$$

As a last definition, we define the minimal operators as the closure of the pre-minimal operators, as usual.

3.4.8 Lemma $T_0'(M_1^2)$ is a symmetric operator in $\ell^2_{w,1}$.

Proof: $D_{T_0'(M_1^2)}$ is obviously a dense domain in $\ell^2_{w,1}$ and, so, a unique adjoint $T_0'^*(M_1^2)$ can be defined. We then show that $T_0'(M_1^2)$ is symmetric: For $x, y \in D_{T_0'(M_1^2)}$,

$\left(T_0'(M_1^2)x, y\right)_1 - \left(x, T_0'(M_1^2)y\right)_1$

$= \sum_{n=1}^{\infty}(M_1^2 x_n \bar{y}_n - x_n \overline{M_1^2 y_n}) w_n$

$= \lim_{m\to\infty} p_m(\bar{y}_{m+1} M_1 x_m - \bar{y}_m M_1 x_{m+1} + \overline{M_1 y}_{m+1} x_m - \overline{M_1 y}_m x_{m+1})$

$= 0$

since $x_m = y_m = 0$ for large m, and hence

$$T_0'(M_1^2) \subseteq T_0'^*(M_1^2) = T_0^*(M_1^2) \tag{3.43}$$

which proves the lemma.

3.4.9 Theorem $T_0^*(M_1^2) = T(M_1^2)$.

Proof: We show that $T_0^*(M_1^2) \subseteq T(M_1^2)$ and $T_0^*(M_1^2) \supseteq T(M_1^2)$. To show the first inclusion, let $y \in D_{T_0^*(M_1^2)}$ and $y^* = T_0^*(M_1^2)y$. Then, since $T_0'(M_1^2) \subseteq T_0(M_1^2) = T_0^{**}(M_1^2)$, for all $x \in D_{T_0'(M_1^2)}$, we have

$$
\begin{aligned}
0 &= \left(T_0^*(M_1^2)y, x\right)_1 - \left(y, T_0'(M_1^2)x\right)_1 \\
&= \sum_{n=1}^{\infty}(\bar{x}_n y_n^* - \overline{M_1^2 x}_n y_n)w_n \\
&= \sum_{n=1}^{\infty}(\bar{x}_n y_n^* - M_1^2 y_n \bar{x}_n)w_n + \sum_{n=1}^{\infty}(M_1^2 y_n \bar{x}_n - y_n \overline{M_1^2 x}_n)w_n \\
&= \sum_{n=1}^{\infty}(y_n^* - M_1^2 y_n)\bar{x}_n w_n \\
&\quad \lim_{m\to\infty} p_m(\bar{x}_{m+1}M_1 y_m - \bar{x}_m M_1 y_{m+1} + \overline{M_1 x}_{m+1} y_m - \overline{M_1 x}_m y_{m+1}) \\
&= \sum_{n=1}^{\infty}(y_n^* - M_1^2 y_n)\bar{x}_n w_n
\end{aligned}
$$

since $x_m = 0$ for large m. Now, taking $x_n = \delta_{n,p}$; $n, p = 1, 2, 3, \ldots$ we get

$$y_n^* = M_1^2 y_n, \ n \geq 1$$

since m can be choosen arbitrarily large. So, $y \in D_{T(M_1^2)}$ and $y^* = T(M_1^2)y$. Hence,

$$y \in D_{T_0^*(M_1^2)} \Rightarrow y \in D_{T(M_1^2)}$$

and the first inclusion follows from this. For the second inclusion, let $x \in D_{T_0'(M_1^2)}$ and $y \in D_{T(M_1^2)}$. Then,

$\left(T_0'(M_1^2)x, y\right)_1 - \left(x, T(M_1^2)y\right)_1$

$= \sum_{n=1}^{\infty}(M_1^2 x_n \bar{y}_n - x_n \overline{M_1^2 y}_n)w_n$

$$= \lim_{m\to\infty} p_m(\bar{y}_{m+1}M_1x_m - \bar{y}_m M_1 x_{m+1} + \overline{M_1y}_{m+1}x_m - \overline{M_1y}_m x_{m+1})$$

$$= 0$$

since $x_m = 0$ for large m, and hence the second inclusion. The theorem is thus proved.

3.4.10 Corollary

(1) Following Theorem 3.4.9,

$$T^*(M_1^2) = \left[T(M_1^2)\right]^* = T_0(M_1^2). \tag{3.44}$$

(2) $T_0(M_1^2)$ is also a symmetric operator following Theorem 3.4.9:

$$T_0(M_1^2) \subseteq T(M_1^2) = T_0^*(M_1^2).$$

3.4.11 Lemma $0 \le def\left[T_0(M_1^2) - \mathrm{i}\right] \le 2.$

Proof: Assume that

$$M_1^2 x_n = \lambda^2 x_n;\ n \ge 1,\ \Im(\lambda^2) \ne 0,\ x \in D_{T(M_1^2)}.$$

More precisely, the equation takes the form

$$\begin{cases} \frac{1}{w_n}[-p_nM_1x_{n+1} + (p_n + p_{n-1} + q_n)M_1x_n - p_{n-1}M_1x_{n-1}] = \lambda^2 x_n, & n \ge 2 \\ \frac{1}{w_1}[-p_1M_1x_2 + (p_1 + p_0 + q_1)M_1x_1] = \lambda^2 x_1, & n = 1. \end{cases} \tag{3.45}$$

Now, since the highest indexed term in M_1x_n is x_{n+1} and the least indexed term is x_{n-1}, from (3.45) it is clear that $M_1^2x_n = \lambda^2 x_n$ is a five term recurence relation. For $n = 1$, x_1 is the least indexed term and x_3 is the biggest indexed term so that x_3 can be determined by x_1 and x_2. For the choice of $x_1^{(1)} = 0$,

$x_2^{(1)} = 1$ there exists a unique solution of (3.45), and for $x_1^{(2)} = 1$, $x_2^{(2)} = 0$ we find another solution so that any solution of (3.45) is of the form

$$x_n = c_1 x_n^{(1)} + c_2 x_n^{(2)},\ x_1 = c_2, x_2 = c_1;\ c_1, c_2 \in \mathbb{C}$$

which is spanned by two linearly independent solutions whether or not they are $\ell^2_{w,1}$. Hence $def\,[T_0(M_1^2) - \mathrm{i}] \leq 2$ and the lemma is therefore proved.

We now give a result similar to that of Theorem 3.4.6 which gives a characterization for M^2 to be LP.

3.4.12 Theorem The following are equivalent:

(1) M^2 is limit-point.

(2) $T_0(M_1^2)$ is self-adjoint.

(3) $T(M_1^2)$ is self-adjoint.

(4) $T(M_1^2)$ is symmetric.

(5) $\forall x, y \in D_{T(M_1^2)}$,

$$\lim_{m\to\infty} p_m(\bar{x}_{m+1} M_1 y_m - \bar{x}_m M_1 y_{m+1} + \overline{M_1 x}_{m+1} y_m - \overline{M_1 x}_m y_{m+1}) = 0.$$

Proof: This theorem is proved in exactly the same way as Theorem 3.2.6. The only difference is the notation we used and this is because we want to get a necessary and sufficient result for the PS condition on M^2.

Back to our main question that under which circumstances $T_0(M_1^2) = [T_0(M_1)]^2$ is true, we use a result by Glazman to be able to see $T_0(M_1^2) \subseteq [T_0(M_1)]^2$. We first give the following equivalent definition for the partial separation condition.

3.4.14 Definition The difference expression M_1^2 is said to be *Partially Separated* (PS) if $x, M_1^2 x \in \ell^2_{w,1}$ together imply $x, M_1 x \in \ell^2_{w,1}$, or equivalently $D_{T(M_1^2)} \subset D_{T(M_1)}$.

Theorem A (Glazman) Let A_j be a closed operator with dense domain in a Hilbert space H and suppose that it has closed range and finite deficiency, $j = 1, 2, ..., k$. Then $\prod_{j=1}^{k} A_j$ is a closed operator with dense domain, closed range and

$$def \prod_{j=1}^{k} A_j = \sum_{j=1}^{k} def A_j$$

(see, Glazman [34], Theorem A, p.26).

Let

$$N_1 = M_1 - w_1, \qquad N_2 = M_1 - w_2 \tag{3.46}$$

where $w_1 = -w_2$, $w_1^2 = w_2^2 =$i. We then have the following important theorem.

3.4.15 Theorem M_1^2 is PS if, and only if, $def\left[T_0(M_1^2) - \mathrm{i}\right] = 2 def\left[T_0(M_1) - \sqrt{\mathrm{i}}\right]$.

Proof: We have

$$\begin{aligned} T_0(N_1 N_2) &= T_0\left[(M_1 - w_1)(M_1 - w_2)\right] \\ &= T_0(M_1^2) - \mathrm{i}. \end{aligned} \tag{3.47}$$

Now, we see that $T_0(N_1)$ and $T_0(N_2)$ are closed, densely defined and have closed range. Then,

$$T_0'(N_1 N_2) = T_0'(N_1) T_0'(N_2) \subseteq T_0(N_1) T_0(N_2) \tag{3.48}$$

$\Rightarrow$

$$T_0(N_1 N_2) \subseteq T_0(N_1) T_0(N_2) \tag{3.49}$$

since $T_0(N_1)T_0(N_2)$ is closed by Glazman and since $T_0(N_1N_2)$ is the minimal closed operator generated by (N_1N_2). Now,

$$M_1^2 \text{ is PS} \Rightarrow D\big(T(N_1)T(N_2)\big) \supseteq D\big(T(N_1N_2)\big)$$

$$\Rightarrow T_0(N_1N_2) \supseteq T_0(N_1)T_0(N_2)$$

$\Rightarrow$

$$T_0(N_1N_2) = T_0(N_1)T_0(N_2) \tag{3.50}$$

by (3.49). By (3.47) and Glazman's theorem again, we have

$$\begin{aligned} def\big[T_0(N_1)T_0(N_2)\big] &= def\big[T_0(N_1)\big] + def\big[T_0(N_2)\big] \\ &= def\big[T_0(M_1^2) - \mathrm{i}\big] \\ &= 2def\big[T_0(N_j)\big] \ (j = 1, 2) \\ &= 2def\big[T_0(M_1) - \sqrt{\mathrm{i}}\big]. \end{aligned} \tag{3.51}$$

Since M is real, we have the same result when we replace i by -i. This gives the proof in one direction. Suppose now that,

$$def\big[T_0(N_1)\big] + def\big[T_0(N_2)\big] = def\big[T_0(N_1N_2)\big]. \tag{3.52}$$

We remind ourselves that, since M_1 is real,

$$def\big[T_0(N_1)\big] = def\big[T_0(N_2)\big] =: def\big[T_0(M_1) - \sqrt{\mathrm{i}}\big].$$

Now, by Glazman,

$$def\big[T_0(N_1N_2)\big] = def\big[T_0(N_1)T_0(N_2)\big]. \tag{3.53}$$

We also have that, by the choice of w_1, w_2,

$$T_0(N_1N_2) = T_0(M_1^2) - \mathrm{i}. \tag{3.54}$$

So, by (3.54), if (3.52) is assumed,

$$\begin{aligned} def\big[T_0(M_1^2) - \mathrm{i}\big] &= def\big[T_0(N_1)T_0(N_2)\big] \\ &= def\big[\big(T_0(M_1) - w_1\big)\big(T_0(M_1) - w_1\big)\big] \\ &= def\big[T_0^2(M_1) - \mathrm{i}\big]. \end{aligned}$$

We also have

$$T_0(M_1^2) - \mathrm{i} \subseteq T_0^2(M_1) - \mathrm{i}. \tag{3.55}$$

Now, if the deficiencies of $T_0(M_1^2) - \mathrm{i}$ and $T_0^2(M_1) - \mathrm{i}$ are the same, then these two operators are the same, otherwise $\big\{Range\big[T_0^2(M_1) - \mathrm{i}\big]\big\}^{\perp}$ would be strictly inside $\big\{Range\big[T_0(M_1^2) - \mathrm{i}\big]\big\}^{\perp}$, i.e. the dimension of $\big\{Range\big[T_0^2(M_1) - \mathrm{i}\big]\big\}^{\perp}$ would be less than the dimension of $\big\{Range\big[T_0(M_1^2) - \mathrm{i}\big]\big\}^{\perp}$. Note that, these dimensions are finite as they are the deficiency indices of $T_0^2(M_1)$ and $T_0(M_1^2)$ respectively. Since, by the hypothesis, we have equality in the deficiencies, we must have equality in (3.55), i.e. additional to (3.55) we must have

$$T_0(M_1^2) - \mathrm{i} \supseteq T_0^2(M_1) - \mathrm{i}. \tag{3.56}$$

Now taking adjoints in (3.56), we find that

$$T(M_1^2) \subseteq T^2(M_1)$$

hence M_1^2 is PS.

The theorem is therefore proved.

Chapter 4

THE INEQUALITY

§4.1 Introduction

The problem we shall investigate is to establish a criterion for finding the best constant in the following analogue ($n = 2$) of the Beynon *et al.* result **[9]**(cf. Chapter 1, §1.1) for the 2nd order formally symmetric difference expression M. This is that,

$$\sum_{n=0}^{\infty} |Mx_n|^2 w_n \leq K\left(\sum_{n=-1}^{\infty} |x_n|^2 w_n\right)^{1/2} \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n\right)^{1/2} \qquad \forall x \in D_{T(M^2)} \tag{4.1}$$

where $D_{T(M^2)}$ is the domain of the maximal operator related to the difference expression M^2 (cf. Chapter 3, §3.1), i.e. the maximal domain that makes the right hand-side of (4.1) finite. We remind ourselves that the difference expressions M and M^2 are defined as before, i.e.

$$Mx_n := \begin{cases} \frac{1}{w_n}[-\Delta(p_{n-1}\Delta x_{n-1}) + q_n x_n], & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}(x_0 - x_{-1}) & n = -1 \end{cases} \tag{4.2}$$

and

$$M^2 x_n = [M(Mx)]_n := \begin{cases} M(Mx_n), & n \geq 1 \\ \frac{1}{w_0}[-\Delta(p_{n-1}\Delta Mx_{n-1}) + q_n Mx_n], & n = 0 \\ -\frac{p_{-1}}{w_{-1}}(Mx_0 - Mx_{-1}), & n = -1. \end{cases} \tag{4.3}$$

Here Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$ and the coefficients p_n, q_n and w_n are real with

$$\{p_n\}_{-1}^{\infty} \quad (p_n \neq 0), \qquad \{q_n\}_0^{\infty}, \qquad \{w_n\}_{-1}^{\infty} \quad (w_n > 0). \tag{4.4}$$

We now give some results in order to get the main inequality. We start with the following lemma, which establishes a basis for the solution spaces of $M^2\psi = \lambda^2\psi$ and $M^2\psi = \bar{\lambda}^2\psi$, and brings up the role of the $m(\lambda)$ function.

4.1.1 Lemma Let M and M^2 be LP. Also, let $m(.)$ be the Hellinger-Nevanlinna $m(\lambda)$ function associated with M, (cf. Chapter 1, §1.3). For $\lambda, \lambda^2 \notin \mathbb{R}$, let $\theta(\lambda)$ and $\phi(\lambda)$ be the solutions of $Mx_n = \lambda x_n$ $(n \geq 0)$ defined by the initial conditions

$$\theta_{-1} = 1, \qquad \theta_0 = 0, \qquad \phi_{-1} = 0, \qquad p_{-1}\phi_0 = 1 \tag{4.5}$$

and set $\psi_n(\lambda) = \theta_n(\lambda) + m(\lambda)\phi_n(\lambda)$ for $n \geq -1$. Then $\psi(\lambda)$, $\psi(-\lambda)$ are linearly independent solutions of

$$M^2 x_n = \lambda^2 x_n, \qquad n \geq 1 \tag{4.6}$$

and satisfy

$$\psi_{-1}(\pm\lambda) = 1, \qquad p_{-1}\psi_0(\pm\lambda) = m(\pm\lambda). \tag{4.7}$$

Hence $N(\lambda^2)$, and $N(\bar{\lambda}^2)$ are the linear spans

$$N(\lambda^2) = \big[< \psi(\lambda), \lambda^2\psi(\lambda) >, < \psi(-\lambda), \lambda^2\psi(-\lambda) > \big] \tag{4.8}$$

$$N(\bar{\lambda}^2) = \big[< \overline{\psi(\lambda)}, \overline{\lambda^2\psi(\lambda)} >, < \overline{\psi(-\lambda)}, \overline{\lambda^2\psi(-\lambda)} > \big]. \tag{4.9}$$

Proof: We see that (4.7) easily follows from (4.5). So we only need to prove (4.8) and (4.9). Since M^2 is LP, and $< u, \lambda^2 u > \in N(\lambda^2)$ implies $M^2 u_n = \lambda^2 u_n$ $(n \geq 1)$, it follows that u is a linear combination of $\psi(\lambda)$ and $\psi(-\lambda)$. Hence (4.8) follows, and (4.9) similarly. The lemma is therefore proved.

From now on, we will use the notation $\psi^{(+)}$ or $\psi^{(1)}$ for the solution $\psi(\lambda)$, and $\psi^{(-)}$ or $\psi^{(2)}$ for the solution $\psi(-\lambda)$ of (4.6) whichever is more convenient to us.

4.1.2 Lemma Let $\psi^{(\pm)} := \psi(\pm\lambda)$ be the sequences defined in Lemma 4.1.1. If M is LP, and $\Im(\lambda) \neq 0$ and $\Re(\lambda) \neq 0$ then

$$\|\psi^{(\pm)}\|^2 = \frac{\Im[m(\pm\lambda)]}{\Im(\pm\lambda)} + w_{-1} \tag{4.10}$$

and

$$(\psi^{(+)}, \psi^{(-)}) = \frac{m(\lambda) - \overline{m(-\lambda)}}{2\Re(\lambda)} + w_{-1}. \tag{4.11}$$

Proof: We insert $x_n = y_n = \psi_n^{(+)}$ in the Green's formula related to M (cf. Chapter 1, §1.4). Since M is LP, we get

$$\sum_{n=0}^{\infty}[\overline{\psi_n^{(+)}}\lambda\psi_n^{(+)} - \psi_n^{(+)}\overline{\lambda\psi_n^{(+)}}]w_n = p_{-1}[\overline{\psi_{-1}^{(+)}}\psi_0^{(+)} - \overline{\psi_0^{(+)}}\psi_{-1}^{(+)}]$$

$\Rightarrow$

$$(\lambda - \bar{\lambda})[(\psi^{(+)}, \psi^{(+)}) - w_{-1}\overline{\psi_{-1}^{(+)}}\psi_{-1}^{(+)}] = m(\lambda) - \bar{m}(\lambda)$$

$\Rightarrow$

$$\|\psi^{(+)}\|^2 = \frac{\Im\big(m(\lambda)\big)}{\Im(\lambda)} + w_{-1}$$

and similarly taking $\psi^{(-)}$ instead of $\psi^{(+)}$ we obtain (4.10). Now, by inserting $x_n = \psi^{(+)}$ and $y_n = \psi^{(-)}$ in the Green's formula related to M we obtain

$$\sum_{n=0}^{\infty} [\overline{\psi_n^{(-)}}\lambda\psi_n^{(+)} - \psi_n^{(+)}\overline{(-\lambda\psi_n^{(-)})}]\, w_n = p_{-1}[\overline{\psi_{-1}^{(-)}}\psi_0^{(+)} - \overline{\psi_0^{(-)}}\psi_{-1}^{(+)}]$$

$\Rightarrow$

$$\begin{aligned}(\psi^{(+)}, \psi^{(-)}) &= \frac{m(\lambda) - \overline{m(-\lambda)}}{\lambda + \bar{\lambda}} + w_{-1} \\ &= \frac{m(\lambda) - \overline{m(-\lambda)}}{2\Re(\lambda)} + w_{-1}\end{aligned}$$

giving (4.11), which completes the proof.

We now turn our attention to our inequality (4.1). Note that, in order to prove (4.1), it is sufficient to consider only real sequences $x \in D_{T(M^2)} \subseteq D_{T(M)}$. To show this, suppose that (4.1) holds true for real $x \in D_{T(M^2)} \subseteq D_{T(M)}$ and let $x_n = X_n + \mathrm{i}Y_n$. Then,

$$\|x\|^2 = \sum_{n=-1}^{\infty} |X_n + \mathrm{i}Y_n|^2 w_n = \|X\|^2 + \|Y\|^2,$$

$$\sum_{n=-1}^{\infty} |Mx_n|^2 w_n = \sum_{n=-1}^{\infty} |MX_n|^2 w_n + \sum_{n=-1}^{\infty} |MY_n|^2 w_n$$

and finally

$$\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n = \sum_{n=1}^{\infty} |M^2 X_n|^2 w_n + \sum_{n=1}^{\infty} |M^2 Y_n|^2 w_n.$$

So,

$$\begin{aligned}\sum_{n=-1}^{\infty} |Mx_n|^2 w_n &= \sum_{n=-1}^{\infty} |MX_n|^2 w_n + \sum_{n=-1}^{\infty} |MY_n|^2 w_n \\ &\leq K\|X\| \left(\sum_{n=1}^{\infty} |M^2 X_n|^2 w_n\right)^{1/2} + K\|Y\| \left(\sum_{n=1}^{\infty} |M^2 Y_n|^2 w_n\right)^{1/2}\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\sum_{n=-1}^{\infty} |Mx_n|^2 w_n &\leq K\left\{\|X\| \left(\sum_{n=1}^{\infty} |M^2 X_n|^2 w_n\right)^{1/2} + \|Y\| \left(\sum_{n=1}^{\infty} |M^2 Y_n|^2 w_n\right)^{1/2}\right\} \\ &\leq K\left(\|X\|^2 + \|Y\|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |M^2 X_n|^2 w_n \right. \\ &\quad \left. + \sum_{n=1}^{\infty} |M^2 Y_n|^2 w_n\right)^{1/2} \\ &= K\|x\| \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n\right)^{1/2},\end{aligned}$$

hence the result follows.

4.1.3 Lemma Let $< x, x^* > \in G_2$ be real. Then, $\exists < y, y^* > \in G_{02}$ and $A_1, A_2 \in \mathbb{C}$, such that

$$< x, x^* > = < y, y^* > + 2\Re \sum_{i=1}^{2} A_i < \psi^{(i)}, \lambda^2 \psi^{(i)} > \qquad \lambda^2 \notin \mathbb{R}. \tag{4.12}$$

Proof: Using the result (3.18),

$$G = G_{02} \oplus N(\lambda^2) \oplus N(\bar{\lambda}^2) \qquad \Im(\lambda) \neq 0$$

for all $< x, x^* > \in G$, there exists $A_i, B_i \in \mathbb{C}$, $i = 1, 2$, and $< y, y^* > \in G_{02}$ such that

$$< x, x^* > = < y, y^* > + \sum_{i=1}^{2} A_i < \psi^{(i)}, \lambda^2 \psi^{(i)} > + \sum_{i=1}^{2} B_i < \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} > . \tag{4.13}$$

Now, since $< x, x^* >$ is real, (4.13) gives

$$< x, x^* > = \overline{< y, y^* >} + \sum_{i=1}^{2} \overline{B_i} < \psi^{(i)}, \lambda^2 \psi^{(i)} > + \sum_{i=1}^{2} \overline{A_i} < \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} > . \tag{4.14}$$

Hence, uniqueness in the orthogonal decomposition gives that $< y, y^* >$ must be real and $A_i = \overline{B_i}$, $i = 1, 2$. This completes the proof.

Before we give the next result, we remind ourselves that the graph inner product on G is given by

$$(< a, a^* >, < b, b^* >)_{\lambda^2} := (a^* - \epsilon a, b^* - \epsilon b) + \eta^2 (a, b), \;\; \lambda^2 = \epsilon + i\eta \;\; (\eta \neq 0)$$

and the norm is the one produced by this inner product (see (3.16) in Chapter 3).

4.1.4 Lemma For real $< x, x^* > \in G$, satisfying (4.12), and $\lambda^2 = \epsilon + i\eta$ we have

$$\| < x, x^* > \|_{\lambda^2}^2 = \| < y, y^* > \|_{\lambda^2}^2 + 4\eta^2 \Re \left\{ \sum_{i=1}^{2} \sum_{k=1}^{2} A_i \overline{A_k} (\psi^{(i)}, \psi^{(k)}) \right\} \qquad \lambda^2 \notin \mathbb{R}. \tag{4.15}$$

Proof: By Lemma 4.1.3

$$< x, x^* > = < y, y^* > + \sum_{i=1}^{2} A_i < \psi^{(i)}, \lambda^2 \psi^{(i)} > + \sum_{i=1}^{2} \overline{A_i} < \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} >,$$

and by using the graph norm,

$$\| < x, x^* > \|_{\lambda^2}^2 = \| < y, y^* > \|_{\lambda^2}^2 + \| \sum_{i=1}^{2} A_i < \psi^{(i)}, \lambda^2 \psi^{(i)} > \|_{\lambda^2}^2$$

$$+ \| \sum_{i=1}^{2} \overline{A_i} < \overline{\psi^{(i)}}, \overline{\lambda^2 \psi^{(i)}} > \|_{\lambda^2}^2$$

$$= \| < y, y^* > \|_{\lambda^2}^2 + 2 \| \sum_{i=1}^{2} A_i < \psi^{(i)}, \lambda^2 \psi^{(i)} > \|_{\lambda^2}^2$$

Now,

$$2 \| \sum_{i=1}^{2} A_i < \psi^{(i)}, \lambda^2 \psi^{(i)} > \|_{\lambda^2}^2$$

$$= 2 \sum_{i=1}^{2} |A_i|^2 \| < \psi^{(i)}, \lambda^2 \psi^{(i)} > \|_{\lambda^2}^2$$

$$+ 4 \Re \{ A_1 \overline{A_2} (< \psi^{(1)}, \lambda^2 \psi^{(1)} >, < \psi^{(2)}, \lambda^2 \psi^{(2)} >)_{\lambda^2} \}$$

and

$$(< \psi^{(i)}, \lambda^2 \psi^{(i)} >, < \psi^{(k)}, \lambda^2 \psi^{(k)} >)_{\lambda^2} = (\lambda^2 \psi^{(i)} - \epsilon \psi^{(i)}, \lambda^2 \psi^{(k)} - \epsilon \psi^{(k)})$$

$$+ \eta^2 (\psi^{(i)}, \psi^{(k)})$$

$$= 2 \eta^2 (\psi^{(i)}, \psi^{(k)}).$$

Hence,

$$\| < x, x^* > \|_{\lambda^2}^2 = \| < y, y^* > \|_{\lambda^2}^2 + 4 \eta^2 \Big\{ \sum_{i=1}^{2} |A_i|^2 \| < \psi^{(i)}, \lambda^2 \psi^{(i)} > \|_{\lambda^2}^2$$

$$+ 2 \Re [A_1 \overline{A_2} (\psi^{(1)}, \psi^{(2)})] \Big\}$$

$$= \| < y, y^* > \|_{\lambda^2}^2 + 4 \eta^2 \Re \Big\{ \sum_{i=1}^{2} \sum_{k=1}^{2} A_i \overline{A_k} (\psi^{(i)}, \psi^{(k)}) \Big\}$$

giving the required result. Therefore the lemma is proved.

§4.2 The main result

4.2.1 Lemma Let $\lambda = \rho e^{i\theta}$ ($\theta \in (0, \frac{\pi}{4}), \rho \in \mathbb{R} - \{0\}$), $\lambda^2 = \epsilon + i\eta$, $< x, x^* > \in G$ be real, M be LP and M^2 be PS, and define

$$Q_{\lambda^2}[x] := \| < x, x^* > \|_{\lambda^2}^2 + 2\epsilon\big[x_0 w_0(x_0^* - M^2 x_0) + x_{-1}(p_{-1}Mx_0 + x_{-1}^* w_{-1}) - p_{-1}x_0 Mx_{-1}\big]. \quad (4.16)$$

Then the following are equivalent.

(a) $2\epsilon \sum_{n=0}^{\infty}(Mx_n)^2 w_n \leq \|x^*\|^2 + |\lambda^2|^2\|x\|^2$ (4.17)

(b) $Q_{\lambda^2}[x] \geq 0.$ (4.18)

Furthermore, there is equality in (a) if and only if there is equality in (b).

Proof: For real $< x, x^* > \in G$,

$$\begin{aligned} \| < x, x^* > \|_{\lambda^2}^2 &= \|x^* - \epsilon x\|^2 + \eta^2\|x\|^2 \\ &= \|x^*\|^2 - 2\epsilon(x^*, x) + |\lambda^2|^2\|x\|^2 \end{aligned}$$

and

$$\begin{aligned} (x^*, x) &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 + \sum_{n=1}^{\infty} x_n M^2 x_n w_n \\ &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - x_0 M^2 x_0 w_0 + \sum_{n=0}^{\infty} x_n M^2 x_n w_n \\ &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - x_0 M^2 x_0 w_0 + \sum_{n=0}^{\infty} (Mx_n)^2 w_n \\ &\quad + p_{-1}(x_{-1}Mx_0 - x_0 M x_{-1}) \\ &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - x_0 M^2 x_0 w_0 \\ &\quad + \sum_{n=0}^{\infty} (Mx_n)^2 w_n + p_{-1}(x_{-1}Mx_0 - x_0 M x_{-1}) \end{aligned}$$

$\Rightarrow$

$$\| < x, x^* > \|^2_{\lambda^2} = \|x^*\|^2 + |\lambda^2|^2\|x\|^2$$

$$- 2\epsilon\big[x^*_{-1}x_{-1}w_{-1} + x^*_0 x_0 w_0 - x_0 M^2 x_0 w_0 + \sum_{n=0}^{\infty}(Mx_n)^2 w_n$$

$$+ p_{-1}(x_{-1}Mx_0 - x_0 Mx_{-1})\big]$$

$\Rightarrow$

$$\|x^*\|^2 + \rho^4\|x\|^2 - 2\epsilon\sum_{n=0}^{\infty}(Mx_n)^2 w_n = Q_{\lambda^2}[x].$$

The proof follows from this fact.

We note that, in order to establish the inequality (4.17) we establish (4.18).

Now, by using (4.5) and (4.7), and by setting $m(\lambda) = m(+\lambda)$ and $n(\lambda) = m(-\lambda)$ for $arg(\lambda) \in (0, \frac{\pi}{4})$, we have the following for real $< x, x^* > \in G$ satisfying (4.12).

$x_{-1} = 2\Re(A_1 + A_2)$,

$x_0 = 2\Re\left(A_1\frac{m(\lambda)}{p_{-1}} + A_2\frac{n(\lambda)}{p_{-1}}\right)$,

$Mx_{-1} = 2\Re\left[A_1\left(-\frac{m(\lambda)}{w_{-1}} + \frac{p_{-1}}{w_{-1}}\right) + A_2\left(-\frac{n(\lambda)}{w_{-1}} + \frac{p_{-1}}{w_{-1}}\right)\right]$, since $My_{-1} = 0$,

$Mx_0 = My_0 + 2\Re\left(A_1\lambda\frac{m(\lambda)}{p_{-1}} - A_2\lambda\frac{n(\lambda)}{p_{-1}}\right)$,

$x^*_{-1} = y^*_{-1} + 2\Re\left(A_1\lambda^2 + A_2\lambda^2\right)$,

$x^*_0 = y^*_0 + 2\Re\left(A_1\lambda^2\frac{m(\lambda)}{p_{-1}} + A_2\lambda^2\frac{n(\lambda)}{p_{-1}}\right)$,

$$M^2x_0 = M^2y_0 + 2\Re\Big\{A_1\left[\lambda^2\frac{m(\lambda)}{p_{-1}} + \frac{p_{-1}}{w_0}\lambda + \frac{p^2_{-1}}{w_0w_{-1}}\left(\frac{m(\lambda)}{p_{-1}} - 1\right)\right]$$
$$+A_2\left[\lambda^2\frac{n(\lambda)}{p_{-1}} - \frac{p_{-1}}{w_0}\lambda + \frac{p^2_{-1}}{w_0w_{-1}}\left(\frac{n(\lambda)}{p_{-1}} - 1\right)\right]\Big\}.$$

We explain the last term as follows, as the rest are easy to evaluate.

We know that $\psi^{(1)} := \psi(\lambda)$ and $\psi^{(2)} := \psi(-\lambda)$ satisfy

$$M\psi^{(1)}_n := \begin{cases} \lambda\psi^{(1)}, & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}(\psi^{(1)}_0 - \psi^{(1)}_{-1}), & n = -1 \end{cases}$$

and

$$M\psi_n^{(2)} := \begin{cases} -\lambda\psi^{(2)}, & n \geq 0 \\ -\frac{p_{-1}}{w_{-1}}(\psi_0^{(2)} - \psi_{-1}^{(2)}), & n = -1 \end{cases}$$

where the difference expression M is defined by (4.3). Then, by using (4.12),

$$M^2 x_0 = M^2 y_0 + 2\Re\left\{A_1 M^2 \psi_0^{(1)} + A_2 M^2 \psi_0^{(2)}\right\}$$

with

$$\begin{aligned}
M^2\psi_0^{(1)} &= \frac{1}{w_0}\big[-p_0 M\psi_1^{(1)} + (p_{-1}+p_0+q_0)M\psi_0^{(1)} - p_{-1}M\psi_{-1}^{(1)}\big] \\
&= \frac{1}{w_0}\big[-p_0 M\psi_1^{(1)} + (p_{-1}+p_0+q_0)M\psi_0^{(1)} - p_{-1}\lambda\psi_{-1}^{(1)}\big] \\
&\quad -\frac{p_{-1}}{w_0}M\psi_{-1}^{(1)} + \frac{p_{-1}}{w_0}\lambda\psi_{-1}^{(1)} \\
&= \frac{1}{w_0}\big[-p_0 \lambda\psi_1^{(1)} + (p_{-1}+p_0+q_0)\lambda\psi_0^{(1)} - p_{-1}\lambda\psi_{-1}^{(1)}\big] \\
&\quad -\frac{p_{-1}}{w_0}M\psi_{-1}^{(1)} + p_{-1}\lambda\psi_{-1}^{(1)} \\
&= \lambda M\psi_0^{(1)} + \frac{p_{-1}^2}{w_{-1}w_0}(\psi_0^{(1)} - \psi_{-1}^{(1)}) + \frac{p_{-1}}{w_0}\lambda\psi_{-1}^{(1)} \\
&= \lambda\psi_0^{(1)} + \frac{p_{-1}^2}{w_{-1}w_0}(\psi_0^{(1)} - \psi_{-1}^{(1)}) + \frac{p_{-1}}{w_0}\lambda\psi_{-1}^{(1)}.
\end{aligned}$$

Now, by (4.7), $\psi_{-1}^{(1)} = \psi_{-1}(\lambda) = 1$ and $p_{-1}\psi_0^{(1)} = m(\lambda)$ implies that

$$M^2\psi_0^{(1)} = 2\Re\left\{A_1\left[\lambda^2\frac{m(\lambda)}{p_{-1}} + \frac{p_{-1}}{w_0}\lambda + \frac{p_{-1}^2}{w_0 w_{-1}}\left(\frac{m(\lambda)}{p_{-1}} - 1\right)\right]\right\}.$$

Similarly $M^2\psi_0^{(2)}$ is obtained, and hence the desired result.

So,

$$\begin{aligned}
x_0 w_0(x_0^* - M^2 x_0) = 4\Re\big[A_1 m(\lambda) + A_2 n(\lambda)\big]\Big\{&\frac{p_{-1}}{w_{-1}}\Re(A_1 + A_2) \\
&+ \Re\{A_1[-\lambda - \frac{m(\lambda)}{w_{-1}}] + A_2[\lambda - \frac{n(\lambda)}{w_{-1}}]\}\Big\},
\end{aligned}$$

$$\begin{aligned}
x_{-1}(p_{-1}Mx_0 + x_{-1}^* w_{-1}) = 4\Re(A_1 + A_2)\Re\Big\{&A_1\big[\lambda m(\lambda) + w_{-1}\lambda^2\big] \\
&+ A_2\big[-\lambda n(\lambda) + w_{-1}\lambda^2\big]\Big\},
\end{aligned}$$

and finally,

$$-p_{-1}x_0Mx_{-1} = 4\Re\big[A_1m(\lambda) + A_2n(\lambda)\big]\Re\Big\{A_1\big[\frac{m(\lambda)}{w_{-1}} - \frac{p_{-1}}{w_{-1}}\big] + A_2\big[\frac{n(\lambda)}{w_{-1}} - \frac{p_{-1}}{w_{-1}}\big]\Big\}.$$

Now, combining these results, we get from (4.16) that

$$Q_{\lambda^2}[x] - \| < x, x^* > \|_{\lambda^2}^2 = 8\epsilon\Re\big[A_1m(\lambda) + A_2n(\lambda)\big]\Re\big[A_1(-\lambda) + A_2\lambda\big] + 8\epsilon\Re(A_1 + A_2)\Re\big\{A_1[\lambda m(\lambda) + \lambda^2 w_{-1}] + A_2[-\lambda n(\lambda) + \lambda^2 w_{-1}]\big\}$$

This implies, by using (4.15) and (4.11) with the notation $\lambda_1 := +\lambda$ and $\lambda_2 := -\lambda$, and $m(\lambda_1) = m(\lambda)$ and $m(\lambda_2) = n(\lambda)$

$$\begin{aligned} &Q_{\lambda^2}[x] - \| < y, y^* > \|_{\lambda^2}^2 \\ &= 4\eta^2\Re\sum_{i=1}^{2}\sum_{k=1}^{2} A_i\bar{A}_k\big[\frac{m(\lambda_i) - \bar{m}(\lambda_k)}{\lambda_i - \bar{\lambda}_k} + w_{-1}\big] \\ &+ 8\epsilon\Re\big[A_1m(\lambda) + A_2n(\lambda)\big]\Re\big[A_1(-\lambda) + A_2\lambda\big] \\ &+ 8\epsilon\Re(A_1 + A_2)\Re\big\{A_1[\lambda m(\lambda) + \lambda^2 w_{-1}] + A_2[-\lambda n(\lambda) + \lambda^2 w_{-1}]\big\} \end{aligned} \tag{4.19}$$

Now, set $A_1 = a_1 + \mathrm{i}b_1$, $A_2 = a_2 + \mathrm{i}b_2$, $m(\lambda) = m_1 + \mathrm{i}m_2$, $n(\lambda) = n_1 + \mathrm{i}n_2$, $\lambda = \mu + \mathrm{i}\nu$ and

$$f(\lambda) = \lambda m(\lambda) + \lambda^2 w_{-1} = f_1 + \mathrm{i}f_2$$

$$g(\lambda) = -\lambda n(\lambda) + \lambda^2 w_{-1} = g_1 + \mathrm{i}g_2.$$

Now, if we name the first expression on the right hand side of the equality (4.19) as I_1, the second as I_2 and the third as I_3, we then obtain the following.

By setting $D_{ik} := \frac{m(\lambda_i) - \overline{m(\lambda_k)}}{\lambda_i - \bar{\lambda}_k} + w_{-1}$, we have

$$I_1 = 4\eta^2\Re\sum_{j=1}^{2}\sum_{k=1}^{2}(a_j + \mathrm{i}b_j)(a_k - \mathrm{i}b_k)D_{jk}.$$

Now, if we define

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

we then have

$$\begin{aligned}
I_1 &= 4\eta^2 \Re\Big\{(A_1, A_2)\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22}\end{pmatrix}\begin{pmatrix}\bar{A}_1 \\ \bar{A}_2\end{pmatrix}\Big\} \\
&= 4\eta^2 \Re\Big\{(a_1 + \mathrm{i}b_1, a_2 + \mathrm{i}b_2)\Re(D)\begin{pmatrix} a_1 - \mathrm{i}b_1 \\ a_2 - \mathrm{i}b_2\end{pmatrix} \\
&\quad + \mathrm{i}(a_1 + \mathrm{i}b_1, a_2 + \mathrm{i}b_2)\Im(D)\begin{pmatrix} a_1 - \mathrm{i}b_1 \\ a_2 - \mathrm{i}b_2\end{pmatrix}\Big\} \\
&= 4\eta^2 \Re\Big\{\big[(a_1, a_2) + \mathrm{i}(b_1, b_2)\big]\Re(D)\Big[\begin{pmatrix} a_1 \\ a_2\end{pmatrix} - \mathrm{i}\begin{pmatrix} b_1 \\ b_2\end{pmatrix}\Big]\Big\} \\
&\quad + 4\eta^2 \Re\Big\{\big[(a_1, a_2) + \mathrm{i}(b_1, b_2)\big]\mathrm{i}\Im(D)\Big[\begin{pmatrix} a_1 \\ a_2\end{pmatrix} - \mathrm{i}\begin{pmatrix} b_1 \\ b_2\end{pmatrix}\Big]\Big\} \\
&= 4\eta^2\Big[(a_1, a_2)\Re(D)\begin{pmatrix} a_1 \\ a_2\end{pmatrix} + (b_1, b_2)\Re(D)\begin{pmatrix} b_1 \\ b_2\end{pmatrix}\Big] \\
&\quad + 4\eta^2\Big[(a_1, a_2)\Im(D)\begin{pmatrix} b_1 \\ b_2\end{pmatrix} - (b_1, b_2)\Im(D)\begin{pmatrix} a_1 \\ a_2\end{pmatrix}\Big]
\end{aligned}$$

$\Rightarrow$

$$I_1 = 4\eta^2\big((a_1, a_2), (b_1, b_2)\big)\begin{pmatrix} \Re(D) & \Im(D) \\ -\Im(D) & \Re(D)\end{pmatrix}\begin{pmatrix}\begin{pmatrix} a_1 \\ a_2\end{pmatrix} \\ \begin{pmatrix} b_1 \\ b_2\end{pmatrix}\end{pmatrix}$$

$\Rightarrow$

$$I_1 = 4\eta^2\begin{pmatrix} a_1 & a_2 & b_1 & b_2\end{pmatrix}\mathbf{O}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2\end{pmatrix} \tag{4.20}$$

where the matrix O is real and symmetric with the following entries.

$O_{11} = \Re(D_{11}) = \frac{\Im[m(\lambda)]}{\nu} + w_{-1} = \frac{m_2}{\nu} + w_{-1}$,

$O_{12} = \Re(D_{12}) = \Re\big[\frac{m(\lambda_1) - \bar{n}(\lambda)}{2\mu} + w_{-1}\big] = \frac{m_1 - n_1}{2\mu} + w_{-1}$,

$O_{13} = \Im(D_{11}) = 0$,

$O_{14} = \Im(D_{12}) = \frac{m_2 + n_2}{2\mu}$,

$O_{22} = \Re(D_{22}) = \frac{\Im[n(\lambda)]}{-\nu} + w_{-1} = \frac{n_2}{-\nu} + w_{-1}$,

$O_{23} = \Im(D_{21}) = \Im\big[\frac{n(\lambda) - \bar{m}(\lambda)}{-2\mu} + w_{-1}\big] = \frac{m_2 + n_2}{-2\mu}$,

$O_{24} = \Im(D_{22}) = 0$,

$O_{33} = \Re(D_{11}) = \frac{m_2}{\nu} + w_{-1}$,

$O_{34} = \Re(D_{12}) = \frac{m_1 - n_1}{2\mu} + w_{-1},$

$O_{44} = \Re(D_{22}) = \frac{n_2}{-\nu} + w_{-1}.$

For the second expression on the right in (4.19) we have

$$I_2 = 8\epsilon\Re\big[A_1 m(\lambda) + A_2 n(\lambda)\big]\Re\big[A_1(-\lambda) + A_2(\lambda)\big]$$

$$= 8\epsilon(a_1 m_1 - b_1 m_2 + a_2 n_1 - b_2 n_2)(-a_1\mu + b_1\nu + a_2\mu - b_2\nu)$$

$$= 8\epsilon\big[a_1^2(-m_1\mu) + a_1 a_2(m_1\mu - n_1\nu) + a_1 b_1(m_1\nu + m_2\mu) + a_1 b_2(-m_1\nu + n_2\mu)$$

$$+ a_2^2 n_1\mu + a_2 b_1(n_1\nu - m_2\mu) + a_2 b_2(-n_1\nu - n_2\mu)$$

$$+ b_1^2(-m_2\nu) + b_1 b_2(m_2\nu - n_2\nu)$$

$$+ b_2^2 n_2\nu\big]$$

$\Rightarrow$

$$I_2 = 8\epsilon\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \end{pmatrix}\mathbf{M}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \tag{4.21}$$

where the matrix M is real and symmetric with the following entries:

$M_{11} = -m_1\mu,\ M_{12} = \frac{m_1\mu - n_1\mu}{2},\ M_{13} = \frac{m_1\nu + m_2\mu}{2},\ M_{14} = \frac{-m_1\nu + n_2\mu}{2},$

$M_{22} = n_1\mu,\ M_{23} = \frac{n_1\nu - m_2\mu}{2},\ M_{24} = \frac{-n_1\nu - n_2\mu}{2},$

$M_{33} = -m_2\nu,\ M_{34} = \frac{m_2\nu - n_2\nu}{2},$

$M_{44} = n_2\nu.$

And finally, for the last expression in (4.19),

$$I_3 = 8\epsilon\Re(A_1 + A_2)\Re\big[A_1 f(\lambda) + A_2 g(\lambda)\big]$$

$$= 8\epsilon(a_1 + a_2)(a_1 f_1 - b_1 f_2 + a_2 g_1 - b_2 g_2)$$

$$= 8\epsilon\big[a_1^2 f_1 + a_1 a_2(f_1 + g_1) + a_1 b_1(-f_2) + a_1 b_2(-g_2)$$

$$+ a_2^2 g_1 + a_2 b_1(-f_2) + a_2 b_2(-g_2)\big].$$

Then,

$$I_3 = 8\epsilon\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \end{pmatrix}\mathbf{N}\begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \tag{4.22}$$

where N is a real and symmetric matrix with the following entries:

$N_{11} = f_1, \; N_{12} = \frac{f_1+g_1}{2}, \; N_{13} = -\frac{f_2}{2}, \; N_{14} = -\frac{g_2}{2},$

$N_{22} = g_1, \; N_{23} = -\frac{f_2}{2}, \; N_{24} = -\frac{g_2}{2},$

$N_{33} = 0, \; N_{34} = 0,$

$N_{44} = 0.$

Now, (4.19), (4.20), (4.21) and (4.22) imply that

$$Q_{\lambda^2}[x] - \| < y, y^* > \|^2_{\lambda^2} = A^T \mathbf{P} A \tag{4.23}$$

where

$$\mathbf{P} := 4\eta^2 O + 8\epsilon(M+N), \qquad A^T = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \end{pmatrix} \tag{4.24}$$

is a real and symmetric matrix of size 4×4 with the following entries:

$P_{11} = 4\eta^2\big(\frac{m_2}{\nu} + w_{-1}\big) + 8\epsilon\big(f_1 - m_1\mu\big)$

$P_{12} = 4\eta^2\big(\frac{m_1-n_1}{2\mu} + w_{-1}\big) + 8\epsilon\big(\frac{f_1+g_1}{2} + \frac{m_1-n_1}{2}\mu\big)$

$P_{13} = 8\epsilon\big(-\frac{f_2}{2} + \frac{m_1\nu+m_2\mu}{2}\big)$

$P_{14} = 4\eta^2\big(\frac{m_2+n_2}{2\mu}\big) + 8\epsilon\big(-\frac{g_2}{2} + \frac{-m_1\nu+n_2\mu}{2}\big)$

$P_{22} = 4\eta^2\big(\frac{n_2}{-\nu} + w_{-1}\big) + 8\epsilon\big(g_1 + n_1\mu\big)$

$P_{23} = 4\eta^2\big(\frac{m_2+n_2}{-2\mu}\big) + 8\epsilon\big(-\frac{f_2}{2} + \frac{n_1\nu-m_2\mu}{2}\big)$

$P_{24} = 8\epsilon\big(-\frac{g_2}{2} - \frac{n_1\nu+n_2\mu}{2}\big)$

$P_{33} = 4\eta^2\big(\frac{m_2}{\nu} + w_{-1}\big) + 8\epsilon\big(-m_2\nu\big)$

$P_{34} = 4\eta^2\big(\frac{m_1-n_1}{2\mu} + w_{-1}\big) + 8\epsilon\big(\frac{m_2-n_2}{2}\nu\big)$

$P_{44} = 4\eta^2\big(\frac{n_2}{-\nu} + w_{-1}\big) + 8\epsilon\big(n_2\nu\big).$

Hence, by (4.23),

$$\begin{aligned} Q_{\lambda^2}[x] &= \|x^*\|^2 + \rho^4\|x\|^2 - 2\epsilon\sum_{n=0}^{\infty}(Mx_n)^2 w_n \\ &= \| < y, y^* > \|^2_{\lambda^2} + A^T \mathbf{P}(\rho,\theta) A \qquad (4.25) \\ &= |x^*_{-1}|^2 w_{-1} + |x^*_0|^2 w_0 + \sum_{n=1}^{\infty}(M^2 x_n)^2 w_n + \rho^4\|x\|^2 - 2\epsilon D[x] \end{aligned}$$

where $D[x] := \sum_{n=0}^{\infty}(Mx_n)^2 w_n$. We then have the following lemma.

4.2.2 Lemma The following are equivalent.

(i) $\|x^*\|^2 + \rho^4\|x\|^2 - 2\rho^2\cos(2\theta)D[x] \geq 0 \qquad \forall$ real $< x, x^* > \in G, \qquad (4.26)$

(ii) $\mathbf{P}(\rho,\theta) \geq 0$, i.e. $\mathbf{P}$ is non-negative definite, (4.27)

(iii) $\sum_{n=1}^{\infty}(M^2 x_n)^2 w_n + \rho^4\|x\|^2 - 2\rho^2\cos(2\theta)D[x] \geq 0, \forall$ real $x \in D_{T(M^2)}$. (4.28)

Proof: The equivalence of (i) and (ii) follows from Lemma 4.2.1 and the subsequent comments. We only need to show that (i)⇒(iii) as (iii)⇒(i) is obvious from

$$\|x^*\|^2 + \rho^4\|x\|^2 - 2\rho^2\cos(2\theta)D[x] = |x^*_{-1}|^2 w_{-1} + |x^*_0|^2 w_0 + F[x] \qquad (4.29)$$

where $F[x] := \sum_{n=1}^{\infty}(M^2 x_n)^2 w_n + \rho^4\|x\|^2 - 2\rho^2\cos(2\theta)D[x]$. So, suppose (i) holds, and let $< x, x^* > \in G$. Then $x \in D_{T(M^2)}$ and so

$$|x^*_{-1}|^2 w_{-1} + |x^*_0|^2 w_0 + \sum_{n=1}^{\infty}(M^2 x_n)^2 w_n + \rho^4\|x\|^2 - 2\rho^2\cos(2\theta)D[x] \geq 0$$

for all $< x, x^* > \in G$. Now, since x^*_{-1} and x^*_0 can be anything by their definition (see §3.1), in case of $x^*_{-1} = x^*_0 = 0$ (4.26) still holds and hence $F[x] \geq 0\ \forall x \in D_{T(M^2)}$. This gives (iii). The lemma is therefore proved.

4.2.3 Remark: Note that, the inequalities in (4.26), (4.27) and (4.28) are strict together, i.e. one holds if and only if each of the other two does.

4.2.4 Lemma For any non-trivial, real $x \in D_{T(M^2)}$ and $K = 2k^{-1}$, the following are equivalent:

(i) $D[x] \leq K\|x\|\Big(\sum_{n=1}^{\infty}(M^2x_n)^2w_n\Big)^{1/2}$

(ii) $J_{\rho^2,k}[x] := \sum_{n=1}^{\infty}(M^2x_n)^2w_n + \rho^4\|x\|^2 - \rho^2kD[x] \geq 0 \quad \forall\rho \in \mathbb{R}\backslash\{0\}$.

Furthermore, there is equality in (ii) for $\rho^2 = \frac{1}{2}kD[x]/\|x\|^2$ iff there is equality in (i).

Proof: We see that,

$$J_{\rho^2,k}[x] = \left(\rho^2 - \frac{k}{2}\frac{D[x]}{\|x\|^2}\right)^2\|x\|^2 + \frac{1}{\|x\|^2}\left(\|x\|^2\sum_{n=1}^{\infty}\left(M^2x_n\right)^2w_n - \frac{k^2}{4}\left(D[x]\right)^2\right) \tag{4.30}$$

and

$$J_{\rho^2,k}[x] \geq 0 \;\forall\rho \in \mathbb{R} - \{0\} \iff D[x] \leq 2k^{-1}\|x\|\left(\sum_{n=1}^{\infty}(M^2x_n)^2w_n\right)^{1/2}. \tag{4.31}$$

Therefore the lemma is proved.

Now, let $\chi(\rho,\theta)$ denote the least eigenvalue of the real, symmetric matrix $\mathbf{P}(\rho,\theta)$ defined in (4.24) with $\lambda = \rho e^{\mathrm{i}\theta}$ $\big(\theta \in (0,\frac{\pi}{4}), \rho \in \mathbb{R}\backslash\{0\}\big)$, and define

$$\theta_0 := \inf\big\{\theta : \theta \in (0,\frac{\pi}{4}), \quad \chi(\rho,\theta) \geq 0 \quad \forall\rho \in \mathbb{R}\backslash\{0\}\big\}. \tag{4.32}$$

The main theorem is as follows.

4.2.5 Theorem: Suppose M is LP, M^2 is PS. Let $\lambda = \rho e^{\mathrm{i}\theta}$ with $\theta \in (0,\frac{\pi}{4})$, $\rho \in \mathbb{R}\backslash\{0\}$. Then,

(A)

$$\theta_0 = \inf\big\{\theta : \theta \in (0,\frac{\pi}{4}), \quad \chi(\rho,\phi) > 0 \quad \forall\phi \in (\theta,\frac{\pi}{4}), \forall\rho \in \mathbb{R}\backslash\{0\}\big\}. \tag{4.33}$$

(B) There exists a constant $K \in (1, \infty)$ such that (4.1) is valid if and only if $0 < \theta_0 < \frac{\pi}{4}$, in which case, the best constant in (4.1) is $K = \sec(2\theta_0)$.

Proof:

(A) Following (4.26), (4.27), (4.28) and Lemma 4.2.4, with $\lambda = \rho e^{i\theta}$ the validity of (4.1) is equivalent to non-negativeness of $\mathbf{P}(\rho, \theta)$, i.e. non-negativeness of the least eigenvalue $\chi(\rho, \theta)$ of the matrix $\mathbf{P}(\rho, \theta)$. Suppose that $\chi(\rho, \theta) \geq 0$ $\forall \rho \in \mathbb{R}\backslash\{0\}$ and for some $\theta \in (0, \frac{\pi}{4})$. Hence,

$$Q_{\lambda^2}[x] \geq 0, \qquad \lambda = \rho e^{i\theta}.$$

Now, since for $\phi \in (\theta, \frac{\pi}{4})$ we have $\cos(2\phi) < \cos(2\theta)$, it follows from (4.28) that, since $\frac{2\epsilon}{\rho^2} = 2\cos(2\theta)$,

$$\begin{aligned} 2\cos(2\phi)D[x] &\leq 2\cos(2\theta)D[x] \qquad (\text{strict if } D[x] \neq 0) \\ &\leq \rho^2 \|x\|^2 + \frac{1}{\rho^2} \sum_{n=1}^{\infty} \left(M^2 x_n\right)^2 w_n \qquad \forall \rho \in \mathbb{R}\backslash\{0\}. \end{aligned}$$

The second inequality above, i.e.

$$2\cos(2\theta)D[x] \leq \rho^2 \|x\|^2 + \frac{1}{\rho^2} \sum_{n=1}^{\infty} \left(M^2 x_n\right)^2 w_n \qquad \forall \rho \in \mathbb{R}\backslash\{0\} \tag{4.34}$$

remains strict if $D[x] = 0$ and $x \neq 0$. Hence, from (4.26), (4.27), (4.28) and Lemma 4.2.4, $\chi(\rho, \phi) > 0$ $\forall \rho \in \mathbb{R}\backslash\{0\}$ and therefore (4.33) is established.

(B) If (4.1) is satisfied for some $K \in (1, \infty)$, then on setting

$$\theta = \frac{1}{2}\cos^{-1}\left(\frac{1}{K}\right) \in \left(0, \frac{\pi}{4}\right),$$

we see that (4.34) is satisfied. So, (4.27) and (4.28) now yield that when $\lambda = \rho e^{i\theta}$

$$\chi(\rho, \theta) \geq 0 \qquad \forall \rho \in \mathbb{R}\backslash\{0\}.$$

In particular, $\theta_0 < \frac{\pi}{4}$ and $\theta_0 = \frac{1}{2}\cos^{-1}(\frac{1}{K})$ where K is the least constant in (4.1).

Conversely, if $\theta_0 < \frac{\pi}{4}$ then with $\lambda = \rho e^{\mathrm{i}\theta}$, for any $\theta \in (\theta_0, \frac{\pi}{4})$,

$$Q_{\lambda^2}[x] \geq 0 \qquad \forall \rho \in \mathbb{R}\backslash\{0\} \text{ and } \forall \text{ real } x \in D_{T(M^2)},$$

and use of Lemma 4.2.4 again yields (4.1) with the constant $\sec(2\theta)$ and this implies that

$$K = \sec(2\theta_0).$$

The proof is therefore completed.

4.2.6 Corollary Suppose for some $\theta_1 \in (0, \frac{\pi}{4})$, $\chi(\rho, \theta_1) \geq 0$ $\forall \rho \in \mathbb{R}\backslash\{0\}$. Then, following Theorem 4.2.5, $\chi(\rho, \theta) > 0$ for any $\theta \in (\theta_1, \frac{\pi}{4})$ and $\rho \in \mathbb{R}\backslash\{0\}$. Therefore, we are able to give an equivalent definition of θ_0 as follows:

$$\theta_0 = \sup\big\{\theta : \theta \in (0, \frac{\pi}{4}), \chi(\rho, \theta) = 0 \text{ for some } \rho \in \mathbb{R}\backslash\{0\}\big\}. \tag{4.35}$$

This means that, if in $\chi(\rho, \theta) = 0$, we can express θ as a function of ρ, $\theta = \theta(\rho)$, then θ_0 is its maximum, possibly attained at $\rho = \infty$. To see (4.35), denote the right hand side by α and the set on the right hand side of (4.35) by S. Then we have that for any $\theta \in S$, $\chi(\rho, \theta) = 0$ for some $\rho \in \mathbb{R}\backslash\{0\}$ and hence $\theta \leq \theta_0$ which gives $\alpha \leq \theta_0$. If $\theta < \theta_0$ then $\chi(\rho, \theta) < 0$ for some $\rho \in \mathbb{R}\backslash\{0\}$ and $\theta_0 \leq \alpha$. Thus the corollary is proved.

4.2.7 Theorem Suppose that the hypothesis of Theorem 4.2.5 holds and $\theta_0 < \frac{\pi}{4}$ so that $K = \sec(2\theta_0)$ is the best constant. Then the sequences $\{x_n\}_{-1}^{\infty} \in D_{T(M^2)}$ which give equality in (4.1) satisfy one of the following distinct conditions:

(a) $x = 0$,

(b) $x \neq 0$, $Mx_n = 0$ $(n = 0, 1, 2, ...)$ in which case, both sides of (4.1) are 0,

(c) $x \neq 0$, $M^2 x_n \neq 0$ for $n = 1, 2, 3, ...$ and

$$E := \{\rho : \rho \in \mathbb{R}\backslash\{0\}, \chi(\rho, \theta_0) = 0\} \neq \emptyset$$

in which case the equalising sequences $\{x_n\}_{-1}^{\infty}$ are given by

$$x = 2\Re \sum_{i=1}^{2} A_i \psi^{(i)}(\lambda_i) \tag{4.36}$$

where $A_i = a_i + ib_i$ and $A^T = (a_1, a_2, b_1, b_2)$ satisfies

$$P(\rho, \theta_0)A = 0 \ (\rho \in E).$$

Proof:

(a) The proof for this case is obvious.

(b) The proof of this case follows from the fact that

$$Mx_n = 0 \qquad \text{for } n \geq 0 \qquad \Rightarrow \qquad M^2 x_n = 0 \qquad \text{for } n \geq 1,$$

which makes both sides of (4.1) zero.

(c) Suppose $J_{\rho^2,k}[x] = 0$ with $\rho^2 = \frac{1}{2}kD[x]/\|x\|^2$. Then, if there is equality in (4.1), i.e.

$$D[x] = sec(2\theta_0)\|x\| \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n \right)^{1/2},$$

we have, with $\lambda = \rho e^{i\theta_0}$ and $\epsilon = \Re(\lambda^2) = \rho^2 cos 2\theta_0$,

$$\frac{2\epsilon}{\rho^2} D[x] = \frac{1}{\rho^2} \sum_{n=1}^{\infty} |M^2 x_n|^2 w_n + \rho^2 \|x\|^2.$$

Hence, by Lemma 4.2.1, $Q_{\lambda^2}[x] = 0$. However, from

$$Q_{\lambda^2}[x] - \| < y, y^* > \|_{\lambda^2}^2 = A^T \mathbf{P}(\rho, \theta_0) A \geq 0, \tag{4.37}$$

it follows that $y = y^* = 0$ and $\mathbf{P}(\rho, \theta_0)A^T = 0$. Then, $\rho \in E$ and, following 3.12, x is given by

$$x = 2\Re \sum_{i=1}^{2} A_i \psi^{(i)}(\lambda_i).$$

Conversely, suppose that $\rho \in E$ and x is given by (4.36). Then $x \neq 0$, $x \in D_{T(M^2)}$, $T(M^2)x = 2\Re\{\lambda^2 \sum_{i=1}^{2} A_i \psi^{(i)}(\lambda_i)\}$, and with $y = y^* = 0$ in (4.37) we get $Q_{\lambda^2}[x] = 0$ in which $\lambda = \rho e^{i\theta_0}$. Then, it follows from Lemma 4.2.1 that

$$2\cos(2\theta_0) D[x] = \frac{1}{\rho^2} \sum_{n=1}^{\infty} |M^2 x_n|^2 w_n + \rho^2 \|x\|^2. \tag{4.38}$$

Now, when $\rho^2 = \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n\right)^{1/2} / \|x\|$ the right hand side of (4.38) attains its minimum, and hence

$$\begin{aligned} \cos(2\theta_0) D[x] &\geq \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n\right)^{1/2} \|x\| \\ &\geq \cos(2\theta_0) D[x]. \end{aligned}$$

Following the last argument there is therefore equality, and the theorem is proved.

§4.3 Another Inequality

In this section we examine the validity of the inequality

$$\sum_{n=-1}^{\infty} |M x_n|^2 w_n \leq K \left(\sum_{n=-1}^{\infty} |x_n|^2 w_n\right)^{1/2} \left(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n\right)^{1/2} \qquad \forall x \in D_{T(M^2)} \tag{4.39}$$

under the same circumstances as (4.1). This inequality is a stronger inequality than that in (4.1), as there is no $|Mx_{-1}|^2 w_{-1}$ term on the left hand side of (4.1). The technique we use in the investigation of the validity of (4.39) is exactly the same as that we use for (4.1). In fact the results we have for (4.1) remain true except that in some of them we need to make a few minor modifications. Hence, we start with the following lemma.

4.3.1 Lemma Let $\lambda = \rho e^{i\theta}$ ($\theta \in (0, \frac{\pi}{4}), \rho \in \mathbb{R} - \{0\}$), $\lambda^2 = \epsilon + i\eta$, $< x, x^* > \in G$ be real, M be LP and M^2 be PS, and define

$$Q_{\lambda^2}[x] := \| < x, x^* > \|_{\lambda^2}^2 + 2\epsilon\big[x_0 w_0 (x_0^* - M^2 x_0) + x_{-1}(p_{-1} M x_0 + x_{-1}^* w_{-1}) - M x_{-1}(M x_{-1} w_{-1} + p_{-1} x_0)\big], \tag{4.40}$$

then the following are equivalent.

(a) $2\epsilon \sum_{n=-1}^{\infty} (Mx_n)^2 w_n \leq \|x^*\|^2 + |\lambda^2|^2 \|x\|^2$ (4.41)

(b) $Q_{\lambda^2}[x] \geq 0$. (4.42)

Furthermore, there is equality in (a) if and only if there is equality in (b).

Proof: For real $< x, x^* > \in G$,

$$\begin{aligned} \| < x, x^* > \|_{\lambda^2}^2 &= \|x^* - \epsilon x\|^2 + \eta^2 \|x\|^2 \\ &= \|x^*\|^2 - 2\epsilon (x^*, x) + |\lambda^2|^2 \|x\|^2 \end{aligned}$$

and

$$\begin{aligned} (x^*, x) &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 + \sum_{n=1}^{\infty} M^2 x_n x_n w_n \\ &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - M^2 x_0 x_0 w_0 + \sum_{n=0}^{\infty} M^2 x_n x_n w_n \\ &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - M^2 x_0 x_0 w_0 + \sum_{n=0}^{\infty} (M x_n)^2 w_n \\ &\quad + p_{-1}(x_{-1} M x_0 - x_0 M x_{-1}) \\ &= x_{-1}^* x_{-1} w_{-1} + x_0^* x_0 w_0 - M^2 x_0 x_0 w_0 - (M x_{-1})^2 w_{-1} \\ &\quad + \sum_{n=-1}^{\infty} (M x_n)^2 w_n + p_{-1}(x_{-1} M x_0 - x_0 M x_{-1}) \end{aligned}$$

$\Rightarrow$

$$\| < x, x^* > \|_{\lambda^2}^2 = \|x^*\|^2 + \rho^4\|x\|^2 - 2\epsilon \sum_{n=-1}^{\infty} (Mx_n)^2 w_n$$
$$- 2\epsilon[x_0 w_0(x_0^* - M^2 x_0) + x_{-1}(p_{-1}Mx_0 + x_{-1}^* w_{-1}) \tag{4.43}$$
$$- Mx_{-1}(Mx_{-1}w_{-1} + p_{-1}x_0)]$$

$\Rightarrow$

$$Q_{\lambda^2}[x] = \|x^*\|^2 + \rho^4\|x\|^2 - 2\epsilon \sum_{n=-1}^{\infty} (Mx_n)^2 w_n.$$

The proof therefore follows.

Similar to the case for the inequality (4.1), we will be dealing with (4.42) in order to obtain the inequality (4.41). We again obtain information on the terms in (4.42) by means of (4.5), (4.7) and (4.12). So,

$$x_0 w_0(x_0^* - M^2 x_0)$$
$$= w_0 2\Re\big[A_1 \frac{m(\lambda)}{p_{-1}} + A_2 \frac{n(\lambda)}{p_{-1}}\big]\Big\{ y_0^* + 2\Re\big[A_1 \lambda^2 \frac{m(\lambda)}{p_{-1}} + A_2 \lambda^2 \frac{n(\lambda)}{p_{-1}}\big]$$
$$- M^2 y_0 - 2\Re\big\{A_1\big[\lambda^2 \frac{m(\lambda)}{p_{-1}} + \frac{p_{-1}}{w_0}\lambda + \frac{p_{-1}^2}{w_0 w_{-1}}(\frac{m(\lambda)}{p_{-1}} - 1)\big]$$
$$+ A_2\big[\lambda^2 \frac{n(\lambda)}{p_{-1}} - \frac{p_{-1}}{w_0}\lambda + \frac{p_{-1}^2}{w_0 w_{-1}}(\frac{n(\lambda)}{p_{-1}} - 1)\big]\big\}\Big\}$$
$$= 4\Re\big[A_1 m(\lambda) + A_2 n(\lambda)\big]\Big\{ \frac{p_{-1}}{w_{-1}}\Re(A_1 + A_2)$$
$$+ \Re\big\{\big[A_1\big[-\lambda - \frac{1}{w_{-1}} m(\lambda)\big] + A_2\big[\lambda - \frac{1}{w_{-1}} n(\lambda)\big]\big\}\Big\}.$$

Also,

$$x_{-1}(p_{-1}Mx_0 + x_{-1}^* w_{-1}) = 4\Re(A_1 + A_2)\Re\big\{A_1\big[\lambda m(\lambda) + w_{-1}\lambda^2\big]$$
$$+ A_2\big[-\lambda n(\lambda) + w_{-1}\lambda^2\big]\big\},$$

and, following (2.13),

$$- Mx_{-1}(Mx_{-1}w_{-1} + p_{-1}x_0)$$
$$= 4\Re(A_1 + A_2)\Re\Big\{A_1\big[\frac{p_{-1}m(\lambda)}{w_{-1}} - \frac{p_{-1}^2}{w_{-1}}\big] + A_2\big[\frac{p_{-1}n(\lambda)}{w_{-1}} - \frac{p_{-1}^2}{w_{-1}}\big]\Big\}.$$

Now, combining these results in (4.40), we get

$$
\begin{aligned}
&Q_{\lambda^2}[x] - \| < x, x^* > \|_{\lambda^2}^2 \\
&= 8\epsilon\Re(A_1 + A_2)\Re\Big\{ A_1\big[\frac{2p_{-1}m(\lambda)}{w_{-1}} + \lambda m(\lambda) + w_{-1}\lambda^2 - \frac{p_{-1}^2}{w_{-1}}\big] \\
&\quad + A_2\big[\frac{2p_{-1}n(\lambda)}{w_{-1}} - \lambda n(\lambda) + w_{-1}\lambda^2 - \frac{p_{-1}^2}{w_{-1}}\big]\Big\} \\
&\quad + 8\epsilon\Re\big[A_1 m(\lambda) + A_2 n(\lambda)\big]\Re\Big\{ A_1\big[-\lambda - \frac{m(\lambda)}{w_{-1}}\big] + A_2\big[\lambda - \frac{n(\lambda)}{w_{-1}}\big]\Big\}.
\end{aligned}
$$

And this implies, by using (4.15), (2.13) and (4.11),

$$
\begin{aligned}
&Q_{\lambda^2}[x] - \| < y, y^* > \|_{\lambda^2}^2 \\
&= 4\eta^2\Re\sum_{i=1}^{2}\sum_{k=1}^{2} A_i\bar{A}_k\big[\frac{m(\lambda_i) - \bar{m}(\lambda_k)}{\lambda_i - \bar{\lambda}_k} + w_{-1}\big] \\
&\quad + 8\epsilon\Re(A_1 + A_2)\Re\Big\{ A_1\big[\frac{2p_{-1}m(\lambda)}{w_{-1}} + \lambda m(\lambda) + w_{-1}\lambda^2 - \frac{p_{-1}^2}{w_{-1}}\big] \\
&\quad + A_2\big[\frac{2p_{-1}n(\lambda)}{w_{-1}} - \lambda n(\lambda) + w_{-1}\lambda^2 - \frac{p_{-1}^2}{w_{-1}}\big]\Big\} \\
&\quad + 8\epsilon\Re\big[A_1 m(\lambda) + A_2 n(\lambda)\big]\Re\Big\{ A_1\big[-\lambda - \frac{m(\lambda)}{w_{-1}}\big] + A_2\big[\lambda - \frac{n(\lambda)}{w_{-1}}\big]\Big\}.
\end{aligned}
\tag{4.44}
$$

Now, set $A_1 = a_1 + \mathrm{i}b_1$, $A_2 = a_2 + \mathrm{i}b_2$, $m(\lambda) = m_1 + \mathrm{i}m_2$, $n(\lambda) = n_1 + \mathrm{i}n_2$, $\lambda = \mu + \mathrm{i}\nu$ and

$$
\begin{aligned}
s(\lambda) &= -\lambda - \frac{m(\lambda)}{w_{-1}} = s_1 + \mathrm{i}s_2 \\
t(\lambda) &= \lambda - \frac{n(\lambda)}{w_{-1}} = t_1 + \mathrm{i}t_2
\end{aligned}
$$

$$
\begin{aligned}
f(\lambda) &= \frac{2p_{-1}m(\lambda)}{w_{-1}} + \lambda m(\lambda) + w_{-1}\lambda^2 - \frac{p_{-1}^2}{w_{-1}} = f_1 + \mathrm{i}f_2 \\
g(\lambda) &= \frac{2p_{-1}n(\lambda)}{w_{-1}} - \lambda n(\lambda) + w_{-1}\lambda^2 - \frac{p_{-1}^2}{w_{-1}} = g_1 + \mathrm{i}g_2.
\end{aligned}
$$

Then, in (4.44)

$$
\begin{aligned}
I_1 &\equiv 8\epsilon\Re\big[A_1 m(\lambda) + A_2 n(\lambda)\big]\Re\big[A_1 s(\lambda) + A_2 t(\lambda)\big] \\
&= 8\epsilon(a_1 m_1 - b_1 m_2 + a_2 n_1 - b_2 n_2)(a_1 s_1 - b_1 s_2 + a_2 t_1 - b_2 t_2)
\end{aligned}
$$

$$= 8\epsilon\big[a_1^2 m_1 s_1 + a_1 a_2(m_1 t_1 + n_1 s_1) + a_1 b_1(-m_2 s_1 - m_1 s_2)$$

$$+ a_1 b_2(-m_1 t_2 - n_2 s_1)$$

$$+ a_2^2 n_1 t_1 + a_2 b_1(-n_1 s_2 - m_2 t_1) + a_2 b_2(-n_1 t_2 - n_2 t_1)$$

$$+ b_1^2 m_2 s_2 + b_1 b_2(m_2 t_2 + n_2 s_2)$$

$$+ b_2^2 n_2 t_2\big]$$

$\Rightarrow$

$$I_1 = 8\epsilon \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \end{pmatrix} \mathbf{M}' \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \tag{4.45}$$

where the matrix M' is real and symmetric with the following entries:

$$M_{11} = m_1 s_1,\ M_{12} = \frac{m_1 t_1 + n_1 s_1}{2},\ M_{13} = -\frac{m_2 s_1 + m_1 s_2}{2},$$

$$M_{14} = -\frac{m_1 t_2 + n_2 s_1}{2},$$

$$M_{22} = n_1 t_1,\ M_{23} = -\frac{n_1 s_2 + m_2 t_1}{2},\ M_{24} = -\frac{n_1 t_2 + n_2 t_1}{2},$$

$$M_{33} = m_2 s_2,\ M_{34} = \frac{m_2 t_2 + n_2 s_2}{2},$$

$$M_{44} = n_2 t_2.$$

Again, in (4.44),

$$I_2 \equiv 8\epsilon \Re(A_1 + A_2) \Re\big[A_1 f(\lambda) + A_2 g(\lambda)\big]$$

$$= 8\epsilon(a_1 + a_2)(a_1 f_1 - b_1 f_2 + a_2 g_1 - b_2 g_2)$$

$$= 8\epsilon\big[a_1^2 f_1 + a_1 a_2(f_1 + g_1) + a_1 b_1(-f_2) + a_1 b_2(-g_2)$$

$$+ a_2^2 g_1 + a_2 b_1(-f_2) + a_2 b_2(-g_2)\big]$$

so that

$$I_2 = 8\epsilon \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \end{pmatrix} \mathbf{N}' \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \tag{4.46}$$

where N' is a real and symmetric matrix with the following entries:

$$N_{11} = f_1,\ N_{12} = \frac{f_1 + g_1}{2},\ N_{13} = -\frac{f_2}{2},\ N_{14} = -\frac{g_2}{2},$$

$$N_{22} = g_1,\ N_{23} = -\frac{f_2}{2},\ N_{24} = -\frac{g_2}{2},$$

$$N_{33} = 0,\ N_{34} = 0,$$

$$N_{44} = 0.$$

Now, (4.45), (4.46) and (4.20) together with (4.44) imply that

$$Q_{\lambda^2}[x] - \| < y, y^* > \|_{\lambda^2}^2 = A^T \mathbf{P}' A \tag{4.47}$$

where

$$\mathbf{P}' := 8\epsilon(M' + N') + 4\eta^2 O, \qquad A^T = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \end{pmatrix}, \tag{4.48}$$

is a real and symmetric matrix of size 4×4. Hence, by (4.46),

$$\begin{aligned} Q_{\lambda^2}[x] &= \|x^*\|^2 + \rho^4\|x\|^2 - 2\epsilon \sum_{n=-1}^{\infty} (Mx_n)^2 w_n \\ &= \| < y, y^* > \|_{\lambda^2}^2 + A^T \mathbf{P}'(\rho, \theta) A \\ &= |x_{-1}^*|^2 w_{-1} + |x_0^*|^2 w_0 + \sum_{n=1}^{\infty} (M^2 x_n)^2 w_n + \rho^4 \|x\|^2 - 2\epsilon D'[x] \end{aligned}$$

where $A^T := \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \end{pmatrix}$ and $D'[x] := \sum_{n=-1}^{\infty} (Mx_n)^2 w_n$. Then, the following are equivalent

(i) $\|x^*\|^2 + \rho^4\|x\|^2 - 2\epsilon D'[x] \geq 0 \quad \forall$ real $< x, x^* > \in G,$ (4.49)

(ii) $\mathbf{P}'(\rho, \theta) \geq 0$, i.e. $\mathbf{P}'$ is non-negative definite. (4.50)

Furthermore, since x_{-1}^* and x_0^* can be chosen to be 0 (see Lemma 4.2.2), (ii) is equivalent to

(iii) $\sum_{n=1}^{\infty} (M^2 x_n)^2 w_n + \rho^4 \|x\|^2 - 2\epsilon D'[x] \geq 0\ \forall$ real $x \in D_{T(M^2)}$. (4.51)

After this point, we face no problem in obtaining the same results for this inequality as that we obtained for (4.1). So, theoretically speaking, there is no reason for (4.39) not to be valid. Namely,

4.3.2 Theorem Suppose M is LP, M^2 is PS. Let $\lambda = \rho e^{i\theta}$ with $\theta \in (0, \frac{\pi}{4})$, $\rho \in \mathbb{R}\backslash\{0\}$. Then,

(A)

$$\theta_0 = \inf\{\theta : \theta \in (0, \frac{\pi}{4}), \quad \chi(\rho, \phi) > 0 \quad \forall \phi \in (\theta, \frac{\pi}{4}), \forall \rho \in \mathbb{R}\backslash\{0\}\}. \qquad (4.52)$$

(B) There exists a constant $K \in (1, \infty)$ such that (4.39) is valid if and only if $0 < \theta_0 < \frac{\pi}{4}$, in which case, the best constant in (4.39) is $K = \sec(2\theta_0)$.

The proof of Theorem 4.3.2 follows from the same steps as Theorem 4.2.5, and so, we omit the proof. But when it comes down to giving an example of a valid inequality, we see that when $p = w = 1$ and $q = 0$, the simplest case, the inequality fails to be valid. We give asymptotic results in Chapter 5 for this case. This may suggest that (4.39) is not the inequality that is the analogue of the Kolmogorov type inequality (1.8) for the differential operators, which was examined by Beynon *et al.* **[9]**.

Chapter 5

EXAMPLES

In this chapter we examine some specific cases of the inequalities (4.1) and (4.39). We also give asymptotic results for the both inequalities in order to build up our confidence on the computational results. In all the following examples, the complex sequences $\{x_n\}_{-1}^{\infty}$ belong to the maximal domain $D_{T(M^2)} \subseteq D_{T(M)}$ that makes the right hand-side of the inequality under consideration finite. The LP and PS conditions that the expressions must satisfy are examined in each example separately. We also use the following notation throughout:

(i) $\lambda = \rho e^{i\theta}$, $\rho \in (0, \infty)$, $\theta \in (0, \frac{\pi}{4})$.

(ii) $\chi(\rho, \theta)$ *is the least eigenvalue of the matrix* $P(\rho, \theta)$.

(iii) $\theta_0 := \inf \{\theta \in (0, \frac{\pi}{4}) : \chi(\rho, \theta) \geq 0, \quad \forall \rho \in \mathbb{R} \backslash \{0\}\}$.

(iv) $K := sec(2\theta_0)$.

(v) $E := \{\rho \in (0, \infty) : \chi(\rho, \theta_0) = 0\}$.

We recall that $E \neq \emptyset$ gives non-trivial equalising sequences (cf. Chapter 4, Theorem 4.2.7).

In each example we give information on θ_0, K, E and the equalising sequences. Each 4×4 real, symmetric matrix that is equivalent to the considered inequality is evaluated by (4.24). The complexity of the characteristic determinant of the matrix forces us to use numerical estimates. For this purpose, we use a numerical algorithm of Beynon (personal communication) in *Fortran*77 that uses a NAG (Numerical Algorithm Group (USA) Inc.) routine.

Following the Theorem 4.2.5, the validity of each example depends on the value of the quantity θ_0,

$$\begin{aligned}\theta_0 :&= \inf\{\theta \in (0, \frac{\pi}{4}) : \chi(\rho, \theta) \geq 0 \quad \forall \rho \in (0, \infty)\} \\ &= \sup\{\theta \in (0, \frac{\pi}{4}) : \chi(\rho, \theta) = 0 \quad \forall \rho \in (0, \infty)\}\end{aligned}$$

in which $\lambda = \rho e^{i\theta}$. This value is found by investigating the non-negativeness of the least eigenvalue $\chi(\rho, \theta)$ of the real, symmetric matrix $P(\rho, \theta)$ and showing that $\chi(\rho, \theta)$ changes sign at some $\theta_0 \in (0, \frac{\pi}{4})$ and for all $\rho \in (0, \infty)$. The process of the numerical algorithm we use is as follows.

The algorithm for the investigation of θ_0 consists of searching for the values of $\theta \in (0, \frac{\pi}{4})$ such that $\chi(\rho, \theta) = 0$ for a finite number of values of $\rho \in (0, \rho_{max})$ in which the value of ρ_{max} is empirically determined. For each ρ a bisection search is performed in order to find $\theta \in (0, \frac{\pi}{4})$ such that $\chi(\rho, \theta) = 0$. Each iteration of the bisection search includes the estimation of the real symmetric matrix $\mathbf{P}(\rho, \theta)$ in order to estimate the value of the least eigenvalue $\chi(\rho, \theta)$. The process of estimating the least eigenvalue $\chi(\rho, \theta)$ of $\mathbf{P}(\rho, \theta)$ consists of reducing the matrix to tridiagonal form by using the Householder procedure, then estimating the value of the least eigenvalue of the reduced matrix by using the QL method.

Thus, the three important sections of the numerical method are:

(1) Calculation of the real symmetric matrix $\mathbf{P}(\rho, \theta)$ for a fixed value of ρ and θ.

(2) Estimating the value of the least eigenvalue $\chi(\rho, \theta)$ of the symmetric matrix $\mathbf{P}(\rho, \theta)$ for a fixed value of ρ and θ using the Householder and QL algorithms.

(3) Using the bisection search method, estimation of the solution of $\chi(\rho, \theta) = 0$ for $0 < \theta < \frac{\pi}{4}$ and $0 < \rho < \rho_{max}$ where ρ_{max} is empirically determined.

§5.1 The case where $p_n = w_n = 1$ and $q_n = 0$

In this case

$$\begin{aligned} Mx_n : &= -\Delta^2 x_{n-1} \\ &= -x_{n+1} + 2x_n - x_{n-1}, \qquad (n \geq 0). \end{aligned} \tag{5.1}$$

As it is shown in Brown-Evans **[12]**, for $\lambda \in \mathbb{C} \backslash \mathbb{R}$, the recurrence relation

$$-\Delta^2 x_{n-1} = \lambda x_n \tag{5.2}$$

has two linearly independent solutions $\left\{\alpha_+^n(\lambda)\right\}_{-1}^{\infty}$, $\left\{\alpha_-^n(\lambda)\right\}_{-1}^{\infty}$ where

$$\alpha_{\pm}(\lambda) := 1 - \frac{\lambda}{2} \pm \frac{1}{2}\sqrt{\lambda(\lambda - 4)}. \tag{5.3}$$

Now, since $\alpha_+(\lambda).\alpha_-(\lambda) = 1$, and for $\Im(\lambda) > 0$ $|\alpha_+(\lambda)| < 1$, this will imply that $|\alpha_-(\lambda)| > 1$ (see Brown-Evans **[12]** for details) and $\left\{\alpha_+^n(\lambda)\right\}_{-1}^{\infty}$ is the unique (up to constant multiples) solution of (5.2) which lies in ℓ^2 and consequently M is in the LP case. Another way to show that M is LP is to apply the LP criterion (cf. Chapter 1, Theorem 1.3.2)

$$\sum_{n=0}^{\infty} \frac{(w_n w_{n+1})^{1/2}}{|p_n|} = \infty,$$

which is satisfied by our expression (5.1) since $p_n = w_n = 1$ (see Hinton-Lewis **[39]**). Another condition that M^2 is required to satisfy is the partial separation

(PS) condition, which is to show the implication

$$x, M^2 x \in \ell^2 \Rightarrow Mx \in \ell^2.$$

This follows immediately from the boundedness of the forward difference operator:

$$\sum_{n=-1}^{\infty} |\Delta x_n|^2 \leq 4||x_n||^2, \qquad \forall x \in \ell^2.$$

In fact, any power of M is bounded, so that any power of M is PS.

Again in the light of Brown-Evans **[12]**, the $m(\lambda)$ and $n(\lambda) := m(-\lambda)$ functions are given by

$$m(\lambda) := 1 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda(\lambda - 4)}, \qquad \lambda \in \mathbb{C}_{+} \tag{5.4}$$

$$n(\lambda) := 1 + \frac{\lambda}{2} - \frac{1}{2}\sqrt{\lambda(\lambda + 4)}, \qquad \lambda \in \mathbb{C}_{+}. \tag{5.5}$$

We are now ready to examine the best possible value of the constant K and the validity of

$$\sum_{n=0}^{\infty} |Mx_n|^2 \leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |M^2 x_n|^2\Big)^{1/2}$$

which is

$$\sum_{n=0}^{\infty} |\Delta^2 x_{n-1}|^2 \leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |\Delta^4 x_{n-2}|^2\Big)^{1/2}$$

or

$$||\Delta^2 x||^2 \leq K||x||\,||\Delta^4 x||, \tag{5.6}$$

which is a norm inequality with $||.||$ being ℓ^2 norm and $(\Delta^2 x)_n = \Delta^2(x_n)$, $(\Delta^4 x)_n = \Delta^4(x_n)$. As we can see in *Figure 5.1*, the graph $\{(\rho, \theta) : \chi(\rho, \theta) = 0\}$ starts at a point (θ_0, ρ) where $0.72896067 < \theta_0 < 0.72896068$ with ρ near 0, then monotonically decreases. Hence, we estimate that the value of the best constant satisfies $8.878200 < K < 8.878201$ and $E = \emptyset$, i.e. the null sequence is the only equalizing sequence in (5.6). We are going to give asymptotic results for $\rho \to \infty$ and $\rho \to 0$ to see a clearer picture.

(1) The case $\rho \to \infty$:

In this case, for $\Im\lambda > 0$, as $|\lambda| \to \infty$

$$\begin{aligned} m(\lambda) &= 1 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda(\lambda-4)} \\ &= 1 - \frac{\lambda}{2} + \frac{\lambda}{2}\sqrt{1 - \frac{4}{\lambda}} \\ &= -\frac{1}{\lambda} + O(\frac{1}{\lambda^2}) \\ &= m_1 + \mathrm{i}m_2 \end{aligned}$$

and

$$\begin{aligned} n(\lambda) &= 1 + \frac{\lambda}{2} - \frac{1}{2}\sqrt{\lambda(\lambda+4)} \\ &= 1 + \frac{\lambda}{2} - \frac{\lambda}{2}\sqrt{1 + \frac{4}{\lambda}} \\ &= \frac{1}{\lambda} + O(\frac{1}{\lambda^2}) \\ &= n_1 + \mathrm{i}n_2 \end{aligned}$$

where $m_1, m_2, n_1, n_2 \in \mathbb{R}$. For $\lambda = \mu + \mathrm{i}\nu = \rho(\cos\theta + \mathrm{i}\sin\theta)$, we obtain the following leading entries for the real, symmetric matrix $P(\rho, \theta)$, with the error of order $O(\rho^3)$,

$$P(\rho,\theta) = 4\rho^4 \begin{pmatrix} 1+\cos^2(2\theta) & 1+\cos^2(2\theta) & -\frac{1}{2}\sin(4\theta) & -\frac{1}{2}\sin(4\theta) \\ \cdot & 1+\cos^2(2\theta) & -\frac{1}{2}\sin(4\theta) & -\frac{1}{2}\sin(4\theta) \\ \cdot & \cdot & \sin^2(2\theta) & \sin^2(2\theta) \\ \cdot & \cdot & \cdot & \sin^2(2\theta) \end{pmatrix} + O(\rho^3)$$

and

$$|P - \ell I| = \ell^2(\ell - 16\rho^4\sin^2\theta)(\ell - 16\rho^4\cos^2\theta)$$

which implies that the least eigenvalue of the leading matrix is $\ell = 0$ for $\theta \in (0, \frac{\pi}{4})$ as $\rho \to \infty$. Thus, in the graph $\{(\rho, \theta) : \chi(\rho, \theta) = 0\}$, $\theta = 0$ for all large enough ρ.

(2) The case $\rho \to 0$:

$$
\begin{aligned}
m(\lambda) &= 1 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda(\lambda - 4)} \\
&= 1 - \frac{\lambda}{2} + \mathrm{i}\sqrt{\lambda}\sqrt{1 - \frac{\lambda}{4}} \\
&= 1 + \mathrm{i}\sqrt{\lambda} + O(\lambda) \\
&= m_1 + \mathrm{i}m_2
\end{aligned}
$$

and

$$
\begin{aligned}
n(\lambda) &= 1 + \frac{\lambda}{2} - \frac{1}{2}\sqrt{\lambda(\lambda + 4)} \\
&= 1 + \frac{\lambda}{2} - \sqrt{\lambda}\sqrt{1 + \frac{\lambda}{4}} \\
&= 1 - \sqrt{\lambda} + O(\lambda) \\
&= n_1 + \mathrm{i}n_2
\end{aligned}
$$

where $m_1, m_2, n_1, n_2 \in \mathbb{R}$. We then obtain the following real, symmetric leading matrix $P(\rho, \theta)$, with the error of order $O(\rho^4)$:

$$
P = 4\rho^3 \begin{pmatrix} a & e & 0 & g \\ e & b & h & 0 \\ 0 & h & a & k \\ g & 0 & k & b \end{pmatrix} + O(\rho^4), \tag{5.7}
$$

where

$$
\begin{aligned}
a &= 2\sin\theta m_2 \\
e &= 2\cos^3\theta(m_1 - n_1) - \cos(2\theta)\sin\theta(m_2 - n_2) \\
g &= 2\cos\theta\sin^2\theta m_2 + 2\cos^3\theta n_2 - \cos(2\theta)\sin\theta(m_1 - n_1) \\
b &= -2\sin\theta n_2 \\
h &= -2\cos\theta\sin^2\theta n_2 - 2\cos^3\theta m_2 - \cos(2\theta)\sin\theta(m_1 - n_1)
\end{aligned}
$$

$$k = 2\cos\theta \sin^2\theta(m_1 - n_1) + \cos(2\theta)\sin\theta(m_2 - n_2).$$

The leading matrix $P(\rho, \theta)$ is exactly the same as the one for the differential equations case, obtained by Beynon *at.al* [**9**]. The complication of the matrix doesn't allow us to find the explicit value of $\theta_0 \in (0, \frac{\pi}{4})$. For this reason, we use numerical estimates as presented in *Figure 5.1*.

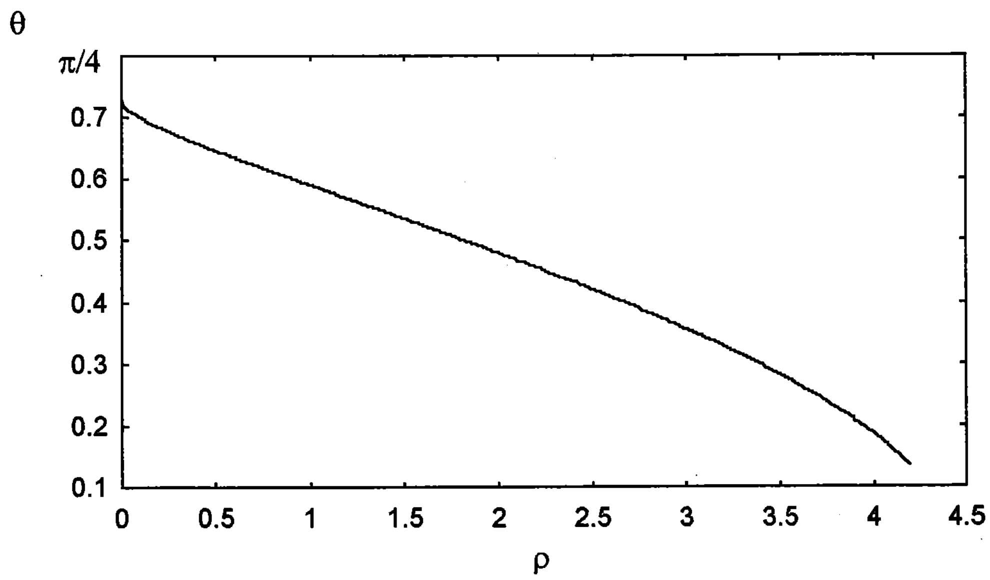

Figure 5.1 The graph $\{(\rho, \theta)\colon \chi(\rho, \theta) = 0\}$, $p_n = w_n = 1, q_n = 0.$

§5.2 The case where $p_n = w_n = 1$ and $q_n = -\tau$

In this case

$$Mx_n := -\Delta^2 x_{n-1} - \tau x_n \qquad (n \geq 0) \tag{5.8}$$

and the recurrence relation is given by (5.2) with τ shift value in λ, i.e.

$$-\Delta^2 x_{n-1} = (\lambda + \tau)x_n \qquad (n \geq 0), \lambda \in \mathbb{C}\backslash\mathbb{R}, \tau \in \mathbb{R} \tag{5.9}$$

and the inequality is

$$\sum_{n=0}^{\infty} |Mx_n|^2 \leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |M^2x_n|^2\Big)^{1/2}$$

which is

$$\sum_{n=0}^{\infty} |-\Delta^2 x_{n-1} - \tau x_n|^2 \leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2}$$
$$\Big(\sum_{n=1}^{\infty} |\Delta^4 x_{n-2} + 2\tau\Delta^2 x_{n-1} + \tau^2 x_n|^2\Big)^{1/2}.$$

M is again in the LP case since otherwise all solutions of (5.9) are in ℓ^2 for all λ. But, this would contradict the fact that in the case $\tau = 0$ (5.9) is LP. M^2 is PS as was proved in §5.1, i.e. the mapping

$$\{x_n\}_{-1}^{\infty} \longmapsto \{Mx_n\}_{-1}^{\infty}$$

is bounded. Similar to the case where $\tau = 0$ before, the $m(\lambda)$ and $n(\lambda)$ functions, which we now denote by $m_\tau(\lambda)$ and $n_\tau(\lambda)$ to indicate their dependence on τ, are given by

$$\begin{aligned} m_\tau(\lambda) &:= m(\lambda+\tau) \\ &= 1 - \frac{\lambda+\tau}{2} + \frac{1}{2}\sqrt{(\lambda+\tau)(\lambda+\tau-4)}, \qquad \lambda \in \mathbb{C}_+ \\ n_\tau(\lambda) &:= n(\lambda-\tau) \\ &= 1 - \frac{-\lambda+\tau}{2} - \frac{1}{2}\sqrt{(-\lambda+\tau)(-\lambda+\tau-4)}, \qquad \lambda \in \mathbb{C}_+. \end{aligned}$$

First of all we present an analytical result, proving that for large enough values of ρ for any $\tau \in \mathbb{R}$, $\theta = 0$ in the graph $\big\{(\rho,\theta) : \chi(\rho,\theta) = 0\big\}$.

Asymptotics as $\rho \to \infty$ $\forall \tau \in \mathbb{R}$:

In this case, the $m(\lambda)$ functions are given by

$$
\begin{aligned}
&m_\tau(\lambda)\\
&= 1 - \frac{\lambda+\tau}{2} + \frac{1}{2}\sqrt{(\lambda+\tau)(\lambda+\tau-4)}\\
&= 1 - \frac{\lambda+\tau}{2} + \frac{\lambda}{2}\sqrt{1+\frac{2(\tau-2)}{\lambda}+\frac{\tau(\tau-4)}{\lambda^2}}\\
&= 1 - \frac{\lambda+\tau}{2} + \frac{\lambda}{2}\left\{1+\frac{1}{2}\left[\frac{2(\tau-2)}{\lambda}+\frac{\tau(\tau-4)}{\lambda^2}\right]-\frac{1}{8}\left[\frac{4(\tau-2)^2}{\lambda^2}+O(\frac{1}{\lambda^3})\right]\right\}\\
&= -\frac{1}{\lambda}+O(\frac{1}{\lambda^2})
\end{aligned}
$$

and

$$
\begin{aligned}
&n_\tau(\lambda)\\
&= 1 - \frac{-\lambda+\tau}{2} - \frac{1}{2}\sqrt{(-\lambda+\tau)(-\lambda+\tau-4)}\\
&= 1 - \frac{-\lambda+\tau}{2} - \frac{\lambda}{2}\sqrt{1-\frac{2(\tau-2)}{\lambda}+\frac{\tau(\tau-4)}{\lambda^2}}\\
&= 1 - \frac{-\lambda+\tau}{2} - \frac{\lambda}{2}\left\{1+\frac{1}{2}\left[-\frac{2(\tau-2)}{\lambda}+\frac{\tau(\tau-4)}{\lambda^2}\right]-\frac{1}{8}\left[\frac{4(\tau-2)^2}{\lambda^2}+O(\frac{1}{\lambda^3})\right]\right\}\\
&= \frac{1}{\lambda}+O(\frac{1}{\lambda^2}).
\end{aligned}
$$

The very important fact is that these functions are independent of choice of τ for $\rho \to \infty$. So, inserting these values in the matrix (4.24), we obtain the real, symmetric matrix $P(\rho,\theta)$, with the error of order $O(\rho^3)$,

$$
P(\rho,\theta) = 4\rho^4 \begin{pmatrix} 1+\cos^2(2\theta) & 1+\cos^2(2\theta) & -\frac{1}{2}\sin(4\theta) & -\frac{1}{2}\sin(4\theta) \\ . & 1+\cos^2(2\theta) & -\frac{1}{2}\sin(4\theta) & -\frac{1}{2}\sin(4\theta) \\ . & . & \sin^2(2\theta) & \sin^2(2\theta) \\ . & . & . & \sin^2(2\theta) \end{pmatrix} + O(\rho^3).
$$

Thus,

$$
|P-\ell I| = \ell^2(\ell-16\rho^4\sin^2\theta)(\ell-16\rho^4\cos^2\theta)
$$

which implies that the least eigenvalue of the leading matrix is $\ell = 0$ for $\theta \in (0, \frac{\pi}{4})$ as $\rho \to \infty$. Thus, in the graph $\{(\rho, \theta) : \chi(\rho, \theta) = 0\}$, $\theta = 0$ for all large enough ρ and for all $\tau \in \mathbb{R}$.

In searching for the value of θ_0 and hence the best possible constant for the different values of τ, there are two cases to examine: the case where $\tau \in [0, 4]$ and the case where $\tau \in \mathbb{R} \setminus [0, 4]$.

(A) The case $\tau \in [0, 4]$:

We give the numerical results first. We see from *Figure 5.2* that, when $\tau = 4$ and ρ is small, $\rho \to 0$, the graph $\{(\rho, \theta) : \chi(\rho, \theta) = 0\}$ starts from a point (θ_0, ρ) where $0.72896067 < \theta_0 < 0.72896068$, and then decreases monotonically. Hence the value of the best constant, K, satisfies $8.878200 < K < 8.878201$. These are the same values as the case $\tau = 0$ (cf. section §5.1). The case $\tau = 0$ is in agreement with Beynon's results (cf. Beynon **[8]**, p.88) in the differential equations case.

For the analytical results, we examine the asymptotics of the real, symmetric matrix $P(\rho, \theta)$ as $\rho \to \infty$: when $\tau = 4$ the $m(\lambda)$ functions are

$$\begin{aligned} m(\lambda) &= 1 - \frac{\lambda + 4}{2} - \frac{1}{2}\sqrt{(\lambda + 4)(\lambda + 4 - 4)} \\ &= -1 + \sqrt{\lambda} + O(\lambda) \\ &= m_1 + \mathrm{i}m_2 \end{aligned}$$

$$\begin{aligned} n(\lambda) &= 1 - \frac{-\lambda + 4}{2} - \frac{1}{2}\sqrt{(-\lambda + 4)(-\lambda + 4 - 4)} \\ &= -1 - \mathrm{i}\sqrt{\lambda} + O(\lambda) \\ &= n_1 + \mathrm{i}n_2 \end{aligned}$$

and we obtain the leading real, symmetric matrix (5.7) with the only difference that m_1, m_2, n_1 and n_2 are now different from the case $\tau = 0$. Because of the complexity of this matrix, we again can not obtain the value of $\theta_0 \in (0, \frac{\pi}{4})$. For this reason, we use numerical estimates as presented in *Figure 5.2*

When $\tau \in [0,4]$, we have *Figure 5.3* and *Figure 5.4* according as τ is taken from the intervals [0,2] and [2,4] respectively.

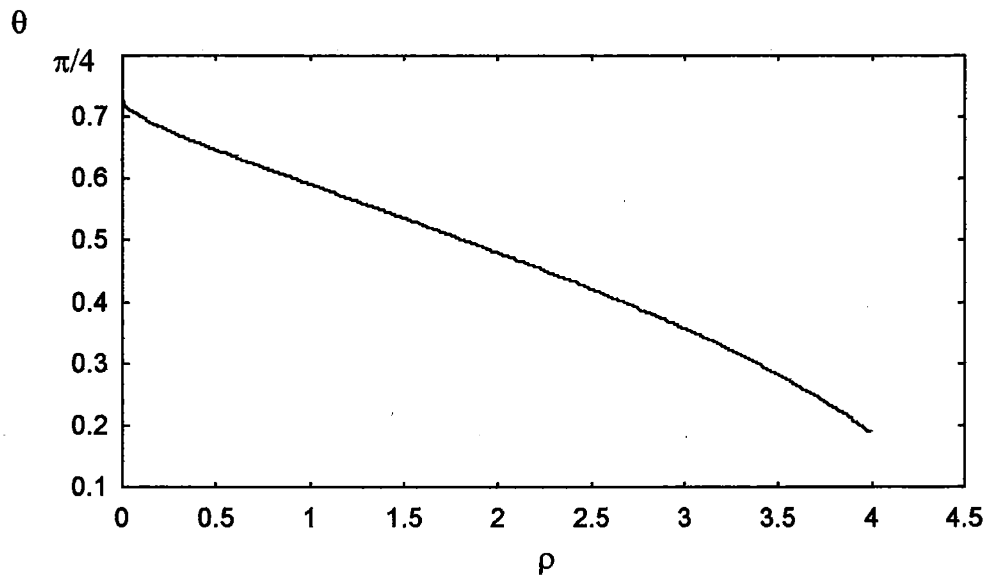

Figure 5.2 The graph $\{(\rho,\theta) \colon \chi(\rho,\theta) = 0\}$, $p_n = w_n = 1, q_n = -4$.

When $\tau \in (0,4)$, we have asymptotic results as $\rho \to 0$ to support the numerical estimates. In this case the $m_\tau(\lambda)$ and $n_\tau(\lambda)$ functions are given by

$$\begin{aligned} n_\tau(\lambda) := & \, 1 - \frac{-\lambda+\tau}{2} - \frac{1}{2}\sqrt{(-\lambda+\tau)(-\lambda+\tau-4)} \\ = & \, 1 - \frac{\tau}{2} - \frac{\mathrm{i}}{2}\sqrt{\tau(4-\tau)} + O(\lambda) \\ = & \, n_1 + \mathrm{i} n_2 \end{aligned}$$

and

$$\begin{aligned}m_\tau(\lambda) :&= 1 - \frac{\lambda+\tau}{2} + \frac{1}{2}\sqrt{(\lambda+\tau)(\lambda+\tau-4)}\\ &= 1 - \frac{\lambda+\tau}{2} + \frac{1}{2}\sqrt{\tau(\tau-4)(1+\frac{\lambda}{\tau})(1+\frac{\lambda}{\tau-4})}\\ &= 1 - \frac{\lambda+\tau}{2} + \frac{\mathrm{i}}{2}\sqrt{\tau(4-\tau)}[1+\frac{\lambda}{2}(\frac{1}{\tau}+\frac{1}{\tau-4})+O(\lambda^2)]\\ &= 1 - \frac{\tau}{2} + \frac{\mathrm{i}}{2}\sqrt{\tau(4-\tau)} + O(\lambda)\\ &= m_1 + \mathrm{i}m_2.\end{aligned}$$

Now, taking $\lambda = \rho e^{\mathrm{i}\theta}$, we obtain the following asymptotic entries for the real-symmetric matrix P, with error of order $O(\rho^4)$:

$$P = 4\rho^3\sqrt{\tau(4-\tau)}\begin{pmatrix}\sin\theta & -\cos(2\theta)\sin\theta & 0 & -\cos(2\theta)\cos\theta\\ \cdot & \sin\theta & -\cos(2\theta)\cos\theta & 0\\ \cdot & \cdot & \sin\theta & \cos(2\theta)\sin\theta\\ \cdot & \cdot & \cdot & \sin\theta\end{pmatrix}. \tag{5.10}$$

After some tedious calculations, we find out that the least eigenvalue of the leading matrix in $P(\rho,\theta)$ is $\ell = 4\rho^3\sqrt{\tau(4-\tau)}[\sin\theta - \cos(2\theta)]$, and the value of ℓ remains non-negative for some $\theta \geq \frac{\pi}{6}$ which proves that as $\rho \to 0$, θ tends to $\frac{\pi}{6}$ in the graphs of *Figure 5.3* and *Figure 5.4* when $\tau \in (0,4)$. This result coincides with the numerical estimates, since for small ρ ($\rho = 2.19464 \times 10^{-12}$) and for $\tau \in (0,4)$ the graph

$$\{(\rho,\theta) : \chi(\rho,\theta) = 0\}$$

always starts from $\frac{\pi}{6} = 0.52359878$ and either decreases monotonically or increases to a value (θ_0, ρ_0) and decreases again. In the first case the best constant in the inequality is given by $K = sec(2\frac{\pi}{6}) = 2$ and the unique equalising sequence from $D_{T(M^2)}$ is the null sequence. The second case occurs for $0 < \tau < \tau_0$ and $4 - \tau_0 < \tau < 4$, say, where $0.610962 < \tau_0 < 0.610963$. The analytic calculation of τ_0 is very complicated and we have had to resort to computational means.

In the second case, the best constant in the inequality is given by $K = \sec(2\theta_0)$, for some $\theta_0 > \frac{\pi}{6}$; also E is a singleton set $E = \{\rho_0\}$. *Table 5.1* gives a summary of what we have discussed.

A very interesting and important observation is that when $\tau = 0, 4$ the leading terms in the asymptotic value of $P(\rho, \theta)$, in (5.10), become zero, and so one has to examine the next terms. This accounts for the discontinuous behaviour of the graphs at $\rho = 0$.

(B) The case $\tau \in \mathbb{R} \backslash [0, 4]$:

In this case, as $\rho \to \infty$ the $m_\tau(\lambda)$ and $n_\tau(\lambda)$ functions satisfy

$$\begin{aligned} m_\tau(\lambda) &= 1 - \frac{\tau}{2} + \frac{1}{2}\sqrt{\tau(\tau-4)} + O(\lambda) \\ &= m_1 + im_2 \\ n_\tau(\lambda) &= 1 - \frac{\tau}{2} - \frac{1}{2}\sqrt{\tau(\tau-4)} + O(\lambda) \\ &= n_1 + in_2. \end{aligned}$$

Now, taking $\lambda = \rho e^{i\theta}$, we obtain the following asymptotic entries for the real-symmetric matrix P, with error of order $O(\rho^4)$:

$$\begin{aligned} P(\rho, \theta) &= 4\rho^3\sqrt{\tau(\tau-4)} \begin{pmatrix} 0 & 2\cos^3\theta & 0 & -\cos(2\theta)\sin\theta \\ \cdot & 0 & -\cos(2\theta)\sin\theta & 0 \\ \cdot & \cdot & 0 & 2\cos\theta\sin^2\theta \\ \cdot & \cdot & \cdot & 0 \end{pmatrix} \\ &= 4\rho^3\sqrt{\tau(\tau-4)}Q. \end{aligned}$$

The characteristic equation of $Q(\rho, \theta)$ is

$$\begin{aligned} det(Q - \ell I) &= \ell^4 - a\ell^2 + b \\ &= k^2 - ak + b \qquad (5.11) \\ &= 0 \end{aligned}$$

where $\ell = \pm\sqrt{k}$ and

$$a = 4\cos^2\theta\sin^4\theta + 2\cos^2(2\theta)\sin^2\theta + 4\cos^6\theta,$$

$$b = 16\cos^8\theta\sin^4\theta - 8\cos^4\theta cos^2(2\theta)\sin^4\theta + \cos^4(2\theta)\sin^4\theta.$$

Let's examine the quadratic in (5.11). Since $a > 0$ $\forall\theta \in (0, \frac{\pi}{4})$, we see that both of the roots can not be zero. Now, looking at the quartic in (5.11), we see that the least ℓ is $-\sqrt{k}$ and hence always remains negative. (Note that k has to be positive in order ℓ to be real since the matrix Q is real and symmetric.) It then follows that one of the eigenvalues of $P(\rho, \theta)$ remains strictly negative for all $\theta \in (0, \frac{\pi}{4})$ as $\rho \to 0$.

Hence there is no valid inequality for this case.

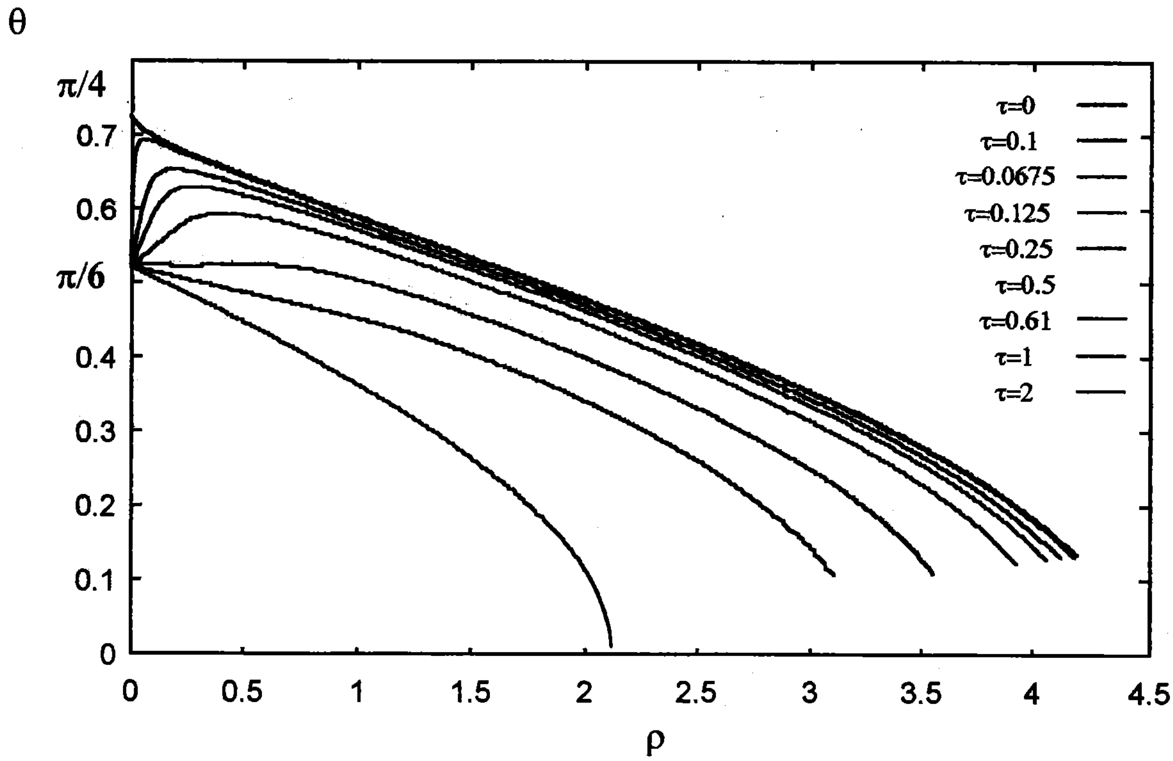

Figure 5.3 The graph $\{(\rho, \theta)\colon \chi(\rho, \theta) = 0\}$, $p_n = w_n = 1, q_n = -\tau$, $\tau \in [0, 2]$.

Table 5.1 The case $p_n = w_n = 1$ and $q_n = -\tau$, $\tau \in [0,4]$.

τ	θ_0	K	$E = \{\rho_0\}$
0.000000	0.72896068	8.8782005	$\emptyset$
0.001000	0.71436994	7.0631870	{0.01330}
0.010000	0.69459334	5.5367007	{0.05913}
0.067500	0.65427425	3.8572481	{0.19380}
0.125000	0.63115740	3.2936753	{0.27880}
0.250000	0.59459489	2.6851985	{0.40915}
0.500000	0.54178273	2.1359270	{0.54807}
0.610962	0.52359888	2.0000007	{0.51756}
1.000000	$\frac{\pi}{6}$	2.0000000	$\emptyset$
2.000000	$\frac{\pi}{6}$	2.0000000	$\emptyset$
3.000000	$\frac{\pi}{6}$	2.0000000	$\emptyset$
3.389038	0.52359888	2.0000007	{0.51756}
3.500000	0.54178273	2.1359270	{0.54807}
3.750000	0.59459489	2.6851985	{0.40916}
3.875000	0.63115740	3.2936753	{0.27880}
3.932500	0.65427425	3.8572481	{0.19380}
0.990000	0.69459334	5.5367007	{0.05913}
3.999000	0.71436994	7.0631870	{0.01330}
4.000000	0.72896068	8.8782005	$\emptyset$

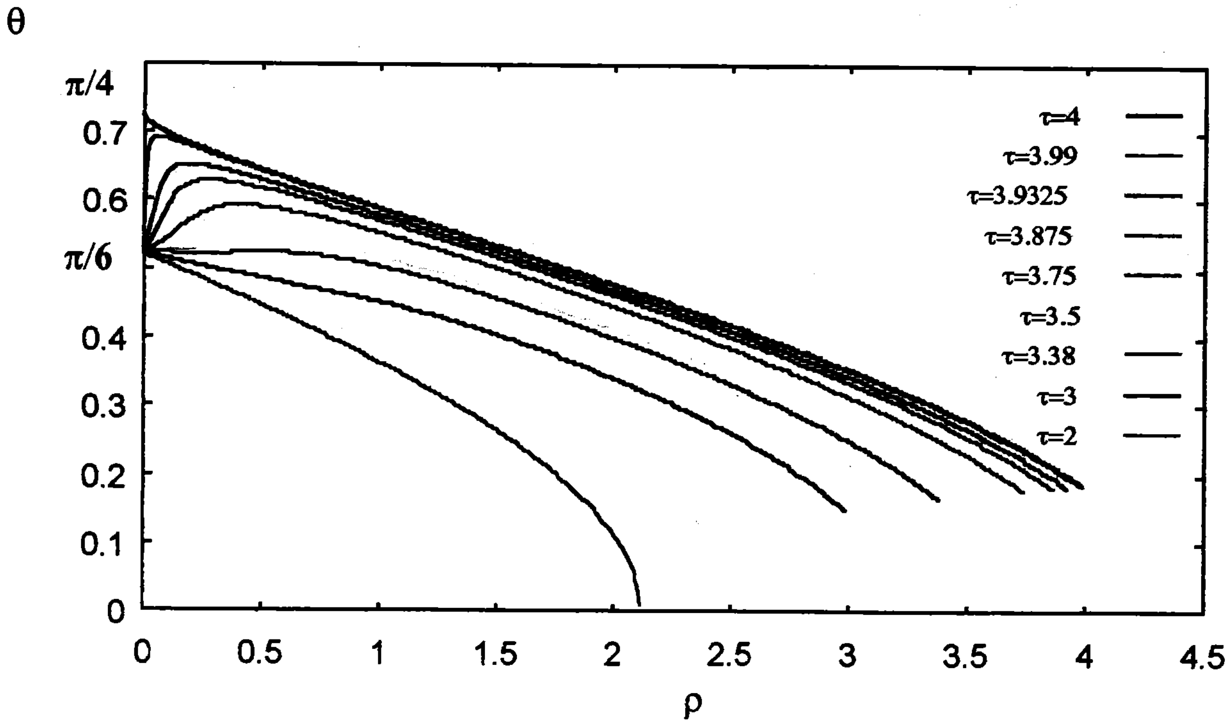

Figure 5.4 The graph $\{(\rho,\theta)\colon \chi(\rho,\theta) = 0\}$, $p_n = w_n = 1, q_n = -\tau$, $\tau \in [2,4]$.

In summary, after our numerical and analytical observations, with $0.610962 < \tau_0 \leq 0.610963$, for $\{x_n\}_{-1}^{\infty} \in D_{T(M^2)} \subseteq D_{T(M)}$ where M is defined by (5.8), we confidently conjecture the following:

(1) In view of the precise value $\frac{\pi}{6}$ for θ_0 when $\tau_0 < \tau < 4 - \tau_0$ (cf. *Table 5.1*), we have the inequality

$$\sum_{n=0}^{\infty} |-\Delta^2 x_{n-1} - \tau x_n|^2 \leq 2\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |\Delta^4 x_{n-2} + 2\tau\Delta^2 x_{n-1} + \tau^2 x_n|^2\Big)^{1/2}.$$

Here, the only equalising sequence is the null sequence.

(2) When $0 \leq \tau < \tau_0$ or $4 - \tau_0 < \tau \leq 4$ we have

$$\sum_{n=0}^{\infty} |-\Delta^2 x_{n-1} - \tau x_n|^2 \leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |\Delta^4 x_{n-2} + 2\tau\Delta^2 x_{n-1} + \tau^2 x_n|^2\Big)^{1/2} \tag{5.11}$$

in which $2 < K < 8.878201$ and E is a singleton set, except for $\tau = 0, 4$ in which case $8.8782 < K < 8.878201$ and $E = \emptyset$.

(3) When $\tau \in \mathbb{R}\backslash[0,4]$, there is no valid inequality in (5.11), i.e. $K = \infty$.

§5.3 The Legendre Problem

The Legendre polynomials $P_n(\lambda), n \geq -1$ satisfy the recurrence relation

$$(n+2)P_{n+1} + (n+1)P_{n-1} = \lambda(2n+3)P_n, \; n \geq 0 \tag{5.12}$$

with

$$P_{-1} = 1, \qquad P_0 = \lambda. \tag{5.13}$$

Let $\{Q_n(\lambda)\}_{-1}^{\infty}$ be another solution of (5.12) with the initial conditions

$$Q_{-1} = 0, \qquad Q_0 = -1. \tag{5.14}$$

Then $[PQ]_0 = 1$ and hence the solutions are linearly independent (cf. Lemma 3.2.5). In terms of the theory of this thesis, the difference expression we are dealing with is defined by

$$Mx_n := \begin{cases} \frac{1}{2n+3}[(n+2)x_{n+1} + (n+1)x_{n-1}] = \lambda x_n, & n \geq 0 \\ \frac{(x_0 - x_{-1})}{w_{-1}}, & n = -1 \end{cases} \tag{5.15}$$

in which $p_n = -(n+2)$, $q_n = w_n = 2n+3$ and $p_{n-1} + p_n + q_n = 0$. Note that, the Legendre Problem now becomes to solve the equation

$$Mx_n = \lambda x_n, \qquad n \geq 0, \lambda \in \mathbb{C}\backslash\mathbb{R} \tag{5.16}$$

with the initial conditions $x_{-1} = 1$, $x_0 = \lambda$.

In Brown, Evans and Littlejohn **[14]**, M is shown to be in the strong limit-point and hence in the limit-point case which is a necessary condition for M to satisfy our inequality. Another condition is that M^2 should be partially separated. This does indeed hold, since, given $x, M^2x \in \ell^2_w$ we can show that $Mx \in \ell^2_w$. For, we see first that

$$\begin{aligned} |Mx_n| &\leq \frac{1}{(2n+3)}\{(n+2)|x_{n+1}| + (n+1)|x_{n-1}|\} \qquad (n \geq -1) \\ &\leq |x_{n+1}| + |x_{n-1}| \end{aligned}$$

and this implies

$$\begin{aligned} \sum_{n=0}^{\infty} |Mx_n|^2 w_n &\leq \sum_{n=0}^{\infty} \left(|x_{n+1}|^2 + 2|x_{n+1}||x_{n-1}| + |x_{n-1}|^2\right) w_n \\ &\leq K \sum_{n=0}^{\infty} \left(|x_{n+1}|^2 w_{n+1} + |x_{n-1}|^2 w_{n-1}\right) \\ &< \infty \end{aligned}$$

since $x \in \ell^2_w$, $w_n = 2n+3$. In fact,

$$\sum_{n=0}^{\infty} |Mx_n|^2 w_n \leq K \sum_{n=0}^{\infty} |x_n|^2 w_n \leq K||x||^2, \qquad K \in (0, \infty)$$

and, so, $x \longmapsto Mx$ is a bounded map where $x = \{x_n\}_{-1}^{\infty}$. Similarly, we see that any power of the expression M is bounded in ℓ^2_w. Hence M^2 is partially separated.

We now introduce the Hellinger-Nevanlinna $m(\lambda)$ function related to the Legendre problem. We follow exactly the same steps as Brown, Evans and Littlejohn ([**14**], p.7) with the only exception that our sequences start from -1: define the solutions $\{P_n\}_{-1}^{\infty}$ and $\{Q_n\}_{-1}^{\infty}$ as in (5.13) and (5.14) respectively. Also define the solutions $\{\theta_n\}_{-1}^{\infty}$ and $\{\phi_n\}_{-1}^{\infty}$ of (5.12) with the initial conditions

$$\theta_{-1} = 1, \qquad \theta_0 = 0, \qquad \phi_{-1} = 0, \qquad p_{-1}\phi_0 = 1. \tag{5.17}$$

The functions $f_k(\lambda, z)$ are defined as in section §**1.3** with $k = 0, 1, 2, ...$ and we now introduce as well

$$F_k(\lambda, z) := -\frac{p_{k-1}Q_k(\lambda) + zQ_{k-1}(\lambda)}{p_{k-1}P_k(\lambda) + zP_{k-1}(\lambda),} \qquad z \in \mathbb{C}, \lambda \in \mathbb{C}_+, k = 0, 1, 2, ... \tag{5.18}$$

The functions $z \mapsto f_k(\lambda, z)$, $z \mapsto F_k(\lambda, z)$ map $\mathbb{R}$ onto circles $c(k, \lambda)$, $C(k, \lambda)$, and as $k \to \infty$ both sequences of circles are nested with the interiors $d(k, \lambda)$, and $D(k, \lambda)$ which are the images of $\mathbb{C}_-$. The counterpart of the $m(\lambda)$ function of section §**1.3** is now either the limit-point of the circles $C(k, \lambda)$ or a point within or on the limit-circle $C(\infty, \lambda)$. Also, there is similarly defined $\mathcal{M}(\lambda)$, say, with respect to the circles $C(k, \lambda)$. In general, if $c(\infty, \lambda)$ is a circle, we have as before that all solutions of (5.12) satisfy $\sum_{n=-1}^{\infty} |x_n|^2 w_n < \infty$ and this is also true if $C(\infty, \lambda)$ is a circle. Hence, $m(\lambda)$ is a limit-point if and only if $\mathcal{M}(\lambda)$ is a limit-point.

We know that $\mathcal{M}(\lambda)$ and hence $m(\lambda)$ is a limit-point, equivalently,

$$lim_{m\to\infty} p_m(x_{m+1}\bar{y}_m - x_m\bar{y}_{m+1}) = 0 \tag{5.19}$$

for all $x, y \in \ell^2_w$ such that $Mx, My \in \ell^2_w$. So, with

$$\psi_n = \theta_n + m(\lambda)\phi_n,$$

we obtain,

$$\begin{aligned}
2i\Im(\lambda)\sum_{n=0}^{\infty}|\psi_n|^2 w_n &= \sum_{n=0}^{\infty}(\overline{\psi_n}M\psi_n - \overline{M\psi_n}\psi_n) \\
&= p_{-1}(\psi_0\overline{\psi_{-1}} - \psi_{-1}\overline{\psi_0}) \\
&= m(\lambda) - \overline{m(\lambda)}
\end{aligned}$$

by (5.17), and we get

$$\sum_{n=0}^{\infty}|\psi_n|^2 w_n = \frac{\Im m(\lambda)}{\Im(\lambda)} \quad (\lambda \in \mathbb{C}_+).$$

Similarly, with $\Psi_n = Q_n + \mathcal{M}P_n$, we obtain from (5.13) and (5.14) that

$$\begin{aligned}
2i\Im(\lambda)\sum_{n=0}^{\infty}|\Psi_n|^2 w_n &= p_{-1}(\Psi_0\overline{\Psi_{-1}} - \Psi_{-1}\overline{\Psi_0}) \\
&= (-1+\lambda\mathcal{M})\overline{\mathcal{M}} - (-1+\bar{\lambda}\overline{\mathcal{M}})\mathcal{M} \\
&= 2i\Im(\mathcal{M}) - (\lambda - \bar{\lambda})|\mathcal{M}|^2
\end{aligned}$$

Hence

$$\sum_{n=-1}^{\infty}|\Psi_n|^2 w_n = \frac{\Im\mathcal{M}(\lambda)}{\Im(\lambda)} \quad (\lambda \in \mathbb{C}_+).$$

As M is in the limit-point case, we have $\Psi(\lambda) = A\psi(\lambda)$ for some $0 \neq A \in \mathbb{C}$ and so,

$$\begin{aligned}
m(\lambda) &= \frac{p_{-1}\psi_0}{\psi_{-1}} \\
&= \frac{p_{-1}\Psi_0}{\Psi_{-1}} \\
&= -\frac{1}{\mathcal{M}(\lambda)} - \lambda.
\end{aligned} \tag{5.20}$$

It can be shown that $\mathcal{M}(\lambda)$ is given by

$$\begin{aligned}\mathcal{M}(\lambda) &= \frac{1}{2}\int_{-1}^{1}\frac{dt}{t-\lambda} \quad (\lambda \in \mathbb{C}_+)\\ &= \frac{1}{2}\log(\frac{\lambda-1}{\lambda+1})\end{aligned} \tag{5.21}$$

(see Brown, Evans and Littlejohn, **[14]**, p.10) and hence $m(\lambda)$ is easily evaluated by the formula (5.20).

We are now ready to give some estimates for the best constant in the inequality

$$\sum_{n=0}^{\infty}|Mx_n|^2 w_n \le K\Big(\sum_{n=-1}^{\infty}|x_n|^2 w_n\Big)^{1/2}\Big(\sum_{n=1}^{\infty}|M^2 x_n|^2 w_n\Big)^{1/2} \tag{5.22}$$

in which M is given by (5.15). The real, symmetric 4×4 matrix that is equivalent to the inequality (5.22) is evaluated by (4.24), in which $w_{-1} = 1$ and

$$m(\lambda) = -\frac{1}{\mathcal{M}(\lambda)} - \lambda, \qquad \lambda \in \mathbb{C}_+$$

$$n(\lambda) = m(-\lambda) = -\frac{1}{\mathcal{M}(-\lambda)} + \lambda, \qquad \lambda \in \mathbb{C}_+.$$

According to the numerical estimates, the graph $\{(\rho,\theta) : \chi(\rho,\theta) = 0\}$, as it is shown by *Figure 5.5*, starts at $\theta_0 = \frac{\pi}{6}$ with ρ near 0 and then decreases monotonically. Hence the best constant is $K = 2$ and $E = \emptyset$, i.e. the only equalising sequence is the null sequence.

We are now going to give asypmtotic results as $\rho \to 0$ to confirm the numerical estimates. The $m(\lambda)$ functions are

$$\begin{aligned}m(\lambda) &= -\frac{1}{\mathcal{M}(\lambda)} - \lambda\\ &= -\frac{2}{\log\left(\frac{\lambda-1}{\lambda+1}\right)} - \lambda\\ &= \frac{2i}{\pi} + O(\rho),\end{aligned}$$

and

$$\begin{aligned} n(\lambda) &= m(-\lambda) \\ &= -\frac{2}{\log\left(\frac{-\lambda-1}{-\lambda+1}\right)} + \lambda \\ &= \frac{-2i}{\pi} + O(\rho). \end{aligned}$$

Using these asymptotics, the real, symmetric matrix $P(\rho, \theta)$ becomes

$$P(\rho,\theta) = \frac{16}{\pi}\rho^3 \begin{pmatrix} \sin\theta & -\cos(2\theta)\sin\theta & 0 & -\cos(2\theta)\cos\theta \\ \cdot & \sin\theta & -\cos(2\theta)\cos\theta & 0 \\ \cdot & \cdot & \sin\theta & \cos(2\theta)\sin\theta \\ \cdot & \cdot & \cdot & \sin\theta \end{pmatrix}$$
$$+ O(\rho^4)$$

which is exactly the same matrix as (5.10). Hence, we have the same conclusions: the least eigenvalue of the leading matrix in $P(\rho,\theta)$ is $\ell = \frac{16\rho^3}{\pi}[\sin\theta - \cos(2\theta)]$, and the value of ℓ remains non-negative for some $\theta \geq \frac{\pi}{6}$ which proves that as $\rho \to 0$

$$\theta_0 = \inf\left\{\theta \in (0, \frac{\pi}{4}) : \ell(\rho,\theta) \geq 0\right\} = \frac{\pi}{6}.$$

Hence, $\theta_0 = \frac{\pi}{6}$ as $\rho \to 0$.

§5.4 The Legendre Problem with a $-\tau$ shift value in q_n

This is the case where $p_n = -(n+2)$, $w_n = 2n+3$ and $q_n = 2n+3-\tau$ in (5.15), i.e.,

$$Mx_n := \begin{cases} \frac{1}{2n+3}[(n+2)x_{n+1} - \tau x_n + (n+1)x_{n-1}] = \lambda x_n, & n \geq 0 \\ \frac{(x_0 - x_{-1})}{w_{-1}}, & n = -1. \end{cases} \tag{5.23}$$

Now, using the criterion of Hinton-Lewis [**39**], that is $\sum_{n=0}^{\infty} \frac{(w_n w_{n+1})^{1/2}}{|p_n|} = \infty$, we see that M again satisfies the required LP condition. In a similar way to the original Legendre problem in section §5.3, M^2 satisfies the PS condition as well.

The $m_\tau(\lambda)$ and $n_\tau(\lambda)$ functions related to this problem are now given by the formulas $m_\tau(\lambda) = m(\lambda + \tau)$ and $n_\tau(\lambda) = m(-\lambda + \tau)$.

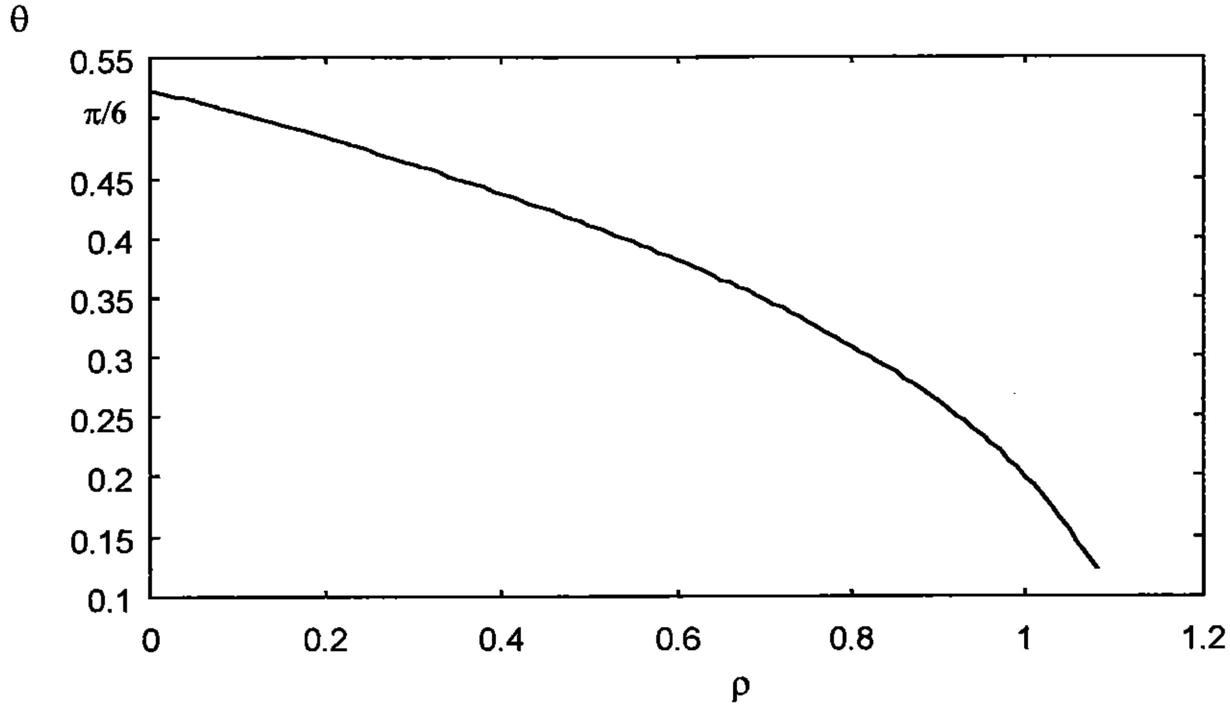

Figure 5.5 The graph $\{(\rho,\theta)\colon \chi(\rho,\theta) = 0\}$ *for the Legendre problem.*

The inequality under consideration is now

$$\sum_{n=0}^{\infty} |Mx_n|^2 w_n \le K\Big(\sum_{n=-1}^{\infty} |x_n|^2 w_n\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |M^2 x_n|^2 w_n\Big)^{1/2} \qquad (5.24)$$

in which M is given by (5.23). The real, symmetric 4×4 matrix that is equivalent to the inequality (5.24) is again evaluated by (4.24). There are two cases to examine now, and the results are presented in (A) and (B) below.

(A) $\tau \in \mathbb{R}\backslash[-1,1]$

In this case, we have no analytical results but numerical estimates show that the graph $\{(\rho,\theta) : \chi(\rho,\theta) = 0\}$ starts from $\frac{\pi}{4}$ and decreases monotonically. This gives no valid inequality in (5.24).

(B) $\tau \in [-1, 1]$

This is examined in two cases:

(i) $\tau \in [-1, 1]\backslash(-0.66118628, 0.66118628)$

In this case, when $\tau = \pm 1$ the graph starts from a θ_0, $0.71552 < \theta_0 < 0.71553$, with ρ near 0 and decreases monotonically. This means that the best constant satisfies $7.1786 < K < 7.1787$ and $E = \emptyset$, i.e. the only equalising sequence is the null sequence (see *Figure 5.6* and *Figure 5.7*). When $\tau \neq \pm 1$, the graph $\{(\rho, \theta) : \chi(\rho, \theta) = 0\}$ starts from $\frac{\pi}{6}$ and increases to a value $\theta_0 < 0.7155203$, giving the best constant $K = sec(2\theta_0) < 7.1786$ and a singleton set E. These results are presented in *Figure 5.8* and *Figure 5.9*, according as $\tau \in [-1, 0]$ or $\tau \in [0, 1]$. See also *Table 5.2* for a summarised information.

(ii) The case $|\tau| < \tau_0$, $0.66118628 \leq \tau_0 < 0.66118629$

In this case, the graph $\{(\rho, \theta) : \chi(\rho, \theta) = 0\}$ starts from $\frac{\pi}{6}$ with ρ near 0 and decreases monotonically, giving the best constant $K = 2$ and $E = \emptyset$, i.e. there is no non-zero equalising sequence. See *Table 5.2* for more information.

We do not have asymptotic results for $\tau \in \mathbb{R}\backslash(-1, 1)$, but we have the following asymptotic results as $\rho \to 0$ for $\tau \in (-1, 1)$: in this case, for $\Im(\lambda) > 0$ and $\lambda = \rho e^{i\theta}$, the $m(\lambda)$ functions are

$$\begin{aligned} m_\tau(\lambda) &= \frac{-2}{\log\left(\frac{\lambda+\tau-1}{\lambda+\tau+1}\right)} - (\lambda + \tau) \\ &= \frac{-2}{\log\left(\frac{\tau-1}{\tau+1}\right)} - \tau + O(\rho) \end{aligned}$$

$$
\begin{aligned}
&= \frac{-2}{\log\left(\frac{1-\tau}{1+\tau}\right) + \pi \mathrm{i}} - \tau + O(\rho) \\
&= m_1 + \mathrm{i} m_2 + O(\rho) \\
n_\tau(\lambda) &= \frac{-2}{\log\left(\frac{-\lambda+\tau-1}{-\lambda+\tau+1}\right)} - (-\lambda + \tau) \\
&= \frac{-2}{\log\left(\frac{1-\tau}{-1-\tau}\right)} - \tau + O(\rho) \\
&= \frac{-2}{\log\left(\frac{1-\tau}{1+\tau}\right) - \pi \mathrm{i}} - \tau + O(\rho) \\
&= n_1 + \mathrm{i} n_2 + O(\rho)
\end{aligned}
$$

where

$$
m_1 = -\tau - \frac{2\log\left(\frac{1-\tau}{1+\tau}\right)}{\log^2\left(\frac{1-\tau}{1+\tau}\right) + \pi^2}, \; m_2 = \frac{2\pi}{\log^2\left(\frac{1-\tau}{1+\tau}\right) + \pi^2}, \; n_1 = m_1, \; n_2 = -m_2.
$$

Inserting these values in (4.24), the leading matrix $P(\rho, \theta)$ becomes

$$
\begin{aligned}
&P(\rho, \theta) \\
&= \frac{16\rho^3\pi}{\log^2\left(\frac{1-\tau}{1+\tau}\right) + \pi^2} \begin{pmatrix} \sin\theta & -\cos(2\theta)\sin\theta & 0 & -\cos(2\theta)\cos\theta \\ . & \sin\theta & -\cos(2\theta)\cos\theta & 0 \\ . & . & \sin\theta & \cos(2\theta)\sin\theta \\ . & . & . & \sin\theta \end{pmatrix} \\
&+ O(\rho^4).
\end{aligned}
$$

Similar to the matrix (5.10), the least eigenvalue $\ell(\rho, \theta)$ of the leading matrix $P(\rho, \theta)$ is

$$
\ell(\rho, \theta) = \frac{16\rho^3\pi}{\log^2\left(\frac{1-\tau}{1+\tau}\right) + \pi^2}[\sin\theta - \cos(2\theta)],
$$

and the value of $\ell(\rho, \theta)$ remains non-negative for some $\theta \geq \frac{\pi}{6}$ which proves that as $\rho \to 0$

$$
\theta_0 = \inf\left\{\theta \in (0, \frac{\pi}{4}) : \ell(\rho, \theta) \geq 0\right\} = \frac{\pi}{6}.
$$

Hence, $\theta_0 = \frac{\pi}{6}$ as $\rho \to 0$ for all $\tau \in (-1, 1)$. Note that, similar to (5.10), when $\tau = -1, 1$ in above matrix, the leading terms become zero, and so, one has to examine the next terms. This again is coused by the discontinuous behaviour of the graphs at $\rho = 0$.

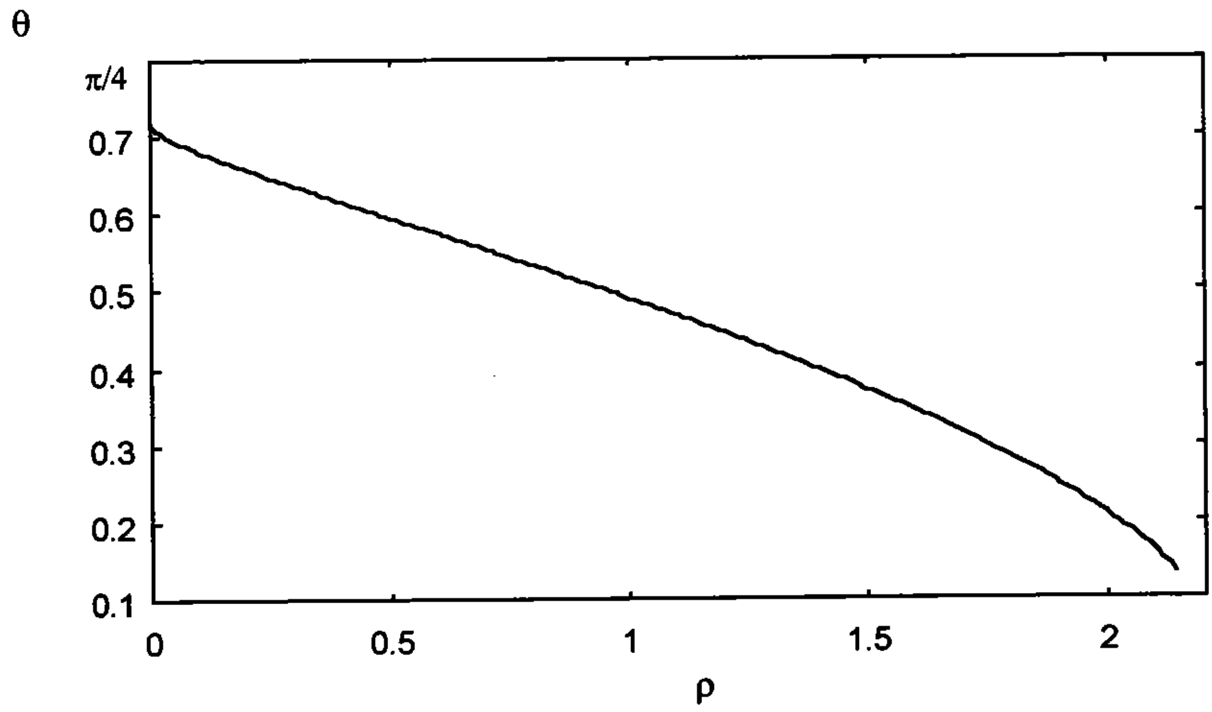

Figure 5.6 The graph $\{(\rho,\theta)\colon\ \chi(\rho,\theta)=0\}$, $q_n = 2n+3-\tau$, $\tau=-1$.

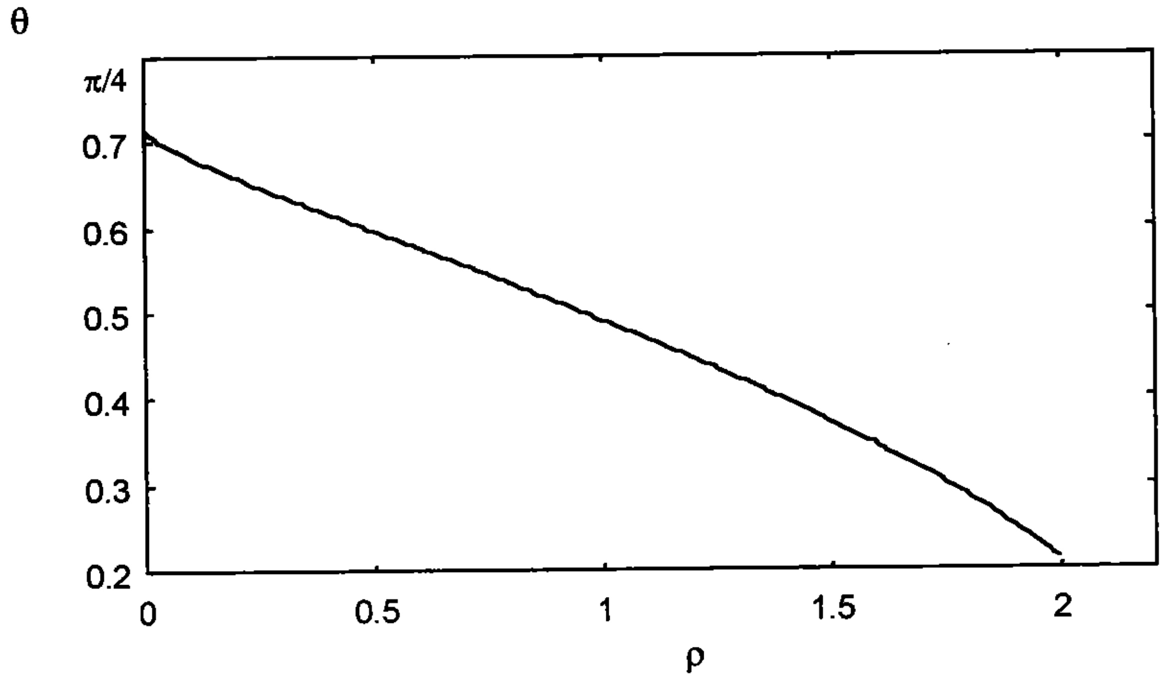

Figure 5.7 The graph $\{(\rho,\theta)\colon\ \chi(\rho,\theta)=0\}$, $q_n = 2n+3-\tau$, $\tau=1$.

Table 5.2 Shifted Legendre problem, $q_n = 2n + 3 - \tau$, $\tau \in [-1, 1]$.

τ	θ_0	K	$E = \{\rho_0\}$
-1.00000	0.7155203	7.178688	$\emptyset$
-0.99900	0.7021850	6.036489	{0.013327}
-0.93250	0.6280650	3.230251	{0.155717}
-0.87500	0.5977690	2.728417	{0.216576}
-0.75000	0.5494347	2.199719	{0.293944}
-0.66119	0.5235988	2.00000003	{0.276030}
-0.50000	$\frac{\pi}{6}$	2.000000	$\emptyset$
0.000000	$\frac{\pi}{6}$	2.000000	$\emptyset$
0.500000	$\frac{\pi}{6}$	2.000000	$\emptyset$
0.661186	0.5235988	2.00000003	{0.276030}
0.750000	0.5494347	2.199719	{0.293944}
0.875000	0.5977690	2.728417	{0.216576}
0.932500	0.6280650	3.230251	{0.155717}
0.999000	0.7021850	6.036489	{0.013327}
1.000000	0.7155203	7.178688	$\emptyset$

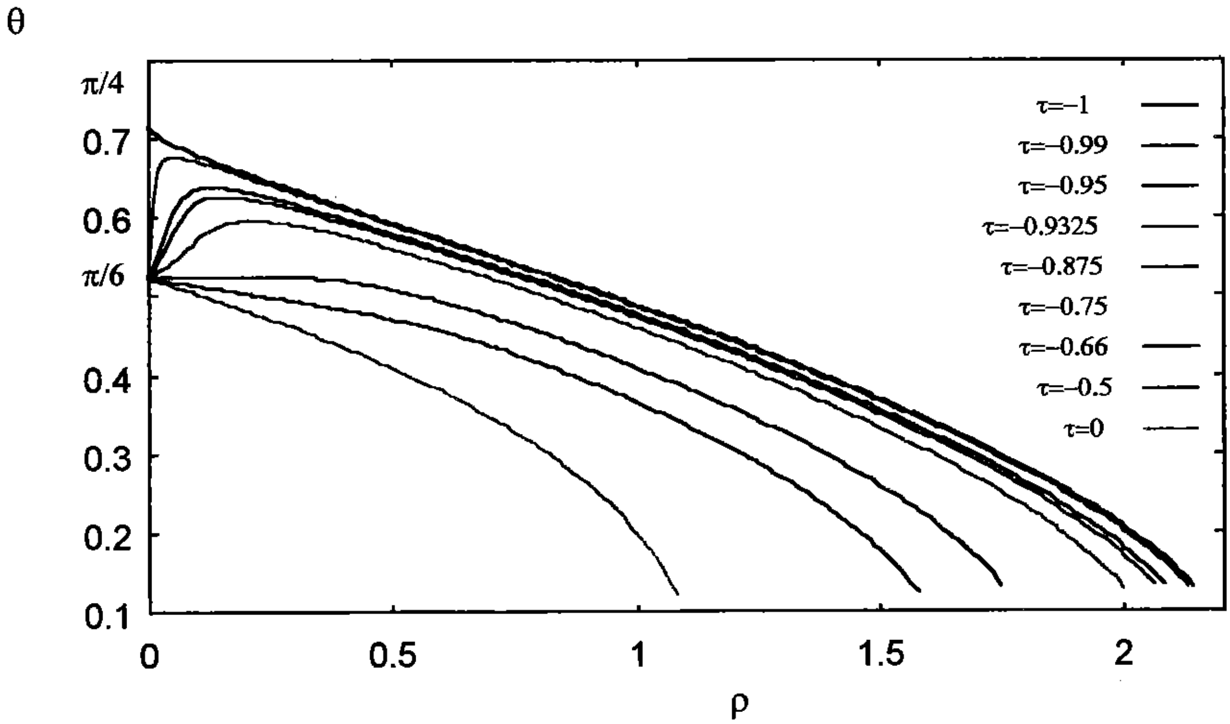

Figure 5.8 The graph $\{(\rho,\theta){:}\chi(\rho,\theta) = 0\}$ for shifted Legendre problem, $\tau \in [-1, 0]$.

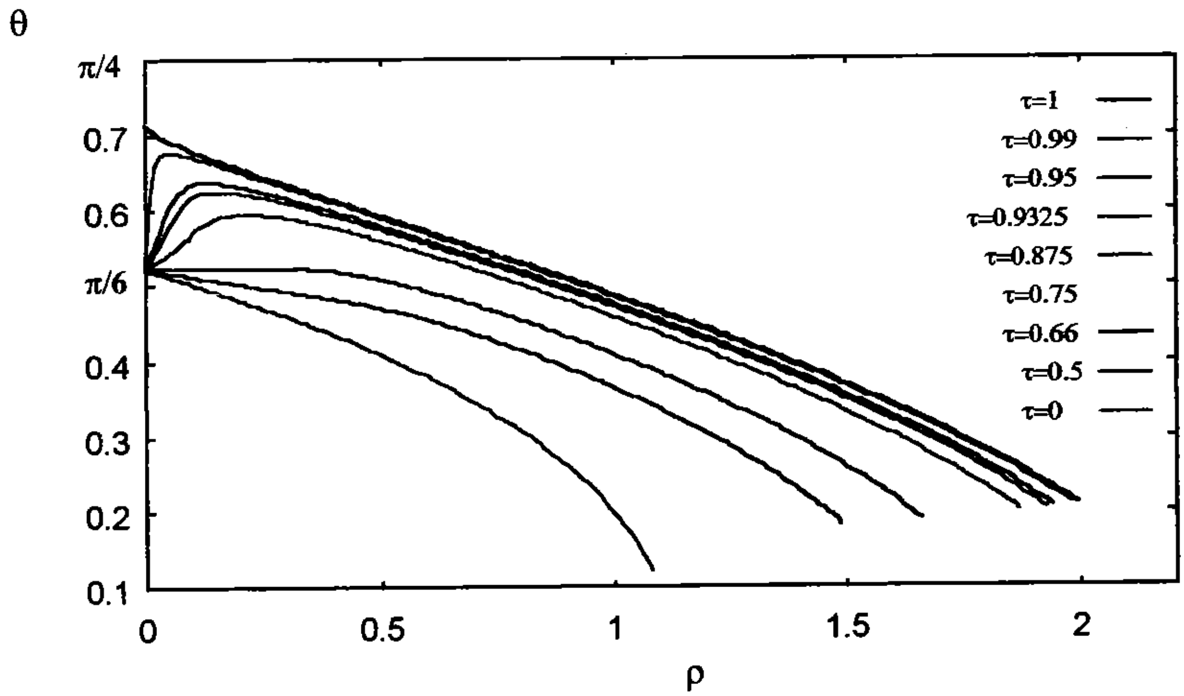

Figure 5.9 The graph $\{(\rho,\theta){:}\chi(\rho,\theta)=0\}$ *for shifted Legendre problem,*$\tau \in [0,1]$.

In summary, with $0.66118628 \le \tau < 0.66118629$ we have the following conjectures for $\{x_n\}_{-1}^{\infty} \in D_{T(M^2)} \subseteq D_{T(M)}$ where M is defined by (5.23):

(1) In view of the precise value $\frac{\pi}{6}$ for θ_0 when $|\tau_0| \le 0.66118628$, the following inequality holds:

$$\sum_{n=0}^{\infty} |\frac{1}{2n+3}[(n+2)x_{n+1} - \tau x_n + (n+1)x_{n-1}]|^2(2n+3)$$
$$\le 2\Big(\sum_{n=-1}^{\infty} |x_n|^2(2n+3)\Big)^{1/2}$$
$$\Big(\sum_{n=1}^{\infty} |\frac{1}{2n+3}[(n+2)Mx_{n+1} - \tau M x_n + (n+1)Mx_{n-1}]|^2(2n+3)\Big)^{1/2}$$

where Mx_n is defined by (5.23). The null sequence is the only equalising sequence.

(2) When $\tau \in [-1,1]\backslash(-\tau_0, \tau_0)$, we have

$$\sum_{n=0}^{\infty} |\frac{1}{2n+3}[(n+2)x_{n+1} - \tau x_n + (n+1)x_{n-1}]|^2(2n+3)$$
$$\leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2(2n+3)\Big)^{1/2}$$
$$\Big(\sum_{n=1}^{\infty} |\frac{1}{2n+3}[(n+2)Mx_{n+1} - \tau Mx_n + (n+1)Mx_{n-1}]|^2(2n+3)\Big)^{1/2}$$

in which $2 < K < 7.178689$ and $E = \{\rho_0\}$ is a singleton set, except for $\tau = \pm 1$ in which case $7.178688 < K < 7.178689$ and $E = \emptyset$.

(3) When $\tau \in \mathbb{R}\backslash[-1,1]$, there is no valid inequality in (5.24).

§5.5 The case where $p_n = w_n = 1$, $q_n = 0$ for the inequality (4.39)

In this case the expression M is given by

$$Mx_n := \begin{cases} [-x_{n+1} + 2x_n - x_{n-1}] = \lambda x_n, & n \geq 0 \\ x_0 - x_{-1}, & n = -1. \end{cases} \tag{5.25}$$

This is the case in section §5.1 with the exception that Mx_{-1} is involved in (5.25). The proofs of the conditions that M and M^2 are required to satisfy remain the same as in section §5.1. The $m(\lambda)$ and $n(\lambda)$ functions are again given by

$$m(\lambda) := 1 - \frac{\lambda}{2} + \frac{1}{2}\sqrt{\lambda(\lambda-4)}, \qquad \lambda \in \mathbb{C}_+$$

$$n(\lambda) := 1 + \frac{\lambda}{2} - \frac{1}{2}\sqrt{\lambda(\lambda+4)}, \qquad \lambda \in \mathbb{C}_+.$$

We are now ready to examine the best possible value of the constant K and the validity of

$$\sum_{n=-1}^{\infty} |Mx_n|^2 \leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2} \Big(\sum_{n=1}^{\infty} |M^2 x_n|^2\Big)^{1/2}$$

which is

$$\sum_{n=-1}^{\infty} |\Delta^2 x_{n-1}|^2 \leq K\Big(\sum_{n=-1}^{\infty} |x_n|^2\Big)^{1/2}\Big(\sum_{n=1}^{\infty} |\Delta^4 x_{n-2}|^2\Big)^{1/2}$$

or

$$||\Delta^2 x||^2 + |x_{-1}|^2 \leq K||x||\,||\Delta^4 x||. \tag{5.26}$$

where $||.||$ is ℓ^2 norm and $(\Delta^2 x)_n = \Delta^2 x_n$, $(\Delta^4 x)_n = \Delta^4 x_n$. This is a stronger inequality than the norm inequality (5.6), as there is an $|x_{-1}|^2$ term on the left hand side of (5.26). As we can see from the *Figure 5.10*, the computational results give no valid inequality as the graph $\{(\rho, \theta)\colon\ \chi(\rho, \theta) = 0\}$ starts from $\frac{\pi}{4}$ and decreases monotonically, which gives $\theta_0 = \frac{\pi}{4}$ and the best constant $K = \infty$.

We are now going to give the asymptotic results which demonstrates a clearer picture on the non-validity of (5.26).

We examine the asymptotics for $|\lambda| \to \infty$ and for $|\lambda| \to 0$ as follows.

(A) The case $\rho \to \infty$: In this case, the asymptotics for $m(\lambda)$ functions take the form

$$m(\lambda) = -\frac{1}{\lambda} + O(\frac{1}{\lambda^2}) = m_1 + im_2$$

and

$$n(\lambda) = \frac{1}{\lambda} + O(\frac{1}{\lambda^2}) = n_1 + in_2$$

so that

$$\begin{aligned} m_1 &= -\frac{\cos\theta}{\rho} + O(\frac{1}{\rho^2}) \\ m_2 &= \frac{\sin\theta}{\rho} + O(\frac{1}{\rho^2}) \\ n_1 &= \frac{\cos\theta}{\rho} + O(\frac{1}{\rho^2}) \\ n_2 &= -\frac{\sin\theta}{\rho} + O(\frac{1}{\rho^2}). \end{aligned}$$

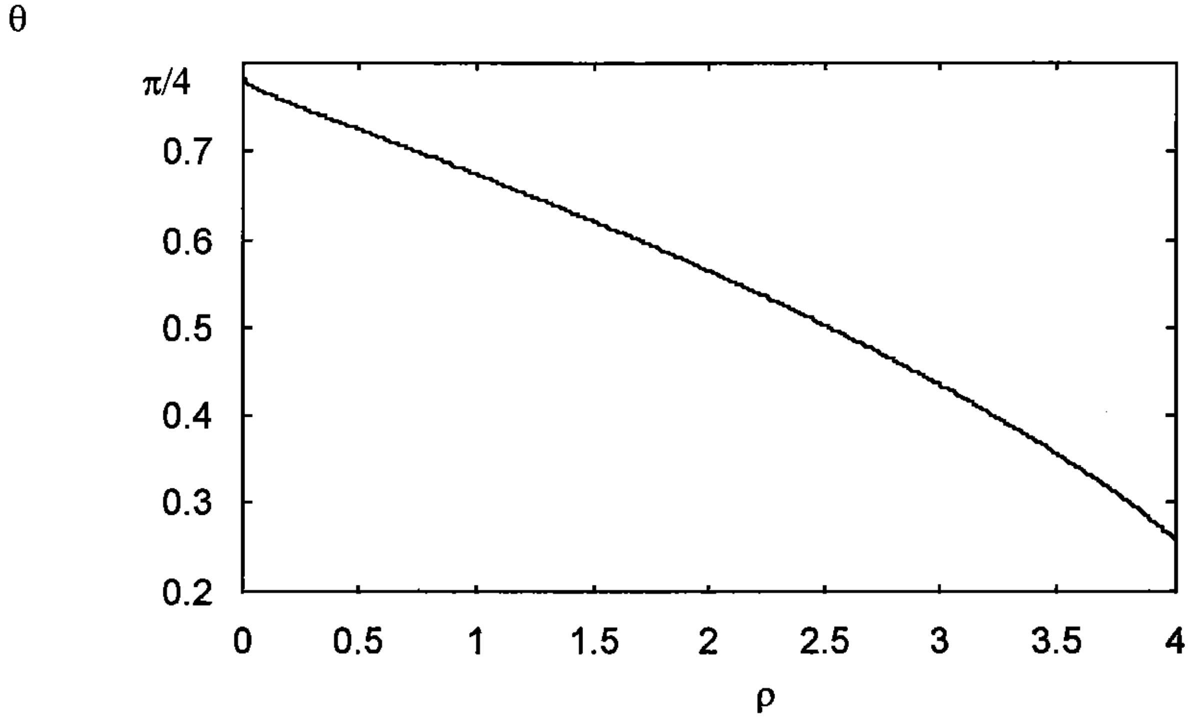

Figure 5.10 The graph $\{(\rho,\theta)\colon \chi(\rho,\theta)=0\}$ *for the inequality (4.39).*

We then obtain the real and symmetric matrix $P'(\rho,\theta)$ as follows:

$$P'(\rho,\theta)=4\rho^4\begin{pmatrix} 1+\cos^2(2\theta) & 1+\cos^2(2\theta) & -\frac{1}{2}\sin(4\theta) & -\frac{1}{2}\sin(4\theta)\\ . & 1+\cos^2(2\theta) & -\frac{1}{2}\sin(4\theta) & -\frac{1}{2}\sin(4\theta)\\ . & . & \sin^2(2\theta) & \sin^2(2\theta)\\ . & . & . & \sin^2(2\theta)\end{pmatrix}$$
$$+\,O(\rho^3)$$

and

$$|P'-\ell I|=\ell^2(\ell-16\rho^4\sin^2\theta)(\ell-16\rho^4\cos^2\theta)$$

which implies that the least eigenvalue of $P'(\rho,\theta)$ is $\ell=0$ for $\theta\in(0,\frac{\pi}{4})$ as $\rho\to\infty$. Thus, in the graph $\{(\rho,\theta):\chi(\rho,\theta)=0\}$, $\theta=0$ for all large enough ρ. For the computational results see *Figure 5.10* which agrees with the asymptotic results.

(B) The case $\rho \to 0$: In this case, the asymptotics for $m(\lambda)$ functions are $m(\lambda) = 1 + \mathrm{i}\sqrt{\lambda} + O(\lambda) = m_1 + \mathrm{i}m_2$ and $n(\lambda) = 1 - \sqrt{\lambda} + O(\lambda) = n_1 + \mathrm{i}n_2$ so that

$$\begin{aligned}
m_1 &= 1 - \sqrt{\rho}\sin(\frac{\theta}{2}) + O(\rho) \\
m_2 &= \sqrt{\rho}\cos(\frac{\theta}{2}) + O(\rho) \\
n_1 &= 1 - \sqrt{\rho}\cos(\frac{\theta}{2}) + O(\rho) \\
n_2 &= -\sqrt{\rho}\sin(\frac{\theta}{2}) + O(\rho).
\end{aligned}$$

We then obtain the real and symmetric matrix $P'(\rho,\theta)$ as follows:

$$\begin{aligned}
P'(\rho,\theta) &= 8\rho^3\cos(2\theta)\begin{pmatrix} -\sin^2(\frac{\theta}{2}) & -\frac{1}{2}\sin\theta & -\frac{1}{2}\sin\theta & \sin^2(\frac{\theta}{2}) \\ \cdot & -\cos^2(\frac{\theta}{2}) & -\cos^2(\frac{\theta}{2}) & \frac{1}{2}\sin\theta \\ \cdot & \cdot & -\cos^2(\frac{\theta}{2}) & \frac{1}{2}\sin\theta \\ \cdot & \cdot & \cdot & -\sin^2(\frac{\theta}{2}) \end{pmatrix} \\
&\quad + O(\rho^4) \\
&= 8\rho^3\cos(2\theta)Q + O(\rho^4)
\end{aligned}$$

and,

$$|Q - \ell I| = \ell^3(\ell + 2).$$

We see that the least eigenvalue of $P'(\rho,\theta)$ is $\ell' = -16\rho^3\cos(2\theta)$, and it remains negative for all $\theta \in (0, \frac{\pi}{4})$. This gives no valid inequality in (5.26).

Chapter 6

CONCLUDING REMARKS

In Chapter 3, we discussed the partial separation condition on the real, symmetric difference expression M^2 and gave necessary and sufficient conditions for M^2 to be partially separated. The problem of giving a general criterion for any power of M to be partially separated hasn't been considered in this doctoral thesis, but we believe that the proof of the general criterion will also follow easily from the same steps as Theorem 3.4.15.

Two reasons for investigating a general criteria for the partial separation of any power of M are because:

(1) it gives necessary and sufficient conditions for any power of M to be LP when M is LP,

(2) it is a necessary condition in the investigation of general Kolmogorov type inequalities of the $2n$th power, namely, for j=-1,0,...,n-1,

$$\left(\sum_{k=j}^{\infty}|M^n x_k|^2 w_k\right)^2 \leq K_n\left(\sum_{k=-1}^{\infty}|x_k|^2 w_k\right)^{1/2}\left(\sum_{k=2n-1}^{\infty}|M^{2n}x_k|^2 w_k\right)^{1/2}. \quad (6.1)$$

Note that, for any value of j, $-1 \leq j \leq n-1$, we obtain a different Kolmogorov type inequality of $2n$th order and both theoretical and computational methods we have used are sure to work for these inequalities. We don't have an answer for the question that which of these is the difference analogue of the inequality (1.7) of Beynon [8]. This needs investigation. But, the way it came out for our inequality, which is the case where $n = 1$ in (6.1), may suggest that the difference analogue of (1.7) is the case where $j = n - 1$ (see the comments at the end of §4.3). We leave these problems for future research.

We examined our inequality for the recurrence relation of Legendre Orthogonal Polynomials (cf. Chapter 5, §5.3). More examples for the recurrence expressions of other classical orthogonal polynomials, such as Hermite and Laguerre, needs investigation. We also leave these problems for future research.

BIBLIOGRAPHY

[**1**] Akhiezer, N.I. (1965)
The Classical Moment Problem. Oliver and Boyd, Edinburg and London. (Tra. from the Russian edition 1960).

[**2**] Akhiezer, N.I. and Glazman, J.M. (1993)
Theory of Linear Operators in Hilbert Space, part 1&2. Dover Publications, Inc. New York. (Originally published: New York: F. Ungar Pub. Co., 1961).

[**3**] Atkinson, F.V. (1964)
Discrete and Continuous Boundary Problems. Academic Press, New York.

[**4**] Ben-Ammar, M. (1992)
Some Problems Associated With Linear Difference Operators. Ph.D. Thesis, University of Wales College Cardiff, UK.

[**5**] Bennewitz, C. (1972)
Symmetric relations on a Hilbert space. Conf. on the Theory of Ordinary and Partial Differential Equations, Dundee, Scotland, March 1972, Lecture Notes in Mathematics 280, Springer, 212-218.

[**6**] Bennewitz, C. (1977)
Spectral theory for pairs of differential operators. Arkiv för Matematik **Vol.15 no:1**, 33-61.

[**7**] Bennewitz, C. and Everitt, W.N. (1980)
Some remarks on the Titchmarsh-Weyl $m-$coefficient. *Tribute to Ake Pleijel.* (Proceedings of the Pleijel Conference Upsala, 1979), pp.49-108.

[**8**] Beynon, M.J. (1993)
Some problems in the investigation of the best constant in certain integral inequalities. Ph.D. Thesis, University of Wales College Cardiff, UK.

[**9**] Beynon, M.J., Brown, B.M. and Evans, W.D. (1993)
On an inequality of Kolmogorov type for a second-order differential expression. Proc. R. Soc. Lond. **A 442**, 555-569.

[10] Billigheimer, C.E. (1987)
Singular self-adjoint fourth-order difference operators. Canadian Math. Soc. Conference proceedings **Vol.8**, 57-86.

[11] Bradley, J.S. and Everitt, W.N. (1974)
On the inequality $\|f''\|^2 \leq K\|f\|\|f^{(4)}\|$. Quart. J. Math. Oxford (2) **25**, 241-252.

[12] Brown, B.M. and Evans, W.D. (1992)
On an extension of Copson's inequality for infinite series. Proc. R. Soc. Edinb., **121A**, 169-183.

[13] Brown, B.M., Evans, W.D. and Everitt, W.N. (1992)
HELP inequalities and serial inequalities. International Series of Numerical Mathematics. **103**, 269-305.

[14] Brown, B.M., Evans, W.D. and Littlejohn, L.L. (1992)
Discrete inequalities, orthogonal polynomials and the spectral theory of difference operators. Proc. R. Soc. Lond. **A 437**, 355-379.

[15] Brown, B.M., Evans, W.D. and Littlejohn, L.L. (1993)
Orthogonal polynomials and extensions of Copson's inequality. Journal of Computational and Applied Mathematics (N. Holland), **48**, 33-48.

[16] Brown, B.M., Kirby, V.G. and Pryce, J.D. (1989)
Numerical determination of the Titchmarsh-Weyl $m-$coefficient and its applications to HELP inequalities. Proc. R. Soc. Lond. **A 426**, 167-188.

[17] Brown, B.M., Kirby, V.G. and Pryce, J.D. (1991)
A numerical method for the determination of the Titchmarsh-Weyl $m-$ coefficient. Proc. R. Soc. Lond. **A 435**, 535-549.

[18] Chaudhuri, J. and Everitt, W.N. (1969)
On the square of a formally self-adjoint differential expression. J. London Math. Soc. (2), **1 (1969)**, 661-673.

[19] Coddington, E.A. and Levinson, N. (1955)
Theory of Ordinary Differential Equations. McGraw-Hill (New York).

[20] Copson, E.T. (1977)
On two integral inequalities. Proc. R. Soc. Edinb. **A 77** (1977a), 325-328.

[21] Copson, E.T. (1979)
Two series inequalities. Proc. R. Soc. Edinb. **83A**, 109- 114.

[22] Ditzian, Z. (1983)
Discrete and shift Kolmogorov type inequalities. Proc. R. Soc. Edinb. **93A**, 307-317.

[23] Edmunds, D.E. and Evans, W.D. (1987)
Spectral Theory and Differential Operators. Oxford University Press.

[24] Evans, W.D. (1977)
On the essential self-adjointness of powers of Shrödinger-type operators. Proc. R. Soc. Edinb. **79A**, 61-77.

[25] Evans, W.D. and Everitt, W.N. (1982)
A return to the Hardy-Littlewood inequality. Proc. R. Soc. Lond. **A 380**, 447-486.

[26] Evans, W.D. and Everitt, W.N. (1991)
Hardy-Littlewood integral inequalities. In inequalities: Fifty years on from Hardy-Littlewood and Pólya. Lecture Notes in Pure and Appl. Math. **Vol.129** (Dekker, 1991).

[27] Evans, W.D., Everitt, W.N., Hayman, W.K. and Ruscheweyh, S. (1986)
On a class of integral inequalities of Hardy-Littlewood type. J. d'Anal. Math. **46**, 118-147.

[28] Evans, W.D. and Zettl, A. (1978)
Norm inequalities involving derivatives. Proc. R. Soc. Edinb. **A 82**, 51-70.

[**29**] Everitt, W.N. (1971/72)

On an extension to an integro-differential inequality of Hardy, Littlewood and Pólya. Proc. R. Soc. Edinb. **A 69**, 295-333.

[**30**] Everitt, W.N. and Jones, D.S. (1977)

On an integral inequality. Proc. R. Soc. Lond. **A 357**, 271-288.

[**31**] Everitt, W.N. and Zettl, A. (1977)

On a class of integral inequalities. J. Lond. Math. Soc. **17**, 291-303.

[**32**] Everitt, W.N. and Zettl, A. (1978)

Products of differential expressions without smoothness assumptions. Quaestiones Mathematicae. **3**, 67-82.

[**33**] Glazman, I.M. (1965)

Direct methods of qualitative spectral analysis of singular differential operators. Israel program for scientific translation., Jerusalem.

[**34**] Glazman, I.M. (1950)

On the theory of singular differential operators. Uspehi Matematičeskih Nauk (N.S.) **5, no:6(40)**, 102-135. (First published by the AMS in 1953 as Trans. no:96, 332-372).

[**35**] Hadamard, J. (1914)

Sur le module maximum d'une fonction et de ses dérivées. C. R. Acad. Sci. Paris. **42**, 68-72.

[**36**] Hardy, G.H and Littlewood, J.E. (1932)

Some integral inequalities connected with the calculus of variations. Quart. J. Math. (2) **3**, 241-252.

[**37**] Hardy, G.H, Littlewood, J.E. and Pólya, G. (1994)

Inequalities. Cambridge Univ. Press. (First published in 1934 by Cambridge Univ. Press.).

[**38**] Hellinger, E. (1922)

Zur Stieltjesschen Kettenbruchtheorie. Math. Ann., **86**, 18-29.

[**39**] Hinton, D.B. and Lewis, R.T. (1978)
Spectral analysis of second-order difference equations. J. Math. Anal. and Appl. **63**, 421-438.

[**40**] Hyslop, J.M. (1959)
Infinite series. Oliver and Boyd, Edinburgh and London.

[**41**] Kaper, H.G. and Spellman, B.E. ((1987)
Best constants in norm inequalities for the difference operator. Transections of the AMS **Vol.299, no:1**, 351-372.

[**42**] Kato, T. (1971)
On an inequality of Hardy, Littlewood and Pólya. Adv. Math. **7**, 217-218.

[**43**] Kauffman, R.M., Read, T.T. and Zettl, A. (1977)
The deficiency index problem for powers of ordinary differential expressions. Berlin: Springer Verlag.

[**44**] Kirby, V.G. (1990)
A numerical method for determining the Titchmarsh-Weyl $m-$ coefficient and its applications to certain integro-differential inequalities. Ph.D. Thesis, University of Wales College Cardiff, UK.

[**45**] Kolmogorov, A.N. (1939)
On inequalities between upper bounds of consecutive derivatives of an arbitrary function defined on an infinite interval. Moskov. Gos. Univ. Učen. Zap. Matematika. **30**, 3-16.

[**46**] Kupcov, N.P. (1975)
Kolmogorov estimates for derivatives in $L^2[0,\infty)$. Proc. Steklov. inst. Mat. **138**, 101-125.

[**47**] Kwong, M.K. and Zettl, A. (1989)
Landau's inequality for the difference operator. Argonne National Laboratory, Mathematics and Comp. Sci. Division. USA, ANL-87-26, **Vol.3**, 127-134.

[**48**] Landau, E. (1913)

Einige ungleichungen für zweimal differentiierbare functionen. Proc. Lond. Math. Soc. (2) **13**, 43-49.

[**49**] Ljubič, Ju.I. (1960)

On inequalities between the powers of a linear operator. Amer. Math. Soc. Transl. (2) **40**, 39-84.

[**50**] Mingarelli, A.B. (1981)

A limit-point criterion for a three-term recurrence relation. C. R. Math. Rep. Acad. Sci. Canada **Vol.3, no:3**, 171-175.

[**51**] Naimark, M.A. (1968)

Linear differential operators, part 1&2. London: Harrap.

[**52**] Neta, B. (1980)

On determination of best-possible constants in integral inequalities involving derivatives. Mathematics of Computation. **35 152**, 1191-1193.

[**53**] Nevanlinna, R. (1922)

Asymptotische entwickelungen beschränkter functionen und das stieltjessche momentenproblem. Ann. Acad. Sci. Fenn. **A 18 (5)**, 1-52.

[**54**] Phóng, V.Q. (1981)

On inequalities for powers of linear operators and for quadratic forms. Proc. R. Soc. Edinb. **89A**, 25-50.

[**55**] Pleijel (1970)

Spectral theory for pairs of formally self-adjoint ordinary differential operators, J. Indian Math. Soc. **34**, 259-268.

[**56**] Pleijel, A. (1974)

A survey of spectral theory for pairs of ordinary differential operators. Spectral Theory and Differential Equations. (Proc. Symp. Dundee, 1974; dedicated to Konrad Jörgens), pp.256-272. Lecture Notes in Math., **Vol.448**, Springer, Berlin, 1975.

[57] Press, W.H., Flannery, B.P., Teukolsky, S.A. and Vetterling, W.T. (1994) Numerical Recipies; The art of scientific Computing. Cambridge University Press.

[58] Race, D. (1985)
On the strong limit-point and Dirichlet properties of second-order differential expressions. Proc. R. Soc. Edinb. **101A**, 283-296.

[59] Read, T.T. (1984)
On the essential self-adjointness of powers of Schrödinger operators. Proc. R. Soc. Edinb. **A 97**, 233-246.

[60] Russel, A. (1975)
Integral inequalities and spectral theory of certain ordinary differential equations. Ph.D. Thesis, University of Dundee.

[61] Russel, A. (1978)
On a fourth-order singular integral inequality. Proc. R. Soc. Edinb. **A 80**, 220-269.

[62] Titchmarsh, E.C. (1941)
On expansions in eigenfunctions (IV), Quart. J. of Math. (Oxford) **12**, 33-50.

[63] Titchmarsh, E.C. (1945)
On expansions in eigenfunctions (VII), Quart. J. of Math. (Oxford) **16**, 103-114.

[64] Titchmarsh, E.C. (1962)
Eigenfunction Expansions. Part 1, 2nd edn. Oxford University Press.

[65] Weidmann, J. (1980)
Spectral theory of ordinary differential operators. Berlin, Heidelberg, New York: Springer Verlag.

[66] Weyl, H. (1910)
Uber gewöhnliche differentialgleichungen mit singularitäten und die zugehörigen entwicklungen willkürlicher funktionen. Math. Ann. **68**, 220-269.